SUBSONIC FREE SURFACE WAVES IN LINEAR ELASTICITY

SÖNKE HANSEN

ABSTRACT. For general anisotropic linear elastic solids with smooth boundaries, Rayleigh-type surface waves are studied. Using spectral factorizations of matrix polynomials, a self-contained exposition of the case of a homogeneous half-space is given first. The main result is about inhomogeneous anisotropic bodies with curved surfaces. The existence of subsonic free surface waves is shown by giving ray series asymptotic expansions, including formulas for the transport equation.

1. Introduction

Rayleigh (1887) discovered waves which propagate along a plane traction-free surface of an isotropic and homogeneous elastic solid. The surface wave speeds are subsonic, that is, they are strictly less than the wave speeds of interior body waves. Furthermore, the amplitudes attenuate exponentially with distance to the surface. Synge (1957) raised doubts about existence of Rayleigh-type waves in anisotropic solids. Stroh (1962) pointed out that these doubts were unfounded, and he introduced a sextic eigenvalue problem which became useful in the theory of free surface waves in anisotropic solids. In the early 1970's, the existence and uniqueness problem of free surface homogeneous plane waves in a semi-infinite half-space was settled by Barnett, Lothe, and coworkers. For any given horizontal propagation direction they showed that there is at most one free surface wave speed, and they gave criteria for the existence of such waves. Lothe & Barnett (1976) rederived their results by the surface impedance tensor method. The surface impedance tensor relates the surface displacement to the surface traction required to sustain it. This tensor was introduced by Ingebrigtsen & Tonning (1969). Detailed presentations of the existence and uniqueness results were given by Chadwick & Smith (1977) and Barnett & Lothe (1985). Much later, Mielke & Fu (2004) simplified some proofs of the Barnett-Lothe theory by using a Ricatti equation satisfied by the impedance tensor. A crucial property of the tensor, the positive definiteness of its real part, follows from an integral identity which, in the original treatments, arises somewhat magically by averaging over rotations in the plane spanned by the normal to the surface and the propagation direction. Existence of subsonic free surface waves was shown by Kamotskii & Kiselev (2009) with a completely different approach based on the variational principle.

Concerning Rayleigh-type waves in inhomogeneous elastic solids with curved boundaries, Petrowsky (1945) exhibited the following locality principle: If a surface wave exists, the velocity of its discontinuity at a given point must be equal to the velocity in the homogeneous elastic half-space which is obtained by freezing the

Date: June 26, 2018.

 $^{2010\ \}textit{Mathematics Subject Classification}.\ \textit{Primary: 74J15},\ \textit{Secondary: 35Q74}.$

elastic parameters at that point. This locality principle is efficiently implemented by asymptotic ray methods, which substitute, in the high-frequency regime, standard plane waves by geometrico-optical 'plane waves'. For inhomogeneous, isotropic elastic solids with curved boundaries, ray methods involving sums of complex plane waves were successfully applied to the free surface wave problem by Babich (1961), Babich & Rusakova (1962), and by Karal, Jr. & Keller (1964). The ray method works, for smooth data, under assumptions on the geometry and on the elastic parameters which are less restrictive than those needed for finding analytic solutions. More importantly, salient features of high-frequency waves such as wave fronts and amplitudes are captured directly by ray methods. Using the existence and regularity theory of linear hyperbolic equations, as was done by Courant & Lax (1956), one can correct asymptotic solutions, without changing the important high-frequency properties, into genuine exact solutions. Gregory (1971) compared some analytic representations of surface waves and corresponding ray approximations. Rayleigh surface waves in inhomogeneous, anisotropic elastic bodies were studied with the complex ray method by Nomofilov (1979). The amplitude of a geometrico-optical wave is governed by a transport equation which, in the case of free surface waves, is quite complicated. Efforts to solve the transport equations culminated in the work of Babich & Kirpichnikova (2004), where detailed formulas for the amplitude and the Berry phase of a Rayleigh surface wave were obtained.

In the present paper we study Rayleigh-type surface waves in anisotropic elastic solids with smooth surfaces and smooth inhomogeneities. We give a self-contained presentation of the theory of Barnett and Lothe, and we incorporate it into a ray theory. Under the same conditions as for homogeneous half-spaces, the existence of subsonic free surface waves is proved. Transport equations for leading amplitudes are established in a way which enables their numerical solution.

Rayleigh wave speeds are frequency dependent if, as happens for coated materials, material parameters vary significantly over one wavelength; see Destrade (2007) for the study of a model case. Our analysis does not deal with this dispersion phenomenon. As in (Babich & Kirpichnikova, 2004), we restrict attention to the high-frequency asymptotics of Rayleigh-type surface waves assuming that the elastic parameters vary slowly in a (thick) surface layer. Dispersive behaviour related to surface waves, not for Rayleigh waves but for whispering galley modes associated with gliding rays of a scalar wave equation, has been shown quite recently by Ivanovici, Lebeau and Planchon, (Ivanovici, 2012) and (Ivanovici et al., 2014).

We use spectral factorizations of the acoustic tensor, which is regarded as a second order matrix polynomial in the variable conormal to the boundary. This allows to conveniently lump together the relevant complex eigenvalues and to avoid some cumbersome and unnecessary considerations. Moreover, the factorizations reduce the elastodynamic system, in the subsonic regime, to a first order system. In general, when the surface is curved and the solid inhomogeneous, the first order system is not differential but pseudo-differential. The zero traction boundary problem is transformed into a pseudo-differential wave equation on the space-time boundary. The associated principal symbol is the surface impedance tensor. The pseudo-differential wave equation is only defined microlocally over the subsonic region. Still it can be treated ray theoretically since it has the structure of a so-called real principal type system (Dencker, 1982). In the microlocal analysis literature such

an approach was carried out for the isotropic case by Taylor (1979), where the subsonic region is called the elliptic region because the theory of elliptic boundary value problems applies in the reduction to the boundary. Nakamura (1991) treated the Barnett–Lothe method from the point of view of microlocal analysis. The present paper uses ideas developed in (Hansen & Röhrig, 2004; Hansen, 2011, 2012). Although the spirit and the techniques of microlocal analysis are key to our approach, we shall give, in order to make the contents readable for a wide audience, a self-contained presentation except for using few basic results from pseudo-differential calculus.

The paper is organized as follows. Section 2 recalls the free surface traction problem in differential geometric tensor notation. In sections 3 and 4 the subsonic free surface wave theory for homogeneous elastic half-spaces is redeveloped using division theory of matrix polynomials as the basic tool. In the appendix of the paper, we give a complete and self-contained presentation of those parts of division theory which we employ. The remaining sections deal with inhomogeneous solids with curved boundaries. For the benefit of readers who are not familiar with pseudo-differential operators, Section 5 contains an account of core properties of pseudo-differential calculus and its relation to asymptotic expansions. A ray theory for pseudo-differential systems is developed in Section 6. In Section 7 the displacement boundary problem is solved for surface displacements concentrated in the subsonic region. A factorization of the elastodynamic operator is constructed and used in that section. Theorem 8.1 in Section 8 is our main result on the existence of Rayleigh-type waves. The transport equation for the leading amplitude of subsonic free surface waves is treated in some detail in Section 9.

2. Equations of linear elastodynamics

Let B be a linearly elastic body with density $\rho > 0$ and stiffness tensor $C = [C^{ijk\ell}]$. The elasticities satisfy the standard symmetries,

$$(1) C^{ijk\ell} = C^{k\ell ij} = C^{jik\ell},$$

and the strong convexity property,

(2)
$$C^{ijk\ell} \epsilon_{ij} \overline{\epsilon_{k\ell}} > 0 \quad \text{if } \epsilon_{ji} = \epsilon_{ij}, \ \epsilon \equiv [\epsilon_{ij}] \neq 0.$$

(We use the summation convention, and we denote a point, if at all, by its coordinates $x = (x^1, x^2, x^3)$. A bar denotes complex conjugation.) Since the elasticities are real, it suffices to assume that (2) holds for symmetric tensors ϵ which are real. Assumptions (1) and (2) say that C defines an inner product on symmetric 2-tensors.

The strain tensor field ϵ measures the deformation of the geometry of B caused by an (infinitesimal) displacement u. The geometry is given by the Riemannian metric tensor $G = [G_{ij}]$. Precisely, strain is the symmetrized covariant derivative of displacement:

$$\epsilon_{k\ell} \equiv (u_{k;\ell} + u_{\ell;k})/2, \quad u_{k;\ell} \equiv u_{k,\ell} - \Gamma_{k\ell}^j u_j.$$

Here $\Gamma_{k\ell}^i \equiv G^{ij}(G_{\ell j,k} + G_{kj,\ell} - G_{k\ell,j})/2$ are the standard Christoffel symbols of differential geometry. We precede a coordinate index by a comma or a semicolon to denote a partial or a covariant derivative, respectively. The stress field, $\sigma = [\sigma^{ij}]$, represents forces. It is related to strain via Hooke's Law:

(3)
$$\sigma^{ij} = C^{ijk\ell} \epsilon_{k\ell} = C^{ijk\ell} u_{k\cdot\ell}.$$

Isotropy is a symmetry property which some elastic bodies have. In this case the elasticity tensor is given by

$$C^{ijk\ell} = \lambda G^{ij} G^{k\ell} + \mu (G^{ik} G^{j\ell} + G^{i\ell} G^{jk}),$$

where λ and μ are the Lamé parameters.

The inverse of G is $G^{-1} = [G^{ij}]$. By raising and lowering indices one switches between covariant and contravariant components, e.g., $u^i = G^{ik}u_k$ and $u_k = G_{kj}u^j$. The metric G defines the length element ds by $ds^2 = G_{ij} dx^i dx^j$, and the volume element $dV \equiv \sqrt{g} dx$, $g \equiv \det G$.

Let S denote the (smooth) boundary surface of B, $\mathrm{d}S$ its surface element, and ν the interior unit normal field. Assume that B is source-free, and that S is traction-free. Then the displacement field satisfies the free surface boundary problem:

(4)
$$\rho \ddot{u}^i - \sigma^{ij}_{;j} = 0, \qquad \sigma^{ij} \nu_j |_S = 0,$$

i=1,2,3. The differential equations are the 3×3 system of elastodynamics. The equations (4) arise as the Euler equations of the Lagrangian which is the functional given by

$$\mathcal{L}(u) \equiv \int \int_{B} (\rho \dot{u}^{i} \overline{\dot{u}_{i}} - \sigma^{ij} \overline{\epsilon_{ij}}) \, dV \, dt.$$

The divergence of the stress tensor is

(5)
$$\sigma^{ij}_{;j} = \sigma^{ij}_{,j} + \Gamma^{i}_{kj}\sigma^{kj} + \Gamma^{k}_{kj}\sigma^{ij} = g^{-1/2}(g^{1/2}\sigma^{ij})_{,i} + \Gamma^{i}_{kj}\sigma^{kj}.$$

The last equality follows because $(\log g)_{,j} = 2\Gamma_{kj}^k$.

The differential geometric formulation of elastodynamics, which we just recalled, does not depend on the choice of a particular coordinate system. Elasticities, strains, stresses, and displacements transform as tensors under changes of coordinates. No generality is lost when we assume that the surface S agrees, near some chosen point, with a coordinate plane, e.g., $x^3 = 0$. In the following, we always use coordinates which are adapted to S in the following sense: The normal coordinate x^3 is the signed distance to S (such that $x^3 < 0$ in the exterior), and the horizontal coordinates x^1 and x^2 are constant along the geodesics which intersect the level surfaces of x^3 orthogonally. The horizontal coordinates are completely determined by their restrictions to S. The metric tensor satisfies $G^{33} = 1$ and $G^{3\lambda} = G^{\lambda 3} = 0$ if $\lambda < 3$. The surface area element equals $dS = \sqrt{g} dx^1 dx^2$, and $\tau^i \equiv \sigma^{i3}$ is the i-th component of the traction τ at a given level surface x^3 =constant.

In the study of surface waves, it is useful to separate, in the elastodynamic equations and in the formula for the traction, differentiations in normal direction from differentiations in the horizontal (and time) directions. We write

$$(6) \quad \sigma^{ij}_{\ ;j} = C^{i3k3} u_{k,33} + \left(C^{i3k\lambda} + C^{k3i\lambda} \right) u_{k,\lambda3} + C^{i\lambda k\mu} u_{k,\lambda\mu} + B^{ik\ell} u_{k,\ell} + \tilde{B}^{ik} u_k,$$

(7)
$$\sigma^{i3} = C^{i3k3}u_{k,3} + C^{i3k\lambda}u_{k,\lambda} - C^{i3j\ell}\Gamma^k_{j\ell}u_k$$

where the implied summations are restricted to $\lambda, \mu < 3$. The coefficients of the first order derivatives are

$$B^{ik\ell} = C^{ijk\ell}_{\quad \ \, ,j} - C^{i\ell mn} \Gamma^k_{mn} + C^{jmk\ell} \Gamma^i_{jm} + C^{ijm\ell} \Gamma^m_{mj}. \label{eq:beta}$$

We have no need to know the \tilde{B}^{ik} 's explicitly, only their vanishing if the metric tensor G is constant.

3. Surface waves in homogeneous half-space

In this section we assume that the elastic solid is homogeneous and occupies a half-space in Euclidean space. Thus the density, the elasticities, and the metric tensor are constant. In particular, the Christoffel symbols are zero, and covariant derivatives are ordinary derivatives.

We use adapted coordinates. Then $x^3 \ge 0$ corresponds the half-space filled by the elastic body. The unit normal field ν at S, which points into the interior, has the components $\nu_1 = \nu_2 = 0$, $\nu_3 = 1$.

The equations of the elastodynamic free boundary problem are as follows:

(8)
$$\rho \ddot{u}_i - C_i^{3k3} u_{k,33} - (C_i^{3k\lambda} + C_i^{k3\lambda}) u_{k,\lambda 3} - C_i^{\lambda k \mu} u_{k,\lambda \mu} = 0,$$

(9)
$$C_i^{3k3}u_{k,3} + C_i^{3k\lambda}u_{k,\lambda} = 0 \text{ at } x^3 = 0.$$

Again summation is restricted to $\lambda, \mu < 3$. We have lowered the index i in order that the elastodynamic and the traction operators map from covariant to covariant components.

Interior and reflected plane waves are found with the time-harmonic ansatz

$$u(x,t) = \exp(i\kappa(\xi x - ct))U, \qquad \xi x \equiv \xi_i x^j, \quad i \equiv \sqrt{-1},$$

where $\kappa > 0$ is the wave number, c > 0 the wave speed, $\xi = [\xi_j]$ the unit propagation direction, and ξ/c the slowness vector, $|\xi| = 1$. The ansatz satisfies the elastodynamic system if and only if the amplitude vector $U = [U_k]$ lies in the null space of the acoustic tensor $[C_i{}^{jk\ell}\xi_j\xi_\ell - c^2\rho\delta_i{}^k]$. In the special case of isotropy there are two wave speeds, $c_S = \sqrt{\mu/\rho}$ and $c_P = \sqrt{(\lambda + 2\mu)/\rho}$, the speeds of shear and pressure waves, respectively. To obtain waves which propagate along the surface, the foregoing ansatz is modified by allowing ξ^3 to be complex with positive imaginary part.

Let $\eta = [\eta_j]$ be a real horizontal unit vector, which means that $|\eta| = 1$ and η is orthogonal to ν , i.e., $\eta_3 = 0$. Consider the acoustic tensor at $\eta + s\nu$:

$$A(s) \equiv s^2 A_0 + s(A_1 + A_1^T) + A_2 - c^2 \rho I.$$

Here $I = [\delta_i^{\ k}]$ denotes the 3×3 unit matrix, and

(10)
$$A_0 \equiv [C_i^{\ 3k3}], \quad A_1 \equiv [C_i^{\ 3k\ell}\eta_\ell], \quad A_2 \equiv [C_i^{\ jk\ell}\eta_j\eta_\ell]$$

are real 3×3 matrices; A_1^T is the transpose of A_1 with respect to the inner product defined by G. Because of the symmetries of the stiffness tensor C, the matrices A_0 , A_2 , and, for real s, A(s) are symmetric with respect to G. The wave speed c > 0 is said to be subsonic if A(s) is non-singular for every real s. By the strong convexity of C, A_0 is positive definite, and there exists a positive limiting wave speed $c_{\infty} = c_{\infty}(\eta)$ such that c is subsonic if and only if $c < c_{\infty}$. Notice that A(s) is positive definite for real s if c is subsonic.

We use the following time-harmonic ansatz to find surface waves which have subsonic horizontal slowness η/c :

(11)
$$u(x,t) = \exp(i\kappa(\eta x - ct))U(\kappa x^3).$$

The amplitude U(z) at $z \equiv \kappa x^3 \geq 0$ is to be determined. We introduce the differential operator $DU(z) \equiv -i\frac{d}{dz}U(z)$. The ansatz (11) satisfies equation (8) if and only if

(12)
$$A_0 D^2 U + (A_1 + A_1^T) D U + (A_2 - c^2 \rho I) U = 0$$

holds.

The matrix polynomial A(s) satisfies the assumptions of Proposition A.1 in the appendix. Therefore, there is a unique complex 3×3 matrix Q with spectrum in the complex upper half-plane and such that, with Q^* denoting the adjoint of Q,

(13)
$$A(s) = (sI - Q^*)A_0(sI - Q)$$

holds for complex s. Now (12) can be rewritten as

$$(DI - Q^*)A_0(DI - Q)U = 0.$$

An analogous formula holds with Q replaced by a matrix P, which has its spectrum in the lower complex half-plane. The solutions of DU = PU and DU = QU span the space of solutions of (12). The solutions of (12), which stay bounded as $\kappa x^3 \to +\infty$, are precisely the solutions of DU = QU, which are given by $U(z) = \exp(izQ)U(0)$. Moreover, as $z \to +\infty$, these solutions decay exponentially.

Remark 3.1. In the argument above, we cannot take $P=Q^*$. In fact, if (13) and $A(s)=(sI-Q)A_0(sI-Q^*)$ were to hold simultaneously, Q would be normal, $QQ^*=Q^*Q$. However, for isotropic media an explicit calculation shows that Q is not normal.

The surface traction of the waves (11) just constructed equals

$$\tau = i\kappa (A_0 DU(0) + A_1 U(0)) = -\kappa ZU(0).$$

Here Z is the 3×3 surface impedance tensor,

$$(15) Z \equiv -i(A_0Q + A_1),$$

first introduced by Ingebrigtsen & Tonning (1969). Summarizing, we have found that, precisely when ZU(0)=0 holds, the ansatz (11) leads to a solution of (8) and (9), which decays into the interior. Since Q depends smoothly on the subsonic wave speed c and on η , so does Z. Suppressing the dependence on η , we often write Z(c) for the impedance tensor.

In the early 1970's the problem of uniqueness and existence of subsonic surface waves in homogeneous half-space was completely solved. The following theorem, which we reprove in the next section, states the solution in terms of the impedance tensor.

Theorem 3.1 (Barnett & Lothe). For a given propagation direction, there exists at most one subsonic free surface wave. A subsonic surface wave exists, if and only if $\det Z(c) < 0$ holds for some c in the range $0 < c < c_{\infty}$. The wave speed c_R is the unique solution, in the range $0 < c < c_{\infty}$, of the secular equation $\det Z(c) = 0$. The surface displacement of a subsonic surface wave belongs to the null-space of $Z(c_R)$. The null-space of $Z(c_R)$ is one-dimensional and contains no non-zero real vector.

4. The surface impedance tensor

The surface impedance tensor method of subsonic surface wave theory, developed by Ingebrigtsen & Tonning (1969) and Barnett & Lothe (1985), relies on the following properties of Z.

Proposition 4.1. For $0 \le c < c_{\infty}$ the following hold:

- (a) Z(c) is Hermitian.
- (b) Z(0) is positive definite.
- (c) The real part of Z(c) is positive definite.

- (d) At most one eigenvalue of Z(c) is non-positive.
- (e) The derivative dZ(c)/dc is negative definite.
- (f) The limit $Z(c_{\infty}) \equiv \lim_{c \uparrow c_{\infty}} Z(c)$ exists.

Theorem 3.1 follows from Proposition 4.1 combined with arguments of the previous section. Indeed, it follows from the proposition that the derivative of the determinant with respect to c is negative at its zeros. Hence $\det Z(c) = 0$ has at most one zero. Since $\det Z(0) > 0$, a zero $c = c_R$ exists if and only if the determinant becomes negative for some subsonic c. The assertions about the null-space of $Z(c_R)$ follow from (c) and (d).

Notice that $\det Z(c) < 0$ holds if and only if $c_R < c < c_\infty$. Barnett & Lothe (1985, Theorem 12) state an existence criterion in terms of $Z(c_\infty)$.

Proof of Proposition 4.1. Following Mielke & Fu (2004), we use the Ricatti equation

(16)
$$(Z - iA_1^T)A_0^{-1}(Z + iA_1) = A_2 - c^2 \rho I.$$

Observing that $Q = iA_0^{-1}(Z + iA_1)$, (16) is seen to be equivalent to

$$A_0Q^2 + (A_1 + A_1^T)Q + A_2 - c^2\rho I = 0,$$

which is the solvency equation (50) of the factorization (13).

Passing to adjoints, one recognizes that (16) also holds when Z is replaced by Z^* . Subtracting the two Ricatti equations, one obtains

$$Q^*(Z - Z^*) - (Z - Z^*)Q = 0.$$

Because Q and Q^* have disjoint spectra, this Sylvester equation is non-singular. Hence $Z^* = Z$, proving (a).

To prove (e), differentiate (16) with respect to c, and get

$$iQ^*\dot{Z} - i\dot{Z}Q = -2c\rho I, \quad \dot{Z} \equiv dZ/dc.$$

This Lyapunov–Sylvester equation has the unique solution

$$\dot{Z} = -2c\rho \int_0^\infty \exp(-isQ^*) \exp(isQ) \, \mathrm{d}s,$$

which is negative definite.

Next we prove the positive definiteness of $Z(0) = [Z^{k\ell}]$. Assume that w is a complex vector such that $Z^{k\ell}w_k\overline{w_\ell} \leq 0$. We have to show that w=0. The exponentially decaying solution of (12) with surface displacement $e^{i\eta x}w$ is

$$u = e^{i\eta x} \exp(ix^3 Q)w.$$

Denote by $\epsilon = [\epsilon_{k\ell}]$, $\epsilon_{k\ell} = (\partial_\ell u_k + \partial_k u_\ell)/2$, the strain tensor and by $\tau = -Z(0)w$ the surface traction of the (complex) displacement field u. Integrate the divergence

$$\partial_3(C^{ijk3}\epsilon_{ij}\overline{u_k}) = \partial_\ell(C^{ijk\ell}\epsilon_{ij}\overline{u_k}) = C^{ijk\ell}\epsilon_{ij}\overline{\epsilon_{k\ell}}$$

over the half-line $x^3 \geq 0$, and get the energy identity

$$\int_0^\infty C^{ijk\ell} \epsilon_{ij} \overline{\epsilon_{k\ell}} \, \mathrm{d}x^3 = -\tau^k \overline{w_k}.$$

By assumption, the right-hand side is non-positive. Hence, in view of (2), the strain vanishes on the half-line $x^3 \ge 0$. In particular,

$$\epsilon_{11} = i\eta_1 u_1$$
, $\epsilon_{12} = i(\eta_1 u_2 + \eta_2 u_1)/2$, and $\epsilon_{33} = i(Qu)_3$

all vanish. Without loss of generality, we assume that $\eta_1 \neq 0$. It follows that $u_1 = u_2 = 0$. Moreover, $\partial_3 |u_3|^2 = 2 \operatorname{Re} \left(i(Qu)_3 \overline{u_3}\right) = 0$. Since u_3 tends to zero as $x^3 \to \infty$, this implies $u_3 = 0$. Therefore w = 0, which proves assertion (b).

Denote by A(s) the polynomial (13). The integral formula for Q, (49) of the appendix, implies

$$iZ \oint_{\gamma_R} A(s)^{-1} ds = \oint_{\gamma_R} (sA_0 + A_1)A(s)^{-1} ds,$$

if the closed contour γ_R consists, with R>0 sufficiently large, of the interval [-R,R] and the arc $Re^{i\varphi}$, $0 \le \varphi \le \pi$. Notice that $A_0A(s)^{-1} = s^{-2}I + \mathcal{O}(s^{-3})$ as $|s| \to \infty$. Therefore, letting $R \to \infty$, we obtain

(17)
$$iZ \int_{-\infty}^{\infty} A(s)^{-1} ds = \pi iI + \int_{-\infty}^{\infty} (sA_0 + A_1)A(s)^{-1} ds,$$

the integral on the right being a principal value integral. The integrals are real matrices. Moreover, the integral on the left-hand side of (17) is positive definite. This proves the assertion (c).

If (d) were not true, then, for some velocity c, the impedance tensor $Z(c) = [Z^{ij}]$ would have only one positive eigenvalue. Furthermore, its eigenspace would be one-dimensional. Since space is three-dimensional, we could then choose a real vector $w \neq 0$ orthogonal to this eigenspace. But then $Z^{ij}w_iw_j \leq 0$, contradicting the positive definiteness of the real part of Z(c). Hence (d) holds.

To prove (f) we first derive a bound on the operator norm ||Z(c)|| of Z(c). Notice that ||Z(c)|| equals the maximum of the absolute values of the eigenvalues of Z(c). By (e), $Z(c) \leq Z(0)$ holds with respect to the usual ordering of Hermitian matrices by the cone of positive definite matrices. Therefore the eigenvalues of Z(c) are not greater than ||Z(0)||. The sum of the eigenvalues of Z(c) is positive because it is equal to the trace of the positive definite matrix $\operatorname{Re} Z(c)$. Therefore the modulus of a negative eigenvalue, if it exists, is less than the sum of two positive eigenvalues. Hence we have $||Z(c)|| \leq 2||Z(0)||$. By compactness, there exist limit points Z_* of Z(c) as $c \uparrow c_{\infty}$. Using (e) again, we find that $Z_* \leq Z(c)$. It follows that all limit points are equal, which implies assertion (f).

Remark 4.1. Formula (17) is a variant of the Barnett–Lothe integral representation. In fact, passing to the adjoint of (17), and making a substitution $s = \tan(\phi)$ to replace the integration variable s by an angle ϕ , one rederives the formula (2.19) of Barnett & Lothe (1985).

Remark 4.2. If the coefficients of the matrix polynomial depend smoothly on parameters, then also the wave speed c_R depends smoothly on these parameters. Since the zeros of the determinant are simple, this follows from the implicit function theorem.

5. Asymptotic expansions and pseudo-differential operators

If the elastic body is inhomogeneous and the boundary surface curved, we do not expect that an exact operator factorization (14) holds. However, using pseudo-differential operators, we shall factorize the elastodynamic operator up to negligible errors, and we shall use this to exhibit asymptotic subsonic surface waves. This will be detailed in the following sections. As a preparation, we summarize standard results on asymptotic expansions and on pseudo-differential operators. Refer

to Hörmander (1965), Hörmander (1985, 18.1), or Alinhac & Gérard (2007) for expositions of the basic pseudo-differential calculus.

We consider vector-valued functions $u(x;\omega)$ of points x in d-dimensional space which oscillate rapidly as the frequency parameter ω tends to $+\infty$. More precisely,

(18)
$$u(x;\omega) \sim e^{i\omega\theta(x)} \sum_{k=0}^{\infty} (i\omega)^{-k} U_{-k}(x),$$

where the series is an asymptotic series in the space of smooth functions x. (Often we simply write = instead of \sim , although the equation may be true only in the sense of asymptotic expansions.) The phase function $\theta(x)$ is real-valued with derivative $\theta'(x) \neq 0$. We use the operators $D_j \equiv -i\partial_j$, where $\partial_j \equiv \partial/\partial x^j$ denotes partial derivative with respect to the j-th coordinate, x^j . By the Leibniz' product rule, $f \equiv D_j u$ has the asymptotic expansion

(19)
$$f(x;\omega) \sim \omega^m e^{i\omega\theta(x)} \sum_{k=0}^{\infty} (i\omega)^{-k} F_{-k}(x)$$

with m=1 and top order coefficient $F_0(x)=\partial_j\theta(x)U_0(x)$. A linear differential operator order m, \hat{A} , is a polynomial in D_j of order m with smooth coefficients. Also $f\equiv \hat{A}u$ has an expansion (19).

The pseudo-differential calculus assigns to certain functions $A(x,\xi)$, which are defined on 2d-dimensional phase space, operators $\hat{A} = A(x,D)$ by

(20)
$$(\hat{A}u)(x) = (2\pi)^{-d} \iint e^{i(x-y)\cdot\xi} A(x,\xi)u(y) \,dy \,d\xi.$$

Here x and ξ denote position and (generalized) momentum variables, respectively. By the Fourier inversion formula, \hat{A} is the identity when A=1. The function $A(x,\xi)$ is called the (full) symbol of the operator \hat{A} . We assume that the symbol A belongs to a standard class $S^m = S^m_{1,0}$ of symbols of order at most m. Moreover, the symbol A admits, as $|\xi| \to \infty$, an asymptotic expansion in homogeneous functions:

$$A(x,\xi) \sim \sum_{j=0}^{\infty} A_{-j}(x,\xi), \quad A_{-j}(x,t\xi) = t^{m-j} A_{-j}(x,\xi) \text{ for } t > 0.$$

(Strictly speaking, homogeneous functions are, unless they are polynomials, not symbols. This technicality is overcome by multiplying with cutoff functions which insure smoothness of symbols at $\xi = 0$.) The symbol $A(x, \xi)$ and the associated operator \hat{A} are said to be of order m if the principal symbol $A_0(x, \xi)$ is not identically zero. Symbols may be matrix-valued. Rather than using $A_{-1}(x, \xi)$ it is preferable to work with the subprincipal symbol $A_{\text{sub}}(x, \xi)$,

(21)
$$A_{\text{sub}} \equiv A_{-1} - \frac{1}{2i} \sum_{i} \partial^{2} A_{0} / \partial x^{j} \partial \xi_{j}.$$

A pseudo-differential operator is a differential operator if and only if its symbol is a polynomial in the ξ variable. In particular, the symbol of D_i is ξ_i .

Every pseudo-differential operator of order m has the property that it maps an asymptotic sum (18) into an asymptotic sum (19). Moreover, this property is characteristic of pseudo-differential operators; see Hörmander (1965). Since we need explicit expressions for F_0 and F_{-1} we indicate how (19) is proved using the method of stationary phase. The method is applied to the summands of (18):

$$e^{-i\omega\theta(x)}(\hat{A}Ue^{i\omega\theta})(x) = (\omega/2\pi)^d \iint e^{i\omega(x-y)\cdot\xi-i\omega(\theta(x)-\theta(y))} A(x,\omega\xi)U(y) \,dy \,d\xi.$$

The stationary point of the phase

$$\Phi \equiv (x - y) \cdot \xi - (\theta(x) - \theta(y)), \quad \Phi'_y = 0 = \Phi'_{\xi},$$

is at y = x, $\xi = \theta'(x)$. The inverse of the Hessian matrix H of Φ at the stationary point is the $2d \times 2d$ -matrix

$$H(x)^{-1} = -\begin{bmatrix} 0 & I \\ I & \theta''(x) \end{bmatrix}$$
, where $H(x) \equiv \Phi''|_{y=x,\xi=\theta'(x)}$.

On a formal level, the stationary phase expansion is given by

$$e^{-i\omega\theta(x)}(\hat{A}Ue^{i\omega\theta})(x) \sim \exp\left((i\omega)^{-1}\left(\partial_y\cdot\partial_\xi + \frac{1}{2}\theta''\partial_y\cdot\partial_y\right)\right)(e^{i\omega\rho}U),$$

where the exponential of differential operators has to be replaced by its formal Taylor series to give the asymptotic expansion (Hörmander, 1990, Theorem 7.7.5). Here ρ is the remainder of the second order Taylor expansion of the phase, it vanishes to third order at the stationary point where the expressions have to be evaluated. Thus the principal coefficients in (19) are

$$F_0(x) = A_0(x, \theta'(x))U_0(x)$$

and, suppressing the arguments x and $\xi = \theta'(x)$ in writing,

$$F_{-1} = \sum_{j} \frac{\partial A_0}{\partial \xi_j} \frac{\partial U_0}{\partial x^j} + \frac{1}{2} \left(\sum_{j,\ell} \frac{\partial^2 \theta}{\partial x^j \partial x^\ell} \frac{\partial^2 A_0}{\partial \xi_j \partial \xi_\ell} \right) U_0 + i A_{-1} U_0 + A_0 U_{-1}.$$

Observe that the double sum enclosed in parenthesis equals

$$\sum_{\ell} \frac{\partial}{\partial x^{\ell}} \left(\frac{\partial A_0}{\partial \xi_{\ell}} (x, \theta'(x)) \right) - \sum_{j,\ell} \frac{\partial^2 A_0}{\partial \xi_j \partial x^{\ell}} (x, \theta'(x)).$$

Therefore,

(22)
$$F_{-1} = \sum_{i} \frac{\partial A_0}{\partial \xi_j} \frac{\partial U_0}{\partial x^j} + \frac{1}{2} \sum_{\ell} \frac{\partial}{\partial x^{\ell}} \left(\frac{\partial A_0}{\partial \xi_{\ell}} \right) U_0 + i A_{\text{sub}} U_0 + A_0 U_{-1}.$$

For $k=2,3,\ldots$ there are formulas for F_{-k} which differ from the formula for F_{-1} by a shifted index and by an additional term which is a sum of (derivatives of) U_{-j} with j < k-1. The additional term arises from the higher order terms in the stationary phase expansion. We emphasize that the asymptotic expansions and the formulas for the coefficients F_{-k} hold for a general system \hat{A} of pseudo-differential operators.

The composition $\hat{C} = \hat{B}\hat{A}$ of pseudo-differential operators \hat{A} and \hat{B} is again a pseudo-differential operator. There is a formula for the asymptotic expansion of the full symbol $C(x,\xi)$. On the principal symbol level, symbols multiply: $C_0 = B_0 A_0$. On the next level,

(23)
$$C_{-1} = B_0 A_{-1} + B_{-1} A_0 - i \sum_{j} (\partial_{\xi_j} B_0) (\partial_{x_j} A_0)$$

holds. Define the Poisson bracket

$$\{P,Q\} \equiv \sum\nolimits_{j} (\partial P/\partial \xi_{j}) (\partial Q/\partial x^{j}) - (\partial P/\partial x^{j}) (\partial Q/\partial \xi_{j}),$$

where the symbols need not be scalar but can take square matrices (of equal dimensions) as their values. It follows that the composition of operators is given on the principal and on the subprincipal level by

(24)
$$C_0 = B_0 A_0, \quad C_{\text{sub}} = B_0 A_{\text{sub}} + B_{\text{sub}} A_0 + \frac{1}{2i} \{B_0, A_0\}.$$

The (formal) adjoint of \hat{A} with respect to a given scalar product is also a pseudo-differential operator, \hat{B} , and its principal and subprincipal symbol are given as follows:

$$B_0 = A_0^*, \quad B_{\text{sub}} = A_{\text{sub}}^*.$$

The star denotes the (Hermitian) adjoint. (The symbol formulas arise more naturally if Weyl quantization is used instead of the Kohn–Nirenberg quantization, $A \mapsto \hat{A}$ as in (20), which we are using here.)

Under ellipticity assumptions, one constructs inverses, square roots, and powers of pseudo-differential operators which are again pseudo-differential operators. The error or remainder terms in such constructions are typically negligible operators with symbols belonging to $S^{-\infty} \equiv \cap_m S^m$. In particular, two pseudo-differential operators with symbols, which are equal when $|\xi| > 1$, differ only by a negligible operator. Negligible operators map distributions into smooth functions. If u has an symptotic expansion (18) and if \hat{A} is a pseudo-differential operator which is negligible in conic neighbourhood of the set of $(x, \omega \theta'(x))$, $\omega > 0$, then the asymptotic expansion of $f = \hat{A}u$ is trivial, i.e., all coefficients $F_{-k} = 0$.

The operator \hat{L} of elastodynamics is a second order differential 3×3 system,

$$(25) \qquad (\hat{L}u)_i \equiv \rho \ddot{u}_i - \sigma_i^{\ j}_{:i}.$$

The principal symbol L_0 of \hat{L} is the acoustic tensor,

(26)
$$L_0(x,\xi) = \left[C_i^{\ jk\ell}(x)\xi_j\xi_\ell - \rho\xi_0^2\delta_i^{\ k}(x) \right].$$

Here we included the time coordinate as $x^0 = t$, and the dual variable (frequency) as ξ_0 . The implied summation extends over the spatial indices, i.e., $j, \ell \geq 1$. The pseudo-differential operator which will be most important to us is the surface impedance operator, which, up to negligible error terms, maps surface displacements to surface tractions of solutions of $\hat{L}u = 0$. This operator is introduced in Section 8.

6. Ray theory for systems

The ray method solves a wave equation $\hat{A}u = 0$ by solving the equations $F_{-j} = 0$ for the amplitudes U_{-k} of an asymptotic expansion (18) of u. For j > 0, the equations $F_{-j} = 0$ are ordinary differential equations along rays, commonly called transport equations.

The equation $F_0 = 0$ is the dispersion equation, $A_0(x, \theta'(x))U_0(x) = 0$. If A_0 is scalar, then this becomes the eikonal equation $A_0(x, \theta'(x)) = 0$. Furthermore, $F_{-1} = 0$ with (22) is a transport equation for U_0 which is solved by the method of characteristics. If A_0 is not scalar, then it is not obvious how (22) and $F_{-1} = 0$ lead to a useful transport equation for U_0 .

For systems of real principal type, introduced by Dencker (1982), there is an efficient ray theory based on the theory of Fourier integral operators. It applies to isotropic elastodynamics, Hansen & Röhrig (2004), and, as we will show, to the

propagation of Rayleigh waves along curved surfaces of general elastic media. We adapt this ray method to our setting.

Let \hat{A} be a (square) system of pseudo-differential operators with principal symbol $A_0(x,\xi)$. The set or zeros of the determinant det A_0 is called the characteristic set of \hat{A} (or of A_0). We assume that the system \hat{A} is of real principal type in the sense of Dencker (1982). This means that there exists a matrix-valued symbol $B(x,\xi)$, homogeneous of degree zero in the ξ variable, and a scalar symbol $a(x,\xi)$ such that $B(x,\xi)A_0(x,\xi)=a(x,\xi)I$ holds with I denoting the unit matrix. Moreover, a is real-valued and its set of zeros equals the characteristic set, on which it vanishes simply, i.e. $\partial_{\xi}a(x,\xi)\neq 0$ holds whenever $a(x,\xi)=0$. We call a a Hamilton function of A_0 (and of the operator \hat{A}). In the special case where the determinant of A_0 vanishes simply on the characteristic set, B is a scalar multiple of the cofactor matrix, and $ab=\det A_0$ with b a nowhere vanishing function.

Next we collect some properties which follow from the real principal type assumption (Dencker, 1982). The simple vanishing of a implies that the complement of the characteristic set is dense in phase space. Moreover, $BA_0 = aI$ and also $A_0B = aI$ holds. This is clear where $a \neq 0$, and, by continuity, this holds in general. The following identities between range spaces and null spaces (kernels) are important:

(27)
$$\operatorname{range}(B) = \ker(A_0), \quad \operatorname{range}(A_0) = \ker(B), \quad \text{where } a = 0.$$

The inclusion of the ranges in the null spaces is clear. Differentiating $BA_0 = aI = A_0B$ in an appropriate direction, with the directional derivative denoted by a prime, we obtain

$$BA'_0 + B'A_0 = a'I = A'_0B + A_0B', \quad a' \neq 0.$$

Now, equality in (27) follows from rank considerations.

Fix a real-valued solution $\theta(x)$ of the Hamilton–Jacobi equation $a(x, \theta'(x)) = 0$. The integral curves $(x(s), \xi(s))$ of Hamilton's canonical equations, $\dot{x} = \partial_{\xi} a(x, \xi)$ and $\dot{\xi} = -\partial_x a(x, \xi)$, satisfy $\xi(s) = \theta'(x(s))$ for all parameters s if this is true for at least one s. These integral curves in phase space are called bicharacteristics. The Poisson bracket $\{a, b\}$ is the derivative of a function $b(x, \xi)$ in bicharacteristic direction,

$$\frac{\mathrm{d}}{\mathrm{d}s}b(x(s),\xi(s)) = \{a,b\}(x(s),\xi(s)).$$

The projections of bicharacteristics to space-time are the space-time rays x(s) associated with θ . The vector field $V(x) \equiv \partial_{\xi} a(x, \theta'(x))$ is called the ray field. We denote by a dot the derivative in ray direction,

$$\dot{U}(x(s)) \equiv (d/ds)U(x(s)) = (V \cdot \partial_x U)(x(s)).$$

Next, we turn to the solution of $F_0=0$ and $F_{-1}=0$. Again we suppress arguments x and ξ , and we assume evaluation of expressions at $\xi=\theta'(x)$. If $A_0U_0=0$ holds, then the leading term in the asymptotic expansion of $\hat{B}f=\hat{B}\hat{A}u$ equals

$$BF_{-1} = \dot{U}_0 + (\operatorname{div} V/2)U_0 + iA'_{\text{sub}}U_0,$$

where div $V \equiv \nabla_x \cdot V$ is the divergence of the ray field, and A'_{sub} denotes the subprincipal symbol of $\hat{A}' \equiv \hat{B}\hat{A}$. To see this, observe that the right-hand equals

the right-hand side in (22) when \hat{A} is replaced by \hat{A}' . Observe that $A'_0 = aI$, and, by (24), $A'_{\text{sub}} = BA_{\text{sub}} + \frac{1}{2i}\{B, A_0\}$ when restricted to the nullspace of A_0 . Therefore,

$$BF_{-1} = \dot{U}_0 + (\operatorname{div} V/2)U_0 + \frac{1}{2}\{B, A_0\}U_0 + iBA_{\operatorname{sub}}U_0.$$

To solve $F_{-1} = 0$ we first solve, using the following lemma, the equations $BF_{-1} = 0$ and $A_0U_0 = 0$, simultaneously.

Lemma 6.1. If U solves the differential equation

(28)
$$\dot{U} + (\operatorname{div} V/2)U + \frac{1}{2}\{B, A_0\}U + BW = 0,$$

then $A_0U=0$ holds along a ray if this holds in at least one point of the ray.

Proof. Multiply (28) by A_0 from the left to get

$$A_0\dot{U} + (\operatorname{div} V/2)A_0U + \frac{1}{2}A_0\{B, A_0\}U = 0$$

along rays. The derivative of $A_0(x,\xi)$ along bicharacteristics is given by

$$2\frac{\mathrm{d}}{\mathrm{d}s}A_0 \equiv 2\{aI, A_0\} = \{A_0B, A_0\} - \{A_0, BA_0\} = A_0\{B, A_0\} - \{B, A_0\}A_0.$$

We obtain a homogeneous linear ordinary differential equation for A_0U ,

$$\frac{\mathrm{d}}{\mathrm{d}s}(A_0U) + (\mathrm{div}\,V/2)A_0U + \frac{1}{2}\{B, A_0\}A_0U = 0.$$

The assertion of the lemma follows from the uniqueness of solutions to initial value problems of this equation. \Box

Using the lemma, find a solution $U_0 \neq 0$ of the transport equation

(29)
$$\dot{U}_0 + (\operatorname{div} V/2)U_0 + \frac{1}{2}\{B, A_0\}U_0 + iBA_{\text{sub}}U_0 = 0$$

such that $A_0U_0=0$. Thus $F_0=0$ and $BF_{-1}=0$. Recall that the null space of B equals the range of A_0 . Therefore, there exists U_{-1} such that $F_{-1}=-A_0U_{-1}$. Set $u=e^{i\omega\theta}(U_0+(i\omega)^{-1}U_{-1})$ to obtain $F_0=0$ and $F_{-1}=0$. To proceede by recursion, assume that $u=e^{i\omega\theta}\sum_{j< k}(i\omega)^{-j}U_{-j}$ has been found such that $F_{-j}=0$ for j< k. Then

$$BF_{-k} = \dot{U}_{-k+1} + (\operatorname{div} V/2)U_{-k+1} + iA'_{\operatorname{sub}}U_{-k+1} + W_{-k},$$

where W_{-k} is a sum of derivatives of U_{-j} , j < k-1. We shall modify U_{-k+1} and add $e^{i\omega\theta}(i\omega)^{-k}U_{-k}$ to u such that $F_{-j}=0$ for $j \leq k$. Using the lemma, solve

$$\dot{U} + (\operatorname{div} V/2)U + \frac{1}{2}\{B, A_0\}U + iBA_{\operatorname{sub}}U + BF_{-k} = 0, \quad A_0U = 0.$$

Replace U_{-k+1} by $U_{-k+1} + U$. Then still $F_{-j} = 0$ for j < k, and, in addition, we now have $BF_{-k} = 0$. Choose U_{-k} as a solution of $F_{-k} + A_0U_{-k} = 0$. With this choice of U_{-k} we arrive at $F_{-k} = 0$, which completes the recursive step of the construction of an asymptotic solution of $\hat{A}u = 0$.

For later reference, we summarize the result just obtained.

Proposition 6.1. Let \hat{A} pseudo-differential system of real principal type with Hamilton function a, and $B(x,\xi)A_0(x,\xi) = a(x,\xi)I$. Let $\theta(x)$ with $\theta'(x) \neq 0$ be a solution of the eikonal equation $a(x,\theta'(x)) = 0$. Denote by $V(x) \equiv \partial_{\xi}a(x,\theta'(x))$ the

ray field, and div $V \equiv \nabla_x \cdot V$ its divergence. Let $U_0(x)$ be a solution of the transport equation (29) which satisfies

$$A_0(x, \theta'(x))U_0(x) = 0.$$

Then there is a solution of $\hat{A}u = 0$ which is given by an asymptotic sum (18) with leading amplitude $U_0(x)$.

7. Subsonic displacement boundary problems

We return to elastodynamics. Given a subsonic phase θ and a vector-valued amplitude W, each defined and smooth on the space-time boundary of the elastic body, we solve, asymptotically as $\omega \to \infty$, the displacement boundary problem

$$\hat{L}u = 0$$
 in B , $u = e^{i\omega\theta}W$ at S .

To construct u we use coordinates adapted to S as introduced in Section 2. Thus the boundary condition reads

(30)
$$u|_{x^3=0} = e^{i\omega\theta(x^0, x^1, x^2)} W(x^0, x^1, x^2).$$

The time variable is denoted t or x^0 , the dual variable is ξ_0 .

First, we define the subsonic region. Recall from (10) the matrices A_j . Since the material parameters are allowed to vary smoothly with spatial position x, we have $A_j = A_j(x, \eta)$, and the acoustic tensor (26) is given as follows:

$$L_0(x,\xi) = \xi_3^2 A_0(x) + \xi_3 (A_1(x,\eta) + A_1(x,\eta)^T) + A_2(x,\eta) - \xi_0^2 \rho(x) I,$$

where $\eta=(\xi_1,\xi_2,0)$. We say that, at the surface point $y=(x^1,x^2)$, the covector $\zeta=(\xi_0,\xi_1,\xi_2)$ is subsonic if $L_0(x,\xi)|_{x^3=0}$ is positive definite for real ξ_3 . If ζ is subsonic, then so are $-\zeta$ and $s\zeta$ for s>0. As in Section 3, we appeal to Proposition A.1 of the appendix, and we get, if ζ is subsonic at y, a factorization

(31)
$$L_0(x,\xi) = (\xi_3 I - Q_0^*(x,\zeta)) A_0(x) (\xi_3 I - Q_0(x,\zeta)),$$

where $Q_0(x,\zeta)$ depends smoothly on x and ζ , and the spectrum of Q_0 is contained in the upper complex half-plane. The factorization (31) holds in a boundary layer $0 \le x^3 < \delta$. The matrices $Q_0(x,\zeta)$ are uniquely determined. Using the uniqueness, it is seen that Q_0 satisfies $Q_0(x,s\zeta) = sQ_0(x,\zeta)$ if s>0. However, note that $Q_0(x,-\zeta) \ne -Q_0(x,\zeta)$ because of the spectral condition. It follows that Q_0 cannot be a polynomial in ζ .

The phase function θ is assumed to be subsonic. By definition, this means that its derivatives $\zeta = \theta'$ are non-zero and subsonic. Moreover, we assume that we have a zeroth order scalar symbol $\chi(x,\xi)$, supported in the subsonic region, such that, for $|\xi| > 1$, $\chi(x,\xi)$ equals unity in an open conic neighbourhood Γ_1 of the set Γ of (x,ξ) , $\xi = s\theta'(x)$, s > 0.

The pseudo-differential operator $(\chi Q_0)(x, D')$, $D' \equiv (D_0, D_1, D_2)$, is defined, has the order one, and is tangential. A pseudo-differential operator is said to be tangential if it commutes with multiplication by x^3 , or, equivalently, the operator pseudo-differentiates only with respect to the coordinates (x^0, x^1, x^2) , and depends smoothly on x^3 as a parameter.

The elastodynamic operator \hat{L} is a second degree polynomial in $D_3 = -i\partial_3 I$ with tangential differential operators as coefficients. Let \hat{Q} be a tangential first

order pseudo-differential operator, say $\hat{Q} = (\chi Q_0)(x, D')$, whose principal symbol is, on Γ_1 , equal to Q_0 . By (31) and the symbol calculus we have

(32)
$$\hat{L} = (D_3 - \hat{P})A_0(D_3 - \hat{Q}) + \hat{R}^0D_3 + \hat{R}^1,$$

where \hat{P} is the adjoint of \hat{Q} , $P_0 = Q_0^*$, and \hat{R}^j is a tangential pseudo-differential operator of order at most j. Moreover, on Γ_1 , the principal symbols R_0^j of \hat{R}^j satisfy

$$\begin{split} R_0^0 &= B_0 + i\partial_3 A_0 + A_0 Q_{-1} + P_{-1} A_0 - i \sum\nolimits_j (\partial_{\xi_j} P_0) (\partial_{x^j} A_0), \\ R_0^1 &= B_1 - i\partial_3 (A_0 Q_0) - P_0 A_0 Q_{-1} - P_{-1} A_0 Q_0 + i \sum\nolimits_j (\partial_{\xi_j} P_0) (\partial_{x^j} A_0 Q_0). \end{split}$$

Here we used (23) and (6), and we have set

(33)
$$B_0(x) = [B^{ik3}], \quad B_1(x,\eta) = [B^{ik\lambda}\eta_{\lambda}].$$

Our next goal is to improve, by modifying \hat{Q} and \hat{P} , the factorization (32) so that the remainders \hat{R}_j are negligible in a conic neighbourhood of Γ . To do so, we first eliminate P_{-1} from the above equations for R_0^j , and we set $R_0^j = 0$. We derive the following equation for Q_{-1} :

(34)
$$Q_0^* A_0 Q_{-1} - A_0 Q_{-1} Q_0$$

$$= B_0 Q_0 + B_1 - i A_0 \partial_3 Q_0 + i \sum_{j} (\partial_{\xi_j} Q_0^*) A_0 (\partial_{x^j} Q_0) A_0 (\partial_{x^j} Q_0)$$

This is a uniquely solvable Sylvester equation for A_0Q_{-1} because Q_0 and its Hermitian adjoint have disjoint spectra. Having found Q_{-1} , we solve for P_{-1} . The symbols Q_{-1} and P_{-1} are only defined in Γ_1 . Fix a new cutoff symbol χ_1 which equals unity in a conic neighbourhood of Γ and is supported in the set where $\chi = 1$. Replace \hat{Q} by $\hat{Q} + (\chi_1 Q_{-1})(x, D')$, similarly for \hat{P} . Then (32) holds with remainders \hat{R}^j which have, where $\chi_1 = 1$, order at most j-1. This construction of reducing the order of the remainders can be iterated. In fact, one recursively solves equations $Q_0^*X - XQ_0 = Y$ for $X = A_0Q_{-k}$. Using asymptotic summation, a factorization (32) is obtained such that the remainders are negligible in a neighbourhood of Γ .

Remark 7.1. The result (32) is a factorization of the elastodynamic operator into a product of pseudo-differential operators. The factorization does not hold everywhere, but only microlocally in (some chosen subregion of) the subsonic region. Such a factorization is not possible using only differential operators,

Our main application of the factorization (32) is the construction of solutions to displacement boundary problems.

Lemma 7.1. Let $\theta(x^0, x^1, x^2)$ be a subsonic phase function, $W(x^0, x^1, x^2)$ a smooth amplitude. There is a solution u,

(35)
$$u \sim e^{i\omega\theta(x^0, x^1, x^2)} \sum_{j=0}^{\infty} (i\omega)^{-j} U_{-j}(x^0, x^1, x^2, \omega x^3),$$

which solves $\hat{L}u = 0$ asymptotically as $\omega \to \infty$ in $x^3 \ge 0$, and satisfies the displacement boundary condition (30). Moreover, $D_3u = \hat{Q}u$, and u decays exponentially in ωx^3 .

The proof consists of finding an asymptotic solution (35) of $(D_3 - \hat{Q})u = 0$ in $x^3 \ge 0$. To satisfy the boundary condition (30) we require $U_0|_{x^3=0} = W$ and

 $U_{-j}|_{x^3=0}=0$ if j>0. By the asymptotic expansion lemma for pseudo-differential operators,

$$i(D_3 - \hat{Q})u \sim \omega e^{i\omega\theta} \sum_{j=0}^{\infty} (i\omega)^{-j} F_{-j}(x^1, x^2, \omega x^3),$$

where, if we set $\zeta = \theta'(x^0, x^1, x^2), y = (x^1, x^2, 0),$

$$F_{-j}(x) = \partial_3 U_{-j}(x) - iQ_0(y,\zeta)U_{-j}(x) + G_{-j}(x).$$

When deriving these formulas, the symbols Q_{-j} of \hat{Q} are replaced by their Taylor series in the x^3 variable, and it is used that a power of x^3 give rise to a power of $1/\omega$ with the same exponent. The remainder term G_{-j} is an expression involving $U_{-\ell}$ for $\ell < j$, and $G_0 = 0$. We recursively solve the equations $F_{-j} = 0$ for U_{-j} imposing, in addition, the above-mentioned boundary conditions at the surface. Since iQ_0 has its spectrum in the left half-plane, the solutions $U_{-j}(x)$ decay exponentially as $x^3 \to \infty$. The factorization remainder $\hat{R}^0 D_3 + \hat{R}^1$ is negligible in a neighbourhood of Γ . Therefore, $(D_3 - \hat{Q})u = 0$ implies $\hat{L}u = 0$.

The assertion of the lemma still holds if (30) is replaced by the more general displacement boundary condition $u|_{x^3=0}=w$,

$$w(x^0, x^1, x^2) \sim e^{i\omega\theta(x^0, x^1, x^2)} \sum_{j=0}^{\infty} (i\omega)^{-j} W_{-j}(x^0, x^1, x^2).$$

In fact, the proof of the lemma is readily adapted to cover this situation too.

Remark 7.2. The methods of this section originate from the theory of elliptic boundary problems. Wloka et al. (1995) systematically use spectral factorizations of matrix polynomials in their treatment of such problems.

8. Existence of subsonic free surface waves

As in (15) we define, in the subsonic region, the surface impedance tensor Z_0 :

(36)
$$iZ_0(y,\zeta) \equiv A_0(y)Q_0(y,\zeta) + A_1(y,\eta).$$

Here $y = (x^1, x^2) = (x^1, x^2, 0)$, $\eta = (\xi_1, \xi_2) = (\xi_1, \xi_2, 0)$, and $\zeta = (\xi_0, \eta)$. Since Q_0 and A_1 are homogeneous of degree 1 in the variables ξ_j , so is Z_0 . The subsonic region is conic. Therefore, there is a function $c_{\infty}(y, \eta) > 0$, homogeneous of degree one in η , such that ζ is subsonic if and only if $c_{\infty}(y, \eta) > |\xi_0|$ holds. If we freeze a point y on the surface and a unit horizontal propagation direction η , $|\eta| = 1$, then Z_0 is the surface impedance tensor of a homogeneous half-space: $Z_0(y, \zeta) = Z(c)$, where $c = -\xi_0$, and $c_{\infty}(y, \eta)$ is the limiting velocity. This shows that Proposition 4.1 and the proof of the Barnett-Lothe Theorem apply mutatis mutandis to $Z_0(y, \zeta)$.

Subsonic free surface waves correspond to solutions of det $Z_0(y,\zeta) = 0$, which is known as the secular equation. We assume from now on that the secular equation has a zero $\xi_0 < 0$, necessarily unique and simple, for every surface point y and every direction $\eta \neq 0$. By the implicit function theorem, there is a smooth function $c_R(y,\eta)$, homogeneous of degree one in η , such that $0 < c_R(y,\eta) < c_\infty(y,\eta)$, and

(37)
$$\det Z_0 = bh, \quad h(y,\zeta) \equiv \xi_0 + c_R(y,\eta), \quad b(y,\zeta) > 0.$$

We call h the Hamilton function of the free surface wave problem. The phase function $\theta \equiv \varphi(y) - t$ is subsonic and solves $h(y, \theta') = 0$ if and only if the eikonal equation $c_R(y, \varphi'(y)) = 1$ is satisfied. The ray field is $V = (1, (\partial c_R/\partial \eta)(y, \varphi'(y)))$.

So we take time t as the parameter along rays, and we regard rays as the spatial curves y(t) which satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = \frac{\partial c_R}{\partial n} (y(t), \varphi'(y(t))).$$

Note that the null-space N(y) of $Z_0(y, -c_R(y, \eta), \eta)$, $\eta = \varphi'(y)$, is one-dimensional. Denote by $r \geq 0$ the distance to the boundary surface S.

Theorem 8.1 (Existence of subsonic free surface waves). Let φ be a solution of the eikonal equation

$$(38) c_R(y, \varphi'(y)) = 1.$$

There is a matrix-valued function H(y), which can be calculated algebraically from the derivatives up to second order of the elasticities, the material density, and the surface S, such that the following holds. Let $W_0(y)$ satisfy, for every ray y(t), the transport equation

(39)
$$\frac{\mathrm{d}}{\mathrm{d}t}W_0(y(t)) + \frac{1}{2}\operatorname{div}V(y(t))W_0(y(t)) + H(y(t))W_0(y(t)) = 0$$

and $W_0(y(t)) \in N(y(t))$ for some t. Then $W_0(y) \in N(y)$ holds for all y. The free surface boundary problem (4) has a non-zero solution

(40)
$$u(t, y, r; \omega) \sim e^{i\omega(\varphi(y)-t)} \sum_{j=0}^{\infty} (i\omega)^{-j} U_{-j}(y, r\omega).$$

The amplitudes $U_{-j}(y, r\omega)$ decay exponentially as $r\omega \to \infty$, and the leading surface amplitude equals $U_0(y, 0) = W_0(y)$.

Let \hat{Q} be the pseudo-differential operator of Lemma 7.1 with the defining cutoff symbol χ chosen equal to unity in a conic neighbourhood of the set Ξ which consist of $(y, s\zeta)$, where $\zeta = (-1, \varphi'(y))$, s > 0. Consider

(41)
$$T = [T_i^k], \quad T_i^k \equiv C_i^{3j\ell} \Gamma_{i\ell}^k,$$

and regard T as a multiplication operator. We define the surface impedance operator \hat{Z} by

(42)
$$\hat{Z} \equiv -i(A_0\hat{Q} + \hat{A}_1 + iT).$$

Notice that \hat{Z} is a first order pseudo-differential operator on the space-time boundary surface. The principal symbol of \hat{Z} is $-i(A_0\chi Q_0 + A_1)$ which equals the surface impedance tensor Z_0 in a conic neighbourhood of Ξ . Let $B(y,\zeta)$ be the cofactor matrix of $Z_0(y,\zeta)$ divided by the factor $b(y,\zeta)$ of (37). By Cramer's rule,

(43)
$$B(y,\zeta)Z_0(y,\zeta) = h(y,\zeta)I.$$

Hence \hat{Z} is (microlocally near Ξ) a real principal type operator with Hamilton function h.

We can now apply Proposition 6.1 to obtain a solution

$$w \sim e^{i\omega(\varphi(y)-t)} \sum_{j=0}^{\infty} (i\omega)^{-j} W_{-j}(y)$$

of $\hat{Z}w = 0$. (Notice that the transport equations can be solved with amplitudes which do not depend explicitly on time t.) From (29) we obtain the transport equation (39), where the coefficient matrix is given by

(44)
$$H = \frac{1}{2} \{B, Z_0\} + iBZ_{\text{sub}}$$

after evaluating at $\eta = \varphi'(y)$ and $\xi_0 = -c_R(y, \eta)$. Here B is defined in (43), and $Z_{\rm sub}$ is the subprincipal symbol of the surface impedance operator. Using Lemma 6.1 it follows that $W_0(y) \in N(y)$. Using Lemma 7.1 and the remark at the end of its proof, we obtain a solution (40) of the following displacement boundary problem: $\hat{L}u = 0$ in $r = x^3 \ge 0$, and u = w at $x^3 = 0$. Moreover, we get that the U_{-j} decay exponentially as $\omega r \to \infty$.

Recall the formula (7) for the surface traction $\tau = [\sigma_i^3]$ of the field u. Since $D_3 u = \hat{Q}u$ holds, we find that, in view of the definition of the surface impedance operator, the surface traction vanishes:

$$-i\tau = A_0 D_3 u + \hat{A}_1 u + iTu = i\hat{Z}w = 0$$
 at $x^3 = 0$.

This proves the existence of a free subsonic surface wave. In the next section we explain how H(y) is evaluated.

Remark 8.1. The theorem says, in particular, that $c_R(y,\eta)/|\eta|$ is the wave speed of a subsonic free surface wave at the surface point y in the horizontal direction $\eta = \varphi'(y)$. Notice that an ansatz (40) will solve (4) only if the phase function φ satisfies the eikonal equation (38).

9. EVALUATION OF THE TRANSPORT EQUATION

Before we discuss the evaluation of the coefficient matrix H(y), let us remark how the transport equation (39) can be reduced to a scalar equation. Assume given a smooth unit field W(y) which spans the one-dimensional null-space N(y). Then the leading amplitude equals $W_0 = \psi W$ with a complex-valued function $\psi(y)$ which satisfies, along rays y(t), the scalar transport equation

(45)
$$\frac{\mathrm{d}}{\mathrm{d}t}\psi + \frac{1}{2}(\mathrm{div}\,V)\psi + \langle W|\frac{\mathrm{d}}{\mathrm{d}t}W + HW\rangle\psi = 0.$$

The angular brackets denote the inner product defined by the metric tensor, conjugate linear in the first slot. If the surface S is not plane, then the inner product will not be constant along rays, in general.

We assume that the elasticity tensor, the material density, and the boundary surface are known, including first and second order derivatives. These data determine the transport equation. We make no attempt to derive analytical solution formulas. Rather we explain how the transport equation can be set up ready for numerical computation. To do actual computations, one chooses coordinates $y=(x^1,x^2)$ on S. The well-known ray coordinates are one convenient choice. For computations at interior points, the unique extensions as adapted coordinates, $x=(x^1,x^2,x^3)$, are used. The data determine the acoustic tensor or principal symbol of the elastodynamic operator, L_0 . At every point of S, the acoustic tensor is viewed as a matrix polynomial in the variable conormal to S at y. A factorization algorithm for matrix polynomials has to produce the factorizations (31). Because of homogeneity it suffices to evaluate $Q_0(y,\xi_0,\eta)$ only at unit vectors η . To actually compute Q_0 one may have to determine eigenvalues, eigenvectors, and in general also Jordan-Keldysh chains, of the matrix polynomial. The result of the computation also gives, by (36), the impedance tensor Z_0 .

First and second order derivatives of Z_0 are also needed. To calculate these, we procede as in the proof of Proposition 4.1(e). The surface impedance tensor

satisfies the Ricatti equation

$$(Z_0(\zeta) - iA_1^T(\eta))A_0^{-1}(Z_0(\zeta) + iA_1(\eta)) = A_2(\eta) - \xi_0^2 \rho I,$$

where we suppressed dependency on x from the notation. Denote differentiation with respect to some chosen coordinate or parameter by a prime. Differentiate the Ricatti equation, and obtain a linear equation

$$(46) Q_0^* X - X Q_0 = Y$$

for the complex 3×3 matrix $X \equiv Z_0'$. The right-hand side Y is evaluated as an algebraic expression in Z_0 , A_0 , A_1 , ρ , and in the derivatives A_0' , A_1' , and ρ' . Since Q_0 and Q_0^* have disjoint spectra, the Sylvester equation (46) is uniquely solvable. Knowing Z_0' we also know the derivative Q_0' of $Q_0 = A_0^{-1}(iZ_0 - A_1)$. Differentiating (46), we obtain an equation for the derivative X':

$$Q_0^*X' - X'Q_0 = Y' - (Q_0')^*X - XQ_0'$$

We conclude that first and second order derivatives of Z_0 can be computed from those of the elasticities and the material density by purely algebraic computations.

Next we indicate how to evaluate the Poisson bracket term (44). Recall that bB is the cofactor matrix of Z_0 . Therefore, a first order derivative (bB)' is an algebraic expression in the elements of Z_0 and Z'_0 . So we can evaluate $\{bB, Z_0\}$. Observe from (37) that

(47)
$$b(y,\eta) = \partial_{\xi_0} \det Z_0(y,\xi_0,\eta) \text{ at } h(y,\xi_0,\eta) = 0.$$

We calculate

$$\{B, Z_0\} = b^{-1}\{bB, Z_0\} + B\{b^{-1}I, Z_0\} = b^{-1}\{bB, Z_0\} - b^{-2}B\{bI, Z_0\}$$
$$= b^{-1}\{bB, Z_0\} + b^{-2}\{bI, B\}Z_0 + b^{-2}\{h, b\}I.$$

Here we have taken the Poisson bracket of (43) with bI to obtain the last equality. Hence

$${B, Z_0}W = b^{-1}{bB, Z_0}W + b^{-2}{h, b}W$$
 if $W(y) \in N(y)$.

It is well-known that $\{h, f\}$ vanishes on the set of zeros of h, if f does. Therefore, in view of (47), the scalar Poisson bracket $\{h, b\}$ remains unchanged at the zero set of h, if we replace b by ∂_{ξ_0} det Z_0 .

By (21) and (23), the subprincipal symbol of \hat{Z} is given by

$$Z_{\text{sub}} = Z_{-1} - \frac{1}{2i} \sum_{j} \partial^2 Z_0 / \partial x^j \partial \xi_j,$$

$$Z_{-1} = A_0 Q_{-1} + iT.$$

We know already how to evaluate the second order derivatives of Z_0 . The equation (34) for A_0Q_{-1} is a linear equation of the form (46). The terms on the right-hand side of (34) can be evaluated. This is also true for the normal derivative $\partial_3 Q_0$. Indeed, the factorization (31), and the Ricatti equation for Z_0 hold also for small $x^3 > 0$, so that differentiation with respect to x^3 is possible. Furthermore, the right-hand side of (34) contains the terms B_0 and B_1 . These are defined in (33) and in the displayed formula after (7). Notice that the formulas for B_0 , B_1 , and for T, defined in (41), contain Christoffel symbols. The curvature of S enters into the transport equation only through these expressions. Summarizing, we have explained how to evaluate $H(y)W_0(y)$ in (39) if, as we can assume, $W_0(y)$ belongs to the null-space N(y) of the surface impedance tensor.

The transport equation involves the ray field V(y) which depends on the eikonal $\varphi(y)$. The zeros of the secular equation are simple, so they can be numerically computed in an efficient way. Differentiating

$$0 = \det Z_0(y, -c_R(y, \eta), \eta)$$

by the chain rule, the derivatives of c_R are found as algebraic expressions in the derivatives of Z_0 . The solution of the eikonal equation for φ can be reduced to the solution of ordinary differential equations by Hamilton–Jacobi theory. This also allows the computation of the ray field V, the divergence div V, and the rays y(t).

Remark 9.1. In the special case of a homogeneous body which fills a half-space, the tensors Q_0 and Z_0 are independent of surface points. So the Poisson bracket vanishes, and so does the subprincipal symbol, because the Christoffel symbols are zero too. The scalar transport equation (45) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi + \left(\frac{1}{2}\operatorname{div}V + i\operatorname{Im}\langle W|\frac{\mathrm{d}}{\mathrm{d}t}W\rangle\right)\psi = 0.$$

If W is constant, this is the differential equation for the spreading factor. The imaginary term corresponds to the fact that a choice of W(y) is unique only up to a phase factor $e^{i\alpha}$, $\alpha(y)$ real. In general, for an inhomogeneous, anisotropic body with curved boundary, we have seen how to evaluate the coefficient $\langle W|HW\rangle$ of the scalar transport equation numerically. It is desirable, however, to also have a good understanding of the coefficients in (45). The real part, if positive, would lead to a damping factor. The imaginary part gives rise to the Berry phase first observed by Babich in the early 1960's; see Babich & Kiselev (2004) and the references therein.

APPENDIX A. SPECTRAL FACTORIZATION OF POSITIVE DEFINITE MATRIX POLYNOMIALS

The purpose of this appendix is to state and prove the spectral factorization theorem for self-adjoint matrix polynomials which is fundamental to our approach to Rayleigh wave theory. Refer to Gohberg *et al.* (1982) for a comprehensive treatment of matrix factorizations.

Let H be a finite-dimensional complex Hilbert space, dim H = n. Let A(s) be a quadratic polynomial with values in the space of linear operators on H,

$$A(s) = A_0 s^2 + (A_1 + A_1^*)s + A_2.$$

(We denote the adjoint by a star.) A number $s \in \mathbb{C}$ is called an eigenvalue of the polynomial if A(s) is singular. The spectrum of A is the set $\sigma(A)$ of its eigenvalues. Assume that, in addition, the polynomial is self-adjoint, i.e., $A(\bar{s}) = A(s)^*$ holds for all s, and that A_0 is positive definite. If s is an eigenvalue of A then so is \bar{s} . If A has no real spectrum, then A(s) is positive definite for real s.

In the following, we abuse language and often call linear operators matrices despite the fact that we do not fix a basis of H.

The following is a special case of Theorem 11.2 in Gohberg et al. (1982).

Proposition A.1. Assume that A(s) has no real eigenvalues. Denote by σ_+ and σ_- the intersection of the spectrum of A(s) with the upper and the lower complex half-plane, respectively. There is a unique matrix Q with spectrum contained in the upper half-plane such that

(48)
$$A(s) = (s - Q^*)A_0(s - Q)$$

holds for all s. If γ_+ is a closed Jordan contour which contains σ_+ in its interior and σ_- in its exterior, then

(49)
$$Q \oint_{\gamma_{+}} A(t)^{-1} dt = \oint_{\gamma_{+}} t A(t)^{-1} dt$$

holds. The integrals are non-singular.

If the coefficient matrices A_j depend continuously or differentiably on some parameters, then, in view of (49), so does Q. This follows from the integral formula (49).

We need some preparations before we can give the proof of the propostion.

Polynomial division of A(s) by s-Q gives $A(s)=(s-Q_-)A_0(s-Q)+F$ with $Q_-A_0+A_0Q+A_1+A_1^*=0$ and $Q_-A_0Q+F=A_2$. A simple calculation shows that the remainder F is zero if and only if

(50)
$$A_0Q^2 + (A_1 + A_1^*)Q + A_2 = 0$$

holds. The divisibility criterion (50) is known as the solvency equation.

The Stroh companion matrix of the polynomial A(s) is the linear operator N on H^2 which is given in block form by

(51)
$$N = \begin{bmatrix} -A_0^{-1}A_1 & A_0^{-1} \\ -A_2 + A_1^*A_0^{-1}A_1 & -A_1^*A_0^{-1} \end{bmatrix}.$$

A direct calculation proves the identity

(52)
$$\begin{bmatrix} sA_0 + A_1^* & I \\ -A_0 & 0 \end{bmatrix} (s - N) = \begin{bmatrix} A(s) & 0 \\ -sA_0 - A_1 & I \end{bmatrix},$$

where s = sI. Obviously, the spectrum of A equals the spectrum of N. Moreover, $(u,t)^T$ is an eigenvector of N with eigenvalue s if and only if A(s)u = 0 and $t = (sA_0 + A_1)u$ hold. Set $L = \begin{bmatrix} I & 0 \end{bmatrix}$ and $R = \begin{bmatrix} 0 & I \end{bmatrix}^T$. Passing to inverses in (52), we find that

(53)
$$A(s)^{-1} = L(s-N)^{-1}R$$

holds for all complex numbers s.

Remark A.1. We call N the Stroh companion matrix of the polynomial A(s) because, in elasticity where n=3, N equals Stroh's sextic matrix, Stroh (1962). In that setting, and in the main part of the present paper, A(s) is the acoustic tensor at $\eta + s\nu$, where ν is a surface normal and η a horizontal propagation direction. The Stroh matrix is a convenient example of a linearization of A(s), which means that (53) holds. The standard companion matrix of A(s) could however also be used for this purpose.

The holomorphic functional calculus of N assigns to a function f, which is holomorphic in a neighbourhood of the spectrum σ of N, the matrix $f(N) = \oint f(s)(s-N)^{-1} \, \mathrm{d}s/2\pi i$. The contour of integration must be chosen such that its winding numbers around the eigenvalues of N in the support of f are equal to one. If σ_0 is a subset of the spectrum of N, and if $f_0 = 1$ and $f_0 = 0$ in neighbourhoods of σ_0 and $\sigma \setminus \sigma_0$, respectively, then $P_0 = f_0(N)$ is a projector, called the Riesz projector associated with σ_0 .

By (53), if f is holomorphic in a neighbourhood of the spectrum of N, then

(54)
$$\frac{1}{2\pi i} \oint f(t)A(t)^{-1} dt = Lf(N)R,$$

where the contour of integration must be admissable for the functional calculus.

Proof of Proposition A.1. Denote by P_{\pm} the Riesz projector of N associated with σ_{\pm} . Clearly, $P_{+}P_{-}=0=P_{-}P_{+}$, and $P_{+}+P_{-}$ is the unit matrix. The range Y_{\pm} of P_{\pm} is an N-invariant subspace of H^{2} , and the direct sum decomposition $H^{2}=Y_{+}\oplus Y_{-}$ holds. Denote by N_{\pm} the restriction of N to Y_{\pm} . Clearly, $\sigma(N_{\pm})=\sigma_{\pm}$.

For sufficiently large $\rho > 0$, we denote by γ_{\pm} the Jordan contour which consists of the segment $[-\rho, \rho]$ and the semicircle $\rho e^{\pm it}$, $0 \le t \le \pi$. By (54) we get the identity

$$\pm LP_{\pm}R = \frac{1}{2\pi i} \oint_{\gamma_{+}} A(t)^{-1} dt = \frac{1}{2\pi i} \int_{-\infty}^{\infty} A(t)^{-1} dt.$$

The last equality follows when letting $\rho \to \infty$. Since $A(t)^{-1}$ is positive definite for t real, the integral defines a non-singular matrix. We infer that $\dim Y_{\pm} = n$. Moreover, LP_{+} has rank n, and there is a unique linear map $L^{\sharp}: H \to H^{2}$ which satisfies $L^{\sharp}LP_{+} = P_{+}$. Clearly, $L^{\sharp} = P_{+}L^{\sharp}$ has rank n.

We prove that the proposition holds with $Q \equiv LNP_+L^{\sharp}$. Observe that $Q^j = LN^jP_+L^{\sharp}$. Using (54), we find that

(55)
$$\frac{1}{2\pi i} \oint_{\gamma_{+}} s^{j} A(s)^{-1} ds = L N^{j} P_{+} R = Q^{j} L P_{+} R.$$

Using Cauchy's integral theorem, we get

$$(A_0Q^2 + (A_1 + A_1^*)Q + A_2)LP_+R = 0.$$

This implies the solvency condition (50). Thus $A(s) = (s - Q_{-})A_{0}(s - Q)$ holds with some matrix Q_{-} .

Observe that LP_+L^{\sharp} equals unity. In view of the definitions, this implies that Q is similar to N_+ . Thus the spectrum of Q lies in the upper half-plane. By (52), we have $\det(A_0)\det(s-N)=\det(A(s))$, and therefore

$$\det(s - N_{-}) \det(s - N_{+}) = \det(s - Q_{-}) \det(s - Q).$$

This implies $\det(s - N_{-}) = \det(s - Q_{-})$. Hence the spectrum of Q_{-} equals the spectrum N_{-} , which is contained in the lower half-plane. This means that the factorization obtained is spectral.

If $A(s) = (s-Q_-)A_0(s-Q)$ holds with the spectra of Q and Q_- contained in σ_+ and σ_- , respectively, then (49) follows by Cauchy's theorem. The integral on the left-hand side of (49) equals $2\pi i L P_+ R$ which is non-singular. The uniqueness of the spectral factorization follows. Since a is self-adjoint, we also have the factorization $A(s) = (s-Q^*)A_0(s-Q_-^*)$. Since the spectra of Q and of Q_-^* are contained in the upper half-plane, this proves that $Q_- = Q^*$.

 $Acknowledgement. \ \,$ The author thanks Aleksei Kiselev and Gerardo Mendoza for valuable remarks and suggestions.

References

Alinhac, S. & Gérard, P. 2007 Pseudo-differential operators and the Nash-Moser theorem, vol. 82 of Graduate Studies in Mathematics. Providence, RI: American Mathematical Society. Translated from the 1991 French original by Stephen S. Wilson.

Babich, V. M. 1961 Propagation of Rayleigh waves along the surface of a homogeneous elastic body of arbitrary shape. *Dokl. Akad. Nauk SSSR*, **137**, 1263–1266.

- Babich, V. M. & Kirpichnikova, N. Y. 2004 A new approach to the problem of the Rayleigh wave propagation along the boundary of a non-homogeneous elastic body. *Wave Motion*, **40**, 209–223.
- Babich, V. M. & Kiselev, A. P. 2004 "Nongeometrical phenomena" in propagation of elastic surface waves. In *Surface waves in anisotropic and laminated bodies and defects detection*, vol. 163 of *NATO Sci. Ser. II Math. Phys. Chem.*, pp. 119–129. Dordrecht: Kluwer Acad. Publ. http://dx.doi.org/10.1007/1-4020-2387-1_5
- Babich, V. M. & Rusakova, N. Y. 1962 The propagation of Rayleigh waves along the surface of a non-homogeneous elastic body of arbitrary shape. *Ž. Vyčisl. Mat. i Mat. Fiz.*, **2**, 652–665. Engl. translation in U.S.S.R. Comput. Math. Math. Phys. **2** (1963), no. 4, 719–735.
- Barnett, D. M. & Lothe, J. 1985 Free surface (Rayleigh) waves in anisotropic elastic half-spaces: The surface impedance method. *Proc. R. Soc. Lond. A*, **402**, 135–152.
- Chadwick, P. & Smith, G. D. 1977 Foundations of the theory of surface waves in anisotropic elastic materials. *Adv. Appl. Mechanics*, **17**, 303–376.
- Courant, R. & Lax, P. D. 1956 The propagation of discontinuities in wave motion. *Proc. Nat. Acad. Sci.*, **42**, 872–876.
- Dencker, N. 1982 On the propagation of polarization sets for systems of real principal type. *J. Funct. Anal.*, **46**, 351–372.
- Destrade, M. 2007 Seismic Rayleigh waves on an exponentially graded, orthotropic half-space. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, **463**(2078), 495–502. http://dx.doi.org/10.1098/rspa.2006.1774
- Gohberg, I., Lancaster, P. & Rodman, L. 1982 *Matrix polynomials*. New York: Academic Press.
- Gregory, R. D. 1971 The propagation of Rayleigh waves over curved surfaces at high frequency. *Math. Proc. Cambridge Phil. Soc.*, **70**, 103–121.
- Hansen, S. 2011 Rayleigh-type surface quasimodes in general linear elasticity. *Anal. PDE*, 4(3), 461–497. http://dx.doi.org/10.2140/apde.2011.4.461
- 2012surface Hansen. S. The impedance tensor and Rayleigh waves. In **Proceedings** oftheInternationalConferenceDaysDiffraction, St.Petersburg,2012,115-118.IEEE. http://ieeexplore.ieee.org/servlet/opac?punumber=6388606
- Hansen, S. & Röhrig, D. 2004 Lagrangian solutions to systems of real principal type. *Math. Nachr.*, **277**, 47–55. http://dx.doi.org/10.1002/mana.200310219
- Hörmander, L. 1965 Pseudo-differential operators. Comm. Pure Appl. Math., 18, 501–517.
- Hörmander, L. 1985 The analysis of linear partial differential operators iii. Berlin and New York: Springer-Verlag.
- Hörmander, L. 1990 The analysis of linear partial differential operators. I, vol. 256 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Berlin: Springer-Verlag, 2nd edn. Distribution theory and Fourier analysis.
- Ingebrigtsen, K. A. & Tonning, A. 1969 Elastic surface waves in crystals. *Phys. Rev.*, **184**, 942–951. http://link.aps.org/doi/10.1103/PhysRev.184.942
- Ivanovici, O. 2012 Counterexamples to the Strichartz inequalities for the wave equation in general domains with boundary. J. Eur. Math. Soc. (JEMS), 14(5),

- 1357-1388. http://dx.doi.org/10.4171/JEMS/335
- Ivanovici, O., Lebeau, G. & Planchon, F. 2014 Dispersion for the wave equation inside strictly convex domains I: the Friedlander model case. *Ann. Math.*, **180**, 323–380.
- Kamotskii, I. V. & Kiselev, A. P. 2009 An energy approach to the proof of the existence of Rayleigh waves in an anisotropic elastic half-space. *Prikl. Mat. Mekh.*, 73(4), 645–654. http://dx.doi.org/10.1016/j.jappmathmech.2009.08.003
- Karal, Jr., F. C. & Keller, J. B. 1964 Geometrical theory of elastic surface-wave excitation and propagation. *J. Acoust. Soc. of America*, **36**(1), 32–40.
- Lothe, J. & Barnett, D. M. 1976 On the existence of surface-wave solutions for anisotropic half-spaces with free surface. *J. Applied Physics*, 47, 428–433.
- Mielke, A. & Fu, Y. B. 2004 Uniqueness of the surface-wave speed: A proof that is independent of the Stroh formalism. *Math. Mech. Solids*, **9**, 5–15.
- Nakamura, G. 1991 Existence and propagation of Rayleigh waves and pulses. In *Modern theory of anisotropic elasticity and applications (Research Triangle Park, NC, 1990)*, pp. 215–231. Philadelphia, PA: SIAM.
- Nomofilov, V. E. 1979 Propagation of quasistationary Rayleigh waves in a nonuniform anisotropic elastic medium. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, **89**, 234–245, 295. Mathematical questions in the theory of wave propagation, 10.
- Petrowsky, I. G. 1945 On the propagation velocity of discontinuities of the displacement derivatives on the surface of an inhomogeneous elastic body of arbitrary form. C. R. (Doklady) Acad. Sci. URSS (N. S.), 47, 255–258.
- Rayleigh, L. 1887 On waves propagated along the plane surface of an elastic solid. *Proc. Lond. Math. Soc.*, **17**, 4–11.
- Stroh, A. N. 1962 Steady state problems in anisotropic elasticity. *J. Math. and Phys.*, **41**, 77–103.
- Synge, J. L. 1957 Elastic waves in anisotropic media. J. Math. and Phys., 35, 323–334.
- Taylor, M. E. 1979 Rayleigh waves in linear elasticity as a propagation of singularities phenomenon. In *Partial differential equations and geometry* (ed. C. Byrnes), pp. 273–291. Marcel Dekker.
- Wloka, J. T., Rowley, B. & Lawruk, B. 1995 Boundary value problems for elliptic systems. Cambridge: Cambridge University Press. http://dx.doi.org/10.1017/CB09780511662850

Institut für Mathematik, Universität Paderborn, 33095 Paderborn, Germany $E\text{-}mail\ address$: soenke@math.upb.de