Nonconvex notions of regularity and convergence of fundamental algorithms for feasibility problems

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Abstract

We consider projection algorithms for solving (nonconvex) feasibility problems in Euclidean spaces. Of special interest are the Method of Alternating Projections (MAP) and the Douglas-Rachford algorithm (DR). In the case of convex feasibility, firm nonexpansiveness of projection mappings is a global property that yields global convergence of MAP and for consistent problems DR. A notion of local sub-firm nonexpansiveness with respect to the intersection is introduced for consistent feasibility problems. This, together with a coercivity condition that relates to the regularity of the collection of sets at points in the intersection, yields local linear convergence of MAP for a wide class of nonconvex problems, and even local linear convergence of nonconvex instances of the Douglas-Rachford algorithm.

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1 Introduction

In the last decade there has been significant progress in the understanding of convergence of algorithms for solving generalized equations for nonmonotone operators, and in particular those arising from variational problems such as minimization or maximization of functions, feasibility, variational inequalities and minimax problems. Early efforts focused on the proximal point algorithm [12,23,32] and notions of *(co)hypomonotonicity* which is closely related to *prox-regularity* of functions [33]. Other works have focused on *metric regularity* and its refinements [2,3,24]. Proximal-type algorithms have been studied for functions satisfying the Kurdyka-Lojasiewicz Inequality in [4]. In a more limited context, the method of alternating projections (MAP) has been investigated in [8,27] with the aim of formulating *dual* characterizations of regularity requirements for linear convergence via the *normal cone* and its variants.

The framework we present here generalizes the tools for the analysis of fixed-point iterations of operators that violate the classical property of firm nonexpansiveness in some quantifiable fashion. As such, our approach is more closely related to the ideas of [32] and hypomonotonicity through the resolvent mapping, however our application to MAP bears a direct resemblance to the more classical *primal* characterizations of regularity described in [5] (in particular what they call "linear regularity"). There are also some parallels between what we call (S, ε) -firm nonexpansiveness and ε -Enlargements of maximal monotone operators [11], though this is beyond the scope of this paper. Our goal is to introduce the essential tools we make use of with a cursory treatment of the connections to other concepts in the literature, and to apply these tools to the MAP and Douglas-Rachford algorithms, comparing our results to the best known results at this time.

We review the basic definitions and classical definitions and results below. In section 2 we introduce our relaxations of the notions of firm-nonexpansiveness and set regularity. Our main abstract result concerning fixed-point iterations of mappings that violate the classical firm-nonexpansive assumptions is in section 3. We specialize in subsequent subsections to MAP and the Douglas-Rachford algorithm. Our statement of linear convergence of MAP is as general as the results reported in [8], with more elementary proofs, although our estimates for the radius of convergence are more conservative. The results on local linear convergence of nonconvex instances of Douglas-Rachford are new and provide some evidence supporting the conjecture that, asymptotically, Douglas-Rachford converges more slowly (albeit still linearly) than simple alternating projections. Our estimates of the rate of convergence for both MAP and Douglas-Rachford are not optimal, but to our knowledge the most general to date.

We also show that strong regularity conditions on the collection of affine

sets are in fact *necessary* for linear convergence of iterates of the Douglas-Rachford algorithm to the intersection, in contrast to MAP where the same conditions are sufficient, but not necessary [8]. This may seem somewhat spurious to experts since, as is well-known, the Douglas-Rachford iterates themselves are not of interest, but rather their shadows or projections onto one of the sets [7, 28]. Indeed, in the convex setting where local is global, the shadows of the iterates of the Douglas-Rachford algorithm could converge even though the iterates themselves diverge. This happens in particular when the sets do not intersect, but have instead best approximation points [7, Theorem 3.13]. The nonconvex setting is much less forgiving, however. Indeed, existence of local best approximation points does not guarantee convergence of the shadows to best approximation points in the nonconvex setting [29], and so convergence of the sequence itself is essential. As nonconvex settings are our principal interest, we focus on convergence of the iterates of the Douglas-Rachford algorithm instead of the shadows, and in particular convergence of these iterates to the intersection of collections of sets. We leave a fuller examination of the shadow sequences to future work.

1.1 Basics/Notation

E is a Euclidean space. We denote the closed unit ball centered at the origin by \mathbb{B} and the ball of radius δ centered at $\overline{x} \in \mathbf{E}$ by $\mathbb{B}_{\delta}(\overline{x}) := \{x \in \mathbf{E} | ||x - \overline{x}|| \leq \delta\}$. When the δ -ball is centered at the origin we write \mathbb{B}_{δ} . The notation " \Rightarrow " indicates that this mapping in general is *multi-valued*. The composition of two *multi-valued* mappings T_1, T_2 is pointwise defined by $T_2 T_1 x = \bigcup_{y \in T_1 x} T_2 y$. A nonempty subset K of \mathbf{E} is a cone if, for all $\lambda > 0$, $\lambda K := \{\lambda k \mid k \in K\} \subseteq K$. The smallest cone containing a set $\Omega \subset \mathbf{E}$ is denoted cone(S).

Definition 1. Let $\Omega \subset \mathbf{E}$ be nonempty, $x \in \mathbf{E}$. The distance of x to Ω is defined by

$$d(x,\Omega) := \inf_{y \in \Omega} \|x - y\|.$$

$$(1.1)$$

Definition 2 (projectors/reflectors). Let $\Omega \subset \mathbf{E}$ be nonempty and $x \in \mathbf{E}$. The (possibly empty) set of all best approximation points from x to Ω denoted $P_{\Omega}(x)$ (or $P_{\Omega}x$), is given by

$$P_{\Omega}(x) := \{ y \in \Omega \mid ||x - y|| = d(x, \Omega) \}.$$
(1.2)

The mapping $P_{\Omega} \rightrightarrows \Omega$ is called the metric projector, or projector, onto Ω . We call an element of $P_{\Omega}(x)$ a projection. The reflector $R_{\Omega} : \mathbf{E} \rightrightarrows \mathbf{E}$ to the set Ω is defined as

$$R_{\Omega}x := 2P_{\Omega}x - x, \tag{1.3}$$

for all $x \in \mathbf{E}$.

Since we are on a Euclidean space \mathbf{E} convexity and closedness of a subset $C \subset \mathbf{E}$ is sufficient for the projector (respectively the reflector) to be single valued. Closedness of a set Ω suffices for the set Ω being *proximinal*, i.e. $P_C x \neq \emptyset$ for all $x \in \mathbf{E}$ (For a modern treatment see [6, Corollary 3.13].

Definition 3 (Method of Alternating Projections). For two sets $A, B \subset \mathbf{E}$ we call the mapping

$$T_{MAP}x = P_A P_B x \tag{1.4}$$

the Method of Alternating Projections operator. We call the MAP algorithm, or simply MAP, the corresponding Picard iteration,

$$x_{n+1} \in T_{MAP} x_n, \quad n = 0, 1, 2, \dots$$
 (1.5)

for x_0 given.

Definition 4 (Averaged Alternating Reflections/Douglas Rachford). For two sets $A, B \subset \mathbf{E}$ we call the mapping

$$T_{DR}x = \frac{1}{2} \left(R_A R_B x + x \right)$$
 (1.6)

the Douglas-Rachford operator. We call the Douglas-Rachford algorithm, or simply Douglas-Rachford, the corresponding Picard iteration,

$$x_{n+1} \in T_{DR}x_n, \quad n = 0, 1, 2, \dots$$
 (1.7)

for x_0 given.

Example 5. The following easy examples will appear throughout this work and serve to illustrate the regularity concepts we introduce and the convergence behavior of the algorithms under consideration.

(i) Two lines in \mathbb{R}^2 :

$$A = \{ (x_1, x_2) \in \mathbb{R}^2 \mid x_2 = 0 \} \subset \mathbb{R}^2$$
(1.8)

$$B = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = x_2\} \subset \mathbb{R}^2.$$
(1.9)

We will see that MAP and Douglas-Rachford converge with a linear rate to the intersection.

(ii) Two lines in \mathbb{R}^3 :

$$A = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 = 0, x_3 = 0 \} \subset \mathbb{R}^3$$
(1.10)

$$B = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = x_2, x_3 = 0 \right\} \subset \mathbb{R}^3.$$
(1.11)

After the first iteration step MAP shows exactly the same convergence behavior as in the first example. Douglas-Rachford does not converge to $\{0\} = A \cap B$. All iterates from starting points on the line $\{t(0, 0, 1) \mid t \in \mathbb{R}\}$ are fixed points of the Douglas Rachford operator. On the other hand, iterates from starting points in A + B stay in A + B, and the case then reduces to Example (i). (iii) A line and a ball intersecting in one point:

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$$A = \{ (x_1, x_2) \in \mathbb{R}^2 \mid x_2 = 0 \} \subset \mathbb{R}^2$$
(1.12)

$$B = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + (x_2 - 1)^2 \le 1 \right\}.$$
 (1.13)

MAP converges to the intersection, but not with a linear rate. Douglas-Rachford has fixed points that lie outside the intersection.

(iv) A cross and a subspace in \mathbb{R}^2 :

$$A = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R} \tag{1.14}$$

$$B = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = x_2\}.$$
 (1.15)

This example relates to the problem of sparse-signal recovery. Both MAP and Douglas-Rachford converge globally to the intersection $\{0\} = A \cap B$, though A is nonconvex. The convergence of both methods is covered by the theory built up in this work.

(v) A circle and a line:

$$A = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid x_2 = \sqrt{2}/2 \right\} \subset \mathbb{R}^2$$
 (1.16)

$$B = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1 \right\}.$$
(1.17)

This example is of our particular interest, since it is a simple model case of the *phase retrieval problem*. So far the only *direct* nonconvex convergence results for Douglas-Rachford are related to this model case, see [1, 10]. Local convergence of MAP is covered by [8, 27] as well as by the results in this work.

1.2 Classical results for (firmly) nonexpansive mappings

To begin, we recall (firmly) nonexpansive mappings and their natural association with projectors and reflectors on convex sets. We later extend this notion to nonconvex settings where the algorithms involve set-valued mappings.

Definition 6. Let $\Omega \subset \mathbf{E}$ be nonempty. $T : \Omega \to \mathbf{E}$ is called nonexpansive, if

$$||Tx - Ty|| \le ||x - y|| \tag{1.18}$$

holds for all $x, y \in \Omega$.

 $T: \Omega \to \mathbf{E}$ is called firmly nonexpansive, if

$$||Tx - Ty||^{2} + ||(\mathrm{Id} - T)x - (\mathrm{Id} - T)y||^{2} \le ||x - y||^{2}$$
(1.19)

holds for all $x, y \in \Omega$.

Lemma 7 (Proposition 4.2 [6]). Let $\Omega \subset \mathbf{E}$ be nonempty and let $T : \Omega \to \mathbf{E}$. The following are equivalent

- (i) T is firmly nonexpansive on Ω
- (ii) 2T Id is nonexpansive on Ω
- (iii) $||Tx Ty||^2 \le \langle Tx Ty, x y \rangle$ for all $x, y \in \Omega$

Theorem 8 (best approximation property - convex case). Let $C \subset \mathbf{E}$ be nonempty and convex, $x \in \mathbf{E}$ and $\overline{x} \in C$. \overline{x} is the best approximation point $\overline{x} = P_C(x)$ if and only if

$$\langle x - \overline{x}, y - \overline{x} \rangle \le 0 \quad \text{for all } y \in C.$$
 (1.20)

If C is a affine subspace then (1.20) holds with equality.

Proof. For (1.20) see Theorem 3.14 of [6], while equality follows from Corollary 3.20 of the same.

Theorem 9 ((firm) nonexpansiveness of projectors/reflectors). Let C be a closed, nonempty and convex set. The projector $P_C : \mathbf{E} \to \mathbf{E}$ is a firmly nonexpansive mapping and hence the reflector R_C is nonexpansive. If, in addition, C is an affine subspace then following conditions hold.

(i) P_C is firmly nonexpansive with equality, i.e.

$$||P_C x - P_C y||^2 + ||(\mathrm{Id} - P_C) x - (\mathrm{Id} - P_C) y||^2 = ||x - y||^2, \quad (1.21)$$

for all $x \in \mathbf{E}$.

(ii) For all $x \in \mathbf{E}$

$$||R_C x - c|| = ||x - c|| \tag{1.22}$$

holds for all $c \in C$.

Proof. For the first part of the statement see [15, Theorems 4.1 and 5.5], [19, Chapter 12], [20, Propositions 3.5 and 11.2] and [35, Lemma 1.1]. The well-known refinement for affine subspaces follows by a routine application of the definitions and Theorem 8. \Box

2 (S, ε) -firm nonexpansiveness

Up to this point, the results have concerned only convex sets, and hence the projector and related algorithms have all been single-valued. In what follows, we generalize to nonconvex sets and therefore allow multi-valuedness of the projectors.

Lemma 10. Let $A, B \subset \mathbf{E}$ be nonempty and closed. Let $x \in \mathbf{E}$. For any element $x_+ \in T_{DR}x$ there is a point $\tilde{x} \in R_A R_B x$ such that $x_+ = \frac{1}{2}(\tilde{x} + x)$. Moreover, T_{DR} satisfies the following properties.

(i)

$$\|x_{+} - y_{+}\|^{2} + \|(x - x_{+}) - (y - y_{+})\|^{2} = \frac{1}{2} \|x - y\|^{2} + \frac{1}{2} \|\tilde{x} - \tilde{y}\|^{2}$$
(2.1)

where x and y are elements of **E**, x_+ and y_+ are elements of $T_{DR}x$ and $T_{DR}y$ respectively, and $\tilde{x} \in R_A R_B x$ and $\tilde{y} \in R_A R_B y$ are the corresponding points satisfying $x_+ = \frac{1}{2}(\tilde{x} + x)$ and $y_+ = \frac{1}{2}(\tilde{y} + y)$.

(ii) For all $x \in \mathbf{E}$

$$T_{DR}x = \{P_A(2z - x) - z + x \mid z \in P_Bx\}.$$
 (2.2)

Proof. By Definition 4

$$x_+ \in T_{DR}x \tag{2.3}$$

$$\iff \qquad x_+ \in \frac{1}{2}(R_A R_B x + x) \tag{2.4}$$

$$\iff 2x_+ - x \in R_A R_B x. \tag{2.5}$$

Defining $\tilde{x} = 2x_+ - x$ yields $x_+ = \frac{1}{2}(\tilde{x} + x)$, where $\tilde{x} \in R_A R_B x$.

(i) For $x_+ \in T_{DR}x$ (respectively $y_+ \in T_{DR}y$) choose $\tilde{x} \in R_A R_B x$ (respectively \tilde{y}) such that $x_+ = (\tilde{x} + x)/2$ (respectively y_+). Then

$$\|x_{+} - y_{+}\|^{2} + \|(x - x_{+}) - (y - y_{+})\|^{2}$$
(2.6)

$$= \left\| \frac{1}{2}\tilde{x} + \frac{1}{2}x - \frac{1}{2}\tilde{y} - \frac{1}{2}y \right\|^{2} + \left\| \frac{1}{2}x - \frac{1}{2}\tilde{x} - \frac{1}{2}y + \frac{1}{2}\tilde{y} \right\|^{2}$$
(2.7)

$$= \frac{1}{2} \|x - y\|^{2} + \frac{1}{2} \|\tilde{x} - \tilde{y}\|^{2} + \frac{1}{2} \langle \tilde{x} - \tilde{y}, x - y \rangle - \frac{1}{2} \langle \tilde{x} - \tilde{y}, x - y \rangle$$
(2.8)

$$= \frac{1}{2} \|x - y\|^2 + \frac{1}{2} \|\tilde{x} - \tilde{y}\|^2.$$
(2.9)

(ii) This follows easily from the definitions. Indeed, we represent $v \in R_B x$ as v = 2z - x for $z \in P_B x$ so that

$$T_{DR}x = \left\{ \frac{1}{2} \left(R_A v + x \right) \middle| v \in R_B x \right\}$$
(2.10)

$$= \left\{ \frac{1}{2} \left(R_A (2z - x) + x \right) \ \middle| \ z \in P_B x \right\}, \tag{2.11}$$

$$= \left\{ \frac{1}{2} \left(2P_A(2z - x) - (2z - x) + x \right) \middle| z \in P_B x \right\}$$
(2.12)

$$= \{ P_A(2z - x) - z + x \mid z \in P_B x \}.$$
(2.13)

Remark 11. In the case where A and B are convex, then as a consequence of Lemma 10 i) and the fact that the reflector R_{Ω} onto a convex set Ω is nonexpansive, we recover the well-known fact that T_{DR} is firmly nonexpansive and (2.1) reduces to

$$|T_{DR}x - T_{DR}y||^{2} + ||(\mathrm{Id} - T_{DR})x - (\mathrm{Id} - T_{DR})y||^{2}$$

= $\frac{1}{2}||x - y||^{2} + \frac{1}{2}||R_{A}R_{B}x - R_{A}R_{B}y||^{2},$ (2.14)

while (2.2) reduces to

$$T_{DR}x = x + P_A R_B x - P_B x. (2.15)$$

We define next an analog to firm nonexpansiveness in the nonconvex case with respect to a set S.

Definition 12 ((S, ϵ)-(firmly-)nonexpansive mappings). Let D and S be nonempty subsets of \mathbf{E} and let T be a (multi-valued) mapping from D to \mathbf{E} .

i) T is called (S, ε) -nonexpansive on D if

$$\begin{aligned} \|x_{+} - \overline{x}_{+}\| &\leq \sqrt{1 + \varepsilon} \, \|x - \overline{x}\| \\ \forall x \in D, \ \forall \overline{x} \in S, \ \forall x_{+} \in Tx, \ \forall \overline{x}_{+} \in T\overline{x}. \end{aligned}$$
(2.16)

If 2.16 holds with $\epsilon = 0$ then we say that T is S-nonexpansive on D.

ii) T is called (S, ε) -firmly nonexpansive on D if

$$\begin{aligned} \|x_{+} - \overline{x}_{+}\|^{2} + \|(x - x_{+}) - (\overline{x} - \overline{x}_{+})\|^{2} &\leq (1 + \varepsilon) \|x - \overline{x}\|^{2} \\ \forall x \in D, \ \forall \overline{x} \in S, \ \forall x_{+} \in Tx, \ \forall \overline{x}_{+} \in T\overline{x}. \end{aligned}$$
(2.17)

If 2.17 holds with $\epsilon = 0$ then we say that T is S-firmly nonexpansive on D.

Note that, as with (firmly) nonexpansive mappings, the mapping T need not be a self-mapping from D to itself. In the special case where S =Fix T, mappings satisfying (2.16) are also called *quasi-(firmly-)nonexpansive* [6]. Quasi-nonexpansiveness is a restriction of another well-known concept, Fejér monotonicity, to Fix T. Equation (2.17) is a relaxed version of firm nonexpansiveness (1.19). The aim of this work is to expand the theory for projection methods (and in particular MAP and Douglas-Rachford) to the setting where one (or more) of the sets are nonconvex. The classical (firmly) nonexpansive operator on D is (D, 0)-(firmly) nonexpansive on D.

Analogous to the relation between firmly nonexpansive mappings and averaged mappings (see [6, Chapter 4] and references therein) we have the following relationship between (S, ε) -firmly nonexpansive mappings and their 1/2-averaged companion mapping. **Lemma 13** (1/2-averaged mappings). Let $D, S \subset \mathbf{E}$ be nonempty and $T : D \rightrightarrows \mathbf{E}$. The following are equivalent

- (i) T is (S, ε) -firmly nonexpansive on D.
- (ii) The mapping $\widetilde{T}: D \rightrightarrows \mathbf{E}$ given by

$$\widetilde{T}x := (2Tx - x) \quad \forall x \in D \tag{2.18}$$

is $(S, 2\varepsilon)$ -nonexpansive on D, i.e. T can be written as

$$Tx = \frac{1}{2} \left(x + \widetilde{T}x \right) \quad \forall x \in D.$$
(2.19)

Proof. For $x \in D$ choose $x_+ \in Tx$. Observe that, by the definition of \widetilde{T} , there is a corresponding $\tilde{x} \in \widetilde{T}x$ such that $x_+ = \frac{1}{2}(x + \tilde{x})$, which is just formula (2.19). Let z be any point in S and select any $z_+ \in Tz$. Then

$$\|x_{+} - z_{+}\|^{2} + \|x - x_{+} - (z - z_{+})\|^{2}$$
(2.20)

$$= \left\| \frac{1}{2} (x + \tilde{x}) - \frac{1}{2} (z + \tilde{z}) \right\|^{2} + \left\| \frac{1}{2} (x - \tilde{x}) - \frac{1}{2} (z - \tilde{z}) \right\|^{2}$$
(2.21)

$$= \frac{1}{4} \left[\|x - z\|^{2} + 2\langle x - z, \tilde{x} - \tilde{z} \rangle + \|\tilde{x} - \tilde{z}\|^{2} \right]$$
(2.22)

$$+\frac{1}{4}\left[\|x-z\|^{2}-2\langle x-z,\tilde{x}-\tilde{z}\rangle+\|\tilde{x}-\tilde{z}\|^{2}\right]$$
(2.23)

$$= \frac{1}{2} \|x - z\|^2 + \frac{1}{2} \|\tilde{x} - \tilde{z}\|^2$$
(2.24)

$$\leq \frac{1}{2} \|x - z\|^2 + \frac{1}{2}(1 + 2\varepsilon) \|x - z\|^2$$
(2.25)

$$= (1+\varepsilon) \|x-z\|^2 \tag{2.26}$$

where the inequality holds if and only if \widetilde{T} is $(S, 2\varepsilon)$ -nonexpansive. By definition, it then holds that T is (S, ϵ) -firmly nonexpansive if and only if \widetilde{T} is $(S, 2\varepsilon)$ -nonexpansive, as claimed.

We state the following theorem to suggest that the framework presented in this work can be extended to a more general setting, for example the adaptive framework discussed in [5]. It shows that the (S, ε) -firm nonexpansiveness is preserved under convex combination of operators.

Theorem 14. Let T_1 be (S, ε_1) -firmly nonexpansive and T_2 be (S, ε_2) -firmly nonexpansive on D. The convex combination $\lambda T_1 + (1 - \lambda)T_2$ is (S, ε) -firmly nonexpansive on D where $\varepsilon = \max{\varepsilon_1, \varepsilon_2}$.

Proof. Let $x, y \in D$. Let

 y_+

$$x_+ \in \lambda T_1 x + (1 - \lambda) T_2 x$$
, and (2.27)

$$\in \lambda T_1 y + (1 - \lambda) T_2 y, \qquad (2.28)$$

$$\Rightarrow \quad x_{+} = \lambda x_{+}^{(1)} + (1 - \lambda) x_{+}^{(2)}, \quad \text{where } x_{+}^{(1)} \in T_{1}x, \ x_{+}^{(2)} \in T_{2}x \tag{2.29}$$

$$y_{+} = \lambda y_{+}^{(1)} + (1 - \lambda) y_{+}^{(2)}, \text{ where } y_{+}^{(1)} \in T_{1}y, \ y_{+}^{(2)} \in T_{2}y.$$
 (2.30)

By Lemma 13 (ii) one has nonexpansiveness of the mappings given by $2T_1x - x$ and $2T_2x - x$, $x \in D$ that is

$$\left\| \left[2x_{+}^{(1)} - x \right] - \left[2y_{+}^{(1)} - y \right] \right\| \le \sqrt{1 + 2\varepsilon_{1}} \|x - y\|, \qquad (2.31)$$

$$\left\| \left[2x_{+}^{(2)} - x \right] - \left[2y_{+}^{(2)} - y \right] \right\| \leq \sqrt{1 + 2\varepsilon_2} \, \|x - y\| \,. \tag{2.32}$$

This implies

$$\|(2x_{+} - x) - (2y_{+} - y)\|$$

$$\|(2.33)$$

$$= \left\| \left(2 \left[\lambda x_{+}^{(1)} + (1 - \lambda) x_{+}^{(2)} \right] - x \right) - \left(2 \left[\lambda y_{+}^{(1)} + (1 - \lambda) y_{+}^{(2)} \right] - y \right) \right\|$$
(2.34)
$$= \left\| \lambda \left(\left[2x_{+}^{(1)} - x \right] - \left[2y_{+}^{(1)} - y \right] \right) - (1 - \lambda) \left(\left[2x_{+}^{(2)} - x \right] - \left[2y_{+}^{(2)} - y \right] \right) \right\|$$
(2.35)

$$\leq \lambda \left\| \left[2x_{+}^{(1)} - x \right] - \left[2y_{+}^{(1)} - y \right] \right\| + (1 - \lambda) \left\| \left[2x_{+}^{(2)} - x \right] - \left[2y_{+}^{(2)} - y \right] \right\|$$
(2.36)

$$\leq \sqrt{1 + 2\varepsilon} \left\| x - y \right\|.$$
(2.37)

Now using Lemma 13 (i) the proof is complete.

2.1 Regularity of Sets

To assure property (2.17) for the projector and the Douglas-Rachford operator, we determine the inheritance of the regularity of the projector and reflectors from the regularity of the sets A and B upon which we project. We begin with some established notions of set regularity and introduce a new, weaker form that will be central to our analysis.

Definition 15 (Prox-regularity). A nonempty (locally) closed set $\Omega \subset \mathbf{E}$ is prox-regular at a point $\overline{x} \in \Omega$ if the projector P_{Ω} is single-valued around \overline{x} .

What we take as the definition of prox-regularity actually follows from the equivalence of prox-regularity of sets as defined in [33, Definition 1.1] and the single-valuedness of the projection operator on neighborhoods of the set [33, Theorem 1.3].

Definition 16 (normal cones). The proximal normal cone $N_{\Omega}^{P}(\overline{x})$ to a set $\Omega \subset \mathbf{E}$ at a point $\overline{x} \in \Omega$ is defined by

$$N_{\Omega}^{P}(\overline{x}) := \operatorname{cone}(P_{\Omega}^{-1}(\overline{x}) - \overline{x}).$$
(2.38)

The the limiting normal cone $N_{\Omega}(\overline{x})$ is defined as any vector that can be written as the limit of proximal normals; that is, $\overline{v} \in N_{\Omega}(\overline{x})$ if and only if there exist sequences $(x_k)_{k\in\mathbb{N}}$ in Ω and $(v_k)_{k\in\mathbb{N}}$ in $N_{\Omega}^P(x_k)$ such that $x_k \to \overline{x}$ and $v_k \to \overline{v}$. The construction of the *limiting normal cone* goes back to Mordukhovich (see [34, Chap. 6 Commentary]).

Proposition 17 (Mordukhovich). *The* limiting normal cone *or* Mordukhovich normal cone *is the smallest cone satisfying the two properties*

- 1. $P_{\Omega}^{-1}(\overline{x}) \subseteq (I + N_{\Omega})(\overline{x})$ where $P_{\Omega}^{-1}(\overline{x})$ is the preimage set of \overline{x} under P_{Ω} .
- 2. for any sequence $x_i \to \overline{x}$ in Ω any limit of a sequence of normals $v_i \in N_{\Omega}(x_i)$ must lie in $N_{\Omega}(\overline{x})$.

Definition 18 ((ε , δ)-(sub)regularity).

i) A nonempty set $\Omega \subset \mathbf{E}$ is (ε, δ) -subregular at \hat{x} with respect to $S \subset \mathbf{E}$ if

$$\langle v_x, \overline{x} - x \rangle \le \varepsilon \|v_x\| \|\overline{x} - x\| \tag{2.39}$$

holds for all $x \in \mathbb{B}_{\delta}(\hat{x}) \cap \Omega$, $\overline{x} \in S \cap \mathbb{B}_{\delta}(\hat{x})$, $v_x \in N_{\Omega}^P(x)$. We simply say Ω is (ε, δ) -subregular at \hat{x} if $S = \{\hat{x}\}$.

- ii) If $S = \Omega$ in i) then we say that the set Ω is (ε, δ) -regular at \hat{x} .
- iii) If for all $\epsilon > 0$ there exists a $\delta > 0$ such that (2.39) holds for all $x, \overline{x} \in \mathbb{B}_{\delta}(\overline{x}) \cap \Omega$ and $v_x \in N_{\Omega}(x)$, then Ω is said to be super-regular.

The definition of (ε, δ) -regularity was introduced in [9, Definition 9.1] and is a generalization of the notion of super-regularity introduced in [27, Definition 4.3]. More details to (ε, δ) -regularity can be seen in [9]. Of particular interest is the following proposition. Preparatory to this, we remind readers of another well-known type of regularity, *Clarke regularity*. To avoid introducing the Féchet normal which is conventionally used to define Clarke regularity, we follow [27] which uses proximal normals.

Definition 19 (Clarke regularity). A nonempty (locally) closed set $\Omega \subset \mathbf{E}$ is Clarke regular at a point $\overline{x} \in \Omega$ if, for all $\varepsilon > 0$, any two points u, z close enough to \overline{x} with $z \in \Omega$, and any point $y \in P_{\Omega}(u)$, satisfy $\langle z - \overline{x}, u - y \rangle \leq \varepsilon ||z - \overline{x}|| ||u - y||$.

Proposition 20 (Prox-regular implies super-regular, Proposition 4.9 of [27]). If a closed set $\Omega \subset \mathbf{E}$ is prox-regular at a point in Ω , then it is super-regular at that point. If a closed set $\Omega \subset \mathbf{E}$ is super-regular at a point in Ω , then it is Clarke regular at that point.

Super-regularity is something between Clarke regularity and amenability or proxregularity. (ϵ, δ) -regularity is weaker still than Clarke regularity (and hence super-regularity) as the next example shows.

Remark 21. (ϵ, δ) -regularity does not imply Clarke regularity

Proof.

$$\Omega := \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid \begin{array}{ccc} x_2 \leq -x_1 & \text{if } x_1 \leq 0 \\ x_2 \leq 0 & \text{if } x_1 > 0 \end{array} \right\}.$$
 (2.40)

For any $x_{-} \in \partial_{-}\Omega := \{(x_{1}, x_{2}) \mid x_{2} = -x_{1}, x_{1} < 0\}$ and $x_{+} \in \partial_{+}\Omega := \{(x_{1}, 0) \mid x_{1} > 0\}$ one has

$$N_{\Omega}(x_{-}) = \left\{ (\lambda, \lambda) \mid \lambda \in \mathbb{R}^{+} \right\}$$

$$(2.41)$$

$$N_{\Omega}(x_{+}) = \left\{ (0,\lambda) \mid \lambda \in \mathbb{R}^{+} \right\}$$

$$(2.42)$$

which implies $N_{\Omega}(0) = N_{\Omega}(x_{-}) \cup N_{\Omega}(x_{+})$. Since $N_{\Omega}^{P}(0) = \{0\}$ the set Ω is not Clarke regular at 0. Define $\nu_{-} = (\sqrt{2}/2, \sqrt{2}/2) \in N_{\Omega}(x_{-}), \nu_{+} = (0, 1) \in N_{\Omega}(x_{+})$ and note

$$\langle \nu_{-}, 0 - x_{-} \rangle = 0 \text{ and } \langle \nu_{+}, 0 - x_{+} \rangle = 0$$
 (2.43)

to show that Ω is $(0, \infty)$ -subregular at 0. By the use of these two inequalities one now has

$$\langle \nu_{-}, x_{+} - x_{-} \rangle = \langle \nu_{-}, x_{+} \rangle \le \sqrt{2}/2 \, \|x_{+}\|$$
 (2.44)

$$\langle \nu_+, x_- - x_+ \rangle = \langle \nu_+, x_- \rangle \le \sqrt{2}/2 \, \|x_-\|$$
 (2.45)

and hence Ω is $(\frac{\sqrt{2}}{2}, \infty)$ -regular.

Remark 22 (Example 5 iv) revisited). The set

$$A := \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R} \tag{2.46}$$

is a particularly easy pathological set that illustrates the distinction between our new notion of subregularity and previous notions found in the literature. Note that for $x_1 \in \mathbb{R} \times \{0\}$, $N_A(x_1) = N_A^P(x_1) = \{0\} \times \mathbb{R}$ and for for $x_2 \in \{0\} \times \mathbb{R}$, $N_A(x_2) = N_A^P(x_2) = \mathbb{R} \times \{0\}$ and that $N_A(0) = A$ and $N_A^P(0) = 0$ which implies that at the origin A is not Clarke regular and therefore neither super-regular nor prox-regular there. In fact, it is not even (ϵ, δ) -regular at the origin for any $\epsilon < 1$ and any $\delta > 0$. The set A is, however, $(0, \infty)$ -subregular at $\{0\}$. Indeed, for any $x_1 \in \mathbb{R} \times \{0\}$ one has $\nu_1 \in N_A(x_1) = \{0\} \times \mathbb{R}$ and therefore $\langle \nu_1, x_1 - 0 \rangle = 0$. Analogously for $x_2 \in \{0\} \times \mathbb{R}$, $\nu_2 \in N_A(x_2) = \mathbb{R} \times \{0\}$ and it follows that $\langle \nu_2, x_2 - 0 \rangle = 0$, which shows that A is $(0, \infty)$ -subregular at 0. \Box

2.2 Projectors and Reflectors

We show in this section how (S, ε) (firm)-nonexpansiveness of projectors and reflectors is a consequence of (sub)regularity of the underlying sets.

Theorem 23 (projectors and reflectors onto (ε, δ) -subregular sets). Let $\Omega \subset$ **E** be nonempty closed and (ε, δ) -subregular at \hat{x} with respect to $S \subseteq \Omega \cap \mathbb{B}_{\delta}(\hat{x})$ and define

$$U := \{ x \in \mathbf{E} \mid P_{\Omega} x \subset \mathbb{B}_{\delta}(\hat{x}) \}.$$
(2.47)

(i) The projector P_{Ω} is $(S, \tilde{\varepsilon}_1)$ -nonexpansive on U, that is, $(\forall x \in U) \ (\forall x_+ \in P_{\Omega}x) \ (\forall \overline{x} \in S)$

$$\|x_{+} - \overline{x}\| \le \sqrt{1 + \tilde{\varepsilon}_{1}} \, \|x - \overline{x}\| \tag{2.48}$$

where $\tilde{\varepsilon}_1 := 2\varepsilon + \varepsilon^2$.

(ii) The projector P_{Ω} is $(S, \tilde{\varepsilon}_2)$ -firmly nonexpansive on $\mathbb{B}_{\delta}(\hat{x})$, that is, $(\forall x \in U) \ (\forall x_+ \in P_{\Omega}x) \ (\forall \overline{x} \in S)$

$$\|x_{+} - \overline{x}\|^{2} + \|x - x_{+}\|^{2} \le (1 + \tilde{\varepsilon}_{2}) \|x - \overline{x}\|^{2}, \qquad (2.49)$$

where $\tilde{\varepsilon}_2 := 2\varepsilon + 2\varepsilon^2$.

(iii) The reflector R_{Ω} is $(S, \tilde{\varepsilon}_3)$ -nonexpansive on $\mathbb{B}_{\delta}(\hat{x})$, that is, $(\forall x \in U) \ (\forall x_+ \in R_{\Omega}x) \ (\forall \overline{x} \in S)$

$$\|x_{+} - \overline{x}\| \le \sqrt{1 + \tilde{\varepsilon}_{3}} \|x - \overline{x}\|, \qquad (2.50)$$

where $\tilde{\varepsilon}_3 := 4\varepsilon + 4\varepsilon^2$.

Proof. (i) The projector is nonempty since Ω is closed. Then by the Cauchy-Schwarz inequality

$$|x_{+} - \overline{x}||^{2} = \langle x - \overline{x}, x_{+} - \overline{x} \rangle + \langle x_{+} - x, x_{+} - \overline{x} \rangle$$
(2.51)

$$\leq \|x - \overline{x}\| \|x_{+} - \overline{x}\| + \langle x_{+} - x, x_{+} - \overline{x} \rangle.$$
 (2.52)

Now for $x \in U$ we have also that $x_+ \in \mathbb{B}_{\delta}(\hat{x})$ and thus, by the definition of (ε, δ) -subregularity with respect to S, $(\forall x \in U)$ $(\forall x_+ \in P_{\Omega}x)$ $(\forall \overline{x} \in S)$

$$\langle x_{+} - x, x_{+} - \overline{x} \rangle \leq \varepsilon \| x - x_{+} \| \| x_{+} - \overline{x} \|$$

$$(2.53)$$

$$\leq \varepsilon \|x - \overline{x}\| \|x_{+} - \overline{x}\|. \tag{2.54}$$

Combining this with (2.52) yields $(\forall x \in U) \ (\forall x_+ \in P_\Omega x) \ (\forall \overline{x} \in S)$

$$|x_{+} - \overline{x}|| \le (1 + \varepsilon) ||x - \overline{x}|| \tag{2.55}$$

$$=\sqrt{1+(2\varepsilon+\varepsilon^2)} \|x-\overline{x}\| \tag{2.56}$$

(2.57)

as claimed.

(ii) Expanding and rearranging the norm yields $(\forall x \in U) (\forall x_+ \in P_{\Omega}x) (\forall \overline{x} \in S)$

$$\begin{aligned} \|x_{+} - \overline{x}\|^{2} + \|x - x_{+}\|^{2} \\ &= \|x_{+} - \overline{x}\|^{2} + \|x - \overline{x} + \overline{x} - x_{+}\|^{2} \\ &= \|x_{+} - \overline{x}\|^{2} + \|x - \overline{x}\|^{2} + 2\langle x - \overline{x}, \overline{x} - x_{+} \rangle + \|x_{+} - \overline{x}\|^{2} \\ &= 2\|x_{+} - \overline{x}\|^{2} + \|x - \overline{x}\|^{2} + 2\langle \underline{x_{+} - \overline{x}, \overline{x} - x_{+} \rangle}_{= -\|x_{+} - \overline{x}\|^{2}} + 2\langle \underline{x_{-} - x_{+}, \overline{x} - x_{+} \rangle}_{\leq \varepsilon \|x - x_{+} \| \|x_{+} - \overline{x}\|} \\ &\leq \|x - \overline{x}\|^{2} + 2\varepsilon \|x_{+} - \overline{x}\| \|x - x_{+}\| \end{aligned}$$
(2.58)

where the last inequality follows from the definition of (ε, δ) -subregularity with respect to S. By definition, $||x - x_+|| = d(x, \Omega) \le ||x - \overline{x}||$. Combining (2.58) and equation (2.48) yields $(\forall x \in U) \ (\forall x_+ \in P_\Omega x) \ (\forall \overline{x} \in S)$

$$\|x_{+} - \overline{x}\|^{2} + \|x - x_{+}\|^{2} \le (1 + 2\varepsilon (1 + \varepsilon)) \|x - \overline{x}\|^{2}.$$
 (2.59)

(iii) By (ii) the projector is $(S, 2\varepsilon + 2\varepsilon^2)$ -firmly nonexpansive on U, and so by Lemma 13 (ii) $R_{\Omega} = 2P_{\Omega} - \text{Id}$ is $(S, 4\varepsilon + 4\varepsilon^2)$ -nonexpansive on U. This completes the proof.

Note that $\tilde{\varepsilon}_1 < \tilde{\varepsilon}_2$ ($\varepsilon > 0$) in the above theorem, in other words, the *degree* to which classical firm nonexpansiveness is violated is greater than the degree to which classical nonexpansiveness is violated. This is as one would expect since firm nonexpansiveness is a stronger property than nonexpansiveness.

We can now characterize the degree to which the Douglas-Rachford operator violates firm-nonexpansiveness on neighborhoods of (ε, δ) -subregular sets.

Theorem 24 ($(S, \tilde{\varepsilon})$ -firm nonexpansiveness of T_{DR}). Let $A, B \subset \mathbf{E}$ be closed and nonempty. Let A and B be (ε_A, δ) - and (ε_B, δ) -subregular respectively at \hat{x} with respect to $S \subset \mathbb{B}_{\delta}(\hat{x}) \cap (A \cap B)$. Let $T_{DR} : \mathbf{E} \rightrightarrows \mathbf{E}$ be the Douglas-Rachford operator defined by (1.6) and define

$$U := \{ z \in \mathbf{E} \mid P_B z \subset \mathbb{B}_{\delta}(\hat{x}) \text{ and } P_A R_B z \subset \mathbb{B}_{\delta}(\hat{x}) \}.$$
(2.60)

Then T_{DR} is $(S, \tilde{\varepsilon})$ -firmly nonexpansive on U where

$$\tilde{\varepsilon} = 2\varepsilon_A(1+\varepsilon_A) + 2\varepsilon_B(1+\varepsilon_B) + 8\varepsilon_A(1+\varepsilon_A)\varepsilon_B(1+\varepsilon_B).$$
(2.61)

That is, $(\forall x \in U) \ (\forall x_+ \in T_{DR}x) \ (\forall \overline{x} \in S)$

$$\|x_{+} - \overline{x}\|^{2} + \|x - x_{+}\|^{2} \le (1 + \tilde{\varepsilon}) \|x - \overline{x}\|^{2}.$$
(2.62)

Proof. Define $U_A := \{z \mid P_A z \subset \mathbb{B}_{\delta}(\hat{x})\}$. By Theorem 23(iii) $(\forall y \in U_A) \ (\forall \tilde{x} \in R_A y) \ (\forall \overline{x} \in S)$

$$\|\tilde{x} - \overline{x}\| \le \sqrt{1 + 4\varepsilon_A (1 + \varepsilon_A)} \|y - \overline{x}\|.$$
(2.63)

Similarly, define $U_B := \{z \mid P_B z \subset \mathbb{B}_{\delta}(\hat{x})\}$ and again apply Theorem 23(iii) to get $(\forall x \in U_B) \ (\forall y \in R_B x) \ (\forall \overline{x} \in S)$

$$\|y - \overline{x}\| \le \sqrt{1 + 4\varepsilon_B (1 + \varepsilon_B)} \|x - \overline{x}\|.$$
(2.64)

Now, we choose any $x \in U_B$ such that $R_B x \in U_A$, that is $x \in U$, so that we can combine (2.63)-2.64 to get $(\forall x \in U)$ $(\forall \tilde{x} \in R_A R_B x)$ $(\forall \overline{x} \in S)$

$$\|\tilde{x} - \overline{x}\| \le \sqrt{1 + 4\varepsilon_A(1 + \varepsilon_A)}\sqrt{1 + 4\varepsilon_B(1 + \varepsilon_B)} \|x - \overline{x}\| = \sqrt{1 + 2\tilde{\varepsilon}} \|x - \overline{x}\|.$$
(2.65)

Note that $R_A R_B \overline{x} = R_B \overline{x} = \overline{x}$ since $\overline{x} \in A \cap B$, so (2.65) says that the operator $\widetilde{T} := R_A R_B$ is $(S, \tilde{\varepsilon})$ -nonexpansive on U. Hence by Lemma 13 $T_{DR} = \frac{1}{2} \left(\widetilde{T} + I \right)$ is $(S, 2\tilde{\varepsilon})$ -firmly nonexpansive on U, as claimed.

If one of the sets above is convex, say B for instance, the constant $\tilde{\varepsilon}$ simplifies to $\tilde{\varepsilon} = 2\varepsilon_A(1 + \varepsilon_A)$ since B is $(0, \infty)$ -subregular at \overline{x} .

3 Linear Convergence of Iterated (S, ε) -firmly nonexpansive Operators

Our goal in this section is to establish the weakest conditions we can (at the moment) under which the MAP and Douglas-Rachford algorithms converge locally linearly. The notions of regularity developed in the previous section are necessary, but not sufficient. In addition to regularity of the operators, we need regularity of the fixed point sets of the operators. This is developed next.

Despite its simplicity, the following Lemma is one of our fundamental tools.

Lemma 25. Let $D \subset \mathbf{E}$, $S \subset \operatorname{Fix} T$, $T : D \rightrightarrows \mathbf{E}$ and $U \subset D$. If

- (a) T is (S, ε) -firmly nonexpansive on U and
- (b) for some $\lambda > 0$, T satisfies the coercivity condition

$$\|x - x_{+}\| \ge \lambda d(x, S) \quad \forall x_{+} \in Tx, \ \forall x \in U.$$

$$(3.1)$$

Then

$$d(x_{+},S) \leq \sqrt{(1+\varepsilon-\lambda^{2})} d(x,S) \quad \forall x_{+} \in Tx, \ \forall x \in U.$$
(3.2)

Proof. For $x \in U$ choose any $x_+ \in Tx$, and define $\overline{x} := P_S x$. Combining equations (3.1) and (2.17) yields

$$\|x_{+} - \overline{x}\|^{2} + (\lambda \|x - \overline{x}\|)^{2} \stackrel{b)}{\leq}$$

$$(3.3)$$

$$\|x_{+} - \overline{x}\|^{2} + \|x - x_{+}\|^{2} \stackrel{a)}{\leq} (1 + \varepsilon) \|x - \overline{x}\|^{2}, \qquad (3.4)$$

which immediately yields

$$\|x_{+} - \overline{x}\|^{2} \leq (1 + \varepsilon - \lambda^{2}) \|x - \overline{x}\|^{2}.$$
(3.5)

Since $\overline{x} \in S$ by definition one has $d(x_+, S) \leq ||x_+ - \overline{x}||$. Inserting this in (3.5) and using the fact $||x - \overline{x}|| = d(x, S)$ then proves (3.2).

3.1 Regularity of Intersections of Collections of Sets

To this point, we have shown how the regularity of sets translates to the degree of violation of (firm) nonexpansiveness of projection-based fixed point mappings. What remains is to develop sufficient conditions for guaranteeing (3.1). For this we define a new notion of regularity of collections of sets which generalizes through localization two well-known concepts. The first concept, which we call *strong regularity* of the collection, has many different names in the literature, among them *linear regularity* [27]. We will use the term *linear regularity* of the collection to denote the second key concept upon which we build. Our generalization is called *local linear regularity*. Both terms "strong" and "linear" are overused in the literature but we have attempted, at the risk of some confusion, to conform to the usage that best indicates the heritage of the ideas.

Definition 26 (strong regularity, Kruger [25]). A collection of m closed, nonempty sets $\Omega_1, \Omega_2, \ldots, \Omega_m$ is strongly regular at \overline{x} if there exists an $\alpha > 0$ and a $\delta > 0$ such that

$$\left(\cap_{i=1}^{m} (\Omega_i - \omega_i - a_i)\right) \cap \mathbb{B}_{\rho} \neq \emptyset \tag{3.6}$$

for all $\rho \in (0, \delta]$, $\omega_i \in \Omega_i \cap \mathbb{B}_{\delta}(\overline{x})$, $a_i \in B_{\alpha\rho}$, $i = 1, 2, \dots, m$.

Theorem 27 (Theorem 1 [26]). A collection of closed, nonempty sets Ω_1 , $\Omega_2, \ldots, \Omega_m$ is strongly regular at \overline{x} if and only if there exists a $\kappa > 0$ and a $\delta > 0$ such that

$$d\left(x, \bigcap_{j=1}^{m} (\Omega_j - x_j)\right) \le \kappa \max_{i=1,\dots,m} d\left(x + x_i, \Omega_i\right), \quad \forall x \in \mathbb{B}_{\delta}(\overline{x}), \tag{3.7}$$

for all $x \in \mathbb{B}_{\delta}(\overline{x}), x_i \in \mathbb{B}_{\delta}, i = 1, \dots, m$.

Theorem 28 (Theorem 1 [25]). A collection of closed sets $\Omega_1, \Omega_2, \ldots, \Omega_m \subset \mathbf{E}$ is strongly regular (3.6) at a point $\overline{x} \in \cap_i \Omega_i$, if the only solution to the system

$$\sum_{i=1}^{m} v_i = 0, \quad \text{with } v_i \in N_{\Omega_i}(\overline{x}) \quad \text{for } i = 1, 2, \dots, m \quad (3.8)$$

is $v_i = 0$ for i = 1, 2, ..., m. For two sets $\Omega_1, \Omega_2 \subset \mathbf{E}$ this can be written as

$$N_{\Omega_1}(\overline{x}) \cap -N_{\Omega_2}(\overline{x}) = \{0\},\tag{3.9}$$

and is equivalent to the previous Definition (3.6).

Definition 29 (linear regularity). A collection of closed, nonempty sets Ω_1 , $\Omega_2, \ldots, \Omega_m$ is locally linearly regular at $\hat{x} \in \bigcap_{j=1}^m \Omega_j$ on $\mathbb{B}_{\delta}(\hat{x})$ ($\delta > 0$) if there exists a $\kappa > 0$ such that, for all $x \in \mathbb{B}_{\delta}(\hat{x})$,

$$d\left(x, \bigcap_{j=1}^{m} \Omega_{j}\right) \le \kappa \max_{i=1,\dots,m} d\left(x, \Omega_{i}\right).$$
(3.10)

The infimum over all κ such that (3.10) holds is called regularity modulus. If there is a $\kappa > 0$ such that (3.10) holds for all $\delta > 0$ (that is, for all $x \in \mathbf{E}$) the collection of sets is called linearly regular at \hat{x} .

Remark 30. Since (3.10) is (3.7) with $x_j = 0$ for all j = 1, 2, ..., m, it is clear that strong regularity implies local linear regularity (for some $\delta > 0$) and is indeed a much more restrictive notion than local linear regularity. What we are calling local linear regularity at \hat{x} has appeared in various forms elsewhere. See for instance [22, Proposition 4], [31, Section 3], and [26, Equation (15)]. Compare this to (bounded) linear regularity defined in [5, Definition 5.6]. Also compare this to the basic constraint qualification for sets in [30, Definition 3.2] and strong regularity of the collection in [26, Proposition 2], also called *linear* regularity in [27].

Remark 31. Based on strong regularity (more specifically, characterization (3.9)) Lewis, Luke and Malick proved local linear convergence of MAP in the nonconvex setting, where both sets A, B are closed and one of the sets is superregular [27]. This was refined later in [8]. The proof of convergence that will be given in this work is different from the one used in [8,27] and more related to the one in [5]. Convergence is achieved using (local) linear regularity (3.10), which is described in [16, Theorem 4.5] as the precise property equivalent to uniform linear convergence of the CPA (Cyclic projections algorithms). However the rate of convergence achieved by the use of linear regularity is not optimal, while the one in [8,27] is in some instances. An adequate description of the relation between the direct/primal techniques used here and the dual approach used in [8,27] is a topic of future research.

Theorem 32 (linear regularity of collections of convex cones). Let Ω_1 , Ω_2 , ..., Ω_m be a collection of closed, nonempty, convex cones. The following statements are equivalent

(i) There is a $\delta > 0$ such that the collection is locally linearly regular at $\hat{x} \in \bigcap_{j=1}^{m} \Omega_j$ on $\mathbb{B}_{\delta}(\hat{x})$.

Π

(ii) The collection is linearly regular at $\hat{x} \in \bigcap_{j=1}^{m} \Omega_j$

Proof. [5, Proposition 5.9]

Example 33 (Example 5 revisited.). The collection of sets in example 5 (i) is strongly regular at 0 ($c = \sqrt{2}/2$) and linearly regular ($\kappa = \sqrt{2}/4$). The same collection of sets embedded in a higher-dimensional space is still linearly regular, but looses its strong regularity. This can be seen by shifting one of the sets in example 5 (ii) in x_3 -direction, as this renders the intersection empty. This shows that *linear regularity does not imply strong regularity*. The collection of sets in example (iii) is neither strongly regular nor linearly regular. The collection of sets in Example 5 (iv) is strongly regular at the intersection. One has $N_B(0) = \{(\lambda, -\lambda) | \lambda \in \mathbb{R}\}$ and by Remark 22 $N_A(0) = A$ and this directly shows $N_A(0) \cap -N_B(0) = \{0\}$. In example 5 (v) one of the sets is nonconvex, but the collection of sets is still well-behaved in the sense that it is both strongly and linearly regular. It is worth emphasizing, however, that the set A in Example 5 (iv) is not Clarke regular at the origin. This illustrates the fact that collections of classically "irregular" sets can still be quite regular at points of intersection.

3.2 Linear Convergence of MAP

In the case of the MAP operator, the connection between local linear regularity of the collection of sets and the coercivity of the operator with respect to the intersection is natural, as the next result shows.

Proposition 34 (coercivity of the projector). Let A, B be nonempty and closed subsets of $\mathbf{E}, \hat{x} \in S := A \cap B$ and let the collection $\{A, B\}$ be locally linearly regular at \hat{x} on $\mathbb{B}_{\delta}(\hat{x})$ with constant κ for some $\delta > 0$. One has

$$\|x - x_{+}\| \ge \gamma d(x, S) \quad \forall x_{+} \in P_{B}x, \ \forall x \in A \cap \mathbb{B}_{\delta}(\hat{x})$$
(3.11)

where $\gamma = 1/\kappa$.

Proof. By the definition of the distance and the projector one has, for $x \in A$ and any $x_+ \in P_B x$,

$$||x - x_{+}|| = d(x, B)$$
(3.12)

$$= \max\left\{ d\left(x,B\right), \underbrace{d\left(x,A\right)}_{=0} \right\}$$
(3.13)

$$\geq \gamma d(x,S). \tag{3.14}$$

The inequality follows by Definition 29 (local linear regularity at \hat{x} on $\mathbb{B}_{\delta}(\hat{x})$ with constant κ), since $x \in \mathbb{B}_{\delta}(\hat{x})$.

Theorem 35 (Projections onto a (ε, δ) -subregular set). Let A, B be nonempty and closed subsets of \mathbf{E} and let $\hat{x} \in S := A \cap B$. If

(a) B is (ε, δ) -subregular at \hat{x} with respect to S and

(b) the collection $\{A, B\}$ is locally linearly regular at \hat{x} on $\mathbb{B}_{\delta}(\hat{x})$ then

$$d(x_{+},S) \leq \sqrt{1 + \tilde{\varepsilon} - \gamma^{2}} \ d(x,S), \quad \forall x_{+} \in P_{B}x, \ \forall x \in U$$
(3.15)

where $\gamma = 1/\kappa$ with κ the regularity modulus on $\mathbb{B}_{\delta}(\hat{x})$, $\tilde{\varepsilon} = 2\varepsilon + 2\varepsilon^2$ and

$$U \subset \{x \in A \cap \mathbb{B}_{\delta}(\hat{x}) \mid P_B x \subset \mathbb{B}_{\delta}(\hat{x})\}.$$
(3.16)

Proof. Since B is (ε, δ) -subregular at \hat{x} with respect to S one can apply Theorem 23 to show that the projector P_B is $(S, 2\varepsilon + 2\varepsilon^2)$ -firmly nonexpansive on U. Moreover, condition (b) and Proposition 34 yield

$$\|x_{+} - x\| \ge \gamma d(x, S) \quad \forall x_{+} \in P_{B}x, \ \forall x \in U.$$
(3.17)

Combining (a) and (b) and applying Lemma 25 then gives

$$d(x_{+},S) \leq \sqrt{1+2\tilde{\varepsilon}-\gamma^{2}}d(x,S), \quad \forall x_{+} \in P_{B}x, \ \forall x \in U.$$
(3.18)

Corollary 36 (Projections onto a convex set [21]). Let A and B be nonempty, closed subsets of \mathbf{E} . If

- (a) the collection $\{A, B\}$ is locally linearly regular at $\hat{x} \in A \cap B$ on $\mathbb{B}_{\delta}(\hat{x})$ with regularity modulus $\kappa > 0$ and
- (b) B is convex

then

$$d(x_+, S) \le \sqrt{1 - \gamma^2} \ d(x, S), \quad \forall x_+ \in P_B x, \ \forall x \in A \cap \mathbb{B}_{\delta}(\hat{x})$$
(3.19)

where $\gamma = 1/\kappa$.

Proof. By convexity of B the projector P_B is nonexpansive and it follows that $P_B x \in \mathbb{B}_{\delta}(\hat{x})$ for all $x \in \mathbb{B}_{\delta}(\hat{x})$. Saying that B is convex equivalent to saying that B is $(0, +\infty)$ -regular and hence $\tilde{\varepsilon} = 0$ in Theorem 35.

Corollary 37 (linear convergence of MAP). Let A, B be closed nonempty subsets of \mathbf{E} and let the collection $\{A, B\}$ be locally linearly regular at $\hat{x} \in$ $S := A \cap B$ on $\mathbb{B}_{\delta}(\hat{x})$ with regularity modulus $\kappa > 0$. Define $\gamma := 1/\kappa$ and let $x_0 \in A$. Generate the sequence $\{x_n\}_{n \in \mathbb{N}}$ by

$$x_{2n+1} \in P_B x_{2n}$$
 and $x_{2n+2} \in P_A x_{2n+1}$ $\forall n = 0, 1, 2, \dots$ (3.20)

(a) If A and B are (ε, δ) -subregular at \hat{x} with respect to S and $\tilde{\varepsilon} := 2\varepsilon + 2\varepsilon^2 \leq \gamma^2$, then

$$d(x_{2n+2}, S) \le (1 - \gamma^2 + \tilde{\varepsilon}) d(x_{2n}, S) \quad \forall n = 0, 1, 2, \dots$$
 (3.21)

for all $x_0 \in \mathbb{B}_{\delta/2}(\hat{x}) \cap A$.

(b) If A is (ε, δ) -subregular with respect to S, B is convex and $\tilde{\varepsilon} := 2\varepsilon + 2\varepsilon^2 \leq (2\gamma - \gamma^2)/(1 - \gamma^2)$, then $d(x_{2n+2}, S) \leq \sqrt{1 - \gamma^2 + \tilde{\varepsilon}} \sqrt{1 - \gamma^2} d(x_{2n}, S) \quad \forall n = 0, 1, 2, \dots,$ (3.22)

for all $x_0 \in \mathbb{B}_{\delta/2}(\hat{x}) \cap A$.

(c) If A and B are convex, then

$$d(x_{2n+2}, S) \le (1 - \gamma^2) d(x_{2n}, S) \quad \forall n = 0, 1, 2, \dots$$
(3.23)

for all $x_0 \in \mathbb{B}_{\delta}(\hat{x}) \cap A$.

Proof. a) First one has to show that all iterates remain close to \hat{x} for x_0 close to \hat{x} , that is, we have to show that all iterates remain in the set U defined by (3.16). Note that for any $x_0 \in \mathbb{B}_{\delta/2}(\hat{x})$, and $x_1 \in P_B x_0$ one has

$$||x_0 - x_1|| = d(x_0, B) \le ||x_0 - \hat{x}||.$$

since $\hat{x} \in B$. Thus

$$||x_1 - \hat{x}|| \le ||x_0 - x_1|| + ||x_0 - \hat{x}|| \le ||x_0 - \hat{x}|| + ||x_0 - \hat{x}|| \le \delta,$$
(3.24)

which shows that $P_B x_0 \subset \mathbb{B}_{\delta}(\hat{x}), \forall x_0 \in \mathbb{B}_{\delta/2}(\hat{x})$. One can now apply Theorem 35 to conclude that

$$d(x_1, S) \le \sqrt{1 - \gamma^2 + \tilde{\varepsilon}} \ d(x_0, S).$$
(3.25)

The last equation then implies that $x_1 \in \mathbb{B}_{\delta/2}(\hat{x})$ as long as $\gamma^2 \geq \tilde{\varepsilon}$ and therefore the same argument can be applied to x_1 to conclude that

$$d(x_2, S) \le \sqrt{1 - \gamma^2 + \tilde{\varepsilon}} d(x_1, S).$$
(3.26)

Combining the last two equations (a) then follow by induction.

b) Applying Corollary 36 yields

$$d(x_1, S) \le \sqrt{1 - \gamma^2} d(x_0, S)$$
 (3.27)

and analogous to a) note that (3.26) is still valid for $\tilde{\varepsilon} \leq (2\gamma - \gamma^2)/(1 - \gamma^2)$. By $\tilde{\varepsilon} \leq (2\gamma - \gamma^2)/(1 - \gamma^2)$ it follows that

$$\sqrt{1-\gamma^2+\tilde{\varepsilon}}\sqrt{1-\gamma^2} \le \sqrt{1-\gamma^2+(2\gamma-\gamma^2)/(1-\gamma^2)}\sqrt{1-\gamma^2} \qquad (3.28)$$

$$\leq \sqrt{1 - 2\gamma + \gamma^2 + (2\gamma - \gamma^2)} \tag{3.29}$$

$$\leq 1 \tag{3.30}$$

and therefore by induction b).

c) is an immediate consequence of Corollary 36.

3.3 Linear Convergence of Douglas-Rachford

We now turn to the Douglas-Rachford algorithm. This algorithm is notoriously difficult to analyze and our results reflect this in considerably more circumscribed conditions than are required for the MAP algorithm. Nevertheless, to our knowledge the following convergence results are the most general to date. The first result gives sufficient conditions for the coercivity condition (3.1) to hold.

Lemma 38. Let the collection of closed subsets A, B of \mathbf{E} be locally linearly regular at $\hat{x} \in S := A \cap B$ on $\mathbb{B}_{\delta}(\hat{x})$ with constant $\kappa > 0$ for some $\delta > 0$. Suppose further that B is a subspace and that for some constant $c \in (0, 1)$ the following condition holds:

$$\begin{array}{ll} x \in \mathbb{B}_{\delta}(\hat{x}), \ y = P_B x, \\ z \in P_A(2y - x) \end{array} \quad and \quad \begin{array}{ll} u \in N_A(z) \cap \mathbb{B} \\ v \in N_B(y) \cap \mathbb{B} \end{array} \} \quad \Rightarrow \quad \langle u, v \rangle \geq -c. \quad (3.31) \end{array}$$

Then T_{DR} satisfies

$$\|x - x_+\| \ge \frac{\sqrt{1 - c}}{\kappa} d(x, S) \quad \forall x_+ \in T_{DR} x, \ \forall \ x \in U,$$
(3.32)

where

$$U \subset \{x \in B_{\delta}(\hat{x}) | P_A R_B x \subset \mathbb{B}_{\delta}(\hat{x})\}.$$
(3.33)

Proof. In what follows we will use the notation $R_B x$ for 2y - x with $y = P_B x$ which is unambiguous, if a slight abuse of notation, since B is convex. We will use (3.31) to show that for all $x \in U$ with $y = P_B x$ and $z \in P_A R_B x$:

$$||x - x_{+}||^{2} = ||z - y||^{2}$$

$$\geq (1 - c) \left[||z - R_{B}x||^{2} + ||R_{B}x - y||^{2} \right]$$
(3.34)

We will then show that, for all $x \in U$ with $y = P_B x$ and $z \in P_A R_B x$

$$||z - R_B x||^2 + ||R_B x - y||^2 \ge \frac{1}{\kappa^2} d(R_B x, S)^2.$$
(3.35)

Combining inequalities (3.34) and (3.35) yields

$$\|x - x_{+}\|^{2} \ge \frac{1 - c}{\kappa^{2}} d(R_{B}x, S)^{2} \quad \forall \ x \in U.$$
(3.36)

Let $\tilde{x} \in P_S(R_B x)$ and note that $d(R_B x, S) = ||R_B x - \tilde{x}||$. Since *B* is a subspace, by (1.22) one has $d(R_B x, S) = ||R_B x - \tilde{x}|| = ||x - \tilde{x}||$. Moreover, $||x - \tilde{x}|| \ge \min_{y \in S} ||x - y|| = d(x, S)$, hence

$$\|x - x_{+}\| \ge \frac{\sqrt{1 - c}}{\kappa} d(x, S) \quad \forall \ x \in U$$
(3.37)

as claimed.

What remains is to prove (3.34) and (3.35) for all $x \in U$ with $y = P_B x$ and $z \in P_A R_B x$.

Proof of (3.34). Using Lemma 10 equation (2.2) one has for $x \in \mathbb{B}_{\delta}(\hat{x})$ with $y = P_B x$ and $z \in P_A R_B x$

$$\begin{aligned} \|x - x_{+}\|^{2} &= \|z - y\|^{2} \\ &= \|z - R_{B}x + R_{B}x - y\|^{2} + 2\langle \underline{z - R_{B}x}, \underbrace{R_{B}x - y}_{\underline{e} - N_{A}(z)} \rangle \\ &= \|z - R_{B}x\|^{2} + \|R_{B}x - y\|^{2} + 2\langle \underline{z - R_{B}x}, \underbrace{R_{B}x - y}_{\underline{e} - N_{B}(y)} \rangle \\ \overset{(3.31)}{\geq} \|z - R_{B}x\|^{2} + \|R_{B}x - y\|^{2} - 2c\|z - R_{B}x\| \|R_{B}x - y\| \\ &= (1 - c) \left[\|z - R_{B}x\|^{2} + \|R_{B}x - y\|^{2} \right] \\ &+ c \left[\|z - R_{B}x\|^{2} - 2\|z - R_{B}x\| \|R_{B}x - y\| + \|R_{B}x - y\|^{2} \right] \\ &= (1 - c) \left[\|z - R_{B}x\|^{2} + \|R_{B}x - y\|^{2} \right] \\ &+ c \left[\|z - R_{B}x\|^{2} + \|R_{B}x - y\|^{2} \right] \\ &+ c \left[\|z - R_{B}x\| - \|R_{B}x - y\|^{2} \right] \end{aligned}$$

$$(3.38)$$

$$\geq (1 - c) \left[\|z - R_{B}x\|^{2} + \|R_{B}x - y\|^{2} \right].$$

Proof of (3.35). First note that if $x \in \mathbb{B}_{\delta}(\hat{x})$, since B is a subspace, by equation (1.22) $R_B x \subset \mathbb{B}_{\delta}(\hat{x})$ and by convexity of $\mathbb{B}_{\delta}(\hat{x})$ it follows that $y = P_B x \subset \mathbb{B}_{\delta}(\hat{x})$ and hence (3.31) is localized to $\mathbb{B}_{\delta}(\hat{x})$ (to which the dividends of linear regularity of the intersection extend) as long as $P_A R_B x \subset \mathbb{B}_{\delta}(\hat{x})$, that is, as long as $x \in U$. By definition of the projector $||R_B x - y|| \ge ||R_B x - P_B(R_B x)||$. Local linear regularity at \hat{x} with radius δ and constant κ yields for $x \in U$ with $y = P_B x$ and $z \in P_A R_B x$

$$||z - R_B x||^2 + ||R_B x - y||^2 \ge ||z - R_B x||^2 + ||R_B x - P_B R_B x||^2$$

= $d (R_B x, A)^2 + d (R_B x, B)^2$
 $\ge \frac{(3.10)}{\kappa^2} \frac{1}{\kappa^2} d (R_B x, S)^2$ (3.39)

This completes the proof of (3.35) and the Theorem.

Remark 39. The coercivity constant in equation (3.32) is not tight, even if κ were chosen to the the regularity modulus of the intersection. Remember, by linearity, that $y = P_B x = P_B R_B x$. In line (3.38) we use the inequality $[\|z - R_B x\| - \|R_B x - y\|]^2 \ge 0$ while we use $\|z - R_B x\|^2 + \|R_B x - y\|^2 \ge \max\{\|z - R_B x\|^2, \|R_B x - y\|^2\}$ in line (3.39). If the first inequality is tight then this is the worst possible result in the second inequality,

since $||z - R_B x|| = ||R_B x - y||$, that is, this second inequality is satisfied not only strictly, but the inequality is as large as it can possibly be. On the other hand if the second inequality is tight this means $||z - R_B x||^2 = 0$ or $||R_B x - y||^2 = 0$ and this means that the first inequality is strict. In any event, it is impossible to achieve equality in the argumentation of the proof. This is a technical limitation of the logic of the proof and does not preclude improvements.

Lemma 38 with the added assumption of (ϵ, δ) -regularity of the nonconvex set yields local linear convergence of the Douglas-Rachford algorithm in this special case.

Theorem 40. Let the collection of closed subsets A, B of \mathbf{E} be locally linearly regular at $\hat{x} \in S := A \cap B$ on $\mathbb{B}_{\delta}(\hat{x})$ with constant $\kappa > 0$ for some $\delta > 0$. Suppose further that B is a subspace and that A is (ε, δ) -regular at \hat{x} with respect to S. Assume that for some constant $c \in (0, 1)$ the following condition holds:

$$z \in A \cap \mathbb{B}_{\delta}(\hat{x}), \quad u \in N_A(z) \cap \mathbb{B} \\ y \in B \cap \mathbb{B}_{\delta}(\hat{x}), \quad v \in N_B(y) \cap \mathbb{B}$$
 $\Rightarrow \quad \langle u, v \rangle \ge -c.$ (3.40)

If $x \in \mathbb{B}_{\delta/2}(\hat{x})$ then

$$d(x_{+},S) \leq \sqrt{1+\tilde{\varepsilon}-\eta} \ d(x,S) \quad \forall \ x_{+} \in T_{DR}x$$
(3.41)

with $\eta := \frac{(1-c)}{\kappa^2}$ and $\tilde{\varepsilon} = 2\varepsilon + 2\varepsilon^2$.

Proof. First one has to show requirement (3.33). Since $\hat{x} \in A \cap B$ note that for any $x \in \mathbb{B}_{\delta/2}(\hat{x})$ for all $z \in P_A R_B x$ by Definition $||z - R_B x|| = d(R_B x, A) \leq$ $||R_B x - \hat{x}||$ and by (1.22) $||R_B x - \hat{x}|| = ||x - \hat{x}||$ holds. This now implies

$$||z - \hat{x}|| \le ||z - R_B x|| + ||R_B x - \hat{x}|| \le 2||x - \hat{x}|| \le \delta,$$
(3.42)

and therefore $z \in \mathbb{B}_{\delta}(\hat{x})$.

Now for B a subspace (3.40) and (3.31) are equivalent, and so by Lemma 38 the coercivity condition (3.1)

$$\|x - x_{+}\| \ge \frac{\sqrt{1 - c}}{\kappa} d(x, S) \tag{3.43}$$

is satisfied on $\mathbb{B}_{\delta/2}(\hat{x})$. Moreover, since A is (ε, δ) -regular and B is $(0, \infty)$ regular, by Theorem 24 T_{DR} is $(S, \tilde{\varepsilon})$ -firmly nonexpansive with $\tilde{\varepsilon} = 2\varepsilon(1+\varepsilon)$,
that is $(\forall x \in \mathbb{B}_{\delta/2}(\hat{x})) \ (\forall x_+ \in T_{DR}x) \ (\forall \overline{x} \in S)$

$$\|x_{+} - \overline{x}\|^{2} + \|x - x_{+}\|^{2} \le (1 + \tilde{\varepsilon}) \|x - \overline{x}\|^{2}.$$
(3.44)

Lemma 25 then applies to yield $(\forall x \in \mathbb{B}_{\delta/2}(\hat{x})) \ (\forall x_+ \in T_{DR}x)$

$$d(x_{+},S) \leq \sqrt{1 + \tilde{\varepsilon} - \eta} \ d(x,S)$$
(3.45)

where $\eta := \frac{1-c}{\kappa^2}$.

The next lemma establishes sufficient conditions under which (3.40) holds.

Lemma 41 ([27] Theorem 5.16). Assume $B \subset \mathbf{E}$ is a subspace and that $A \subset \mathbf{E}$ is closed and super-regular at $\hat{x} \in A \cap B$. If the collection $\{A, B\}$ is strongly regular at \hat{x} , then there is a $\delta > 0$ and a constant $c \in (0, 1)$ such that (3.40) holds on $\mathbb{B}_{\delta}(\hat{x})$.

Proof. Condition (3.31) can be shown using (3.9). For more details see [27]. \Box

We summarize this discussion with the following convergence result for the Douglas-Rachford algorithm in the case of an affine subspace and a superregular set.

Theorem 42. Assume $B \subset \mathbf{E}$ is a subspace and that $A \subset \mathbf{E}$ is closed and super-regular at $\hat{x} \in S := A \cap B$. If the collection $\{A, B\}$ is strongly linearly regular at S, then there is a $\delta > 0$ such that,

$$\frac{(1-c)}{\kappa^2} > 2\varepsilon + 2\varepsilon^2 \tag{3.46}$$

and hence

$$d(x_+, S) \le \tilde{c} \ d(x, S) \quad \forall \ x_+ \in T_{DR}x, \tag{3.47}$$

with $\tilde{c} = \sqrt{1 + 2\varepsilon + 2\varepsilon^2 - \frac{(1-c)}{\kappa^2}} < 1$ for all $x \in \mathbb{B}_{\delta/2}(\hat{x})$.

Proof. Strong regularity of the collection implies linear regularity with constant κ on $\mathbb{B}_{\delta_1}(\hat{x})$ (see Remark 30). Lemma 41 guaranties the existence of constants $\delta_2 > 0$ and $c \in (0,1)$ such that (3.40) holds on $\mathbb{B}_{\delta_2}(\hat{x})$. Now, by super-regularity at \hat{x} , for any ε there exists a δ_3 such that A is (ε, δ_3) subregular at \hat{x} . In other words, for c and κ determined by the regularity of the collection $\{A, B\}$ at \hat{x} , we can always choose ε (generating a corresponding δ_3 radius) so that (3.46) is satisfied on $\mathbb{B}_{\delta_3}(\hat{x})$. Then for $\delta := \min \{\delta_1, \delta_2, \delta_3\}$, the requirements of Theorem 40 are satisfied on $\mathbb{B}_{\delta}(\hat{x})$, which completes the proof of linear convergence on $\mathbb{B}_{\delta/2}(\hat{x})$.

Remark 43. The example Example 5 (v) has been studied by Borwein and coauthors [1, 10] where they achieve global characterizations of convergence with rates. Our work does not directly overlap with [1, 10] since our results are local, and the order of the reflectors is reversed: we must reflect first across the subspace, then reflect across the nonconvex set; Borwein and coauthors reflect first across the circle.

3.4 Douglas-Rachford on Subspaces

We finish this section with the fact that strong regularity of the intersection is *necessary*, not just sufficient for convergence of the iterates of the Douglas-Rachford algorithm to the intersection in the affine case.

Corollary 44. Let A, B be two affine subspaces with $A \cap B \neq \emptyset$. Douglas-Rachford converges for any starting point $x_0 \in \mathbf{E}$ with linear rate to the intersection $A \cap B$ if and only if $A^{\perp} \cap B^{\perp} = \{0\}$.

Proof. Without loss of generality for $\hat{x} \in A \cap B$ by shifting the subspaces by \hat{x} we consider the case of linear subspaces. By (3.9), on subspaces the condition $A^{\perp} \cap B^{\perp} = \{0\}$ is equivalent to strong regularity of the collection $\{A, B\}$.

If the intersection is strongly regular and A and B are subspaces, then the requirements of Theorem 42 are globally satisfied, so Douglas-Rachford converges with linear rate

$$\tilde{c} = \sqrt{1 - \frac{(1-c)}{\kappa^2}} < 1$$
 (3.48)

where $c \in [0, 1)$ (compare (3.31)) now becomes

$$c = \max\langle u, v \rangle, \qquad u \in A^{\perp}, \ \|u\| = 1, \ v \in B^{\perp}, \ \|v\| = 1,$$
 (3.49)

and κ is an associated *global* constant of linear regularity (see Theorem 32).

On the other hand for $\hat{x} \in A \cap B$ by [7, Thm 3.5] we get the characterization

Fix
$$T_{DR} = (A \cap B) + N_{A-B}(0)$$
 (3.50)

$$= (A \cap B) + (N_A(\hat{x}) \cap -N_B(\hat{x}))$$
(3.51)

$$= (A \cap B) + A^{\perp} \cap B^{\perp}. \tag{3.52}$$

and so the fix point set of T_{DR} does not coincide with the intersection unless the collection $\{A, B\}$ is strongly regular. In other words, if the intersection is not strongly regular, then convergence to the intersection cannot be linear, thus proving the reverse implication by the contrapositive.

Remark 45 (Friedrichs angle [18]). We would like to make a final remark about connection between the notion of the angle of the sets at the intersection and the regularity of the collection of sets at points in the intersection. The operative notion of angle is the *Friedrichs angle*. For two subspaces A and B the *Friedrichs angle* is the angle $\alpha(A, B)$ in $[0, \pi/2]$ whose cosine is defined by

$$c_F(A,B) := \sup\left\{ |\langle x, y \rangle| \; \middle| \; \begin{array}{l} x \in A \cap (A \cap B)^{\perp}, \; \|x\| \le 1, \\ y \in B \cap (A \cap B)^{\perp}, \; \|y\| \le 1. \end{array} \right\}$$
(3.53)

The Friedrichs angle being less than 1 is not sufficient for convergence of Douglas-Rachford. This can be seen by example 5 (ii). The Friedrichs angle

in this example is the same as in 5 (i), but for $x_0 \notin \mathbb{R}^2 \times \{0\}$ the Douglas-Rachford algorithm does not converge to $\{0\} = A \cap B$. Another interesting observation is that if, on the other hand, $A^{\perp} \cap B^{\perp} = \{0\}$ then $(A^{\perp} \cap B^{\perp})^{\perp} = \mathbf{E}$ which implies that $c_F(A^{\perp}, B^{\perp})$ coincides with (3.49) and by [14, Theorem 2.16] $c_F(A^{\perp}, B^{\perp})$ then coincides with $c_F(A, B)$. So if the Douglas-Rachford algorithm on subspaces converges linearly, then the rate of convergence is dependent on the Friedrichs angle. A detailed analysis regarding the relation between the Friedrichs angle and linear convergence of MAP can be found in [16, 17].

4 Concluding Remarks

In the time that has passed since first submitting our manuscript for publication we learned about an optimal linear convergence result for Douglas-Rachford applied to ℓ_1 optimization with an affine constraint using different techniques [13]. The modulus of linear regularity does not recover optimal convergence results (see Remark 31), but we suspect this is an artifact of our proof technique. The question remains whether there is a quantitative primal definition of a angle between two sets that recovers the same results for MAP as [8]. This could also be useful to achieve optimal linear convergence results for Douglas-Rachford in general. Also, as we noted in the introduction, it is well-known that the fixed-point set of the Douglas-Rachford operator is in general bigger than the intersection of the sets, and Corollary 44 stating that the iterates converge to the intersection if and only if the collection of sets is strongly regular is a consequence of this. In the convex case, the *shadows* of the iterates still converge. We leave a fuller investigation of the shadows of the iterates and the angles between the sets at the intersection in the nonconvex setting to future work.

Another direction of future work will be to extend this analysis more generally to fixed point mappings built upon functions and more general set-valued mappings, but also in particular *proximal* operators and reflectors. The generality of our approach makes such extensions quite natural. Indeed, local linear regularity of collections of sets can be shown to be related to *strong metric subregularity* of set-valued mappings which guarantees that the condition (3.1) of Lemma 25 is satisfied. Of course, the difficulty remains to show that the the mappings are indeed metrically subregular.

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