# 4-critical graphs on surfaces without contractible ( $\leq 4$ )-cycles 

Zdeněk Dvořák* Bernard Lidický ${ }^{\dagger}$

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#### Abstract

We show that if $G$ is a 4-critical graph embedded in a fixed surface $\Sigma$ so that every contractible cycle has length at least 5 , then $G$ can be expressed as $G=G^{\prime} \cup G_{1} \cup G_{2} \cup \ldots \cup G_{k}$, where $\left|V\left(G^{\prime}\right)\right|$ and $k$ are bounded by a constant (depending linearly on the genus of $\Sigma$ ) and $G_{1}, \ldots, G_{k}$ are graphs (of unbounded size) whose structure we describe exactly. The proof is computer-assisted-we use computer to enumerate all plane 4-critical graphs of girth 5 with a precolored cycle of length at most 16 , that are used in the basic case of the inductive proof of the statement.


## 1 Introduction

The problem of 3 -coloring triangle-free graphs embedded in a fixed surface, motivated by the celebrated Grötzsch theorem [7], has drawn much attention. Thomassen [10] showed that if a graph $G$ is embedded in the torus or the projective plane so that every contractible cycle has length at least 5, then $G$

[^0]is 3 -colorable. Thomas and Walls [9] showed that graphs of girth at least 5 embedded in the Klein bottle are 3-colorable and gave a description of all 4critical graph on the Klein bottle without contractible cycles of length at most 4. Gimbel and Thomassen [6] showed that graphs of girth 6 embedded in the double torus are 3-colorable and described triangle-free projective plane graphs that are not 3-colorable.

Recently, Dvořák, Král' and Thomas [5] gave a structural description of 4 -critical (i.e., minimal non-3-colorable) triangle-free graphs embedded in a fixed surface, and used this result to give a linear-time algorithm to decide 3 -colorability of such graphs. In particular, this description implies the following.

Theorem 1 (Dvořák, Král' and Thomas [5]). There exists an absolute constant $K$ such that every 4 -critical graph of girth 5 embedded in a surface of genus $g$ has at most $K g$ vertices.

This improves a doubly-exponential bound by Thomassen [11]. Let us note that the linear bound was proved by Postle [8] also for girth 5 and 3list coloring. Somewhat unsatisfactorily, the bound on $K$ given by Dvořák et al. [5] is rather weak, proving that $K<10^{28}$ (we are not aware of any non-trivial lower bound, and suspect that $K \approx 100$ should suffice). One of the reasons why this bound is so large is hidden in the handling of the basic case of the induction, where they prove that if $G$ is a plane graph with exactly two faces $C_{1}$ and $C_{2}$ of length at most 4 , all other cycles have length at least 5 and the distance between $C_{1}$ and $C_{2}$ is at least 1500000 , then any precoloring of $C_{1}$ and $C_{2}$ extends to a proper coloring of $G$ by three colors. In this paper, we give a computer-assisted proof showing that it suffices to assume that the distance between $C_{1}$ and $C_{2}$ is at least 4 , which can be used to show that $K<10^{21}$. We were originally hoping in a bigger improvement on $K$.

Theorem 2. Let $G$ be a plane graph and let $C_{1}$ and $C_{2}$ be faces of $G$ of length at most 4, such that every cycle in $G$ distinct from $C_{1}$ and $C_{2}$ has length at least 5. If the distance between $C_{1}$ and $C_{2}$ is at least 4, then every precoloring of $C_{1} \cup C_{2}$ extends to a proper 3 -coloring of $G$.

Combining these results, we give a more precise description of the structure of the 4-critical graphs without contractible cycles of length at most 4 (a cycle is contractible if it separates the surface to two parts and at least
one of them is homeomorphic to the open disc). Thomassen [10] showed that every graph of girth at least 5 embedded in the projective plane or in the torus is 3 -colorable. Actually, he proved a stronger claim that enables him to apply induction: every graph embedded in the projective plane or in the torus so that all contractible cycles have length at least 5 (but there may be non-contractible triangles or 4 -cycles) is 3 -colorable. Thus, it might seem possible to strengthen Theorem 1 by allowing non-contractible triangles or 4-cycles. However, Thomas and Walls [9] exactly characterized 4-critical graphs embedded in the Klein bottle so that no contractible cycle has length at most 4 , showing that there are infinitely many such graphs.

Let $\mathcal{C}$ be the class of plane graphs that can be obtained from a cycle of length 4 by a finite number of repetitions of the following operation: given a graph with the outer face $v_{1} v_{2} v_{3} v_{4}$ of length 4 such that $v_{1}$ and $v_{3}$ have degree two, add new vertices $v_{2}^{\prime}, v_{3}^{\prime}$ and $v_{4}^{\prime}$ and edges $v_{1} v_{2}^{\prime}, v_{2}^{\prime} v_{3}^{\prime}, v_{3}^{\prime} v_{4}^{\prime}, v_{4}^{\prime} v_{1}$ and $v_{3} v_{3}^{\prime}$, and let $v_{1} v_{2}^{\prime} v_{3}^{\prime} v_{4}^{\prime}$ be the outer face of the resulting graph. Examples of elements of $\mathcal{C}$ are the 4 -cycle, and the graphs $Z_{3}$ in Figure 1, $A_{11}$ in Figure 11 and $Z_{3} A_{11} a$ and $Z_{3} A_{11} b$ in Figure 15. Let $\mathcal{C}^{\prime}$ be the class of graphs obtained from those in $\mathcal{C}$ by adding a chord joining the pair of vertices of degree two in both of the 4 -faces (thus introducing 4 triangles). Each graph in $\mathcal{C}^{\prime}$ can be embedded in the Klein bottle by putting crosscaps on both newly added chords; the 4 -faces of a graph in $\mathcal{C}$ thus become 6 -faces in such an embedding of the corresponding graph in $\mathcal{C}^{\prime}$.

Thomas and Walls [9] proved that a graph embedded in the Klein bottle without contractible $(\leq 4)$-cycles is 4 -critical if and only if it belongs to $\mathcal{C}^{\prime}$. We extend this result to other surfaces.

Theorem 3. There exists a function $f(g)=O(g)$ with the following property. Let $G$ be a 4-critical graph embedded in a surface $\Sigma$ of genus $g$ so that every contractible cycle has length at least 5 . Then $G$ contains a subgraph $H$ such that

- $|V(H)| \leq f(g)$, and
- if $F$ is a face of $H$ that is not equal to a face of $G$, then $F$ has exactly two boundary walks, each of the walks has length 4, and the subgraph of $G$ drawn in the closed region corresponding to $F$ belongs to $\mathcal{C}$.


## 2 Preliminaries

In order to state more technical results necessary to prove Theorems 2 and 3 , we need a few definitions. The graphs considered in this paper are undirected and without loops and parallel edges. By a coloring of a graph we always mean a proper 3 -coloring. By the genus $g(\Sigma)$ of a surface $\Sigma$ we mean the Euler genus, i.e., $2 h+c$, where $h$ is the number of handles and $c$ is the number of crosscaps attached to the sphere in order to create $\Sigma$. If $G$ is a graph embedded in $\Sigma$, a face $F$ of $G$ is a maximal connected open subset of $\Sigma-G$. Sometimes, we also let $F$ stand for the subgraph of $G$ consisting of the edges of $G$ contained in the closure of $F$. We let $\ell(F)$ be the sum of the lengths of the boundary walks of $F$ in $G$.

A graph $G$ is $k$-critical if $G$ is not $(k-1)$-colorable, but every proper subgraph of $G$ is $(k-1)$-colorable. A well-known result of Grötzsch [7] states that all triangle-free planar graphs are 3-colorable, i.e., there are no planar triangle-free 4-critical graphs. Since the cycles of length 4 can be easily eliminated, the main part of the proof of Grötzsch's theorem concerns graphs of girth 5. Generalizing this result, Thomassen [11] proved that there exists a function $f$ such that every 4-critical graph of girth 5 and genus $g$ has at most $f(g)$ vertices (where $f$ is double-exponential in $g$ ), and thus the number of such graphs is finite. This was later improved by Dvořák et al. [5], by showing that the number of vertices of such a graph is at most linear in $g$ (Theorem 11). Both the original result of Thomassen and its improvement allow a bounded number of vertices to be precolored. To state this generalization, we need to extend the notion of a 4-critical graph.

There are two natural ways one can define a critical graph with precolored vertices. Consider a graph $G$ and a subgraph (not necessarily induced) $S \subseteq$ $G$. We call $G$ strongly $S$-critical if there exists a coloring of $S$ that does not extend to a coloring of $G$, but extends to a coloring of every proper subgraph $G^{\prime} \subset G$ such that $S \subseteq G^{\prime}$. We say that $G$ is $S$-critical if for every proper subgraph $G^{\prime} \subset G$ such that $S \subseteq G^{\prime}$, there exists a coloring of $S$ that does not extend to a coloring of $G$, but extends to a coloring of $G^{\prime}$. We call a (strongly) $S$-critical graph $G$ nontrivial if $G \neq S$. Note that every strongly $S$-critical graph is also $S$-critical, but the converse is false (for example, if $G$ is a cycle $S$ with two chords, then $G$ is $S$-critical, but not strongly $S$-critical). Also, $G$ is $\emptyset$-critical (or strongly $\emptyset$-critical) if and only if $G$ is 4 -critical.

Dvořák et al. 5] bounded the size of critical graphs as follows:

Theorem 4 (Dvořák et al. [5]). Let $K=10^{28}$. Let $G$ be a graph embedded in a surface $\Sigma$ of genus $g$ and let $\left\{F_{1}, F_{2}, \ldots, F_{k}\right\}$ be a set of faces of $G$ such that the open region corresponding to $F_{i}$ is homeomorphic to the open disk for $1 \leq i \leq k$. If $G$ is $\left(F_{1} \cup F_{2} \cup \ldots \cup F_{k}\right)$-critical and every cycle of length of at most 4 in $G$ is equal to $F_{i}$ for some $1 \leq i \leq k$, then

$$
|V(G)| \leq \ell\left(F_{1}\right)+\ldots+\ell\left(F_{k}\right)+K(g+k)
$$

Let us note that such a claim does not hold without the restriction on the cycles of length 4, since Youngs [13] gave a construction of an infinite family of 4-critical triangle-free graphs that can be embedded in any surface distinct from the sphere.

Analogously, we will prove a generalization of Theorem 3 allowing a bounded number of precolored vertices (Theorem 25). It is easy to reduce the proof to the case that $\Sigma$ is the sphere and exactly two cycles are precolored. In this case, we say that the graph is embedded in the cylinder, and we call the precolored cycles the boundaries of the cylinder. In fact, it suffices to consider the case that both boundaries have length at most 4. By cutting along cycles of length at most 4 , such a graph decomposes to a possibly large number of graphs embedded in the cylinder such that the only cycles of length at most 4 are the boundaries. The main part of our proof is based on an enumeration of such graphs:

Theorem 5. Let $G$ be a connected graph embedded on the cylinder with distinct boundaries $C_{1}$ and $C_{2}$ such that $\ell\left(C_{1}\right), \ell\left(C_{2}\right) \leq 4$ and every cycle in $G$ distinct from $C_{1}$ and $C_{2}$ has length at least 5. If $G$ is $\left(C_{1} \cup C_{2}\right)$-critical, then $G$ is isomorphic to one of the graphs drawn in Figures 1 and 2.

It is straightforward to check that the distance between the boundaries in the described critical graphs is at most three; hence, Theorem 5 implies Theorem 2.

Theorem 5 is proved by the method of reducible configurations: Considering a graph $G$ on the cylinder in that the distance between the boundaries $C_{1}$ and $C_{2}$ is at least 5 , we find a reducible configuration - a subgraph that enables us to transform $G$ to a smaller graph $H$ that is nontrivial $\left(C_{1} \cup C_{2}\right)$ critical if and only if $G$ is nontrivial $\left(C_{1} \cup C_{2}\right)$-critical. If $H$ does not contain cycles of length at most 4 distinct from $C_{1}$ and $C_{2}$, then we argue that $H$ is not one of the graphs enumerated in Theorem 5, thus showing that $G$ is not $\left(C_{1} \cup C_{2}\right)$-critical. Otherwise, we cut $H$ along the cycles of length at most


Figure 1: Critical graphs on the cylinder, bounded by 4-cycles.


Figure 2: Critical graphs on the cylinder, with precolored triangle.


Figure 3: Nontrivial critical graphs with precolored face of length at most 12.

4, use Theorem 5 to describe the resulting pieces, and conclude that every precoloring of $C_{1}$ and $C_{2}$ extends to a coloring of $H$, again implying that $H$ (and thus also $G$ ) is not ( $C_{1} \cup C_{2}$ )-critical.

This leaves us with the case that the distance between $C_{1}$ and $C_{2}$ in $G$ is at most 4. In that case, we color the shortest path between $C_{1}$ and $C_{2}$ and cut the graph along it, obtaining a graph of girth 5 with a precolored face of length at most 16. Such critical graphs with a precolored face of length at most 11 were enumerated by Walls [12] and independently by Thomassen [11], who also gives some necessary conditions for graphs with a precolored face of length 12. The exact enumeration of graphs with a precolored face of length 12 appears in Dvořák and Kawarabayashi [4]. These results can be summarized as follows.

Given a plane graph with the outer face $B$, a chord of $B$ is an edge in $E(G) \backslash E(B)$ incident with two vertices of $B$. A $t$-chord of $B$ is a path $Q=$ $q_{0} q_{1} \ldots q_{t}$ of length $t(t \geq 2)$ such that $q_{0} \neq q_{t}$ and $V(Q) \cap V(B)=\left\{q_{0}, q_{t}\right\}$. Sometimes, we refer to a chord as a 1-chord. A shortcut is a $t$-chord of $B$ such that $t$ is smaller than the distance between $u$ and $v$ in $B$.

Theorem 6 (Dvořák and Kawarabayashi [4). Let $G$ be a plane graph of girth at least 5 and $B$ the outer face of $G$ of length at most 12 . If $B$ is a cycle, $G$ contains no shortcut of length at most two, no two vertices of $G$ of degree two are adjacent and $G$ is nontrivial $B$-critical, then $G$ is isomorphic to one of the graphs in Figure 3 .


Figure 4: Nontrivial critical graphs with precolored face of length at most 10.


Figure 5: Nontrivial critical graphs with precolored faces of length 14 and 16, respectively.

Let us note that all other critical graphs with the precolored face of length at most 12 can be constructed from the graphs in Figure 3 and a 5 -cycle by a sequence of subdividing the edges of the outer face and gluing pairs of graphs along paths of length at most two in their outer faces. For instance, all such nontrivial critical graphs with $\ell(B) \leq 10$ are drawn in Figure 4 .

The number of critical graphs grows exponentially with the length of the precolored face, and enumerating all the graphs becomes increasingly difficult. We implemented an algorithm to generate such graphs based on the results of Dvořák and Kawarabayashi [4], and used the computer to enumerate the graphs with the outer face of length at most 16. There are 108 such graphs with the precolored face of length 13,427 for length 14 , 1746 for length 15 and 7969 for length 16, up to isomorphism (including the case that $G=B$ ). Even excluding the trivial cases that $G$ has a shortcut of length at most 2 or contains two adjacent vertices of degree two as in Theorem 6, there still remain 8 graphs with the precolored face of length 14 (there are none with a precolored face with length 13), 13 with the length 15 and 76 with the length 16 , thus we do not include their list in this paper. Here, let us point out only the following claim, which still makes it possible to enumerate all the graphs easily:

Theorem 7. Let $G$ be a plane graph of girth 5 and $B$ the outer face of $G$ of length at most 16. If $G$ has no shortcut of length at most 4 and $G$ is nontrivial B-critical, then $G$ is isomorphic to the graph in Figure 3(a) or to the graphs in Figure 5.

The complete list of the graphs, as well as programs used to generate
them can be found at http://arxiv.org/abs/1305.2670. A description of the programs can be found in Section 7 .

In Section 4 we give a proof of Theorem 7. Section 5 is devoted to Theorem 5. Finally, Section 6 contains a proof of Theorem 3 .

## 3 Properties of the critical graphs

Let $G$ be a $T$-critical graph, for some $T \subseteq G$. For $S \subseteq G$, a graph $G^{\prime} \subseteq G$ is an $S$-component of $G$ if $S \subseteq G^{\prime}, T \cap G^{\prime} \subseteq S$ and all edges of $G$ incident with vertices of $V\left(G^{\prime}\right) \backslash V(S)$ belong to $G^{\prime}$. When we use $S$-components, $T$ will always be clear from the context. For example, if $G$ is a plane graph with $T$ contained in the boundary of its outer face and $S$ is a cycle in $G$, then the subgraph of $G$ consisting of the vertices and edges drawn in the closed disk bounded by $S$ is an $S$-component of $G$.

Lemma 8. Let $G$ be a $T$-critical graph. If $G^{\prime}$ is an $S$-component of $G$, for some $S \subseteq G$, then $G^{\prime}$ is $S$-critical.

Proof. Since $G$ is $T$-critical, every isolated vertex of $G$ belongs to $T$, and thus every isolated vertex of $G^{\prime}$ belongs to $S$. Suppose for a contradiction that $G^{\prime}$ is not $S$-critical. Then, there exists an edge $e \in E\left(G^{\prime}\right) \backslash E(S)$ such that every coloring of $S$ that extends to $G^{\prime}-e$ also extends to $G^{\prime}$. Note that $e \notin E(T)$. Since $G$ is $T$-critical, there exists a coloring $\psi$ of $T$ that extends to a coloring $\varphi$ of $G-e$, but does not extend to a coloring of $G$. However, by the choice of $e$, the restriction of $\varphi$ to $S$ extends to a coloring $\varphi^{\prime}$ of $G^{\prime}$. Let $\varphi^{\prime \prime}$ be the coloring that matches $\varphi^{\prime}$ on $V\left(G^{\prime}\right)$ and $\varphi$ on $V(G) \backslash V\left(G^{\prime}\right)$. Observe that $\varphi^{\prime \prime}$ is a coloring of $G$ extending $\psi$, which is a contradiction.

Let us remark that Lemma 8 would not hold if we replaced "critical" with "strongly critical", see Figure 6 for an example. This is the main reason why we (unlike some previous works in the area, e.g. Thomassen [11) consider critical rather than strongly critical graphs. However, since every strongly critical graph is also critical, all the characterizations and enumerations that we provide for critical graphs apply to strongly critical graphs as well.

Lemma 8 in conjunction with Theorem 6 describes the subgraphs drawn inside cycles in plane critical graphs. Let us state a few useful special cases of this claim explicitly:

(a)

(b)

Figure 6: (a) A strongly critical graph, with a precolored path on three vertices; (b) not a strongly critical graph with a precolored 5-cycle.

Corollary 9. Let $G$ be a plane graph and $T$ a subgraph of $G$ such that $G$ is $T$-critical. Suppose that every cycle in $G$ that is not contained in $T$ has length at least 5. Let $C$ be a cycle in $G$ and $H$ the subgraph of $G$ drawn in the closed disk bounded by $C$. Suppose that $H \cap T \subseteq C$. If $H \neq C$, then $\ell(C) \geq 8$. If $|V(H) \backslash V(C)| \geq 1$, then $\ell(C) \geq 9$. Finally, if $|V(H) \backslash V(C)| \geq 2$, then $\ell(C) \geq 10$.

## 4 Graphs with one precolored face

In this section we describe an algorithm for enumerating all $B$-critical graphs of girth 5 with outer face $B$. First, we describe a previously know recursive description. Then we show that it can be turned into an algorithm for enumerating $B$-critical graphs. We implemented the resulting algorithm and we provide its source code.

Dvoráak and Kawarabayashi [4] proved the following claim (in a more general setting of list-coloring):

Theorem 10 (Dvořák and Kawarabayashi [4]). Let $G$ be a plane graph of girth at least 5 with the outer face $B$ bounded by a cycle of length at least 10 . If $G$ is $B$-critical, then $|E(G)| \leq 18 \ell(B)-160$ and $|V(G)| \leq \frac{37 \ell(B)-320}{3}$.

The obvious algorithm to enumerate the critical graphs by trying all the graphs of the size given by Theorem 10 is too slow. However, the proof of Theorem 10 identifies a list of configurations such that at least one of them must appear in each plane critical graph of girth at least 5 with the precolored
outer face. For each such configuration, a reduction is provided that makes it possible to obtain $G$ from critical graphs with a shorter precolored outer face. This leads to a practical algorithm to generate such graphs. For the algorithm, it turns out to be simpler to use the following easy corollary of the structural result of Dvořák and Kawarabayashi [4].

Theorem 11 (Dvořák and Kawarabayashi [4]). Let $G$ be a plane graph of girth at least 5 with the outer face $B$ bounded by a cycle. If $G$ is a $B$-critical graph, then $G$ is 2-connected and at least one of the following holds:
(a) G has a shortcut of length at most 4, or
(b) $G$ contains two adjacent vertices of degree two (belonging to $B$ ), or
(c) there exists a path $P=v_{0} v_{1} v_{2} v_{3} v_{4} \subseteq B$ and a 4 -chord $Q=v_{0} w_{1} w_{2} w_{3} v_{4}$ of $B$ such that $v_{2} w_{2} \in E(G)$, or
(d) there exists a 4-chord $Q=w_{0} w_{1} w_{2} w_{3} w_{4}$ of $B$ and 5 -faces $C_{1}$ and $C_{2}$ such that a cycle $C \subseteq B \cup Q$ distinct from $B$ bounds a face of $G$, $\left|V\left(C_{1} \cap B\right)\right|=\left|V\left(C_{2} \cap B\right)\right|=3, C_{1} \cap C=w_{0} w_{1}$ and $C_{2} \cap C=w_{3} w_{4}$.

See Figure 7.
While these configurations are not sufficient to prove Theorem 10, each of the more complicated configurations considered in the proof of Theorem 10 contains one of the configurations of Theorem 11 as a subgraph. For the reduction in case (d), we also need the following result, which is shown for strongly critical graphs in Thomassen [11], and explicitly for critical graphs in Dvořák and Kawarabayashi [4]. For a plane graph $G$ with the outer face $B$, let $m(G)$ be the length of the longest face of $G$ distinct from $B$.

Theorem 12 ([4, 11). Let $G$ be a plane graph of girth at least 5 with the outer face $B$ bounded by a cycle. If $G$ is a nontrivial $B$-critical graph, then $m(G) \leq \ell(B)-3$.

We now define several graph generating operations roughly corresponding to the cases (a)-(d) of Theorem 11. Let $G_{1}$ and $G_{2}$ be plane graphs with outer faces $B_{1}$ and $B_{2}$, respectively.
(a) Let $P_{i}=v_{0}^{i} v_{1}^{i} \ldots v_{t}^{i}$ be paths such that $P_{i} \subseteq B_{i}$ for $i \in\{1,2\}$ and some $t>0$. We let $U\left(G_{1}, P_{1}, G_{2}, P_{2}\right)$ be the graph obtained from the disjoint union of $G_{1}$ and $G_{2}$ by identifying $v_{j}^{1}$ with $v_{j}^{2}$ for $j=0,1, \ldots, t$ and suppressing the arising parallel edges.


Figure 7: Cases of Theorem 11.
(b) For an edge $e \in E\left(G_{1}\right)$, let $S\left(G_{1}, e\right)$ be the graph obtained from $G_{1}$ by subdividing the edge $e$ by one vertex.
(c) For a path $P=v_{0} w_{1} w_{2} w_{3} v_{4} \subseteq B_{1}$, let $J\left(G_{1}, P\right)$ be the graph obtained from $G_{1}$ by adding new vertices $v_{1}, v_{2}$ and $v_{3}$ and edges $v_{0} v_{1}, v_{1} v_{2}$, $v_{2} w_{2}, v_{2} v_{3}$ and $v_{3} v_{4}$.
(d) Let $P=u_{0} u_{1} u_{2} u_{3} u_{4} \subseteq B_{1}$ be a path, let $y_{1}=u_{1}, y_{2}, \ldots, y_{k}=u_{3}$ be the vertices adjacent to $u_{2}$ in the cyclic order according to their drawing around $u_{2}$ and let $f_{i}$ be the edge $u_{2} y_{i}$ for $1 \leq i \leq k$. For $2 \leq i \leq k$ and $0 \leq j \leq 1$, let $X\left(G_{1}, P, f_{i}, j\right)$ be the plane graph obtained from $G_{1}$ by splitting $u_{2}$ to two vertices $u_{2}^{\prime}$ and $u_{2}^{\prime \prime}$ so that $u_{2}^{\prime}$ is adjacent to $y_{1}$, $y_{2}, \ldots, y_{i-1}$ and $u_{2}^{\prime \prime}$ is adjacent to $y_{i}, \ldots, y_{k}$, adding vertices $x_{1}, x_{2}$, $\ldots, x_{4+j}$ and edges $u_{0} x_{1}, x_{1} x_{2}, x_{2} x_{3}, \ldots, x_{4+j} u_{4}, u_{2}^{\prime} x_{2}$ and $u_{2}^{\prime \prime} x_{3+j}$. See Figure 8.

Note that $m\left(X\left(G_{1}, P, f_{i}, j\right)\right) \leq m\left(G_{1}\right)+j+3$ and the length $\ell$ of the outer face of $X\left(G_{1}, P, f_{i}, j\right)$ is equal to $\ell\left(B_{1}\right)+j+1$. By Theorem 12, if $G_{1}$ is a nontrivial $B_{1}$-critical graph, then $m\left(X\left(G_{1}, P, f_{i}, j\right)\right) \leq \ell\left(B_{1}\right)+j<\ell$.


Figure 8: $X\left(G_{1}, P, f_{i}, j\right)$.

For $i \geq 5$, let $\mathcal{K}_{i}$ be the set of all (up to isomorphism) plane graphs $G$ of girth 5 with the outer face $B$ bounded by a cycle, such that $G$ is $B$-critical and $\ell(B)=i$. By Theorem 10 and Theorem6, $\mathcal{K}_{i}$ is finite. For a 2 -connected plane graph $G$ with the outer face $B$, let $\mathcal{K}(G)$ be the set of all graphs $H \supseteq G$ with the outer face $B$ such that for every face $C$ of $G$, the subgraph of $H$ drawn in the closed disk bounded by $C$ belongs to $\mathcal{K}_{\ell(C)}$. In other words, $\mathcal{K}(G)$ consists of graphs obtained from $G$ by pasting a critical graph to each face distinct from the outer one. Let us remark that we do not exclude the case that the pasted graph is trivial, i.e., a face of $G$ may also be a face of some graphs in $\mathcal{K}(G)$. Note that $\mathcal{K}(G)$ is finite, and can be constructed by a straightforward algorithm if the sets $\mathcal{K}_{i}$ are provided for $5 \leq i \leq m(G)$.

For some $\ell$, suppose that $\mathcal{S}$ is a finite set of plane graphs $G$ of girth at least 5 with the outer face $B(G)$ bounded by a cycle of length $\ell$. Let $\mathcal{K}(\mathcal{S})=\bigcup_{G \in \mathcal{S}} \mathcal{K}(G)$. Let $\mathcal{T}(\mathcal{S}) \subseteq \mathcal{S}$ be the set consisting of all $B(G)$-critical graphs $G$ in $\mathcal{S}$. Let $\mathcal{J}^{\prime}(\mathcal{S})=\{J(G, P): G \in \mathcal{S}, P \subseteq B(G), \ell(P)=4\}$. Note that the outer face of each graph $G$ in $\mathcal{J}^{\prime}(\mathcal{S})$ has length $\ell$. Let $\mathcal{S}_{0}, \mathcal{S}_{1}, \mathcal{S}_{2}, \ldots$, be the sequence of sets of graphs such that $\mathcal{S}_{0}=\mathcal{T}(\mathcal{S})$ and $\mathcal{S}_{i+1}=\mathcal{T}\left(\mathcal{J}^{\prime}\left(\mathcal{S}_{i}\right)\right)$ for $i \geq 0$. Let $\mathcal{J}(\mathcal{S})=\bigcup_{i>0} \mathcal{S}_{i}$. Since $\mathcal{J}(\mathcal{S}) \subseteq \mathcal{K}_{\ell}, \mathcal{J}(\mathcal{S})$ is finite, and there exists $k$ such that $\mathcal{S}_{i}=\emptyset$ for each $i \geq k$. Therefore, the set $\mathcal{J}(\mathcal{S})$ can be constructed algorithmically by finding the sets $\mathcal{S}_{0}, \mathcal{S}_{1}, \ldots$, as long as they are non-empty.

Let
$\mathcal{K}_{i}^{(a)}=\left\{\begin{array}{cc} & 1 \leq t \leq 4, i+2 t=i_{1}+i_{2}, 5 \leq i_{1} \leq i_{2} \leq i-1, \\ U\left(G_{1}, P_{1}, G_{2}, P_{2}\right): & G_{1} \in \mathcal{K}_{i_{1}}, G_{2} \in \mathcal{K}_{i_{2}} \\ & P_{j} \text { is a path in the outer face of } G_{j} \\ \text { with } \ell\left(P_{j}\right)=t, \text { for } j \in\{1,2\}\end{array}\right\}$.
Let

$$
\mathcal{K}_{i}^{(b)}=\left\{S(G, e): G \in \mathcal{K}_{i-1}, e \text { is an edge of the outer face of } G\right\} .
$$

Let
$\mathcal{K}_{i}^{(d)}=\left\{\begin{array}{c}0 \leq j \leq 1, G \in \mathcal{K}_{i-j-1}, \\ X(G, P, e, j): \\ P=u_{0} u_{1} u_{2} u_{3} u_{4} \text { is a path in the outer face of } G, \\ e \neq u_{1} u_{2} \text { is incident with } u_{2}\end{array}\right\}$.
Let $\mathcal{K}_{i}^{\prime \prime}$ consist of all graphs in $\mathcal{K}_{i}^{(a)} \cup \mathcal{K}_{i}^{(b)} \cup \mathcal{K}\left(\mathcal{K}_{i}^{(d)}\right)$ that have girth at least 5. Let $\mathcal{K}_{i}^{\prime}=\mathcal{J}\left(\mathcal{K}_{i}^{\prime \prime}\right)$. Note that the set $\mathcal{K}_{i}^{\prime}$ is finite and can be constructed algorithmically, given the sets $\mathcal{K}_{j}$ for $5 \leq j<i$.

Theorem 13. The following holds for $i>5: \mathcal{K}_{i}=\mathcal{K}_{i}^{\prime}$.
Proof. Note that every graph $G \in \mathcal{K}_{i}^{\prime}$ is a plane $B$-critical graph of girth at least 5 , where $B$ is the outer face of $G$ and $\ell(B)=i$ and thus $\mathcal{K}_{i}^{\prime} \subseteq \mathcal{K}_{i}$. Therefore, we only need to show that $\mathcal{K}_{i} \subseteq \mathcal{K}_{i}^{\prime}$. By Theorem 10, there exists a constant $N$ such that $|V(H)| \leq N$ for every $H \in \mathcal{K}_{i}$.

Consider a graph $G \in \mathcal{K}_{i}$ with the outer face $B$. If there exists a path $P=v_{0} v_{1} v_{2} v_{3} v_{4} \subseteq B$ and a 4 -chord $Q=v_{0} w_{1} w_{2} w_{3} v_{4}$ of $B$ such that $v_{2} w_{2} \in$ $E(G)$, then let $B^{\prime}$ be the cycle obtained from $B$ by replacing $P$ by $Q$. By Corollary 9, $v_{0} v_{1} v_{2} w_{2} w_{1}$ and $v_{4} v_{3} v_{2} w_{2} w_{3}$ are 5 -faces, and by Lemma 8, $G^{\prime}=$ $G-\left\{v_{1}, v_{2}, v_{3}\right\}$ is $B^{\prime}$-critical. It follows that $G=J\left(G^{\prime}, Q\right)$. We conclude that there exists a sequence (of length at most $N / 3$ ) of plane graphs $G=G_{0}$, $G_{1}, \ldots, G_{k}$ of girth at least 5 with the outer faces $B=B_{0}, B_{1}, \ldots, B_{k}$, respectively, and paths $P_{1}, \ldots, P_{k}$ such that $G_{j}$ is $B_{j}$-critical for $0 \leq j \leq k$, $P_{j} \subseteq B_{j}$ and $G_{j-1}=J\left(G_{j}, P_{j}\right)$ for $1 \leq j \leq k$, and $G_{k}$ does not contain the configuration (c) of Theorem 11. In other words, as long as $G_{j}$ contains the configuration (c), we keep reducing the graph and when there is no configuration (c), we stop. We claim that $G_{k} \in \mathcal{K}_{i}^{\prime \prime}$, implying that $G \in \mathcal{K}_{i}^{\prime}$.

Since $G_{k}$ is a plane $B_{k}$-critical graph of girth at least 5 , Theorem 11 implies that it contains one of the configurations (a), (b) or (d). If it contains
the configuration (a) (a shortcut $Q$ of length at most 4), then let $C_{1}$ and $C_{2}$ be the cycles in $B_{k} \cup Q$ distinct from $B_{k}$, and let $H_{j}$ be the subgraph of $G_{k}$ drawn in the closed disk bounded by $C_{j}$ for $j \in\{1,2\}$. By Lemma 8, $H_{j}$ is $C_{j}$-critical, which implies that $H_{j} \in \mathcal{K}_{\ell\left(C_{j}\right)}$. Since $Q$ is a shortcut, $\ell\left(\mathbb{C}_{j}\right)<i$. We conclude that $G_{k} \in \mathcal{K}_{i}^{(a)} \subseteq \mathcal{K}_{i}^{\prime \prime}$. Therefore, we may assume that $G_{k}$ has no shortcut of length at most 4.

Suppose that $G_{k}$ contains two adjacent vertices $u$ and $v$ of degree two (the configuration (b)). Since $G_{k}$ is $B_{k}$-critical, both $u$ and $v$ belong to $V\left(B_{k}\right)$. The edge $u v$ is not contained in any cycle of length 5 , since otherwise there would exist a shortcut of length at most two. Let $H$ be the graph obtained from $G_{k}$ by identifying the vertices $u$ and $v$ to a new vertex $w$, with the outer face $B^{\prime}$, and note that $H$ has girth at least 5. Furthermore, $H$ is $B^{\prime}$-critical, since each precoloring of $B$ corresponds to a precoloring of $B^{\prime}$ matching it on $V(B) \backslash\{u, v\}$. Observe that $G=S(H, e)$, where $e$ is an edge incident with $w$, and thus $G_{k} \in \mathcal{K}_{i}^{(b)} \subseteq \mathcal{K}_{i}^{\prime \prime}$. Thus, we may assume that no two vertices of degree two are adjacent in $G_{k}$. In particular, $G_{k}$ is a nontrivial $B_{k}$-critical graph, and there exists a precoloring $\varphi$ of $B_{k}$ that does not extend to a coloring of $G_{k}$.

Finally, consider the case that $G_{k}$ contains the configuration (d). That is, there exists a 4 -chord $Q=w_{0} w_{1} w_{2} w_{3} w_{4}$ of $B_{k}$ and 5 -faces $C_{1}$ and $C_{2}$ such that a cycle $C \subseteq B_{k} \cup Q$ distinct from $B_{k}$ bounds a face of $G,\left|V\left(C_{1} \cap B_{k}\right)\right|=$ $\left|V\left(C_{2} \cap B_{k}\right)\right|=3, C_{1} \cap C=w_{0} w_{1}$ and $C_{1} \cap C=w_{3} w_{4}$. Since $G_{k}$ does not contain adjacent vertices of degree two, we have $\ell(C) \leq 6$. Let $j=\ell(C)-5$. Let $H$ be the graph obtained from $G_{k}$ by removing $w_{0}, w_{4}$ and their neighbors in $V(B)$ and by identifying $w_{1}$ with $w_{3}$ to a new vertex $w$, and let $B^{\prime}$ be the outer face of $H$. Since $w_{2}$ has degree at least thre\& ${ }^{1} w_{1} w_{2} w_{3}$ is not a subpath of the boundary of a face $F \neq C$ in $G_{k}$; hence, Corollary 9 implies that the girth of $H$ is at least 5 . Indeed, if there is a cycle $Z$ in $H$ of length at most 4 , it must contain $w$. We can replace $w$ by $w_{1}, w_{2}, w_{3}$ and obtain a cycle $Z^{\prime}$ of length at most 6 in $G$. Since $w$ has degree at least three, the cycle is not a face which contradicts Corollary 9 .

Observe that the precoloring $\varphi$ of $B_{k}$ (which does not extend to $G_{k}$ ) extends to a coloring $\psi$ of $\left(B_{k} \cup C_{1} \cup C_{2} \cup C\right)-\left\{w_{2}\right\}$ such that $w_{1}$ and $w_{3}$ have the same color. Since $\varphi$ does not extend to a coloring of $G_{k}$, we conclude

[^1]that the precoloring of $B^{\prime}$ given by $\psi$ does not extend to a coloring of $H$. Therefore, $H$ has a nontrivial $B^{\prime}$-critical subgraph $H^{\prime}$. Let $P \subseteq B^{\prime}$ be the path of length 4 such that $w$ is the middle vertex of $P$. Lemma 8 implies that $G_{k} \in \mathcal{K}\left(X\left(H^{\prime}, P, e, j\right)\right)$ for some edge $e \in E\left(H^{\prime}\right)$ incident with $w$. Thus, $G_{k} \in \mathcal{K}\left(\mathcal{K}_{i}^{(d)}\right) \subseteq \mathcal{K}_{i}^{\prime \prime}$.

It follows that $G_{k} \in \mathcal{K}_{i}^{\prime \prime}$, and thus $G \in \mathcal{K}_{i}^{\prime}$. Since the choice of $G$ was arbitrary, this implies that $\mathcal{K}_{i} \subseteq \mathcal{K}_{i}^{\prime}$ and hence $\mathcal{K}_{i}^{\prime}=\mathcal{K}_{i}$.

The sets $\mathcal{K}_{5}, \ldots, \mathcal{K}_{12}$ are given by Theorem 6. Theorem 13 gives an algorithm that we used to construct the sets $\mathcal{K}_{13}, \ldots, \mathcal{K}_{16}$ (we also used the program to generate the sets $\mathcal{K}_{8}, \ldots, \mathcal{K}_{12}$, to give it a better testing). Theorem 7 follows by the inspection of the graphs in $\mathcal{K}_{5} \cup \ldots \cup \mathcal{K}_{16}$ (which was also computer assisted).

## 5 Graphs on the cylinder

Let us now turn our attention to graphs drawn in the cylinder. Our goal is to describe plane graphs that are critical for two precolored $(\leq 4)$-faces, such that all other cycles have length at least 5 . Such graphs can be thought of as embedded in the cylinder so that the two short faces are on the top and bottom of the cylinder.

First, in Lemma 14 we use $\mathcal{K}_{5} \cup \ldots \cup \mathcal{K}_{16}$ to generate critical graphs on cylinder with two precolored $(\leq 4)$-cycles at distance at most 4 from each other and all other cycles of length at least 5 . This part is computer assisted. Next, we glue pairs of these graphs together to obtain critical graphs with one non-precolored separating $(\leq 4)$-cycle, see Lemma 15 . We discuss the outcomes of gluing three such graphs in Lemma 16. Finally, in Lemma 17 we give a general description of the critical graphs created by from those of Lemma 14 by gluing. We complete the description by Lemma 22, which shows that a plane graph with two precolored $(\leq 4)$-faces at distance at least 5 and all other cycles of length at least 5 is never critical.

Lemma 14. Let $G$ be a connected graph embedded on the cylinder with distinct boundaries $C_{1}$ and $C_{2}$ such that $\ell\left(C_{1}\right), \ell\left(C_{2}\right) \leq 4$ and every cycle in $G$ distinct from $C_{1}$ and $C_{2}$ has length at least 5 . If $G$ is $\left(C_{1} \cup C_{2}\right)$-critical and the distance between $C_{1}$ and $C_{2}$ is at most 4 , then $G$ is isomorphic to one of the graphs drawn in Figures 1 or 2.


Figure 9: Splitting of $\mathrm{T}_{7}$ into $B$-critical graph $H$ where $B$ is the outer face of $H$.

Proof. Let us first consider the case that $\ell\left(C_{1}\right)=\ell\left(C_{2}\right)=4$. If $C_{1}$ and $C_{2}$ share an edge, then $C_{1} \cup C_{2}$ contains a 6 -cycle which bounds a face by Corollary 9. Thus $G$ is graph $Z_{1}$ in Figure 1. Therefore, we may assume $C_{1}$ and $C_{2}$ share no edges.

Let $P$ be a shortest path between $C_{1}$ and $C_{2}$. By Lemma $8, G$ is $\left(C_{1} \cup C_{2} \cup\right.$ $P)$-critical. Let $H$ be the graph obtained from $G$ by cutting along the path $P$, splitting the vertices of $P$ into two and duplicating the edges of $P$, and let $B$ be the resulting face. See Figure 9 for the splitting of $T_{7}$. Observe that $H$ is $B$-critical and $B$ is a cycle. Furthermore, $\ell(B)=\ell\left(C_{1}\right)+\ell\left(C_{2}\right)+2 \ell(P) \leq 16$, thus $H$ is one of the graphs in $\mathcal{K}_{5} \cup \ldots \cup \mathcal{K}_{16}$, which we enumerated using a computer in the previous section.

Note that $G$ can be obtained from $H$ by identifying appropriate paths in the face $B$. Using a computer, we checked all possible choices of $H$ (as described in Section (4) and the paths, and checked whether the resulting graph satisfies the assumptions of this lemma. This way, we proved that $G$ must be one of the graphs depicted in Figure 1.

If $\ell\left(C_{1}\right)=3$ or $\ell\left(C_{2}\right)=3$, then we subdivide edges of $C_{1}$ or $C_{2}$, so that the new precolored cycles $C_{1}^{\prime}$ and $C_{2}^{\prime}$ have length exactly 4 . This does not change the distance between the cycles, and the resulting graph $G^{\prime}$ is $\left(C_{1}^{\prime} \cup C_{2}^{\prime}\right)$-critical. Therefore, $G^{\prime}$ is one of the graphs depicted in Figure 1. Inspection of these graphs shows that $G$ is one of the graphs in Figure 2.

Let us remark that there are no graphs satisfying the assumptions of Lemma 14 where the distance between $C_{1}$ and $C_{2}$ is exactly 4 , and only one such graph $R$ where the distance is exactly three. This is the last computer-
based result of this paper (although we used the computer to check the correctness of the case analyses on several other places in the paper, all of them were also performed independently by hand). In particular, if someone proved Lemma 14 without the use of a computer, this would give computerfree proofs of Theorems 2 and 3 .

For the graphs depicted in Figures 1 and 2, let $B$ denote the outer face and $T$ the other face of length at most 4 . For a plane graph $G$ with faces $F_{1}$ and $F_{2}$ and a precoloring $\psi$ of $F_{1}$, let $c\left(G, F_{1}, \psi, F_{2}\right)$ be the number of colorings $\varphi$ of $F_{2}$ such that $\psi \cup \varphi$ does not extend to a coloring of $G$ (in case that $F_{1}$ and $F_{2}$ intersect, this includes the colorings that assign the common vertices colors different from those given by $\psi$; say in $Z_{2}$ if $v$ is a common vertex of $F_{1}$ and $F_{2}$, we count colorings where $\left.\psi(v) \neq \varphi(v)\right)$. Let $c\left(G, F_{1}, F_{2}\right)$ be the maximum of $c\left(G, F_{1}, \psi, F_{2}\right)$ over all precolorings $\psi$ of $F_{1}$. By a straightforward inspection of the listed graphs, we find that the values of $c(G, B, T)$ and $c(G, T, B)$ for the graphs in Figures 1 and 2 are as follows:

| $G$ | $c(G, B, T)$ | $c(G, T, B)$ | $G$ | $c(G, B, T)$ | $c(G, T, B)$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $Z_{1}$ | 15 | 15 | $O_{6}$ | 4 | 11 |
| $Z_{2}$ | 12 | 12 | $O_{7}$ | 2 | 2 |
| $Z_{3}$ | 16 | 16 | $T_{1}$ | 8 | 8 |
| $Z_{4}$ | 5 | 15 | $T_{2}$ | 1 | 2 |
| $Z_{5}$ | 4 | 12 | $T_{3}$ | 4 | 4 |
| $Z_{6}$ | 4 | 4 | $T_{4}$ | 3 | 3 |
| $O_{1}$ | 6 | 6 | $T_{5}$ | 2 | 2 |
| $O_{2}$ | 12 | 11 | $T_{6}$ | 2 | 2 |
| $O_{3}$ | 11 | 11 | $T_{7}$ | 2 | 2 |
| $O_{4}$ | 12 | 12 | $T_{8}$ | 2 | 2 |
| $O_{5}$ | 2 | 6 | $R$ | 4 | 4 |

As an example, let us compute $c\left(T_{1}, B, T\right)$. Let $B=b_{1} b_{2} b_{3} b_{4}$ and $T=$ $t_{1} t_{2} t_{3} t_{4}$ Denote the two not precolored vertices by $z_{1}$ and $z_{3}$, where $z_{i}$ is adjacent to $t_{i}$ and $b_{i}$ for $i \in\{1,3\}$. Observe that a precoloring $\psi$ of $B$ and $\varphi$ of $T$ do not extend to a coloring of $T_{1}$ if and only if $\left\{\psi\left(b_{1}\right), \varphi\left(t_{1}\right)\right\}=$ $\left\{\psi\left(b_{3}\right), \varphi\left(t_{3}\right)\right\}$ and $\psi\left(b_{1}\right) \neq \varphi\left(t_{1}\right)$. Hence we need to consider only two cases for $\psi$ : either $\psi\left(t_{1}\right)=\psi\left(t_{3}\right)$ or $\psi\left(t_{1}\right) \neq \psi\left(t_{3}\right)$. Assume first that $\psi\left(t_{1}\right) \neq \psi\left(t_{3}\right)$. Then $\varphi\left(b_{1}\right)=\psi\left(t_{3}\right)$ and $\varphi\left(b_{3}\right)=\psi\left(t_{1}\right)$. This leaves only one possibility for $\varphi$ of $b_{2}$ and $b_{4}$. Hence there is one coloring $\varphi$ such that $\phi \cup \varphi$ does not extend to $T_{1}$. For the second case assume that $\psi\left(t_{1}\right)=\psi\left(t_{3}\right)$. There are two possibilities for assigning $\varphi\left(b_{1}\right)=\varphi\left(b_{3}\right)$ such that $\psi\left(t_{1}\right) \neq \varphi\left(b_{1}\right)$. Each of
these two possibilities can be extended to a coloring of $T$ if four ways. Hence the total number of precolorings $\varphi$ such that $\psi \cup \varphi$ does not extend is 8 .

Based on these numbers, we characterize critical graphs obtained by pasting two such cylinders together. A cycle $C$ in a plane graph separates subgraphs $G_{1}$ and $G_{2}$ if neither of the closed regions of the plane bounded by $C$ contains both $G_{1}$ and $G_{2}$.

Let $G$ be a plane graph and $C_{1}$ and $C_{2}$ be two cycles in $G$ such that $C_{2}$ is drawn in the closed interior of $C_{1}$. A graph $H$ drawn between $C_{1}$ and $C_{2}$ is a graph obtained from $G$ by removing open exterior of $C_{1}$ and open interior of $C_{2}$. In particular, $C_{i}$ bounds a face in $H$ for $i \in\{1,2\}$.

Lemma 15. Let $G$ be a connected graph embedded on the cylinder with distinct boundaries $C_{1}$ and $C_{2}$ such that $\ell\left(C_{1}\right), \ell\left(C_{2}\right) \leq 4$. Let $C \subseteq G$ be a cycle of length at most 4 separating $C_{1}$ from $C_{2}$. Assume that every cycle in $G$ distinct from $C, C_{1}$ and $C_{2}$ has length at least 5, and that the distance between $C_{1}$ and $C$, as well as the distance between $C$ and $C_{2}$, is at most 4 . If $G$ is $\left(C_{1} \cup C_{2}\right)$-critical, then $G$ is isomorphic to one of the graphs drawn in Figures 10,11 and 12.

Proof. Let $G_{i}$ be the subgraph of $G$ drawn between $C_{i}$ and $C$, for $i \in\{1,2\}$. By Lemma 8 and Lemma 14, $G_{i}$ is equal to one of the graphs drawn in Figures 1 and 2. Let $\varphi$ be a precoloring of $C_{1} \cup C_{2}$ that does not extend to a coloring of $G$. Suppose first that $\ell(C)=3$, i.e., the situation depicted in Figure 10. There exist 6 colorings of $C$ by three colors. Observe that for every coloring $\psi$ of $C$ there exists $i \in\{1,2\}$ such that the precoloring of $C_{i}$ and $C$ given by $\varphi \cup \psi$ does not extend to $G_{i}$, and thus $c\left(G_{1}, C_{1}, C\right)+c\left(G_{2}, C_{2}, C\right) \geq 6$. By symmetry, we may assume that $c\left(G_{1}, C_{1}, C\right) \geq 3$ and hence $G_{1}$ is one of $Z_{4}, Z_{5}, Z_{6}$ and $O_{6}$. If $G_{1} \in\left\{Z_{5}, Z_{6}\right\}$, then $C$ contains two vertices that have degree two in $G_{1}$, and since $G$ is critical, they must either belong to $C_{2}$ or have degree at least three, implying that $G_{2} \in\left\{Z_{4}, O_{6}\right\}$. Hence, $G$ is one of the graphs $D_{1}, D_{2}, D_{3}$ or $D_{4}$. If $G_{1}=Z_{4}$, then we conclude similarly that $G$ is one of $D_{1}, D_{2}, D_{5}, D_{6}, D_{7}$ or $D_{8}$, and if $G_{1}=O_{6}$, then $G$ is one of $D_{3}$, $D_{4}, D_{7}, D_{9}, D_{10}$ or $D_{11}$.

Let us now consider the case that $\ell\left(C_{1}\right)=\ell\left(C_{2}\right)=\ell(C)=4$, as depicted in Figure 11. Since $C$ has 18 colorings, we have $c\left(G_{1}, C_{1}, C\right)+c\left(G_{2}, C_{2}, C\right) \geq$ 18. We may assume that $c\left(G_{1}, C_{1}, C\right) \geq 9$, i.e., $G_{1} \in\left\{Z_{1}, Z_{2}, Z_{3}, O_{2}, O_{3}, O_{4}\right\}$. If $G_{1} \in\left\{Z_{1}, Z_{2}\right\}$ or $G_{1}=O_{2}$ with $C_{1}$ being the outer face of $O_{2}$, then $C$ contains two adjacent vertices whose degree is two in $G_{1}$. These vertices must either belong to $C_{2}$, or their degree must be at least three in


Figure 10: Critical graphs on the cylinder, one separating triangle.


Figure 11: Critical graphs on the cylinder, one separating 4-cycle. Note that $\mathrm{A}_{5}$ and $\mathrm{A}_{5}$ ' is the same graph but different embedding. We use primes to distinguish different embeddings.


Figure 12: Critical graphs on the cylinder, one separating 4-cycle, precolored triangle.
$G_{2}$. We conclude (also taking into account that $c\left(G_{2}, C_{2}, C\right) \geq 3$ ) that $G_{2} \in\left\{Z_{1}, O_{2}, O_{4}, T_{3}, T_{4}\right\}$, and (excluding the combinations that do not result in a critical graph), $G$ is one of the graphs $A_{1}, A_{2}$ or $A_{3}$. From now on, assume that $G_{1}, G_{2} \notin\left\{Z_{1}, Z_{2}\right\}$. If $G_{1}=O_{3}$ or $G_{1}=O_{2}$ with $C$ being the outer face of $O_{2}$, then we similarly conclude that $G_{2} \in\left\{Z_{3}, O_{2}, O_{3}, O_{4}, T_{1}\right\}$ and $G$ is one of the graphs $A_{4}, A_{5}, A_{5}^{\prime}, A_{6}$ or $A_{7}$. We may assume that $G_{1}, G_{2} \notin\left\{O_{2}, O_{3}\right\}$. If $G_{1}=O_{4}$, then $G_{2} \in\left\{Z_{3}, O_{1}, O_{4}, T_{1}\right\}$, and $G$ is $A_{8}, A_{9}$ or $A_{10}$. Finally, if $G_{1}=Z_{3}$, then $G$ is $A_{11}, A_{12}, A_{12}^{\prime}$ or $A_{13}$.

If $\ell(C)=4$ and $\ell\left(C_{1}\right)=3$ or $\ell\left(C_{2}\right)=3$, then $G$ is one of the graphs in Figure 12, obtained from those in Figure 11 by suppressing vertices of degree two.

Again, let us summarize the values of $c(G, B, T)$ and $c(G, T, B)$ for these graphs:

| $G$ | $c(G, B, T)$ | $c(G, T, B)$ | $G$ | $c(G, B, T)$ | $c(G, T, B)$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $D_{1}$ | 12 | 12 | $A_{5}, A_{5}^{\prime}$ | 4 | 6 |
| $D_{2}$ | 4 | 12 | $A_{6}$ | 2 | 1 |
| $D_{3}$ | 8 | 12 | $A_{7}$ | 2 | 3 |
| $D_{4}$ | 4 | 8 | $A_{8}$ | 10 | 10 |
| $D_{5}$ | 15 | 15 | $A_{9}$ | 9 | 8 |
| $D_{6}$ | 6 | 6 | $A_{10}$ | 2 | 2 |
| $D_{7}$ | 11 | 12 | $A_{11}$ | 14 | 14 |
| $D_{8}$ | 2 | 6 | $A_{12}, A_{12}^{\prime}$ | 4 | 4 |
| $D_{9}$ | 2 | 4 | $A_{13}$ | 4 | 4 |
| $D_{10}$ | 6 | 4 | $X_{1}$ | 9 | 3 |
| $D_{11}$ | 6 | 6 | $X_{2}$ | 3 | 3 |
| $A_{1}$ | 9 | 9 | $X_{3}$ | 1 | 1 |
| $A_{2}$ | 1 | 3 | $X_{4}$ | 2 | 1 |
| $A_{3}$ | 2 | 3 | $X_{5}$ | 4 | 2 |
| $A_{4}$ | 4 | 2 | $X_{6}$ | 2 | 1 |

We proceed by listing the graphs with two separating cycles of length at most 4.

Lemma 16. Let $G$ be a connected graph embedded on the cylinder with distinct boundaries $C_{1}$ and $C_{2}$ such that $\ell\left(C_{1}\right), \ell\left(C_{2}\right) \leq 4$. Let $C, C^{\prime} \subseteq G$ be distinct cycles of length at most 4 separating $C_{1}$ from $C_{2}$, such that $C$ separates $C_{1}$ from $C^{\prime}$. Assume that every cycle in $G$ distinct from $C, C^{\prime}, C_{1}$ and


Figure 13: Critical graphs on the cylinder, two separating triangles.


Figure 14: Critical graphs on the cylinder, separating triangle and a 4-cycle.


Figure 15: Critical graphs on the cylinder, two separating 4-cycles.
$C_{2}$ has length at least 5, and that the distances between $C_{1}$ and $C$, between $C$ and $C^{\prime}$, and between $C^{\prime}$ and $C_{2}$ are at most 4. If $G$ is $\left(C_{1} \cup C_{2}\right)$-critical, then $G$ is isomorphic to one of the graphs drawn in Figures 13 , 14 and 15.

Proof. By symmetry between $C_{1}, C$ and $C_{2}, C$, assume that $\ell(C) \leq \ell\left(C^{\prime}\right)$. Also, assume that $\ell\left(C_{1}\right)=\ell\left(C_{2}\right)=4$-the graphs bounded by triangles follow by suppressing the precolored vertices of degree two. Let $G_{i}$ be the subgraph of $G$ drawn between $C_{i}$ and $C$, for $i \in\{1,2\}$. By Lemmas 8, 14 and 15, $G_{1}$ is equal to one of the graphs drawn in Figures 1 and 2 and $G_{2}$ is equal to one of the graphs in Figures 10, 11 and 12 .

Suppose first that $\ell(C)=\ell\left(C^{\prime}\right)=3$. It suffices to consider the graphs $G_{1}$ and $G_{2}$ such that $G_{1}$ is one of the graphs in Figure 2 and $G_{2}$ is one of the graphs in Figure 10, that is, $G_{1} \in\left\{Z_{4}, Z_{5}, O_{5}, O_{6}\right\}$ and $G_{2} \in\left\{D_{2}, D_{4}, D_{8}, D_{9}\right\}$. Furthermore, it suffices to consider the pairs satisfying $c\left(G_{1}, C_{1}, C\right)+c\left(G_{2}, C_{2}, C\right) \geq$ 6. All critical graphs arising from these combinations are depicted in Figure 13. Let us remark that combination $G_{1}=O_{6}$ and $G_{2}=D_{2}$ is the same as $Z_{4} D_{4}$ and combination $G_{1}=O_{6}$ and $G_{2}=D_{8}$ is the same as $Z_{4} D_{9}$.

If $\ell(C)=3$ and $\ell\left(C^{\prime}\right)=4$, then we combine graphs $G_{1}$ from Figure 2 with graphs $G_{2}$ from Figure 12 such that $c\left(G_{1}, C_{1}, C\right)+c\left(G_{2}, C_{2}, C\right) \geq 6$, i.e., $G_{1}=Z_{4}$ and $G_{2} \in\left\{X_{1}, X_{3}, X_{4}, X_{5}, X_{5}^{\prime}, X_{6}\right\}$, or $G_{1} \in\left\{Z_{5}, O_{6}\right\}$ and $G_{2} \in\left\{X_{1}, X_{5}, X_{5}^{\prime}\right\}$. All critical graphs arising from these combinations are depicted in Figure 14 (let us remark that the combination $G_{1}=O_{6}$ and $G_{2}=X_{1}$ is not critical, since the set of precolorings that extend to it is equal to that of $D_{10}$, which is its subgraph).

Finally, if $\ell(C)=\ell\left(C^{\prime}\right)=4$, then we combine graphs $G_{1}$ from Figure 1 with graphs $G_{2}$ from Figure 11 such that $c\left(G_{1}, C_{1}, C\right)+c\left(G_{2}, C_{2}, C\right) \geq 18$ and $C$ does not contain a non-precolored vertex of degree two. Furthermore, if $G_{1}=Z_{1}$, we can exclude from consideration the graphs such that $C$ is a cycle of non-precolored vertices of degree three, as an even cycle of vertices of degree three cannot appear in any critical graph. That is, for $G_{1}=Z_{1}$ we need to consider $G_{2} \in\left\{A_{1}, A_{3}, A_{8}, A_{9}, A_{13}\right\}$ (only $G_{2}=A_{1}$ results in a critical graph). Almost all combinations need to be considered for $G_{1}=Z_{3}$, where $G_{2} \in\left\{A_{8}, A_{9}, A_{11}, A_{13}\right\}$ result in a critical graph. Once these combinations are considered, we may assume that $G_{2} \notin\left\{A_{1}, A_{11}\right\}$ by symmetry, since in these graphs the subgraph drawn between $C^{\prime}$ and $C_{2}$ would be $Z_{1}$ or $Z_{3}$. Finally, we need to consider the combinations $G_{1} \in\left\{O_{2}, O_{3}, O_{4}\right\}$ and $G_{2} \in\left\{A_{5}, A_{5}^{\prime}, A_{8}, A_{9}\right\}$ or $G_{1}=T_{1}$ and $G_{2}=A_{8}$. All the critical graphs obtained by these combinations are in Figure 15.

The numbers of non-extending colorings for these graphs are

| $G$ | $c(G, B, T)$ | $c(G, T, B)$ | $G$ | $c(G, B, T)$ | $c(G, T, B)$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $Z_{4} D_{2}$ | 12 | 12 | $Z_{4} X_{2}$ | 3 | 9 |
| $Z_{4} Z_{4}$ | 12 | 8 | $Z_{6} X_{5}, Z_{6} X_{5}^{\prime}$ | 4 | 2 |
| $Z_{4} D_{8}$ | 6 | 6 | $Z_{1} A_{1}$ | 3 | 3 |
| $Z_{4} Z_{9}$ | 6 | 4 | $Z_{4} A_{1}$ | 3 | 1 |
| $O_{6} D_{4}$ | 8 | 8 | $Z_{4} X_{1} a$ | 1 | 1 |
| $O_{6} D_{9}$ | 4 | 4 | $Z_{3} A_{8} a$ | 4 | 4 |
| $Z_{4} X_{1} b$ | 9 | 9 | $Z_{3} A_{8} b$ | 8 | 8 |
| $Z_{4} X_{3}$ | 1 | 3 | $Z_{3} A_{9} a$ | 4 | 4 |
| $Z_{4} X_{4} a$ | 2 | 3 | $Z_{3} A_{9} b$ | 4 | 4 |
| $Z_{4} X_{4} b$ | 2 | 3 | $Z_{3} A_{11} a$ | 12 | 12 |
| $Z_{4} X_{5}, Z_{4} X_{5}^{\prime}$ | 4 | 6 | $Z_{3} A_{11} b$ | 12 | 12 |
| $Z_{4} X_{6} a$ | 2 | 3 | $Z_{3} A_{13}$ | 4 | 4 |
| $Z_{4} X_{6} b$ | 2 | 3 | $O_{4} A_{8}$ | 2 | 2 |
| $Z_{5} X_{5}, Z_{5} X_{5}^{\prime}$ | 4 | 6 | $O_{4} A_{9}$ | 2 | 2 |

This rather tedious case analysis concludes with the next lemma.
Lemma 17. Let $G$ be a connected graph embedded in the cylinder with distinct boundaries $C_{1}$ and $C_{2}$ such that $\ell\left(C_{1}\right), \ell\left(C_{2}\right) \leq 4$. Let $C_{1}=K_{0}, K_{1}, \ldots$, $K_{k}=C_{2}$ be a sequence of distinct cycles of length at most 4 in $G$ such that $K_{i}$ separates $K_{i-1}$ from $K_{i+1}$ for $1 \leq i \leq k-1$ and the distance between $K_{i}$ and $K_{i+1}$ is at most 4 for $0 \leq i \leq k-1$. Assume that every cycle of length at most 4 in $G$ is equal to $K_{i}$ for some $i \in\{0, \ldots, k\}$. If $G$ is $\left(C_{1} \cup C_{2}\right)$-critical, then one of the following holds:

- $G$ is one of the graphs described by Lemmas 14,15 or 16, or
- $G \in \mathcal{C}$, or
- $G$ is one of the graphs drawn in Figure 16.

Proof. By Lemmas 14, 15 or 16, we may assume that $k \geq 4$. The graphs described by Lemma 15 satisfy that if $\ell(C)=\ell\left(C^{\prime}\right)=3$, then $\ell\left(C_{1}\right)=$ $\ell\left(C_{2}\right)=4$. Therefore, Lemma 8 implies that at least one of $K_{i}, K_{i+1}$ and $K_{i+2}$ has length 4, for $0 \leq i \leq k-2$. For $1 \leq i \leq k-1$, let $P_{i}$ (respectively $N_{i}$ ) be the subgraphs of $G$ drawn between $K_{i}$ and $C_{1}$ (respectively $C_{2}$ ).


Figure 16: Other critical graphs on the cylinder.

Suppose first that $k=4$, and assume that $\ell\left(C_{1}\right)=\ell\left(C_{2}\right)=4$. If $\ell\left(K_{2}\right)=$ $\ell\left(K_{3}\right)=3$, then $P_{2} \in\left\{X_{1}, X_{3}, X_{4}, X_{5}, X_{5}^{\prime}, X_{6}\right\}$ and $N_{2} \in\left\{D_{2}, D_{4}, D_{8}, D_{9}\right\}$. Furthermore, $c\left(P_{2}, C_{1}, K_{2}\right)+c\left(N_{2}, C_{2}, K_{2}\right) \geq 6$, implying that $P_{2} \in\left\{X_{1}, X_{5}, X_{5}^{\prime}\right\}$ and $N_{2} \in\left\{D_{2}, D_{4}\right\}$. The critical graphs arising this way are $X_{5} D_{2}, X_{5}^{\prime} D_{2}$, $X_{5} D_{4}$ and $X_{5}^{\prime} D_{4}$. The case that $\ell\left(K_{1}\right)=\ell\left(K_{2}\right)=3$ is symmetric. If $\ell\left(K_{1}\right)=$ $\ell\left(K_{3}\right)=3$, then $P_{3}=Z_{4} X_{2}$ and $N_{1}=Z_{4} X_{2}$, and thus $N_{3}=Z_{4}$. It follows that $G \in\left\{Z_{4} X_{4} Z_{4} a, Z_{4} X_{4} Z_{4} b\right\}$. If $\ell\left(K_{2}\right)=3$ and $\ell\left(K_{1}\right)=\ell\left(K_{3}\right)=4$, then $P_{2}, N_{2} \in\left\{X_{1}, X_{3}, X_{4}, X_{5}, X_{5}^{\prime}, X_{6}\right\}$, and since $c\left(P_{2}, C_{1}, K_{2}\right)+c\left(N_{2}, C_{2}, K_{2}\right) \geq$ 6 , we conclude that $P_{2}=N_{2}=X_{1}$. However, the graph obtained by combining $X_{1}$ with itself is not critical. If $\ell\left(K_{1}\right)=3$ and $\ell\left(K_{2}\right)=\ell\left(K_{3}\right)=4$, then $N_{1}=Z_{4} A_{1}$ and $P_{1} \in\left\{Z_{4}, Z_{5}, O_{5}, O_{6}\right\}$. Since $c\left(P_{1}, C_{1}, K_{1}\right)+c\left(N_{1}, C_{2}, K_{1}\right) \geq$ 6, it follows that $P_{1}=Z_{4}$ and $G=Z_{4} Z_{4} A_{1}$. The case that $\ell\left(K_{3}\right)=3$ and $\ell\left(K_{1}\right)=\ell\left(K_{2}\right)=4$ is symmetric.

Finally, consider the case that $\ell\left(K_{1}\right)=\ell\left(K_{2}\right)=\ell\left(K_{3}\right)=4$. Then $P_{3}$ is one of the graphs in Figure 15, implying that $P_{1} \in\left\{Z_{1}, Z_{3}, O_{4}\right\}$, and by symmetry, $N_{3} \in\left\{Z_{1}, Z_{3}, O_{4}\right\}$. If $N_{3} \neq Z_{3}$, we have $c\left(P_{3}, C_{1}, K_{3}\right) \geq 6$, and thus $P_{3} \in\left\{Z_{3} A_{8} b, Z_{3} A_{11} a, Z_{3} A_{11} b\right\}$. For all these choices of $P_{3}$, we have $P_{1}=Z_{3}$. Therefore, by symmetry we may assume $N_{3}=Z_{3}$. The combinations of $Z_{3}$ with the graphs in Figure 15 that result in a critical graph are $Z_{3} A_{8} Z_{3}, Z_{3} A_{9} Z_{3}$ and the graphs belonging to $\mathcal{C}$.

The only graph with $k=4$ and $\ell\left(C_{1}\right) \leq 3$ or $\ell\left(C_{2}\right) \leq 3$ is $Z_{4} Z_{4} X_{1}$, obtained by suppressing a vertex of degree two in $Z_{4} Z_{4} A_{1}$. Thus, all the graphs with $k=4$ satisfy the conclusion of this lemma.

Suppose now that $k=5$. The graphs $P_{4}$ and $N_{1}$ are among the graphs described by this lemma for $k=4$. This implies that $P_{1} \in\left\{Z_{1}, Z_{3}, Z_{4}\right\}$. If $\ell\left(K_{1}\right)=3$, then $P_{1}=Z_{4}$ and $N_{1}=Z_{4} Z_{4} X_{1}$ and $G=Z_{4} Z_{4} Z_{1} Z_{4} Z_{4} a$ or $G=Z_{4} Z_{4} Z_{1} Z_{4} Z_{4} b$. The case that $\ell\left(K_{4}\right)=3$ is symmetric. Therefore, assume that $\ell\left(K_{1}\right)=\ell\left(K_{4}\right)=4$. This implies that $N_{1} \notin\left\{Z_{4} X_{2} Z_{4} a, Z_{4} X_{2} Z_{4} b\right\}$. Neither $Z_{1}$ nor $Z_{4}$ can be combined with a graph from $\mathcal{C}$ to form a critical graph, as the resulting graph would contain a non-precolored vertex of degree two. The same argument shows that if $P_{1} \in\left\{Z_{1}, Z_{4}\right\}$, then $N_{1} \notin\left\{X_{5} D_{4}, X_{5}^{\prime} D_{4}, Z_{3} A_{8} Z_{3}, Z_{3} A_{9} Z_{3}\right\}$. The combinations of $Z_{1}$ or $Z_{4}$ with $X_{5} D_{2}, X_{5}^{\prime} D_{2} . Z_{4} Z_{4} A_{1}$ or $Z_{4} Z_{4} X_{1}$ are not critical. We conclude that $P_{1}=Z_{3}$, and by symmetry, $N_{4}=Z_{3}$. Since $G$ does not contain non-precolored vertices of degree two, we have $N_{1} \notin\left\{X_{5} D_{2}, X_{5}^{\prime} D_{2}, Z_{4} Z_{4} A_{1}, Z_{4} Z_{4} X_{1}\right\}$. If $N_{1} \in \mathcal{C}$, then $G \in \mathcal{C}$. Otherwise, $N_{1} \in\left\{X_{5} D_{4}, X_{5}^{\prime} D_{4}, Z_{3} A_{8} Z_{3}, Z_{3} A_{9} Z_{3}\right\}$. However, the combinations of $Z_{3}$ with these graphs are not critical.

Therefore, we may assume that $k \geq 6$. Let $G_{i}$ be the subgraph of $G$ drawn


Figure 17: Critical graphs with a precolored vertex.
between $K_{i}$ and $K_{i+5}$, for $0 \leq i \leq k-5$. By the previous paragraph, $G_{i} \in$ $\left\{Z_{4} Z_{4} Z_{1} Z_{4} Z_{4} a, Z_{4} Z_{4} Z_{1} Z_{4} Z_{4} b\right\} \cup \mathcal{C}$, hence $\ell\left(K_{i}\right)=\ell\left(K_{i+5}\right)=4$. Furthermore, considering $G_{i-1}($ if $i>0)$ or $G_{i+1}$ (if $i<k-5$ ), we conclude that $\ell\left(K_{i+1}\right)=4$ or $\ell\left(K_{i+4}\right)=4$, implying that $G_{i} \in \mathcal{C}$. This implies that $G \in \mathcal{C}$.

Let us remark that if $G$ is a graph in $\mathcal{C}$ with 4 -faces $C_{1}$ and $C_{2}$, then $G$ is $\left(C_{1} \cup C_{2}\right)$-critical - to see this observe that that the precolorings of $C_{1}$ and $C_{2}$ in that the vertices of $C_{i}$ of degree two have different colors for each $i \in\{1,2\}$ do not extend to a coloring of $G$. Let us now point out some consequences of Lemma 17 that are useful in the proof of Theorem 5 .

Lemma 18. Let $G$ be a connected plane graph, $v$ a vertex of $G$ and $C \subseteq G$ either a vertex of $G$, or a cycle bounding a face of length at most 4. Assume that every cycle of length at most 4 distinct from $C$ separates $v$ from $C$. Furthermore, assume that for every two subgraphs $K_{1}, K_{2} \subseteq G$ such that $K_{i} \in\{v, C\}$ or $K_{i}$ is a cycle of length at most 4 for $i \in\{1,2\}$, either the distance between $K_{1}$ and $K_{2}$ is at most 4, or there exists a cycle of length at most 4 separating $K_{1}$ from $K_{2}$. If $G$ is nontrivial $(v \cup C)$-critical, then $G$ is
one of the graphs $J_{1}, J_{2}, \ldots, J_{5}$ drawn in Figure 17 .
Proof. Let $G^{\prime}$ be the graph obtained from $G$ in the following way: Add new vertices $v^{\prime}$ and $v^{\prime \prime}$ and edges of the triangle $C_{1}=v v^{\prime} v^{\prime \prime}$. If $C$ is a single vertex, add also new vertices $c^{\prime}$ and $c^{\prime \prime}$ and edges of the triangle $C_{2}=C c^{\prime} c^{\prime \prime}$, otherwise set $C_{2}=C$. Observe that $G^{\prime}$ is $\left(C_{1} \cup C_{2}\right)$-critical and satisfies assumptions of Lemma 17. The claim follows by the inspection of the graphs enumerated by Lemma 17, using the fact that $C_{1}$ and $C_{2}$ are disjoint, $\ell\left(C_{1}\right)=3$ and $v^{\prime}$ and $v^{\prime \prime}$ have degree two.

The following claims follow by a straightforward inspection of the graphs listed in Lemmas 17 and 18:

Corollary 19. Let $G$ be a connected plane graph and $C_{1}$ and $C_{2}$ distinct subgraphs of $G$ such that $C_{i}$ is either a single vertex or a cycle of length at most 4 bounding a face, for $i \in\{1,2\}$. Assume that $G$ is nontrivial $\left(C_{1} \cup C_{2}\right)$ critical and that every cycle of length at most 4 distinct from $C$ separates $C_{1}$ from $C_{2}$. Furthermore, assume that for every two subgraphs $K_{1}, K_{2} \subseteq G$ such that $K_{i} \in\left\{C_{1}, C_{2}\right\}$ or $K_{i}$ is a cycle of length at most 4 for $i \in\{1,2\}$, either the distance between $K_{1}$ and $K_{2}$ is at most 4, or there exists a cycle of length at most 4 separating $K_{1}$ from $K_{2}$.
(a) If the distance between $C_{1}$ and $C_{2}$ is at least three and $G$ has a face of length at least 7, then
$G \in\left\{D_{9}, D_{10}, A_{12}, A_{12}^{\prime}, Z_{4} D_{8}, Z_{4} D_{9}, O_{6} D_{9}, Z_{5} X_{5}, Z_{5} X_{5}^{\prime}\right\}$.
(b) If the distance between $C_{1}$ and $C_{2}$ is at least three and all cycles of length at most 4 in $G$ distinct from $C_{1}$ and $C_{2}$ intersect in a non-precolored vertex, then
$G \in\left\{R, J_{5}, D_{9}, D_{10}, A_{10}, A_{12}, A_{12}^{\prime}, Z_{4} D_{4}, O_{6} D_{4}, Z_{4} X_{4} b, Z_{3} A_{9} b, O_{4} A_{9}\right\}$.
(c) If the distance between $C_{1}$ and $C_{2}$ is at least three and all cycles of length at most 4 in $G$ distinct from $C_{1}$ and $C_{2}$ intersect in a precolored vertex, then $G \in\left\{R, A_{12}, A_{12}^{\prime}, Z_{4} X_{4} b\right\}$.
(d) If the distance between $C_{1}$ and $C_{2}$ is at least two and $G$ has a face of length at least 9 , then $G \in\left\{D_{6}, D_{10}\right\}$.
(e) If the distance between $C_{1}$ and $C_{2}$ is at least two and $G$ is not 2-edgeconnected, then $G \in\left\{J_{4}, J_{5}, D_{6}, D_{8}, D_{9}, D_{10}, Z_{4} D_{8}, Z_{4} D_{9}, O_{6} D_{9}\right\}$.
(f) If the distance between $C_{1}$ and $C_{2}$ is at least two, $G$ has a face $M^{\prime}$ of length at least 7, and there exists an edge e, a vertex $x$ and a face $M \neq M^{\prime}$ distinct from $C_{1}$ and $C_{2}$ such that

- $M$ and $M^{\prime}$ share the edge $e$, and $x$ is incident with $M$,
- every path of length two between $C_{1}$ and $C_{2}$ contains the edge e, and
- every path of length at most 4 between $C_{1}$ and $C_{2}$ contains e or $x$ or both, and
- every cycle of length at most 4 distinct from $C_{1}$ and $C_{2}$ contains $e$ or $x$ or both,
then $G \in\left\{D_{9}, D_{10}, A_{7}, A_{12}, A_{12}^{\prime}\right\}$.
Aksenov [1] proved that every planar graph with at most three triangles is 3 -colorable. Let us note that the result was recently reproved with a significantly simpler proof [3] and the description of planar graphs with 4 triangles that are not 3 -colorable is known [2]. In the origianl proof, Aksenov showed that for any plane graph $G$, if a face $B$ of length at most 4 is precolored and $G$ contains at most one triangle distinct from $F$, then the precoloring of $F$ extends.

Theorem 20 (Aksenov [1]). Let $G$ be a plane graph with the outer face $B$ of length at most 4 . If $G$ is nontrivial $B$-critical, then $G$ contains at least two triangles distinct from $B$.

The next lemma (following from Theorem 20) enables us to consider only connected graphs in the proof of Theorem 5.

Lemma 21. Let $G$ be a plane graph and $C_{1}$ and $C_{2}$ distinct subgraphs of $G$ such that $C_{i}$ is either a single vertex or a cycle of length at most 4 bounding a face, for $i \in\{1,2\}$. Assume that every cycle in $G$ of length at most 4 separates $C_{1}$ from $C_{2}$. If $G$ is nontrivial $\left(C_{1} \cup C_{2}\right)$-critical, then $G$ is connected.

Proof. We may assume that $C_{1}$ and $C_{2}$ are faces, since otherwise we can add a new cycle of length three to $G$ to replace $C_{1}$ or $C_{2}$ if they were single vertices. If $G$ were not connected, then there would exist a cycle $K$ of length at most 4 and a nontrivial $K$-component $G^{\prime}$ of $G$ such that $G^{\prime}$ contains at most one triangle distinct from $K$. By Lemma 8, $G^{\prime}$ is $K$-critical, contradicting Theorem 20.

The following lemma finishes the proof of Theorem 5:
Lemma 22. Let $G$ be a connected plane graph and $C_{1}$ and $C_{2}$ distinct subgraphs of $G$ such that $C_{i}$ is either a single vertex or a cycle of length at most 4 bounding a face, for $i \in\{1,2\}$. Assume that every cycle in $G$ distinct from $C_{1}$ and $C_{2}$ has length at least 5 . If $G$ is $\left(C_{1} \cup C_{2}\right)$-critical, then the distance between $C_{1}$ and $C_{2}$ is at most 4 .

Proof. Suppose for a contradiction that $G$ is a smallest counterexample to this claim, i.e., the distance between $C_{1}$ and $C_{2}$ is at least 5 , and if $H$ is a graph satisfying the assumptions of Lemma 22 with $|E(H)|<|E(G)|$, then the distance between its precolored cycles is at most 4. For future references, let us note that
the distance between $C_{1}$ and $C_{2}$ in $G$ is at least 5,
and
all cycles distinct from $C_{1}$ and $C_{2}$ in $G$ have length at least 5.
Let us now show some properties of $G$.
$G$ is 2-connected.

Proof. Suppose that $v$ is a cut-vertex in $G$. Since $G$ is $\left(C_{1} \cup C_{2}\right)$-critical, Grötzsch's theorem implies that $v$ separates $C_{1}$ from $C_{2}$. Let $G_{1}$ and $G_{2}$ be induced subgraphs of $G$ such that $G=G_{1} \cup G_{2}, V\left(G_{1}\right) \cap V\left(G_{2}\right)=\{v\}$, $C_{1} \subseteq G_{1}$ and $C_{2} \subseteq G_{2}$. By Lemma 8, $G_{i}$ is $\left(C_{i} \cup v\right)$-critical, for $i \in\{1,2\}$. Furthermore, $\left|E\left(G_{i}\right)\right|<|E(G)|$, thus the distance between $C_{i}$ and $v$ is at most 4. By (1) and symmetry, we may assume that the distance between $C_{1}$ and $v$ is at least three. However, since $G_{1}$ does not contain a cycle of length at most 4 distinct from $C_{1}$, this contradicts Lemma 18 .

No two vertices of degree two in $G$ are adjacent.

Proof. Suppose that vertices $v_{1}$ and $v_{2}$ of degree two in $G$ are adjacent. Since $G$ is critical, both $v_{1}$ and $v_{2}$ are precolored, and by symmetry, we may assume that they belong to $C_{1}$. Since $G$ is 2 -connected, it follows that $\ell\left(C_{1}\right)=4$. Let $C_{1}=v_{1} v_{2} v_{3} v_{4}$. By (2), $v_{3} v_{4}$ and $v_{3} v_{2} v_{1} v_{4}$ are the only paths of length at most three between $v_{3}$ and $v_{4}$. It follows that the graph $G^{\prime}$ obtained from $G$ by identifying $v_{1}$ with $v_{2}$ to a new vertex $v$ does not contain a cycle of length at most 4 distinct from $v v_{3} v_{4}$ and $C_{2}$. Furthermore, observe that $G^{\prime}$ is $\left(v v_{3} v_{4} \cup C_{2}\right)$-critical, $\left|E\left(G^{\prime}\right)\right|<|E(G)|$ and the distance between $v v_{3} v_{4}$ and $C_{2}$ is at least 5 . This contradicts the minimality of $G$.

Let us fix a precoloring $\varphi$ of $C_{1} \cup C_{2}$ that does not extend to a coloring of $G$. By the minimality of $G, \varphi$ extends to every proper subgraph of $G$ that contains $C_{1} \cup C_{2}$.

Let $v_{1} v_{2} v_{3}$ be a path in $C_{1} \cup C_{2}$ such that $v_{2}$ has degree two and is incident with a face of length 5 . Then $\varphi\left(v_{1}\right)=\varphi\left(v_{3}\right)$.

Proof. By symmetry, assume that $v_{1} v_{2} v_{3} \subseteq C_{1}$. Since $v_{2}$ is incident with a 5 -face and no cycle in $G$ distinct from $C_{1}$ and $C_{2}$ has length 4 , we conclude that $\ell\left(C_{1}\right)=4$. Let $C_{1}=v_{1} v_{2} v_{3} v_{4}$. Suppose for a contradiction that $\varphi\left(v_{1}\right) \neq$ $\varphi\left(v_{3}\right)$. Let $v_{1} v_{2} v_{3} x y$ be a 5 -face, and let $G^{\prime}$ be the graph obtained from $G-v_{2}$ by identifying $v_{1}$ and $x$ to a new vertex $z$. Let $C_{1}^{\prime}=v_{3} z v_{4}$. Note that the precoloring of $C_{1}^{\prime} \cup C_{2}$ given by $\varphi$ does not extend to a coloring of $G^{\prime}$, thus $G^{\prime}$ contains a nontrivial $\left(C_{1}^{\prime} \cup C_{2}\right)$-critical subgraph $G^{\prime \prime}$ such that $\varphi$ does not extend to a coloring of $G^{\prime \prime}$. The distance between $C_{1}^{\prime}$ and $C_{2}$ is at least 4. Observe that $G^{\prime \prime}$ contains a cycle $C$ of length at most 4 distinct from $C_{1}^{\prime}$ and $C_{2}$, as otherwise we would obtain a contradiction with Lemmas 14 or 18 or with the minimality of $G$.

Since $C$ does not exist in $G$, we have $z \in V(C)$. Let $K$ be a cycle in $G$ induced by $(V(C) \backslash\{z\}) \cup\left\{v_{1} y x\right\}$. Note that $V(K)$ indeed induces a cycle since it cannot have any chords. Moreover, $K$ does not bound a face, since $y$ has degree at least three. Suppose that the exterior of $K$ contains both $C_{1}$ and $C_{2}$. Corollary 9 applied on $K$ and its nonempty interior contradicts the criticality of $G$. Hence $K$ separates $C_{1}$ from $C_{2}$, and thus $C$ separates $C_{1}^{\prime}$ from $C_{2}$ in $G^{\prime \prime}$. Choose $C$ among the cycles of length at most 4 in $G^{\prime \prime}$ distinct from $C_{1}^{\prime}$ and $C_{2}$ so that the subgraph $G_{2}^{\prime \prime} \subseteq G^{\prime \prime}$ drawn between $C$ and $C_{2}$ is as small as possible. This implies that all cycles in $G_{2}^{\prime \prime}$ distinct from $C$ and
$C_{2}$ have length at least 5. By Lemma 8, $G_{2}^{\prime \prime}$ is $\left(C \cup C_{2}\right)$-critical, and by the minimality of $G$, the distance between $C$ and $C_{2}$ is at most 4 . Lemmas 14 and 18 imply that the distance between $C$ and $C_{2}$ is at most three.

On the other hand, since $z \in V(C)$, by (1) the distance between $C$ and $C_{2}$ is at least three. Therefore, the distance between $C$ and $C_{2}$ is exactly three. By Lemma 14, $\ell(C)=\ell\left(C_{2}\right)=4$ and $G_{2}^{\prime \prime}=R$. The graph $G^{\prime \prime}$ contradicts Lemma 17.

We call a vertex $v$ light if $v$ is not precolored and the degree of $v$ is exactly three.

The graph $G$ does not contain the following configuration: A 5-face $F=$ $v_{1} v_{2} v_{3} v_{4} v_{5}$ such that $v_{1}, v_{3}, v_{4}$ and $v_{5}$ are light, and either $v_{2}$ is light, or both $v_{4}$ and $v_{5}$ have a precolored neighbor.

Proof. If $v_{i}$ is a light vertex of $F$, then let $x_{i}$ be the neighbor of $v_{i}$ that is not incident with $F$, for $1 \leq i \leq 5$. By (2), the vertices $x_{1}, \ldots, x_{5}$ are distinct. By (1), we may assume that all precolored neighbors of the vertices of $F$ belong to $C_{1}$.

If both $x_{4}$ and $x_{1}$ are precolored, then there exists a path $P \subseteq C_{1}$ joining $x_{1}$ and $x_{4}$ and a closed region $\Delta$ of the plane bounded by the cycle $K$ formed by $P$ and $x_{1} v_{1} v_{5} v_{4} x_{4}$ such that $\Delta$ contains neither $C_{1}$ nor $C_{2}$. Since $\ell(K) \leq 7$, Corollary 9 implies that the open interior of $\Delta$ is a face. However, $\Delta \neq$ $F$, which implies that $v_{5}$ has degree two. This is a contradiction, thus at most one of $x_{1}$ and $x_{4}$ is precolored. Similarly, at most one of $x_{3}$ and $x_{5}$ is precolored.

If $v_{2}$ is light, then by the symmetry of $F$ we may assume that either no vertex of $F$ has a precolored neighbor, or that $x_{4}$ is precolored and $x_{3}$ is not. By the previous paragraph, this also implies that $x_{1}$ is not precolored. If $v_{2}$ is not light, then both $x_{4}$ and $x_{5}$ are precolored, and thus neither $x_{1}$ nor $x_{3}$ are precolored.

Let $G^{\prime}$ be the graph obtained from $G$ by removing the light vertices of $F$ and adding the edge $x_{1} x_{3}$. Since $x_{1} \neq x_{3}, G^{\prime}$ has no loops. Suppose that $\varphi$ extends to a coloring $\psi$ of $G^{\prime}$. If $v_{2}$ is light, then each vertex of $F$ has one precolored neighbor, thus it has two available colors. Furthermore, the lists of colors available at $v_{1}$ and $v_{3}$ are not the same, thus $\psi$ extends to a coloring of $F$, giving a coloring of $G$ that extends $\varphi$. Suppose that $v_{2}$ is
not light. If $\psi\left(x_{1}\right)=\psi\left(v_{2}\right)$, then we can color vertices of $F$ in order $v_{3}, v_{4}$, $v_{5}, v_{1}$. Similarly, $\psi$ extends to $F$ if $\psi\left(x_{3}\right)=\psi\left(v_{2}\right)$. Therefore, assume that $\psi\left(x_{1}\right)=1, \psi\left(v_{2}\right)=2$ and $\psi\left(x_{3}\right)=3$. Then, color $v_{1}$ by $3, v_{3}$ by 1 and extend the coloring to $v_{4}$ and $v_{5}$. This is possible, since by (5), $\varphi\left(x_{4}\right)=\varphi\left(x_{5}\right)$. We conclude that $\varphi$ extends to a coloring of $G$, which is a contradiction.

Therefore, $\varphi$ does not extend to a coloring of $G^{\prime}$, and $G^{\prime}$ has a nontrivial $\left(C_{1} \cup C_{2}\right)$-critical subgraph $G^{\prime \prime}$. By the minimality of $G$, we have $x_{1} x_{3} \in$ $E\left(G^{\prime \prime}\right)$. Note that the distance between $C_{1}$ and $C_{2}$ in $G^{\prime \prime}$ is at least three, since neither $x_{1}$ nor $x_{3}$ is precolored. Also, every cycle in $G^{\prime \prime}$ of length at most 4 distinct from $C_{1}$ and $C_{3}$ contains the edge $x_{1} x_{3}$. If $x_{1} x_{3}$ were an edge-cut, then $G^{\prime \prime}$ contains no such cycle, and thus $G^{\prime \prime}$ would contradict Lemmas 14 and 18 or the minimality of $G$. It follows that $x_{1} x_{3}$ is incident with two distinct faces in $G^{\prime \prime}$. For a cycle $M$ in $G^{\prime \prime}$ containing $x_{1} x_{3}$, let $\bar{M}$ be the closed walk in $G$ obtained from $M$ by replacing $x_{1} x_{3}$ by the path $x_{1} v_{1} v_{2} v_{3} x_{3}$. Let $F^{\prime}$ be the face of $G^{\prime \prime}$ incident with $x_{1} x_{3}$ such that the interior of the corresponding region bounded by $\overline{F^{\prime}}$ in $G$ contains $v_{4}$ and $v_{5}$. By Corrolary $9, \ell\left(\overline{F^{\prime}}\right) \geq 10$, and thus $\ell\left(F^{\prime}\right) \geq 7$.

Consider a cycle $C$ of length at most 4 in $G^{\prime \prime}$ distinct from $C_{1}$ and $C_{2}$. If the cycle $\bar{C}$ of length at most 7 does not separate $C_{1}$ from $C_{2}$, then by Corollary 9 it bounds a face. Since $\bar{C} \neq F$, we conclude that $v_{2}$ has degree two. This is a contradiction, since $v_{2}$ is not precolored. We conclude that $C$ separates $C_{1}$ from $C_{2}$. By Lemma 21, $G^{\prime \prime}$ is connected. Furthermore, by the minimality of $G$, the graph $G^{\prime \prime}$ satisfies the assumptions of Corollary 19. Since the distance between $C_{1}$ and $C_{2}$ in $G^{\prime \prime}$ is at least three and $G^{\prime \prime}$ has a face $F^{\prime}$ of length at least 7, Corollary 19(a) implies that $G^{\prime \prime} \in\left\{D_{9}, D_{10}, A_{12}, A_{12}^{\prime}, Z_{4} D_{8}, Z_{4} D_{9}, O_{6} D_{9}, Z_{5} X_{5}, Z_{5} X_{5}^{\prime}\right\}$. Furthermore, since all cycles of length at most 4 distinct from $C_{1}$ and $C_{2}$ in $G^{\prime \prime}$ contain a common edge $x_{1} x_{3}$, we conclude that $G^{\prime \prime} \in\left\{D_{9}, D_{10}, A_{12}, A_{12}^{\prime}\right\}$. Since neither $x_{1}$ nor $x_{3}$ is precolored, the inspection of the possible choices for $G^{\prime \prime}$ shows that $G^{\prime \prime}$ contains a path $Q$ of length at most 3 joining a vertex of $C_{1}$ with a vertex of $C_{2}$, such that $x_{1} x_{3} \notin E(Q)$. However, $Q$ is a subgraph of $G$, contradicting (1).

The graph $G$ does not have any face $F$ of length at least 7 .

Proof. Suppose for a contradiction that $F=v_{1} v_{2} \ldots v_{k}$ is a face of length
$k \geq 7$ in $G$. Since the distance between $C_{1}$ and $C_{2}$ is at least 5 , we may assume that $v_{1}, v_{2}$ and $v_{3}$ are not precolored. Let $G^{\prime}$ be the graph obtained from $G$ by identifying $v_{1}$ with $v_{3}$ to a new vertex $v$. Observe that $\varphi$ does not extend to a coloring of $G^{\prime}$, thus $G^{\prime}$ has a nontrivial ( $C_{1} \cup C_{2}$ )-critical subgraph $G^{\prime \prime}$. The distance between $C_{1}$ and $C_{2}$ in $G^{\prime \prime}$ is at least three. Furthermore, since $v_{2}$ has degree at least three and every cycle $C$ in $G^{\prime \prime}$ of length at most 4 distinct from $C_{1}$ and $C_{2}$ corresponds to a cycle of length at most 6 in $G$ containing the path $v_{1} v_{2} v_{3}$, Corollary 9 implies that each such cycle $C$ separates $C_{1}$ from $C_{2}$ and satisfies $v \in V(C)$. By the minimality of $G$, the graph $G^{\prime \prime}$ satisfies the assumptions of Corollary 19(b). By (1), all paths of length at most 4 in $G^{\prime \prime}$ between $C_{1}$ and $C_{2}$ contain $v$. This implies that $G^{\prime \prime} \in\left\{J_{5}, D_{9}, D_{10}, Z_{4} D_{4}, O_{6} D_{4}\right\}$. However, in these graphs, it is not possible to split $v$ to two vertices $\left(v_{1}\right.$ and $\left.v_{3}\right)$ in such a way that the resulting graph contains neither a path of length at most 4 between $C_{1}$ and $C_{2}$ nor a cycle of length at most 4 distinct from $C_{1}$ and $C_{2}$, which is a contradiction.

All faces of $G$ distinct from $C_{1}$ and $C_{2}$ have length 5.

Proof. Let $F=v_{1} v_{2} \ldots v_{k}$ be a face of $G$ distinct from $C_{1}$ and $C_{2}$. By (7), $k \leq 6$. Suppose for a contradiction that $k=6$. Let us first consider the case that $v_{1}, v_{3}$ and $v_{5}$ are not precolored. Then consider the graph $G^{\prime}$ obtained from $G$ by identifying $v_{1}, v_{3}$ and $v_{5}$ to a single vertex $v$. Note that $\varphi$ does not extend to a coloring of $G^{\prime}$, thus $G^{\prime}$ has a non-trivial $\left(C_{1} \cup C_{2}\right)$-critical subgraph $G^{\prime \prime}$.

Consider now a cycle $C$ of length at most 4 in $G^{\prime \prime}$ distinct from $C_{1}$ and $C_{2}$. Note that $v \in V(C)$, and by symmetry, we may assume that a cycle $K \subseteq G$ can be obtained from $C$ by replacing $v$ by $v_{1} v_{2} v_{3}$. Suppose that $C$ does not separate $C_{1}$ from $C_{2}$. Then $K$ does not separate $C_{1}$ from $C_{2}$, and by Corollary $9, K$ bounds a face distinct from $F$, hence $v_{2}$ has degree two. Since neither $v_{1}$ nor $v_{3}$ is precolored, we conclude that $C_{1}=v_{2}$ or $C_{2}=v_{2}$. But that implies that $C$ separates $C_{1}$ from $C_{2}$ in $G^{\prime \prime}$, which is a contradiction. Therefore, every cycle of length at most 4 distinct from $C_{1}$ and $C_{2}$ separates $C_{1}$ from $C_{2}$ in $G^{\prime \prime}$.

As in the proof of (7), we conclude that $G^{\prime \prime} \in\left\{J_{5}, D_{9}, D_{10}, Z_{4} D_{4}, O_{6} D_{4}\right\}$. Furthermore, since it is possible to split $v$ to three vertices $v_{1}, v_{3}$ and $v_{5}$ so that the resulting graph contains neither a cycle of length at most 4 distinct


Figure 18: A graph with a 6-face.
from $C_{1}$ and $C_{2}$ nor a path between $C_{1}$ and $C_{2}$ of length at most 4 , we have $G^{\prime \prime} \in\left\{J_{5}, D_{9}, D_{10}\right\}$. Furthermore, we may assume that $C_{1}=c_{1} c_{2} c_{3} c_{4}$ has length 4 , there exists a path $c_{1} w_{1} w_{2} c_{3}, v_{1}$ is adjacent to $w_{1}, v_{3}$ is adjacent to $w_{2}$ and $v_{5}$ is adjacent to a vertex of $C_{2}$ in $G$. We choose the labels of $c_{2}$ and $c_{4}$ so that the 8 -cycle $c_{1} w_{1} v_{1} v_{2} v_{3} w_{2} c_{3} c_{4}$ does not separate $C_{1}$ from $C_{2}$. Since $v_{2}$ cannot be a non-precolored vertex of degree two, Corollary 9 implies that $v_{2}$ is adjacent to $c_{4}$, and it is not precolored. Corollary 9 also implies that $c_{1} c_{2} c_{3} w_{2} w_{1}$ is a face. By (1), $v_{4}$ and $v_{6}$ are not precolored. Therefore, we may identify $v_{2}, v_{4}$ and $v_{6}$ instead, and by a symmetric argument, we conclude that $\ell\left(C_{2}\right)=4$ and $G$ is the graph depicted in Figure 18. However, this graph is not $\left(C_{1} \cup C_{2}\right)$-critical.

It follows that at least one of $v_{1}, v_{3}$ and $v_{5}$ is precolored, and by symmetry, at least one of $v_{2}, v_{4}$ and $v_{6}$ is precolored. If $v_{1}$ and $v_{4}$ were precolored and the rest of the vertices of $F$ were internal, then Corollary 9 implies that $v_{1} v_{2} v_{3} v_{4}$ or $v_{1} v_{6} v_{5} v_{4}$ together with a path in $C_{1} \cup C_{2}$ bounds a face, implying that $v_{2}$ or $v_{6}$ have degree two. This is a contradiction, thus by symmetry, we may assume that $v_{1}, v_{2} \in V\left(C_{1}\right)$. Since $G$ does not contain a cycle of length at most 4 distinct from $C_{1}$ and $C_{2}$, at least one of $v_{3}$ and $v_{6}$, say $v_{3}$, is not precolored. Also, Corollary 9 implies that $v_{4}$ and $v_{5}$ are not precolored. Let us consider the graph $G^{\prime}$ obtained by identifying $v_{1}, v_{3}$ and $v_{5}$ to a single vertex $v$ and its $\left(C_{1} \cup C_{2}\right)$-critical subgraph $G^{\prime \prime}$. By Corollary 19(c), we have $G^{\prime \prime} \in\left\{R, A_{12}, A_{12}^{\prime}, Z_{4} X_{4} b\right\}$. However, all these graphs contain a path of length at most 4 joining $C_{1}$ and $C_{2}$ that does not contain $v$, contradicting (1).


Figure 19: Configuration from (9).

No face of $G$ is incident with 4 light vertices.

Proof. Suppose for a contradiction that $F=v_{1} v_{2} v_{3} v_{4} v_{5}$ is a face of $G$ such that $v_{1}, v_{3}, v_{4}$ and $v_{5}$ are light. For $i \in\{1,3,4,5\}$, let $x_{i}$ be the neighbor of $v_{i}$ that is not incident with $F$. By (22), the vertices $x_{i}$ are distinct and $x_{4}$ is not adjacent to $x_{5}$. Also, by (6), we may assume that $x_{4}$ is not precolored and that $v_{2}$ is not light. See Figure 19 .

Let $G^{\prime}$ be the graph obtained from $G$ by removing $v_{1}, v_{3}, v_{4}$ and $v_{5}$, identifying $x_{4}$ with $x_{5}$ to a new vertex $x$, and adding the edge $x_{1} x_{3}$. Consider a coloring $\psi$ of $G^{\prime}$. We show that $\psi$ extends to a coloring of $G$ : Color both $x_{4}$ and $x_{5}$ by $\psi(x)$. If $\psi\left(v_{2}\right)=\psi\left(x_{1}\right)$, then color $v_{3}, v_{4}, v_{5}$ and $v_{1}$ in this order; each vertex has at least one available color. The case that $\psi\left(v_{2}\right)=\psi\left(x_{3}\right)$ is symmetric. Finally, if $\psi\left(x_{1}\right), \psi\left(x_{3}\right)$ and $\psi\left(v_{2}\right)$ are pairwise different, then color $v_{1}$ by $\psi\left(x_{3}\right)$ and $v_{3}$ by $\psi\left(x_{1}\right)$, and extend this coloring to $v_{4}$ and $v_{5}$ (this is possible, since $x_{4}$ and $x_{5}$ are both colored by the same color $\left.\psi(x)\right)$. We conclude that $\varphi$ does not extend to a coloring of $G^{\prime}$, and thus $G^{\prime}$ has a nontrivial $\left(C_{1} \cup C_{2}\right)$-critical subgraph $G^{\prime \prime}$.

Let $C$ be a cycle of length at most 4 in $G^{\prime \prime}$ distinct from $C_{1}$ and $C_{2}$, and let $K$ be the corresponding cycle in $G$, obtained by replacing the edge $x_{1} x_{3}$ by the path $P_{1}=x_{1} v_{1} v_{2} v_{3} x_{3}$ or the vertex $x$ by the path $P_{2}=x_{4} v_{4} v_{5} x_{5}$ (or both). Suppose that $C$ does not separate $C_{1}$ from $C_{2}$. If $\ell(K) \leq 7$, then Corollary 9 implies that $K$ bounds a face, and by (8), $\ell(K)=5$. However, that implies $\ell(C) \leq \ell(K)-3 \leq 2$, which is a contradiction. Therefore,
$\ell(K) \geq 8$, and thus $P_{1}, P_{2} \subseteq K$. By planarity, $K-\left(P_{1} \cup P_{2}\right)$ consists of paths $Q_{1}$ between $x_{1}$ and $x_{5}$ and $Q_{2}$ between $x_{3}$ and $x_{4}$. However, since $\ell(C) \leq 4$, at least one of $Q_{1}$ and $Q_{2}$ has length one, contradicting (2). We conclude that $C$ separates $C_{1}$ from $C_{2}$. By Lemma 21, $G^{\prime \prime}$ is connected.

By (1), if $x_{1} \in V\left(C_{i}\right)$, then $x_{3} \notin V\left(C_{3-i}\right)$ for $i \in\{1,2\}$. Also, if $x_{5} \in$ $V\left(C_{i}\right)$, then $x_{4}$ has no neighbor in $V\left(C_{3-i}\right)$. It follows that the distance between $C_{1}$ and $C_{2}$ in $G^{\prime \prime}$ is at least two. Let us also note that by (8), the distance between $x_{4}$ and $x_{5}$ in $G$ is two, thus if $x_{1} x_{3} \notin E\left(G^{\prime \prime}\right)$, then the distance between $C_{1}$ and $C_{2}$ in $G^{\prime \prime}$ is at least three.

Suppose now that $v_{2}$ has degree at most three in $G$. Since $v_{2}$ is not light and $v_{1}$ and $v_{3}$ are light, this implies that $v_{2}=C_{1}$ or $v_{2}=C_{2}$. We assume the former. In $G^{\prime \prime}, v_{2}$ has degree at most one. By Lemma $18, G^{\prime \prime} \in\left\{J_{4}, J_{5}\right\}$. Let $x_{2}$ be the neighbor of $v_{2}$ in $G^{\prime \prime}$. Note that $x_{2} \notin\left\{x_{1}, x_{3}, x\right\}$ by (2). Let $x_{2}^{\prime}$ and $x_{2}^{\prime \prime}$ be the neighbors of $x_{2}$ distinct from $v_{2}$. Similarly, we conclude that $\left\{x_{2}^{\prime}, x_{2}^{\prime \prime}\right\} \cap\left\{x_{1}, x_{3}\right\}=\emptyset$. Since $x_{2} x_{2}^{\prime} x_{2}^{\prime \prime}$ is a triangle, we have $x \in\left\{x_{2}^{\prime}, x_{2}^{\prime \prime}\right\}$, say $x=x_{2}^{\prime \prime}$. Then a path starting with $v_{2} x_{2} x_{2}^{\prime}$ shows that the distance between $C_{1}$ and $C_{2}$ in $G$ is at most three, which is a contradiction. We conclude that $v_{2}$ has degree at least 4 .

For a face $M$ of $G^{\prime \prime}$, let $G_{M}$ be the subgraph of $G$ drawn in the region of the plane corresponding to $M$, bounded by the closed walk $\bar{M}$ obtained from the boundary walk of $M$ by replacing $x_{1} x_{3}$ by $P_{1}$ or $x$ by $P_{2}$ (or both). Let us note that the open interior of this region is either an open disk, or a union of two open disks (the latter is the case when both $x_{1} x_{3}$ and $v_{2}$ are incident with $M$ ).

Suppose first that $x_{1} x_{3} \notin E\left(G^{\prime \prime}\right)$. Let us recall that in this case, the distance between $C_{1}$ and $C_{2}$ in $G^{\prime \prime}$ is at least three. Since $\varphi$ extends to every proper subgraph of $G$, we have $x \in V\left(G^{\prime \prime}\right)$. Let $M$ be the face of $G^{\prime \prime}$ such that $v_{1} v_{5}, v_{3} v_{4} \in E\left(G_{M}\right)$. If $\ell(\bar{M})=\ell(M)+6$, then $x$ forms a cut in $G^{\prime \prime}$ and $G^{\prime \prime}$ contains no cycle of length at most 4 distinct from $C_{1}$ and $C_{2}$. We conclude that $G^{\prime \prime}=R$ because the distance between $C_{1}$ and $C_{2}$ in $G^{\prime \prime}$ is at least three. However, $R$ is 2 -connected, which is a contradiction. Therefore, $\ell(\bar{M})=\ell(M)+3$. If $v_{2} \in V\left(G^{\prime \prime}\right)$, then $G_{M}$ contains two 2 -chords $v_{5} v_{1} v_{2}$ and $v_{4} v_{3} v_{2}$. If $v_{2} \notin V\left(G^{\prime \prime}\right)$, then $G_{M}$ contains a vertex $v_{2}$ of degree at least 4 not contained in $\bar{M}$. In both cases, Theorem 6 implies that $\ell(\bar{M}) \geq 12$, and thus $\ell(M) \geq 9$. By Corollary $19(\mathrm{~d}), G^{\prime \prime} \in\left\{D_{6}, D_{10}\right\}$. The former is not possible, since the distance between $C_{1}$ and $C_{2}$ in $G^{\prime \prime}$ is at least three, thus $G^{\prime \prime}=D_{10}$. Note that $x$ must lie in the triangle in $D_{10}$. Observe that it is not possible to split such a vertex to two vertices $x_{4}$ and $x_{5}$ so that the distance
between $C_{1}$ and $C_{2}$ is more than three, contradicting (1). This implies that $x_{1} x_{3} \in E\left(G^{\prime \prime}\right)$.

Let $M$ be the face of $G^{\prime \prime}$ incident with $x_{1} x_{3}$ such that $v_{1} v_{5}, v_{3} v_{4} \in E\left(G_{M}\right)$, and $M^{\prime}$ the other face incident with $x_{1} x_{3}$. Suppose first that $M=M^{\prime}$. Then by Corollary 19 (e), $G^{\prime \prime} \in\left\{J_{4}, J_{5}, D_{6}, D_{8}, D_{9}, D_{10}, Z_{4} D_{8}, Z_{4} D_{9}, O_{6} D_{9}\right\}$. Since $x_{1} x_{3}$ is not contained in any cycle, any cycle of length at most 4 in $G^{\prime \prime}$ distinct from $C_{1}$ and $C_{2}$ contains $x$. This implies that $G^{\prime \prime} \notin\left\{Z_{4} D_{8}, Z_{4} D_{9}, O_{6} D_{9}\right\}$. By inspection of the remaining choices for $G^{\prime \prime}$ we conclude that $x_{1}$ or $x_{3}$ is precolored, and belongs to say $C_{2}$. Since the distance between $C_{1}$ and $C_{2}$ in $G$ is at least 5 , neither $x_{4}$ nor $x_{5}$ belongs to $C_{1}$, thus $G^{\prime \prime} \notin\left\{J_{4}, D_{6}, D_{8}\right\}$ and $x$ is not precolored. By symmetry, assume that $x_{3} \in V\left(C_{2}\right)$. Then $x_{1}$ and $x$ both belong to a triangle in $G^{\prime \prime}$. Note that $x_{1}$ and $x_{5}$ are not adjacent, thus $x_{1}$ is adjacent to $x_{4}$. By (1), $x_{4}$ is not adjacent to a vertex in $C_{1}$. The 5 -cycle $x_{1} v_{1} v_{5} v_{4} x_{4}$ in $G$ does not bound a face, thus by Corollary 9 it separates $C_{1}$ from $C_{2}$. Let $y$ be the common neighbor of $x_{1}$ and $x_{5}$ in $G$. Since $G^{\prime \prime} \in\left\{J_{5}, D_{9}, D_{10}\right\}, x_{5}$ and $y$ have neighbors in $C_{1}$. By (8), $x_{4}$ and $x_{5}$ have a unique common neighbor $z$ in $G$; also, the distance between $x_{3}$ and $x_{4}$ is two. By (11), $z \notin V\left(C_{1}\right)$. Note that $z \neq x_{1}$, since otherwise the 4-cycle $x_{1} y x_{5} z$ would contradict (2). Observe that $G$ contains an 8-cycle $K$ consisting of $x_{5} z x_{4} x_{1} y$, edges between $y$ and $C_{1}$ and between $x_{5}$ and $C_{1}$ and a path in $C_{1}$ such that $K$ does not separate $C_{1}$ from $C_{2}$. Since $z$ has degree at least three, Corollary 9 implies that $z$ is adjacent to a vertex of $C_{1}$. However, this implies that the distance of $x_{4}$ is two both from $x_{3} \in V\left(C_{2}\right)$ and from $C_{1}$, contradicting (11).

Therefore, $M \neq M^{\prime}$. Let us note that every path of length two between $C_{1}$ and $C_{2}$ in $G^{\prime \prime}$ contains $x_{1} x_{3}$, and every path of length at most 4 contains $x_{1} x_{3}$ or $x$, and both $x_{1} x_{3}$ and $x$ are incident with $M$. Suppose now that $\ell\left(M^{\prime}\right) \geq 7$. Corollary $19(\mathrm{f})$ implies that $G^{\prime \prime} \in\left\{D_{9}, D_{10}, A_{7}, A_{12}, A_{12}^{\prime}\right\}$. If $G^{\prime \prime} \in\left\{D_{9}, D_{10}\right\}$, then $x$ is adjacent both to $x_{1}$ and $x_{3}$, contradicting (2) or planarity. Otherwise, let $C \subseteq G^{\prime \prime}$ be the cycle of length 4 distinct from $C_{1}$ and $C_{2}$. Note that $x_{1} x_{3} \notin E(C)$, thus $x \in V(C)$. But if $x$ is split to two vertices ( $x_{4}$ and $x_{5}$ ) so that $C$ is not a cycle on the resulting graph, then the resulting graph contains a path of length at most 4 between $C_{1}$ and $C_{2}$ that does not contain $x_{1} x_{3}$, contradicting (11). It follows that $\ell\left(M^{\prime}\right) \leq 6$.

By (2), every path between $x_{1}$ and $v_{2}$ other than $x_{1} v_{1} v_{2}$ and every path between $x_{3}$ and $v_{2}$ other than $x_{3} v_{3} v_{2}$ has length at least three. Since $\ell\left(M^{\prime}\right) \leq 6$, we conclude that $v_{2}$ is not incident with $M^{\prime}$, and thus $v_{2} \notin V\left(G^{\prime \prime}\right)$. Therefore, $G_{M^{\prime}}$ is not a union of two cycles intersecting in $v_{2}$. Since $v_{2}$ has degree at least

4, Theorem 6 implies that $\ell\left(\overline{M^{\prime}}\right) \geq 10$. It follows that $\ell\left(M^{\prime}\right)<\ell\left(\overline{M^{\prime}}\right)-3$, and thus $P_{2} \subseteq \overline{M^{\prime}}$.

Let us again consider a cycle $C$ of length at most 4 in $G^{\prime \prime}$ distinct from $C_{1}$ and $C_{2}$, and let $K$ be the corresponding cycle in $G$, obtained by replacing $x_{1} x_{3}$ by $P_{1}$ or $x$ by $P_{2}$ or both. Since $M$ and $M^{\prime}$ are both incident with both $x$ and $x_{1} x_{3}$, there is a cut in $G^{\prime \prime}$ formed by $x$ and $x_{1} x_{3}$. Thus $P_{1} \cup P_{2} \subseteq K$. However, this contradicts (2) or planarity. It follows that $G^{\prime \prime}$ does not contain a cycle of length at most 4 distinct from $C_{1}$ and $C_{2}$. Since $M$ and $M^{\prime}$ are incident with both $x_{1} x_{3}$ and $x$, the minimality of $G$ and Lemma 14 imply that $G^{\prime \prime}=T_{1}$. But, $T_{1}$ contains two edge-disjoint paths of length two between $C_{1}$ and $C_{2}$, and at most one of them contains $x_{1} x_{3}$. It follows that the distance between $C_{1}$ and $C_{2}$ in $G$ is at most 4 , contradicting (1).

Let us assign the initial charge $c_{0}(v)=\operatorname{deg}(v)-4$ to each vertex and $c_{0}(F)=\ell(F)-4$ to each face of $G$ (including $C_{1}$ and $C_{2}$ ). By Euler's formula, the sum of these charges is -8 . Now, each face of $G$ distinct from $C_{1}$ and $C_{2}$ sends a charge of $1 / 3$ to each incident light vertex. This way we obtain the final charge $c$. Clearly, $c(v) \geq 0$ for each non-precolored vertex $v$, and $c(v)>0$ if $\operatorname{deg}(v)>4$.

The final charge of each face $F$ of $G$ distinct from $C_{1}$ and $C_{2}$ is non-negative. Furthermore, if $F$ is incident with less than three light vertices, then $c(F)>$ 0 .

Proof. By (8), $\ell(F)=5$, and thus $c_{0}(F)=1$. If $F$ is incident with $k$ light vertices, then $c(F)=1-k / 3$. Furthermore, (6) and (9) imply that $k \leq 3$, hence $c(F) \geq 0$, and if $k<3$, then $c(F)>0$.

A face $F$ distinct from $C_{1}$ and $C_{2}$ is $C_{i}$-close (for $i \in\{1,2\}$ ) if $F$ shares an edge with $C_{i}$. By (1) and (8), a $C_{1}$-close face cannot share a vertex with a $C_{2}$-close face.

For $i \in\{1,2\}$, the sum $S_{i}$ of the final charges of $C_{i}$ (if $C_{i}$ bounds a face), the vertices of $C_{i}$ and the $C_{i}$-close faces is at least -4 , and if it is equal to -4, then $V(G) \backslash V\left(C_{1} \cup C_{2}\right)$ contains a vertex of degree at least 5 .

Proof. If $C_{i}$ is equal to a single vertex $v$, then by (3) its degree is at least 2 , thus $c(v)=-2>-4$.

Assume now that $C_{i}$ is a triangle $v_{1} v_{2} v_{3}$. Then $c\left(C_{i}\right)=-1$. For $1 \leq$ $j<k \leq 3$, let $F_{j k}$ be the $C_{i}$-close face that shares the edge $v_{j} v_{k}$ with $C_{i}$. If all vertices of $C_{i}$ have degree at least three, then the final charge of each of them is at least -1 , and by (10), $S_{i} \geq-4$. Furthermore, if $S_{i}=-4$, then all vertices of $C_{i}$ have degree exactly three, and all non-precolored vertices of $C_{i}$-near faces are light. However, this implies that $V\left(F_{12} \cup F_{23} \cup F_{13}\right) \backslash V\left(C_{i}\right)$ induces a cycle $C$ of length 6 consisting of light vertices. Observe that every coloring of $G-V(C)$ extends to a coloring of $G$, contradicting the criticality of $G$.

Let us consider the case that say $v_{1}$ has degree two. By (8), $F_{12}=F_{13}$ is a 5 -face. Replacing the path $v_{2} v_{1} v_{3}$ in $F_{12}$ by $v_{2} v_{3}$ results in a 4 -cycle, contradicting (2).

Finally, assume that $C_{i}=v_{1} v_{2} v_{3} v_{4}$ has length 4 , and thus $c\left(C_{i}\right)=0$. For $1 \leq j \leq 4$, let $F_{j}$ be the $C_{i}$-close face that shares the edge $v_{j} v_{j+1}$ with $C_{i}$ (where $v_{5}=v_{1}$ ). If all vertices of $C_{i}$ have degree at least three, then $c\left(v_{j}\right) \geq-1$ for $1 \leq j \leq 4$, and $S_{i} \geq-4$. Furthermore, $S_{i}=-4$ only if $\operatorname{deg}\left(v_{j}\right)=3$ for $1 \leq j \leq 4$ and all non-precolored vertices of $C_{i}$-close faces are light. However, then $\left(F_{1} \cup F_{2} \cup F_{3} \cup F_{4}\right)-V\left(C_{i}\right)$ is a cycle $C$ of 8 light vertices. Observe that any coloring of $G-V(C)$ extends to a coloring of $G$, contradicting the criticality of $G$.

Therefore, we may assume that $\operatorname{deg}\left(v_{1}\right)=2$. By (4), we have $\operatorname{deg}\left(v_{2}\right), \operatorname{deg}\left(v_{4}\right) \geq$ 3. Suppose now that $\operatorname{deg}\left(v_{3}\right) \geq 3$. If at least one vertex of $C_{i}$ has degree greater than 3 , then $S_{i} \geq c\left(v_{1}\right)+c\left(v_{2}\right)+c\left(v_{3}\right)+c\left(v_{4}\right)+c\left(F_{1}\right)>-4$. Let us consider the case that $\operatorname{deg}\left(v_{2}\right)=\operatorname{deg}\left(v_{3}\right)=\operatorname{deg}\left(v_{4}\right)=3$. By (8), $\left(F_{1} \cup F_{2} \cup F_{3}\right)-V\left(C_{i}\right)$ is a 5 -cycle $w_{1} w_{2} w_{3} w_{4} w_{5}$, where $w_{1}$ is adjacent to $v_{2}$, $w_{3}$ is adjacent to $v_{3}$ and $w_{5}$ is adjacent to $v_{4}$. If $w_{1}$ and $w_{5}$ are light, then (8) implies that $w_{2}$ and $w_{4}$ have a common neighbor $x$ such that $w_{4} w_{5} w_{1} w_{2} x$ is a 5 -face. Since $w_{2}$ has degree at least three, $x \neq w_{3}$, and the 4 -cycle $w_{2} w_{3} w_{4} x$ contradicts (2). Therefore, assume that say $\operatorname{deg}\left(w_{1}\right) \geq 4$. Then $S_{i} \geq-5+c\left(F_{1}\right)+c\left(F_{2}\right) \geq-4$, and $S_{i}=-4$ only if $w_{2}, w_{3}, w_{4}$ and $w_{5}$ are light. If that were the case and all vertices of $V(G) \backslash V\left(C_{1} \cup C_{2}\right)$ had degree at most 4 , then $\operatorname{deg}\left(w_{1}\right)=4$. Let $x$ be the neighbor of $w_{1}$ distinct from $w_{2}, w_{5}$ and $v_{2}$. Let $G^{\prime}=G-V\left(C_{i}\right)-\left\{w_{1}, w_{2}, w_{3}, w_{4}, w_{5}\right\}$, and let $G^{\prime \prime}$ be a $\left(C_{3-i} \cup x\right)$-critical subgraph of $G^{\prime}$ such that every precoloring of $C_{3-i} \cup x$ that extends to $G^{\prime \prime}$ also extends to $G^{\prime}$. Note that the distance between $x$ and $C_{3-i}$ is at least three and $G^{\prime \prime}$ does not contain a cycle of length at most 4 distinct
from $C_{3-i}$, and thus by the minimality of $G$ and Lemma $18, G^{\prime \prime}$ is trivial. It follows that every precoloring of $x$ and $C_{3-i}$ extends to $G^{\prime}$. Let $\varphi^{\prime}$ be a coloring of $G^{\prime}$ that matches $\varphi$ on $C_{3-i}$, such that $\varphi^{\prime}(x)=\varphi\left(v_{2}\right)$. Then $\varphi^{\prime} \cup \varphi$ extends to a coloring of $G$, since every vertex of the 5 -cycle $w_{1} w_{2} w_{3} w_{4} w_{5}$ has two available colors, and the lists of colors available at $w_{3}$ and $w_{5}$ are not the same. This is a contradiction.

Finally, consider the case that $\operatorname{deg}\left(v_{3}\right)=2$. If $\operatorname{deg}\left(v_{2}\right)=3$, then by (8), $\left(F_{1} \cup F_{3}\right)-\left\{v_{1}, v_{2}, v_{3}\right\}$ is a 4 -cycle, contradicting (2). We conclude that $\operatorname{deg}\left(v_{2}\right) \geq 4$, and by symmetry, $\operatorname{deg}\left(v_{4}\right) \geq 4$. It follows that $S_{i} \geq$ $c\left(v_{1}\right)+c\left(v_{2}\right)+c\left(v_{3}\right)+c\left(v_{4}\right)+c\left(F_{1}\right)+c\left(F_{2}\right)>-4$.

By (10) and (11), we have

$$
-8=\sum_{v \in V(G)} c(v)+\sum_{F \in F(G)} c(F) \geq c(w)+S_{1}+S_{2}>-8
$$

where $w$ is the vertex of $V(G) \backslash V\left(C_{1} \cup C_{2}\right)$ of maximum degree. This is a contradiction.

Theorem 5 follows from Lemmas 14 and 22. Together with Lemmas 17 and 21 have the following corollary:

Corollary 23. Let $G$ be a graph embedded in the cylinder with boundaries $C_{1}$ and $C_{2}$ such that $\ell\left(C_{1}\right), \ell\left(C_{2}\right) \leq 4$. If $G$ is nontrivial $\left(C_{1} \cup C_{2}\right)$-critical and every cycle of length at most 4 distinct from $C_{1}$ and $C_{2}$ separates $C_{1}$ from $C_{2}$, then $G \in \mathcal{C}$ or $G$ is one of the graphs drawn in Figures 1, 2, 10, 11, 12, 13, 14, 15 or 16 .

## 6 The main result

Theorem 3 follows easily from Corollary 23 and Theorem 4. Let $G$ be a graph embedded in a surface $\Sigma$, and let $\mathcal{F}=\left\{F_{1}, F_{2}, \ldots, F_{k}\right\}$ be a subset of faces of $G$. We say that a subgraph $H$ of $G$ is $\mathcal{F}$-contractible if $H \notin \mathcal{F}$ and there exists a closed disk $\Delta \subseteq \Sigma$ such that $\Delta$ contains $H$, but $\Delta$ does not contain any face of $\mathcal{F}$. For $F \in \mathcal{F}$, we say that $H$ surrounds $F$ if $H$ is not $\mathcal{F}$-contractible and there exists a closed disk $\Delta \subseteq \Sigma$ such that $\Delta$ contains $H$ and $F$, but no other face of $\mathcal{F}$. We say that a subgraph $H \subseteq G$ is $\mathcal{F}$-good if $F_{1} \cup \ldots \cup F_{k} \subseteq H$ and if $F$ is a face of $H$ that is not equal to a face of $G$, then $F$ has exactly two boundary walks, each of the walks has length 4, and
the subgraph of $G$ drawn in the closed region corresponding to $F$ belongs to $\mathcal{C}$.

Let $K$ be the constant from Theorem 4. Let us note that $K>8$. Theorem 3 follows trivially from Grötzsch's theorem if $g=0$. The following holds for graphs embedded in the cylinder:

Lemma 24. Let $G$ be a plane graph and $F_{1}$ and $F_{2}$ faces of $G$. If $G$ is $\left(F_{1} \cup F_{2}\right)$-critical and every cycle of length at most 4 separates $F_{1}$ from $F_{2}$, then $G$ contains an $\left\{F_{1}, F_{2}\right\}$-good subgraph with at most $\ell\left(F_{1}\right)+\ell\left(F_{2}\right)+4 K+20$ vertices.

Proof. If $G$ does not contain a cycle of length at most 4 distinct from $F_{1}$ and $F_{2}$, then by Theorem 4 we have $|V(G)| \leq \ell\left(F_{1}\right)+\ell\left(F_{2}\right)+2 K$, and we may set $H=G$. Otherwise, let $C_{i}$ be the cycle of length at most 4 in $G$ such that the subgraph $G_{i} \subseteq G$ drawn between $F_{i}$ and $C_{i}$ is as small as possible, for $i \in\{1,2\}$. By Theorem 4, $\left|V\left(G_{i}\right)\right| \leq \ell\left(F_{i}\right)+\ell\left(C_{i}\right)+2 K$. Let $M$ be the subgraph of $G$ drawn between $C_{1}$ and $C_{2}$. If $|V(M)| \leq 20$, then $|V(G)| \leq \ell\left(F_{1}\right)+\ell\left(F_{2}\right)+4 K+20$, and we set $H=G$. Suppose that $|V(M)|>20$. If $C_{1}$ and $C_{2}$ are not vertex-disjoint, then there exists a subset $\Delta$ of the plane, disjoint with $F_{1}$ and $F_{2}$ and homeomorphic to an open disk, such that the boundary of $\Delta$ is formed by a closed walk (of length at most 8) in $C_{1} \cup C_{2}$ and all vertices of $M$ are contained in the closure of $\Delta$. By a variant of Corollary 9, we would conclude that $V(M)=V\left(C_{1} \cup C_{2}\right)$, contrary to the assumption that $|V(M)|>20$. If $C_{1}$ and $C_{2}$ are vertex-disjoint, then Corollary 23 implies that $M \in \mathcal{C}$, and we set $H=G_{1} \cup G_{2}$.

Let $\alpha=21 K+104$ and $\beta=15 K+76$. For other surfaces, we prove the following generalization of Theorem 3 ,

Theorem 25. Let $G$ be a graph embedded in a surface $\Sigma$ of genus $g$ and let $\mathcal{F}=\left\{F_{1}, F_{2}, \ldots, F_{k}\right\}$ be a set of faces of $G$ such that the open region corresponding to $F_{i}$ is homeomorphic to the open disk for $1 \leq i \leq k$. Assume that $g \geq 1$ or $k \geq 3$. If $G$ is $\left(F_{1} \cup F_{2} \ldots \cup F_{k}\right)$-critical and every $\mathcal{F}$-contractible cycle has length at least 5 , then $G$ has an $\mathcal{F}$-good subgraph $H$ with at most $\ell\left(F_{1}\right)+\ldots+\ell\left(F_{k}\right)+\alpha g+\beta(k-2)-4$ vertices.

Proof. Let us prove the claim by the induction. Let us assume that the claim is true for all graphs embedded in surfaces of genus smaller than $g$, or embedded in $\Sigma$ with fewer than $k$ precolored faces. Let $\ell=\ell\left(F_{1}\right)+\ldots+\ell\left(F_{k}\right)$.

Suppose first that $G$ contains a cycle $C \notin \mathcal{F}$ of length at most 4 that does not surround any face in $\mathcal{F}$. Cut $\Sigma$ along $C$ and cap the resulting hole(s) by disk(s); the vertices and edges of $C$ are duplicated, resulting in a graph $G^{\prime}$. Let us now discuss several cases:

- If the curve given by the drawing of $C$ in $\Sigma$ is one-sided, then $G^{\prime}$ is embedded in a surface $\Sigma^{\prime}$ of genus $g-1$. Let $C_{1}$ be the face of $G^{\prime}$ corresponding to $C$; note that $\ell\left(C_{1}\right)=2 \ell(C)$. Observe that $G^{\prime}$ is $\left(\mathcal{F} \cup\left\{C_{1}\right\}\right)$-critical. Thomassen [10] proved that every graph embedded in the projective plane without contractible cycles of length at most 4 is 3 -colorable, and thus if $g=1$, then $k \geq 1$. We conclude that if $g\left(\Sigma^{\prime}\right)=0$, then $\left|\mathcal{F} \cup\left\{C_{1}\right\}\right| \geq 2$.
If $g\left(\Sigma^{\prime}\right)=0$ and $\left|\mathcal{F} \cup\left\{C_{1}\right\}\right|=2$, then by Lemma 24, $G^{\prime}$ has an $(\mathcal{F} \cup$ $\left.\left\{C_{1}\right\}\right)$-good subgraph $H^{\prime}$ with at most $\ell+\ell\left(C_{1}\right)+4 K+20 \leq \ell+\alpha g+$ $\beta(k-2)-4$ vertices.

Otherwise, we may apply the induction hypothesis, hence $G^{\prime}$ has an $\left(\mathcal{F} \cup\left\{C_{1}\right\}\right)$-good subgraph $H^{\prime}$ with at most $\ell+\ell\left(C_{1}\right)+\alpha(g-1)+\beta(k-$ 1) $-4 \leq \ell+\alpha g+\beta(k-2)-4$ vertices.

In both cases, the graph $H \subseteq G$ obtained from $H^{\prime}$ by identifying the corresponding vertices of $C_{1}$ is $\mathcal{F}$-good, and has at most $\ell+\alpha g+\beta(k-$ 2) -4 vertices.

- If $C$ is two-sided, then let $C_{1}$ and $C_{2}$ be the faces of $G^{\prime}$ corresponding to $C$. If $C$ is not separating, then $G^{\prime}$ is embedded in a surface of genus $g-2$. If $g=2$ and $k=0$, then by Lemma 24, $G^{\prime}$ has a $\left(\left\{C_{1}, C_{2}\right\}\right)$-good subgraph $H^{\prime}$ with at most $\ell\left(C_{1}\right)+\ell\left(C_{2}\right)+4 K+20 \leq \ell+\alpha g+\beta(k-$ $2)-4$ vertices. Otherwise, we can apply induction hypothesis to $G^{\prime}$ and conclude that it has a $\left(\mathcal{F} \cup\left\{C_{1}, C_{2}\right\}\right)$-good subgraph $H^{\prime}$ with at most $\ell+\ell\left(C_{1}\right)+\ell\left(C_{2}\right)+\alpha(g-2)+\beta k-4 \leq \ell+\alpha g+\beta(k-2)-4$ vertices. The graph $H \subseteq G$ obtained from $H^{\prime}$ by identifying the corresponding vertices of $C_{1}$ and $C_{2}$ is $\mathcal{F}$-good and has at most $\ell+\alpha g+\beta(k-2)-4$ vertices.
- Finally, if $C$ is two-sided and separating, then $G^{\prime}$ consists of subgraphs $G_{1}$ and $G_{2}$ embedded in surfaces $\Sigma_{1}$ and $\Sigma_{2}$, respectively, such that $g=g\left(\Sigma_{1}\right)+g\left(\Sigma_{2}\right)$. Let $\mathcal{F}_{i}$ be the subset of $\mathcal{F}$ contained in $\Sigma_{i}$ and $k_{i}=\left|\mathcal{F}_{i}\right|$, for $i \in\{1,2\}$. Let $\ell_{i}=\sum_{F \in \mathcal{F}_{i}} \ell(F)$. Since $C$ is not $\mathcal{F}$ contractible and does not surround a face of $\mathcal{F}$, we have either $g\left(\Sigma_{i}\right)<$
$g$, or $\left|\mathcal{F}_{i} \cup\left\{C_{i}\right\}\right|<k$ for $i \in\{1,2\}$, and furthermore, if $g\left(\Sigma_{i}\right)=0$, then $\left|\mathcal{F}_{i} \cup\left\{C_{i}\right\}\right| \geq 3$. By the induction hypothesis, $G_{i}$ has an $\left(\mathcal{F}_{i} \cup\left\{C_{i}\right\}\right)$ good subgraph $H_{i}$ with at most $\ell_{i}+\ell\left(C_{i}\right)+\alpha g\left(\Sigma_{i}\right)+\beta\left(k_{i}-1\right)-4$ vertices. The graph $H \subseteq G$ obtained from $H_{1}$ and $H_{2}$ by identifying the corresponding vertices of $C_{1}$ and $C_{2}$ has at most $\ell+\ell\left(C_{1}\right)+\ell\left(C_{2}\right)-$ $\ell(C)+\alpha g+\beta(k-2)-8 \leq \ell+\alpha g+\beta(k-2)-4$ vertices.

Therefore, we may assume that every cycle of length at most 4 in $G$ surrounds a face in $\mathcal{F}$. Then, there exist cycles $C_{1}, \ldots, C_{k} \subseteq G$ such that

- for $1 \leq i \leq k$, either $C_{i}=F_{i}$, or $\ell\left(C_{i}\right) \leq 4$ and $C_{i}$ surrounds $F_{i}$,
- if $\Delta_{i}$ is the open disk bounded by $C_{i}$ that contains the face $F_{i}$, then $\Delta_{i} \cap \Delta_{j}=\emptyset$ for $1 \leq i<j \leq k$, and
- the graph $G^{\prime}$ obtained from $G$ by removing all vertices and edges contained in $\Delta_{1} \cup \ldots \cup \Delta_{k}$ contains no cycle of length at most 4 distinct from $C_{1}, C_{2}, \ldots, C_{k}$.

Let $G_{i}$ be the subgraph of $G$ drawn in the closure of $\Delta_{i}$, for $1 \leq i \leq k$. Note that $G_{i}$ is $\left(F_{i} \cup C_{i}\right)$-critical, and by Lemma 24, $G_{i}$ has an $\left(F_{i} \cup C_{i}\right)$-good subgraph $H_{i}$ with at most $\ell\left(F_{i}\right)+\ell\left(C_{i}\right)+4 K+20$ vertices. By Theorem 4 , $\left|V\left(G^{\prime}\right)\right| \leq \sum_{i=1}^{k} \ell\left(C_{i}\right)+K(g+k)$. Note that $H=G^{\prime} \cup H_{1} \cup \ldots \cup H_{k}$ is $\mathcal{F}$-good, and it has at most $\ell+(4 K+24) k+K(g+k) \leq \ell+\alpha g+\beta(k-2)-4$ vertices. The previous inequality does not hold for $k=0$ and $g=1$. However, in this case $G$ is a projective planar graph without contractible cycles of length at most 4 and hence $G$ is 3 -colorable by a result of Thomassen [10].

Proof of Theorem [3. Follows from Grötzsch's theorem and Theorem 25, with $f(g)=\alpha g$.

## 7 Programs

Both authors of the paper wrote independent programs implementing the algorithm following from Theorem [13, as well as the programs to verify the claims of Theorem 7 and Lemma 14 . The complete lists of the graphs, as well as programs used to generate them can be found at http://arxiv.org/ abs/1305.2670. For the technical details describing the programs and their usage, see README files in the subdirectories. The subdirectory dvorak also
contains the programs used to verify the claims of Section 5, which were first derived manually without computer.

The most time-consuming part of the graph generation is criticality testing. We applied the straightforward algorithm following from the definition of the critical graph: given a planar graph $G$ with the outer face $B$, for each edge $e$ not incident with $B$ we tested whether there exists a precoloring of $B$ that does not extend to $G$, but extends to $G-e$. We augmented this algorithm with a few simple heuristics to speed it up (e.g., all vertices in $V(G) \backslash V(B)$ must have degree at least three). Generating the set $\mathcal{K}_{16}$ took about 10 minutes on a 2.67 GHz machine. We believe that by parallelization and possibly using a more clever criticality testing algorithm, it would be possible to generate the graphs at least up to $\mathcal{K}_{20}$, if someone would need them.

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[^0]:    *Charles University, Prague, Czech Republic. E-mail: rakdver@iuuk.mff.cuni.cz Supported by the Center of Excellence - Inst. for Theor. Comp. Sci., Prague (project P202/12/G061 of Czech Science Foundation), and by project LH12095 (New combinatorial algorithms - decompositions, parameterization, efficient solutions) of Czech Ministry of Education.
    ${ }^{\dagger}$ Charles University, Prague, Czech Republic and University of Illinois at UrbanaChampaign, Urbana, USA. E-mail: lidicky@illinois.edu. Supported by NSF grant DMS-1266016.

[^1]:    ${ }^{1}$ Note that vertices of degree one or two in $C$-critical graph $G$ must be in $C$. For every vertex $v$ of degree at most two, every coloring of $G-v$ extends to a coloring of $G$, which contradicts $C$-criticality if $v \notin V(C)$.

