Verified Error Bounds for Isolated Singular Solutions of Polynomial Systems

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In this paper, we generalize the algorithm described by Rump and Graillat, as well as our previous work on certifying breadth-one singular solutions of polynomial systems, to compute verified and narrow error bounds such that a slightly perturbed system is guaranteed to possess an isolated singular solution within the computed bounds. Our new verification method is based on deflation techniques using smoothing parameters. We demonstrate the performance of the algorithm for systems with singular solutions of multiplicity up to hundreds.

1 Introduction

It is a challenge problem to solve polynomial systems with singular solutions. In [28], Rall studied some convergence properties of Newton's method for singular solutions, and many modifications of Newton's method to restore the quadratic convergence for singular solutions have been proposed in [1, 5, 6, 7, 11, 12, 13, 25, 27, 29, 30, 34, 38]. Recently, some symbolic-numeric methods have also been proposed for refining approximate isolated singular solutions to high accuracy [2, 3, 4, 9, 10, 17, 18, 19, 23, 36, 37]. In [21, 22], we described an algorithm based on the regularized Newton iterations and the computation of differential conditions satisfied at given approximate singular solutions to compute isolated singular solutions accurately to the full machine precision when its Jacobian matrix has corank one (the breadth-one case).

Since arbitrary small perturbations of coefficients may transform an isolated singular solution into a cluster of simple roots or even make it disappear, it is more difficult to certify that a polynomial system or a nonlinear system has a multiple root, if not the entire computation is performed without any rounding error.

In [33], by introducing a smoothing parameter, Rump and Graillat described a verification method for computing guaranteed (real or complex) error bounds such that a slightly perturbed system is proved to have a double root within the computed bounds. In [20], by adding a perturbed univariate polynomial in one selected variable with some smoothing parameters to one selected equation of the original system, we generalized the algorithm in [33] to compute guaranteed error bounds, such that a slightly perturbed system is proved to possess an isolated singular solution whose Jacobian matrix has corank one within the computed bounds.

In [23], Mantzaflaris and Mourrain proposed a one-step deflation method, and by applying a well-chosen symbolic perturbation, they verified a multiple root of a nearby system with a given multiplicity structure, which depends on the accuracy of the given approximate singular solution. The size of the deflated system is equal to the multiplicity times the size of the original system, which might be large (e.g. DZ1 and KSS in Table 1).

In [39], based on deflated square systems proposed by Yamamoto in [38], Kanzawa and Oishi presented a numerical method for proving the existence of "imperfect singular solutions" of nonlinear equations with guaranteed accuracy. In [38], if the second-order deflation is applied, then smoothing parameters are added not only to the original system but also to differential systems independently (see (20)). Therefore, one can only prove the existence of an isolated solution of a slightly perturbed system which satisfies the first-order differential condition approximately.

In [8, 14, 15], Kearfott et al. presented completely different and extremely interesting methods based on verifying a nonzero topological degree to certify the existence of singular zeros of nonlinear systems.

Main contribution Suppose a polynomial system F and an approximate singular solution are given. Stimulated by our previous work on certifying breadth-one singular solutions [20], we show firstly that the number of deflations used by Yamamoto to obtain a regular system is bounded by the depth of the singular solution. Then we show how to move the independent perturbations in the first-order differential system (20) appeared in [38] back to the original system. We prove that the modified deflations will terminate after a finite number of steps which is bounded by the depth as well, and return a regular and square augmented system, which can be used to prove the existence of an isolated singular solution of a slightly perturbed system exactly, see Theorem 3.7 and 3.8. Finally, we present an algorithm for computing verified (real or complex) error bounds, such that a slightly perturbed system is guaranteed to possess an isolated singular solution within the computed bounds. The algorithm has been implemented in Maple and Matlab, and narrow error bounds of the order of the relative rounding error are computed efficiently for examples given in literature.

Structure of the paper Section 2 is devoted to recall some notations and well-known facts. In Section 3, we present a new deflation method by adding smoothing parameters properly to the original system, which will return a regular and square augmented system within a finite number of steps bounded by the depth. In Section 4, we propose an algorithm for computing verified (real or complex) error bounds, such that a slightly perturbed system is guaranteed to possess an isolated singular solution within the computed bounds. Some numerical results are given to demonstrate the performance of our algorithm in Section 5.

2 Preliminaries

Let $F = \{f_1, \ldots, f_n\}$ be a polynomial system in $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \ldots, x_n]$ and $I \in \mathbb{C}[\mathbf{x}]$ be the ideal generated by polynomials in F.

Definition 2.1 *An isolated solution of $F(\mathbf{x}) = \mathbf{0}$ is a point $\hat{\mathbf{x}} \in \mathbb{C}^n$ which satisfies:*

$$\text{for a small enough } \varepsilon > 0 : \{\mathbf{y} \in \mathbb{C}^n : \|\mathbf{y} - \hat{\mathbf{x}}\| < \varepsilon\} \cap F^{-1}(\mathbf{0}) = \{\hat{\mathbf{x}}\}.$$

Definition 2.2 *We call $\hat{\mathbf{x}}$ a singular solution of $F(\mathbf{x}) = \mathbf{0}$ if and only if*

$$\operatorname{rank}(F_{\mathbf{x}}(\hat{\mathbf{x}})) < n, \tag{1}$$

where $F_{\mathbf{x}}(\mathbf{x})$ is the Jacobian matrix of $F(\mathbf{x})$ with respect to $\mathbf{x}$.

Definition 2.3 *Let $Q_{\hat{\mathbf{x}}}$ be the isolated primary component of the ideal $I = (f_1, \ldots, f_n)$ whose associate prime is $m_{\hat{\mathbf{x}}} = (x_1 - \hat{x}_1, \ldots, x_n - \hat{x}_n)$, then the multiplicity μ of $\hat{\mathbf{x}}$ is defined as $\mu = \dim(\mathbb{C}[\mathbf{x}]/Q_{\hat{\mathbf{x}}})$, and the index ρ of $\hat{\mathbf{x}}$ is defined as the minimal nonnegative integer ρ such that $m_{\hat{\mathbf{x}}}^{\rho} \subseteq Q_{\hat{\mathbf{x}}}$ [35].*

Let $\mathbf{d}_{\hat{\mathbf{x}}}^{\alpha} : \mathbb{C}[\mathbf{x}] \to \mathbb{C}$ denote the differential functional defined by

$$\mathbf{d}_{\hat{\mathbf{x}}}^{\alpha}(g) = \frac{1}{\alpha_1! \cdots \alpha_n!} \cdot \frac{\partial^{|\alpha|} g}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}(\hat{\mathbf{x}}), \quad \forall g(\mathbf{x}) \in \mathbb{C}[\mathbf{x}], \tag{2}$$

for a point $\hat{\mathbf{x}} \in \mathbb{C}^n$ and an array $\alpha \in \mathbb{N}^n$. The normalized differentials have a useful property: when $\hat{\mathbf{x}} = \mathbf{0}$, we have $\mathbf{d}_{\mathbf{0}}^{\alpha}(\mathbf{x}^{\beta}) = 1$ if $\alpha = \beta$ or 0 otherwise.

Definition 2.4 *The local dual space of I at $\hat{\mathbf{x}}$ is the subspace of elements of $\mathfrak{D}_{\hat{\mathbf{x}}} = \operatorname{Span}_{\mathbb{C}}\{\mathbf{d}_{\hat{\mathbf{x}}}^{\alpha}, \alpha \in \mathbb{N}^n\}$ that vanish on all the elements of I*

$$\mathcal{D}_{\hat{\mathbf{x}}} := \{\Lambda \in \mathfrak{D}_{\hat{\mathbf{x}}} \mid \Lambda(f) = 0, \ \forall f \in I\}. \tag{3}$$

It is clear that $\dim(\mathcal{D}_{\hat{\mathbf{x}}}) = \mu$ and the maximal degree of an element $\Lambda \in \mathcal{D}_{\hat{\mathbf{x}}}$ is equal to the index $\rho - 1$, which is also known as the depth of $\mathcal{D}_{\hat{\mathbf{x}}}$.

A singular solution $\hat{\mathbf{x}}$ of a square system $F(\mathbf{x}) = \mathbf{0}$ satisfies equations

$$\begin{cases} F(\mathbf{x}) = \mathbf{0}, \\ \det(F_{\mathbf{x}}(\mathbf{x})) = 0. \end{cases} \tag{4}$$

The above augmented system forms the basic idea for the deflation method [25, 26, 27]. But the determinant is usually of high degree, so it is numerically unstable to evaluate the determinant of the Jacobian matrix.

In [18], Leykin et al. modified (4) by adding new variables and equations. Let $r = \mathrm{rank}(F_{\mathbf{x}}(\hat{\mathbf{x}}))$, then there exists a unique vector $\hat{\boldsymbol{\lambda}} = (\hat{\lambda}_1, \hat{\lambda}_2 \ldots, \hat{\lambda}_{r+1})^T$ such that $(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}})$ is an isolated solution of

$$\begin{cases} F(\mathbf{x}) = \mathbf{0}, \\ F_{\mathbf{x}}(\mathbf{x}) B \boldsymbol{\lambda} = \mathbf{0}, \\ \mathbf{h}^T \boldsymbol{\lambda} = 1, \end{cases} \tag{5}$$

where $B \in \mathbb{C}^{n \times (r+1)}$ is a random matrix, $\mathbf{h} \in \mathbb{C}^{r+1}$ is a random vector and $\boldsymbol{\lambda}$ is a vector consisting of $r + 1$ extra variables $\lambda_1, \lambda_2 \ldots, \lambda_{r+1}$. If $(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}})$ is still a singular solution of (5), the deflation is repeated. Furthermore, they proved that the number of deflations needed to derive a regular root of an augmented system is strictly less than the multiplicity of $\hat{\mathbf{x}}$. Dayton and Zeng showed that the depth of $\mathcal{D}_{\hat{\mathbf{x}}}$ is a tighter bound for the number of deflations [4].

Let $\mathbb{IR}$ be the set of real intervals, and $\mathbb{IR}^n$ and $\mathbb{IR}^{n \times n}$ be the set of real interval vectors and real interval matrices, respectively. Standard verification methods for nonlinear systems are based on the following theorem [16, 24, 31].

Theorem 2.5 *Let $F(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial system, and $\tilde{\mathbf{x}} \in \mathbb{R}^n$. Given $\mathbf{X} \in \mathbb{IR}^n$ with $\mathbf{0} \in \mathbf{X}$ and $M \in \mathbb{IR}^{n \times n}$ satisfies $\nabla f_i(\tilde{\mathbf{x}} + \mathbf{X}) \subseteq M_{i,:}$, for $i = 1, \ldots, n$. Denote by I the $n \times n$ identity matrix and assume*

$$-F_{\mathbf{x}}^{-1}(\tilde{\mathbf{x}}) F(\tilde{\mathbf{x}}) + (I - F_{\mathbf{x}}^{-1}(\tilde{\mathbf{x}}) M) \mathbf{X} \subseteq \mathrm{int}(\mathbf{X}). \tag{6}$$

Then there is a unique $\hat{\mathbf{x}} \in \mathbf{X}$ with $F(\hat{\mathbf{x}}) = 0$. Moreover, every matrix $\tilde{M} \in M$ is nonsingular. In particular, the Jacobian matrix $F_{\mathbf{x}}(\hat{\mathbf{x}})$ is nonsingular.

Naturally the non-singularity of the Jacobian matrix $F_{\mathbf{x}}(\hat{\mathbf{x}})$ restricts the application of Theorem 2.5 to regular solutions of square systems. Notice that Theorem 2.5 is valid mutatis mutandis over complex numbers as well. Next we will use this theorem to derive a verification method to prove the existence of an isolated singular solution of a slightly perturbed system.

3 A Square and Regular Augmented System

Let a polynomial system $F = \{f_1, \ldots, f_n\} \in \mathbb{C}[\mathbf{x}]$ be given and $\hat{\mathbf{x}} = (\hat{x}_1, \ldots, \hat{x}_n)$ is an isolated singular solution satisfying $F(\hat{\mathbf{x}}) = \mathbf{0}$.

The augmented systems (4) and (5) have been used to restore the quadratic convergence of Newton's method. But notice that these extended systems are always over-determined, which are not applicable by Theorem 2.5. Hence, a natural thought of modifications is, whether we could add several smoothing parameters to derive a square system with a nonsingular Jacobian matrix.

In [38], by introducing smoothing parameters, Yamamoto derived square deflated systems. These systems were used successfully by Kanzawa and Oishi in [39] to certify the existence of *"imperfect singular solutions"* of polynomial systems. However, for isolated singular solutions with high singularities, the smoothing parameters are added not only to the original system but also to differential systems independently (see (20)). Therefore, according to (21), one can only prove the existence of an isolated solution of a slightly perturbed system which satisfies the first-order differential condition approximately.

In the following, we rewrite the deflation techniques in [38] in our setting, and prove that the number of deflations needed to obtain a regular system is bounded by the depth of $\mathcal{D}_{\hat{\mathbf{x}}}$, see Theorem 3.2. Then we show how to lift the independent perturbations in the first-order differential system appeared in (20) back to the original system. We prove that the modified deflations will terminate after a finite number of steps bounded by the depth of $\mathcal{D}_{\hat{\mathbf{x}}}$ as well, and return a regular and square augmented system, which can be used to verify the existence of an isolated singular solution of a slightly perturbed system exactly, see Theorem 3.7 and 3.8.

3.1 The first-order deflation

Let $\hat{\mathbf{x}} \in \mathbb{C}^n$ be an isolated singular solution of $F(\mathbf{x}) = \mathbf{0}$, and

$$\operatorname{rank}(F_{\mathbf{x}}(\hat{\mathbf{x}})) = n - d, \ \ (1 < d \leq n). \tag{7}$$

Let $\mathbf{c} = \{c_1, c_2, \ldots, c_d\}$ $(1 \leq c_1 \leq c_2 \leq \ldots \leq c_d \leq n)$ and $F_{\mathbf{x}}^{\mathbf{c}}(\hat{\mathbf{x}})$ be obtained from $F_{\mathbf{x}}(\hat{\mathbf{x}})$ by deleting its $c_1, c_2, \ldots, c_d$-th columns which satisfies

$$\operatorname{rank}(F_{\mathbf{x}}^{\mathbf{c}}(\hat{\mathbf{x}})) = n - d. \tag{8}$$

There exists a positive-integer set $\mathbf{k} = \{k_1, k_2, \ldots, k_d\}$ such that

$$\operatorname{rank}(F_{\mathbf{x}}^{\mathbf{c}}(\hat{\mathbf{x}}), I_{\mathbf{k}}) = n, \tag{9}$$

where

$$I_{\mathbf{k}} = (\mathbf{e}_{k_1}, \mathbf{e}_{k_2}, \ldots, \mathbf{e}_{k_d}), \tag{10}$$

and $\mathbf{e}_{k_i}$ is the k_i-th unit vector of dimension n.

Similar to the augmented system (2.34) in [38], we introduce d smoothing parameters $\mathbf{b}_0 = (b_1, b_2, \ldots, b_d)^T$ and consider the following square system

$$G(\mathbf{x}, \boldsymbol{\lambda}_1, \mathbf{b}_0) = \begin{cases} F(\mathbf{x}) - \sum_{i=1}^{d} b_i \mathbf{e}_{k_i} = \mathbf{0}, \\ F_{\mathbf{x}}(\mathbf{x})\mathbf{v}_1 = \mathbf{0}, \end{cases} \tag{11}$$

where $\mathbf{v}_1$ is a vector consisting of $n-d$ extra variables $\boldsymbol{\lambda}_1 = (\lambda_1, \lambda_2, \ldots, \lambda_{n-d})^T$ and its entries at the positions $c_1, c_2, \ldots, c_d$ are fixed to be 1 rather than random nonzero numbers used in [38]. According to (8), the rank of $F_{\mathbf{x}}^{\mathbf{c}}(\hat{\mathbf{x}})$ is $n-d$, the linear system $F_{\mathbf{x}}(\hat{\mathbf{x}})\mathbf{v}_1 = \mathbf{0}$ has a unique solution, denoted by $\hat{\boldsymbol{\lambda}}_1$. Therefore, $(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1, \mathbf{0})$ is an isolated solution of (11). If $(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1, \mathbf{0})$ is still a singular solution, as proposed in [38], the deflation process mentioned above is repeated to the first-order deflated system G and the solution $(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1, \mathbf{0})$.

Note that Yamamoto did not prove explicitly the termination of the above-mentioned deflation process. Motivated by the results in [18, 4], we show below that the number of deflations needed to derive a regular and square augmented system is also bounded by the depth of $\mathcal{D}_{\hat{\mathbf{x}}}$.

Let $\mathbf{h} = (\underbrace{0, \ldots, 0}_{n-d}, 1)^T$, $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{n-d}, \lambda_{n-d+1})^T$ and

$$B = (\hat{\mathbf{e}}_1, \ldots, \underset{c_1}{\hat{\mathbf{e}}_{n-d+1}}, \ldots, \underset{c_d}{\hat{\mathbf{e}}_{n-d+1}}, \ldots, \hat{\mathbf{e}}_{n-d})^T \in \mathbb{C}^{n\times(n-d+1)},$$

where $\hat{\mathbf{e}}_i$ is the i-th unit vector of dimension $n-d+1$. Then the augmented system (5) used in [18] is equivalent to

$$\widetilde{G}(\mathbf{x}, \boldsymbol{\lambda}_1) = \begin{cases} F(\mathbf{x}) = \mathbf{0}, \\ F_{\mathbf{x}}(\mathbf{x})\mathbf{v}_1 = \mathbf{0}, \end{cases} \tag{12}$$

which has an isolated solution at $(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1)$, and the Jacobian matrix of $\widetilde{G}(\mathbf{x}, \boldsymbol{\lambda}_1)$ at $(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1)$ is

$$\widetilde{G}_{\mathbf{x},\boldsymbol{\lambda}_1}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1) = \begin{pmatrix} F_{\mathbf{x}}(\hat{\mathbf{x}}) & \mathcal{O}_{n,n-d} \\ F_{\mathbf{xx}}(\hat{\mathbf{x}})\hat{\mathbf{v}}_1 & F_{\mathbf{x}}^{\mathbf{c}}(\hat{\mathbf{x}}) \end{pmatrix}, \tag{13}$$

where $\mathcal{O}_{i,j}$ denotes the $i \times j$ zero matrix and $F_{\mathbf{xx}}(\mathbf{x})$ is the Hessian matrix of $F(\mathbf{x})$. On the other hand, the Jacobian matrix of $G(\mathbf{x}, \boldsymbol{\lambda}_1, \mathbf{b}_0)$ computes to

$$G_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1, \mathbf{0}) = \begin{pmatrix} F_{\mathbf{x}}(\hat{\mathbf{x}}) & \mathcal{O}_{n,n-d} & -I_{\mathbf{k}} \\ F_{\mathbf{xx}}(\hat{\mathbf{x}})\hat{\mathbf{v}}_1 & F_{\mathbf{x}}^{\mathbf{c}}(\hat{\mathbf{x}}) & \mathcal{O}_{n,d} \end{pmatrix}. \tag{14}$$

Lemma 3.1 *The null spaces of the Jacobian matrices (13) and (14) satisfy*

$$\mathrm{Null}\left(G_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1, \mathbf{0})\right) = \left\{ \begin{pmatrix} \mathbf{y} \\ \mathbf{0} \end{pmatrix} \in \mathbb{C}^{2n} \mid \mathbf{y} \in \mathrm{Null}\left(\widetilde{G}_{\mathbf{x},\boldsymbol{\lambda}_1}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1)\right) \right\}.$$

Proof. If $\mathbf{y} \in \text{Null}\left(\widetilde{G}_{\mathbf{x},\boldsymbol{\lambda}_1}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1)\right)$ then $\begin{pmatrix} \mathbf{y} \\ \mathbf{0} \end{pmatrix} \in \text{Null}\left(G_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1, \mathbf{0})\right)$. Suppose $\begin{pmatrix} \mathbf{y} \\ \mathbf{z} \end{pmatrix}$ is a null vector of $G_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1, \mathbf{0})$. We divide $\mathbf{y}$ into $\begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix}$ corresponding to the blocks $F_{\mathbf{x}}(\hat{\mathbf{x}})$ and $\mathcal{O}_{n,n-d}$. Therefore, we have

$$F_{\mathbf{x}}(\hat{\mathbf{x}})\mathbf{y}_1 - I_{\mathbf{k}}\mathbf{z} = \mathbf{0}.$$

By (9), we have

$$\text{rank}(F_{\mathbf{x}}^{\mathbf{c}}(\hat{\mathbf{x}}), -I_{\mathbf{k}}) = n.$$

It is clear that $\mathbf{z}$ must be a zero vector. □

If $(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1)$ is still an isolated singular solution of the deflated system (12), as proposed in [18], the deflation process is repeated for $\widetilde{G}(\mathbf{x}, \boldsymbol{\lambda}_1)$ and $(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1)$. Then as shown in [4], if the s-th deflated system is singular, there exists at least one differential functional of the order $s+1$ in $\mathcal{D}_{\hat{\mathbf{x}}}$. However, the order of differential functionals in $\mathcal{D}_{\hat{\mathbf{x}}}$ is bounded by its depth which is equal to $\rho - 1$. Therefore, after at most $\rho - 1$ steps of deflations, one will obtain a regular deflated system, i.e., the corank of the Jacobian matrix of the deflated system will be zero.

As a consequence, based on Lemma 3.1, we claim the finite termination of Yamamoto's deflation method.

Theorem 3.2 *The number of Yamamoto's deflations needed to derive a regular solution of a square augmented system is bounded by the depth of* $\mathcal{D}_{\hat{\mathbf{x}}}$.

Proof. By Lemma 3.1, we have

$$\text{corank}\left(\widetilde{G}_{\mathbf{x},\boldsymbol{\lambda}_1}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1)\right) = \text{corank}\left(G_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1, \mathbf{0})\right). \tag{15}$$

Therefore, the smoothing parameters we added in the deflated system (11) do not change rank-deficient information of the Jacobian matrix of the deflated system (12). If $\text{corank}\left(\widetilde{G}_{\mathbf{x},\boldsymbol{\lambda}_1}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1)\right) = \text{corank}\left(G_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1, \mathbf{0})\right) > 0$, then we repeat the deflation steps to (11) and (12) accordingly. Inductively, we know that coranks of Jacobian matrices of two different kinds of deflated systems remain equal at every step. Moreover, we have shown that, after as most $\rho - 1$ steps, the corank of the Jacobian matrix of the deflated system corresponding to (12) will become zero. Therefore, the deflated system corresponding to (11) will also become regular after at most $\rho - 1$ steps. □

3.2 The second-order deflation

Suppose the Jacobian matrix $G_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}}_1,\mathbf{0})$ is singular, i.e.,

$$\operatorname{rank}(G_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}}_1,\mathbf{0})) = 2n - d',\ \ (d' \geq 1). \tag{16}$$

Let $\mathbf{c}' = \{c'_1, c'_2, \ldots, c'_{d'}\}$ and $\mathbf{k}' = \{k'_1, k'_2, \ldots, k'_{d'}\}$ be two positive-integer sets such that

$$\operatorname{rank}(G^{\mathbf{c}'}_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}}_1,\mathbf{0})) = 2n - d', \tag{17}$$

$$\operatorname{rank}\left(G^{\mathbf{c}'}_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}}_1,\mathbf{0}), I_{\mathbf{k}'+n}\right) = 2n, \tag{18}$$

where $G^{\mathbf{c}'}_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}}_1,\mathbf{0})$ is a matrix obtained from $G_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}}_1,\mathbf{0})$ by deleting its $c'_1, c'_2, \ldots, c'_{d'}$-th columns, and

$$I_{\mathbf{k}'+n} = \begin{pmatrix} \mathcal{O}_{n,d'} \\ I_{\mathbf{k}'} \end{pmatrix},\ I_{\mathbf{k}'} = (\mathbf{e}_{k'_1}, \mathbf{e}_{k'_2}, \ldots, \mathbf{e}_{k'_{d'}}). \tag{19}$$

Theorem 3.3 *Comparing to $F_{\mathbf{x}}(\hat{\mathbf{x}})$, the corank of $G_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}}_1,\mathbf{0})$ does not increase, i.e., $d' \leq d$. Moreover, we can choose $\mathbf{c}'$ and $\mathbf{k}'$ such that $\mathbf{c}' \subseteq \mathbf{c}$, $\mathbf{k}' \subseteq \mathbf{k}$ and satisfy (17) and (18) respectively.*

Proof. Let

$$G^{\mathbf{c}}_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}}_1,\mathbf{0}) = \begin{pmatrix} F^{\mathbf{c}}_{\mathbf{x}}(\hat{\mathbf{x}}) & \mathcal{O}_{n,n-d} & -I_{\mathbf{k}} \\ \star & F^{\mathbf{c}}_{\mathbf{x}}(\hat{\mathbf{x}}) & \mathcal{O}_{n,d} \end{pmatrix},$$

be the matrix obtained from $G_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}}_1,\mathbf{0})$ by deleting its $c_1, c_2, \ldots, c_d$-th columns. By (8) and (9) we claim that

$$\operatorname{rank}(G^{\mathbf{c}}_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}}_1,\mathbf{0})) = 2n - d.$$

Hence $d' \leq d$. Besides there exists a positive-integer set $\mathbf{c}' \subseteq \mathbf{c}$ such that the condition (17) is satisfied.

According to (9), it is clear that

$$\operatorname{rank}(G^{\mathbf{c}}_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}}_1,\mathbf{0}), I_{\mathbf{k}+n}) = 2n,$$

where $I_{\mathbf{k}+n} = \begin{pmatrix} \mathcal{O}_{n,d} \\ I_{\mathbf{k}} \end{pmatrix}$. Hence we can choose $\mathbf{k}' \subseteq \mathbf{k}$ such that the condition (18) is satisfied. $\square$

If $d' \geq 1$, then Yamamoto repeated the first-order deflation to $G(\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0)$ defined by (11). By Theorem 3.3, we notice that Yamamoto's second-order deflation is equivalent to adding d' new smoothing parameters denoted by

$\mathbf{b}_1$ to the first-order differential system $F_{\mathbf{x}}(\mathbf{x})\mathbf{v}_1 = \mathbf{0}$, to derive a square system

$$H(\mathbf{x},\boldsymbol{\lambda},\mathbf{b}) = \begin{cases} F(\mathbf{x}) - I_{\mathbf{k}}\mathbf{b}_0 = \mathbf{0}, \\ F_{\mathbf{x}}(\mathbf{x})\mathbf{v}_1 - I_{\mathbf{k}'}\mathbf{b}_1 = \mathbf{0}, \\ G_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0)\mathbf{v}_2 = \mathbf{0}, \end{cases} \tag{20}$$

where $\mathbf{v}_2$ is a vector consisting of $2n-d'$ extra variables $\boldsymbol{\lambda}_2$ and its entries at the positions $c_1', c_2', \ldots, c_{d'}'$ are all 1, and $\mathbf{b} = (\mathbf{b}_0^T, \mathbf{b}_1^T)^T$, $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_1^T, \boldsymbol{\lambda}_2^T)^T$. Let $\hat{\boldsymbol{\lambda}}_2$ denote the unique solution of the linear system $G_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}}_1,\mathbf{0})\mathbf{v}_2 = \mathbf{0}$, then $(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}},\mathbf{0})$ is an isolated solution of (20).

Suppose Theorem 2.5 is applicable to the deflated system $H(\mathbf{x},\boldsymbol{\lambda},\mathbf{b})$, and yields inclusions for $\hat{\mathbf{x}}$, $\hat{\boldsymbol{\lambda}}$, $\hat{\mathbf{b}}_0$ and $\hat{\mathbf{b}}_1$. Then we have

$$\widetilde{F}(\hat{\mathbf{x}}) = F(\hat{\mathbf{x}}) - I_{\mathbf{k}}\hat{\mathbf{b}}_0 = \mathbf{0} \text{ and } \widetilde{F}_{\mathbf{x}}(\hat{\mathbf{x}})\hat{\mathbf{v}}_1 = F_{\mathbf{x}}(\hat{\mathbf{x}})\hat{\mathbf{v}}_1 = I_{\mathbf{k}'}\hat{\mathbf{b}}_1, \tag{21}$$

where smoothing parameters $\hat{\mathbf{b}}_1$ might be very small, but are not guaranteed to be zeros. Therefore, one can only prove the existence of an isolated solution $\hat{\mathbf{x}}$ of a perturbed system $\widetilde{F}(\mathbf{x})$, which satisfies the first-order differential condition approximately.

In order to verify the existence of an isolated singular solution of a slightly perturbed system, we should add the smoothing parameters $\mathbf{b}_1$ back to the original system. Let us consider the modified system:

$$\widetilde{H}(\mathbf{x},\boldsymbol{\lambda},\mathbf{b}) = \begin{cases} F(\mathbf{x}) - I_{\mathbf{k}}\mathbf{b}_0 - X_1\mathbf{b}_1 = \mathbf{0}, \\ F_{\mathbf{x}}(\mathbf{x})\mathbf{v}_1 - I_{\mathbf{k}'}\mathbf{b}_1 = \mathbf{0}, \\ \widetilde{G}_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0,\mathbf{b}_1)\mathbf{v}_2 = \mathbf{0}, \end{cases} \tag{22}$$

where $X_1 = (x_{c_1'}\mathbf{e}_{k_1'}, \ldots, x_{c_{d'}'}\mathbf{e}_{k_{d'}'})$ and

$$\widetilde{G}(\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0,\mathbf{b}_1) = \begin{cases} F(\mathbf{x}) - I_{\mathbf{k}}\mathbf{b}_0 - X_1\mathbf{b}_1 = \mathbf{0}, \\ F_{\mathbf{x}}(\mathbf{x})\mathbf{v}_1 - I_{\mathbf{k}'}\mathbf{b}_1 = \mathbf{0}. \end{cases} \tag{23}$$

Theorem 3.4 *Let*

$$\widetilde{F}(\mathbf{x},\mathbf{b}) = F(\mathbf{x}) - I_{\mathbf{k}}\mathbf{b}_0 - X_1\mathbf{b}_1, \tag{24}$$

then we have

$$F_{\mathbf{x}}(\mathbf{x})\mathbf{v}_1 - I_{\mathbf{k}'}\mathbf{b}_1 = \mathbf{0} \Longleftrightarrow \widetilde{F}_{\mathbf{x}}(\mathbf{x},\mathbf{b})\mathbf{v}_1 = \mathbf{0}. \tag{25}$$

Proof. Let

$$\mathbf{b}_1 = (b_{1,1}, b_{1,2}, \ldots, b_{1,d'})^T \text{ and } \mathbf{v}_1 = (\lambda_1, \cdots, \underset{c_1}{1}, \cdots, \underset{c_d}{1}, \cdots, \lambda_{n-d})^T,$$

then

$$
\begin{aligned}
\widetilde{F}_{\mathbf{x}}(\mathbf{x},\mathbf{b})\mathbf{v}_1 &= F_{\mathbf{x}}(\mathbf{x})\mathbf{v}_1 - (\mathbf{0},\cdots,\underset{c'_1}{b_{1,1}\mathbf{e}_{k'_1}},\cdots,\underset{c'_{d'}}{b_{1,d'}\mathbf{e}_{k'_{d'}}},\cdots,\mathbf{0})\,\mathbf{v}_1 \\
&= F_{\mathbf{x}}(\mathbf{x})\mathbf{v}_1 - (\mathbf{e}_{k'_1},\ldots,\mathbf{e}_{k'_{d'}})\,\mathbf{b}_1 \text{ (since } \mathbf{c}' \subseteq \mathbf{c}) \\
&= F_{\mathbf{x}}(\mathbf{x})\mathbf{v}_1 - I_{\mathbf{k}'}\mathbf{b}_1
\end{aligned}
$$

□

According to Theorem 3.4, we can rewrite the system (22) as

$$
\widetilde{H}(\mathbf{x},\boldsymbol{\lambda},\mathbf{b}) = \begin{cases} \widetilde{F}(\mathbf{x},\mathbf{b}) = \mathbf{0}, \\ \widetilde{F}_{\mathbf{x}}(\mathbf{x},\mathbf{b})\mathbf{v}_1 = \mathbf{0}, \\ \widetilde{G}_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0,\mathbf{b}_1)\mathbf{v}_2 = \mathbf{0}. \end{cases} \tag{26}
$$

Therefore, if we can certify that $(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}},\hat{\mathbf{b}})$ is a regular solution of the augmented system $\widetilde{H}(\mathbf{x},\boldsymbol{\lambda},\mathbf{b})$ based on Theorem 2.5, then by (26), $\hat{\mathbf{x}}$ is guaranteed to be an isolated singular solution of $\widetilde{F}(\mathbf{x},\hat{\mathbf{b}})$.

Theorem 3.5 *Jacobian matrices of (20) and (22) share the same null space.*

Proof. The Jacobian matrix $H_{\mathbf{x},\boldsymbol{\lambda},\mathbf{b}}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}},\mathbf{0})$ of (20) computes to

$$
\left(\begin{array}{ccccccc}
F_{\mathbf{x}}(\hat{\mathbf{x}}) & \mathcal{O}_{n,n-d} & -I_{\mathbf{k}} & & \mathcal{O}_{2n,2n-d'} & & \mathcal{O}_{n,d'} \\
F_{\mathbf{x}\mathbf{x}}(\hat{\mathbf{x}})\hat{\mathbf{v}}_1 & F^{\mathbf{c}}_{\mathbf{x}}(\hat{\mathbf{x}}) & \mathcal{O}_{n,d} & & & & -I_{\mathbf{k}'} \\
 & & \mathcal{O}_{n,d} & F^{\mathbf{c}'}_{\mathbf{x}}(\hat{\mathbf{x}}) & \mathcal{O}_{n,n-d} & -I_{\mathbf{k}} & \mathcal{O}_{n,d} \\
 & \star & \mathcal{O}_{n,d} & \star & F^{\mathbf{c}}_{\mathbf{x}}(\hat{\mathbf{x}}) & \mathcal{O}_{n,d} & \mathcal{O}_{n,d}
\end{array}\right), \tag{27}
$$

while the Jacobian matrix $\widetilde{H}_{\mathbf{x},\boldsymbol{\lambda},\mathbf{b}}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}},\mathbf{0})$ of (22) computes to

$$
\left(\begin{array}{ccccccc}
F_{\mathbf{x}}(\hat{\mathbf{x}}) & \mathcal{O}_{n,n-d} & -I_{\mathbf{k}} & & \mathcal{O}_{2n,2n-d'} & & -\hat{X} \\
F_{\mathbf{x}\mathbf{x}}(\hat{\mathbf{x}})\hat{\mathbf{v}}_1 & F^{\mathbf{c}}_{\mathbf{x}}(\hat{\mathbf{x}}) & \mathcal{O}_{n,d} & & & & -I_{\mathbf{k}'} \\
 & & \mathcal{O}_{n,d} & F^{\mathbf{c}'}_{\mathbf{x}}(\hat{\mathbf{x}}) & \mathcal{O}_{n,n-d} & -I_{\mathbf{k}} & -I_{\mathbf{k}'} \\
 & \star & \mathcal{O}_{n,d} & \star & F^{\mathbf{c}}_{\mathbf{x}}(\hat{\mathbf{x}}) & \mathcal{O}_{n,d} & \mathcal{O}_{n,d'}
\end{array}\right), \tag{28}
$$

where the matrix $\hat{X}$ consists of vectors $\hat{x}_{\mathbf{c}'(i)}\mathbf{e}_{\mathbf{k}'(i)}$, $i = 1,\ldots,d'$. Since $\mathbf{k}' \subseteq \mathbf{k}$, we can reduce the last column of the block matrix (28) by its third and sixth columns to get the block matrix (27). Therefore, two Jacobian matrices (27) and (28) are of the same corank and share the same null space. □

Suppose the Jacobian matrix $H_{\mathbf{x},\boldsymbol{\lambda},\mathbf{b}}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}},\mathbf{0})$ is still singular, i.e.,

$$
\operatorname{rank}(H_{\mathbf{x},\boldsymbol{\lambda},\mathbf{b}}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}},\mathbf{0})) = 4n - d'', \ (d'' \geq 1). \tag{29}
$$

Let $\mathbf{c}'' = \{c''_1, c''_2, \ldots, c''_{d''}\}$ and $\mathbf{k}'' = \{k''_1, k''_2, \ldots, k''_{d''}\}$ be two positive-integer sets such that

$$\operatorname{rank}(H^{\mathbf{c}''}_{\mathbf{x},\boldsymbol{\lambda},\mathbf{b}}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}, \mathbf{0})) = 4n - d'' \tag{30}$$

$$\operatorname{rank}\left(H^{\mathbf{c}''}_{\mathbf{x},\boldsymbol{\lambda},\mathbf{b}}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}, \mathbf{0}), I_{\mathbf{k}''+3n}\right) = 4n, \tag{31}$$

where $H^{\mathbf{c}''}_{\mathbf{x},\boldsymbol{\lambda},\mathbf{b}}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}, \mathbf{0})$ is a matrix obtained from $H_{\mathbf{x},\boldsymbol{\lambda},\mathbf{b}}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}, \mathbf{0})$ by deleting its $c''_1, c''_2, \ldots, c''_{d''}$-th columns, and

$$I_{\mathbf{k}''+3n} = \begin{pmatrix} \mathcal{O}_{3n,d''} \\ I_{\mathbf{k}''} \end{pmatrix}, \ I_{\mathbf{k}''} = (\mathbf{e}_{k''_1}, \mathbf{e}_{k''_2}, \ldots, \mathbf{e}_{k'_{d''}}). \tag{32}$$

Theorem 3.6 *Comparing to $G_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1, \mathbf{0})$, the corank of $H_{\mathbf{x},\boldsymbol{\lambda},\mathbf{b}}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}, \mathbf{0})$ does not increase, i.e., $d'' \leq d'$. Moreover, we can choose $\mathbf{c}''$ and $\mathbf{k}''$ such that $\mathbf{c}'' \subseteq \mathbf{c}'$, $\mathbf{k}'' \subseteq \mathbf{k}'$ and satisfy (30) and (31) respectively.*

Proof. Similar to the proof of Theorem 3.3, let $H^{\mathbf{c}'}_{\mathbf{x},\boldsymbol{\lambda},\mathbf{b}}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}, \mathbf{0})$ be the matrix obtained from $H_{\mathbf{x},\boldsymbol{\lambda},\mathbf{b}}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}, \mathbf{0})$ by deleting its $c'_1, c'_2, \ldots, c'_d$-th columns. By (17) and (18), we claim that

$$\operatorname{rank}(H^{\mathbf{c}'}_{\mathbf{x},\boldsymbol{\lambda},\mathbf{b}}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}, \mathbf{0})) = 4n - d'.$$

Therefore, $d'' \leq d'$, and there exists a positive-integer set $\mathbf{c}'' \subseteq \mathbf{c}'$ such that the condition (30) is satisfied.

Meanwhile, we know that $\operatorname{rank}(G^{\mathbf{c}'}_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1, \mathbf{0}), I_{\mathbf{k}'+n}) = 2n$, then

$$\operatorname{rank}(H^{\mathbf{c}'}_{\mathbf{x},\boldsymbol{\lambda},\mathbf{b}}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}, \mathbf{0}), I_{\mathbf{k}'+3n}) = 4n,$$

where $I_{\mathbf{k}'+3n} = \begin{pmatrix} \mathcal{O}_{3n,d'} \\ I_{\mathbf{k}'} \end{pmatrix}$. Therefore, we can choose $\mathbf{k}'' \subseteq \mathbf{k}'$ such that the condition (31) is satisfied. $\square$

EXAMPLE 3.1 *[4, DZ1] Consider a polynomial system*

$$F = \{x_1^4 - x_2x_3x_4, x_2^4 - x_1x_3x_4, x_3^4 - x_1x_2x_4, x_4^4 - x_1x_2x_3\}.$$

The system F has $(0, 0, 0, 0)$ as a 131-fold isolated zero.

Since $F_{\mathbf{x}}(\hat{\mathbf{x}}) = \mathcal{O}_{4,4}$, we derive $d = 4$, $\mathbf{c} = \mathbf{k} = \{1, 2, 3, 4\}$ and $\mathbf{v}_1 = (1, 1, 1, 1)^T$

$$G(\mathbf{x}, \mathbf{b}_0) = \begin{cases} F(\mathbf{x}) - I_{\mathbf{k}}\mathbf{b}_0 = \mathbf{0}, \\ 4x_1^3 - x_3x_4 - x_2x_4 - x_2x_3 = 0, \\ 4x_2^3 - x_3x_4 - x_1x_4 - x_1x_3 = 0, \\ 4x_3^3 - x_2x_4 - x_1x_4 - x_1x_2 = 0, \\ 4x_4^3 - x_2x_3 - x_1x_3 - x_1x_2 = 0. \end{cases}$$

The Jacobian matrix of $G(\mathbf{x},\mathbf{b}_0)$ at $(\mathbf{0},\mathbf{0})$ is

$$G_{\mathbf{x},\mathbf{b}_0}(\mathbf{0},\mathbf{0})=\begin{pmatrix}\mathcal{O}_{4,4} & -I_{\mathbf{k}}\\ \mathcal{O}_{4,4} & \mathcal{O}_{4,4}\end{pmatrix},$$

Hence, $d'=4$, $\mathbf{c}'=\mathbf{k}'=\{1,2,3,4\}$ and

$$H(\mathbf{x},\boldsymbol{\lambda},\mathbf{b})=\begin{cases} F(\mathbf{x})-I_{\mathbf{k}}\mathbf{b}_0-X_1\mathbf{b}_1=\mathbf{0},\\ F_{\mathbf{x}}(\mathbf{x})\mathbf{v}_1-I_{\mathbf{k}'}\mathbf{b}_1=\mathbf{0},\\ \widetilde{G}_{\mathbf{x},\mathbf{b}_0}(\mathbf{x},\mathbf{b}_0,\mathbf{b}_1)\mathbf{v}_2=\mathbf{0},\end{cases}\tag{33}$$

where $\mathbf{v}_2=(1,1,1,1,\lambda_1,\lambda_2,\lambda_3,\lambda_4)^T$, and $\widetilde{G}_{\mathbf{x},\mathbf{b}_0}(\mathbf{0},\mathbf{0},\mathbf{0})\mathbf{v}_2=\mathbf{0}$ has a unique solution $(\hat{\lambda}_1,\hat{\lambda}_2,\hat{\lambda}_3,\hat{\lambda}_4)=(0,0,0,0)$. The Jacobian matrix of $H(\mathbf{x},\boldsymbol{\lambda},\mathbf{b})$ at $(\mathbf{0},\mathbf{0},\mathbf{0})$ is

$$H_{\mathbf{x},\boldsymbol{\lambda},\mathbf{b}}(\mathbf{0},\mathbf{0},\mathbf{0})=\begin{pmatrix}\mathcal{O}_{4,4} & -I_{\mathbf{k}} & \mathcal{O}_{4,4} & \mathcal{O}_{4,4}\\ \mathcal{O}_{4,4} & \mathcal{O}_{4,4} & \mathcal{O}_{4,4} & -I_{\mathbf{k}'}\\ \mathcal{O}_{4,4} & \mathcal{O}_{4,4} & -I_{\mathbf{k}'} & -I_{\mathbf{k}'}\\ A & \mathcal{O}_{4,4} & \mathcal{O}_{4,4} & \mathcal{O}_{4,4}\end{pmatrix}, A=\begin{pmatrix}0 & -2 & -2 & -2\\ -2 & 0 & -2 & -2\\ -2 & -2 & 0 & -2\\ -2 & -2 & -2 & 0\end{pmatrix}.$$

The Jacobian matrix $H_{\mathbf{x},\boldsymbol{\lambda},\mathbf{b}}(\mathbf{0},\mathbf{0},\mathbf{0})$ is nonsingular. Therefore we obtain a regular and square system $H(\mathbf{x},\boldsymbol{\lambda},\mathbf{b})$ and a perturbed system

$$\widetilde{F}(\mathbf{x},\mathbf{b})=\begin{cases} x_1^4-x_2x_3x_4-b_1-b_5x_1=0,\\ x_2^4-x_1x_3x_4-b_2-b_6x_2=0,\\ x_3^4-x_1x_2x_4-b_3-b_7x_3=0,\\ x_4^4-x_1x_2x_3-b_4-b_8x_4=0.\end{cases}$$

Applying the verification method based on Theorem 2.5 to $H(\mathbf{x},\boldsymbol{\lambda},\mathbf{b})$, we show in Section 4 that a slightly perturbed polynomial system $\widetilde{F}(\mathbf{x},\hat{\mathbf{b}})$ for

$$|\hat{b}_i|\leq 1.0e-321, i=1,2,\ldots,8$$

has an isolated singular solution $\hat{\mathbf{x}}$ within

$$|\hat{x}_i|\leq 1.0e-321, i=1,2,3,4.$$

3.3 Higher-order deflations

For higher-order deflations, in the following, we show inductively how to add new smoothing parameters properly to the original system in order to derive a square and regular deflated system for certifying the existence of an isolated singular solution of a slightly perturbed system.

Let $H^{(0)}(\mathbf{x})=F(\mathbf{x})$, then for the $(s+1)$-th deflation, we add smoothing parameters $\mathbf{b}^{(s)}=(\mathbf{b}_0^T,\ldots,\mathbf{b}_s^T)^T$ and consider the following square system

$$H^{(s+1)}(\mathbf{x},\boldsymbol{\lambda}^{(s+1)},\mathbf{b}^{(s)})=\begin{cases} \widetilde{F}(\mathbf{x},\mathbf{b}^{(s)}) &=\mathbf{0},\\ \widetilde{F}_{\mathbf{x}}(\mathbf{x},\mathbf{b}^{(s)})\mathbf{v}_1 &=\mathbf{0},\\ \vdots & \\ G^{(s)}_{\mathbf{x},\boldsymbol{\lambda}^{(s)},\mathbf{b}^{(s-1)}}(\mathbf{x},\boldsymbol{\lambda}^{(s)},\mathbf{b}^{(s)})\mathbf{v}_{s+1} &=\mathbf{0},\end{cases}\tag{34}$$

where $\boldsymbol{\lambda}^{(s+1)} = (\boldsymbol{\lambda}_1^T, \ldots, \boldsymbol{\lambda}_{s+1}^T)^T$ are extra variables corresponding to the vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_{s+1}\}$, $G^{(s)}(\mathbf{x}, \boldsymbol{\lambda}^{(s)}, \mathbf{b}^{(s)})$ consists of the first $2^s n$ polynomials in $H^{(s+1)}(\mathbf{x}, \boldsymbol{\lambda}^{(s+1)}, \mathbf{b}^{(s)})$, and

$$\widetilde{F}(\mathbf{x}, \mathbf{b}^{(s)}) = F(\mathbf{x}) - X_0\mathbf{b}_0 - X_1\mathbf{b}_1 - \cdots - X_s\mathbf{b}_s, \tag{35}$$

the matrix X_j $(0 \le j \le s)$ consists of vectors $\frac{1}{j!} \cdot x^j_{\mathbf{c}^{(j)}(i)} \cdot \mathbf{e}_{\mathbf{k}^{(j)}(i)}$, $i = 1, \ldots, d_j$, where $\mathbf{c}^{(j)}$ and $\mathbf{k}^{(j)}$are two positive-integer sets selected at the j-th order deflation satisfying conditions obtained by replacing the polynomial system $F(\mathbf{x})$ in (8) and (9) by the j-th deflated system $H^{(j)}(\mathbf{x}, \boldsymbol{\lambda}^{(j)}, \mathbf{b}^{(j-1)})$ and replacing $I_{\mathbf{k}}$ by the matrix $I_{\mathbf{k}^{(j)}+(2^j-1)n} = \begin{pmatrix} \mathcal{O}_{(2^j-1)n, d_j} \\ I_{\mathbf{k}^{(j)}} \end{pmatrix}$, $I_{\mathbf{k}^{(j)}} = (\mathbf{e}_{k_1^{(j)}}, \mathbf{e}_{k_2^{(j)}}, \ldots, \mathbf{e}_{k_{d_j}^{(j)}})$, where d_j is the corank of $H^{(j)}_{\mathbf{x}, \boldsymbol{\lambda}^{(j)}, \mathbf{b}^{(j-1)}}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}^{(j)}, \mathbf{0})$.

Theorem 3.7 *The corank d_{s+1} of $H^{(s+1)}_{\mathbf{x}, \boldsymbol{\lambda}^{(s+1)}, \mathbf{b}^{(s)}}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}^{(s+1)}, \mathbf{0})$ does not increase and the number of deflations needed to derive a regular solution of an augmented system (34) is less than the depth of $\mathcal{D}_{\hat{\mathbf{x}}}$, i.e., we have*

$$d_0 \ge d_1 \ge \cdots \ge d_{s+1} \ge \cdots \ge d_{\rho-1} = 0. \tag{36}$$

Moreover, we can choose $\mathbf{c}^{(j)}$ and $\mathbf{k}^{(j)}$ satisfying

$$\mathbf{c}^{(s)} \subseteq \cdots \subseteq \mathbf{c}^{(0)} \text{ and } \mathbf{k}^{(s)} \subseteq \cdots \subseteq \mathbf{k}^{(0)}. \tag{37}$$

Proof. Applying Theorem 3.3, 3.5 and 3.6 inductively, we can show that the above deflation process (34) produces a decreasing nonnegative-integer sequence $d_0 \ge d_1 \ge \cdots \ge d_{s+1} \ge \cdots$, which is as same as the sequence consisting of coranks of the Jacobian matrices of the deflated systems by Yamamoto's method. According to Theorem 3.2, the number of Yamamoto's deflations to derive a regular solution of an augmented system is bounded by the depth of $\mathcal{D}_{\hat{\mathbf{x}}}$. Hence the number of the modified deflations (34) is also bounded by the depth of $\mathcal{D}_{\hat{\mathbf{x}}}$. The proof of (37) is similar to the proofs of Theorem 3.3 and 3.6. $\square$

Theorem 3.8 *Suppose Theorem 2.5 is applicable to the augmented system (34), and yields inclusions for $\hat{\mathbf{x}}$, $\hat{\boldsymbol{\lambda}}$ and $\hat{\mathbf{b}}$. Then the perturbed system $\widetilde{F}(\mathbf{x}, \hat{\mathbf{b}})$ has an isolated singular solution at $\hat{\mathbf{x}}$.*

Proof. Since $(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}, \hat{\mathbf{b}})$ is the unique solution of the augmented system (34), we have

$$\widetilde{F}(\hat{\mathbf{x}}, \hat{\mathbf{b}}) = \mathbf{0} \text{ and } \widetilde{F}_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{\mathbf{b}})\hat{\mathbf{v}}_1 = \mathbf{0},\ \hat{\mathbf{v}}_1 \ne \mathbf{0}.$$

Hence, $\hat{\mathbf{x}}$ is an isolated singular solution of the slightly perturbed system

$$\widetilde{F}(\mathbf{x}, \hat{\mathbf{b}}) = F(\mathbf{x}) - X_0\hat{\mathbf{b}}_0 - X_1\hat{\mathbf{b}}_1 - \cdots - X_s\hat{\mathbf{b}}_s.$$

$\square$

EXAMPLE 3.2 *[4, DZ2] Consider a polynomial system*

$$F = \{x^4, x^2y + y^4, z + z^2 - 7x^3 - 8x^2\}.$$

The system F has $(0, 0, -1)$ as a 16-fold isolated zero.

The Jacobian matrix of F at $\hat{\mathbf{x}} = (0, 0, -1)$ is

$$F_{\mathbf{x}}(\hat{\mathbf{x}}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \text{ so that } d_0 = 2 \text{ and we choose } \mathbf{c}^{(0)} = \mathbf{k}^{(0)} = \{1, 2\}.$$

The first-order deflated system is

$$H^{(1)}(\mathbf{x}, \boldsymbol{\lambda}_1, \mathbf{b}_0) = \begin{cases} F(\mathbf{x}) - X_0\mathbf{b}_0 = \mathbf{0}, \\ 4x^3 = 0, \\ 2xy + x^2 + 4y^3 = 0, \\ -21x^2 - 16x + \lambda_1 + 2z\lambda_1 = 0, \end{cases}$$

where

$$X_0 = (\mathbf{e}_1, \mathbf{e}_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{b}_0 = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad \mathbf{v}_1 = (1, 1, \lambda_1)^T, \quad \boldsymbol{\lambda}_1 = (\lambda_1).$$

The Jacobian matrix of $H^{(1)}(\mathbf{x}, \boldsymbol{\lambda}_1, \mathbf{b}_0)$ at $(0, 0, -1, 0, 0, 0)$ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -16 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}, \ d_1 = 2 \text{ and we choose } \mathbf{c}^{(1)} = \mathbf{k}^{(1)} = \{1, 2\}.$$

Therefore, we derive the second-order deflated system

$$H^{(2)}(\mathbf{x}, \boldsymbol{\lambda}^{(2)}, \mathbf{b}^{(1)}) = \begin{cases} F(\mathbf{x}) - X_0\mathbf{b}_0 - X_1\mathbf{b}_1 = \mathbf{0}, \\ F_{\mathbf{x}}(\mathbf{x})\mathbf{v}_1 - X_1'\mathbf{b}_1 = \mathbf{0}, \\ G^{(1)}_{\mathbf{x}, \boldsymbol{\lambda}_1, \mathbf{b}_0}(\mathbf{x}, \boldsymbol{\lambda}_1, \mathbf{b}^{(1)})\mathbf{v}_2 = \mathbf{0}, \end{cases}$$

where

$$X_1 = \begin{pmatrix} x & 0 \\ 0 & y \\ 0 & 0 \end{pmatrix}, \quad \mathbf{b}_1 = \begin{pmatrix} b_3 \\ b_4 \end{pmatrix}, \quad \mathbf{b}^{(1)} = (b_1, b_2, b_3, b_4)^T, \quad X_1' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$\mathbf{v}_2 = (1, 1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)^T, \quad \boldsymbol{\lambda}^{(2)} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)^T.$$

Moreover, $G^{(1)}_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}^{(1)}, \mathbf{0})\mathbf{v}_2 = \mathbf{0}$ has a unique solution $\hat{\boldsymbol{\lambda}}_2 = (0, -16, 0, 0)^T$.

For the third-order deflation, we have $d_2 = 1$, $\mathbf{c}^{(2)} = \mathbf{k}^{(2)} = \{1\}$, so

$$H^{(3)}(\mathbf{x}, \boldsymbol{\lambda}^{(3)}, \mathbf{b}^{(2)}) = \begin{cases} F(\mathbf{x}) - X_0\mathbf{b}_0 - X_1\mathbf{b}_1 - X_2\mathbf{b}_2 = \mathbf{0}, \\ F_{\mathbf{x}}(\mathbf{x})\mathbf{v}_1 - X_1'\mathbf{b}_1 - X_2'\mathbf{b}_2 = \mathbf{0}, \\ F_{\mathbf{x}}(\mathbf{x})\mathbf{v}_2' - X_0\mathbf{v}_2'' - X_1'\mathbf{b}_1 - X_2'\mathbf{b}_2 = \mathbf{0}, \\ F_{\mathbf{x}\mathbf{x}}(\mathbf{x})\mathbf{v}_1\mathbf{v}_2' + F_{\mathbf{x}}^{\mathbf{c}^{(0)}}(\mathbf{x})\lambda_3 - X_2''\mathbf{b}_2 = \mathbf{0}, \\ G^{(2)}_{\mathbf{x},\boldsymbol{\lambda}^{(2)},\mathbf{b}^{(1)}}(\mathbf{x}, \boldsymbol{\lambda}^{(2)}, \mathbf{b}^{(2)})\mathbf{v}_3 = \mathbf{0}, \end{cases} \tag{38}$$

where

$$X_2 = \begin{pmatrix} \frac{1}{2}x^2 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{b}_2 = (b_5), \quad X_2' = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}, \quad X_2'' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{v}_2' = \begin{pmatrix} 1 \\ 1 \\ \lambda_2 \end{pmatrix}, \mathbf{v}_2'' = \begin{pmatrix} \lambda_4 \\ \lambda_5 \end{pmatrix},$$

$$\mathbf{v}_3 = (1, \lambda_6, \lambda_7, \ldots, \lambda_{16})^T, \quad \boldsymbol{\lambda}^{(3)} = (\lambda_1, \ldots, \lambda_{16})^T.$$

Moreover, $G^{(2)}_{\mathbf{x},\boldsymbol{\lambda}^{(2)},\mathbf{b}^{(1)}}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}^{(2)}, \mathbf{0})\mathbf{v}_3 = \mathbf{0}$ has a unique solution

$$\hat{\boldsymbol{\lambda}}_3 = (-2, 0, 0, 0, -16, 0, 0, -16, 0, 0, -42)^T.$$

Finally, the Jacobian matrix of $H^{(3)}(\mathbf{x}, \boldsymbol{\lambda}^{(3)}, \mathbf{b}^{(2)})$ is nonsingular, and we obtain a perturbed polynomial system

$$\begin{aligned} \widetilde{F}(\mathbf{x}, \mathbf{b}) &= F(\mathbf{x}) - X_0\mathbf{b}_0 - X_1\mathbf{b}_1 - X_2\mathbf{b}_2 \\ &= \{x^4 - b_1 - b_3x - \frac{1}{2}b_5x^2, x_2y + y^2 - b_2 - b_4y, z + z^2 - 7x^3 - 8x^2\}. \end{aligned} \tag{39}$$

Note that

$$F_{\mathbf{x}}(\mathbf{x})\mathbf{v}_1 - X_0'\mathbf{b}_1 - X_1'\mathbf{b}_2 = \mathbf{0} \Leftrightarrow \widetilde{F}_{\mathbf{x}}(\mathbf{x}, \mathbf{b})\mathbf{v}_1 = \mathbf{0},$$

after applying the verification method to the above regular augmented system (38), we are able to verify that a slightly perturbed system $\widetilde{F}(\mathbf{x}, \hat{\mathbf{b}})$ defined in (39) for

$$|\hat{b}_i| \leq 1.0e-14, \quad i = 1, 2, \ldots, 5$$

has an isolated singular solution $\hat{\mathbf{x}}$ within

$$|\hat{x}_i| \leq 1.0e-14, \; i = 1, 2, \text{ and } |1 + \hat{x}_3| \leq 1.0e-14.$$

4 An Algorithm for Verifying Multiple Roots

Based on Theorem 3.7 and 3.8, we propose below an algorithm for computing verified error bounds such that, a slightly perturbed system is guaranteed to possess an isolated singular solution within the computed bounds.

Algorithm 4.1 *VISS*
Input: *A square polynomial system* $F \in \mathbb{C}[x_1, \ldots, x_n]$*, a point* $\tilde{\mathbf{x}} \in \mathbb{C}^n$ *and a tolerance* ε*.*

Output: *A perturbed system* $\widetilde{F}(\mathbf{x}, \mathbf{b})$*, inclusions* $\mathbf{X}$ *and* $\mathbf{B}$ *for* $\hat{\mathbf{x}}$ *and* $\hat{\mathbf{b}}$ *such that* $\widetilde{F}(\hat{\mathbf{x}}, \hat{\mathbf{b}}) = \mathbf{0}$ *and* $\widetilde{F}_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{\mathbf{b}})$ *is singular.*

1. *Set* $s := 0$*,* $m := n$*,* $\widetilde{F} := F$*,* $G := \widetilde{F}$*,* $\mathbf{y} := \mathbf{x}$*, and* $\tilde{\mathbf{y}} := \tilde{\mathbf{x}}$*.*

2. *Compute* $d := n - \operatorname{rank}(F_{\mathbf{x}}(\tilde{\mathbf{x}}), \varepsilon)$*, select integer sets* $\mathbf{c}$ *and* $\mathbf{k}$ *satisfying (8) and (9) respectively.*

3. *Set* $\widetilde{F} := \widetilde{F} + X_s \mathbf{b}_s$*, where the matrix* X_s *consists of vectors* $\frac{1}{s!} \cdot x^s_{\mathbf{c}(i)} \cdot \mathbf{e}_{\mathbf{k}(i)}$*,* $i = 1, \ldots, d$*.*

   (a) *If* $s \geq 1$*, then set* $G := \widetilde{F}$*; for* j *from 1 to* s *do* $G := \{G, G_{\mathbf{y}} \mathbf{v}_j\}$*;* $\mathbf{y} := (\mathbf{y}, \boldsymbol{\lambda}_j, \mathbf{b}_{j-1})$*.*

   (b) *Compute* $\tilde{\mathbf{y}} := (\tilde{\mathbf{y}}, \operatorname{LeastSquares}(G_{\mathbf{y}}(\tilde{\mathbf{y}})\mathbf{v}_{s+1} = \mathbf{0}), \mathbf{0})$*;*

   (c) *Set* $G := \{G, G_{\mathbf{y}} \mathbf{v}_{s+1}\}$*;* $\mathbf{y} := (\mathbf{y}, \boldsymbol{\lambda}_{s+1}, \mathbf{b}_s)$*;* $m := 2m$*.*

4. *Compute* $d := m - \operatorname{rank}(G_{\mathbf{y}}(\tilde{\mathbf{y}}), \varepsilon)$*;*

   (a) *If* $d = 0$*, apply* `verifynlss` *to* G *and* $\tilde{\mathbf{y}}$ *to compute inclusions* $\mathbf{X}$ *and* $\mathbf{B}$ *for* $\hat{\mathbf{x}}$ *and* $\hat{\mathbf{b}}$*.*

   (b) *Otherwise, select* $\mathbf{c}$*,* $\mathbf{k}$ *satisfying (8),(9) for the polynomial system* G*, set* $s := s + 1$*,* $\mathbf{y} = \mathbf{x}$ *and go back to Step 3.*

Example 3.1 (continued) Given an approximate singular solution $\tilde{\mathbf{x}} = (.0003445, .0009502, .0003171, .0006948)$ and a tolerance $\varepsilon = 0.005$, we obtain the augmented system (33) and a point

$$\tilde{\mathbf{y}} = (\tilde{\mathbf{x}}, 0.8009 \times 10^{-6}, 0.4236 \times 10^{-7}, 0.8859 \times 10^{-7}, 0.5374 \times 10^{-7}, 0, \ldots, 0).$$

After running `verifynlss`$(H, \tilde{\mathbf{y}})$ in Matlab [32], it yields

$$-1.0e - 321 \leq \hat{x}_i \leq 1.0e - 321, \text{ for } i = 1, 2, 3, 4,$$

$$-1.0e - 321 \leq \hat{b}_i \leq 1.0e - 321, \text{ for } i = 1, 2, \ldots, 8.$$

By Theorem 3.8, this proves that the perturbed polynomial system $\widetilde{F}(\mathbf{x}, \hat{\mathbf{b}})$ ($|\hat{b}_i| \leq 1.0e - 321, i = 1, 2, \ldots, 8$) has an isolated singular solution $\hat{\mathbf{x}}$ within $|\tilde{x}_i| \leq 1.0e - 321, i = 1, 2, 3, 4$.

Special case The breadth-one case where the corank of the Jacobian matrix equals one occurs frequently, and can be treated more efficiently.

In fact, we have shown in [22, Theorem 3.8] that each step of deflation described by (5) only reduces the multiplicity μ of the singular solution $\hat{\mathbf{x}}$ by 1. According to Theorem 3.7, the number of deflations described by (34) will be $\mu - 1$. Hence, Algorithm VISS generates an augmented regular system of the size $(2^{\mu-1}n) \times (2^{\mu-1}n)$. However, in [20], we introduced a more efficient method based on the parameterized multiplicity structure, to obtain a deflated regular system $G(\mathbf{x}, \mathbf{b}, \boldsymbol{\lambda})$ which is of the size $(\mu n) \times (\mu n)$ and can be used to verify not only the existence of an isolated singular solution, but also its multiplicity structure.

Let us introduce briefly the method in [20] for the special case of breadth one. By adding $\mu - 1$ smoothing parameter $b_0, b_1, \ldots, b_{\mu-2}$ to a well selected polynomial, assumed to be f_1, we derive a square augmented system

$$G(\mathbf{x}, \mathbf{b}, \boldsymbol{\lambda}) = \begin{pmatrix} \widetilde{F}(\mathbf{x}, \mathbf{b}) \\ L_1(\widetilde{F}) \\ \vdots \\ L_{\mu-1}(\widetilde{F}) \end{pmatrix} = \mathbf{0}, \text{ where } \widetilde{F}(\mathbf{x}, \mathbf{b}) = \begin{pmatrix} f_1(\mathbf{x}) - \sum_{\nu=0}^{\mu-2} \frac{b_\nu x_1^\nu}{\nu!} \\ f_2(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{pmatrix},$$

and $L_1, \ldots, L_{\mu-1}$ are parameterized bases of the local dual space in variables $\boldsymbol{\lambda}$. Furthermore, we proved that if Theorem 2.5 is applicable to G and yields inclusions for $\hat{\mathbf{x}} \in \mathbb{R}^n$, $\hat{\mathbf{b}} \in \mathbb{R}^{\mu-1}$ and $\hat{\boldsymbol{\lambda}} \in \mathbb{R}^{(\mu-1)\times(n-1)}$ such that $G(\hat{\mathbf{x}}, \hat{\mathbf{b}}, \hat{\boldsymbol{\lambda}}) = \mathbf{0}$, then $\hat{\mathbf{x}}$ is a breadth-one singular solution of $\widetilde{F}(\mathbf{x}, \hat{\mathbf{b}}) = \mathbf{0}$ with multiplicity μ and $\{1, L_1, \ldots, L_{\mu-1}\}$ with $\boldsymbol{\lambda} = \hat{\boldsymbol{\lambda}}$ is a basis of $\mathcal{D}_{\hat{\mathbf{x}}}$.

EXAMPLE 4.1 *[33, Example 4.11] Consider a polynomial system*

$$F = \{x_1^2 x_2 - x_1 x_2^2, x_1 - x_2^2\}.$$

The system F has $(0,0)$ as a 4-fold isolated zero.

We choose x_2 as the perturbed variable and add the univariate polynomial $-b_1 - b_2 x_2 - \frac{b_3}{2} x_2^2$ to the first equation in F to obtain an augmented system

$$\left\{ \begin{array}{r} x_1^2 x_2 - x_1 x_2^2 - b_1 - b_2 x_2 - \frac{b_3}{2} x_2^2 = 0, \\ x_1 - x_2^2 = 0, \\ 2\lambda_1 x_1 x_2 - \lambda_1 x_2^2 + x_1^2 - 2x_1 x_2 - b_2 - b_3 x_2 = 0, \\ \lambda_1 - 2x_2 = 0, \\ \lambda_1^2 x_2 + 2\lambda_1 x_1 - 2\lambda_1 x_2 + 2\lambda_2 x_1 x_2 - \lambda_2 x_2^2 - x_1 - \frac{b_3}{2} = 0, \\ \lambda_2 - 1 = 0, \\ \lambda_1^2 + 2\lambda_1\lambda_2 x_2 - \lambda_1 + 2\lambda_2 x_1 - 2\lambda_2 x_2 + 2\lambda_3 x_1 x_2 - \lambda_3 x_2^2 = 0, \\ \lambda_3 = 0, \end{array} \right.$$

which is of the size 8×8 while Algorithm VISS generates a system of the size 16×16. Applying verifynlss with an initial approximation

$$(0.002, 0.003, -0.001, 0.0015, -0.002, 0.002, 1.001, -0.01),$$

we obtain inclusions

$$-1.0e - 14 \leq \hat{x}_i \leq 1.0e - 14, \text{ for } i = 1, 2, 3,$$

$$-1.0e - 14 \leq \hat{b}_i \leq 1.0e - 14, \text{ for } i = 1, 2, 3.$$

This proves that the perturbed system $\widetilde{F}(\mathbf{x}, \hat{\mathbf{b}})$ ($|\hat{b}_i| \leq 10^{-14}, i = 1, 2, 3$) has a 4-fold breadth-one root $\hat{\mathbf{x}}$ within $|\hat{x}_i| \leq 10^{-14}, i = 1, 2, 3$.

5 Experiments

We can generate an augmented square and regular system and initial values for $\tilde{\mathbf{y}}$ in Maple or Matlab, then apply INTLAB function verifynlss in Matlab [32] to obtain the verified error bounds. The following experiments are done in Maple 15 for Digits := 14 and Matlab R2011a with INTLAB_V6 under Windows 7. Let n be the number of polynomials and variables, μ be the multiplicity. The fourth and fifth column show the decrease of the corank and the increase of the smallest singular values of the Jacobian matrix respectively. The last two columns give qualities of the verified error bounds.

The first three examples DZ1, DZ2, DZ3 are cited from [4]. It should be noticed that the coefficients of polynomials in the example DZ3 have algebraic numbers $\sqrt{5}, \sqrt{7}$. These irrational coefficients are rounded to fourteen digits in Maple or Matlab. The other examples are quoted from the PHCpack demos by Jan Verschelde. Codes of Algorithm VISS and examples are available at `http://www.mmrc.iss.ac.cn/~lzhi/Research/hybrid/VISS`.

Acknowledgements The authors are grateful to Yijun Zhu for helping us implement Algorithm VISS in Matlab. The first author is grateful to Anton Leykin for fruitful discussions during the IMA summer program at Georgia Tech, 2012.

This research is supported by NKBRPC 2011CB302400 and the Chinese National Natural Science Foundation under Grants: 91118001, 60821002/F02, 60911130369 and 10871194.

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Table 1: Algorithm Performance

System	n	μ	corank($G_{\mathbf{y}}(\tilde{\mathbf{y}})$)	Smallest σ	$\|\mathbf{X}\|$	$\|\mathbf{B}\|$
DZ1	4	131	$4 \to 4 \to 0$	1.1e-07 $\to$ 6.2e-01	e-321	e-321
DZ2	3	16	$2 \to 2 \to 1 \to 0$	7.1e-11 $\to$ 5.3e-03	e-14	e-14
DZ3	2	4	$1 \to 1 \to 1 \to 0$	2.2e-04 $\to$ 9.6e-03	e-7	e-7
cbms1	3	11	$3 \to 0$	5.5e-04 $\to$ 1.0e-00	e-321	e-321
cbms2	3	8	$3 \to 0$	3.2e-04 $\to$ 1.0e-00	e-321	e-321
mth191	3	4	$2 \to 0$	2.5e-04 $\to$ 3.7e-01	e-14	e-14
KSS	10	638	$9 \to 0$	6.5e-05 $\to$ 3.0e-01	e-14	e-14
Caprasse	4	4	$2 \to 0$	1.4e-03 $\to$ 9.9e-01	e-14	e-14
cyclic9	9	4	$2 \to 0$	2.1e-10 $\to$ 3.8e-01	e-13	e-13
RuGr09	2	4	$1 \to 1 \to 1 \to 0$	3.0e-07 $\to$ 1.0e-00	e-14	e-14
LiZhi12	100	3	$1 \to 1 \to 0$	3.6e-12 $\to$ 2.2e-05	e-14	e-14
Ojika1	2	3	$1 \to 1 \to 0$	3.7e-04 $\to$ 5.6e-02	e-14	e-14
Ojika2	3	2	$1 \to 0$	9.9e-04 $\to$ 4.6e-01	e-14	e-14
Ojika3	3	2	$1 \to 0$	9.6e-05 $\to$ 5.0e-02	e-14	e-14
Ojika4	3	3	$1 \to 1 \to 0$	1.2e-04 $\to$ 2.0e-00	e-14	e-14
Decker2	3	4	$1 \to 1 \to 1 \to 0$	2.2e-09 $\to$ 1.0e-00	e-14	e-14

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