# Graphs with $\chi = \Delta$ have big cliques

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#### Abstract

Brooks' Theorem implies that if a graph has  $\Delta \geq 3$  and and  $\chi > \Delta$ , then  $\omega = \Delta + 1$ . Borodin and Kostochka conjectured that if  $\Delta \geq 9$  and  $\chi \geq \Delta$ , then  $\omega \geq \Delta$ . We show that if  $\Delta \geq 13$  and  $\chi \geq \Delta$ , then  $\omega \geq \Delta - 3$ . For a graph G, let  $\mathcal{H}(G)$  denote the subgraph of G induced by vertices of degree  $\Delta$ . We also show that if  $\chi \geq \Delta$ , then  $\omega \geq \Delta$  or  $\omega(\mathcal{H}(G)) \geq \Delta - 5$ .

### 1 Introduction

Our goal in this paper is to prove the following two main results. For a graph G, we write  $\Delta(G)$ ,  $\omega(G)$ , and  $\chi(G)$  to denote (respectively) the maximum degree, clique number, and chromatic number of G. When the context is clear, we simply write  $\Delta$ ,  $\omega$ , and  $\chi$ .

**Theorem 1.** If G is a graph with  $\chi \geq \Delta \geq 13$ , then  $\omega \geq \Delta - 3$ .

**Theorem 2.** Let G be a graph and let  $\mathcal{H}(G)$  denote the subgraph of G induced by vertices of degree  $\Delta$ . If  $\chi \geq \Delta$ , then  $\omega \geq \Delta$  or  $\omega(\mathcal{H}(G)) \geq \Delta - 5$ .

The proofs of Theorems 1 and 2 are both somewhat detailed, so we first prove Theorem 3, which plays a central role in proving our two main theorems. (For a less formal and less notationally dense presentation of these results, see an earlier version of this paper [11].) Brooks' Theorem states that if G is connected and  $\chi > \Delta$ , then G is a complete graph on  $\Delta + 1$  vertices (in particular,  $\omega = \Delta + 1$ ) or G is an odd cycle; so if  $\Delta \geq 3$ , then  $\chi > \Delta$  implies  $\omega = \Delta + 1$ . Thus, the interesting case of Theorems 1 and 2 is when  $\chi = \Delta$ .

**Theorem 3.** If G is a graph with  $\chi \geq \Delta$ , then  $\omega \geq \Delta - 3$  if  $\Delta \equiv 1 \pmod{3}$  and  $\omega \geq \Delta - 4$  otherwise.

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When  $\Delta = 13$ , Theorem 3 implies that either G is 12-colorable or G contains a  $K_{10}$ . This result will serve as the base case for a proof of Theorem 1 by induction on  $\Delta$ . To prove Theorem 2, we will further analyze the proof of Theorem 3, and show that we can continue a certain recoloring process unless  $\mathcal{H}(G)$  contains a big clique.

Borodin and Kostochka [5] conjectured in 1977 that if G is a graph with  $\Delta \geq 9$  and  $\omega \leq \Delta - 1$ , then  $\chi \leq \Delta - 1$ . The hypothesis  $\Delta \geq 9$  is needed, as witnessed by the following example. Form G from five disjoint copies of  $K_3$ , say  $D_1, \ldots, D_5$ , by adding edges between u and v if  $u \in D_i$ ,  $v \in D_j$ , and  $i - j \equiv 1 \mod 5$ . This graph is 8-regular with  $\omega = 6$  and  $\chi \geq \lceil 15/2 \rceil = 8$ , since each color is used on at most 2 of the 15 vertices; by Brooks' Theorem G is 8-colorable, so  $\chi(G) = 8$ . Various other examples with  $\chi = \Delta$  and  $\omega < \Delta$  are known for  $\Delta \leq 8$  (see for example [12]). The Borodin-Kostochka Conjecture has been proved for various families of graphs. Reed [30] used probabilistic arguments to prove it for graphs with  $\Delta \geq 10^{14}$ . The present authors [12] proved it for claw-free graphs (those with no induced  $K_{1,3}$ ).

The contrapositive of the conjecture states that if  $\chi \geq \Delta \geq 9$ , then  $\omega \geq \Delta$ . The first result in this direction was due to Borodin and Kostochka [5], who proved that  $\omega \geq \lfloor \frac{\Delta+1}{2} \rfloor$  when  $\chi \geq \Delta$ . Subsequently, Mozhan [25] improved this to  $\omega \geq \lfloor \frac{2\Delta+1}{3} \rfloor$  when  $\Delta \geq 10$  and Kostochka [20] showed that  $\chi \geq \Delta$  implies that  $\omega \geq \Delta - 28$ . Finally, Mozhan proved that  $\omega \geq \Delta - 3$  when  $\chi \geq \Delta \geq 31$  (this result was in his Ph.D. thesis, which unfortunately is not readily accessible [30]). Theorem 1 strengthens Mozhan's result, by weakening the condition to  $\Delta \geq 13$ . Work in the direction of Theorem 2 began in [16], where Kierstead and Kostochka proved that if  $\chi \geq \Delta \geq 7$  and  $\omega \leq \Delta - 1$ , then  $\omega(\mathcal{H}(G)) \geq 2$ . This was strengthened in [21] to the conclusion  $\omega(\mathcal{H}(G)) \geq \lfloor \frac{\Delta-1}{2} \rfloor$ . We further strengthen the conclusion to  $\omega(\mathcal{H}(G)) \geq \Delta - 5$ . We give more background in the introduction to Section 3.

Most of our notation is standard, as in [32]. We write  $K_t$  and  $E_t$  to denote the complete and edgeless graphs on t vertices, respectively. A subset of vertices S is a clique if S induces a complete graph. We write [n] to denote  $\{1,\ldots,n\}$ . The join of disjoint graphs G and H, denoted  $G \vee H$ , is formed from the disjoint union of G and G and G in a graph G are joined if for every pair of distinct vertices G with G and G in a graph G contains the edge G in G and G in a graph G contains the edge G in G and G in ed not be disjoint.) Subgraphs G and G are joined if G are joined if G and G are joined. If G and G are joined to each other, we may also say that G is complete to G.

For a vertex v and a set S (containing v or not) we write  $d_S(v)$  to denote  $|S \cap N(v)|$ . When vertices x and y are adjacent, we write  $x \leftrightarrow y$ ; otherwise  $x \not\leftrightarrow y$ . If  $\mathcal{Z}$  is a set of graphs, we let  $V(\mathcal{Z}) = \bigcup_{G \in \mathcal{Z}} V(G)$ . A graph G is k-critical if  $\chi(G) = k$  and  $\chi(H) < k$  for every proper induced subgraph H. (When we say simply that a graph G is critical, we mean that is  $\chi(G)$ -critical.) A vertex v in a graph G is critical if  $\chi(G \setminus \{v\}) < \chi(G)$ . Note that in a  $\Delta$ -critical graph, every vertex has degree  $\Delta$  or  $\Delta - 1$ . A vertex v is high if  $d(v) = \Delta$  and low otherwise.

# 2 Mozhan's Partitioned Colorings

In [25], Mozhan used a partition of a graph into groups of color classes to prove bounds on the chromatic number in terms of the degree and clique number. These ideas trace all the way back to the 1966 paper of Lovász [22] where he proves that if G is a graph and  $r_1, \ldots, r_k \in \mathbb{N}$  with  $\sum_{i \in [k]} r_i \geq \Delta(G) + 1 - k$ , then V(G) has a partition  $\{V_1, \ldots, V_k\}$  where  $\Delta(G[V_i]) \leq r_i$  for all  $i \in [k]$ . The proof idea is simple; just take a partition minimizing the number of edges within parts (with an appropriate weighting depending on  $r_i$ ). In [7], Catlin took this idea further by starting with such a minimum partition and then moving vertices around (while preserving minimality) until he achieved a desired property. To get the ability to move vertices around like this, he needed to strengthen the condition on the  $r_i$  to  $\sum_{i \in [k]} r_i \geq \Delta(G) + 2 - k$ .

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Mozhan's idea is very similar to Catlin's, but not equivalent. As we will see below, Mozhan considers partitions of V(G) minimizing the number of edges within parts, just like Lovász and Catlin, but he adds the restriction that each part is the disjoint union of color classes in some fixed  $\chi(G)$ -coloring of G. With this added restriction we get a weaker bound on the degrees within parts, but more information about the coloring. Because of this trade-off Mozhan's method excels when all we care about is coloring the parts, but if we require the parts to have more structure (for example, for them to be degenerate as in Borodin's result [4]), we need to use Catlin's method or some other technique (see [6] for example). In some cases either technique will work; Mozhan's method was used in [28] and [21], but the same results were derived in [29] using Catlin's method. The results in this paper require the use of Mozhan's more restrictive partitions, which we define now.

In our proofs of Theorems 1, 2, and 3, we assume that G is critical, so we only need the partition in the following definition when G is critical. However, we include non-critical graphs as well because the more general concept is needed to extract an efficient algorithm from our proofs. We discuss algorithmic considerations in the final section of the paper. Since the proof of Theorem 3 is long, we provide a proof sketch as soon as we have the necessary definitions. This immediately follows Definition 5.

**Definition 1.** For  $s \in \mathbb{N}_{\geq 2}$  and  $r_1, \ldots, r_s \in \mathbb{N}_{\geq 3}$ , an  $(r_1, \ldots, r_s)$ -partition P of a graph G is a partition  $(P_1, \ldots, P_s)$  of V(G), together with an integer  $j \in [s]$  and vertex  $v \in P_j$ , such that

- (1)  $\chi(G[P_i]) = r_i$  for all  $i \in [s] \setminus \{j\}$ ; and
- (2)  $\chi(G[P_j] \setminus \{v\}) \le r_j$ .

We refer to j and v by j(P) and v(P) respectively.

For example, if G is critical and  $\Delta(G) = 13$ , then we get a (3, 3, 3, 3)-partition of G by removing any  $v \in V(G)$ , partitioning the color classes of a 12-coloring of G - v into four equal parts and then adding v to one part, called part j.

We are interested in  $(r_1, \ldots, r_s)$ -partitions that minimize the total number of edges within parts (without v(P)). More precisely, for an  $(r_1, \ldots, r_s)$ -partition P of a graph G, let  $\sigma(P) = \|G[P_{j(P)}] \setminus \{v(P)\}\| + \sum_{i \in [s] \setminus \{j(P)\}} \|G[P_i]\|$ ; here  $\|H\|$  denotes the number of edges in subgraph H. A minimum  $(r_1, \ldots, r_s)$ -partition of G is an  $(r_1, \ldots, r_s)$ -partition P minimizing  $\sigma(P)$  and, subject to that, minimizing  $d_{j(P)}(v(P)) - r_{j(P)}$ .

**Lemma 4.** If P is a minimum  $(r_1, ..., r_s)$ -partition of a graph G with  $\chi(G) = \Delta(G) = 1 + \sum_{i \in [s]} r_i$ , then

- (1)  $G[P_{j(P)}]$  has a component  $\mathcal{A}(P)$ , called the active component, that is  $K_{r_{j(P)}+1}$  and  $\chi(G[P_{j(P)}] \setminus V(\mathcal{A}(P))) \leq r_{j(P)}$ ; and
- (2) for each  $u \in V(\mathcal{A}(P))$  and  $i \in [s] \setminus \{j(P)\}$  with  $d_{P_i}(u) = r_i$ , the graph  $G[P_i \cup \{u\}]$  has a  $K_{r_{i+1}}$  component (which contains u); and
- (3) for each  $u \in V(\mathcal{A}(P))$  and  $i \in [s] \setminus \{j(P)\}$ , if u has at least  $d_{P_i}(u) + 1 r_i$  neighbors in the same component D of  $G[P_i]$ , then  $\chi(G[V(D) \cup \{u\}]) = r_i + 1$ ; and
- (4) if  $u \in V(G)$  and  $a \in [s]$  so that  $d_{P_a}(u) > r_a + 1$ , then there is  $i \in [s]$  where  $d_{P_i}(u) < r_i$ . In particular, any  $r_i$ -coloring of  $G[P_i]$  can be extended to an  $r_i$  coloring of  $G[P_i \cup \{u\}]$ ; and
- (5) for each  $u \in V(A(P))$  and  $i \in [s] \setminus \{j(P)\}$ , we have  $d_{P_i}(u) \leq r_i + 1$ .

Proof. Let P be a minimum  $(r_1, \ldots, r_s)$ -partition of a graph G with  $\chi(G) = \Delta(G) = 1 + \sum_{i \in [s]} r_i$ . Let j = j(P) and v = v(P). Let  $\mathcal{A}(P)$  be the component of  $G[P_j]$  containing v. Fix a  $(\Delta(G) - 1)$ -coloring of G - v consistent with the partition. Since  $\sum_{i \in [k]} r_i = \Delta(G) - 1$ , and since  $d_{j(P)}(v) - r_j$  is minimized in the choice of the partition, we must have  $d_j(v) \leq r_j$ . Equality must hold, since otherwise we could extend the  $(\Delta(G) - 1)$ -coloring of G - v to v.

By construction,  $G[P_j \setminus \{v\}]$  has an  $r_j$ -coloring. So we may assume that  $\chi(\mathcal{A}(P)) = r_j + 1$ , since otherwise we get an  $r_j$ -coloring of  $G[P_j]$ , and hence a  $(\Delta - 1)$ -coloring of G.

To prove (1), it suffices to show that  $\mathcal{A}(P)$  is  $K_{r_j+1}$ . By Brooks' Theorem, it is enough to show that  $\Delta(\mathcal{A}(P)) \leq r_j$ . Suppose instead that there exists  $u \in V(\mathcal{A}(P))$  with  $d_{\mathcal{A}(P)}(u) > r_j$ ; choose u to minimize the distance in  $\mathcal{A}(P)$  from u to v. Uncolor the vertices on a shortest path Q in  $\mathcal{A}(P)$  from u to v; move u to some  $P_k$  where it has at most  $r_k$  neighbors. Color the vertices of Q, starting at v and working along Q; this is possible since each vertex of Q has at most  $r_j - 1$  colored neighbors in  $\mathcal{A}(P)$  when we color it. The resulting new partition R (with v(R) = u) has fewer edges within color classes, since we lost at least  $r_j + 1$  edges incident to u and gained at most  $r_j$  incident to v. This contradiction implies that  $\Delta(\mathcal{A}(P)) \leq r_j$ , so  $\mathcal{A}(P)$  must be  $K_{r_j+1}$  by Brooks' Theorem. Thus (1) holds.

Now we prove (2). Choose such a vertex  $u \in V(\mathcal{A}(P))$  and such an  $i \in [s] \setminus \{j\}$ . Form a new partition R by deleting u from  $P_j$  and adding it to  $P_i$  (now u = v(R)); this maintains the total number of edges within parts, so R is another minimum  $(r_1, \ldots, r_s)$ -partition. By the above proof of (1), u lies in a component of  $G[P_i]$  that is  $K_{r_j+1}$ . Thus, (2) holds.

- If (3) is false, then u has at most  $r_i 1$  neighbors in  $G[P_i] \setminus D$ , so we may choose an  $r_i$ -coloring of  $G[P_i] \setminus D$  so that the neighbors of u in  $P_i \setminus V(D)$  each get a color in  $[r_i 1]$ . Together with an  $r_i$ -coloring of  $G[V(D) \cup \{u\}]$  where u is colored  $r_i$ , this gives an  $r_i$ -coloring of  $G[V(P_i) \cup \{u\}]$ . But then we have a  $(\chi(G) 1)$ -coloring of G, a contradiction.
  - (4) is immediate, since  $d_G(u) \leq 1 + \sum_{i \in [s]} r_i$

If (5) is false, then apply (4) and move u to  $P_i$  to get a  $(\chi(G)-1)$ -coloring of G, a contradiction.

**Definition 2.** A move is a quadruple (P, u, i, P') where

- (1) P is an  $(r_1, \ldots, r_s)$ -partition of a graph G; and
- (2)  $u \in V(\mathcal{A}(P))$ ; and
- (3)  $i \in [s] \setminus \{j(P)\}$  with  $d_{P_i}(u) = r_i$ ; and
- (4) P' is obtained from P by moving u from  $P_{j(P)}$  to  $P_i$ .

In P', vertex v(P) is in the part containing  $V(\mathcal{A}(P) \setminus \{u\})$ . Also j(P') = i and v(P') = u.

In the proof of part (2) of Lemma 4, we showed that if P is a minimum  $(r_1, \ldots, r_s)$ -partition and (P, v, i, P') is a move, then P' is a minimum  $(r_1, \ldots, r_s)$ -partition as well. Moreover, for each  $k \in [s]$ , the number of components in  $G[P_k]$  equals the number of components in  $G[P'_k]$ .

**Definition 3.** Let P be an  $(r_1, \ldots, r_s)$ -partition of a graph G. A move sequence starting at P is a sequence of moves  $((P^1, v_1, i_1, P^2), \ldots, (P^q, v_q, i_q, P^{q+1}))$  where  $P^1 = P$ .

**Definition 4.** Let P be an  $(r_1, \ldots, r_s)$ -partition of a graph G and

$$S = ((P^1, v_1, i_1, P^2), \dots, (P^q, v_q, i_q, P^{q+1}))$$

a move sequence starting at P. For each  $i \in [s]$  and component X of  $G[P_i]$ , let the club of X, written  $\mathcal{C}_{\mathcal{S}}(X)$ , be the sequence  $(X_1, X_2, X_3, \dots, X_{q+1})$  where  $X_1 = X$  and for  $t \in [q] \setminus \{1\}$ 

- $X_t = X_{t-1} \setminus \{v_{t-1}\}$  if  $X_{t-1}$  is the active component in  $P^{t-1}$ ; otherwise
- $X_t = X_{t-1} \cup \{v_{t-1}\}$  if  $G[V(X_{t-1}) \cup \{v_{t-1}\}]$  is the active component in  $P^t$ ; otherwise
- $X_t = X_{t-1}$ .

We need to extend the domain of  $\mathcal{C}_{\mathcal{S}}$  to all components at all times in a given sequence. To do this consistently, we will let  $\mathcal{C}_{\mathcal{S}}(Y)$  be the club that Y appears in most recently. Now we give a precise definition of this extension. Let P be an  $(r_1, \ldots, r_s)$ -partition of a graph G and

$$S = ((P^1, v_1, i_1, P^2), \dots, (P^q, v_q, i_q, P^{q+1}))$$

a move sequence starting at P. For each  $t \in [q+1]$  and Y a component of  $G[P_i^t]$  for some  $i \in [s]$ , we define  $\mathcal{C}_{\mathcal{S}}^t(Y)$  to be  $\mathcal{C}_{\mathcal{S}}(X) = (X_1, X_2, X_3, \dots, X_{q+1})$  where X is the component of  $G[P_i^1]$  such that  $V(Y) = V(X_t)$ . Often, the time t will be clear from context, so we can write simply  $\mathcal{C}_{\mathcal{S}}(X)$ .

When the move sequence is clear from context, we write  $\mathcal{C}(X)$  in place of  $\mathcal{C}_{\mathcal{S}}(X)$ . We say R is a club of  $\mathcal{S}$  if  $R = \mathcal{C}_{\mathcal{S}}(X)$  for a component X of  $G[P_i]$  for some  $i \in [s]$ . For a club R, we write  $R_t$  for the t-th element of R.

We observe a few basic facts about clubs; we omit formal proofs by induction, which are easy exercises.

**Observation 1.** Let P be a minimum  $(r_1, \ldots, r_s)$ -partition of a graph G with  $\chi(G) = \Delta(G) = 1 + \sum_{i \in [s]} r_i$ . If

$$S = ((P^1, v_1, i_1, P^2), \dots, (P^q, v_q, i_q, P^{q+1}))$$

is a move sequence starting at P, then for a club R of S, we have

- (1) if  $V(R_1) \subseteq P_i^1$ , then  $V(R_t) \subseteq P_i^t$  for all  $t \in [q+1]$ . We call this i the part of R, written  $\rho_{\mathcal{S}}(R)$  (or  $\rho(R)$  when context allows).
- (2) if  $a, b \in [q+1]$ , then  $R_a$  is complete if and only if  $R_b$  is complete.
- (3) if  $R_t$  is complete and  $|R_t| \ge r_{\rho(R)}$  for all  $t \in [q+1]$ , then  $|R_a| = r_{\rho(R)} + 1$  when  $R_a$  is active and otherwise  $|R_a| = r_{\rho(R)}$ .

The notion introduced in (3) of the previous observation is important, so, in the following definition, we name it.

**Definition 5.** Let P be a minimum  $(r_1, \ldots, r_s)$ -partition of a graph G with  $\chi(G) = \Delta(G) = 1 + \sum_{i \in [s]} r_i$ . Let

$$S = ((P^1, v_1, i_1, P^2), \dots, (P^q, v_q, i_q, P^{q+1}))$$

be a move sequence starting at P. A club R of S is full if  $R_t$  is complete and  $|R_t| \ge r_{\rho(R)}$  for all  $t \in [q+1]$ .

At this point we have enough definitions to outline the plan for proving Theorem 3. We start with a Mozhan partition (as in Definition 1) and repeatedly move a vertex from the active component; our goal is either to find a  $(\Delta - 1)$ -coloring or a copy of  $K_{\Delta-4}$  (before we reach a club with no unmoved vertices). Our move sequence will satisfy the following criteria: each vertex moves at most once; a vertex never moves from a club R to a club S if R and S are joined; if possible the active club sends a vertex to a club to which it has already sent a vertex.

Now each vertex in the active component can be sent to any of all but at most one other clubhouse due to degree considerations. If it cannot be sent to some additional clubhouse, this is because the active component, say R, is joined to a full club, say S, in that clubhouse (Definition 6 defines two full clubs, R and S, being joined, but it implies that V(R) is joined to V(S), which is enough for now). The main idea is that when a maximal such move sequence stops, it is because the active component is joined to full clubs in all but at most one of the other clubhouses. The final ingredient is to show that full clubs being joined is a transitive relation; that is, if clubs R, S, and T are full and R is joined to S and T, then also S is joined to T. This implies that at the end of a maximal move sequence all of the full clubs joined to the active component are joined to each other, and thus induce a big clique (in fact, size at least  $\Delta - 4$ ).

**Lemma 5.** Let H be a graph with induced subgraphs  $A_1, \ldots, A_k$  where  $\{V(A_1), \ldots, V(A_k)\}$  partitions V(H) and  $\chi(H) = \sum_{i \in [k]} \chi(A_i)$  where  $\chi(A_1) \geq 4$  and  $\chi(A_i) \geq 3$  for all  $i \in [k] \setminus \{1\}$ . Let  $u \in V(A_1)$  be such that  $\chi(A_1 \setminus \{u\}) < \chi(A_1)$  and let  $T_1$  be the component of  $A_1$  containing u. Now  $\chi(T_1) = \chi(A_1)$  and the next three statements hold.

- (a) For each  $i \in [k] \setminus \{1\}$  there is a component  $T_i$  of  $A_i$  such that  $\chi(T_i) = \chi(A_i)$  and  $d_{V(T_i)}(u) \geq \chi(A_i)$ .
- (b) Define  $T_i$ , for all  $i \in [k]$ , as above. Suppose  $d_{V(T_k)}(u) = \chi(A_k)$  and  $d_{V(A_k)}(u) \le \chi(A_k) + 1$ . Put  $A^* = V(\{A_1, \dots, A_{k-1}\})$  and  $T^* = V(\{T_1, \dots, T_{k-1}\})$ . Further suppose there is  $v \in N(u) \cap V(T_k)$  with  $d_{A^*}(v) \le 1 + \sum_{i \in [k-1]} \chi(A_i)$  and  $d_{T^*}(v) \ge 3$ . Now there exists  $q \in [k-1]$  such that  $d_{V(T_q)}(v) \ge \chi(A_q)$ .
- (c) Define  $A^*$ ,  $T^*$ , and v as in (b). If  $T^*$  induces a clique,  $T_k$  is complete, and  $d_{A^*}(w) \leq |T^*|$  for all  $w \in T^*$ , then  $T^* \cup \{v\}$  induces a clique.

*Proof.* First we prove (a). Pick  $i \in [k] \setminus \{1\}$ . Since  $\chi(A_1 \setminus \{u\}) < \chi(A_1)$ , we must have  $\chi(A_i') = \chi(A_i) + 1$ , where  $A_i' = G[V(A_i) \cup \{u\}]$ . So, u has at least  $\chi(A_i)$  neighbors in some component  $T_i$  of  $A_i$ , for otherwise we get a  $\chi(A_i)$ -coloring of  $A_i'$  from a  $\chi(A_i)$ -coloring of  $A_i$  by permuting colors in components of  $A_i$ . This proves (a).

Now we prove (b). Our plan is to move u to part  $A_k$  and move v to some other part, and show that if (b) fails, then we have a  $(\chi(G)-1)$ -coloring. Put  $A'_1 = G[V(A_1 \setminus \{u\}) \cup \{v\}]$  and  $A'_k = G[V(A_k) \cup \{u\}]$  and  $A'_i = G[V(A_i) \cup \{v\}]$  for each  $i \in [k-1] \setminus \{1\}$ . Since  $\chi(A_1 \setminus \{u\}) < \chi(A_1)$ , we must have  $\chi(A'_k) = \chi(A_k) + 1$  and u is critical in  $A'_k$ . Also v is critical in  $A'_k$  since we can  $\chi(A_k)$ -color  $G[A_k \setminus \{v\}]$ , and extend the coloring to u, since  $d_{V(T_k)}(u) = \chi(A_k)$  but v is removed from  $T_k$  (if u has one other neighbor in  $A_k$ , then we may possibly have to permute the colors on that component of  $G[A_k]$  to avoid the color used on u in  $G[V(T_k \setminus \{v\}) \cup \{u\}]$ ). Since v is critical in  $A'_k$ , we conclude that  $d_{V(A_i)}(v) \geq \chi(A_i)$  for each  $i \in [k-1]$ .

Since  $\chi(A'_k \setminus \{v\}) < \chi(A'_k)$ , we must have  $\chi(A'_1) \ge \chi(A_1)$  and  $\chi(A'_i) \ge \chi(A_i) + 1$  for each  $i \in [k-1] \setminus \{1\}$ . In particular, v is critical in  $A'_i$  for each  $i \in [k-1]$ . Note that  $d_{V(A_i)}(v) \le \chi(A_i) + 1$  for each  $i \in [k-1]$  since  $d_{A^*}(v) \le 1 + \sum_{i \in [k-1]} \chi(A_i)$ . Moreover, there is at most one  $i \in [k-1]$  for which  $d_{V(A_i)}(v) = \chi(A_i) + 1$ . Now the remainder of (b) consists of the following claim.

Claim 1. There exists  $q \in [k-1]$  such that  $d_{V(T_q)}(v) \ge \chi(A_q)$ .

Pick  $w, x \in N(v) \cap T^* \setminus \{u\}$ . First, suppose there is  $i \in [k-1]$  with  $w, x \in V(T_i)$ . Since v is critical in  $A_i'$ , it has at least  $\chi(A_i') - 1$  neighbors in some component C of  $A_i' \setminus \{v\}$ . Since v has two neighbors in  $T_i$ , our bounds on  $d_{V(A_i)}(v)$  and  $\chi(A_i')$  imply that  $C = T_i$ . Since  $\chi(A_i') \geq \chi(A_i) + 1$  for each  $i \in [k-1] \setminus \{1\}$  (and if i = 1, v gets u as an extra neighbor), the claim is satisfied.

So, we may assume there are different  $i, j \in [k-1]$  with  $w \in V(T_i)$  and  $x \in V(T_j)$ . Since there is at most one  $p \in [k-1]$  for which  $d_{V(A_p)}(v) = \chi(A_p) + 1$ , by symmetry we may assume that  $d_{V(A_j)}(v) = \chi(A_j)$ . Since v is critical in  $A'_j$ , it has at least  $\chi(A'_j) - 1$  neighbors in some component C of  $A'_j \setminus \{v\}$ . Since v has at least one neighbor in  $T_j$ , our bounds on  $d_{V(A_j)}(v)$  and  $\chi(A'_j)$  imply that  $C = T_j$ . This proves the claim, and completes the proof of (b).

Now we prove (c), which we restate as the following claim.

Claim 2. If  $T^*$  induces a clique,  $T_k$  is complete, and  $d_{A^*}(w) \leq |T^*|$  for all  $w \in T^*$ , then  $T^* \cup \{v\}$  induces a clique.

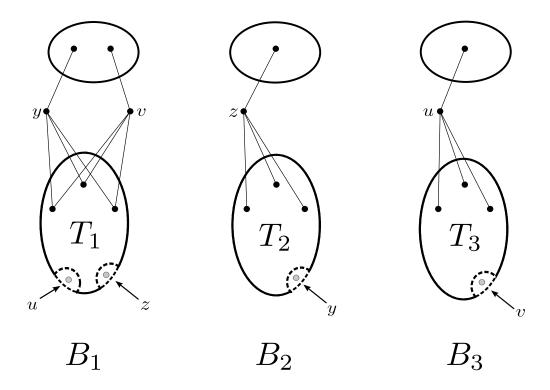


Figure 1: The partition in Claim 2 of Lemma 5. To form  $B_1$ ,  $B_2$ , and  $B_3$  from  $A_1$ ,  $A_2$ , and  $A_3$  (respectively), the vertices circled with dotted lines (and shown in gray) have now been moved to other parts, where they are shown above the  $T_i$ 's.

Suppose otherwise that  $T^*$  induces a clique,  $T_k$  is complete, and  $d_{A^*}(w) \leq |T^*|$  for all  $w \in T^*$  but  $T^* \cup \{v\}$  does not induce a clique. By (b) we have  $q \in [k-1]$  such that  $d_{V(T_q)}(v) \geq \chi(A_q)$ . If  $u \notin V(A_q)$ , then we could move u into  $A_q$  without violating any hypotheses. So, we may assume that q = 1. Since  $T^* \cup \{v\}$  does not induce a clique, there is some  $A_p$  to which v is not joined.

By hypothesis  $d_{V(T_1)}(v) \geq \chi(A_1)$  and  $T_1$  is complete, so v must be joined to  $T_1$ . So, by considering only the indices 1, p, k, we can assume that k = 3 and p = 2. More precisely, in what follows we will move some vertices between parts  $A_1$ ,  $A_p$ , and  $A_q$  and color the graph  $H[V(A_1) \cup V(A_p) \cup V(A_q)]$  with at most  $\chi(A_1) + \chi(A_p) + \chi(A_k) - 1$  colors. By combining this coloring with one that uses  $\chi(A_i)$  colors on each other part  $A_i$ , we show that  $\chi(H) < \sum_{i=1}^k \chi(A_i)$ . This contradiction proves Claim 2.

Pick  $y \in V(T_2) \setminus N(v)$  and  $z \in V(T_1 \setminus \{u\})$ . Let  $B_1 = G[(A_1 \cup \{v, y\}) \setminus \{u, z\}]$ ,  $B_2 = G[(A_2 \cup \{z\}) \setminus \{y\}]$ , and  $B_3 = G[(A_3 \cup \{u\}) \setminus \{v\}]$ . We derive a contradiction by showing that  $\chi(B_1) < \chi(A_1)$  and  $\chi(B_2) \le \chi(A_2)$  and  $\chi(B_3) \le \chi(A_3)$ .

Since,  $d_{A^*}(z) \leq |T^*|$  and  $T^*$  is complete, we have  $d_{V(A_2)}(z) \leq \chi(A_2) + 1$  and hence  $d_{V(B_2)}(z) = d_{V(A_2)}(z) - 1 \leq \chi(A_2)$  since  $z \leftrightarrow y$ . Since z has exactly  $\chi(A_2) - 1$  neighbors in  $T_2 \setminus \{y\}$ , we see that z has at most  $\chi(A_2) - 1$  neighbors in each component of  $B_2 \setminus \{z\}$  and hence  $\chi(B_2) \leq \chi(A_2)$ . Since, by assumption,  $d_{V(A_k)}(u) \leq \chi(A_k) + 1$  and  $T_k$  is complete, the proof that  $\chi(B_3) \leq \chi(A_3)$  is nearly identical (if u has a neighbor in  $A_k \setminus T_k$ , then we may need to permute colors on its component, so that this neighbor does not use the same color as u).

Since  $\{u,z\}$  is joined to  $\{v,y\}$ , we have  $d_{V(B_1)}(v) = d_{V(A_1)}(v) - 2 \le \chi(A_1) + 1 - 2 = \chi(A_1) - 1$ . Similarly,  $d_{V(B_1)}(y) \le \chi(A_1) - 1$ . Let  $K = G[T_1 \cup \{v,y\} \setminus \{u,z\}]$ . Then K is a copy of  $K_{\chi(A_1)}$  with the edge vy deleted. First, color  $B_1 \setminus V(K)$  with  $\chi(A_1) - 1$  colors. Since v and y each have at most one neighbor outside of K in  $B_1$  and  $\chi(A_1) \ge 4$ , we can finish the coloring on K by choosing the same color for v and y, different from the colors used on their at most 2 (collective) neighbors in  $B_1 \setminus V(K)$ , and then coloring  $K \setminus \{v,y\}$  with the  $\chi(A_1) - 2$  other colors (see Figure 1).

In proving our next few lemmas, we repeatedly use the following helper lemma, which is an easy corollary of Lemma 5.

**Lemma 6.** Let P be a minimum  $(r_1, \ldots, r_s)$ -partition of a graph G with  $\chi(G) = \Delta(G) = 1 + \sum_{i \in [s]} r_i$ . Let

$$S = ((P^1, v_1, i_1, P^2), \dots, (P^q, v_q, i_q, P^{q+1}))$$

be a move sequence starting at P. Let R and S be distinct full clubs of S and  $t \in [q+1]$ . If  $R_t = \mathcal{A}(P^t)$ , then

- (a) if  $u \in V(R_t)$  and u has at least 2 neighbors in  $S_t$ , then u is joined to  $S_t$ .
- (b) if  $u \in V(R_t)$  and  $v \in V(S_t)$  and u has at least 2 neighbors in  $S_t$  and v has at least 2 neighbors in  $R_t \setminus \{u\}$ , then v is joined to  $R_t$ .

*Proof.* First we prove (a). By symmetry, assume that  $V(R_t) \subseteq P_1^t$  and  $V(S_t) \subseteq P_2^t$ . We apply Lemma 5 (a) with  $A_i = G[P_i^t]$  for  $i \in [2]$ ,  $H = G[V(A_1) \cup V(A_2)]$  and  $T_1 = R_t$ . By Lemma 4,  $\chi(H) = r_1 + r_2 + 1 = \chi(A_1) + \chi(A_2)$  and  $\chi(A_1 - x) < \chi(A_1)$  for all

 $x \in V(T_1)$ . Also by Lemma 4,  $d_{A_2}(x) \leq \chi(A_2) + 1$  for all  $x \in V(T_1)$ . By Lemma 5, u has at least  $\chi(A_2)$  neighbors in some component  $T_2$  of  $A_2$ . Since  $d_{A_2}(u) \leq \chi(A_2) + 1$  and u has at least two neighbors in  $S_t$ , we must have  $T_2 = S_t$ . Since  $S_t$  is a  $K_{\chi(A_2)}$  this proves (a).

Now we prove (b). If  $d_{A_1}(v) > \chi(A_1) + 1$ , then there exists some part  $P_k^t$  with  $d_{P_k^t}(v) < r_k$ . By moving v to  $P_k^t$  and any vertex in  $T_1$  to  $P_2^t$ , we get a  $(\chi(G)-1)$ -coloring of G, a contradiction. So  $d_{A_1}(v) \leq \chi(A_1) + 1$ . By (a),  $|N(u) \cap V(T_2)| = \chi(A_2)$  and  $v \in N(u) \cap V(T_2)$ . So, we may apply Lemma 5 (b) to conclude that  $|N(v) \cap V(T_1)| \geq \chi(A_1)$ . Since  $T_1$  is a  $K_{\chi(A_1)}$  this proves (b).

**Lemma 7.** Let P be a minimum  $(r_1, \ldots, r_s)$ -partition of a graph G with  $\chi(G) = \Delta(G) = 1 + \sum_{i \in [s]} r_i$ . Let S be a move sequence starting at P and let R and S be distinct full clubs of S. Then, for any  $t_1, t_2 \geq 1$ , we have that  $R_{t_1}$  is joined to  $S_{t_1}$  if and only if  $R_{t_2}$  is joined to  $S_{t_2}$ .

*Proof.* Suppose the lemma is false and let

$$S = ((P^1, v_1, i_1, P^2), \dots, (P^q, v_q, i_q, P^{q+1}))$$

be the shortest move sequence for which it fails. There must be a  $t \in [q]$  such that either  $R_t$  is not joined to  $S_t$ , but  $R_{t+1}$  is joined to  $S_{t+1}$  or else  $R_t$  is joined to  $S_t$ , but  $R_{t+1}$  is not joined to  $S_{t+1}$ . Note that q = 1, for otherwise, the move sequence  $((P^t, v_t, i_t, P^{t+1}))$  is a shorter counterexample. Hence  $S = ((P^1, v_1, i_1, P^2))$ . Since the reverse sequence  $(P^2, v_1, j(P^1), P^1)$  is also a counterexample, we may assume that  $R_1$  is not joined to  $S_1$ , but  $R_2$  is joined to  $S_2$ .

By symmetry between R and S, we may assume that  $R_1$  is the active component. Since  $R_1$  is not joined to  $S_1$ , but  $R_2$  is joined to  $S_2$ , it must be that  $R_2 = R_1 \setminus \{v_1\}$  is joined to  $S_2 = S_1$  and there is  $u \in V(S_1)$  with  $v_1 \nleftrightarrow u$ . Pick  $w \in V(R_1 \setminus \{v_1\})$ . Now applying Lemma 6(b) to w and u shows that  $S_1$  is joined to  $R_1$ , a contradiction.  $\square$ 

Lemma 7 makes it possible for us to talk about full clubs being joined or not joined as follows.

**Definition 6.** Let P be a minimum  $(r_1, \ldots, r_s)$ -partition of a graph G with  $\chi(G) = \Delta(G) = 1 + \sum_{i \in [s]} r_i$ . Let S be a move sequence starting at P and let R and S be distinct full clubs of S. Then R and S are joined if  $R_t$  and  $S_t$  are joined for all  $t \geq 1$ . Also R and S are not joined if  $R_t$  and  $S_t$  are not joined for all  $t \geq 1$ . Note that by Lemma 7 R and S are either joined or not joined.

**Definition 7.** Let P be a minimum  $(r_1, \ldots, r_s)$ -partition of a graph G. For a club R of a move sequence

$$S = ((P^1, v_1, i_1, P^2), \dots, (P^q, v_2, i_2, P^{q+1}))$$

starting at P, we say that R is active k times if the number of  $t \in [q+1]$  such that  $R_i$  is active is k.

**Lemma 8.** Let G be a graph with  $\chi(G) = \Delta(G) = 1 + \sum_{i \in [s]} r_i$  and let S be a move sequence starting at a minimum  $(r_1, \ldots, r_s)$ -partition of G. If S is a full club of S that is active at least once and R and W are different full clubs of S such that R is joined to S and S is joined to W, then R is joined to W.

Proof. Pick t such that  $S_t$  is active and say the t-th move of S is  $(P, v_t, i_t, P')$ . Put  $T_1 = S_t$ ,  $T_2 = R_t$ , and  $T_3 = W_t$ . By symmetry, we assume that  $V(T_1) \subseteq P_1$ ,  $V(T_2) \subseteq P_2$ , and  $V(T_3) \subseteq P_3$ . We will apply Lemma 5 with  $A_i = G[P_i]$  for all  $i \in [3]$  and  $H = G[V(A_1) \cup V(A_2) \cup V(A_3)]$ . Define  $A^*$  and  $T^*$  as in Lemma 5.

Pick  $u \in V(T_1)$ . We have  $\chi(H) = r_1 + r_2 + r_3 + 1 = \chi(A_1) + \chi(A_2) + \chi(A_3)$  and  $\chi(A_1 \setminus \{u\}) < \chi(A_1)$ . Also by Lemma 4,  $d_{V(A_3)}(u) \leq \chi(A_3) + 1$ . Since  $T_3$  is a  $K_{r_3}$ , we also have  $d_{V(T_3)}(u) = \chi(A_3)$ . For any  $v \in V(T_3)$ , we have  $d_{A^*}(v) \leq 1 + \chi(A_1) + \chi(A_2)$ , for otherwise there exists some part  $P_q$  with  $d_{P_q}(v) < r_q$ . By moving v to  $P_q$  and u to  $P_3$ , we get a  $(\chi(G) - 1)$ -coloring of G, a contradiction. Also,  $d_{T^*}(v) \geq 3$  since  $T_1$  is joined to  $T_3$ . Additionally,  $T^*$  induces a clique and  $T_k$  is complete. To apply Lemma 5, it remains to check that  $d_{A^*}(w) \leq |T^*|$  for all  $w \in T^*$ . If not, then we could move w to some part  $P_q$  with  $d_{P_q}(w) < r_q$  and get a  $(\chi(G) - 1)$ -coloring of G. So, we apply Lemma 5 with each  $v \in V(T_3)$  and conclude that  $T_3$  is joined to  $T_2$  as desired.  $\square$ 

**Definition 8.** Let P be a minimum  $(r_1, \ldots, r_s)$ -partition of a graph G. For a club R of a move sequence S starting at P, the *spread* of R is the set of indices of parts to which R sends vertices; more formally,

$$\operatorname{sp}_{\mathcal{S}}(R) = \{i \mid (Q, v, i, Q') \in \mathcal{S} \text{ with } \mathcal{C}(\mathcal{A}(Q)) = R\}.$$

The spread of S is  $sp(S) = max_R |sp(R)|$  where the max is over all clubs R of S.

**Lemma 9.** Let P be a minimum  $(r_1, \ldots, r_s)$ -partition of a graph G with  $\chi(G) = \Delta(G) = 1 + \sum_{i \in [s]} r_i$ . If

$$S = ((P^1, v_1, i_1, P^2), \dots, (P^q, v_q, i_q, P^{q+1}))$$

is a move sequence with  $sp(S) \leq 2$ , then one of the following holds:

- (1)  $v_i = v_j$  for some distinct  $i, j \in [q]$  (i.e. a vertex moves more than once); or
- (2) there is  $t \in [q]$  such that the active component in  $P^t$  is joined to the active component in  $P^{t+1}$ ; or
- (3) every club of S is active at most 3 times.

*Proof.* Suppose the lemma is false and choose a move sequence

$$S = ((P^1, v_1, i_1, P^2), \dots, (P^q, v_q, i_q, P^{q+1}))$$

for which it fails minimizing q. By minimality of q, we have a length three subsequence  $((P^a, v_a, i_a, P^{a+1}), (P^b, v_b, i_b, P^{b+1}), (P^{c-1}, v_{c-1}, i_{c-1}, P^c))$  of S such that

- $\text{(i)} \ \ \mathcal{C}^a(\mathcal{A}(P^a)) = \mathcal{C}^b(\mathcal{A}(P^b)) = \mathcal{C}^c(\mathcal{A}(P^c)) \ \text{and} \ \ \mathcal{C}^{a+1}(\mathcal{A}(P^{a+1})) = \mathcal{C}^{b+1}(\mathcal{A}(P^{b+1})); \ \text{and} \ \ \mathcal{C}^{a+1}(\mathcal{A}(P^a)) = \mathcal{C}^{b+1}(\mathcal{A}(P^b)) = \mathcal{C}^b(\mathcal{A}(P^b)) = \mathcal{C}^b$
- (ii) there is at most one  $(P^d, v_d, i_d, P^{d+1})$  in S with a < d < b such that  $C^d(A(P^d)) = C^a(A(P^a))$ ; and

(iii)  $C^{a+1}(A(P^{a+1}))$  is active at most 3 times.

Let  $X = \mathcal{C}(\mathcal{A}(P^a))$  and  $Y = \mathcal{C}(\mathcal{A}(P^{a+1}))$ . We may choose c to be the smallest index in  $\{b+1,\ldots,q+1\}$  such that X is active at stage c. We will show that X is joined to Y, which gives a contradiction, since we are assuming (2) does not hold. If there does not exist  $(P^d, v_d, i_d, P^{d+1})$  in S with a < d < b such that  $\mathcal{C}(\mathcal{A}(P^d)) = \mathcal{C}(\mathcal{A}(P^a))$ , then let d = b.

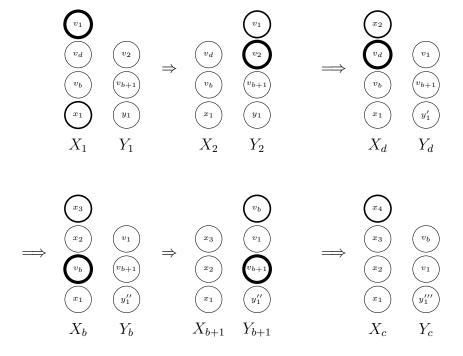


Figure 2: The six key partitions  $X_i, Y_i$  in the proof of Lemma 9. In each partition, the next vertex that will move is marked in bold, and the vertex that most recently moved is marked in semi-bold. If a vertex is unnamed in the proof, we denote it as  $x_i$  or  $y_i$  based on whether it appears in  $X_j$  or  $Y_j$ .

#### Claim 1. $\{v_b\}$ is joined to $V(Y_d)$ .

Since Y becomes active at most once (by (iii)) between move d and move b+1, we have  $|V(Y_d) \cap V(Y_b)| \geq 2$ . One vertex in this intersection is  $v_a$ , and another is  $v_{b+1}$  (since no vertex is moved twice, by (1)). So  $v_b$  is adjacent to  $v_a$  and  $v_{b+1}$ , since  $v_a, v_b, v_{b+1} \in V(Y_{b+1})$  and Y is full. Applying Lemma 6(a) to X and Y with t=d, shows that  $v_b$  is joined to  $V(Y_d)$ , since  $v_a, v_{b+1} \in V(Y_d)$ .

Claim 2.  $\{v_a\}$  is joined to  $V(X_d)$ .

Since  $|V(X_d) \cap V(X_a)| \geq 3$ ,  $v_a$  has at least 3 neighbors in  $X_d$ . Now we show that  $v_a$  is joined to  $V(X_d)$  by Lemma 6(b). Specifically, we apply the lemma to X and Y with t = d. We let  $u = v_b$  and  $v = v_a$ . Claim 1 states that  $v_b$  is joined to  $Y_d$ , so, in particular,  $v_b$  has at least two neighbors in  $V(Y_d)$ . Since  $X_a$  is full,  $v_a$  is adjacent to both  $v_d$  and  $x_a$  (the final vertex moved before S began). So Lemma 6(b) implies that  $\{v_a\}$  is joined to  $V(X_d)$ , as desired.

Claim 3.  $\{v_a\}$  is joined to  $V(X_b)$ .

Since Y is full,  $v_b$  is joined to  $V(Y_b)$ . Since  $|V(X_d) \cap V(X_b)| \geq 3$  and  $v_a$  is joined to  $V(X_d)$ ,  $v_a$  has at least 3 neighbors in  $X_b$ . So  $v_a$  is joined to  $V(X_b)$  by Lemma 6(b) applied to X and Y with t = b.

Claim 4.  $V(X_{b+1})$  is joined to  $V(Y_c)$ .

Since  $V(X_{b+1}) \subset V(X_b)$ , Claim 3 shows that  $\{v_a, v_b\}$  is joined to  $V(X_{b+1})$ . But,  $\{v_a, v_b\} \subset V(Y_c)$ , so applying Lemma 6(a) to X and Y with t = c shows that  $V(X_{b+1})$  is joined to  $V(Y_c)$ .

Claim 5.  $V(X_c)$  is joined to  $V(Y_c)$ . In particular, X is joined to Y.

Since  $|X_{b+1}| \geq 3$ , Claim 4 and an application of and Lemma 6(b) to X and Y with t = c shows that  $V(X_c)$  is joined to  $V(Y_c)$ .

**Theorem 3.** If G is a graph with  $\chi(G) \geq \Delta(G)$ , then  $\omega(G) \geq \Delta(G) - 3$  if  $\Delta(G) \equiv 1 \pmod{3}$  and  $\omega(G) \geq \Delta(G) - 4$  otherwise.

Proof. The theorem is trivially true if  $\Delta(G) \leq 6$ , so we assume that  $\Delta(G) \geq 7$ . It suffices to consider critical graphs, since any graph G contains a critical subgraph H with  $\chi(H) = \chi(G)$ . By Brooks' Theorem, we may assume  $\chi(G) = \Delta(G)$ . Let  $s = \left\lfloor \frac{\Delta(G)-1}{3} \right\rfloor$  and  $r_1, \ldots, r_s \in \{3,4\}$  such that  $\Delta(G) = 1 + \sum_{i \in [s]} r_i$ . Now G has an  $(r_1, \ldots, r_s)$ -partition, so we can let P be a minimum  $(r_1, \ldots, r_s)$ -partition of G. Let

$$S = ((P^1, v_1, i_1, P^2), \dots, (P^q, v_q, i_q, P^{q+1}))$$

be a move sequence starting at P with  $\operatorname{sp}(\mathcal{S}) \leq 2$  of maximum length such that  $v_i \neq v_j$  for all pairs of distinct  $i, j \in [q]$  and for each  $t \in [q]$  the active component in  $P^t$  is not joined to the active component in  $P^{t+1}$ . Let  $A = \mathcal{A}(P^{q+1})$ . Lemma 9 implies that  $\mathcal{C}^{q+1}(A)$  is active at most 3 times in  $\mathcal{S}$ . Since  $r_{i_q} \geq 3$ , there is  $x \in V(A)$  such that  $x \notin \{v_t \mid t \in [q]\}$ , i.e., x has never moved during  $\mathcal{S}$ .

Let  $T = \operatorname{sp}(\mathcal{C}(A))$ . If there is  $i \in T$  with  $d_{P_i^{q+1}}(x) = r_i$ , then we have a move  $(P^{q+1}, x, i, Q^i)$  and by the maximality condition on  $\mathcal{S}$ , it must be that A is joined to  $\mathcal{A}(Q^i)$ . But, by assumption, A is not joined to  $\mathcal{A}(Q^i)$  for any  $i \in T$ , so this is impossible.

Since  $d_G(x) \leq 1 + \sum_{i \in [s]} r_i$  and x has exactly  $r_{i_q}$  neighbors in  $P_{i_q}^{q+1}$ , there is at most one  $i \in [s] \setminus \{i_q\}$  for which  $d_{P_i^{q+1}}(x) > r_i$ . So,  $|T| \leq 1$  and if |T| = 1, then T contains the one i with  $d_{P_i^{q+1}}(x) > r_i$ . By the maximality condition on  $\mathcal{S}$ , it must be that A is joined to clubs in  $P_i^{q+1}$  for all but one  $i \in [s] \setminus \{i_q\}$ . By Lemma 4(2), we know that each club joined to A is full. By Lemma 8, all of these full clubs must be

pairwise joined to each other. Thus, together, they induce a large clique. Specifically, they induce a clique of size  $1 + \sum_{j \in [s] \setminus \{i\}} r_j$ , which is size  $\Delta(G) - r_i$ . Since  $r_j = 3$  if  $\Delta(G) \equiv 1 \pmod{3}$  and  $r_j \leq 4$  otherwise, we have the desired large clique.

As an immediate consequence of Theorem 3, we get the following corollary.

Corollary 10. If G is a graph with  $\chi(G) \geq \Delta(G) = 13$ , then G contains  $K_{10}$ .

### 3 The First Main Theorem

A hitting set is an independent set that intersects every maximum clique. If I is a hitting set and also a maximal independent set, then  $\Delta(G-I) \leq \Delta(G)-1$  and  $\chi(G-I) \geq \chi(G)-1$ . (In our applications, we can typically assume that  $\Delta(G-I) = \Delta(G)-1$ , since otherwise we get a good coloring or a big clique from Brooks' Theorem. We give more details in the proof of Theorem 1.) So if G-I has a clique of size  $\Delta(G-I)-t$ , for some constant t, then also G has a clique of size  $\Delta(G)-t$ . We repeatedly remove hitting sets to reduce a graph with  $\Delta \geq 13$  to one with  $\Delta = 13$ . Since we proved in Corollary 10 that every graph with  $\chi \geq \Delta = 13$  contains  $K_{10}$ , this repeated removal of hitting sets allows us to prove that every G with  $\chi \geq \Delta \geq 13$  contains  $K_{\Delta-3}$ .

This idea is not new. Kostochka [20] proved that every graph with  $\omega \geq \Delta - \sqrt{\Delta} + \frac{3}{2}$  has a hitting set. Rabern [27] extended this result to the case  $\omega \geq \frac{3}{4}(\Delta+1)$ , and King [17] strengthened his argument to prove that G has a hitting set if  $\omega > \frac{2}{3}(\Delta+1)$ . This condition is optimal, as illustrated by the lexicographic product of an odd cycle and a clique. Finally, King's argument was refined by Christofides, Edwards, and King [8] to show that these lexicographic products of odd cycles and cliques are the only sharpness examples; that is, G has a hitting set if  $\omega \geq \frac{2}{3}(\Delta+1)$  and G is not the lexicographic product of an odd cycle and a clique. Hitting set reductions have application to other vertex coloring problems. Using this idea (and others), King and Reed [18] gave a short proof that there exists  $\epsilon > 0$  such that  $\chi \leq \lceil (1-\epsilon)(\Delta+1) + \epsilon \omega \rceil$ .

To keep this paper largely self-contained, we prove our own hitting set lemma. In the present context, it suffices to find a hitting set when G is a minimal counterexample to Theorem 1 with  $\Delta \geq 14$ . Such a G is  $\Delta$ -critical, which facilitates a shorter proof. In [10], we proved a number of results about so called  $d_1$ -choosable graphs (defined below), which are certain graphs that cannot appear as induced subgraphs in a  $\Delta$ -critical graph. We leverage these  $d_1$ -choosability results to prove our hitting set lemma, then we use the hitting set lemma to reduce the problem to the case  $\Delta = 13$ , which we handled in Corollary 10. Since the proofs of the  $d_1$ -choosability results in [10] are lengthy, we give a short proof of the special case that we need here.

A list assignment L is an assignment L(v) of a set of allowable colors to each vertex  $v \in V(G)$ . An L-coloring is a proper coloring such that each vertex v is colored from L(v). An f-assignment is a list assignment L such that |L(v)| = f(v) for all  $v \in V(G)$ . In particular, a  $d_1$ -assignment is an f-assignment with f(v) = d(v) - 1 for all v. A graph G is f-choosable if G has an L-coloring for every f-assignment L. No  $\Delta$ -critical graph contains an induced  $d_1$ -choosable subgraph H (by criticality, color  $G \setminus H$ , then extend the coloring to H, since it is  $d_1$ -choosable). For a list assignment

L, let  $Pot(L) = \bigcup_{v \in V(G)} L(v)$ . The following lemma is central in proving each of our  $d_1$ -choosability results.

**Lemma 11** (Small Pot Lemma, [15, 31]). For a list size function  $f: V(G) \rightarrow \{0, \ldots, |G|-1\}$ , a graph G is f-choosable if and only if G is L-colorable for each list assignment L such that |L(v)| = f(v) for all  $v \in V(G)$  and  $|\bigcup_{v \in V(G)} L(v)| < |G|$ .

Proof. Fix G and f, and let V = V(G). The "only if" direction is true by definition. Now we prove the "if" direction. Assume that G is L-colorable for each list assignment L such that |L(v)| = f(v) for all v and  $|\cup_{v \in V} L(v)| < |G|$ . For any  $U \subseteq V$  and any list assignment L, let L(U) denote  $\cup_{v \in U} L(v)$ . Let L be an f-assignment such that  $|L(V)| \ge |G|$  and G is not L-colorable. For each  $U \subseteq V$ , let g(U) = |U| - |L(U)|. Let  $\mathcal{B}$  be a bipartite graph, where one part consists of vertices in V and the other part consists of colors in Pot(L), and a vertex v is adjacent to a color c if  $c \in L(v)$ . Since G is not L-colorable,  $\mathcal{B}$  has no matching saturating V, so Hall's Theorem implies there exists U with g(U) > 0. Choose U to maximize g(U). Let A be an arbitrary set of |G| - 1 colors containing L(U). Construct L' as follows. For  $v \in U$ , let L'(v) = L(v). Otherwise, let L'(v) be an arbitrary subset of A of size f(v). Now |L'(V)| < |G|, so by hypothesis, G has an L'-coloring. This gives an L-coloring of U. By the maximality of g(U), for all  $W \subseteq (V \setminus U)$ , we have  $|L(W) \setminus L(U)| \ge |W|$ . Let  $\mathcal{B}' = \mathcal{B} \setminus (\cup_{u \in U} \{u\} \cup N_{\mathcal{B}}(u))$ . Thus, by Hall's Theorem,  $\mathcal{B}'$  has a matching saturating  $V \setminus U$ ; so we can extend the L-coloring of U to all of V.

**Lemma 12** ([10]). For  $t \ge 4$ ,  $K_t \lor B$  is not  $d_1$ -choosable if and only if  $\omega(B) \ge |B| - 1$ ; or t = 4 and B is  $E_3$  or  $K_{1,3}$ ; or t = 5 and B is  $E_3$ .

Proof. If  $\omega(B) \geq |B|-1$ , then assign each  $v \in V(K_t \vee B)$  a subset of  $\{1, \ldots, t+|B|-2\}$ ; since  $\omega(K_t \vee B) \geq t+|B|-1$ , clearly G is not colorable from this list assignment. Now let  $G = K_5 \vee E_3$ , and note that  $K_4 \vee K_{1,3} \cong K_5 \vee E_3$ . Consider the following list assignment L for G: each dominating vertex has list  $\{1, \ldots, 6\}$  and the three other vertices get distinct lists among  $\{1, 2, 3, 4\}, \{1, 2, 5, 6\}, \{3, 4, 5, 6\}$ . If G has a proper L-coloring, then the dominating vertices use five distinct colors; this leaves only one color for the three remaining vertices, but no color appears in all three lists. Hence, G has no L-coloring. Now form G' from G by deleting one dominating vertex (note that  $G' = K_4 \vee E_3$ ), and let  $L' = L \setminus \{6\}$ . Since G has no L-coloring, also G' has no L'-coloring. This proves one direction of the lemma; now we consider the other.

Suppose the only if direction of the lemma is false, and let G and L be a minimal counterexample, where  $G = K_t \vee B$  and L is a  $d_1$ -assignment. Since  $\omega(B) \leq |B| - 2$ , subgraph B contains either (i) an independent set  $S = \{x_1, x_2, x_3\}$  or (ii) a set  $S = \{x_1, x_2, x_3, x_4\}$  with  $x_1x_2, x_3x_4 \notin E(B)$ . If B contains only (i), then  $G[S] = E_3$  and  $t \geq 6$  (by moving any dominating vertices from B to  $K_t$ ). Let  $T = V(K_t)$  and denote T by  $\{y_1, \ldots, y_t\}$ . In Cases (i) and (ii) we assume by minimality that t = 6 and t = 4, respectively (for larger t, we can greedily color all but 6 (resp. 4) vertices; each vertex has enough colors to be colored greedily, since at least 4 of its neighbors remain uncolored). Also by minimality, we assume that V(B) = S (as for larger t, we can greedily color vertices of B not in S).

By definition |L(v)| = d(v) - 1; specifically,  $|L(x_i)| = d_S(x_i) + t - 1$  and  $|L(y_j)| = |S| + t - 2$  for all  $x_i \in S$  and  $y_j \in T$ . When we have i, j, k with  $x_i \not\hookrightarrow x_j$  and  $|L(x_i)| + |L(x_j)| > |L(y_k)|$ , we often use the following technique, called saving a color on  $y_k$  via  $x_i$  and  $x_j$ . If there exists  $c \in L(x_i) \cap L(x_j)$ , then use c on  $x_i$  and  $x_j$ . Otherwise, color just one of  $x_i$  and  $x_j$  with some  $c \in (L(x_i) \cup L(x_j)) \setminus L(y_k)$ . For a set U, let  $L(U) = \bigcup_{v \in U} L(v)$ .

Case (i) By the Small Pot Lemma, assume that  $|L(G)| \leq 8$ . This implies  $|L(x_i) \cap L(x_j)| \geq 2$  for all  $i, j \in [3]$ . If there exist  $x_i$  and  $y_k$  with  $L(x_i) \not\subseteq L(y_k)$ , then color  $x_i$  to save a color on  $y_k$ . Color the remaining x's with a common color; this saves an additional color on each y. Now finish greedily, ending with  $y_k$ . Thus, we have  $L(x_i) \subset L(y_k)$  for all  $i \in [3]$  and  $k \in [6]$ . This gives  $\left|\bigcup_{i=1}^3 L(x_i)\right| \leq 7$ . Since  $\sum_{i=1}^3 |L(x_i)| = 15 > 2|\bigcup_{k=1}^3 L(x_k)|$ , we have a color  $c \in \bigcap_{i=1}^3 L(x_i)$ . Use c on every  $x_i$  and finish greedily.

Case (ii) By the Small Pot Lemma, assume that  $|L(G)| \leq 7$ . If S induces at least two edges, then  $|L(x_1)| + |L(x_2)| \geq 8$ . So  $L(x_1) \cap L(x_2) \neq \emptyset$ . Color  $x_1$  and  $x_2$  with a common color c. If  $|L(y_1) \setminus \{c\}| \leq 5$ , then save a color on  $y_1$  via  $x_3$  and  $x_4$ . Now finish greedily, ending with  $y_1$ .

Suppose S induces exactly one edge; by symmetry, say it is  $x_1x_3$ . Suppose that  $L(x_1) \cap L(x_2) \neq \emptyset$ . Similar to the previous argument, use a common color on  $x_1$  and  $x_2$ , possibly save on  $y_1$  via  $x_3$  and  $x_4$ , then finish greedily. So instead, assume that  $L(x_1) \cap L(x_2) = \emptyset$ . Since  $|L(G)| \leq 7$  and  $L(x_1) \cap L(x_2) = \emptyset$ , by symmetry (between  $x_1$  and  $x_3$  and also between  $x_2$  and  $x_4$ ), we may assume that  $L(x_1) = L(x_3) = \{a, b, c, d\}$  and  $L(x_2) = L(x_4) = \{e, f, g\}$ . Also by symmetry, a or e is missing from  $L(y_1)$ . So color  $x_1$  with a and  $x_2$  and  $x_4$  with e and  $x_3$  arbitrarily; this saves one color on each  $y_i$  and a second color on  $y_1$ . Now finish greedily, ending with  $y_1$ .

So instead  $G[S] = E_4$ . If a common color appears on 3 vertices of S, use it there, then finish greedily. If not, then by pigeonhole, at least 5 colors appear on pairs of vertices; so, two colors appear on disjoint pairs. Color two such disjoint pairs, each with a common color. Now finish the coloring greedily.

The following lemma of King enables us to find an independent transversal.

**Lemma 13** (Lopsided Transversal Lemma [17]). Let H be a graph and  $V_1 \cup \cdots \cup V_r$  a partition of V(H). If there exists  $s \geq 1$  such that for each  $i \in [r]$  and each  $v \in V_i$  we have  $d(v) \leq \min\{s, |V_i| - s\}$ , then H has an independent transversal I of  $V_1, \ldots, V_r$ .

Now we have all the tools to prove our hitting set lemma.

**Lemma 14.** Every  $\Delta$ -critical graph with  $\chi \geq \Delta \geq 14$  and  $\omega = \Delta - 4$  has a hitting set.

Proof. Suppose the lemma is false, and let G be a counterexample minimizing |G|. Consider distinct intersecting maximum cliques A and B. Since a vertex in their intersection has degree at most  $\Delta$ , we have  $|A \cap B| \geq |A| + |B| - (\Delta + 1) = \Delta - 9 \geq 5$ . Since G contains no induced  $d_1$ -choosable subgraph, letting  $A \cap B = K_t$  in Lemma 12 implies that  $\omega(G[A \cup B]) \geq |A \cup B| - 1$ . Hence  $|A \cap B| = \omega - 1 = \Delta - 5$ . Suppose C is another maximum clique intersecting A; let  $U = A \cup B \cup C$  and  $J = A \cap B \cap C$ . We use inclusion-exclusion to bound |U| and |J|. First,  $|U| = |A \cup B \cup C| = |A \cup B| + |C \setminus A| = |A \cup B| + |C| - |C \cap A| \leq (\Delta - 5 + 1 + 1) + (\Delta - 4) - |C|$ 

 $(\Delta - 5) = \Delta - 2$ . Second,  $|J| = |A \cap B| + |C| - |(A \cap B) \cup C| \ge |A \cap B| + |C| - |U| \ge (\Delta - 5) + (\Delta - 4) - (\Delta - 2) = \Delta - 7 \ge 7$ .

Since  $|J| \geq 7$ , by Lemma 12,  $\omega(G[U]) \geq |U|-1$ ; so C = A or C = B, a contradiction. Thus, every maximum clique intersects at most one other maximum clique. Hence we can partition the union of the maximum cliques into sets  $D_1, \ldots, D_r$  such that either  $D_i$  is a  $(\Delta - 4)$ -clique  $C_i$  or  $D_i = C_i \cup \{x_i\}$  for a  $(\Delta - 4)$ -clique  $C_i$ , where  $x_i$  is adjacent to all but one vertex of  $C_i$ .

For each  $D_i$ , if  $D_i = C_i$ , then let  $K_i = C_i$ . If  $D_i = C_i \cup \{x_i\}$ , then let  $K_i = C_i \cap N(x_i)$ . Consider the subgraph F of G formed by taking the subgraph induced on the union of the  $K_i$  and then making each  $K_i$  independent. We apply Lemma 13 to F with  $s = \frac{\Delta}{2} - 2$ . We have two cases to check, when  $K_i = C_i$  and when  $K_i = C_i \cap N(x_i)$ . In the former case,  $|K_i| = \Delta - 4$  and for each  $v \in K_i$  we have  $d_F(v) \leq \Delta(G) + 1 - (\Delta - 4) = 5$ . Hence  $d_F(v) \leq \frac{\Delta}{2} - 2 = \min\{\frac{\Delta}{2} - 2, \Delta - 4 - (\frac{\Delta}{2} - 2)\}$  since  $\Delta \geq 14$ . In the latter case, we have  $|K_i| = \Delta - 5$  and since every  $v \in K_i$  is adjacent to  $x_i$  and to the vertex in  $C_i \setminus K_i$ , neither of which is in F, we have  $d_F(v) \leq \Delta - (\Delta - 4) = 4$ . This gives  $d_F(v) \leq \frac{\Delta}{2} - 3 = \min\{\frac{\Delta}{2} - 2, \Delta - 5 - (\frac{\Delta}{2} - 2)\}$  since  $\Delta \geq 14$ . Now Lemma 13 gives an independent transversal I of the  $K_i$ , which is a hitting set.

Now we can prove the first of our two main results. For convenience, we restate it.

#### **Theorem 1.** Every graph with $\chi \geq \Delta \geq 13$ contains $K_{\Delta-3}$ .

Proof. Let G be a counterexample minimizing |G|; note that G is vertex critical. By Corollary 10, we have  $\Delta(G) \geq 14$ . By Theorem 3, we know that  $\omega(G) \geq \Delta(G)-4$ . Since G contains no  $K_{\Delta-3}$ , we know that  $\omega(G) = \Delta(G)-4$ . So let I be a hitting set given by Lemma 14, expanded to a maximal independent set. Now  $\omega(G-I) < \Delta(G)-4$ ,  $\Delta(G-I) \leq \Delta(G)-1$ , and  $\chi(G-I) \geq \chi(G)-1$ . If  $\Delta(G-I) \leq \Delta(G)-3$ , then greedy coloring gives  $\chi(G-I) \leq \Delta(G-I)+1 \leq \Delta(G)-2$ , so  $\chi(G) \leq \Delta(G)-1$ . If  $\Delta(G-I) = \Delta(G)-2$ , then  $\chi(G-I) \leq \Delta(G-I)$  by Brooks' Theorem (since  $\omega(G-I) < \Delta(G)-4$ ), so  $\chi(G) \leq \Delta(G)-1$ . So instead  $\Delta(G-I) = \Delta(G)-1$ . Now  $\chi(G-I) \geq \Delta(G-I) \geq 13$  and  $\omega(G-I) < \Delta(G-I)-3$ , contradicting the minimality of |G|.

We suspect that Theorem 1 holds for all  $\Delta$ . By Theorem 3 and Theorem 1, the following conjecture is only open when  $\Delta \in \{6, 8, 9, 11, 12\}$ .

#### Conjecture 1. Every graph with $\chi \geq \Delta$ contains $K_{\Delta-3}$ .

We conclude this section with a nice application of Theorem 1 to the Borodin-Kostochka conjecture for vertex-transitive graphs. Suppose G is a vertex-transitive graph with  $\chi(G) \geq \Delta(G) \geq 15$ . Now  $\omega(G) \geq \Delta(G) - 3$  by Theorem 1. Since G is vertex-transitive, every vertex of G is in a  $K_{\Delta(G)-3}$ . In [26], it was proved that the Borodin-Kostochka conjecture holds for graphs where every vertex is in a  $K_{\frac{2}{3}\Delta(G)+2}$ . Now  $\Delta(G) - 3 \geq \frac{2}{3}\Delta(G) + 2$  since  $\Delta(G) \geq 15$ , so we have proved the following.

Corollary 15. Every vertex-transitive graph with  $\chi \geq \Delta \geq 15$  contains  $K_{\Delta}$ .

Corollary 15 should hold for  $\Delta \geq 9$  and this may be much easier to prove than the full Borodin-Kostochka conjecture. In a short note [9], we explore these ideas further and prove Corollary 15 for  $\Delta \geq 13$ . A more general conjecture comes out of these considerations which is worth mentioning because it implies Corollary 15 for  $\Delta \geq 9$ .

Conjecture 2. Every vertex-transitive graph satisfies  $\chi \leq \max \{\omega, \lceil \frac{5\Delta+3}{6} \rceil \}$ .

### 4 The Second Main Theorem

In this section, we prove our second main theorem. First, we prove a lemma that follows from [10] about list coloring (we use it to forbid a certain subgraph in a  $\Delta$ -critical graph).

**Lemma 16** ([10]). Let  $G = K_3 \vee E_2$ . If L is a list assignment such that  $|L(v)| \geq d(v) - 1$  for all  $v \in V(G)$  and for some  $w \in V(K_3)$  and some  $x \in V(E_2)$  we have  $|L(w)| \geq d(w)$  and  $|L(x)| \geq d(x)$ , then G has an L-coloring.

*Proof.* Denote  $V(E_2)$  by  $\{x,y\}$ . By the Small Pot Lemma, we assume  $|Pot(L)| \le 4 < 5 \le |L(x)| + |L(y)|$ . After coloring x and y the same, finish greedily, ending with w.  $\square$ 

In the rest of this section, we extend and refine the ideas in Section 2.

**Definition 9.** Let P be a minimum  $(r_1, \ldots, r_s)$ -partition of a graph G with  $\chi(G) = \Delta(G) = 1 + \sum_{i \in [s]} r_i$ . Let S be a move sequence starting at P. For a full club S with respect to S, the *clubgroup*  $\mathcal{G}_S(S)$  of S is the set consisting of S and the full clubs to which S is joined.

When the move sequence is clear from context, we write  $\mathcal{G}(S)$  in place of  $\mathcal{G}_{\mathcal{S}}(S)$ . Clearly if R and S are full clubs and  $R \in \mathcal{G}(S)$ , then  $S \in \mathcal{G}(R)$ . By Lemma 8, we know that if R, S, and T are full clubs, and  $R \in \mathcal{G}(S)$  and  $S \in \mathcal{G}(T)$ , then  $R \in \mathcal{G}(T)$ . So, the set of full clubs with respect to S is partitioned into clubgroups. We need a way of differentiating moves that are internal to a clubgroup and moves that go from one clubgroup to another. This motivates the following definition of *internal* and *external* moves.

With the notation we have at this point, referring to objects like "the clubgroup of the club of the active component" is a bit unwieldy. So, we allow ourselves to write  $\mathcal{G}_{\mathcal{S}}(A)$  in place of  $\mathcal{G}_{\mathcal{S}}(\mathcal{C}_{\mathcal{S}}(A))$ .

**Definition 10.** Let P be a minimum  $(r_1, \ldots, r_s)$ -partition of a graph G with  $\chi(G) = \Delta(G) = 1 + \sum_{i \in [s]} r_i$ . Let S be a move sequence starting at P. Let  $M = (P^a, v, i, P^b)$  be a move in S,  $A^a$  the active component in  $P^a$  and  $A^b$  the active component in  $P^b$ . Then move M is *internal* if  $\mathcal{G}_S(A^a) = \mathcal{G}_S(A^b)$ . Otherwise, M is *external*. We write  $\mathcal{E}(S)$  for the subsequence of S consisting of all the external moves of S.

**Definition 11.** Let P be a minimum  $(r_1, \ldots, r_s)$ -partition of a graph G with  $\chi(G) = \Delta(G) = 1 + \sum_{i \in [s]} r_i$ . Let  $S = ((P^1, v_1, i_1, P^2), \ldots, (P^q, v_q, i_q, P^{q+1}))$  be a move sequence starting at P. Let R be a full club of S. We say that the clubgroup  $\mathcal{G}_S(R)$  is activated at least k times if there is a subsequence  $((P^{a_1}, v_{a_1}, i_{a_1}, P^{a_1+1}), \ldots, (P^{a_k}, v_{a_k}, i_{a_k}, P^{a_k+1})$  of  $\mathcal{E}(S)$  where the active club in  $P^{a_i+1}$  is in  $\mathcal{G}_S(R)$  for  $i \in [k]$ .

**Definition 12.** Let P be a minimum  $(r_1, \ldots, r_s)$ -partition of a graph G with  $\chi(G) = \Delta(G) = 1 + \sum_{i \in [s]} r_i$ . Let  $S = ((P^1, v_1, i_1, P^2), \ldots, (P^q, v_q, i_q, P^{q+1}))$  be a move sequence starting at P. Let R be a full club of S. The external spread of R is

$$\operatorname{esp}_{\mathcal{S}}(R) = \{i \mid (Q, v, i, Q') \in \mathcal{E}(\mathcal{S}) \text{ with } \mathcal{C}^{i}(\mathcal{A}(Q)) \in \mathcal{G}_{\mathcal{S}}(R)\}.$$

The external spread of S is  $\exp(S) = \max_{R} |\exp(R)|$  where the max is over all full clubs R of S.

In an  $(r_1, \ldots, r_s)$ -partition of a graph G a clubgroup containing s-1 clubs is called a big clubgroup. A clubgroup with fewer than s-1 clubs is small. Our next key lemma is Lemma 18, which is an analogue of Lemma 9. Intuitively, it says that clubgroups can be thought of much like clubs: in a move sequence with external spread at most 2 (and each vertex moved at most once), each clubgroup is activated at most 3 times. The proof is similar to that of Lemma 9. Not suprisingly, we must first prove an analogue of the helper lemma that played a key role in that proof. This is Lemma 17 which follows quickly from Lemma 5.

**Lemma 17.** Let P be a minimum  $(r_1, \ldots, r_s)$ -partition of a graph G with  $\chi(G) = \Delta(G) = 1 + \sum_{i \in [s]} r_i$ . Let

$$S = ((P^1, v_1, i_1, P^2), \dots, (P^q, v_q, i_q, P^{q+1}))$$

be a move sequence starting at P. Let R and S be full clubs of S and  $t \in [q+1]$ . If  $R_t = \mathcal{A}(P^t)$ , then

- (a) if  $u \in V(R_t)$  and u has at least 2 neighbors in  $S_t$ , then u is joined to  $S_t$ .
- (b) if  $u \in V(R_t)$  and  $v \in V(S_t)$  and u has at least 2 neighbors in  $S_t$  and v has at least 2 neighbors in  $V(\mathcal{G}(R_t)) \setminus \{u\}$ , then v is joined to  $V(\mathcal{G}(R_t))$ .

*Proof.* (a) is the same as (a) in Lemma 6; we only restate it here for convenience.

(b): By symmetry, we may assume that  $V(\mathcal{G}(R_t))$  intersects each of  $P_1^t, \ldots, P_{k-1}^t$  and none of  $P_k^t, \ldots, P_s^t$ . Moreover, we assume that  $V(S_t) \subseteq P_k^t$ . Let  $A_i = G\left[P_i^t\right]$  for  $i \in [k]$ . Let  $H = G\left[V(\{A_1, \ldots, A_k\})\right]$  and let  $T_1$  be the component of  $A_1$  containing u. Plainly,  $\chi(H) = \sum_{i \in [k]} \chi(A_i)$ . By Lemma 4,  $\chi(A_1 \setminus \{u\}) < \chi(A_1)$  and  $d_{A_k}(u) \le \chi(A_k) + 1$ . By Lemma 5 (a), vertex u has at least  $\chi(A_k)$  neighbors in some component  $T_k$  of  $A_k$ . Since  $d_{A_k}(u) \le \chi(A_k) + 1$  and u has at least two neighbors in  $S_t$ , we must have  $T_k = S_t$ .

If  $d_{A^*}(v) > 1 + \sum_{i \in [k-1]} \chi(A_i)$ , then there exists some part  $P_q^t$  with  $d_{P_q^t}(v) < r_q$ . By moving v to  $P_q^t$  and u to  $P_k^t$ , we get a  $(\chi(G) - 1)$ -coloring of G, a contradiction. So  $d_{A^*}(v) \le 1 + \sum_{i \in [k-1]} \chi(A_i) \le |T^*|$ . Similarly,  $d_{A^*}(w) \le |T^*|$  for all  $w \in T^*$ . To finish the proof of (b), we now apply Lemma 5 (c), with  $T^* = V(\mathcal{G}(R_t))$ .

**Lemma 18.** Let P be a minimum  $(r_1, \ldots, r_s)$ -partition of a graph G with  $\chi(G) = \Delta(G) = 1 + \sum_{i \in [s]} r_i$ . If

$$S = ((P^1, v_1, i_1, P^2), \dots, (P^q, v_q, i_q, P^{q+1}))$$

is a move sequence with  $esp(S) \leq 2$  and  $v_i \neq v_j$  for all distinct  $i, j \in [q+1]$ , then:

- (1) every clubgroup of S is activated at most 3 times; and
- (2) every big clubgroup of S is activated at most 2 times.

*Proof.* Suppose the lemma is false and choose a move sequence

$$S = ((P^1, v_1, i_1, P^2), \dots, (P^q, v_q, i_q, P^{q+1}))$$

for which it fails minimizing q. By minimality of q (and since  $\exp(S) \leq 2$ ), we have a length three subsequence  $((P^a, v_a, i_a, P^{a+1}), (P^b, v_b, i_b, P^{b+1}), (P^{c-1}, v_{c-1}, i_{c-1}, P^c))$  of S such that

- (i)  $\mathcal{G}(\mathcal{A}(P^a)) = \mathcal{G}(\mathcal{A}(P^b)) = \mathcal{G}(\mathcal{A}(P^c))$  and  $\mathcal{C}^{a+1}(\mathcal{A}(P^{a+1})) = \mathcal{C}^{b+1}(\mathcal{A}(P^{b+1}))$ ; and
- (ii) there is at most one  $(P^d, v_d, i_d, P^{d+1})$  in S with a < d < b such that  $\mathcal{G}(\mathcal{A}(P^d)) = \mathcal{G}(\mathcal{A}(P^a))$ ; and
- (iii)  $C^{a+1}(A(P^{a+1}))$  is active at most 3 times.

Let  $X = \mathcal{G}(\mathcal{A}(P^a))$  and  $Y = \mathcal{C}^{a+1}(\mathcal{A}(P^{a+1}))$ . We will show that X is joined to Y; this gives a contradiction, since we are assuming Y is not in the clubgroup of X. We may choose c to be the smallest index in  $\{b+1,...,q+1\}$  such that X is active at stage c. If there does not exist  $(P^d, v_d, i_d, P^{d+1})$  in S with a < d < b such that  $\mathcal{C}^d(\mathcal{A}(P^d)) = \mathcal{C}^a(\mathcal{A}(P^a))$ , then let d = b. The proof of (1) is nearly identical to the proof of Lemma 9. The only difference is that each instance of Lemma 6 in that proof is now replaced by Lemma 17; so we omit the proof.

Now for the proof of (2). If a clubgroup is big, then each of its external moves goes to the same part  $X_i$  of the partition. Thus, if a big clubgroup becomes active 3 times, then we again have the move subsequence  $((P^a, v_a, i_a, P^{a+1}), (P^b, v_b, i_b, P^{b+1}), (P^{c-1}, v_{c-1}, i_{c-1}, P^c))$ , with properties (i), (ii), and (iii) above. Hence, the proof of (1) is also valid in this context, and yields a proof of (2).

Now we can prove our second main theorem (we restate it for convenience), which strengthens Theorem 19 for  $\Delta \geq 10$ .

**Theorem 19** (Kostochka, Rabern, and Stiebitz [21]). If G is a critical graph with  $\chi(G) \geq \Delta(G)$  and  $\omega(G) < \Delta(G)$ , then  $\omega(\mathcal{H}(G)) \geq \left\lfloor \frac{\Delta(G)-1}{2} \right\rfloor$ .

**Theorem 2.** If G is a graph with  $\chi(G) \geq \Delta(G)$  and  $\omega(G) < \Delta(G)$ , then  $\omega(\mathcal{H}(G)) \geq \Delta(G) - 4$  if  $\Delta(G) \equiv 1 \pmod{3}$  and  $\omega(\mathcal{H}(G)) \geq \Delta(G) - 5$  otherwise.

*Proof.* Suppose the theorem is false and let G be a minimal counterexample. Note that G is a critical graph with  $\chi(G) \geq \Delta(G)$ ,  $\omega(G) < \Delta(G)$ , and  $\omega(\mathcal{H}(G)) < \Delta(G) - 4$  if  $\Delta(G) \equiv 1 \pmod{3}$  and  $\omega(\mathcal{H}(G)) < \Delta(G) - 5$  otherwise. By Brooks' Theorem, we have  $\chi(G) = \Delta(G)$ . By Theorem 19,  $\Delta(G) \geq 10$ .

Let  $s = \left\lfloor \frac{\Delta(G)-1}{3} \right\rfloor$  and  $r_1, \ldots, r_s \in \{3,4\}$  such that  $\Delta(G) = 1 + \sum_{i \in [s]} r_i$ . Since G has an  $(r_1, \ldots, r_s)$ -partition, we can let P be a minimum  $(r_1, \ldots, r_s)$ -partition of G. Let  $\mathcal{S} = ((P^1, v_1, i_1, P^2), \ldots, (P^q, v_q, i_q, P^{q+1}))$  be a move sequence starting at P that never moves a low vertex within a clubgroup, with  $\exp(\mathcal{S}) \leq 2$ , with  $v_i \neq v_j$  for all distinct  $i, j \in [q+1]$ , and, subject to that, P has the maximum number of external moves. Let  $A = \mathcal{A}(P^{q+1})$ . Hereafter  $\mathcal{G}(\mathcal{C}(A))$  denotes  $\mathcal{G}(\mathcal{C}^{q+1}(A))$ .

Suppose  $\mathcal{G}(\mathcal{C}(A))$  is small. By Lemma 18,  $\mathcal{G}(\mathcal{C}(A))$  is activated at most 3 times in  $\mathcal{S}$ . Since  $r_{i_q} \geq 3$ , there is  $x \in V(A)$  such that  $x \notin \{v_t \mid t \in [q]\}$ , i.e., since A has at least 4 vertices, some  $x \in V(A)$  has not yet moved. Since  $\mathcal{G}(\mathcal{C}(A))$  is small, there exist at least two indices  $j_1, j_2 \in [s]$  such that  $\mathcal{G}(\mathcal{C}(A))$  has no club in part  $j_1$  and no club in part  $j_2$ . Now for some  $i \in \{1, 2\}$ , we have  $d_{P_{j_i}}(x) \leq r_{j_i}$ . By Lemma 4(2), we know that x is joined to a full club in part  $j_i$ , so moving x to part  $j_i$  is a valid move.

We show that in all cases, we can extend the move sequence  $\mathcal{S}$  to a sequence  $\mathcal{S}'$  by moving x to part  $j_i$ ; we need only to verify that after moving x to part  $j_i$ , the resulting sequence  $\mathcal{S}'$  satisfies  $\exp(\mathcal{S}') \leq 2$ . If we presently have  $\exp(\mathcal{S}) \leq 1$ , then clearly  $\exp(\mathcal{S}') \leq 2$ . If instead the present sequence has  $\exp(\mathcal{S}) \leq 2$ , then we can choose  $j_1$  and  $j_2$  such that  $\mathcal{S}$  contains external moves to both. In that case, moving x to one of the parts will not increase the external spread. So in all cases, we contradict the maximality of the move sequence.

Hence  $\mathcal{G}(\mathcal{C}(A))$  is big. By Lemma 18,  $\mathcal{G}(\mathcal{C}(A))$  is activated at most 2 times in  $\mathcal{S}$ . Consider  $K = \bigcup_{Z \in \mathcal{G}(\mathcal{C}(A))} V(Z_{q+1})$ . Since  $\mathcal{G}(\mathcal{C}(A))$  is big, K is a clique that has vertices in all but one part of  $P^{q+1}$ . By renumbering if necessary, we may assume that K has vertices in each of  $P_1^{q+1}, \ldots, P_{s-1}^{q+1}$ ; so  $|K| = 1 + \sum_{i \in [s-1]} r_i$ . Hence  $|K| = \Delta(G) - 3$  if  $\Delta(G) \equiv 1 \pmod{3}$  and  $|K| \geq \Delta(G) - 4$  otherwise. In either case, K has at least two low vertices, since G is a counterexample to the theorem.

If K contains a low vertex x that has not moved, i.e.,  $x \in K \setminus \{v_t \mid t \in [q]\}$ , then we can move x to part s (by Lemma 4(2)), which contradicts the fact that  $\mathcal{S}$  has maximum length. Hence, K does not contain an unmoved low vertex. Since low vertices are not moved within clubgroups, and each low vertex in K has already moved, we know that each was moved externally. So, since  $\mathcal{G}(\mathcal{C}(A))$  is activated at most 2 times in  $\mathcal{S}$ , we know that K has exactly two low vertices, v and v. Since both v and v have moved,  $\mathcal{G}(\mathcal{C}(A))$  is activated exactly 2 times; one time when v was moved in and one time when v was moved in. Therefore,  $\mathcal{S}$  contains external moves  $(P^{a_1}, v, i_{a_1}, P^{a_1+1})$  and  $(P^{a_2}, w, i_{a_2}, P^{a_2+1})$  and in both  $P^{a_1+1}$  and  $P^{a_2+1}$  the clubgroup  $\mathcal{G}(\mathcal{C}(A))$  contains the active club (possibly different each time). By symmetry, assume v0 and v1 and v2 and so v3 and v4 are the clubgroup  $\mathcal{G}(\mathcal{C}(A))$  contains the

Let B be the active component in  $P^q$ . Since  $w \in V(B)$  and w is adjacent to at least  $\Delta(G) - 5$  vertices in K, we see that  $\mathcal{C}(B)$ 's clubgroup is  $\{\mathcal{C}(B)\}$  (otherwise w would be adjacent to more than 5 vertices coming from  $\mathcal{C}(B)$ 's clubgroup, which is too many). Suppose that V(B) contains a high vertex that is unmoved, i.e.,  $z \in V(B) \setminus \{v_t \mid t \in [q-1]\}$ . Since  $\Delta(G) \geq 10$ , we have  $s \geq 3$ . So there is an external move  $M = (P^q, z, i, Q)$  where  $i \in [s-1]$ . Consider the move sequence formed from  $\mathcal{S}$  by removing the last move and appending M. By our considerations in the previous paragraph, this move sequence can be extended (the active club now contains an unmoved low vertex, since the last vertex moved is high), contradicting the maximality condition on  $\mathcal{S}$ . So, every  $z \in V(B) \setminus \{v_t \mid t \in [q-1]\}$  is low.

Since w is low, for every move (Q, z, i, Q') in S where C(B) is active in Q, we must have  $z \in K$ ; otherwise w would have at least  $\Delta$  neighbors. In particular, there are at most two such moves since  $\mathcal{G}(C(A))$  is activated at most twice. So B contains an unmoved vertex, i.e.,  $|V(B) \setminus \{v_t \mid t \in [q]\}| \geq 1$ .

Let  $(P^{a_3}, u, i_{a_3}, P^{a_3+1})$  be the first external move in  $\mathcal{S}$  after  $(P^{a_1}, v, i_{a_1}, P^{a_1+1})$ . Let A' be the active component in  $P^{a_3}$  and consider  $K' = \bigcup_{Z \in \mathcal{G}(\mathcal{C}(A'))} V(Z_{a_3})$ . Since |K'| = |K|, as we saw before for K, also K' has at least two low vertices v, w'. If u is high, then K would contain low vertices v, w, w', a contradiction. So u is low; in fact, u = w'.

We show that  $\mathcal{C}(\mathcal{A}(P^{a_3+1})) = \mathcal{C}(B)$ . Since v is low, we have the move  $M' = (P^{a_3}, v, s, Q')$ . Let  $B' = \mathcal{A}(Q') \setminus \{v\}$ . Since v is adjacent to w (and v is low), we must have  $w \in V(B')$ . So  $\mathcal{C}(B) = \mathcal{C}(B')$ . Since  $\mathcal{C}(B')$  is active at most twice, v has at least |B'|-2>0 neighbors in  $\mathcal{C}(B')_{q+1}$ . Since v is low, we have the move  $M=(P^{q+1}, v, s, Q)$ . Now Lemma 4, part (2) shows that  $\{v\} \cup V(\mathcal{C}(B')_{q+1})$  induces a  $K_{r_s+1}$ . But  $u \in P_s^{q+1}$  and v is adjacent to u, so  $u \in V(\mathcal{C}(B')_{q+1})$ . Therefore,  $\mathcal{C}(\mathcal{A}(P^{a_3+1})) = \mathcal{C}(B') = \mathcal{C}(B)$ .

Now we have the  $K_3$  on  $\{u, v, w\}$  joined to a set of vertices T with  $|T| = \Delta(G) - 3$ . Namely,  $T = (V(K) \setminus \{v, w\}) \cup (V(B) \setminus \{u\})$ . Moreover, since  $|V(B) \setminus \{v_t \mid t \in [q]\}| \ge 1$ , there is a low vertex in  $V(B \setminus \{v_q, u\})$  and  $V(B \setminus \{v_q, u\}) \subseteq T$ . So, by Lemma 16,  $\{u, v, w\} \cup T$  induces a  $K_{\Delta(G)}$ , a contradiction.

We conjecture that the previous theorem actually holds with  $\omega(\mathcal{H}(G)) \geq \Delta - 5$  replaced by  $\omega(\mathcal{H}(G)) \geq \Delta - 4$ . In [28], the second author proved this result for  $\Delta = 6$ ; later in [21] it was proved for  $\Delta = 7$ . The condition  $\omega(\mathcal{H}(G)) \geq \Delta - 4$  would be tight since the graph  $O_5$  in Figure 3 is a counterexample to  $\omega(\mathcal{H}(G)) \geq \Delta - 3$  when  $\Delta = 5$ . In fact, it was shown in [21] that  $O_5$  is the only counterexample to  $\omega(\mathcal{H}(G)) \geq \Delta - 3$  when  $\Delta = 5$ .

Conjecture 3. Let G be a graph. If  $\chi \geq \Delta$ , then  $\omega \geq \Delta$  or  $\omega(\mathcal{H}(G)) \geq \Delta - 4$ .

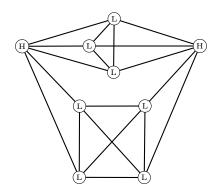


Figure 3: The graph  $O_5$  is a  $\Delta$ -critical graph with  $\Delta = 5$  and  $\omega(\mathcal{H}(G)) = 1$ .

## 5 Algorithms

All of our coloring proofs do translate into algorithms to construct the colorings. However these algorithms cannot obviously be made to run in polynomial time. Attempts to do so encounter two main obstacles. The first comes in our proof of Theorem 3, when we consider a critical subgraph H of our given graph G. We do not know an efficient algorithm to find such a critical subgraph; however, we will see how to overcome this difficulty. Our second obstacle comes from King's Lopsided Transversal Lemma. While his proof is constructive, the algorithm it implies may require exponential time.

We are not aware of any workaround to efficiently find our hitting set; however, when  $\Delta$  is sufficiently large, we can use an idea of Alon instead. We implement a modified version of the algorithm from Theorem 3.

**Theorem 20.** There is a  $\mathcal{O}(V^2E^2)$  time graph algorithm that finds either a  $(\Delta - 1)$ -coloring or a clique on  $\Delta - 4$  vertices  $(\Delta - 3)$  vertices if  $\Delta \equiv 1 \pmod{3}$ .

Proof. Let G be an n-vertex graph with  $\Delta \geq 10$ , and let I be a maximal independent set in G. Let  $G_0 = G - I$ , and note that  $\Delta(G_0) \leq \Delta(G) - 1$ . Lovász's proof of Brooks' theorem [23] can be implemented in time  $\mathcal{O}(V + E)$  (see [3]). Applying this to  $G_0$  we either get a  $\Delta(G)$ -clique or a  $(\Delta(G) - 1)$ -coloring of  $G_0$ . In the former case, we are done, so suppose we have a  $(\Delta(G) - 1)$ -coloring  $\phi$  of  $G_0$ .

Let v be an arbitrary vertex in I and put  $G_1 = G[V(G_0) \cup \{v\}]$ . We give an algorithm that either finds a  $(\Delta(G) - 1)$ -coloring of  $G_1$  or a clique on  $\Delta(G) - 4$  vertices  $(\Delta(G) - 3)$  vertices if  $\Delta(G) \equiv 1 \pmod{3}$ . Iterating this gives the desired algorithm.

Note that  $G_1$  has an  $(r_1, \ldots, r_s)$ -partition P, where  $s = \left\lfloor \frac{\Delta(G)-1}{3} \right\rfloor$  and  $r_1, \ldots, r_s \in \{3,4\}$ ; choose an arbitrary such partition which respects the color classes of  $\phi$ . Now we will construct a move sequence as in the proof of Theorem 3, treating the resulting partitions as if they were minimum partitions. For each partition arising from the move sequence, we check whether any property in Lemma 4 is violated; if some property is violated for a partition P, then we can modify P to form a new partition P' such that P' has fewer edge within parts, i.e.,  $\sigma(P') < \sigma(P)$ . When this happens, we begin our move sequence anew, starting from P'. Eventually, we will reach a partition and a move sequence that does not allow us to reduce the number of edges within parts. Such a move sequence will terminate with either (1) a clique on  $\Delta(G) - 4$  vertices  $(\Delta(G) - 3)$  vertices if  $\Delta(G) \equiv 1 \pmod{3}$  or (2) a  $(\Delta(G) - 1)$ -coloring of  $G_1$ . In the case of (1), our algorithm halts. In the case of (2), we add a new vertex v' from  $I \setminus \{v\}$  and continue.

So, we need only analyze the running time. Each move sequence has length at most n, since each vertex moves at most once. After adding a vertex, we can reduce the number of edges within parts at most |E(G)| times. Hence, after we add a new vertex from I to our partition, we need at most n |E(G)| moves until we find either a big clique or a  $(\Delta(G) - 1)$ -coloring. After each move, we can verify that the resulting partition satisfies all the properties of Lemma 4 (or doesn't) and find a vertex to swap with in  $\mathcal{O}(V + E)$  time. Since we need to do this at most n|I||E(G)| times, the running time of the algorithm is  $\mathcal{O}(V^2E^2)$ .

When  $\Delta \not\equiv 1 \pmod{3}$ , Theorem 20 only finds a  $K_{\Delta-4}$ ; but Theorem 1 guarantees a  $K_{\Delta-3}$  when  $\Delta \geq 13$ . To get an algorithmic version of this result, we need to efficiently find a hitting set when  $\chi = \Delta$  and  $\omega = \Delta - 4$ . We will show how to do this when  $\Delta$  is sufficiently large. The proof we present here works for  $\Delta \geq 37$ . We also sketch how to refine this idea to work for  $\Delta \geq 33$ . Further, using a result of Kolipaka, Szegedy and Xu [19], we show how to get down to  $\Delta \geq 26$ . The general idea is to find a set of disjoint cliques  $\mathcal{A} = \{A_1, A_2, \ldots\}$  such that  $|A_i|$  is large for all i and each maximum clique contains some  $A_i$ . Following an idea of Alon, we choose one vertex uniformly at random from each  $A_i$  and use the Lovasz Local Lemma to prove that with positive

probability the chosen vertices form an independent set. Our proof uses one classical lemma each from Hajnal [13] and Kostochka [20].

**Lemma 21** (Hajnal [13]). If S is a collection of maximum cliques in a graph G, then

$$\left|\bigcup \mathcal{S}\right| + \left|\bigcap \mathcal{S}\right| \ge 2\omega.$$

*Proof.* We use induction on  $|\mathcal{S}|$ ; the base case  $|\mathcal{S}| = 1$  is trivial. Let  $S_1 \in \mathcal{S}$  and  $\mathcal{S}' = \mathcal{S} - S_1$ . Consider the set  $(\cap \mathcal{S}' \setminus S_1) \cup (S_1 \cap (\cup \mathcal{S}'))$ , which induces a clique. Since  $S_1$  is a maximum clique,  $|S_1| \geq |(\cap \mathcal{S}' \setminus S_1) \cup (S_1 \cap (\cup \mathcal{S}'))|$ , which yields  $|S_1 \setminus (\cup \mathcal{S}')| \geq |(\cap \mathcal{S}') \setminus S_1|$ . By hypothesis,  $|\cup \mathcal{S}'| + |\cap \mathcal{S}'| \geq 2\omega$ . Adding this to the previous inequality gives the desired result.

Now we need the following definition. Given a collection S of sets, the *intersection* graph  $X_S$  has one vertex for each set of S and two vertices are adjacent if their sets intersect.

**Lemma 22** (Kostochka [20]). Let G be a graph with  $\omega(G) > \frac{2}{3}(\Delta(G) + 1)$ . If S is a collection of maximum cliques in G and the intersection graph  $X_S$  is connected, then  $|\bigcap S| \geq 2\omega(G) - (\Delta(G) + 1)$ .

Proof. We use induction on  $|\mathcal{S}|$ ; the base case |S|=1 is trivial. The key is to show that  $|\bigcap \mathcal{S}| > 0$ , for then  $|\bigcup \mathcal{S}| \leq \Delta(G)+1$ , so the lemma follows directly from Lemma 21. Let  $S_1 \in \mathcal{S}$  be a noncutvertex of  $X_{\mathcal{S}}$ , and choose  $S_2 \in \mathcal{S}$  that intersects  $S_1$ . Lemma 21 for the set  $\{S_1, S_2\}$  implies  $|S_1 \setminus S_2| = |S_1| - |S_1 \cap S_2| \leq \omega(G) - (2\omega(G) - (\Delta(G) + 1)) = \Delta(G) + 1 - \omega(G)$ . Let  $\mathcal{S}' = \mathcal{S} - S_1$ . Now  $X_{\mathcal{S}'}$  is connected, so by hypothesis, the lemma holds for  $\mathcal{S}'$ . Choose  $v \in \bigcap \mathcal{S}'$ . Now  $|\bigcup \mathcal{S}'| \leq d_G(v) + 1 \leq \Delta(G) + 1$ . Thus,  $|\bigcup \mathcal{S}| \leq |\bigcup \mathcal{S}'| + |S_1 \setminus S_2| \leq (\Delta(G) + 1) + (\Delta(G) + 1 - \omega(G)) < 2\omega(G)$ . By Lemma 21,  $|\bigcap \mathcal{S}| > 0$ , so the lemma follows.

In [20], Kostochka used Lemma 21 and Lemma 22 to prove that a hitting set always exists when  $\omega \geq \Delta + \frac{3}{2} - \sqrt{\Delta}$ . Using an independent transversal result of Haxell [14], this was improved to  $\omega \geq \frac{3}{4}(\Delta+1)$  in [27] and finally to the best possible  $\omega > \frac{2}{3}(\Delta+1)$  in [17]. Using an independent transversal result of Alon [1] (see also [2], p. 70), we get  $\omega \geq \frac{2e+1}{2e+2}(\Delta+1)$ . Since Alon's proof is based on the Local Lemma, we can use the efficient algorithms developed by Moser and Tardos [24].

**Lemma 23.** If G is a graph with  $\omega \geq \frac{2e+1}{2e+2}(\Delta+1)$ , then G contains an independent set I such that I intersects every maximum clique in G.

*Proof.* Let S be the set of maximum cliques in G and let  $S_i$  be the set of vertices in one component  $C_i$  of  $X_S$ . For each i, Lemma 22 gives  $|\bigcap S_i| \ge 2\omega - (\Delta + 1) \ge \frac{e}{e+1}(\Delta + 1)$ .

Let  $k = \lceil \frac{e}{e+1}(\Delta+1) \rceil$ . For each component  $C_i$ , let  $A_i$  be a set of k vertices that lie in every clique of  $C_i$ . Use the Local Lemma (see [2], p. 64–65) to choose the desired independent set. From each  $A_i$ , choose a vertex uniformly at random. For each edge uv with  $u \in A_i$  and  $v \in A_j$  (and  $i \neq j$ ), let  $E_{uv}$  be the bad event that both u and v are chosen for I; event  $E_{uv}$  occurs with probability  $p = 1/(|A_i| |A_j|) = k^{-2}$ . Each  $E_{uv}$  is independent of all other bad events except for those corresponding to edges with an

endpoint in  $A_i$  or  $A_j$ . Since each u has at least  $\omega - 1$  neighbors in  $S_i$  and v has at least  $\omega - 1$  neighbors in  $S_j$ , the degree d of  $E_{uv}$  in the dependency graph is at most  $(\Delta + 1 - \omega)(|A_i| + |A_j|) - 1 \le \frac{2k}{2e+2}(\Delta + 1) - 1 = \frac{k}{e+1}(\Delta + 1) - 1$ . This gives  $ep(d+1) \le 1$ , so the desired independent set I exists.

**Corollary 24.** If G is a graph with  $\Delta \geq 37$  and  $\omega = \Delta - 4$ , then G contains an independent set I such that I intersects every maximum clique in G. Furthermore, I can be found in polynomial time.

Proof. If  $\Delta \geq 37$ , then we have  $\omega = \Delta - 4 \geq \frac{2e+1}{2e+2}(\Delta+1)$ , so we can apply Lemma 23. All that remains is to show that we can implement its proof in polynomial time. We can find the set of all maximum cliques by considering each  $(\Delta - 4)$ -element subset of the closed neighborhood of each vertex. We use a union-find algorithm to find the components of the intersection graph of this set of maximum cliques. Now consider a set  $\mathcal{S}$  of maximum cliques such that the intersection graph  $X_{\mathcal{S}}$  is connected. We can slightly modify the union-find algorithm so that it also returns  $\cap \mathcal{S}$ . To now find our hitting set, we apply the algorithm for the Local Lemma from Moser and Tardos [24].

With a more complicated algorithm we can do better. Specifically, instead of using Lemma 21 and Lemma 22, we use Lemma 12 as in the proof of Lemma 14. Basically, we just need to do a preprocessing step where we find and remove all  $d_1$ -choosable induced subgraphs on at most 9 vertices (we can color them after coloring the rest). Once we have a graph with none of these  $d_1$ -choosable induced subgraphs, we know, as in the proof of Lemma 14, that the components of  $X_{\mathcal{S}}$  have at most two vertices. So, we can replace our estimate  $|\bigcap \mathcal{S}_i| \geq 2\omega - (\Delta + 1)$  with  $|\bigcap \mathcal{S}_i| \geq \omega - 1$ . This improves the needed condition in Lemma 23 to  $\omega \geq \frac{2e}{2e+1}\Delta + 1$  and thus allows Corollary 24 to work for  $\Delta \geq 33$ .

Using a recent result of Kolipaka, Szegedy and Xu [19] we can do a bit better. The idea is that the local lemma can be strengthened when the dependency graph has nice structure. In our case, the dependency graph is the line graph of a multigraph (the multigraph formed by contracting all the  $A_i$  in  $G[\bigcup_i A_i]$ ). Because of this structure, we may apply the Clique Lovász Local Lemma from [19] to prove Lemma 23 with  $\omega \geq \frac{4}{5}\Delta + 1$ . Since there is an efficient algorithm for the Clique Lovász Local Lemma as well, we get Corollary 24 for  $\Delta \geq 26$ . So, we can prove the following conjecture for  $\Delta \geq 26$ .

Conjecture 4. For  $\Delta \geq 13$ , there is a polynomial time graph algorithm that finds either a  $(\Delta - 1)$ -coloring or a clique on  $\Delta - 3$  vertices.

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