MODELING AND STABILIZABILITY OF VOLTAGE-ACTUATED PIEZOELECTRIC BEAMS WITH MAGNETIC EFFECTS

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Abstract. Models for piezoelectric beams and structures with piezoelectric patches generally ignore magnetic effects. This is because the magnetic energy has a relatively small effect on the overall dynamics. Piezoelectric beam models are known to be exactly observable, and can be exponentially stabilized in the energy space by using a mechanical feedback controller. In this paper, a variational approach is used to derive a model for a piezoelectric beam that includes magnetic effects. It is proven that the partial differential equation model is well-posed. Magnetic effects have a strong effect on the stabilizability of the control system. For almost all system parameters the piezoelectric beam can be strongly stabilized, but is not exponentially stabilizable in the energy space. Strong stabilization is achieved using only electrical feedback. Furthermore, using the same electrical feedback, an exponentially stable closed-loop system can be obtained for a set of system parameters of zero Lebesgue measure. These results are compared to those of a beam without magnetic effects.

Key words. Voltage-controlled piezoelectric beam, strongly coupled wave system, exact observability, stabilizability, current feedback.

1. Introduction. Piezoelectric actuators have a unique characteristic of converting mechanical energy to electrical and *magnetic energy*, and vice versa. Therefore they could be used as actuators or sensors. Piezoelectric actuators are generally scalable, smaller, less expensive and more efficient than traditional actuators, and hence, a competitive choice for many tasks in industry, particularly those involving control of structures. Piezoelectric materials been employed in civil, industrial, automotive, aeronautic, and space structures.

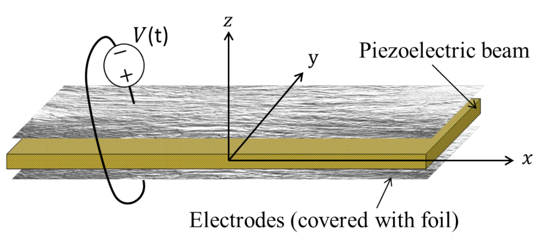

Fig. 1.1: For a voltage-actuated beam/plate, when voltage $V(t)$ is supplied to the electrodes, an electric field is created between the electrodes, and therefore the beam/plate either shrinks or extends.

In modeling of piezoelectric systems, three major effects and their interrelations need to be considered: mechanical, electrical, and magnetic. Mechanical effects are generally modeled through Kirchhoff, Euler-Bernoulli, or Mindlin-Timoshenko small displacement assumptions; see, for instance, [3], [12], [28], [39]. To include electrical and magnetic effects, there are mainly three approaches: *electrostatic, quasi-static,* and *fully dynamic* [29]. Electrostatic and quasi-static approaches are widely used - see, for instance, [9], [12], [14], [17], [25], [28], [29], [35]. These models completely exclude magnetic effects and their coupling with electrical and mechanical effects. In a electrostatic approach, electrical effects are stationary, even though the mechanical equations are dynamic. In the case of quasi-static approach, magnetic effects are still ignored but electric charges have time dependence. The electromechanical coupling is not dynamic.

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A piezoelectric beam is an elastic beam with electrodes at its top and bottom surfaces, insulated at the edges (to prevent fringing effects), and connected to an external electric circuit. (See Figure 1.1). These are the simplest structures on which to study the interaction between the electrical and mechanical energy in these systems. It is experimentally observed that the magnetic effects are minor in the overall dynamics for polarized ceramics (see the review article [40]), and therefore these effects are ignored in piezoelectric beam models. A single piezoelectric beam either shrinks or extends when the electrodes are subjected to a voltage source. For a beam of length L and thickness h, models derived by electrostatic and quasi-static approaches with the Euler-Bernoulli small displacement assumptions (no damping) describe the stretching motion as

$$\begin{cases} \rho v_{tt} - \alpha_1 v_{xx} = 0, & (x,t) \in (0,L) \times \mathbb{R}^+ \qquad (1.1a) \\ v(0,t) = 0, \ \alpha_1 v_x(L,t) = -\dfrac{\gamma V(t)}{h}, & t \in \mathbb{R}^+ \qquad (1.1b) \\ (v, \dot{v})(x,0) = (v^0, v^1), & x \in [0,L] \qquad (1.1c) \end{cases}$$

where ρ, α_1, γ denote mass density, elastic stiffness, and piezoelectric coefficients of the beam, respectively, $V(t)$ denotes the voltage applied at the electrodes, and v denotes the longitudinal displacement of the beam. In these models an elliptic-type differential equation for the electrical component is obtained due to Gauss' law 2.14. Solving this equation and then substituting into the mechanical equations leads to the wave equation (1.1). (See (2.20) with $\mu\ddot{p} \equiv 0$.) The system (1.1) is a well-posed boundary control problem on an appropriate Sobolev space. As a side note, both Kirchhoff and Mindlin-Timoshenko small displacement assumptions yield the same stretching equations (1.1). From the control theory point of view, it is well-known that a single wave equation (1.1) can be exactly controlled in the energy space (therefore the uncontrolled system, i.e. $V(t) \equiv 0$, is exactly observable). With a mechanical feedback controller in the form of boundary damping $V(t) = v_t(L,t)$, the solutions of the closed-loop system are exponentially stable in the energy space (i.e. [15] and references therein).

Exact observability and exponential stabilizability if magnetic effects are included in the mathematical models is investigated in this paper. In the fully dynamic approach, magnetic effects are included, and hence the wave behavior of the electromagnetic fields. We obtain a strongly coupled system of wave equations, one for stretching and one for magnetic effects. Voltage control comes into the play through only one boundary condition at one end. The problem of exponentially stabilizability is essentially one of simultaneous stabilizability since a single control needs to stabilize two coupled wave equations. Simultaneous control problems for wave and beam systems have been studied by a number of researchers, including [6], [15], [19], [26], [32]. In [32] conditions for simultaneous exact controllability are obtained for decoupled systems with the same input function. In [15] the controllability of coupled strings of different lengths connected at one end point is considered. It is shown that controllability in finite time, in a smaller space than the natural energy space, is determined by the ratios of the string lengths. Simultaneous controllability for general networks are considered in [6].

It is proven here that for almost all choices of system parameters a simple electrical feedback controller (current flowing through the electrodes) yields strong stability. However, for almost all system parameters, the uncontrolled system is not exactly observable in the energy space, and therefore there is no feedback $V(t)$ that makes the system exponentially stabilizable in the energy space. Finally, it is shown that the system can be exponentially stabilized only for a set of system parameters of Lebesque measure zero. This behavior is qualitatively very different from the electrostatic or quasi-static models.

This paper is organized as follows. In Section 2, a variational approach is used to derive the model; a system of partial differential equations that include magnetic effects. In section 3, well-posedness of the model is shown and also and strong stabilizability for a class of parameters. Strong stabilizability is achieved with a feedback operator that is dual to the control operator. This feedback is purely electrical. Finally in Section 4 observability and exponential stabilizability is shown to depend on system parameters. If the system is exponentially stabilizable, exponential stability is achieved with the same electrical feedback.

Throughout this paper, dots ($\dot{w}$) indicates differentiation with respect to time, $\frac{d}{dx_1} = \frac{d}{dx}$ and $\partial\Omega$ indicates the boundary of the beam Ω.

A	Magnetic potential vector	i_b	Volume current density
B	Magnetic flux density vector	n	Surface unit outward normal vector
β	Impermittivity coefficients	σ_s	Surface charge density
c, α	Elastic stiffness coefficients	σ_b	Volume charge density
γ	Piezoelectric coefficients	S	Strain tensor
D	Electric displacement vector	T	Stress tensor
E	Electric field intensity vector	v	Longitudinal displacement
ε	Permittivity coefficients	h	Thickness of the beam
f_1	Lateral force resultant in x_1 direction	H	Magnetic field intensity vector
$\tilde{f}_1$	Lateral force in x_1 direction	V	Voltage (constant in space)
f_3	Transverse force resultant in x_3 direction	w	Transverse displacement
$\tilde{f}_3$	Transverse force in x_3 direction	μ	Magnetic permeability of beam
U_1	x_1 component of the displacement field	ρ	Mass density per unit volume
U_3	x_3 component of the displacement field	ϕ	Electric potential
i_s	Surface current density		

Table 1.1: Notation

2. Piezoelectric beam model with magnetic effects. Let x_1, x_3 be the longitudinal and transverse directions, respectively. Let the piezoelectric beam occupy the region $\Omega = [0, L] \times [-\frac{h}{2}, \frac{h}{2}]$ where $h << L$. A very widely-used linear constitutive relationship [29] for piezoelectric beams is

$$\begin{pmatrix} T \\ D \end{pmatrix} = \begin{bmatrix} c & -\gamma^{\mathrm{T}} \\ \gamma & \varepsilon \end{bmatrix} \begin{pmatrix} S \\ E \end{pmatrix} \tag{2.1}$$

where $T = (T_{11}, T_{22}, T_{33}, T_{23}, T_{13}, T_{12})^{\mathrm{T}}$ is the stress vector, $S = (S_{11}, S_{22}, S_{33}, S_{23}, S_{13}, S_{12})^{\mathrm{T}}$ is the strain vector, $D = (D_1, D_2, D_3)^{\mathrm{T}}$ and $E = (E_1, E_2, E_3)^{\mathrm{T}}$ are the electric displacement and the electric field vectors, respectively, and moreover, the matrices $[c], [\gamma], [\varepsilon]$ are the matrices with elastic, electro-mechanic and dielectric constant entries (for more details the reader can refer to [29]). A list of all notation used for the piezoelectric beam model is in Table 1. Under the assumption of transverse isotropy and polarization in x_3-direction, these matrices reduce to

$$c = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{21} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{31} & c_{32} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix}, \quad \gamma = \begin{bmatrix} 0 & 0 & 0 & 0 & \gamma_{15} & 0 \\ 0 & 0 & 0 & -\gamma_{15} & 0 & 0 \\ \gamma_{31} & \gamma_{31} & \gamma_{33} & 0 & 0 & 0 \end{bmatrix}$$

$$\varepsilon = \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix}.$$

Since $h << L$, assume that all forces acting in the x_2 direction are zero. Moreover, T_{33} is also assumed to be zero. Therefore

$$T = (T_{11}, T_{13})^{\mathrm{T}}, S = (S_{11}, S_{13})^{\mathrm{T}}, D = (D_1, D_3)^{\mathrm{T}}, E = (E_1, E_3)^{\mathrm{T}}$$

and (2.1) reduces to

$$\begin{pmatrix} T_{11} \\ T_{13} \\ D_1 \\ D_3 \end{pmatrix} = \begin{pmatrix} c_{11} & 0 & 0 & -\gamma_{31} \\ 0 & c_{55} & -\gamma_{15} & 0 \\ 0 & \gamma_{15} & \varepsilon_{11} & 0 \\ \gamma_{31} & 0 & 0 & \varepsilon_{33} \end{pmatrix} \begin{pmatrix} S_{11} \\ S_{13} \\ E_1 \\ E_3 \end{pmatrix}.$$

Finally, for an Euler-Bernoulli beam, the shear stress $S_{13} = 0$. (See 2.12.) The linear constitutive equations for an Euler-Bernoulli piezoelectric beam are thus

$$\begin{cases} T_{11} = c_{11} S_{11} - \gamma_{31} E_3 & \text{(2.2a)} \\ T_{13} = -\gamma_{15} E_1 & \text{(2.2b)} \\ D_1 = \varepsilon_{11} E_1 & \text{(2.2c)} \\ D_3 = \gamma_{31} S_{11} + \varepsilon_{33} E_3. & \text{(2.2d)} \end{cases}$$

Lagrangian. Let $\mathbf{K}, \mathbf{P}, \mathbf{E}$ and $\mathbf{B}$ denote kinetic, potential, electrical, magnetic energies of the beam, respectively, and $\mathbf{W}$ is the work done by the external forces. Moreover, $\mathbf{P} - \mathbf{E} + \mathbf{B}$ is often called electrical enthalpy.

To model charge or current-controlled piezoelectric beams, that is, charge density or current density are prescribed at the electrodes, the pair (S, E) are taken to be the independent variables. The Lagrangian [18, 21]

$$\mathbf{L} = \int_0^T [\mathbf{K} - (\mathbf{P} - \mathbf{E} + \mathbf{B}) + \mathbf{W}] \, dt \tag{2.3}$$

with constitutive equations (2.2) is appropriate. For the Lagrangian $\mathbf{L}$, the work done by the external forces is

$$\mathbf{W} = \int_\Omega \left(\tilde{f}_1 U_1 + \tilde{f}_3 U_3 \right) \, dX + \int_{\partial\Omega} \bar{\sigma}_s \phi \, d\Gamma$$

where $\tilde{f}_1, \tilde{f}_3$ are external lateral and transverse forces respectively, $\bar{\sigma}_s$ is the surface charge prescribed at the electrodes, and (U_1, U_3) is the displacement field (see (2.11)); the external forces are as defined in [16]. Therefore

$$\delta\mathbf{W} = \int_\Omega \left(\tilde{f}_1 \delta U_1 + \tilde{f}_3 \delta U_3 \right) \, dX + \int_{\partial\Omega} \bar{\sigma}_s \delta\phi \, d\Gamma.$$

For voltage-driven electrodes, voltage is prescribed at the boundaries, and a different Lagrangian is needed so that the applied voltage appears in the work term. Applying a Legendre transformation to $\mathbf{L}$ yields

$$\tilde{\mathbf{L}} = \int_0^T [\mathbf{K} - (\mathbf{P} + \mathbf{E}) + \mathbf{B} + \mathbf{W}] \ dt \tag{2.4}$$

where $\mathbf{P}+\mathbf{E}$ is the total stored energy of the beam. The new Lagrangian $\tilde{L}$ is a function of independent variables (S, D). The constitutive relationship (2.2) transforms to the following relationship for (T, E)

$$\begin{cases} T_{11} = \alpha S_{11} - \gamma\beta D_3 & \text{(2.5a)} \\ T_{13} = -\gamma_1\beta_1 D_1 & \text{(2.5b)} \\ E_1 = \beta_1 D_1 & \text{(2.5c)} \\ E_3 = -\gamma\beta S_{11} + \beta D_3 & \text{(2.5d)} \end{cases}$$

where

$$\gamma = \gamma_{31}, \quad \gamma_1 = \gamma_{15} \quad \alpha = \alpha_1 + \gamma^2\beta, \quad \alpha_1 = c_{11}, \quad \beta = \frac{1}{\varepsilon_{33}}, \quad \beta_1 = \frac{1}{\varepsilon_{11}}. \tag{2.6}$$

Calling $\delta(\cdot)$ the variation of the corresponding quantity, $\tilde{\mathbf{L}}$ in (2.4) is obtained by applying the Legendre transformation to $\mathbf{L}$:

$$\delta \mathbf{L} = \int_0^T \left(\delta \mathbf{K} - \delta \left(\mathbf{H} + \int_\Omega D_i E_i \ dX \right) + \delta \left(\mathbf{W} + \int_{\partial\Omega} \sigma_s \phi \ d\Gamma \right) \right) dt = 0, \tag{2.7}$$

where ϕ is the electric potential, $\mathbf{H}$ is the enthalpy [18] and

$$\delta \mathbf{H} = \int_\Omega (T_{ij}\delta S_{ij} - D_k \delta E_k + M \cdot \delta B) \ dX$$

where $M = \frac{1}{\mu} B$ is the magnetic flux vector and μ is the permeability of the beam. The new Lagrangian $\tilde{\mathbf{L}}$ essentially remains the same since

$$\int_\Omega \delta(D_i E_i) \ dX = -\int_{\partial\Omega} \delta(\phi D_i n_i) \ d\Gamma + \int_\Omega \delta(\phi \nabla \cdot D) - \int_\Omega \delta(M \cdot B) \ dX \tag{2.8}$$

However, for the Lagrangian $\tilde{\mathbf{L}}$, the work done by the external forces is given by

$$\mathbf{W} = \int_\Omega \left(\tilde{f}_1 U_1 + \tilde{f}_3 U_3 \right) \ dX + \int_{\partial\Omega} \sigma_s \bar{\phi} \ d\Gamma \tag{2.9}$$

where $\bar{\phi}$ (namely voltage) is the electric potential prescribed at the electrodes, and therefore, using (2.15a),

$$\begin{aligned} \delta \mathbf{W} &= \int_\Omega \left(\tilde{f}_1 \delta U_1 + \tilde{f}_3 \delta U_3 \right) \ dX + \int_{\partial\Omega} \bar{\phi} \ \delta\sigma_s \ d\Gamma \\ &= \int_\Omega \left(\tilde{f}_1 \delta U_1 + \tilde{f}_3 \delta U_3 \right) \ dX - \int_{\partial\Omega} \bar{\phi} \ (\delta D_i) n_i \ d\Gamma \, . \end{aligned} \tag{2.10}$$

Therefore, depending on the prescribed quantity at the electrodes, Lagrangian can be chosen either $\mathbf{L}$ or $\tilde{\mathbf{L}}$. In this paper, the voltage at the electrodes is controlled.

Returning to the linear theory of Euler-Bernoulli beam small-displacement assumptions, the displacement field is

$$U_1 = v - x_3 \frac{\partial w}{\partial x_1}, \; U_3 = w \tag{2.11}$$

where $v = v(x_1)$ and $w = w(x_1)$ denote the longitudinal displacement of the center line, and transverse displacement of the beam, respectively. Since

$$S_{13} = \frac{1}{2}\left(\frac{\partial U_1}{\partial x_3} + \frac{\partial U_3}{\partial x_1}\right) = 0, \tag{2.12}$$

then the only strain component is given by

$$S_{11} = \frac{\partial U_1}{\partial x_1} = \frac{\partial v}{\partial x_1} - x_3 \frac{\partial^2 w}{\partial x_1^2}. \tag{2.13}$$

Magnetic effects. The magnetic energy is added to the Lagrangian $\tilde{\mathbf{L}}$ through Maxwell's equations. Let B denote magnetic field vector, and $\sigma_b, i_b, \sigma_s, i_s, V, \mu, n$ denote body charge density, body current density, surface charge density, surface current density, voltage, magnetic permeability, and unit normal vector respectively. Maxwell's equations are

$$\begin{cases} \nabla \cdot D = \sigma_b \quad \text{in} \quad \Omega \times \mathbb{R}^+ & \text{(Electric Gauss's law)} \quad (2.14\text{a}) \\ \nabla \cdot \mathrm{B} = 0 \quad \text{in} \quad \Omega \times \mathbb{R}^+ & \text{(Gauss's law of magnetism)} \quad (2.14\text{b}) \\ \nabla \times E = -\dot{\mathrm{B}} \quad \text{in} \quad \Omega \times \mathbb{R}^+ & \text{(Faraday's law)} \quad (2.14\text{c}) \\ \frac{1}{\mu}(\nabla \times \mathrm{B}) = i_b + \dot{D} \quad \text{in} \quad \Omega \times \mathbb{R}^+ & \text{(Ampére-Maxwell law)} \quad (2.14\text{d}) \end{cases}$$

with one of the essential electric boundary conditions prescribed on the electrodes

$$\begin{cases} -D \cdot n = \sigma_s \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+ & \text{(Charge)} \quad (2.15\text{a}) \\ \frac{1}{\mu}(\mathrm{B} \times n) = i_s \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+ & \text{(Current)} \quad (2.15\text{b}) \\ \phi = V \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+ & \text{(Voltage)} \quad (2.15\text{c}) \end{cases}$$

and with a chosen mechanical boundary condition at the edges of the beam (the beam is clamped, hinged, free, etc.). Since the electrodes are voltage-driven, (2.15c) is appropriate.

In modeling piezoelectric beams, there are mainly three approaches to include electric and magnetic effects [29]:

1) Electrostatic electric field: An electrostatic electric field is the most widely-used approach. It completely ignores magnetic effects: $\mathrm{B} = \dot{D} = i_b = \sigma_b = 0$. Maxwell's equations (2.14) reduce to $\nabla \cdot D = 0$ and $\nabla \times E = 0$. Therefore, by Poincaré's theorem [8] there exist a scalar electric potential such that $E = -\nabla\phi$ and ϕ is determined up to a constant.
2) Quasi-static electric field: approach ignores some of the magnetic effects (polarizable but non-magnetizable materials) [29]: it is allowed that $\dot{D}$ and B are non-zero, however $\sigma_b = i_b = 0$. Therefore, (2.14) reduces to

$$\nabla \cdot D = 0, \quad \nabla \cdot \mathrm{B} = 0, \quad \dot{\mathrm{B}} = -\nabla \times E, \quad \dot{\mathrm{D}} = \frac{1}{\mu}(\nabla \times \mathrm{B}).$$

The equation $\nabla \cdot \mathrm{B} = 0$ implies that there exists a magnetic potential vector A such that $\mathrm{B} = \nabla \times A$, by Poincáre's theorem. It follows from substituting B to $\dot{\mathrm{B}} = -\nabla \times E$ that there exists a scalar electric potential ϕ such that $E = -\nabla\phi - \dot{A}$. One simplification in this approach is to set $A = 0$ and $\dot{A} = 0$ since $A, \dot{A} \ll \phi$. Note that $\dot{D}$ non-zero.

3) Fully dynamic electric field: Unlike the quasi-static assumption, A and $\dot{A}$ are left in the model. Depending on the type of material, body charge density σ_b and body current density i_b can also be non-zero. Note that even though the displacement current $\dot{D}$ is assumed to be non-zero in both quasi-static and fully dynamic approaches, the term $\ddot{D}$ is zero in quasi-static approach since $\dot{A} = 0$.

In this paper, the third, dynamic, approach is used for the modeling of a piezoelectric beam. Assume that there is neither external body charges nor body currents, i.e., $\sigma_b = i_b \equiv 0$. The magnetic field B is perpendicular to the $x_1 - x_3$ plane due to (2.15b), and therefore B has only the $y-$component B_2, and it is only a function of $x_1 = x$. This is simply because the surface current i_s at the electrodes have only $x-$component (tangential) and B is perpendicular to both the outward normal vector ($n = (0, 0, 1)$ or $n = (0, 0, -1)$) at the electrodes and i_s. Also assume that $E_1 = 0$, and thus $D_1 = 0$ by (2.5c). Therefore, Maxwell's equations including the effects of B become

$$\nabla \cdot \mathrm{B} = 0, \quad \dot{\mathrm{B}} = -\nabla \times E, \quad \dot{D} = \frac{1}{\mu}(\nabla \times \mathrm{B}).$$

It follows from the last equation that $\frac{d\mathrm{B}_2}{dx} = -\mu \dot{D}_3$, and so

$$\mathrm{B}_2 = -\mu \int_0^x \dot{D}_3(\xi, x_3, t)\ d\xi.$$

The magnetic energy, which can be regarded as the "electric kinetic energy", is

$$\mathbf{B} = \frac{1}{2\mu}\int_\Omega \|\mathrm{B}\|^2\ dX = \frac{1}{2\mu}\int_\Omega (\mathrm{B}_2)^2\ dX = \frac{\mu}{2}\int_\Omega \left[\int_0^x \dot{D}_3(\xi, x_3, t) d\xi\right]^2\ dX.$$

The next assumption is that D_3 does not vary in the thickness direction

$$D_3(x, x_3, t) = D_3(x, t).$$

This assumption lines up with choice of electrical potential $\varphi(x, z, t)$ defined above to be linear in the thickness direction [25], i.e. $\varphi(x, z, t) = \varphi^0(x, t) + z\varphi^1(x, t)$. Therefore the electric field component in the thickness direction satisfies

$$E_3 = \frac{\partial \varphi}{\partial z} = \beta D_3(x, t).$$

Now define

$$p = \int_0^x D_3(\xi, t)\ d\xi \tag{2.16}$$

to be the total electric charge at point x. Therefore $p_x = D_3$.

Hamilton's Principle. Using (2.5) (with $D_1 = 0$), (2.2), (2.13), and the definition (2.16) of p, the stored energy (potential+ electric) $\mathbf{P} + \mathbf{E}$, magnetic energy $\mathbf{B}$ and kinetic energy $\mathbf{K}$ of the beam are

$$\begin{aligned}\mathbf{P}+\mathbf{E} &= \frac{1}{2}\int_{\Omega} (T_{11}S_{11} + D_3E_3)\ dX \\ &= \frac{h}{2}\int_0^L \left[\alpha\left(v_x^2 + \frac{h^2}{12}w_{xx}^2\right) - 2\gamma\beta v_x p_x + \beta p_x^2\right]\ dx, \end{aligned} \tag{2.17}$$

$$\mathbf{B} = \frac{1}{2\mu}\int_{\Omega} \|\mathrm{B}\|^2\ dX = \frac{\mu h}{2}\int_0^L \dot{p}^2\ dx, \tag{2.18}$$

$$\mathbf{K} = \frac{\rho}{2}\int_{\Omega}\left(\dot{U}_1^2 + \dot{U}_3^2\right)\ dX = \frac{\rho h}{2}\int_0^L \left[\dot{v}^2 + \frac{h^2}{12}\dot{w}_x^2 + \dot{w}^2\right]\ dx. \tag{2.19}$$

Defining

$$f_1(x,t) = \int_{-h/2}^{h/2} \tilde{f}_1(x,z,t)\ dz, \quad f_3(x,t) = \int_{-h/2}^{h/2} \tilde{f}_3(x,z,t)\ dz$$

to be the external force resultants defined as in [16], and $V(t)$ the voltage applied at the electrodes, the work done by the external forces is

$$\begin{aligned}\mathbf{W} &= \int_{\Omega}\left(\tilde{f}_1 U_1 + \tilde{f}_3 U_3\right)\ dX - \int_{\partial\Omega} D_3\ \bar{\phi}\ d\Gamma \\ &= \int_0^L (f_1 v + f_3 w - p_x V(t))\ dx \\ &= \int_0^L -p_x V(t)\ dx \end{aligned}$$

since there is no applied external force $\tilde{f}_1$ or lateral force $\tilde{f}_2$.

Application of Hamilton's principle, setting the variation of admissible displacements $\{v, w, p\}$ of $\tilde{\mathbf{L}}$ to zero, yields two sets of equations one for stretching and one for bending with associated boundary conditions

$$\text{Stretching:}\quad \begin{cases} \rho\ddot{v} - \alpha v_{xx} + \gamma\beta p_{xx} = 0 & \text{(2.20a)} \\ \mu\ddot{p} - \beta p_{xx} + \gamma\beta v_{xx} = 0, & \text{(2.20b)} \end{cases}$$

$$\begin{cases} v(0) = p(0) = \alpha v_x(L) - \gamma\beta p_x(L) = 0, \quad \beta p_x(L) - \gamma\beta v_x(L) = -\dfrac{V(t)}{h} & \text{(2.20c)} \\ (v, p, \dot{v}, \dot{p})(x, 0) = (v^0, p^0, v^1, p^1). & \text{(2.20d)} \end{cases}$$

$$\text{Bending:}\ \left\{ \rho h\ddot{w} + \frac{\rho h^3}{12}\ddot{w}_{xx} + \frac{\alpha h^3}{12}w_{xxxx} = 0, \right. \tag{2.21a}$$

$$\begin{cases} w(0) = w_x(0) = w_{xx}(L) = w_{xxx}(L) = 0 & \text{(2.21b)} \\ (w, \dot{w})(x, 0) = (w^0, w^1). & \text{(2.21c)} \end{cases}$$

Equation (2.21) is the Rayleigh beam equation for bending. Neglecting the moment of inertia term $\frac{\rho h^3}{12}\ddot{w}_{xx}$ in (2.21), leads to the familiar Euler-Bernoulli beam equation. Use of Mindlin-Timoshenko small displacement assumptions instead of Euler-Bernoulli leads to the same stretching equation (2.20) [24]. However, the equations for the bending and rotation of the beam are different:

$$\begin{cases} \rho h\ddot{w} - \varsigma h(\psi + w_x)_x = 0, & (2.22a)\\ \dfrac{\rho h^3}{12}\ddot{\psi} - \dfrac{\alpha h^3}{12}\psi_{xx} + \varsigma h(\psi + w_x) = 0, & (2.22b)\end{cases}$$

$$\begin{cases} \psi(0) = \psi_x(L) = w(0) = (\psi + w_x)(L) = 0 & (2.22c)\\ (w, \psi, \dot{w}, \dot{\psi})(x, 0) = (w^0, \psi^0, w^1, \psi^1) & (2.22d)\end{cases}$$

where ψ and ς denote the angle of rotation of the beam and shear stiffness coefficient, respectively.

Note that the bending equation (2.21) in the Euler-Bernoulli beam case, and the bending and rotation equations (2.22) in the Mindlin-Timoshenko case are completely decoupled from the stretching equations (2.20). The applied voltage $V(t)$ affects only the stretching motion. Therefore throughout the rest of the paper only the stretching equations (2.20) are considered.

Note that in the case of static magnetic effects, then $\mu\ddot{p} = 0$ in (2.20b) and (2.20b) can be solved for p_{xx}. Elimination of p_{xx} in (2.20a) yields the system (1.1). This is the stretching equation obtained for a single piezoelectric beam in all of the classical models, i.e. [3], [28], [29]. This model is known to be exactly observable and stabilizable, i.e. see [15]. Similarly, the case of no electro-mechanical coupling, $\gamma = 0$, the voltage V only affects p. We will assume throughout this paper that $\gamma > 0$ and $\mu > 0$ so that the stretching equations (2.20) are coupled.

3. Well-posedness. Define

$$H^1_L(0, L) = \{v \in H^1(0, L) : v(0) = 0\}, \qquad \mathbb{X} = (\mathbb{L}^2(0, L))^2,$$

and the complex linear space

$$\mathrm{H} = \left(H^1_L(0, L)\right)^2 \times \mathbb{X}.$$

Since we are neglecting the bending terms, the energy associated with (2.20) is, recalling from (2.6) that $\alpha = \alpha_1 + \gamma^2\beta$,

$$\mathrm{E} = \frac{1}{2}\int_0^L \left\{\rho|\dot{v}|^2 + \mu|\dot{p}|^2 + \alpha_1|v_x|^2 + \beta\left|\gamma v_x - p_x\right|^2\right\}\, dx. \tag{3.1}$$

This motivates definition of the inner product on H

$$\left\langle \begin{pmatrix} u_1\\ u_2\\ u_3\\ u_4\end{pmatrix}, \begin{pmatrix} v_1\\ v_2\\ v_3\\ v_4\end{pmatrix}\right\rangle_{\mathrm{H}} = \left\langle \begin{pmatrix} u_3\\ u_4\end{pmatrix}, \begin{pmatrix} v_3\\ v_4\end{pmatrix}\right\rangle_{(\mathbb{L}^2(0,L)^2} + \left\langle \begin{pmatrix} u_1\\ u_2\end{pmatrix}, \begin{pmatrix} v_1\\ v_2\end{pmatrix}\right\rangle_{\left(H^1_L(0,L)\right)^2}$$

$$= \int_0^L \{\rho u_3\bar{v}_3 + \mu u_4\bar{v}_4\}\, dx + \int_0^L \{\alpha_1(u_1)_x(\bar{v}_1)_x$$
$$+\beta\left(\gamma(u_1)_x - (u_2)_x\right)\left(\gamma(\bar{v}_1)_x - (\bar{v}_2)_x\right)\} dx$$

$$= \int_0^L \left\{\rho u_3\bar{v}_3 + \mu u_4\bar{v}_4 + \left\langle \begin{pmatrix} \alpha_1 + \gamma^2\beta & -\gamma\beta\\ -\gamma\beta & \beta\end{pmatrix}\begin{pmatrix} u_{1x}\\ u_{2x}\end{pmatrix}, \begin{pmatrix} v_{1x}\\ v_{2x}\end{pmatrix}\right\rangle_{\mathbb{C}^2}\right\}\, dx \tag{3.2}$$

where $\langle\cdot,\cdot\rangle_{\mathbb{C}^2}$ is the inner product on $\mathbb{C}^2$.

Rewriting the last term,

$$\left\langle \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \right\rangle_{\mathrm{H}} = \int_0^L \left\{\rho|v_3|^2 + \mu|v_4|^2 + \alpha_1|v_{1x}|^2 + \beta\left|\gamma v_{1x} - v_{2x}\right|^2\right\}\, dx,$$

and so $\langle\,,\,\rangle$ does indeed define an inner product, with induced norm

$$\left\| \begin{pmatrix} v \\ p \\ \dot{v} \\ \dot{p} \end{pmatrix} \right\|^2 = \frac{2}{h}E.$$

Define the operator

$$A : \mathrm{Dom}(A) \subset \mathbb{X} \to \mathbb{X} \quad A = \begin{pmatrix} -\frac{\alpha}{\rho}D_x^2 & \frac{\gamma\beta}{\rho}D_x^2 \\ \frac{\gamma\beta}{\mu}D_x^2 & -\frac{\beta}{\mu}D_x^2 \end{pmatrix}, \tag{3.3}$$

where

$$\mathrm{Dom}(A) = \{(w_1, w_2)^{\mathrm{T}} \in (H^2(0,L)\ \cap\ H_L^1(0,L))^2\ ;\ w_{1x}(L) = w_{2x}(L) = 0\}. \tag{3.4}$$

The operator A can be easily shown to be a positive and self-adjoint operator.

For $\theta \geq 0$ define $\mathbb{X}_\theta = \mathrm{Dom}(A^\theta)$ with the norm $\|\cdot\|_\theta = \|A^\theta\cdot\|_{\mathbb{X}}$ The space $\mathbb{X}_{-\theta}$ is the dual of $\mathbb{X}_\theta$ pivoted with respect to $\mathbb{X}$. For example, the inner product on $\mathbb{X}_{-1/2}$ is

$$\langle z_1, z_2\rangle_{\mathbb{X}_{-1/2}} := \left\langle A^{-1/2}z_1, A^{-1/2}z_2\right\rangle_{\mathbb{X}}.$$

Using the definition of inner product $\langle\cdot,\cdot\rangle_{(H_L^1(0,L))^2}$ in (3.2) yields

$$\langle z_1, z_2\rangle_{\mathbb{X}_{1/2}} = \left\langle A^{1/2}z_1, A^{1/2}z_2\right\rangle_{\mathbb{X}} = \langle Az_1, z_2\rangle_{\mathbb{X}} = \langle z_1, z_2\rangle_{(H_L^1(0,L))^2},$$

and therefore

$$\mathbb{X}_0 = \mathbb{X}, \quad \mathbb{X}_{1/2} = (H_L^1(0,L))^2, \quad \mathbb{X}_{-1/2} = ((H_L^1(0,L))^*)^2 \tag{3.5}$$

where $(H_L^1(0,L))^*$ is the dual space of $H_L^1(0,L)$ pivoted with respect to $\mathbb{L}^2(0,L)$. Moreover, $\mathbb{X}_1 = \mathrm{Dom}(A)$.

Let $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^{\mathrm{T}}$. Note that $\mathrm{H} = \mathbb{X}_{1/2} \times \mathbb{X}$ and define $\mathcal{A} : \mathrm{Dom}(\mathcal{A}) \subset \mathrm{H} \to \mathrm{H}$ by

$$\mathcal{A} = \begin{pmatrix} 0 & I_{2\times 2} \\ -A & 0 \end{pmatrix},$$

$$\begin{aligned} \mathrm{Dom}(\mathcal{A}) &= \mathrm{X}_1 \times \mathrm{X}_{1/2} \\ &= \{\psi \in \mathrm{H} \cap ((H^2(0,L))^2 \times (H_L^1(0,L))^2);\ \psi_{1x}(L) = \psi_{2x}(L) = 0\} \end{aligned} \tag{3.6}$$

which is densely defined in H. Also define the control operator B

$$B_0 \in \mathcal{L}(\mathbb{C}, \mathbb{X}_{-1/2}),\ \text{with}\ B_0 = \begin{pmatrix} 0 \\ -\frac{1}{h}\delta(x-L) \end{pmatrix},$$

$$B \in \mathcal{L}(\mathbb{C}, \mathrm{H}_{-1}),\ \text{with}\ B = \begin{pmatrix} 0 \\ B_0 \end{pmatrix} \tag{3.7}$$

where H_{-1} is the dual of the space $\mathrm{Dom}(\mathcal{A}) = \mathbb{X}_1 \times \mathbb{X}_{1/2}$ pivoted with respect to $\mathrm{H} = \mathbb{X}_{1/2} \times \mathbb{X}$. By (3.5). We have $\mathrm{H}_{-1} = \mathbb{X}_0 \times \mathbb{X}_{-1/2}$. The dual operators of B_0 and B are

$$B_0^* \in \mathcal{L}(\mathbb{X}_{1/2}, \mathbb{C}), \quad B_0^*\psi = -\frac{1}{h}\psi_4(L), \quad \text{with} \quad B^*\psi = (0_{2\times 2} \quad B_0^*)^{\mathrm{T}}\psi = -\frac{1}{h}\psi_4(L).$$

Writing $\varphi = (v, p, \dot{v}, \dot{p})^{\mathrm{T}}$ and defining the output

$$y(t) = \frac{1}{h}\dot{p}(L,t),$$

the control system (2.20) with this output can be put into the state-space form

$$\begin{cases} \dot{\varphi} = \underbrace{\begin{pmatrix} 0 & I_{2\times 2} \\ -A & 0 \end{pmatrix}}_{\mathcal{A}} \varphi + \underbrace{\begin{pmatrix} 0 \\ B_0 \end{pmatrix}}_{B} V(t), & \text{(3.8a)} \\ \varphi(x,0) = \varphi^0, & \text{(3.8b)} \\ y(t) = -B^*\varphi(t). & \text{(3.8c)} \end{cases}$$

Lemma 3.1. *The operator $\mathcal{A}$ satisfies $\mathcal{A}^* = -\mathcal{A}$ on* H, *and*

$$\mathrm{Re}\, \langle \mathcal{A}\psi, \psi \rangle_{\mathrm{H}} = \mathrm{Re}\, \langle \mathcal{A}^*\psi, \psi \rangle_{\mathrm{H}} = 0. \tag{3.9}$$

Also, $\mathcal{A}$ has a compact resolvent.

Proof: Choose any $u = [u_1, u_2, u_3, u_4]^{\mathrm{T}},\ v = [v_1, v_2, v_3, v_4]^{\mathrm{T}} \in \mathrm{Dom}(\mathcal{A})$. A simple calculation using integration by parts and the boundary conditions (3.6) shows

$$\begin{aligned} \langle \mathcal{A}u, v \rangle_{\mathrm{H}} = \int_0^L & \{(-\alpha(\bar{v}_1)_{xx} + \gamma\beta(\bar{v}_2)_{xx})u_3 + (\beta(\bar{v}_2)_{xx} - \gamma\beta(\bar{v}_1)_{xx})u_4 \\ & -\alpha(\bar{v}_3)_x(u_1)_x + \gamma\beta(\bar{v}_4)_x(u_1)_x + \gamma\beta(u_2)_x(\bar{v}_3)_x - \beta(u_2)_x(\bar{v}_4)_x\}\, dx \\ = \langle u, -\mathcal{A}v \rangle_{\mathrm{H}}\, . & \qquad \text{(3.10)} \\ & \qquad \text{(3.11)} \end{aligned}$$

This shows that $\mathcal{A}$ is skew-symmetric. To prove that $\mathcal{A}$ is skew-adjoint on H, i.e. $\mathcal{A}^* = -\mathcal{A}$ on H, with the same domains it is required to show that for any $v \in \mathrm{H}$ there is $u \in \mathrm{Dom}(\mathcal{A})$ so that $\mathcal{A}u = v$. This is equivalent to solving the system of equations for $u \in \mathrm{Dom}(\mathcal{A})$. Using (2.6) to simplify the equations leads to

$$\begin{aligned} u_3 &= v_1 \\ u_4 &= v_2 \\ -(u_1)_{xx} &= \frac{\rho}{\alpha_1}v_3 + \frac{\mu\gamma}{\alpha_1}v_4 \\ -(u_2)_{xx} &= -\frac{(\alpha+\alpha_1)\rho}{\alpha_1\gamma\beta}v_3 - \frac{\alpha\mu}{\beta\alpha_1}v_4\, . \end{aligned} \tag{3.12}$$

Since the Greens function corresponding to the operator $-\frac{d^2}{dx^2}$ with the boundary conditions $(\cdot)(0) = \frac{d(\cdot)}{dx}(L) = 0$ is $K(x,r) = \begin{cases} r, & x > r \\ x, & x < r, \end{cases}$ the solution of (3.12) is

$$\begin{aligned} u_1 &= \frac{1}{\alpha_1}\int_0^L K(x,r)\left(\rho v_3(r) + \mu\gamma v_4(r)\right)\,dr \\ u_2 &= -\frac{1}{\alpha_1}\int_0^L K(x,r)\left(\frac{(\alpha+\alpha_1)\rho}{\gamma\beta}v_3(r) + \frac{\alpha\mu}{\beta}v_4(r)\right)\,dr \\ u_3 &= v_1, \quad u_4 = v_2. \end{aligned} \tag{3.13}$$

Using $v \in \mathrm{H}$, i.e. $v_1, v_2 \in H^1_L(0,L)$ and $v_3, v_4 \in L^2(0,L)$, implies that $u_3, u_4 \in H^1_L(0,L)$ and $u_1, u_2 \in H^2(0,L) \cap H^1_L(0,L)$ with $(u_1)_x(L) = (u_2)_x(L) = 0$. Therefore, $u \in \mathrm{Dom}(\mathcal{A})$ is uniquely defined. Using Proposition 3.7.3 in [33] leads to the conclusion that $\mathcal{A}^* = -\mathcal{A}$ on H. Since then for $u \in \mathrm{Dom}(\mathcal{A})$, with a similar expression for A^*, (3.9) follows.

Moreover, $\mathrm{Dom}(\mathcal{A})$ is densely defined and compact in H by Sobolev's embedding theorem. Therefore, for any $\lambda \in \rho(\mathcal{A})$, $(\lambda I - \mathcal{A})^{-1}$ is a compact operator. □

The transfer function corresponding to the control system (3.8) is (see [37] for the calculation for a similar system)

$$\mathbf{G}(s) = sB_0^*(s^2 I + A)^{-1}B_0 \tag{3.14}$$

for s, $\mathrm{Re}\, s > 0$.

Lemma 3.2. *Define the set $\mathcal{C}_{s_1} = \{s \in \mathbb{C} \;:\; s = s_1 + is_2, \quad s_1 > 0\}$. We have*

$$\sup_{s\in\mathcal{C}_{s_1}} \|\mathbf{G}(s)\|_{\mathcal{L}(\mathbb{C})} < \infty. \tag{3.15}$$

Proof: See Appendix A.

Definition 3.3. *The operator $B \in \mathcal{L}(\mathbb{C}, \mathrm{H}_{-1})$ is an admissible control operator for $\{e^{\mathcal{A}t}\}_{t\geq 0}$ if there exists a positive constant $c(T)$ such that for all $u \in H^1(0,T)$,*

$$\left\| \int_0^T e^{\mathcal{A}(T-t)}Bu(t)dt \right\|_{\mathrm{H}} \leq c(T)\|u\|_{\mathbb{L}^2(0,T)}.$$

Definition 3.4. *The operator $B^* \in \mathcal{L}(\mathrm{Dom}(\mathcal{A}), \mathbb{C})$ is an admissible observation operator for $\{e^{\mathcal{A}^*}t\}_{t\geq 0}$ if there exists a positive constant $c(T)$ such that for all $\varphi^0 \in$* $\mathrm{Dom}(\mathcal{A})$

$$\int_0^T \|B^* e^{\mathcal{A}^* t}\varphi^0\|^2\,dt \leq c(T)\|\varphi^0\|^2_{\mathrm{H}}.$$

The operator B^* is an admissible observation operator for $\{e^{\mathcal{A}^*t}\}_{t\geq 0}$, if and only if B is an admissible control operator for $\{e^{\mathcal{A}t}\}_{t\geq 0}$ [33, pg. 127]).

Consider the uncontrolled system

$$\begin{aligned} \dot{\varphi}(t) &= \mathcal{A}\varphi(t), \\ \varphi(x,0) &= \varphi^0, \\ y(t) &= -B^*\varphi. \end{aligned} \tag{3.16}$$

The following theorem on well-posedness of (3.8) and (3.16) is now immediate. It proves that for any $T > 0$, the map from the input $V(t) \in \mathbb{L}^2(0,T)$ to the solution $\psi \in \mathrm{H}$, and the map from the input $V(t)$ to the output $y(t)$ of (3.8) are bounded.

THEOREM 3.5. *Let $T > 0$, and $V(t) \in \mathbb{L}^2(0,T)$. For any $\varphi^0 \in \mathrm{H}$, there exists positive constants $c_1(T), c_2(T)$ such that*

$$\|\varphi(T)\|_{\mathrm{H}}^2 \leq c_1(T) \left\{ \|\varphi^0\|_{\mathrm{H}}^2 + \|V\|_{\mathbb{L}^2(0,T)}^2 \right\}, \tag{3.17}$$

$$\|y\|_{\mathbb{L}^2(0,T)}^2 \leq c_2(T) \left\{ \|\varphi^0\|_{\mathrm{H}}^2 + \|V\|_{\mathbb{L}^2(0,T)}^2 \right\}. \tag{3.18}$$

Proof: The operator B^* defined above is an admissible observation operator for the system (3.16) by Lemma 3.2 (see Proposition 3.2 and 3.3 in [1]). Therefore B is an admissible control operator for the semigroup $\{e^{\mathcal{A}t}\}_{t\geq 0}$ corresponding to (3.8). Lemma 3.2 and the Paley-Wiener Theorem implies that the map from the input V to the output y is bounded from $\mathbb{L}^2(0,T)$ to $\mathbb{L}^2(0,T)$ [5, Thm. 5.1]. The conclusions (3.17) and (3.18) follow. □

Alternatively, the state could be defined as

$$(\sqrt{\rho} v_t, \sqrt{\alpha_1} v_x, \sqrt{\mu} p_t, \sqrt{\beta}(p_x - \gamma v_x)).$$

With this choice of state, the control system is well-posed on $[\mathbb{L}^2(0,L)]^4$ and is a port-Hamiltonian system [20].

Damped system. Setting the control signal in (3.8) to be $V(t) = -\frac{1}{2}B^* z + u(t)$ where $u(t)$ is a new controlled input and modifying the output slightly leads to the system

$$\begin{cases} \dot{z}(t) = \mathcal{A}_d z(t) + Bu(t) = \begin{pmatrix} 0 & I_{2\times 2} \\ -A & -\frac{1}{2}B_0 B_0^* \end{pmatrix} z + \begin{pmatrix} 0_{2\times 2} \\ B_0 \end{pmatrix} u(t), & (3.19\text{a}) \\ z(x,0) = z^0, & (3.19\text{b}) \\ y(t) = -B^* z(t) + u(t) & (3.19\text{c}) \end{cases}$$

where $\mathcal{A}_d : \mathrm{Dom}(\mathcal{A}_d) \subset \mathrm{H} \to \mathrm{H}$ and $\mathrm{Dom}(\mathcal{A}_d)$ is defined by

$$\begin{aligned} \mathrm{Dom}(\mathcal{A}_d) = \big\{ z \in (H^2(0,L))^2 \times (H_L^1(0,L))^2 \ : \ z_1(0) = z_2(0) = 0, \\ \alpha z_{1x}(L) - \gamma\beta z_{2x}(L) = 0, \quad \beta z_{2x}(L) - \gamma\beta z_{1x}(L) = -\frac{z_4(L)}{2h^2} \Big\}. \end{aligned} \tag{3.20}$$

This system can also be written in second-order form as

$$\begin{cases} \begin{pmatrix} \ddot{v} \\ \ddot{p} \end{pmatrix} + A \begin{pmatrix} v \\ p \end{pmatrix} + \frac{1}{2} B_0 B_0^* \begin{pmatrix} \dot{v} \\ \dot{p} \end{pmatrix} = B_0 u(t), & (3.21\text{a}) \\ \begin{pmatrix} v \\ p \end{pmatrix}(x,0) = \begin{pmatrix} v^0 \\ p^0 \end{pmatrix}, \quad \begin{pmatrix} \dot{v} \\ \dot{p} \end{pmatrix}(x,0) = \begin{pmatrix} v^1 \\ p^1 \end{pmatrix} & (3.21\text{b}) \\ y(t) = -B_0^* \begin{pmatrix} \dot{v} \\ \dot{p} \end{pmatrix} + u(t). & (3.21\text{c}) \end{cases}$$

This system is a member of the class studied in [37].

Let H_{-1}^d is the dual of the space $\mathrm{Dom}(\mathcal{A}_\mathrm{d})$ pivoted with respect to $\mathrm{H} = \mathbb{X}_{1/2} \times \mathbb{X}$.

THEOREM 3.6. *Let $T > 0$. The system (3.19) defines a well-posed and conservative linear system with the input $u(t) \in \mathbb{L}^2(0,T)$, the output $y(t) \in \mathbb{L}^2(0,T)$, the state space* H, *the semigroup $\{e^{\mathcal{A}_d t}\}_{t\geq 0}$, and the transfer function $\mathbf{G}_d$. Then $\mathcal{A}_d$ is the generator of a contraction semigroup on $\mathcal{H}$, $B \in \mathcal{L}(\mathbb{C}, \mathrm{H}_{-1}^d)$ and $B^* \in \mathcal{L}(\mathrm{Dom}(\mathcal{A}_d), \mathbb{C})$ are admissible control and observation operators, respectively, and $\|\mathbf{G}_d(s)\| \leq 1$ for all $s \in C_s = \{s \in \mathbb{C} \ : \ s = s_1 + is_2, \quad s_1 > 0\}$.*

Proof: Since $E(t) = \frac{1}{2}\|z(t)\|_{\mathrm{H}}^2$ by (3.1), a direct calculation by using (3.19) reads

$$\begin{aligned} E(T) - E(0) &= \int_0^T \left(-\frac{1}{2} \left\langle B^* z, B^* z \right\rangle_{\mathbb{C}^2} + \frac{1}{2} \left\langle u(t), B^* z \right\rangle_{\mathbb{C}^2} + \frac{1}{2} \left\langle B^* z, u(t) \right\rangle_{\mathbb{C}^2} \right) \, dx \\ &= \frac{1}{2} \left(\int_0^T |u(t)|^2 \, dt - \int_0^T |y(t)|^2 \, dt \right), \end{aligned}$$

and therefore

$$\|z(t)\|_{\mathrm{H}}^2 + \int_0^T |y(t)|^2 = \|z^0\|_{\mathrm{H}}^2 + \int_0^T |u(t)|^2. \tag{3.22}$$

By Proposition 4.5 in [37], the conclusion of the theorem follows. □

We now show that the semigroup $\{e^{\mathcal{A}_d t}\}_{t\geq 0}$ is strongly stable for almost all choices of system parameters.

THEOREM 3.7. *The spectrum $\sigma(\mathcal{A}_d)$ of $\mathcal{A}_d$ has all isolated eigenvalues, and $0 \in \sigma(\mathcal{A}_d)$.*

Proof: First show that $0 \in \rho(\mathcal{A}_d)$. Let $G = (g_1, g_2, g_3, g_4) \in \mathrm{H}$ and find $U = (u_1, u_2, u_3, u_4)$ such that $U \in \mathrm{Dom}(\mathcal{A}_d)$ and $\mathcal{A}_d U = G$. Similar to (3.13), the solution of $\mathcal{A}_d U = G$ is

$$\begin{aligned} u_3 &= g_1 \\ u_4 &= g_2 \\ u_1 &= \frac{1}{\alpha_1} \int_0^L \left(\rho g_3(r) + \gamma\mu g_4(r)\right) K(x,r) \, dr - \frac{\gamma}{2h^2\alpha_1} g_2(L) x \\ u_2 &= -\frac{1}{\alpha_1} \int_0^L \left(\frac{(\alpha+\alpha_1)\rho}{\gamma\beta} g_3(r) + \frac{\mu\alpha}{\beta} g_4(r) \right) K(x,r) \, dr - \frac{1}{2h^2} \left(\frac{\gamma^2}{\alpha_1} + \frac{1}{\beta} \right) g_2(L) x \end{aligned}$$

where $K(x,r) = \begin{cases} x, & x \leq r \\ r, & x \geq r. \end{cases}$ Since $G \in \mathrm{H}$, $g_1, g_2 \in H_L^1(0,L)$ and $g_3, g_4 \in \mathbb{L}^2(0,L)$, and by the Trace theorem $g_2(L) \in \mathbb{L}^2(0,L)$. Note that u_1 and u_2 satisfy the boundary conditions in (3.20). Therefore $U \in \mathrm{Dom}(\mathcal{A}_\mathrm{d})$. Also, there is a unique solution U. Thus $0 \in \rho(\mathcal{A}_d)$.

Moreover, $\mathrm{Dom}(\mathcal{A}_d)$ is densely defined and compact in H by Sobolev's embedding theorem. This together with $0 \in \rho(\mathcal{A}_d)$ implies that $(\lambda I - \mathcal{A}_d)^{-1}$ is compact at $\lambda = 0$, thus compact for all $\lambda \in \rho(\mathcal{A}_d)$. Hence the spectrum of $\mathcal{A}_d$ contains all isolated eigenvalues. □

THEOREM 3.8. *$\{e^{\mathcal{A}_d t}\}_{t\geq 0}$ is strongly stable in* H *if and only if $\frac{\zeta_1}{\zeta_2} \neq \frac{2n-1}{2m-1}$, for* *some $n, m \in \mathbb{N}$ where*

$$\zeta_1 = \frac{1}{\sqrt{2}}\sqrt{\frac{\gamma^2\mu}{\alpha_1} + \frac{\mu}{\beta} + \frac{\rho}{\alpha_1} + \sqrt{\left(\frac{\gamma^2\mu}{\alpha_1} + \frac{\mu}{\beta} + \frac{\rho}{\alpha_1}\right)^2 - \frac{4\rho\mu}{\beta\alpha_1}}} \tag{3.23}$$

$$\zeta_2 = \frac{1}{\sqrt{2}}\sqrt{\frac{\gamma^2\mu}{\alpha_1} + \frac{\mu}{\beta} + \frac{\rho}{\alpha_1} - \sqrt{\left(\frac{\gamma^2\mu}{\alpha_1} + \frac{\mu}{\beta} + \frac{\rho}{\alpha_1}\right)^2 - \frac{4\rho\mu}{\beta\alpha_1}}}. \tag{3.24}$$

Proof: By Theorems 3.6 and 3.7, the spectrum consists of only eigenvalues, and $\mathrm{Re}\lambda \leq 0$. The eigenvalue problem

$$\mathcal{A}_d z = \lambda z$$

with $z = (v, p, \tilde{v}, \tilde{p})$ can be written

$$\begin{cases} \alpha v_{xx} - \gamma\beta p_{xx} = \rho\lambda^2 v & (3.25a)\\ \beta p_{xx} - \gamma\beta v_{xx} = \mu\lambda^2 p, & (3.25b)\\ \tilde{v} = \lambda v, & (3.25c)\\ \tilde{p} = \lambda p & (3.25d) \end{cases}$$

with the boundary conditions

$$\begin{array}{lcl} v(0) = p(0) & = & 0 \\ \alpha v_x(L) - \gamma\beta p_x(L) & = & 0 \\ \beta p_x(L) - \gamma\beta v_x(L) & = & -\frac{1}{2h^2}\lambda p(L). \end{array} \tag{3.26}$$

Since $0 \in \rho(\mathcal{A}_d)$, if we can show that there are no eigenvalues on the imaginary axis, or in other words, the set

$$\left\{ Y \in \mathrm{H} \mid \mathrm{Re}\,\langle \mathcal{A}_d Y, Y\rangle_{\mathrm{H}} = -\frac{1}{2h^2}|\tilde{p}(L)|^2 = 0 \right\} \tag{3.27}$$

has only the trivial $Y = 0$ solution, then by Arendt-Batty's stability theorem [2], $e^{\mathcal{A}_d(t)}$ is a strongly stable semigroup. Since $\tilde{p} = \lambda p$ where $\lambda \neq 0$ by Theorem 3.7, (3.27) implies that $p(L) = 0$.

Let $\lambda = i\tau$ where $\tau \in \mathbb{R}\backslash\{0\}$. The eigenvalue problem (3.25)-(3.26) can be written

$$\begin{cases} v_{xx} = \dfrac{-\tau^2}{\alpha_1}\left(\rho v + \gamma\mu p\right) & (3.28a)\\ p_{xx} = -\tau^2\left(\dfrac{\gamma\rho}{\alpha_1}v + \left(\dfrac{\gamma^2\mu}{\alpha_1} + \dfrac{\mu}{\beta}\right)p\right) & (3.28b) \end{cases}$$

with the over-determined boundary conditions

$$v(0) = p(0) = p_x(L) = v_x(L) = p(L) = 0. \tag{3.29}$$

Proving strong stability reduces to showing that (3.28,3.29) has only the trivial solution. Let $Z = [v, v_x, p, p_x]$. We write the system (3.28) in the form

$$\frac{dZ}{dx} = \mathcal{D}Z = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{-\rho\tau^2}{\alpha_1} & 0 & \frac{-\gamma\mu\tau^2}{\alpha_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-\gamma\rho\tau^2}{\alpha_1} & 0 & -\left(\frac{\gamma^2\mu}{\alpha_1} + \frac{\mu}{\beta}\right)\tau^2 & 0 \end{pmatrix} Z. \tag{3.30}$$

The solution to (3.30) is

$$Z = e^{\mathcal{D}x} K \tag{3.31}$$

where $K = [k_1, k_2, k_3, k_4]$ is the vector with arbitrary coefficients. The characteristic equation, $\mathrm{Det}(\mathcal{D}Z - \tilde{\lambda}Z) = 0$, is

$$\tilde{\lambda}^4 + \tilde{\lambda}^2 \left(\frac{\gamma^2 \mu}{\alpha_1} + \frac{\mu}{\beta} + \frac{\rho}{\alpha_1} \right) \tau^2 + \frac{\rho\mu}{\beta\alpha_1} \tau^4 = 0.$$

This can be regarded as a quadratic equation of $\tilde{\lambda}^2$. Since $\left(\frac{\gamma^2\mu}{\alpha_1} + \frac{\mu}{\beta} + \frac{\rho}{\alpha_1} \right)^2 - \frac{4\rho\mu}{\beta\alpha_1} = \left(\frac{\gamma^2\mu}{\alpha_1} + \frac{\mu}{\beta} - \frac{\rho}{\alpha_1} \right)^2 + \frac{4\rho\gamma^2\mu}{\alpha_1^2} > 0$, there are four roots $\{\tilde{\lambda}_1^+, -\tilde{\lambda}_1^+, \tilde{\lambda}_2^-, -\tilde{\lambda}_2^-\}$, where defining

$$a_1 = \tau\zeta_1, \quad a_2 = \tau\zeta_2, \quad \tilde{\lambda}_1^+ = ia_1, \quad \tilde{\lambda}_2^- = ia_2. \tag{3.32}$$

The solution of (3.30) is written $Z = Pe^{Jx}P^{-1}K$ where

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 \\ ia_1 & -ia_1 & ia_2 & -ia_2 \\ b_1 & b_1 & b_2 & b_2 \\ ia_1 b_1 & -ia_1 b_1 & ia_2 b_2 & -ia_2 b_2 \end{pmatrix} \tag{3.33}$$

and $e^{Jx} = \mathrm{diag}(e^{ia_1 x}, e^{-ia_1 x}, e^{ia_2 x}, e^{-ia_2 x})$,

$$b_1 = \frac{1}{\gamma\mu}(\alpha_1 \zeta_1^2 - \rho), \quad b_2 = \frac{1}{\gamma\mu}(\alpha_1 \zeta_2^2 - \rho), \tag{3.34}$$

or explicitly,

$$\begin{aligned} b_1 &= \frac{1}{2} \left(\gamma + \frac{\alpha_1}{\gamma\beta} - \frac{\rho}{\gamma\mu} + \sqrt{\left(\gamma + \frac{\alpha_1}{\gamma\beta} - \frac{\rho}{\gamma\mu} \right)^2 + \frac{4\rho}{\mu}} \right) \\ b_2 &= \frac{1}{2} \left(\gamma + \frac{\alpha_1}{\gamma\beta} - \frac{\rho}{\gamma\mu} - \sqrt{\left(\gamma + \frac{\alpha_1}{\gamma\beta} - \frac{\rho}{\gamma\mu} \right)^2 + \frac{4\rho}{\mu}} \right). \end{aligned} \tag{3.35}$$

Note that $b_1, b_2 \neq 0$, $b_1 \neq b_2$, and $b_1 b_2 = -\frac{\rho}{\mu}$. The solution of (3.30) can be written $Z = Pe^{Jx}P^{-1}K$ where

$$Pe^{Jx}P^{-1} = \left(\begin{matrix} \frac{b_1 \cos a_2 x - b_2 \cos a_1 x}{b_1 - b_2} & \frac{a_1 b_1 \sin a_2 x - a_2 b_2 \sin a_1 x}{a_1 a_2 (b_1 - b_2)} \\ \frac{-a_2 b_1 \sin a_2 x + a_1 b_2 \sin a_1 x}{b_1 - b_2} & \frac{b_1 \cos a_2 x - b_2 \cos a_1 x}{(b_1 - b_2)} \\ \frac{(-\cos a_1 x + \cos a_2 x) b_1 b_2}{b_1 - b_2} & \frac{(a_1 \sin a_2 x - a_2 \sin a_1 x) b_1 b_2}{a_1 a_2 (b_1 - b_2)} \\ \frac{(a_1 \sin a_1 x - a_2 \sin a_2 x) b_1 b_2}{b_1 - b_2} & \frac{(-\cos a_1 x + \cos a_2 x) b_1 b_2}{(b_1 - b_2)} \end{matrix} \right. \cdots$$

$$\left. \begin{matrix} \frac{\cos a_1 x - \cos a_2 x}{b_1 - b_2} & \frac{-a_1 \sin a_2 x + a_2 \sin a_1 x}{a_1 a_2 (b_1 - b_2)} \\ \frac{-a_1 \sin a_1 x + a_2 \sin a_2 x}{b_1 - b_2} & \frac{-\cos a_2 x + \cos a_1 x}{(b_1 - b_2)} \\ \frac{b_1 \cos a_1 x - b_2 \cos a_2 x}{b_1 - b_2} & \frac{a_2 b_1 \sin a_1 x - a_1 b_2 \sin a_2 x}{a_1 a_2 (b_1 - b_2)} \\ \frac{-a_1 b_1 \sin a_1 x + a_2 b_2 \sin a_2 x}{b_1 - b_2} & \frac{b_1 \cos a_1 x - b_2 \cos a_2 x}{(b_1 - b_2)} \end{matrix} \right),$$

and K is the vector of arbitrary coefficients defined in (3.31). Note that $\mathrm{Det}(Pe^{Jx}P^{-1}) = 4a_1a_2(b_1-b_2)^2 \neq 0$ since $a_1, a_2 \neq 0$ and $b_1 \neq b_2$. Using the boundary conditions $v(0) = p(0)$ implies $b_1 \neq b_2$ and so $k_1 = k_3 = 0$. Thus the solution of the eigenvalue problem (3.28) is

$$
\begin{aligned}
v(x) &= k_2 \frac{a_1b_1 \sin a_2x - a_2b_2 \sin a_1x}{a_1a_2(b_1-b_2)} + k_4 \frac{-a_1 \sin a_2x + a_2 \sin a_1x}{a_1a_2(b_1-b_2)} \\
p(x) &= k_2 \frac{(a_1 \sin a_2x - a_2 \sin a_1x)b_1b_2}{a_1a_2(b_1-b_2)} + k_4 \frac{a_2b_1 \sin a_1x - a_1b_2 \sin a_2x}{a_1a_2(b_1-b_2)}.
\end{aligned} \tag{3.36}
$$

Using the other two boundary conditions $v_x(L) = p_x(L) = 0$ leads to

$$
\tilde{K}\begin{pmatrix} k_2 \\ k_4 \end{pmatrix} = \begin{pmatrix} \frac{b_1 \cos a_2L - b_2 \cos a_1L}{(b_1-b_2)} & \frac{-\cos a_2L + \cos a_1L}{(b_1-b_2)} \\ \frac{(-\cos a_1L + \cos a_2L)b_1b_2}{(b_1-b_2)} & \frac{b_1 \cos a_1L - b_2 \cos a_2L}{(b_1-b_2)} \end{pmatrix}\begin{pmatrix} k_2 \\ k_4 \end{pmatrix} = 0. \tag{3.37}
$$

Observe that $\mathrm{Det}\tilde{K} = \cos a_1L \cos a_2L = 0$ if and only if $\cos a_1L = 0$ or $\cos a_2L = 0$. If for some integers n, m, $a_1 = (2n+1)\frac{\pi}{2L}$ and $a_2 = (2m+1)\frac{\pi}{2L}$ then $v_x(L) = p_x(L) = 0$ for all choices of k_2, k_4. We can choose k_4 so that $p(L) = 0$. Hence the controlled system has an imaginary eigenvalue and is not strongly stable. On the other hand, if $a_1 \neq (2n+1)\frac{\pi}{2L}$ and $a_2 \neq (2m+1)\frac{\pi}{2L}$, then there is only the trivial solution $Z = 0$. It follows from the Arendt-Batty's Theorem that the controlled system is strongly stable. Suppose now that $a_1 = (2n+1)\frac{\pi}{2L}$ and $a_2 \neq (2m+1)\frac{\pi}{2L}$. Then $\mathrm{Det}\tilde{K} = 0$ and $k_4 = b_1k_2$. The solution with a parameter k_2 is

$$
v(x) = k_2 \frac{\sin a_1x}{a_1}, \quad p(x) = k_2 \frac{b_1 \sin a_1x}{a_1}.
$$

However, since $p(L) = 0$, the only way to obtain $p(L) = 0$ is to choose $k_2 = 0$ and so $Z = 0$. The argument is identical if $a_1 \neq (2n+1)\frac{\pi}{2L}$ and $a_2 = (2m+1)\frac{\pi}{2L}$. Thus, if $a_1 \neq (2n+1)\frac{\pi}{2L}$ or $a_2 \neq (2m+1)\frac{\pi}{2L}$, the system is strongly stable. Thus, the system is strongly stable if and only if there are not integers n, m so that $\frac{a_1}{a_2} = \frac{2n+1}{2m+1}$. This proves the theorem. $\square$

The following theorem about the original control system (3.8) is immediate.

THEOREM 3.9. *For any $k > 0$, the control system (3.8) with feedback control $V(t) = k\dot{p}(L,t)$, is strongly stable if and only if $\frac{\varsigma_1}{\varsigma_2} \neq \frac{2n-1}{2m-1}$, for some $n, m \in \mathbb{N}$.*

The feedback signal $V(t) = k\dot{p}(L) = k\int_0^L \dot{D}_3(\xi,t)\,d\xi$ is physical since $\dot{p}(L)$ denotes the current flowing through the electrodes of the beam (2.16). The limiting case of static magnetic effects corresponds to $\mu = 0$. In this case the boundary value problem (3.28) with over-determined boundary conditions has only the trivial solution and the controlled system is strongly stable, as is well-known. However, in this case $B^*\varphi(x,t) = \frac{\gamma}{h}\dot{v}(L)$ where $\dot{v}(L)$ represents the velocity of the beam at $x = L$.

Note that strong stability is achieved with the feedback $V(t) = k\dot{p}(L,t)$ except for a set of coefficients $\frac{\varsigma_1}{\varsigma_2}$ with Lebesgue measure zero.

4. Exact observability and exponential stabilizability. The stabilizability of the controlled piezoelectric beam will be shown to be determined by the observability of the same system. We start with standard definitions of exact observability, exponential stability and stabilizability, and optimizability.

DEFINITION 4.1. *The pair (A, B^*) is exactly observable in time $T > 0$ if there exists a positive constant $C(T)$ such that for all $\varphi^0 \in \mathrm{H}$*

$$
\int_0^T \|B^* e^{\mathcal{A}t}\varphi^0\|^2 \, dt \geq C(T)\|\varphi^0\|^2_{\mathrm{H}}.
$$

DEFINITION 4.2. *The semigroup* $\{e^{\mathcal{A}t}\}_{t\geq 0}$ *with the generator* $\mathcal{A}$ *is exponentially stable on* H *if there exists constants* $M, \mu > 0$ *such that* $\|e^{\mathcal{A}t}\|_{\mathrm{H}} \leq Me^{-\mu t}$ *for all* $t \geq 0$.

DEFINITION 4.3. *We say that the scalar* k *is an admissible feedback for transfer function* $G(s) = C(sI - \mathcal{A})B$ *if* $I - G(s)k$ *has an inverse that is uniformly bounded on some right-half-plane.*

DEFINITION 4.4. *The pair* $(\mathcal{A}, B)$ *is exponentially stabilizable on* H *if there exists* $F \in \mathcal{L}(\mathrm{Dom}(\mathcal{A}), \mathbb{C})$ *such that* $(\mathcal{A}, B, F)$ *is a regular triple (the transfer function* $F_\Lambda(sI - \mathcal{A})^{-1}B$ *is well-defined),* 1 *is an admissible feedback for the transfer function* $F_\Lambda(sI - \mathcal{A})^{-1}B$, *and* $A + BF_\Lambda$ *with the domain* $Dom(A + BF_\Lambda) = \{z \in Dom(F_\Lambda) \ : \ Az + BF_\Lambda z \in \mathrm{H}\}$ *generates an exponentially stable semigroup* $\{e^{(\mathcal{A}+BF_\Lambda)t}\}_{t\geq 0}$ *on* H. *In the above, the operator* F_Λ *is the* $\Lambda-$*extension of* F :

$$F_\Lambda z = \lim_{\lambda\to\infty} F\lambda(\lambda I - A)^{-1} z$$

for all $z \in \mathrm{H}$ *for which the limit makes sense.*

DEFINITION 4.5. *The pair* $(\mathcal{A}, B)$ *is optimizable if for any* $z^0 \in \mathrm{H}$ *there exists a control* $u(t) \in \mathbb{L}^2(0, T)$ *such that* $z(t) \in \mathbb{L}^2(0, T; \mathrm{H})$ *where*

$$z(t) = e^{\mathcal{A}t} z^0 + \int_0^t e^{\mathcal{A}(t-\tau)} Bu(\tau)\ d\tau. \tag{4.1}$$

It is clear from the definitions that if $(\mathcal{A}, B)$ is stabilizable, then it is optimizable. The converse of this statement is in general false for unbounded B[36].

A result in [1] implies that exact observability of the pair $(\mathcal{A}, B^*)$ in finite time on H is equivalent to exponential stability of the semigroup $\{e^{\mathcal{A}_d t}\}_{t\geq 0}$ on H.

THEOREM 4.6. *The semigroup* $\{e^{\mathcal{A}_d t}\}_{t\geq 0}$ *is exponentially stable on* H *if and only if the pair* $(\mathcal{A}, B^*)$ *(defined in (3.16)) is exactly observable in finite time on* H.

Proof: Recall that the operator A (3.3) is self-adjoint and positive definite and also $B_0 \in \mathcal{L}(\mathbb{C}, \mathbb{X}_{-\frac{1}{2}})$ where $\mathbb{X}_1$ is $\mathrm{Dom}A$ with the norm $\|A \cdot \|$ and the state-space $\mathrm{H} = \mathbb{X}_{\frac{1}{2}} \times \mathbb{X}$. (3.5).Furthermore, the transfer function $G(s)$ is uniformly bounded in any right-hand plane with $\mathrm{Re}\, s \geq s_1 > 0$. Thus, the assumptions of [1, Thm. 2.2] are satisfied and the conclusion follows. □

THEOREM 4.7. *The semigroup* $\{e^{\mathcal{A}_d t}\}_{t\geq 0}$ *is exponentially stable on* H *if and only if the pair* $(\mathcal{A}_d, B)$ *is optimizable, i.e. for any* $z^0 \in \mathrm{H}$ *there exists* $u(t) \in \mathbb{L}^2(0, T)$ *such that* $z(t) \in \mathbb{L}^2(0, T; \mathrm{H})$ *where* $z(t)$ *is defined by (4.1).*

Proof: Since (3.19) defines a well-posed and conservative system by Theorem 3.6, and it is of the class studied in [34], the conclusion of the theorem follows from [34, Thm. 1.3]. □

The following result is an immediate consequence of the two preceding results.

THEOREM 4.8. *The control system* $(\mathcal{A}, B)$ *is optimizable if and only if* $(\mathcal{A}, B)$ *is exactly observable.*

Proof: Theorems 4.6 and 4.7 imply that $(\mathcal{A}_d, B)$ is optimizable if and only if $(\mathcal{A}, B)$ is exactly observable. Optimizability of a well-posed system is invariant under an admissible feedback [36, Thm. 6.3]. Since $(\mathcal{A}, B, B^*)$ is well-posed (Thm. 3.5), the pair $(\mathcal{A}, B)$ is optimizable if and only if $(\mathcal{A}, B)$ is exactly observable. □

Since stabilizability implies optimizability, if $(\mathbb{A}, B)$ is not exactly observable there is no feedback controller $V(t) \in \mathbb{L}^2(0, T)$ that makes the system (3.8) exponentially stable on H. We now turn our attention to the observability in the energy space H of the pair $(\mathcal{A}, B^*)$. The following result on the eigenvalues and eigenvectors of $\mathcal{A}$ will be needed.

THEOREM 4.9. *Let* $\sigma_j = \frac{(2j-1)\pi}{2L}, \quad j \in \mathbb{N}$. *The operator* $\mathcal{A}$ *has eigenvalues*

$$\lambda_{1j}^{\mp} = \frac{\mp i\sigma_j}{\zeta_1}, \quad \lambda_{2j}^{\mp} = \frac{\mp i\sigma_j}{\zeta_2}, \quad j \in \mathbb{N}. \tag{4.2}$$

The corresponding eigenfunctions are, using $\lambda_{1j}^- = -\lambda_{1j}^+, \quad \lambda_{2j}^- = -\lambda_{2j}^+$,

$$\Psi_{1j} = \begin{pmatrix} \frac{1}{\lambda_{1j}^+} \\ \frac{b_1}{\lambda_{1j}^+} \\ 1 \\ b_1 \end{pmatrix} \sin\sigma_j x, \ \Psi_{-1j} = \begin{pmatrix} \frac{1}{\lambda_{1j}^+} \\ \frac{b_1}{\lambda_{1j}^+} \\ -1 \\ -b_1 \end{pmatrix} \sin\sigma_j x,$$

$$\Psi_{2j} = \begin{pmatrix} \frac{1}{\lambda_{2j}^+} \\ \frac{b_2}{\lambda_{2j}^+} \\ 1 \\ b_2 \end{pmatrix} \sin\sigma_j x, \ \Psi_{-2j} = \begin{pmatrix} \frac{1}{\lambda_{2j}^+} \\ \frac{b_2}{\lambda_{2j}^+} \\ -1 \\ -b_2 \end{pmatrix} \sin\sigma_j x, \quad j \in \mathbb{N} \tag{4.3}$$

where $\zeta_1, \zeta_2, b_1, b_2$ *are defined by (3.23), (3.24) and (3.34), respectively.*

The eigenfunctions form an orthogonal basis for H *and so every* $\varphi^0 \in$ H *can be written, for some choice of constants* $\{c_{kj}, d_{kj} \in \mathbb{C}, \quad k = 1, 2, \quad j \in \mathbb{N}\}$,

$$\begin{aligned} \varphi^0 &= \sum_{j\in\mathbb{N}} [c_{1j}\Psi_{1j} + d_{1j}\Psi_{-1j} + c_{2j}\Psi_{2j} + d_{2j}\Psi_{-2j}] \\ &= \sum_{j\in\mathbb{N}} \begin{pmatrix} \frac{1}{\lambda_{1j}^+}(c_{1j}+d_{1j}) + \frac{1}{\lambda_{2j}^+}(c_{2j}+d_{2j}) \\ \frac{b_1}{\lambda_{1j}^+}(c_{1j}+d_{1j}) + \frac{b_2}{\lambda_{2j}^+}(c_{2j}+d_{2j}) \\ (c_{1j}-d_{1j}) + (c_{2j}-d_{2j}) \\ b_1(c_{1j}-d_{1j}) + b_2(c_{2j}-d_{2j}) \end{pmatrix} \sin\sigma_j x. \end{aligned} \tag{4.4}$$

Also, there are positive constants $\tilde{C}_1, \tilde{C}_2$ *independent of the choice of* $\Psi^0 \in$ H *so that*

$$\tilde{C}_1 \ \|\varphi^0\|_{\mathrm{H}}^2 \le \sum_{j\in\mathbb{N}} \left(|c_{1j}|^2 + |d_{1j}|^2 + |c_{2j}|^2 + |d_{2j}|^2\right) \le \tilde{C}_2 \ \|\varphi^0\|_{\mathrm{H}}^2. \tag{4.5}$$

The function

$$\varphi = \sum_{j\in\mathbb{N}} \left[c_{1j}\Psi_{1j}e^{\lambda_{1j}^+ t} + d_{1j}\Psi_{-1j}e^{-\lambda_{1j}^+ t} + c_{2j}\Psi_{2j}e^{\lambda_{2j}^+ t} + d_{2j}\Psi_{-2j}e^{-\lambda_{2j}^+ t}\right] \tag{4.6}$$

solves (3.16) for the initial data (4.4).

Proof: See Appendix A.

We now prove that the pair $(\mathcal{A}, B^*)$ corresponding to (3.16) is not exactly observable for almost all choices of parameters. The following lemma from [27] is needed to prove this result.

LEMMA 4.10. *For every irrational number* ζ *there exists increasing sequences of coprime odd integers* $\{\tilde{p}_m\}, \{\tilde{q}_m\}$ *and a constant* $C_\zeta \ge 1$ *satisfying the asymptotic relation*

$$\left| \ \zeta - \frac{\tilde{p}_m}{\tilde{q}_m} \ \right| \le \frac{C_\zeta}{\tilde{q}_m^2}, \quad m \to \infty. \tag{4.7}$$

THEOREM 4.11. *Assume that* $\frac{\zeta_2}{\zeta_1} \in \mathbb{R} - \mathbb{Q}$. *Then the pair* $(\mathcal{A}, B^*)$ *corresponding to (3.16) is not exactly observable on* H.

Proof: Let the sequences $\{\tilde{p}_m\}$ and $\{\tilde{q}_m\}$ be chosen as in Lemma 4.10 with $\zeta = \frac{\zeta_2}{\zeta_1}$ and

$$\left| \frac{\zeta_2}{\zeta_1} - \frac{\tilde{p}_m}{\tilde{q}_m} \right| \leq \frac{C_\zeta}{\tilde{q}_m^2}. \tag{4.8}$$

Define

$$\kappa_{1m} = \begin{cases} -1, & \text{if } \tilde{q}_m + 1 \equiv 0 \pmod 4 \\ 1, & \text{otherwise} \end{cases}, \quad \kappa_{2m} = \begin{cases} -1, & \text{if } \tilde{p}_m + 1 \equiv 0 \pmod 4 \\ 1, & \text{otherwise} \end{cases} \tag{4.9}$$

so that $\kappa_{1m} \sin\left(\frac{\tilde{q}_m \pi}{2}\right) = \kappa_{2m} \sin\left(\frac{\tilde{p}_m \pi}{2}\right) = 1$, and

$$\lambda_{1m} = i\left(\frac{\tilde{q}_m \pi}{2L\zeta_1}\right), \ \lambda_{2m} = i\left(\frac{\tilde{p}_m \pi}{2L\zeta_2}\right). \tag{4.10}$$

Defining

$$\Phi_{1m}^0 = \frac{\kappa_{1m}}{b_1} \begin{pmatrix} \frac{1}{\lambda_{1m}} \\ \frac{b_1}{\lambda_{1m}} \\ 1 \\ b_1 \end{pmatrix} \sin\left(\frac{\tilde{q}_m \pi x}{2L}\right), \quad \Phi_{2m}^0 = \frac{\kappa_{2m}}{b_2} \begin{pmatrix} \frac{1}{\lambda_{2m}} \\ \frac{b_2}{\lambda_{2m}} \\ 1 \\ b_2 \end{pmatrix} \sin\left(\frac{\tilde{p}_m \pi x}{2L}\right) \tag{4.11}$$

where b_1 and b_2 are defined by (3.34),

$$\begin{aligned} \Phi_{1m} &= = \frac{\kappa_{1m}}{b_1} \begin{pmatrix} \frac{1}{\lambda_{1m}} \\ \frac{b_1}{\lambda_{1m}} \\ 1 \\ b_1 \end{pmatrix} \sin\left(\frac{\tilde{q}_m \pi x}{2L}\right) e^{\lambda_{1j} t}, \\ \Phi_{2m} &= \frac{\kappa_{2m}}{b_2} \begin{pmatrix} \frac{1}{\lambda_{2m}} \\ \frac{b_2}{\lambda_{2m}} \\ 1 \\ b_2 \end{pmatrix} \sin\left(\frac{\tilde{p}_m \pi x}{2L}\right) e^{\lambda_{2j} t} \end{aligned} \tag{4.12}$$

are the solutions of (3.16) with the initial conditions $\Phi_{1m}(x,0) = \Phi_{1m}^0$ and $\Phi_{2m}(x,0) = \Phi_{2m}^0$, respectively. (This follows easily from (4.4) with the choices of $c_{2j} = d_{2j} = d_{1j} \equiv 0$ for all $j \in \mathbb{N}$, $c_{1j} \equiv 0$ for $j \neq m$, and $c_{1m} = \frac{\kappa_{1m}}{b_1}$ for the first solution, and $c_{1j} = d_{1j} = d_{2j} \equiv 0$ for all $j \in \mathbb{N}$, $c_{2j} \equiv 0$ for $j \neq m$, and $c_{2m} = \frac{\kappa_{2m}}{b_2}$ for the second solution.) By linearity, $\Phi_m = \Phi_{1m} - \Phi_{2m}$ is the solution of (3.16) corresponding to the initial condition $\Phi_m(x,0) = \Phi_{1m}(x,0) - \Phi_{2m}(x,0) = \Phi_{1m}^0 - \Phi_{2m}^0$. Using (3.2) and (4.11)

$$\begin{aligned} \|\Phi_m(x,0)\|_{\mathrm{H}}^2 &= \|\Phi_{1m}^0\|_{\mathrm{H}}^2 + \|\Phi_{2m}^0\|_{\mathrm{H}}^2 \\ &= \frac{L}{2}\left[\frac{\rho}{b_1^2} + \frac{\rho}{b_2^2} + 2\mu + \zeta_1^2\left(\frac{\alpha_1}{b_1^2} + \frac{\beta}{b_1^2}(\gamma - b_1^2)\right) + \zeta_2^2\left(\frac{\alpha_1}{b_2^2} + \frac{\beta}{b_2^2}(\gamma - b_2^2)\right)\right] \\ &= \text{constant}. \end{aligned} \tag{4.13}$$

Recalling the definition of the operator B (3.7), (4.8), and (4.9), leads to

$$\begin{aligned}
|B^*\Phi_m| &= \frac{1}{h}\left|\ \kappa_{1m}\sin\left(\frac{\tilde{q}_m\pi}{2}\right)e^{i\left(\frac{\tilde{q}_m\pi t}{2L\zeta_1}\right)} - \kappa_{2m}\sin\left(\frac{\tilde{p}_m\pi}{2}\right)e^{i\left(\frac{\tilde{p}_m\pi t}{2L\zeta_2}\right)}\ \right| \\
&= \frac{1}{h}\left|\ e^{i\left(\frac{\tilde{q}_m\pi t}{2L\zeta_1}\right)} - e^{i\left(\frac{\tilde{p}_m\pi t}{2L\zeta_2}\right)}\ \right| \\
&\leq \frac{\pi t}{2hL}\left|\frac{\tilde{q}_m}{\zeta_1} - \frac{\tilde{p}_m}{\zeta_2}\ \right| \\
&\leq \frac{\pi t C_\zeta}{2Lh\zeta_2\tilde{q}_m}
\end{aligned} \tag{4.14}$$

where the Mean Value Theorem was used to obtain the third line from the second line. Therefore, defining $M = \frac{\pi^2 T^3 C_\zeta^2}{12L^2h^2\zeta_2^2}$,

$$\int_0^T |B^*\Phi_m|^2\ dt \leq \frac{M}{\tilde{q}_m^2}.$$

Now $\|\Phi_m(x,0)\|_{\mathrm{H}}^2$ is constant while $\int_0^T |B^*\Phi_m|^2\ dt = O(\tilde{q}_m^{-2})$. Therefore the pair $(\mathcal{A}, B^*)$ is not exactly observable on H if $\frac{\zeta_2}{\zeta_1} \in \mathbb{R} - \mathbb{Q}$. □

Corollary 4.12. *If* $\frac{\zeta_2}{\zeta_1} \in \mathbb{R} - \mathbb{Q}$ *then* $(\mathcal{A}, \mathcal{B})$ *is not exponentially stabilizable on* H.

Proof: Since the pair $(\mathcal{A}, B^*)$ is not exactly observable on H by Theorem 4.11, Theorem 4.8 implies that $(\mathcal{A}, B)$ is not optimizable. Finally, since stabilizability implies optimizability, there is no admissible feedback operator that makes the system exponentially stable on H. □

Corollary 4.13. *Let* $\frac{\zeta_2}{\zeta_1} \in \mathbb{Q}$ *such that* $\frac{\zeta_2}{\zeta_1} = \frac{\tilde{p}}{\tilde{q}}$ *where* $\gcd(\tilde{p}, \tilde{q}) = 1$ *and* $\tilde{p}, \tilde{q}$ *are both odd integers. Then the pair* $(\mathcal{A}, B^*)$ *corresponding to (3.16) is not exactly observable on* H. *Therefore the system is not exponentially stabilizable on* H.

Proof: We can choose $m \in \mathbb{N}$ such that $\tilde{q}_m = \tilde{q}$ and $\tilde{p}_m = \tilde{p}$. For this particular choice of $\tilde{p}_m$ and $\tilde{q}_m$,

$$|\lambda_{1m} - \lambda_{2m}| = \frac{\pi}{2L}\left|\frac{i\tilde{q}}{\zeta_1} - \frac{i\tilde{p}}{\zeta_2}\right| \equiv 0. \tag{4.15}$$

This implies that some eigenvalues coincide, and therefore there is no gap between the eigenvalues. With the identical choice of Φ_m^0 (4.11) with $\tilde{q}_m = \tilde{q}$ and $\tilde{p}_m = \tilde{p}$, as in the proof of Thm. 4.11, $|B^*\Phi_m| \equiv 0$ by (4.14) and (4.15) while $\|\Phi_m^0\|_{\mathrm{H}} = \text{constant}$ by (4.13). Thus $(\mathcal{A}, B^*)$ is not exactly observable on H. The conclusion then follows from Theorem 4.8 □

Note that if $\frac{\zeta_2}{\zeta_1}$ can be written as a ratio of odd integers, the system is not even approximately observable.

The only remaining case to consider is when $\frac{\zeta_2}{\zeta_1}$ can be written as a ratio of coprime integers where one is odd and one is even. In this case eigenvalues (4.2) have a uniform gap and the system is exactly observable. We will use the following theorem.

Theorem 4.14. *(Ingham's Theorem) [38, page 162] If the strictly increasing sequence* $\{s_n\}_{n\in\mathbb{N}}$ *of real numbers satisfies the gap condition*

$$s_{n+1} - s_n \geq \gamma \tag{4.16}$$

for all $n \in \mathbb{N}$, for some $\gamma > 0$, then there exists positive constants $\tilde{c}_3(T)$ and $\tilde{c}_4(T)$ such that for all $T > \frac{2\pi}{\gamma}$

$$\tilde{c}_3(T) \sum_{n\in\mathbb{N}} |g_n|^2 \le \int_0^T \left| \sum_{n\in\mathbb{N}} g_n e^{is_n t} \right|^2 dt \le \tilde{c}_4(T) \sum_{n\in\mathbb{N}} |g_n|^2 \tag{4.17}$$

for all functions $\sum\limits_{n\in\mathbb{N}} g_n e^{is_n t} \ : \ \sum\limits_{n\in\mathbb{N}} |g_n|^2 < \infty$.

THEOREM 4.15. *Let $\frac{\zeta_2}{\zeta_1} \in \mathbb{Q}$ such that $\frac{\zeta_2}{\zeta_1} = \frac{\tilde{p}}{\tilde{q}}$ where $\gcd(\tilde{p}, \tilde{q}) = 1$ and $\tilde{p}, \tilde{q}$ are even and odd integers, respectively; or the other way around. Choose*

$$T > 2L \, \min(\zeta_1, \zeta_2, 2\tilde{q}\zeta_2).$$

Then the pair $(\mathcal{A}, B^*)$ is exactly observable on H, i.e. there exists a constant $C(T) > 0$ such that solutions φ of the system (3.16) satisfy the following observability estimate:

$$\int_0^T |B^*\varphi|^2 \, dt \ge C(T) \|\varphi^0\|^2_{\mathrm{H}}. \tag{4.18}$$

Proof: Let $s_{1j} = \frac{\sigma_j}{\zeta_1} = \frac{(2j-1)\pi}{2L\zeta_1}$ and $s_{2j} = \frac{\sigma_j}{\zeta_2} = \frac{(2j-1)\pi}{2L\zeta_2}$ for $j \in \mathbb{N}$. The set of eigenvalues (4.2) can be rewritten as

$$\lambda^{\mp}_{kj} = \mp i s_{kj}, \quad k = 1, 2, \quad j \in \mathbb{N}. \tag{4.19}$$

Letting φ be any solution to (3.16), with initial condition expanded as in (3.7). By (4.2)-(4.5)

$$|B^*\varphi| = \left| \frac{1}{h} \sum_{j\in\mathbb{N}} \left[b_1 \left(c_{1j} e^{is_{1j}t} - d_{1j} e^{-is_{1j}t} \right) + b_2 \left(c_{2j} e^{is_{2j}t} - d_{2j} e^{-is_{2j}t} \right) \right] (-1)^j \right|.$$

Showing (4.18) is equivalent to finding a constant $C(T)$, independent of the initial condition, so that

$$\begin{aligned} &\frac{1}{h^2} \int_0^T \left| \sum_{j\in\mathbb{N}} \left[b_1 \left(c_{1j} e^{is_{1j}t} - d_{1j} e^{-is_{1j}t} \right) + b_2 \left(c_{2j} e^{is_{2j}t} - d_{2j} e^{-is_{2j}t} \right) \right] (-1)^j \right|^2 dt \\ &\qquad \ge C(T) \|\varphi^0\|^2_{\mathrm{H}}. \end{aligned} \tag{4.20}$$

We first show that the gap condition (4.16) in Theorem 4.14 holds. If $k = n$,

$$|s_{kj} - s_{nm}| \ge \begin{cases} \frac{\pi}{L\zeta_1}, & k = n = 1; \\ \frac{\pi}{L\zeta_2}, & k = n = 2. \end{cases}, \quad j, m \in \mathbb{N} \tag{4.21}$$

by (4.19). Now let $k \ne n$. Without loss of generality, assume that $\tilde{p}$ is even and $\tilde{q}$ is odd. By (4.19)

$$\begin{aligned} |s_{1j} - s_{2m}| &= \frac{\pi}{2L} \left| \frac{2j-1}{\zeta_1} - \frac{2m-1}{\zeta_2} \right| \\ &= \frac{\pi}{2L} \frac{1}{\zeta_2 \tilde{q}} \left| (2j-1)\tilde{p} - (2m-1)\tilde{q} \right| \\ &\ge \frac{\pi}{2L} \frac{1}{\zeta_2 \tilde{q}} \end{aligned} \tag{4.22}$$

using $|(2j-1)\tilde{p}-(2m-1)\tilde{q}| \geq 1$ since $(2j-1)\tilde{p}$ is an even number and $(2m-1)\tilde{q}$ is an odd number. Similarly $|s_{2j}-s_{1m}| \geq \frac{\pi}{2L}\frac{1}{\zeta_2\tilde{q}}$.

Let's rearrange the set $\{\mp s_{kj} : k=1,2,\ j\in\mathbb{N}\}$ into an increasing sequence of $\{s_n,\ n\in\mathbb{N}\}$, and denote the coefficients $\{(-1)^j b_k c_{kj}, (-1)^{j+1} b_k d_{kj} : k=1,2,\ j\in\mathbb{N}\}$ by $\{g_n,\ n\in\mathbb{N}\}$. Then (4.21) and (4.22) yields

$$s_{n+1}-s_n \geq \gamma := \frac{\pi}{L}\min\left(\frac{1}{\zeta_1}, \frac{1}{\zeta_2}, \frac{1}{2\zeta_2\tilde{q}}\right),$$

and therefore the gap condition (4.16) holds. By Theorem 4.14, for $T > \frac{2\pi}{\gamma} = 2L\,\min(\zeta_1,\zeta_2,2\zeta_2\tilde{q})$

$$\begin{aligned}
\int_0^T |B^*\varphi|^2\,dt &= \frac{1}{h^2}\int_0^T \Bigg|\sum_{j\in\mathbb{N}} \left[b_1\left(c_{1j}e^{is_{1j}t}-d_{1j}e^{-is_{1j}t}\right)\right. \\
&\qquad\qquad \left. + b_2\left(c_{2j}e^{is_{2j}t}-d_{2j}e^{-is_{2j}t}\right)\right](-1)^j\Bigg|^2 dt \\
&= \frac{1}{h^2}\int_0^T \left|\sum_{j\in\mathbb{N}} g_n e^{is_nt}\right|^2 dt \\
&\geq \tilde{c}_3(T)\sum_{n\in\mathbb{N}}|g_n|^2 \\
&= \tilde{c}_3(T)\sum_{j\in\mathbb{N}}\left(b_1^2\left(|c_{1j}|^2+|d_{1j}|^2\right)+b_2^2\left(|c_{2j}|^2+|d_{2j}|^2\right)\right) \\
&\geq \tilde{c}_3(T)\min(b_1^2,b_2^2)\sum_{j\in\mathbb{N}}\left(|c_{1j}|^2+|d_{1j}|^2+|c_{2j}|^2+|d_{2j}|^2\right) \\
&\geq C(T)\|\varphi^0\|_{\mathrm{H}}^2
\end{aligned} \tag{4.23}$$

where $C(T)=\frac{\tilde{c}_3(T)\min(b_1^2,b_2^2)}{\tilde{C}_1}$. The constants $\tilde{C}_1$ and $\tilde{c}_3(T)$ are due to (4.5) and (4.17), respectively. Hence (4.20) holds and the system is exactly observable. □

Corollary 4.16. *Let $\frac{\zeta_2}{\zeta_1}\in\mathbb{Q}$ such that $\frac{\zeta_2}{\zeta_1}=\frac{\tilde{p}}{\tilde{q}}$ where $\gcd(\tilde{p},\tilde{q})=1$ and $\tilde{p},\tilde{q}$ are even and odd integers, respectively; or the other way around. The semigroup $\{e^{\mathcal{A}_d t}\}_{t\geq 0}$ is exponentially stable on* H, *and so $(\mathcal{A},B)$ is exponentially stabilizable on* H.

Proof: The pair $(\mathcal{A},B^*)$ is exactly observable on H by Theorem 4.15. Therefore the semigroup $\{e^{\mathcal{A}_d t}\}_{t\geq 0}$ is exponentially stable on H by Theorem 4.6. □

5. Conclusions. The main result of this paper is to show that magnetic effects in piezoelectric beams, even though small, have a dramatic effect on observability and stabilizability. The piezoelectric beam model, without magnetic effects is exactly observable and exponentially stabilizable, by $-B^*$. However, when magnetic effects are included, the beam is only observable and stabilizable when the parameter $\frac{\zeta_2}{\zeta_1}$ is coprime ratio of odd and even integers. In this case, the beam can be stabilized by the feedback $-B^*$. If this parameter is an irrational number, the beam can be strongly stabilized by the $-B^*$ feedback, but not exponentially stabilized. Explicit polynomial estimates for this situation have been obtained [23] .

Another difference between the model with magnetic effects and without is the physical nature of the $-B^*$ feedback. In models without magnetic effects (1.1) this observation corresponds to the measurement of velocity of the beam at the end. However, in the model with magnetic effects (2.20) B^* corresponds to the total current at the electrodes. It is typically easy to measure the current at the electrodes, much easier than to measure velocity.

However, voltage-controlled systems exhibit hysteresis when they are actuated at high-frequencies [7, e.g.]. Experimental evidence shows that current and charge actuation leads to much less hysteresis than voltage actuation, for instance see [10, 11, 22]. A model for piezo-electric beams with magnetic effects and current control has been derived [21]. The model for current control is quite different and the control operator is bounded.

No damping was considered in this paper. Including damping would of course make the system stable. However, the electrical nature, as opposed to mechanical, of B^* would still remain, as would the basic conclusions of the restricted effectiveness of control. As noted at the end of Section 2, modifying the Euler-Bernoulli beam to a Mindlin-Timoshenko beam makes no fundamental difference to the model since the bending and rotation parts of the model are decoupled from the stretching.

The extension to including magnetic effects in structures with piezoelectric patches is studied in [24] for both Euler-Bernoulli and Mindlin-Timoshenko beam models. For patches, bending and rotation equations are coupled to the stretching equation. Previous research on control of structures with piezo-electric patches, without magnetic effects, [13, 30, 31] showed that the location of the patch(es) on the beam/plate strongly determines the controllability/stabilizability. The recent research discussed in [24] and [13, 30, 31] suggest that controllability/stabilizability depends on not only the location of the patches but also the system parameters. This is currently being studied.

Parameter $\frac{\zeta_2}{\zeta_1}$	Strongly Stabzble	Exactly Obs.	Exp. Stabzble
irrational	✓	X	X
$\frac{\tilde{p}}{\tilde{q}}$, $\tilde{p}, \tilde{q}$ odd	X	X	X
$\frac{\tilde{p}}{\tilde{q}}$, $\tilde{p}$ odd , $\tilde{q}$ even	✓	✓	✓
$\frac{\tilde{p}}{\tilde{q}}$, $\tilde{p}$ odd , $\tilde{q}$ even	✓	✓	✓

Table 5.1: Summary of results. In the first column, $(\tilde{p}, \tilde{q})$ are coprime integers, except for the first line where $\frac{\zeta_2}{\zeta_1}$ is irrational.

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Appendix A. Proof of Lemma 3.2. Our goal is to prove that for any real $\tilde{s} > 0$, there is a constant $M > 0$ such that for all $\operatorname{Re} s \geq \tilde{s}$,

$$|\mathbf{G}(s)| = |sB_0^*(s^2 I + A)^{-1} B_0| \leq M. \tag{A.1}$$

The transfer function $G(s)$ can be found as the solution to the elliptic problem corresponding to the boundary control problem (2.20); see, for instance, [4]. Define for any scalar V

$$\begin{pmatrix} Y \\ Z \end{pmatrix} = (s^2 I + A)^{-1} B_0 V$$

where Y and Z satisfy

$$\begin{cases} \alpha Y_{xx} - \gamma\beta Z_{xx} = \rho s^2 Y & \text{(A.2a)} \\ \beta Z_{xx} - \gamma\beta Y_{xx} = \mu s^2 Z, & \text{(A.2b)} \end{cases}$$

with the boundary conditions

$$Y(0) = Z(0) = \alpha Y_x(L) - \gamma\beta Z_x(L) = \beta Z_x(L) - \gamma\beta Y_x(L) + \frac{V}{h} = 0. \qquad \text{(A.3)}$$

The system (A.2)-(A.3) is similar to the system (3.25)-(3.26) with a slight change in the boundary conditions (3.26). We follow the same approach used in the proof of Theorem 3.8 to solve the system (A.2). By (3.36), the first two boundary conditions in (A.3) yield the general solution

$$Y = k_1 \frac{\zeta_1 b_1 \sinh(\zeta_2 s x) - \zeta_2 b_2 \sinh(\zeta_1 s x)}{\zeta_1\zeta_2(b_1 - b_2)} + k_2 \frac{-\zeta_1 \sinh(\zeta_2 s x) + \zeta_2 \sinh(\zeta_1 s x)}{\zeta_1\zeta_2(b_1 - b_2)}$$
$$Z = k_1 \frac{(\zeta_1 \sinh(\zeta_2 s x) - \zeta_2 \sinh(\zeta_1 s x)) b_1 b_2}{\zeta_1\zeta_2(b_1 - b_2)} + k_2 \frac{\zeta_2 b_1 \sinh(\zeta_1 s x) - \zeta_1 b_2 \sinh(\zeta_2 s x)}{\zeta_1\zeta_2(b_1 - b_2)}$$

with two arbitrary constants k_1 and k_2. In the above ζ_1, ζ_2, b_1, and b_2 are the same nonzero constants defined by (3.23), (3.24), and (3.34), respectively. k_1 and k_2 are determined by applying the last two boundary conditions in (A.3)

$$k_1 = \frac{\frac{V}{\alpha_1 s}}{\cosh \zeta_1 L \cosh \zeta_2 L} \frac{\gamma \left(b_2 \cosh(\zeta_2 L) - b_1 \cosh(\zeta_1 L)\right) + \frac{\alpha}{\beta}\left(\cosh(\zeta_1 L) - \cosh(\zeta_2 L)\right)}{b_1 - b_2}$$
$$k_2 = \frac{\frac{V}{\alpha_1 s}}{\cosh \zeta_1 L \cosh \zeta_2 L} \frac{\frac{\alpha}{\beta}\left(b_2 \cosh(\zeta_1 L) - b_1 \cosh(\zeta_2 L)\right) + \gamma\left(\cosh(\zeta_2 L) - \cosh(\zeta_1 L)\right)}{b_1 - b_2}.$$

After simplifications,

$$Z(x) = \frac{V}{\alpha_1 h (b_1 - b_2) s} \left[\frac{b_2(\frac{\alpha}{\beta} - b_1\gamma)}{\zeta_2} \frac{\sinh(\zeta_2 s x)}{\cosh(\zeta_2 s L)} + \frac{b_1(b_2\gamma - \frac{\alpha}{\beta})}{\zeta_1} \frac{\sinh(\zeta_1 s L)}{\cosh(\zeta_1 s x)} \right]$$

and therefore

$$\begin{aligned} \mathbf{G}(s) &= \frac{s}{V} B_0^* Z \\ &= -\frac{s}{Vh} Z(L) \\ &= \frac{1}{\alpha_1 h^2 (b_1 - b_2)} \left[\frac{b_2(b_1\gamma - \frac{\alpha}{\beta})}{\zeta_2} \tanh(\zeta_2 s L) - \frac{b_1(b_2\gamma - \frac{\alpha}{\beta})}{\zeta_1} \tanh(\zeta_1 s L) \right]. \quad \text{(A.4)} \end{aligned}$$

Now bounds for the functions $|\tanh s\zeta_1 L|$ and $|\tanh s\zeta_2 L|$ are calculated. Writing $s = s_1 + i s_2$ where $s_1 \geq \tilde{s}$ for some real $\tilde{s} > 0$,

$$|\tanh(\zeta_1 s L)| = \left| \frac{e^{s\zeta_1 L} - e^{s\zeta_1 L}}{e^{s\zeta_1 L} + e^{s\zeta_1 L}} \right| = \frac{\left|1 - e^{-2s\zeta_1 L}\right|}{\left|1 + e^{-2s\zeta_1 L}\right|} \leq \frac{2}{1 - e^{-2s_1\zeta_1 L}} \leq \frac{2}{1 - e^{-2\tilde{s}\zeta_1 L}}.$$

A similar bound holds for $|\tanh(\zeta_2 s L)|$. Finally, since $\frac{b_2(b_1\gamma - \frac{\alpha}{\beta})}{\alpha_1 h^2 \zeta_2 (b_1 - b_2)}$ and $\frac{b_1(b_2\gamma - \frac{\alpha}{\beta})}{\alpha_1 h^2 \zeta_1 (b_1 - b_2)}$ in (A.4) are all nonzero constants, there exists a positive constant $M(\tilde{s}) < \infty$ such that (A.1) holds.

Multiplying both numerator and denominator in $\mathbf{G}(\mathbf{s})$ by γ and noting that in the case of $\gamma = 0$, i.e. the system (A.2)-(A.3) is completely decoupled, $\zeta_1 = \sqrt{\frac{\rho}{\alpha_1}}$, $\zeta_2 = \sqrt{\frac{\mu}{\beta}}$, $b_1\gamma = 0$, $b_2\gamma = \frac{\alpha}{\beta} - \frac{\rho}{\mu}$ where $\alpha_1 = \alpha$. Then the transfer function (A.4) of the decoupled system is $\mathbf{G}(s) = \frac{1}{h^2\sqrt{\beta\mu}} \tanh\left(\sqrt{\frac{\mu}{\beta}} Ls\right)$, the same transfer function as obtained from (A.2)-(A.3) with $\gamma = 0$. $\square$

Proof of Theorem 4.9: Let $\Psi = [\, z_1, z_2, z_3, z_4 \,]^{\mathrm{T}}$. Solving the eigenvalue problem $\mathcal{A}\Psi = \lambda\Psi$ corresponding to (3.16) is equivalent to solving

$$\begin{pmatrix} z_3 \\ z_4 \end{pmatrix} = \lambda \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad -A \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = -\lambda \begin{pmatrix} z_3 \\ z_4 \end{pmatrix} = -\lambda^2 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

By using $\alpha = \alpha_1 + \gamma^2\beta$, (2.6) and (3.4), the eigenvalue problem can be rewritten

$$\begin{cases} z_3 = \lambda z_1, \quad z_4 = \lambda z_2 & \text{(A.5a)} \\ z_{1xx} = \dfrac{\lambda^2}{\alpha_1}\left(\rho z_1 + \gamma\mu z_2\right) & \text{(A.5b)} \\ z_{2xx} = \lambda^2 \left(\dfrac{\gamma\rho}{\alpha_1} z_1 + \left(\dfrac{\gamma^2\mu}{\alpha_1} + \dfrac{\mu}{\beta}\right) z_2\right) & \text{(A.5c)} \end{cases}$$

with the boundary conditions

$$z_1(0) = z_2(0) = z_{1x}(L) = z_{2x}(L) = 0. \tag{A.6}$$

First, find the eigenvalues. Since the solution of (A.5) with $\lambda = 0$ is $z_1 = z_2 = z_3 = z_4 \equiv 0$; $\lambda = 0$ is not an eigenvalue.

Define

$$z_{1j} = f_j \sin\sigma_j x, \quad z_{2j} = g_j \sin\sigma_j x, \quad \sigma_j = \frac{(2j-1)\pi}{2L}, \quad j \in \mathbb{N}. \tag{A.7}$$

Solutions of this form satisfy all the homogeneous boundary conditions (A.6). We seek f_j, g_j and λ_j so that the system (A.5) is satisfied. Upon substitution of (A.7) into (A.5) we obtain

$$\begin{cases} -\sigma_j^2 f_j = \dfrac{\lambda^2}{\alpha_1}\left(\rho f_j + \gamma\mu g_j\right) \\ -\sigma_j^2 g_j = \lambda^2 \left(\dfrac{\gamma\rho}{\alpha_1} f_j + \left(\dfrac{\gamma^2\mu}{\alpha_1} + \dfrac{\mu}{\beta}\right) g_j\right). \end{cases}$$

Letting $y_j = \frac{\sigma_j^2}{\lambda^2}$, this linear system has nontrivial solutions if and only if the following characteristic equation is satisfied:

$$y_j^2 + \left(\frac{\gamma^2\mu}{\alpha_1} + \frac{\mu}{\beta} + \frac{\rho}{\alpha_1}\right) y_j + \frac{\rho\mu}{\beta\alpha_1} = 0.$$

Since $\left(\frac{\gamma^2\mu}{\alpha_1} + \frac{\mu}{\beta} + \frac{\rho}{\alpha_1}\right)^2 - \frac{4\rho\mu}{\beta\alpha_1} = \left(\frac{\gamma^2\mu}{\alpha_1} + \frac{\mu}{\beta} - \frac{\rho}{\alpha_1}\right)^2 + \frac{4\rho\gamma^2\mu}{\alpha_1^2} > 0$, the roots are $y_{1j} = -\zeta_1^2$, $y_{2j} = -\zeta_2^2$ where $\zeta_1, \zeta_2 \in \mathbb{R}$ are defined by (3.23) and (3.24), respectively. Therefore $\lambda_{1j}^{\mp} = \mp\frac{i\sigma_j}{\zeta_1}$ and $\lambda_{2j}^{\mp} = \mp\frac{i\sigma_j}{\zeta_2}$ for $j \in \mathbb{N}$, and (4.2) follows. Observe that

$$\lambda_{1j}^- = -\lambda_{1j}^+, \quad \lambda_{2j}^- = -\lambda_{2j}^+. \quad j \in \mathbb{N}. \tag{A.9}$$

Setting $f_j = \frac{1}{\lambda_{1j}^+}$ yields $g_j = \frac{b_1}{\lambda_{1j}^+}$. By (A.5a) and (A.9), the first two sets of eigenvectors Ψ_{1j}, Ψ_{-1j} are, using the fact that $\lambda_{1j}^- = -\lambda_{1j}^+$

$$\Psi_{1j} = \begin{pmatrix} \frac{1}{\lambda_{1j}^+} \sin \sigma_j x \\ \frac{b_1}{\lambda_{1j}^+} \sin \sigma_j x \\ \sin \sigma_j x \\ b_1 \sin \sigma_j x \end{pmatrix}, \qquad \Psi_{-1j} = \begin{pmatrix} \frac{1}{\lambda_{1j}^+} \sin \sigma_j \\ \frac{b_1}{\lambda_{1j}^+} \sin \sigma_j x \\ -\sin \sigma_j x \\ -b_1 \sin \sigma_j x \end{pmatrix}, \quad j \in \mathbb{N}.$$

Similarly, setting $f_j = \frac{1}{\lambda_{2j}^+}$ yields $g_j = \frac{b_2}{\lambda_{2j}^+}$. By (A.5a) and (A.9), the last two eigenvectors Ψ_{2j}, Ψ_{-2j} in (4.3)are

$$\Psi_{2j} = \begin{pmatrix} \frac{1}{\lambda_{2j}^+} \sin \sigma_j x \\ \frac{b_2}{\lambda_{2j}^+} \sin \sigma_j x \\ \sin \sigma_j x \\ b_2 \sin \sigma_j x \end{pmatrix}, \qquad \Psi_{-2j} = \begin{pmatrix} \frac{1}{\lambda_{2j}^+} \sin \sigma_j x \\ \frac{b_2}{\lambda_{2j}^+} \sin \sigma_j x \\ -\sin \sigma_j x \\ -b_2 \sin \sigma_j x \end{pmatrix}, \quad j \in \mathbb{N}.$$

The fact that the eigenfunctions $\{\Psi_{-1j}, \Psi_{1j}, \Psi_{-2j}, \Psi_{2j}\}_{j\in\mathbb{N}}$ are mutually orthogonal and form a basis of H follows from the fact that $\mathcal{A}$ is skew-symmetric and has a compact resolvent (Lemma 3.1).

Finally, prove (4.5). Since

$$\rho + b_1^2\mu = \zeta_1^2 \left(\alpha_1 + \beta(\gamma - b_1^2)\right), \quad \rho + b_2^2\mu = \zeta_2^2 \left(\alpha_1 + \beta(\gamma - b_2^2)\right), \tag{A.10}$$

by (3.2) and (3.35), a direct calculation leads to

$$\begin{aligned} \|\Psi^0\|_{\mathrm{H}}^2 = \frac{L}{2} \Bigg(\sum_{j\in\mathbb{N}} & (\rho + b_1^2\mu + \alpha_1\zeta_1^2 + \beta(\gamma - b_1)^2\zeta_1^2) \left(|c_{1j}|^2 + |d_{1j}|^2\right) \\ & + (\rho + b_2^2\mu + \alpha_1\zeta_2^2 + \beta(\gamma - b_2)^2\zeta_2^2) \left(|c_{2j}|^2 + |d_{2j}|^2\right)\Big) \\ = L \Bigg(\sum_{j\in\mathbb{N}} & (\rho + b_1^2\mu) \left(|c_{1j}|^2 + |d_{1j}|^2\right) + (\rho + b_2^2\mu) \left(|c_{2j}|^2 + |d_{2j}|^2\right) \Bigg). \end{aligned}$$

Setting $\tilde{C}_1 = L \min\left(\rho + b_1^2\mu, \rho + b_2^2\mu\right)$, $\tilde{C}_2 = L \max\left(\rho + b_1^2\mu^2, \rho + b_2^2\mu\right)$ results in (4.5). $\square$

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