ASYMPTOTICS OF THE FIRST LAPLACE EIGENVALUE WITH DIRICHLET REGIONS OF PRESCRIBED LENGTH

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ABSTRACT. We consider the problem of maximizing the first eigenvalue of the p-laplacian (possibly with non-constant coefficients) over a fixed domain Ω , with Dirichlet conditions along $\partial\Omega$ and along a supplementary set Σ , which is the unknown of the optimization problem. The set Σ , that plays the role of a supplementary stiffening rib for a membrane Ω , is a compact connected set (e.g. a curve or a connected system of curves) that can be placed anywhere in $\overline{\Omega}$, and is subject to the constraint of an upper bound L to its total length (one-dimensional Hausdorff measure). This upper bound prevents Σ from spreading throughout Ω and makes the problem well-posed. We investigate the behavior of optimal sets Σ_L as $L \to \infty$ via Γ -convergence, and we explicitly construct certain asymptotically optimal configurations. We also study the behavior as $p \to \infty$ with L fixed, finding connections with maximum-distance problems related to the principal frequency of the ∞ -laplacian.

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1. INTRODUCTION

Variational problems for the eigenvalues of the Laplace operator have a long history, which dates back to 1877 when Lord Rayleigh observed and conjectured that of all membranes with a given area, the circle has the minimum principal frequency. Lord Rayleigh's conjecture was proved many years later by G. Faber in 1923 and, independently, by E. Krahn in 1924, and many other contributions to similar problems were given ever since (see [11] for other references on the subject).

Here we consider a new optimization problem for the first eigenvalue of an isotropic elliptic operator with nonconstant coefficients in two dimensions, where the unknown is the location of the Dirichlet condition. The general setting consists of:

- a bounded connected open set $\Omega \subset \mathbb{R}^2$, with Lipschitz boundary $\partial \Omega$ (we do not assume that Ω is simply connected);
- two functions ρ, σ , continuous over $\overline{\Omega}$ and strictly positive;
- a real number $p \ge 1$.

With these ingredients, for any open set $A \subset \Omega$ one can define the first Dirichlet eigenvalue $\lambda_p^{\sigma,\rho}(A)$ through the variational formula

(1)
$$\lambda_p^{\sigma,\rho}(A) := \inf_{\substack{u \in W_0^{1,p}(A)\\ u \neq 0}} \frac{\int_A \sigma(x) |\nabla u(x)|^p dx}{\int_A \rho(x) |u(x)|^p dx}$$

(when p > 1 the infimum is attained by the so called first eigenfunction, while for p = 1 it is not attained and is tightly related to the *Cheeger constant* see [4]). Note that, if $\rho = \sigma \equiv 1$, we obtain the first eigenvalue of the *p*-laplacian, denoted for

simplicity by λ_p :

(2)
$$\lambda_p(A) := \inf_{\substack{u \in W_0^{1,p}(A) \\ u \neq 0}} \frac{\int_A |\nabla u(x)|^p dx}{\int_A |u(x)|^p dx}$$

For every L > 0, as in [5] we define the class of admissible sets

(3)
$$\mathcal{A}_L(\Omega) := \{ \Sigma \subset \overline{\Omega} : \Sigma \text{ is a continuum, and } \mathcal{H}^1(\Sigma) \leq L \}$$

where "continuum" stands for "connected, compact and non-empty set" and \mathcal{H}^1 denotes the one-dimensional Hausdorff measure.

The variational problem we consider is

(4)
$$\max\left\{\lambda_p^{\sigma,\rho}(\Omega \setminus \Sigma): \ \Sigma \in \mathcal{A}_L(\Omega)\right\},$$

that is, to find the best location $\Sigma \in \mathcal{A}_L(\Omega)$ for an *extra Dirichlet condition* (in addition to that along $\partial\Omega$), in order to maximize the first eigenvalue $\lambda_p^{\sigma,\rho}$. In particular, we shall study the behavior of the optimal sets, as $L \to \infty$, via Γ -convergence.

A possible physical interpretation of (4), at least when p = 2, is the following. The domain Ω represents an elastic structure (e.g. a membrane) in the plane (with density ρ and Young modulus σ) which is fixed along its boundary $\partial\Omega$ and, as such, has a fundamental frequency given by $\sqrt{\lambda_p^{\sigma,\rho}(\Omega)}$. One is interested in augmenting and possibly maximizing this fundamental frequency, by fixing the membrane along a supplementary curve (or system of curves) Σ of given length, which may be placed anywhere in $\overline{\Omega}$. One may think of Σ as a sort of stiffening rib, to obtain a reinforced structure $\Omega \setminus \Sigma$. Note that not only the location, but also the shape of Σ is free (cf. [10], where the shape is a ball of given radius and only its placement is to be optimized). The role of the parameter L is an upper bound to the total resources available: if fixing the structure along a system of curves Σ has a cost proportional to its total length $\mathcal{H}^1(\Sigma)$, then L is the maximum cost one is willing to spend to reinforce the membrane.

Another problem, in the same spirit but with the compliance functional instead of the first eigenvalue, has been treated in [5]. Among the main differences is the fact that (4) is not driven by an *a priori* given PDE, which reflects into the nonlocality of the resulting Γ -limit (see below). The class of admissible sets $\mathcal{A}_L(\Omega)$ is also typical of the well-studied *average distance problems*, where the distance function to Σ is to be minimized. These were first introduced by Buttazzo, Oudet and Stepanov in [7, 6], while the asymptotics of minimizers was studied in [18] (see also [15]).

As in [7, 5], the geometric restrictions on Σ entailed by (3) are useful in view of an existence result for (4) (connectedness of Σ can be relaxed to a bound on the number of connected components, but some control is needed to prevent Σ from spreading throughout Ω and prejudice existence). By classical results of Blaschke and Gołab (see [1]) the space $\mathcal{A}_L(\Omega)$ is compact in the Hausdorff metric, and the map $\Sigma \mapsto \lambda_p^{\sigma,\rho}(\Omega \setminus \Sigma)$ is continuous (see [22] for p = 2 and [2] for the general case). This leads to

Theorem 1.1 (Existence). For every L > 0, there exists a maximizer in (4). Moreover, every maximizer Σ satisfies $\mathcal{H}^1(\Sigma) = L$. The second claim follows from the elementary consideration that, if $\mathcal{H}^{1}(\Sigma) < L$, then one could increase $\lambda_{p}^{\sigma,\rho}(\Omega \setminus \Sigma)$ by attaching to Σ some short segments, thus reducing all the connected components of $\Omega \setminus \Sigma$ (cf. [7, 5]). Also note that the *minimization* problem analogous to (4) would be trivial, since $\lambda_{p}^{\sigma,\rho}(\Omega \setminus \Sigma) \geq \lambda_{p}^{\sigma,\rho}(\Omega)$ for every $\Sigma \in \mathcal{A}_{L}(\Omega)$ and equality is achieved by any admissible Σ hidden inside $\partial\Omega$.

Contrary to average distance problems where regularity results are available for the optimal sets Σ (see [6, 20]), similar questions for (4) are open (except for Ahlfors regularity which we will not discuss here).

In this paper we mainly focus on the asymptotic behavior of optimal sets Σ_L of problem (4) as $L \to \infty$, with the goal of studying the limit distribution of Σ_L within Ω . Of course, as L increases, the optimal configurations Σ_L tend to saturate Ω (i.e. $\Sigma_L \to \overline{\Omega}$ in the Hausdorff distance), and any information concerning the density (length of Σ_L per unit area in Ω) is lost in the limit. To retrieve this information we prove a Γ -convergence result, in the space $\mathcal{P}(\overline{\Omega})$ of probability measures in $\overline{\Omega}$, identifying each admissible $\Sigma \in \mathcal{A}_L(\Omega)$ with the probability measure

(5)
$$\mu_{\Sigma} = \frac{\mathcal{H}^{1} \mathsf{L} \Sigma}{\mathcal{H}^{1}(\Sigma)},$$

where $\mathcal{H}^1 \sqsubseteq \Sigma$ denotes the restriction of the one-dimensional Hausdorff measure to Σ (this expedient is natural in this kind of problems, see [18, 5]). Thus, our problem (4) becomes equivalent to the *minimization* of the functional $F_L : \mathcal{P}(\overline{\Omega}) \to [0, \infty]$ defined as

(6)
$$F_L(\mu) = \begin{cases} \frac{L^p}{\lambda_p^{\sigma,\rho}(\Omega \setminus \Sigma)} & \text{if } \mu = \mu_{\Sigma} \text{ for some } \Sigma \in \mathcal{A}_L(\Omega), \\ \infty & \text{otherwise.} \end{cases}$$

The scaling factor L^p , as we will see, is natural as the maximum achieved in (4) grows as L^p for large L (and, of course, rescaling does not alter the original problem anyhow). The Γ -convergence result we will prove is then the following.

Theorem 1.2. As $L \to \infty$, the functionals F_L defined in (6) Γ -converge, with respect to the weak* topology on $\mathcal{P}(\overline{\Omega})$, to the functional $F : \mathcal{P}(\overline{\Omega}) \mapsto [0, \infty]$ defined as

(7)
$$F(\mu) := \frac{1}{\Lambda_p} \operatorname{ess\,sup}_{x \in \Omega} \frac{\rho(x)}{\sigma(x) f(x)^p}$$

where $f \in L^1(\Omega)$ is the density (Radon-Nikodym derivative) of μ with respect to the Lebesgue measure, while Λ_p is the numerical constant

$$\Lambda_p := (p-1) \left(\frac{2\pi}{p\sin(\pi/p)}\right)^p \quad \text{if } p > 1, \qquad \Lambda_1 := 2$$

Remark 1.3. The constant Λ_p (see [3, 16]) is just the first Dirichlet eigenvalue for the *p*-laplacian in one variable, namely

(8)
$$\Lambda_p = \inf_{\substack{u \in W_0^{1,p}(0,1)\\ u \neq 0}} \frac{\int_0^1 |u'(z)|^p dz}{\int_0^1 |u(z)|^p dz}.$$

If p > 1, the infimum is attained by the first eigenfunction u_1 that solves the equation

(9)
$$-(|u_1'|^{p-2}u_1')' = \Lambda_p |u_1|^{p-2}u_1, \quad u_1(0) = u_1(1) = 0.$$

When p = 1, the infimum is not attained and $\Lambda_1 = 2$, as one can see letting u approximate the characteristic function $\chi_{(0,1)}$. See [3, 16] for more details.

The Γ -limit functional F defined in (7) has a unique minimizer. Indeed

$$\min_{\mu \in \mathcal{P}(\overline{\Omega})} F(\mu) = \frac{1}{\Lambda_p} \min_{\substack{f \ge 0\\ \int_{\Omega} f \le 1}} \operatorname{ess\,sup}_{x \in \Omega} \frac{\rho(x)}{\sigma(x)f(x)^p} = \frac{1}{\Lambda_p} \Big(\int_{\Omega} \left(\frac{\rho(x)}{\sigma(x)} \right)^{1/p} dx \Big)^p$$

achieved only when $\mu = \mu_{\infty}$, the absolutely continuous measure with density

(10)
$$f(x) = \frac{(\rho(x)/\sigma(x))^{1/p}}{\int_{\Omega} (\rho(y)/\sigma(y))^{1/p} \, dy}$$

(note that μ_{∞} reduces to normalized Lebesgue measure, if ρ and σ are constant). As the space $\mathcal{P}(\overline{\Omega})$ is compact in the weak^{*} topology, from standard Γ -convergence theory (see [8]) we can recover the limiting distribution of the optimal sets Σ_L for large L:

Corollary 1.4. If Σ_L is a maximizer of problem (4), then as $L \to \infty$ the probability measures μ_{Σ_L} converge, in the weak* topology of $\mathcal{P}(\overline{\Omega})$, to the probability measure μ_{∞} , absolutely continuous with respect to the Lebesgue measure, having the density given in (10). In particular, for every square $Q \subset \Omega$,

(11)
$$\lim_{L \to \infty} \frac{\mathcal{H}^1(\Sigma_L \cap Q)}{\mathcal{H}^1(\Sigma_L)} = \int_Q f(x) \, dx$$

and, moreover,

(12)
$$\lim_{L \to \infty} \frac{L^p}{\lambda_p^{\sigma, \rho}(\Omega \setminus \Sigma_L)} = F(\mu_{\infty}) = \frac{\left(\int_{\Omega} (\rho(x)/\sigma(x))^{1/p} dx\right)^p}{\Lambda_p}.$$

This corollary formalizes the ansatz that, in order to maximize the principal frequency of a membrane with density ρ and Young modulus σ , it is convenient to concentrate the stiffening rib Σ in those regions with higher ratio ρ/σ , with a density proportional to $(\rho/\sigma)^{1/p}$. In particular, for a homogeneous membrane with constant ρ and σ , it is convenient to distribute Σ with, roughly speaking, a constant ratio of length per unit area in Ω .

Moreover, the proof of Theorem 1.2 is constructive: it turns out that certain comb-shaped patterns (see Definition 3.6), periodically reproduced inside Ω at different scales, can be used to build examples of asymptotically optimal sets (that is, those sets Σ_L satisfying (12)).

In section 4 we will also see that the eigenvalue problems (4), as $p \to \infty$ with L being fixed, converge to the so called *maximum distance problem*, for which several qualitative results on the minimizers have been proved in [17] and [19].

Finally, let us mention that Theorem 1.2 may be considered as the "eigenvalue counterpart" of the Γ -convergence results obtained in [18] and [5] for average distance and compliance problems.

Throughout the paper, we denote by $d(x,C) = \min_{y \in C} |y - x|$ the distance function to the set C, a generic closed subset of \mathbb{R}^2 . Moreover, we denote by meas(E) the two-dimensional Lebesgue measure and by $\mathcal{H}^1(E)$ the one-dimensional Hausdorff measure of a Borel set $E \subset \mathbb{R}^2$. We will deal with the level sets of the distance function, and in particular we need the following result proved in [21] (see also Lemma 4.2 in [18]).

Lemma 2.1. Fix L > 0 and $\Sigma \in \mathcal{A}_L(\Omega)$. For $t \ge 0$ let

(13)
$$A_t = \{ x \in \Omega \mid d(x, \Sigma \cup \partial \Omega) < t \}$$

be the sublevel set of the distance function to $\Sigma \cup \partial \Omega$. If κ is the number of connected components of $\partial \Omega$ and

(14)
$$\overline{t} := \frac{\operatorname{meas}(\Omega)}{\left(\mathcal{H}^1(\Sigma \cup \partial \Omega) + \sqrt{\mathcal{H}^1(\Sigma \cup \partial \Omega)^2 + (\kappa + 1)\pi \operatorname{meas}(\Omega)}\right)}$$

is the positive root of the quadratic equation $2\mathcal{H}^1(\Sigma \cup \partial \Omega)\overline{t} + (\kappa + 1)\pi\overline{t}^2 = \operatorname{meas}(\Omega)$, then for every $t \ge 0$

(15)
$$\operatorname{meas}(A_t) \le H(t) := \begin{cases} 2\mathcal{H}^1(\Sigma \cup \partial\Omega)t + (\kappa + 1)\pi t^2 & \text{if } t \le \overline{t}, \\ \operatorname{meas}(\Omega) & \text{if } t > \overline{t}. \end{cases}$$

Remark 2.2. The number of connected components of $\partial\Omega$ is necessarily finite, since $\partial\Omega$ is Lipschitzian and compact. Similarly, also $\mathcal{H}^1(\partial\Omega)$ is finite.

Remark 2.3. From the definition of \overline{t} in Lemma 2.1 we see that the function H(t) in (15) is Lipschitzian and increasing. Letting $T = \max_{x \in \overline{\Omega}} d(x, \Sigma \cup \partial \Omega)$, since $\overline{A_T} = \overline{\Omega}$ while $H(t) < \operatorname{meas}(\Omega)$ for $t < \overline{t}$, we see that

(16)
$$0 < \overline{t} \le T := \max_{x \in \overline{\Omega}} d(x, \Sigma \cup \partial \Omega)$$

and

(17)
$$\operatorname{meas}(A_T) = \operatorname{meas}(\Omega) = H(T).$$

Moreover, $H'(t) \equiv 0$ for $t > \overline{t}$.

We start by proving an upper bound for the first eigenvalue $\lambda_p(\Omega \setminus \Sigma)$ (defined as in (2)) in terms of the length $\mathcal{H}^1(\Sigma)$. As we will see in Theorem 3.7, this estimate is sharp when $\mathcal{H}^1(\Sigma)$ is large. Some of the techniques that we use are refinements of those in [21], where a similar bound was proved for the compliance functional.

Theorem 2.4. Let $\Omega \subset \mathbb{R}^2$ be a bounded, connected open set with a Lipschitz boundary $\partial\Omega$ made of κ connected components. For any L and $\Sigma \in \mathcal{A}_L(\Omega)$, it holds

(18)
$$\lambda_p(\Omega \setminus \Sigma) \le \frac{\Lambda_p}{\left(2\overline{t}\right)^p} \left(1 + \frac{(\kappa + 1)\pi\overline{t}}{\mathcal{H}^1(\Sigma \cup \partial\Omega)}\right),$$

where \overline{t} is the number defined in (14).

Proof. By (2), for any non-zero function $u \in W_0^{1,p}(\Omega \setminus \Sigma)$ we have

(19)
$$\lambda_p(\Omega \setminus \Sigma) \le \frac{\int_{\Omega} |\nabla u(x)|^p dx}{\int_{\Omega} |u(x)|^p dx},$$

and a suitable choice of u will lead to (18). More precisely, we choose u depending on the distance function

(20)
$$u(x) := g(d(x, \Sigma \cup \partial \Omega)), \qquad x \in \Omega,$$

where $g: \mathbb{R}^+ \to \mathbb{R}^+$ is $C^{1,1}$, increasing and concave, and such that g(0) = 0. Then u vanishes along $\Sigma \cup \partial \Omega$, is Lipschitzian and, in particular, it is an admissible function for the Rayleigh quotient in (19).

We first estimate the numerator in (19). Since $|\nabla d(x, \Sigma \cup \partial \Omega)| = 1$ a.e., from the coarea formula (see [9]) we have

(21)
$$\int_{\Omega} |\nabla u(x)|^p dx = \int_{\Omega} g'(d(x, \Sigma \cup \partial \Omega))^p dx = \int_0^T g'(t)^p P(A_t, \Omega) dt,$$

where $T = \max_{x \in \overline{\Omega}} d(x, \Sigma \cup \partial \Omega)$, A_t is as in (13) and $P(A_t, \Omega)$ is the perimeter of A_t in Ω (see [9] for more details on perimeters). Recall that still from the coarea formula one has $\operatorname{meas}(A_t) = \int_0^t P(A_t, \Omega) dt$ for every t > 0 and hence

$$P(A_t, \Omega) = \frac{d}{dt} \operatorname{meas}(A_t) \text{ for a.e. } t > 0.$$

Letting $G(t) = g'(t)^p$, as $G'(t) \le 0$ by assumption, we can integrate by parts in (21) and use (15). Since meas $(A_0) = 0$, we obtain

$$\int_{\Omega} |\nabla u(x)|^p dx = -\int_0^T G'(t) \operatorname{meas}(A_t) dt + G(T) \operatorname{meas}(A_T)$$
$$\leq -\int_0^T G'(t)H(t) dt + G(T)H(T).$$

Since H(0) = 0, we can integrate by parts the other way round, thus obtaining

(22)
$$\int_{\Omega} |\nabla u(x)|^p dx \le \int_0^T g'(t)^p H'(t) \, dt = \int_0^{\overline{t}} g'(t)^p H'(t) \, dt$$

where the last equality follows from Remark 2.3

Similarly, letting $G(t) = g(t)^p$ and observing that now $G'(t) \ge 0$, recalling (15) and (17) we can estimate from below the denominator of (19) as follows

$$\int_{\Omega} |u(x)|^p dx = \int_0^T g(t)^p P(A_t, \Omega) \, dt = -\int_0^T G'(t) \operatorname{meas}(A_t) \, dt + G(T) \operatorname{meas}(A_T) \, dt = -\int_0^T G'(t) H(t) \, dt + G(T) H(T) = \int_0^T G(t) H'(t) \, dt = \int_0^{\overline{t}} g(t)^p H'(t) \, dt.$$

Now we plug the last estimate and (22) into (19): observing that

$$2\mathcal{H}^1(\Sigma \cup \partial \Omega) \le H'(t) \le 2\mathcal{H}^1(\Sigma \cup \partial \Omega) + 2(\kappa + 1)\pi \overline{t}, \quad t \in (0, \overline{t})$$

and changing variable $z = t/(2\overline{t})$ in the two integrals, from (19) we find that

(23)
$$\lambda_p(\Omega \setminus \Sigma) \le \left(1 + \frac{(\kappa + 1)\pi \overline{t}}{\mathcal{H}^1(\Sigma \cup \partial\Omega)}\right) \frac{1}{(2\overline{t})^p} \frac{\int_0^{1/2} g'(z)^p dz}{\int_0^{1/2} g(z)^p dz},$$

now valid for every $C^{1,1}$ function g increasing and concave on [0, 1/2], and such that g(0) = 0. In fact, by a density argument, we can relax $g \in C^{1,1}(0, 1/2)$ to $g \in W^{1,p}(0, 1/2)$. If p > 1, the first Dirichlet eigenfunction $u_1(z)$ of the p-laplacian on (0, 1) is symmetric with respect to y = 1/2 and, from (9) it follows that u_1 is also

increasing and concave on [0, 1/2]. This means that one can choose $g(z) = u_1(z)$ in (23) and this gives (18), since the ratio of the two integrals then reduces to Λ_p .

Finally, if p = 1, it suffices to let $g(z) = \min\{1, nz\}$ in (23) and then let $n \to \infty$ (recall that $\Lambda_1 = 2$).

Remark 2.5. It is clear from the proof that, in order to obtain (18), there is nothing special with $\Sigma \cup \partial \Omega$ except that this set supports the Dirichlet condition associated with $\lambda_p(\Omega \setminus \Sigma)$ through the function space $W_0^{1,p}(\Omega \setminus \Sigma)$. Indeed, the estimate still holds if one considers the first eigenvalue with a Dirichlet condition prescribed along any compact set $D \subset \overline{\Omega}$ such that $0 < \mathcal{H}^1(D) < \infty$, having K connected components. Of course, in this case, one has to replace $\mathcal{H}^1(\Sigma \cup \partial \Omega)$ with $\mathcal{H}^1(D)$ and $\kappa + 1$ with K.

3. Proof of the Γ -convergence result

In this section we will prove Theorem 1.2, first proving the Γ -limit and the Γ -limit inequalities up to a multiplicative constant θ_p , defined as follows:

(24)
$$\theta_p := \inf\left(\liminf_{n \to \infty} \frac{L_n^p}{\lambda_p(Y \setminus \Sigma_n)}\right),$$

where $Y = (0,1)^2$ is the unit square and the infimum is over all sequences of numbers $L_n \to \infty$ and all sequences of sets $\Sigma_n \in \mathcal{A}_{L_n}$ such that $\mathcal{H}^1(\Sigma_n) = L_n$. Then we will compute explicitly this constant, showing that it is the inverse of the first Dirichlet eigenvalue of the *p*-laplacian on the unit interval (Theorem 3.7).

In the sequel we will often use the following well-known properties of the first Dirichlet eigenvalue:

- Monotonicity: for any two bounded open sets $A \subset B$, $\lambda_p^{\sigma,\rho}(A) \ge \lambda_p^{\sigma,\rho}(B)$.

- Splitting over connected components: if A can be written as $\bigcup A_i$ with pairwise disjoint open sets $A_i \neq \emptyset$ (e.g., its connected components), then $\lambda_p^{\sigma,\rho}(A) = \min_i \lambda_p^{\sigma,\rho}(A_i)$.

- Comparison with the homogeneous case: for any open set $D \subset \Omega$, on comparing (1) and (2) we have

(25)
$$\frac{\inf_D \sigma}{\sup_D \rho} \lambda_p(D) \le \lambda_p^{\sigma,\rho}(D) \le \frac{\sup_D \sigma}{\inf_D \rho} \lambda_p(D).$$

3.1. The Γ -limit inequality. We start proving that the Γ -limit functional F_L is minorized by the limit functional F defined, up to θ_p , by (7). We shall use some of the ideas that were introduced in [18], see also [5].

Proposition 3.1. For every probability measure $\mu \in \mathcal{P}(\overline{\Omega})$ and every sequence $\{\mu_L\} \subset \mathcal{P}(\overline{\Omega})$ such that $\mu_L \rightharpoonup^* \mu$, it holds

(26)
$$\liminf_{L \to \infty} F_L(\mu_L) \ge \theta_p \operatorname{ess\,sup}_{x \in \Omega} \frac{\rho(x)}{\sigma(x) f(x)^p}$$

Proof. Consider an arbitrary subsequence (still denoted by $\{\mu_L\}$ for simplicity, but L should be regarded as L_n etc.) for which the

$$\lim_{L \to \infty} F_L(\mu_L) \stackrel{\text{by (6)}}{=} \lim_{L \to \infty} \frac{L^p}{\lambda_p^{\sigma,\rho}(\Omega \setminus \Sigma_L)}$$

exists and is finite. By finiteness of the limit, recalling (6) we can assume that each μ_L has the form (5) for some $\Sigma_L \in \mathcal{A}_L$, so that

(27)
$$\mu_L(E) = \frac{\mathcal{H}^1(\Sigma_L \cap E)}{\mathcal{H}^1(\Sigma_L)} \quad \text{for all Borel sets } E \subset \overline{\Omega}.$$

Moreover, finiteness also entails that $\lambda_p(\Omega \setminus \Sigma_L)$ (being comparable with $\lambda_p^{\sigma,\rho}(\Omega \setminus \Sigma_L)$) tends to infinity: hence, if we choose any open square $Q \subset \Omega$, by monotonicity also $\lambda_p(Q \setminus \Sigma_L) \to \infty$, and in particular the distance function $d(x, Q \setminus \Sigma_L)$ uniformly tends to zero (as $L \to \infty$) over Q (otherwise $Q \setminus \Sigma_L$ would contain a ball B of radius bounded away from zero, and $\lambda_p(Q \setminus \Sigma_L)$ would be bounded from above). Hence, applying Lemma 2.1 with $\Omega := Q$ and $\Sigma := \Sigma_L$, we see from (16) that $\overline{t} \to 0$ as $L \to \infty$, and by (14) this means that

(28)
$$\lim_{L \to \infty} \mathcal{H}^1(\Sigma_L \cap Q) = \infty \quad \text{(for every open square } Q \subset \Omega\text{)}.$$

Now fix $\varepsilon > 0$, and consider an open square $Q \subset \Omega$, whose role is to localize the estimate on F_L . From the monotonicity of λ_p and (25) it follows that,

$$\frac{L^p}{\lambda_p^{\sigma,\rho}(\Omega \setminus \Sigma_L)} \ge \frac{L^p}{\lambda_p^{\sigma,\rho}(Q \setminus \Sigma_L)} \ge \frac{\inf_Q \rho}{\sup_Q \sigma} \frac{L^p}{\lambda_p(Q \setminus \Sigma_L)}$$

which, using $L \geq \mathcal{H}^1(\Sigma_L)$ and (27), gives

(29)
$$\frac{L^p}{\lambda_p^{\sigma,\rho}(\Omega \setminus \Sigma_L)} \ge \frac{\inf_Q \rho}{\sup_Q \sigma} \frac{1}{\mu_L(Q)^p} \frac{\mathcal{H}^1(\Sigma_L \cap Q)^p}{\lambda_p(Q \setminus \Sigma_L)}.$$

From $\mu_L \rightharpoonup^* \mu$, we have that $\limsup \mu_L(Q) \le \mu(\overline{Q})$ and hence

(30)
$$\liminf_{L \to \infty} \frac{1}{\mu_L(Q)^p} \ge \frac{1}{\mu\left(\overline{Q}\right)^p + \varepsilon \operatorname{meas}(Q)^p}$$

where the quantity $\varepsilon \operatorname{meas}(Q)^p$ serves to avoid vanishing denominator, for the moment. Moreover, if a is the side-length of Q and $Y = a^{-1}Q$ is a unit square, by scaling

(31)
$$\frac{\mathcal{H}^1(\Sigma_L \cap Q)^p}{\lambda_p(Q \setminus \Sigma_L)} = \frac{a^p \mathcal{H}^1(a^{-1}\Sigma_L \cap Y)^p}{a^{-p}\lambda_p(Y \setminus a^{-1}\Sigma_L)} = \operatorname{meas}(Q)^p \frac{\mathcal{H}^1(a^{-1}\Sigma_L \cap Y)^p}{\lambda_p(Y \setminus a^{-1}\Sigma_L)}.$$

Now let $\Sigma_n := \partial Y \cup (a^{-1}\Sigma_L \cap Y)$, and observe that Σ_n is connected, since by (28) Σ_L must cross the boundary of Q. Hence, using (28) again, by translation invariance we can use (24) and, from (31), estimate

(32)
$$\liminf_{L \to \infty} \frac{\mathcal{H}^1(\Sigma_L \cap Q)^p}{\lambda_p(Q \setminus \Sigma_L)} \ge \theta_p \operatorname{meas}(Q)^p.$$

Now combining (30) and (32) with (29), we obtain the estimate

$$\liminf_{L \to \infty} F_L(\mu_L) = \liminf_{L \to \infty} \frac{L^p}{\lambda_p^{\sigma, \rho}(\Omega \setminus \Sigma_L)} \ge \frac{\inf_Q \rho}{\sup_Q \sigma} \frac{\theta_p \operatorname{meas}(Q)^p}{\mu(\overline{Q})^p + \varepsilon \operatorname{meas}(Q)^p}$$

for every open square $Q \subset \Omega$. Thus, if $f \in L^1(\Omega)$ is the density of μ and $x \in \Omega$ is a Lebesgue point for f, letting Q shrink towards x, from Radon-Nikodym Theorem we find that

$$\liminf_{L \to \infty} F_L(\mu_L) \ge \theta_p \frac{\rho(x)}{\sigma(x)(f(x)^p + \varepsilon)} \quad \text{for a.e. } x \in \Omega.$$

Finally, letting $\varepsilon \downarrow 0$ one obtains (26).

3.2. The Γ -limsup inequality. As in (24), we denote by Y the unit square $(0, 1)^2$. Lemma 3.2. Given $\varepsilon > 0$, there exists a compact connected set $\Sigma \subset \overline{Y}$ such that

(33)
$$\frac{\mathcal{H}^1(\Sigma)^p}{\lambda_p(Y \setminus \Sigma)} < (1+\varepsilon)\theta_p$$

and

$$(34) \qquad \qquad \partial Y \subset \Sigma.$$

Proof. According to how θ_p was defined in (24), we can take a sequence of sets $\{\Sigma_n\}$ satisfying $L_n := \mathcal{H}^1(\Sigma_n) \to \infty, \Sigma_n \in \mathcal{A}_{L_n}(Y)$ and

$$\frac{\mathcal{H}^1(\Sigma_n)^p}{\lambda_p(Y \setminus \Sigma_n)} < (1 + \varepsilon/2)\theta_p \quad \forall n \ge 1.$$

If n is large enough, then clearly

$$\frac{\left(\mathcal{H}^1(\Sigma_n) + \mathcal{H}^1(\partial Y) + 1/2\right)^p}{\lambda_p(Y \setminus \Sigma_n)} < (1 + \varepsilon)\theta_p,$$

and defining $\Sigma := \Sigma_n \cup \partial Y \cup S$, where S is any segment of length at most 1/2 that connects Σ_n to ∂Y , we can guarantee (33) and (34) (note that $\lambda_p(Y \setminus \Sigma) \ge \lambda_p(Y \setminus \Sigma_n)$ by monotonicity). \Box

We start proving the Γ -limsup inequality for a particular class of measures.

Definition 3.3. For s > 0, let \mathcal{Q}_s denote the collection of all those open squares $Q_i \subset \mathbb{R}^2$, with side-length s and corners on the lattice $(s\mathbb{Z})^2$, such that $Q_i \cap \Omega \neq \emptyset$. We say that a probability measure $\mu \in \mathcal{P}(\overline{\Omega})$ is *fitted to* \mathcal{Q}_s if μ is absolutely continuous, with a density f(x) > 0 which is *constant* on each set of the form $Q_i \cap \Omega$ with $Q_i \in \mathcal{Q}_s$. In formulae,

(35)
$$d\mu = f(x)dx, \qquad f(x) = \sum_{i} \alpha_i \chi_{\Omega_i}(x), \quad \Omega_i = \Omega \cap Q_i, \quad \mathcal{Q}_s = \{Q_i\}$$

where the constants $\alpha_i > 0$ satisfy (since $\mu(\Omega) = 1$) the normalization condition

(36)
$$\sum_{i} \alpha_i \operatorname{meas}(\Omega_i) = 1.$$

Proposition 3.4. If $\mu \in \mathcal{P}(\overline{\Omega})$ is fitted to \mathcal{Q}_s for some s > 0, then for every $\varepsilon > 0$ there exists a sequence $\{\mu_L\}$ in $\mathcal{P}(\overline{\Omega})$ such that $\mu_L \rightharpoonup^* \mu$ and

(37)
$$\limsup_{L \to \infty} F_L(\mu_L) \le (1+\varepsilon)\theta_p \operatorname{ess\,sup}_{x \in \Omega} \frac{\rho(x)}{\sigma(x)f(x)^p}.$$

Proof. Consider μ fitted to \mathcal{Q}_s , with the same notation as in (35). As $\partial\Omega$ is Lipschitzian, by replacing (if necessary) s with $s/2^k$ for some k > 1 (thus keeping μ fitted to \mathcal{Q}_s), we may assume that s is so small that

(38) no connected component of $\partial\Omega$ is strictly contained in any square $Q_i \in Q_s$.

Given a small $\varepsilon > 0$, we will construct the measures μ_L of the form (27) for suitable sets $\Sigma_L \in \mathcal{A}_L(\Omega)$. We will call "tile" the set Σ obtained from Lemma (3.2), satisfying (33) and (34). We define its "effective length" L_e as the number

(39)
$$L_e := \mathcal{H}^1\left(\Sigma \cap [0,1)^2\right).$$

Given L large enough, the set Σ_L is obtained through the following construction:

(i) fix a set Ω_i in (35) and scale down the tile Σ to a factor s/k_i , where the integer k_i is defined as

(40)
$$k_i = k_i(L) = \left\lfloor \frac{s\alpha_i(L - \sqrt{L})}{L_e} \right\rfloor$$

- (ii) Put k_i^2 copies of the rescaled tile inside the closed square $\overline{Q_i}$ corresponding to Ω_i , as to form a $k_i \times k_i$ chalkboard, and intersect with Ω (the resulting set is contained in $\overline{\Omega_i}$).
- (iii) Repeat for each Ω_i , and take the union. Finally, add $\partial\Omega$ to the resulting set.

Formally, if $l_i \in \mathbb{R}^2$ denotes the lower-left corner of the square Q_i , this construction amounts to defining

$$\Sigma_L := \partial \Omega \cup \left(\bigcup_{i \ 0 \le m, n < k_i} \Sigma_{i,m,n} \cap \Omega \right), \quad \Sigma_{i,m,n} = l_i + s(m/k_i, n/k_i) + (s/k_i)\Sigma.$$

The main idea is to put several *microtiles* $\Sigma_{i,m,n}$ side by side within each square $\overline{Q_i}$, with a density (length per unit area) therein roughly proportional to α_i , in such a way that the total length is about L (subtracting \sqrt{L} in (40) serves to save some o(L) of length, to compensate for $\mathcal{H}^1(\partial\Omega)$ and boundary effects due to cut-out microtiles close to $\partial\Omega$).

Building on (38) and (34), a little thought reveals that the set Σ_L thus constructed is connected.

Each set $\overline{\Omega_i}$ contains a certain number $W_i(L)$ of whole microtiles $\Sigma_{i,m,n}$ and, if Q_i crosses $\partial\Omega$, also a certain number of *incomplete* microtiles $\Sigma_{i,m,n} \cap \Omega$ (those that have really been cut out by intersection with Ω , in step (ii) above). As $\partial\Omega$ is Lipschitzian, however, and diam $(\Sigma_{i,m,n}) = O(1/L)$, for large L there are at most C_1L incomplete microtiles, where C_1 depends on μ , $\mathcal{H}^1(\Sigma)$ and $\partial\Omega$ but not on L. Moreover, as $\mathcal{H}^1(\Sigma_{i,m,n}) = O(1/L)$, the incomplete tiles contribute to $\mathcal{H}^1(\Sigma_L)$ by, at most, a constant length C_2 independent of L. Hence, since clearly $W_i(L) \leq \max(\Omega_i)k_i^2/s^2$,

$$\mathcal{H}^{1}(\Sigma_{L}) \leq \mathcal{H}^{1}(\partial\Omega) + C_{2} + \sum_{i} W_{i}(L) \frac{sL_{e}}{k_{i}} \leq C_{3} + \sum_{i} \operatorname{meas}(\Omega_{i}) \frac{k_{i}L_{e}}{s}$$
$$\leq C_{3} + (L - \sqrt{L}) \sum_{i} \operatorname{meas}(\Omega_{i}) \alpha_{i} \leq L$$

provided L is large enough (recall (36)). This shows that $\Sigma_L \in \mathcal{A}_L(\Omega)$ for large L, hence defining μ_L as in (27), from (6) we obtain

(41)
$$\limsup_{L \to \infty} F_L(\mu_L) = \limsup_{L \to \infty} \frac{L^p}{\lambda_p^{\sigma,\rho}(\Omega \setminus \Sigma_L)}.$$

Note that, in fact, $\mathcal{H}^1(\Sigma_L) \sim L$ as $L \to \infty$. Indeed, since $\overline{\Omega_i}$ contains $W_i(L)$ whole microtiles (each contributing an effective length sL_e/k_i) and $W_i(L) \sim \text{meas}(\Omega_i)k_i^2/s^2$ as $L \to \infty$, from (40) and (36)

$$\mathcal{H}^{1}(\Sigma_{L}) \geq \sum_{i} W_{i}(L) \frac{sL_{e}}{k_{i}} \sim \sum_{i} \operatorname{meas}(\Omega_{i}) \frac{k_{i}L_{e}}{s} \sim L \sum_{i} \operatorname{meas}(\Omega_{i})\alpha_{i} = L.$$

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Now, as the restriction of μ_L to each Ω_i is a periodic homogenization of the same pattern with period s/k_i , it is clear that the μ_L converge, as $L \to \infty$, to some measure in $\mathcal{P}(\overline{\Omega})$ which is fitted to \mathcal{Q}_s . More precisely, recalling (35)

$$\mu_L\left(\overline{\Omega_i}\right) = \frac{\mathcal{H}^1\left(\Sigma_L \cap \overline{\Omega_i}\right)}{\mathcal{H}^1(\Sigma_L)} \sim \frac{W_i(L)sL_e/k_i}{L} \sim \frac{\left(\operatorname{meas}(\Omega_i)k_i^2/s^2\right)sL_e/k_i}{L}$$
$$= \frac{\operatorname{meas}(\Omega_i)k_iL_e}{sL} \sim \operatorname{meas}(\Omega_i)\alpha_i = \mu(\Omega_i),$$

and we see that in fact $\mu_L \rightharpoonup^* \mu$ as $L \rightarrow \infty$.

To estimate $\lambda_p^{\sigma,\rho}(\Omega \setminus \Sigma_L)$ in (41), fix L and observe that $\Omega \setminus \Sigma_L$ consists, by construction, of several small connected components (at least one for each microtile $\Sigma_{i,m,n}$, due to (34)). Therefore, since the first Dirichlet eigenvalue splits over connected components, for a suitable triplet i, m, n,

$$\lambda_p^{\sigma,\rho}(\Omega \setminus \Sigma_L) = \lambda_p^{\sigma,\rho}(D), \quad D := \Omega \cap (Y_{i,m,n} \setminus \Sigma_{i,m,n})$$

where $Y_{i,m,n}$ is the open square of side s/k_i , contained in Q_i , that frames $\Sigma_{i,m,n}$. Therefore, using (25),

$$\frac{L^p}{\lambda_p^{\sigma,\rho}(\Omega \setminus \Sigma_L)} = \frac{L^p}{\lambda_p^{\sigma,\rho}(D)} \le \frac{\sup_D \rho}{\inf_D \sigma} \frac{L^p}{\lambda_p(D)}$$

Moreover, by monotonicity, scaling, (33) and (39),

$$\lambda_p(D) \ge \lambda_p(Y_{i,m,n} \setminus \Sigma_{i,m,n}) = \frac{k_i^p}{s^p} \lambda_p(Y \setminus \Sigma) \ge \frac{\left(k_i \mathcal{H}^1(\Sigma)\right)^p}{s^p(1+\varepsilon)\theta_p} \ge \frac{\left(k_i L_e\right)^p}{s^p(1+\varepsilon)\theta_p}$$

which plugged into the previous estimate gives

(42)
$$\frac{L^p}{\lambda_p^{\sigma,\rho}(\Omega \setminus \Sigma_L)} \le \frac{\sup_D \rho}{\inf_D \sigma} \left(\frac{sL}{k_i L_e}\right)^p (1+\varepsilon)\theta_p.$$

Now $D \subset \Omega_i$ and, by (40), diam(D) = O(1/L). Hence, by positivity and uniform continuity of ρ and σ over Ω ,

$$\frac{\sup_D \rho}{\inf_D \sigma} \le (1 + \delta_L) \sup_{\Omega_i} \rho / \sigma, \quad \text{and} \quad \left(\frac{sL}{k_i L_e}\right)^p \le \frac{1 + \delta_L}{\alpha_i^p}$$

where δ_L is independent of *i* and tends to zero as $L \to \infty$. Therefore, (37) follows from (41) and (42), taking the limsup there, and observing that

$$\operatorname{ess\,sup}_{x\in\Omega} \frac{\rho(x)}{\sigma(x)f(x)^p} = \max_j \left(\frac{1}{\alpha_j^p} \sup_{\Omega_j} \rho/\sigma\right),\,$$

as f(x) is piecewise constant according to (35).

Finally, we prove the density in energy of those measures $\mu \in \mathcal{P}(\overline{\Omega})$ that are fitted to \mathcal{Q}_s for some s > 0. Then, by a general result of Γ -convergence theory [8], the Γ -limsup inequality (37) will be established for every probability measure $\mu \in \mathcal{P}(\overline{\Omega})$.

Proposition 3.5. For every $\mu \in \mathcal{P}(\overline{\Omega})$ there exists a sequence $\{\mu_n\} \subset \mathcal{P}(\overline{\Omega})$ such that every μ_n is fitted to \mathcal{Q}_s for some s > 0, $\mu_n \rightharpoonup^* \mu$ and

(43)
$$\limsup_{n \to \infty} F(\mu_n) \le F(\mu)$$

Proof. Consider an arbitrary measure $\mu \in \mathcal{P}(\overline{\Omega})$. Keeping the notation of Definition 3.3, we construct $\mu_n \in \mathcal{P}(\overline{\Omega})$, fitted to $\mathcal{Q}_{1/n}$, as follows. By analogy with (35), we set

$$d\mu_n = f_n(x)dx, \qquad f_n(x) = \sum_i \alpha_i^{(n)} \chi_{\Omega_i^{(n)}}(x), \quad \Omega_i^{(n)} = \Omega \cap Q_i^{(n)}, \quad \mathcal{Q}_{1/n} = \{Q_i^{(n)}\}$$

where the numbers $\alpha_i^{(n)}$ are chosen as to satisfy the conditions

(44)
$$\frac{\mu\left(\Omega_{i}^{(n)}\right)}{\max\left(\Omega_{i}^{(n)}\right)} \leq \alpha_{i}^{(n)} \leq \frac{\mu\left(\Omega_{i}^{(n)}\right)}{\max\left(\Omega_{i}^{(n)}\right)}, \quad \sum_{i} \alpha_{i}^{(n)} \max\left(\Omega_{i}^{(n)}\right) = 1.$$

Note that, for fixed n, the $\{\Omega_i^{(n)}\}$ are pairwise disjoint while their closures $\{\overline{\Omega_i^{(n)}}\}$ cover $\overline{\Omega}$. Then the double inequality above means that μ_n is a sort of sampling of μ , and it is easy to see that $\mu_n \rightharpoonup^* \mu$ as $n \rightarrow \infty$.

Let $f \in L^1(\Omega)$ be the density of μ with respect to the Lebesgue measure. Recalling (7), passing to reciprocals we see that (43) reduces to

(45)
$$\liminf_{n \to \infty} \left(\operatorname{ess\,inf}_{x \in \Omega} g(x) f_n(x) \right) \ge \operatorname{ess\,inf}_{x \in \Omega} g(x) f(x), \quad \text{where } g(x) := \frac{\sigma(x)^{1/p}}{\rho(x)^{1/p}}.$$

Define the quantity $\tau_n = \min_i \left(\inf_{\Omega_i^{(n)}} g / \sup_{\Omega_i^{(n)}} g \right)$ and observe that, as $\operatorname{diam}(\Omega_i^{(n)}) = O(1/n)$, by uniform continuity and positivity of σ, ρ over $\overline{\Omega}, \tau_n \to 1$ as $n \to \infty$. To estimate the first ess inf in (45), as the $\partial \Omega_i^{(n)}$ are Lebesgue-negligible, we can restrict ourselves to consider $x \in \Omega_i^{(n)}$ for some *i*. Then, using (44),

$$g(x)f_n(x) = g(x)\alpha_i^{(n)} \ge \frac{g(x)\mu\left(\Omega_i^{(n)}\right)}{\max\left(\Omega_i^{(n)}\right)} \ge \frac{g(x)\int_{\Omega_i^{(n)}}f(y)\,dy}{\max\left(\Omega_i^{(n)}\right)} \ge \frac{\tau_n\int_{\Omega_i^{(n)}}g(y)f(y)\,dy}{\max\left(\Omega_i^{(n)}\right)}$$

and hence, from the arbitrariness of i,

$$\operatorname{ess\,inf}_{x\in\Omega} g(x)f_n(x) \ge \tau_n \operatorname{ess\,inf}_{x\in\Omega} g(x)f(x).$$

Taking the limit and using $\tau_n \to 1$, one obtains (45) as claimed.

3.3. Computation of θ_p and optimal sequences. In this section we prove that the constant θ_p , defined by (24), is the inverse of the first Dirichlet eigenvalue of the *p*-laplacian on the unit interval. To this purpose, we define the following class of admissible sets.

Definition 3.6. Let $\overline{Y} = [0, 1]^2$ be the closed unit square and let $n \ge 1$ be an integer. We define the set $C_n \subset \overline{Y}$ (called *comb configuration*) as the union of n+1 equispaced vertical segments of length one, a distance of 1/n apart, together with the lower base of \overline{Y} (the role of the latter is to make C_n connected, see Figure 1).

Theorem 3.7. Recalling (8) and (24), there holds

$$\theta_p = \frac{1}{\Lambda_p}.$$

Moreover, the constant θ_p is achieved, in (24), when Σ_n is the comb-structure C_n of Definition 3.6.

Proof. Consider any sequence of sets $\Sigma_n \subset \overline{Y}$ such that $L_n := \mathcal{H}^1(\Sigma_n) \to \infty$, as described after (24). Applying Theorem 2.4 with $\Omega := Y$, $\Sigma := \Sigma_n$ (and $\kappa = 1$ as ∂Y is connected), we find the bound (18), namely

(46)
$$\lambda_p(Y \setminus \Sigma_n) \le \frac{\Lambda_p}{2^p \overline{t}_n^p} \left(1 + \frac{2\pi \overline{t}_n}{\mathcal{H}^1(\Sigma_n \cup \partial Y)} \right),$$

where \overline{t}_n is defined as \overline{t} in (14). Note that, since $\mathcal{H}^1(\Sigma_n) = L_n \to \infty$,

(47)
$$\overline{t}_n = \frac{1}{\left(\mathcal{H}^1(\Sigma_n \cup \partial Y) + \sqrt{\mathcal{H}^1(\Sigma_n \cup \partial Y)^2 + 2\pi}\right)} \sim \frac{1}{2L_n} \quad \text{as } n \to \infty.$$

Plugging (46) into (24) and using (47), from the arbitrariness of Σ_n one obtains that $\theta_p \geq 1/\Lambda_p$.

To prove the opposite inequality, let C_n be the comb structure of Definition 3.6. Note that $\mathcal{H}^1(C_n) = n + 2$, hence $C_n \in \mathcal{A}_{n+2}(Y)$. Moreover, the set $Y \setminus C_n$ is the union of *n* rectangles of size $1/n \times 1$, hence

(48)
$$\lambda_p(Y \setminus C_n) = \lambda_p(R_n), \quad R_n = (0, 1/n) \times (0, 1).$$

Even though $\lambda_p(R_n)$ is not known explicitly, a lower bound is obtained by relaxing the boundary condition, from Dirichlet on the whole ∂R_n to Dirichlet on the two long sides of R_n (and Neumann on the short, horizontal sides). With these boundary conditions, it is well known that the first eigenvalue coincides with the corresponding eigenvalue in one variable, on the interval (0, 1/n). Thus, if W is the subspace of those functions $w \in W^{1,p}(R_n)$ with null trace at x = 0 and at x = 1/n, recalling (8)

$$\lambda_p(R_n) \ge \inf_{\substack{w \in W \\ w \neq 0}} \frac{\int_0^1 \int_0^{1/n} |\nabla w(x,y)|^p \, dx dy}{\int_0^1 \int_0^{1/n} |w(x,y)|^p \, dx dy} = \inf_{\substack{u \in W_0^{1,p}(0,1/n) \\ u \neq 0}} \frac{\int_0^{1/n} |u'(x)|^p \, dx}{\int_0^{1/n} |u(x)|^p \, dx} = n^p \Lambda_p,$$

that is $\lambda_p(Y \setminus C_n) \ge n^p \Lambda_p$. Now, if we choose $\Sigma_n = C_n$ (and $L_n = n+2$) in (24), we find the optimal upper bound

$$\theta_p \leq \liminf_{n \to \infty} \frac{(n+2)^p}{\lambda_p(Y \setminus C_n)} \leq \liminf_{n \to \infty} \frac{(n+2)^p}{n^p \Lambda_p} = \frac{1}{\Lambda_p}.$$

Remark 3.8. If p = 2 or p = 1, $\lambda_p(R_n)$ in (48) is known explicitly. More precisely, when p = 2 it is well known that $\lambda_2(R_n) = \pi^2(n^2 + 1)$. Moreover, when p = 1 the first Dirichlet eigenvalue $\lambda_1(R_n)$ is just the Cheeger constant of R_n , namely

$$h(R_n) = \frac{4 - \pi}{1 + 1/n - \sqrt{(1 - 1/n)^2 + \pi/n}} = \left(1 + 1/n + \sqrt{(1 - 1/n)^2 + \pi/n}\right)n$$

(see [13, 14] for more details).

Remark 3.9. We may call "asymptotically optimal" (for the unit square Y) those sequences of admissible configurations Σ_n (such as the comb configurations C_n) that achieve the infimum θ_p in (24). Other examples of asymptotically optimal configurations are provided by *oblique* comb structures, that is, the intersection of Y with a (thicker and thicker) family of equispaced parallel lines (plus ∂Y to make the structure connected): the reason for asymptotic optimality is that (much like the vertical combs C_n) these sets disconnect Y into the union of (approximate) thin rectangles, and the first eigenvalue of a thin rectangle is mainly governed by

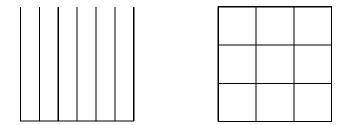


FIGURE 1. A comb-shaped configuration as opposite to a grid structure.

its short side (a detailed proof would follow the same lines as the proof concerning C_n).

We point out, however, that for large length L a comb configurations is *strictly* more performant (at least when p = 2) than a grid structure of about the same length (see Figure 1). Indeed, while for C_n the ratio $\mathcal{H}^1(C_n)^2/\lambda_2(Y \setminus C_n)$ is about $1/\Lambda_2 = 1/\pi^2$, one can easily check that, replacing C_n with a grid structure of about the same length, the new ratio would approach $2/\pi^2$, hence a comb structure is twice more performant than a grid structure.

Remarkably, the same comb structures are asymptotically optimal also for averagedistance problems (as proved in [18]) and for the compliance optimization (this was conjectured by Buttazzo and Santambrogio in [5] and later proved in [21]).

4. The asymptotics as $p \to \infty$ and maximum distance problems

In this section we investigate problem (4) as p tends to ∞ (with fixed L), showing that it converges to the problem

(49)
$$\max\Big\{\frac{1}{\max_{x\in\Omega}d(x,\Sigma\cup\partial\Omega)}:\ \Sigma\in\mathcal{A}_L(\Omega)\Big\}.$$

This is not surprising, since for every bounded domain D

(50)
$$\lim_{p \to \infty} \lambda_p(D)^{1/p} = \frac{1}{\max_{x \in D} d(x, \partial D)}$$

(and the right-hand side can be taken as the definition of $\lambda_{\infty}(D)$, the principal frequency of the " ∞ -laplacian", see [12]). In other words, problem (4) in the limiting case $p = \infty$ reduces to the so-called maximum distance problem

(51)
$$\min\{\max_{x\in\Omega} d(x,\Sigma\cup\partial\Omega): \ \Sigma\in\mathcal{A}_L(\Omega)\},\$$

that is, to search for those configurations $\Sigma \in \mathcal{A}_L(\Omega)$ that minimize the radius of the largest ball that fits in $\Omega \setminus \Sigma$ (see [19]).

In view of letting $p \to \infty$ in (4), since now L is fixed, scaling by L^p as in (6) would be pointless: the proper normalization, suggested by (50), is raising the eigenvalue to the power 1/p. Moreover, it is no longer necessary to work in the space of probability measures, as the set $\mathcal{A}_L(\Omega)$ provides the natural common domain for the relevant functionals.

The precise Γ -convergence result is then the following.

Theorem 4.1. Fix L > 0. As $p \to \infty$, the functionals $F_L^{(p)} : \mathcal{A}_L(\Omega) \mapsto (0,\infty)$ defined for p > 1 by

$$F_L^{(p)}(\Sigma) := \frac{1}{\lambda_p^{\sigma,\rho} (\Omega \setminus \Sigma)^{1/p}}$$

 Γ -converge, with respect to the Hausdorff distance on $\mathcal{A}_L(\Omega)$, to the Γ -limit

$$F_L^{\infty}(\Sigma) := \max_{x \in \Omega} d(x, \Sigma \cup \partial \Omega).$$

Proof. Since the functions ρ, σ are uniformly positive and bounded, by (25) we may assume that $\rho, \sigma \equiv 1$, and work with λ_p in place of $\lambda_p^{\sigma,\rho}$. We start with the Γ -liminf inequality, proving that for every $\Sigma \in \mathcal{A}_L(\Omega)$ and every sequence $\{\Sigma_p\} \subset \mathcal{A}_L(\Omega)$ such that $\Sigma_p \to \Sigma$ in the Hausdorff distance, it holds

(52)
$$\liminf_{p \to \infty} F_L^{(p)}(\Sigma_p) \ge \max_{x \in \Omega} d(x, \Sigma \cup \partial \Omega).$$

For a fixed $\Sigma \in \mathcal{A}_L(\Omega)$ and a sequence $\{\Sigma_p\}$ converging in the Hausdorff distance to Σ , choose a number r > 0 such that $r < \max_{x \in \Omega} d(x, \Sigma \cup \partial \Omega)$. From the Hausdorff convergence of $\{\Sigma_p\}$ to Σ , we see that $r < d(x, \Sigma_p \cup \partial \Omega)$, and hence there exists a ball B_r of radius r such that $B_r \subset \Omega \setminus \Sigma_p$, provided p is large enough. Therefore, by monotonicity of λ_p and (50)

$$\liminf_{p \to \infty} F_L^{(p)}(\Sigma_p) \ge \liminf_{p \to \infty} \frac{1}{\lambda_p(B_r)} = \max_{x \in B_r} d(x, \partial B_r) = r,$$

and letting $r \to \max_{x \in \Omega} d(x, \Sigma \cup \partial \Omega)$ we obtain (52). Finally, the Γ -limsup inequality follows immediately from the pointwise convergence of $F_L^{(p)}$ to F_L^{∞} , i.e. from (50). Indeed, given $\Sigma \in \mathcal{A}_L(\Omega)$ one can define the constant sequence $\Sigma_p := \Sigma$, which gives

$$\limsup_{p \to \infty} F_L^{(p)}(\Sigma_p) = \lim_{p \to \infty} F_L^{(p)}(\Sigma) = F_L^{\infty}(\Sigma).$$

Remark 4.2. As a consequence of this Γ -convergence result and the compactness of the space $\mathcal{A}_L(\Omega)$ with respect to the Hausdorff convergence, we get the stability of the maximizers Σ_p , as p converges to ∞ . If Σ_p is a maximizer of problem (4) and $p \to \infty$, then, up to subsequences, the sets Σ_p converge in the Hausdorff distance to a minimizer Σ_{∞} of problem (51). Moreover

$$\lim_{p \to \infty} \lambda_p^{\sigma,\rho} (\Omega \setminus \Sigma_p)^{1/p} = \frac{1}{\max_{x \in \Omega} d(x, \Sigma_\infty \cup \partial \Omega)}.$$

The next Γ -convergence result is the analogue of Theorem 1.2, for the case $p = \infty$. **Theorem 4.3.** As $L \to \infty$ the functionals $F_L^{\infty} : \mathcal{P}(\overline{\Omega}) \to [0, \infty]$ defined as

(53)
$$F_L^{\infty}(\mu) = \begin{cases} L \max_{x \in \Omega} d(x, \Sigma \cup \partial \Omega) & \text{if } \mu = \mu_{\Sigma} \text{ for some } \Sigma \in \mathcal{A}_L(\Omega), \\ \infty & \text{otherwise} \end{cases}$$

 Γ -converge, with respect to the weak* topology of $\mathcal{P}(\overline{\Omega})$, to the functional F_{∞}^{∞} : $\mathcal{P}(\overline{\Omega}) \mapsto [0, \infty]$ defined by

(54)
$$F_{\infty}^{\infty}(\mu) = \frac{1}{2} \operatorname{ess\,sup}_{x \in \Omega} \frac{1}{f(x)},$$

where $f \in L^1(\Omega)$ is the density of μ with respect to the Lebesgue measure.

Proof. One can adapt, with only minor changes and several simplifications, the proof of Theorem 1.2, the details are omitted. The only relevant change concerns the analogue of the constant θ_p and its computation (the analogue of Theorem 3.7). For $p = \infty$, we adapt (24) by defining

(55)
$$\theta_{\infty} := \inf \left\{ \liminf_{n \to \infty} L_n \max_{x \in Y} d(x, \Sigma_n \cup \partial Y) \right\}, \quad Y = (0, 1) \times (0, 1),$$

where, as in (24), the infimum is over all sequences $L_n \to \infty$ and all sequences of sets $\Sigma_n \in \mathcal{A}_{L_n}$ such that $\mathcal{H}^1(\Sigma_n) = L_n$. To complete the proof, we will show that $\theta_{\infty} = 1/2$.

Consider two sequences L_n , Σ_n as described above. Applying Lemma 2.1 with $\Omega = Y$ (hence $\kappa = 1$) and $\Sigma = \Sigma_n$, we can use (16) with \overline{t} defined as in (14). Since clearly $\mathcal{H}^1(\Sigma_n \cup \partial Y) \leq L_n + 4$, (16) gives

$$L_n \max_{x \in Y} d(x, \Sigma_n \cup \partial Y) \ge \frac{L_n}{(L_n + 4) + \sqrt{(L_n + 4)^2 + 2\pi}}.$$

Letting $n \to \infty$, from the arbitrariness of Σ_n and L_n , we see from (55) that $\theta_{\infty} \ge 1/2$.

On the other hand, choosing $\Sigma_n = C_n$ (the comb-shaped structure of Definition 3.6) and $L_n = \mathcal{H}^1(C_n) = n + 2$, we clearly have

$$L_n \max_{x \in Y} d(x, \Sigma_n \cup \partial Y) = \frac{L_n}{2n} = \frac{n+2}{2n},$$

and letting $n \to \infty$ we see from (55) that $\theta_{\infty} \leq 1/2$. Thus, $\theta_{\infty} = 1/2$.

Remark 4.4. The Γ -limit functional F_{∞}^{∞} has a unique minimizer, given by normalized Lebesgue measure over Ω . As a consequence (cf. Corollary 1.4), if Σ_L is a maximizer of problem (49), as $L \to \infty$ the probability measures μ_{Σ_L} converge in the weak* topology to the uniform measure $dx/\text{meas}(\Omega)$ (the minimizer of F_{∞}^{∞}).

Finally, for completeness, we prove that the functionals defined in (7), after renormalization, Γ -converge to the functional in (54).

Theorem 4.5. As $p \to \infty$ the functionals $F_{\infty}^{(p)}$ defined by

$$F_{\infty}^{(p)}(\mu) = \frac{1}{\Lambda_p^{-1/p}} \operatorname{ess\,sup}_{x \in \Omega} \frac{\rho(x)^{1/p}}{\sigma(x)^{1/p} f(x)}, \qquad \mu \in \mathcal{P}(\overline{\Omega})$$

 Γ -converge, in the weak* topology on $\mathcal{P}(\overline{\Omega})$, to the Γ -limit F_{∞}^{∞} defined by (54).

Proof. Since for all $\mu \in \mathcal{P}(\overline{\Omega})$ we have $m_p F_{\infty}^{\infty}(\mu) \leq F_{\infty}^{(p)}(\mu) \leq M_p F_{\infty}^{\infty}(\mu)$ where

$$m_p = \frac{2}{\Lambda_p^{1/p}} \min_{x \in \overline{\Omega}} \frac{\rho(x)^{1/p}}{\sigma(x)^{1/p}}, \quad M_p = \frac{2}{\Lambda_p^{1/p}} \max_{x \in \overline{\Omega}} \frac{\rho(x)^{1/p}}{\sigma(x)^{1/p}},$$

and the constants $m_p, M_p \to 1$ as $p \to \infty$, it suffices to prove that the sequence of functionals $\{F_{\infty}^{\infty}\}_p$ (independent of p) Γ -converges to F_{∞}^{∞} itself. But this is true, since F_{∞}^{∞} (already obtained as a Γ -limit in $\mathcal{P}(\overline{\Omega})$ by Theorem 4.3) is a fortiori lower semicontinuous (see Prop. 6.8 and Rem. 4.5 in [8]).

An overall picture of these Γ -convergence results is given in the following commutative diagram, where the first line is an equivalent formulation (see [8], Prop. 6.16) of Theorem 1.2.

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