# ON-LINE VERTEX RANKING OF TREES 

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#### Abstract

A $k$-ranking of a graph $G$ is a labeling of its vertices from $\{1, \ldots, k\}$ such that any nontrivial path whose endpoints have the same label contains a larger label. The least $k$ for which $G$ has a $k$ ranking is the ranking number of $G$, also known as tree-depth. Applications of rankings include VLSI design, parallel computing, and factory scheduling. The on-line ranking problem asks for an algorithm to rank the vertices of $G$ as they are presented one at a time along with all previously ranked vertices and the edges between them (so each vertex is presented as the lone unranked vertex in a partially labeled induced subgraph of $G$ whose final placement in $G$ is not specified). The on-line ranking number of $G$ is the minimum over all such algorithms of the largest label that algorithm can be forced to use. We give bounds on the on-line ranking number of trees in terms of maximum degree, diameter, and number of interior vertices.


## 1. Introduction

We consider a special type of proper vertex coloring using positive integers, called "ranking." As with proper colorings, there exist variations on the original ranking problem. In this paper we consider the on-line ranking problem, introduced by Tuza and Voigt in 1995 [14].

Definition 1.1. A ranking of a finite simple graph $G$ is a function $f: V(G) \rightarrow\{1,2, \ldots\}$ with the property that if $u \neq v$ but $f(u)=f(v)$, then every $u$, $v$-path contains a vertex $w$ satisfying $f(w)>f(u)$ (equivalently, every path $P$ contains a unique vertex with largest label, where $f(v)$ is called the label of $v$ ). A $k$-ranking of $G$ is a ranking $f: V(G) \rightarrow\{1, \ldots, k\}$. The ranking number of a graph $G$, denoted here by $\rho(G)$ (though in the literature often as $\chi_{r}(G)$ ), is the minimum $k$ such that $G$ has a $k$-ranking.

Vertex rankings of graphs were introduced in [5], and results through 2003 are surveyed in [7]. Their study was motivated by applications to VLSI layout, cellular networks, Cholesky factorization, parallel processing, and computational geometry. For example, vertex ranking models the efficient assembly of a graph from vertices, where each stage of construction consists of individual vertices being added in such a way that no component ever has more than one new vertex. Vertex rankings are sometimes called ordered colorings, and the ranking number of a graph is trivially equal to its "tree-depth," a term introduced by Nes̆etřil and Ossona de Mendez in 2006 [11] in developing their theory of graph classes having bounded expansion.

The vertex ranking problem has spawned multiple variations, including list ranking [10] and on-line ranking, studied here. The on-line ranking problem is to vertex ranking as the on-line coloring problem is to ordinary vertex coloring.
1.1. The on-line vertex ranking problem. The on-line vertex ranking problem is a game between two players, Presenter and Ranker. A class $\mathcal{G}$ of unlabeled graphs is shown to both players at the beginning of the game. In round 1, Presenter presents to Ranker the graph $G_{1}$ consisting of a single vertex $v_{1}$, to which Ranker assigns a positive integer label $f\left(v_{i}\right)$. In round $i$ for $i>1$, Presenter extends $G_{i-1}$ to an $i$-vertex induced subgraph $G_{i}$ of a graph $G \in \mathcal{G}$ by presenting an unlabeled vertex $v_{i}$ (without specifying which copy of $G_{i}$ among all induced subgraphs of graphs in $\mathcal{G}$ ). Ranker must then extend the ranking $f$ of $G_{i-1}$ to a ranking of $G_{i}$ by assigning $f\left(v_{i}\right)$.

Presenter seeks to maximize the largest label assigned during the game, while Ranker seeks to minimize it. The on-line ranking number of $\mathcal{G}$, denoted here by $\stackrel{\rho}{\rho}(\mathcal{G})$ (though in the literature often as $\chi_{r}^{*}(\mathcal{G})$ ), is the resulting maximum assigned value under optimal play. If Presenter can guarantee that arbitrarily high labels are used, then $\rho(\mathcal{G})=\infty$. If $\mathcal{G}$ is the class of induced subgraphs of a graph $G$, then we define $\rho(G)=\rho(\mathcal{G})$.

[^0]Note that $\stackrel{\rho}{\rho}\left(\mathcal{G}^{\prime}\right) \leq \stackrel{\rho}{\rho}(\mathcal{G})$ if every graph in $\mathcal{G}^{\prime}$ is an induced subgraph of a graph in $\mathcal{G}$, since any strategy for Ranker on $\mathcal{G}$ includes a strategy on $\mathcal{G}^{\prime}$. Also $\rho(G) \leq \stackrel{\rho}{\rho}(G)$ trivially.

Several papers have been written about the on-line ranking number of graphs, including [2], [12], and [13]; some of the results from these papers will be mentioned later. On-line ranking has also been looked at from an algorithmic perspective, in the sense that one seeks a fast algorithm for determining the smallest label Ranker is allowed to use on a given turn; see [3, [6, [8, and [9]. Our paper is of the former variety.

A minimal ranking of $G$ is a ranking $f$ with the property that decreasing $f$ on any set of vertices produces a non-ranking. Let $\psi(G)$ be the largest label used in any minimal ranking of $G$. Isaak, Jamison, and Narayan [4] showed that the minimal rankings of $G$ are precisely the rankings produced when Ranker plays greedily, so $\stackrel{\rho}{\rho}(G) \leq \psi(G)$. For the $n$-vertex path $P_{n}$, this yields $\stackrel{\circ}{\rho}\left(P_{n}\right) \leq \psi\left(P_{n}\right)=$ $\left\lfloor\log _{2}(n+1)\right\rfloor+\left\lfloor\log _{2}\left(n+1-2^{\left\lfloor\log _{2} n\right\rfloor-1}\right)\right\rfloor$. Bruoth and Hornák [2 gave the best known lower bound for paths $\stackrel{\circ}{\rho}\left(P_{n}\right) \geq 1.619\left(\log _{2} n\right)-1$.
1.2. Our Results. Recall that the distance between two vertices $u$ and $v$ in a connected graph $G$ is the number of edges in a shortest $u, v$-path in $G$. The eccentricity of $v$ is the greatest distance between $v$ and any other vertex in $G$. The diameter of $G$ is the maximum eccentricity of any vertex in $G$.

In Section 2, we give bounds on the on-line ranking number of $T_{k, d}$, defined for $k \geq 2$ and $d \geq 0$ to be the largest tree having maximum degree $k$ and diameter $d$, i.e., the tree all of whose internal vertices have degree $k$ and all of whose leaves have eccentricity $d$. Since the family of trees with maximum degree at most $k$ and diameter at most $d$ is precisely the set of connected induced subgraphs of $T_{k, d}$, our upper bound on $\stackrel{\rho}{\rho}\left(T_{k, d}\right)$ also serves as an upper bound for the on-line ranking number of this class of graphs.

Theorem 1.2. There exist positive constants $c$ and $c^{\prime}$ such that if $d \geq 0$ and $k \geq 3$, then $c(k-1)^{\lfloor d / 4\rfloor} \leq$ $\stackrel{\circ}{\rho}\left(T_{k, d}\right) \leq c^{\prime}(k-1)^{\lfloor d / 3\rfloor}$.

We find it informative to compare the on-line ranking number of $T_{k, d}$ to the regular ranking number of $T_{k, d}$.
Proposition 1.3. For $k \geq 3$, we have $\rho\left(T_{k, d}\right)=\lceil d / 2\rceil+1$.
Proof. The construction for the upper bound assigns label $i+1$ to vertices at distance $i$ from the nearest leaf, with the exception of labeling one of the vertices in the central edge of $T_{k, d}$ with $(d+3) / 2$ if $d$ is odd. For the lower bound, note that choosing the unique highest ranked vertex $v$ of a tree $T$ reduces the ranking problem to individually ranking the components of $T-v$. Thus if there exists $u \in V(T)$ such that for every $w \in V(T)$ each component of $T-u$ is isomorphic to a subtree of some component of $T-w$, then $T$ can be optimally ranked by optimally ranking each component of $T-u$ and labeling $u$ one greater than the largest label used on those components. Letting $F_{i}$ denote the subforest of $T_{k, d}$ induced by the set of vertices within distance $i$ of a leaf, we conclude by induction on $i$ that for $1 \leq i \leq\lceil d / 2\rceil$, each component of $F_{i}$ is optimally ranked by the upper bound construction.

Setting $n=\left|V\left(T_{k, d}\right)\right|$ and using Theorem 1.2 and Proposition 1.3, we see that $\stackrel{\circ}{\rho}\left(T_{k, d}\right)=\Omega(\sqrt{n})$ while $\rho\left(T_{k, d}\right)=O(\log n)$. Thus $\stackrel{\rho}{\rho}$ is exponentially larger than $\rho$ on these trees. Theorem 1.5 shows that this large separation between $\rho$ and $\rho$ does not hold for all trees. Nevertheless we conjecture a general upper bound like that of Theorem 1.2 ,
Conjecture 1.4. There exist universal constants a and batisfying $0<a<1<b$ such that $\stackrel{\circ}{\rho}(T) \leq$ $b(k n)^{a}$ for any $n$-vertex tree $T$ with maximum degree $k$.

In Section 3, we consider the on-line ranking number of trees with few internal vertices. Let $\mathcal{T}^{p, q}$ be the family of trees having at most $p$ internal vertices and diameter at most $q$. The main result of that section is an upper bound on $\stackrel{\rho}{\rho}\left(\mathcal{T}^{p, q}\right)$ for any $p$ and $q$.
Theorem 1.5. $\stackrel{\rho}{\rho}\left(\mathcal{T}^{p, q}\right) \leq p+q+1$.
Since $q \leq p+1$, this establishes $\stackrel{\rho}{\rho}\left(\mathcal{T}^{p, q}\right) \leq 2 p+2$. We also compute $\stackrel{\circ}{\rho}\left(\mathcal{T}^{2,3}\right)=4$. This extends the work of Schiermeyer, Tuza, and Voigt [13], who characterized the families of graphs having on-line ranking number 1,2 , and 3 .

## 2. Strategies for Presenter and Ranker on $T_{k, d}$

In this section, we obtain upper and lower bounds on $\left.\stackrel{\rho}{( } T_{k, d}\right)$, where $T_{k, d}$ is the largest tree having maximum degree $k$ and diameter $d$. For convenience, we let $T_{k, r}^{*}$ denote the tree with unique root vertex $v^{*}$ such that every internal vertex has $k$ children and every leaf is distance $r$ from $v^{*}$. For $U \subseteq V(G)$, let $G[U]$ denote the subgraph of $G$ induced by $U$.

### 2.1. A strategy for Presenter. We first develop a tool for proving lower bounds.

Theorem 2.1. Let $G$ be a connected graph. Suppose for some $U \subset V(G)$ that $G-U$ has components $G^{0}, G^{1}, \ldots, G^{a}$, all isomorphic to some graph $F$. If also $U$ contains disjoint subsets $U^{1}, \ldots, U^{a}$ so that each $U^{i}$ consists of the internal vertices of a path joining $G^{0}$ and $G^{i}$, then $\stackrel{\circ}{\rho}(G) \geq \stackrel{\circ}{\rho}(F)+a$.

Proof. Presenter has a strategy to produce a copy of $F$ on which Ranker must use a label at least $\rho(F)$. Begin by playing this strategy $a+1$ times on distinct sets of vertices. Index the resulting copies of $F$ as $G^{0}, G^{1}, \ldots, G^{a}$ so that $G^{0}$ is a copy whose largest label is smallest (in the labeling by Ranker) among the copies of $F$. Present $U$ in any order to complete $G$.

Let $m_{0}$ denote the largest label given to a vertex in $V\left(G^{0}\right)$. For $1 \leq i \leq a$, let $m_{i}$ denote the largest label given to a vertex in $V\left(G^{i}\right) \cup U^{i}$. Set $H^{i}=G\left[V\left(G^{0}\right) \cup U^{i} \cup V\left(G^{i}\right)\right]$ for $1 \leq i \leq a$. Each $H^{i}$ is a connected subgraph of $G$, so $m_{0}<m_{i}$. For $i \neq j, H^{i} \cup H^{j}$ is a connected subgraph of $G$, so $m_{i} \neq m_{j}$. Thus the largest $m_{i}$ satisfies $m_{i} \geq m_{0}+a \geq \circ(F)+a$.


Figure 1. The graph $G$ of Theorem 2.1.
Note that $T_{k, 2 r}$ consists of a copy of $T_{k-1, r}^{*}$ and a copy of $T_{k-1, r-1}^{*}$ with an edge joining their roots, and $T_{k, 2 r+1}$ consists of two copies of $T_{k-1, r}^{*}$ with an edge joining their roots. Hence $T_{k-1,|d / 2|}^{*}$ is an induced subgraph of $T_{k, d}$, so a lower bound on $\stackrel{\circ}{\rho}\left(T_{k-1,\lfloor d / 2\rfloor}^{*}\right)$ also serves as a lower bound on $\stackrel{\circ}{\rho}\left(T_{k, d}\right)$.

Corollary 2.2. If $k \geq 2$ and $r \geq 0$, then $\stackrel{\circ}{\rho}\left(T_{k, r}^{*}\right) \geq k^{\lfloor r / 2\rfloor}$.
Proof. Since $T_{k, r}^{*}$ is an induced subgraph of $T_{k, r+1}^{*}$, we have $\stackrel{\rho}{\rho}\left(T_{k, r}^{*}\right) \leq \stackrel{\circ}{\rho}\left(T_{k, r+1}^{*}\right)$, so we may assume that $r$ is even. Set $a=k^{r / 2}$, and let $U$ be the set of vertices $u_{1}, \ldots, u_{a}$ at distance $r / 2$ from $v^{*}$. Define $G$ to be the subtree of $T_{k, r}^{*}$ obtained by deleting, for each $u_{i} \in U$, the descendants of all but one child of $u_{i}$. Now $G-U$ consists of $a+1$ disjoint copies of $T_{k, r / 2-1}^{*}$. Let $G^{0}$ be the component rooted at $v^{*}$, and for $1 \leq i \leq a$ let $G^{i}$ be the component rooted at the child of $u_{i}$. Setting $U^{i}=\left\{u_{i}\right\}$ for $1 \leq i \leq a$, we see that $U^{i}$ contains the lone vertex of the path joining $G^{0}$ and $G^{i}$. By Theorem [2.1] $\stackrel{\rho}{\rho}\left(T_{k, r}^{*}\right) \geq \stackrel{\rho}{\rho}(G) \geq a$.

Corollary 2.3. If $k \geq 3$ and $d \geq 0$, then $\stackrel{\rho}{\rho}\left(T_{k, d}\right) \geq(k-1)^{\lfloor d / 4\rfloor}$.
We finish this subsection with a comment on Conjecture 1.4. Subdivide each edge of the star $K_{1, a}$ to get a $(2 a+1)$-vertex tree $G$. Letting $G^{0}, G^{1}, \ldots, G^{a}$ correspond to the vertices of the unique maximum independent set of $G$, Theorem 2.1 yields $\stackrel{\rho}{\rho}(G) \geq a+1>|V(G)| / 2$. Thus Conjecture 1.4 cannot be strengthened to the statement "There exist universal constants $a$ and $b$ satisfying $0<a<1<b$ such that $\stackrel{\circ}{\rho}(T) \leq b n^{a}$ for any tree $n$-vertex tree $T$."
2.2. A strategy for Ranker. We now exhibit a strategy for Ranker to establish an upper bound on $\stackrel{\circ}{\rho}\left(T_{k, d}\right)$. In Section 3 we shall see $\stackrel{\circ}{\rho}\left(T_{k, 0}\right)=1, \stackrel{\circ}{\rho}\left(T_{k, 1}\right)=2, \stackrel{\circ}{\rho}\left(T_{k, 2}\right)=3, \stackrel{\circ}{\rho}\left(T_{k, 3}\right)=4, \stackrel{\circ}{\rho}\left(T_{k, 4}\right) \leq k+6$, and $\stackrel{\circ}{\rho}\left(T_{k, 5}\right) \leq 2 k+6$, so here we only consider $d \geq 6$. In specifying a strategy for Ranker on $T_{k, d}$, we will give a procedure for ranking the presented vertex $v$ based solely on the component containing $v$ in the graph presented so far.

Definition 2.4. Let $T(v)$ denote the component containing $v$ when $v$ is presented. Given two sets $A$ and $B$ of labels, not necessarily disjoint, let $T_{B}(v)$ be the largest subtree of $T(v)$ containing $v$ all of whose other vertices are labeled from $B$. Should it exist, let $f_{B}^{A}(v)$ denote the smallest element of $A$ that would complete a ranking of $T_{B}(v)$.

The following lemmas analyze when $f_{B}^{A}(v)$ exists and, if it does exist, when $f_{B}^{A}(v)$ provides a valid label that Ranker can give $v$.

Lemma 2.5. Suppose that each vertex $u \in V\left(T_{B}(v)\right)$ labeled from $A$ was given label $f_{B}^{A}(u)$ when it arrived. If $\min A>\max (B-A)$, and every component of $T_{B}(v)-v$ lacks some label in $A$, then $f_{B}^{A}(v)$ exists.

Proof. Let $A=\left\{a_{1}, \ldots, a_{m}\right\}$, with $a_{1}<\ldots<a_{m}$. For a component $T$ of $T_{B}(v)-v$ having $q$ distinct labels from $A$, we claim that the largest label used on $T$ is $a_{q}$. Each vertex $u \in V\left(T_{B}(v)\right)$ labeled from $A$ was given label $f_{B}^{A}(u)$ when it arrived, with $\min A>\max (B-A)$, so if $f_{B}^{A}(u)=a_{i}$ then either $i=1$ or $a_{i-1}$ was already used in $T_{B}(u)$ (since otherwise $a_{i-1}$ would complete a ranking). Hence all used labels are less than all missing labels in $A$. Since every component of $T_{B}(v)-v$ lacks some label in $A$, we thus have $a_{q}<a_{m}$. Therefore $a_{m}$ is a valid label for $v$ in $T_{B}(v)$ because the largest label on any path through $v$ would be used only at $v$. Hence $f_{B}^{A}(v)$ exists.

Lemma 2.6. Suppose that $f_{B}^{A}(v)$ exists. If $T(v)=T_{B}(v)$ or if all vertices of $T(v)-V\left(T_{B}(v)\right)$ having a neighbor in $T_{B}(v)$ are in the same component of $T(v)-v$ and have labels larger than $\max (A \cup B)$, then setting $f(v)=f_{B}^{A}(v)$ is a valid move by Ranker.
Proof. Set $f(v)=f_{B}^{A}(v)$. Let $P$ be an $x, y$-path in $T(v)$ such that $x \neq y, f(x)=f(y)=\ell$, and $v \in V(P)$. We show that $P$ has an internal vertex $z$ satisfying $f(z)>\ell$. Since $f_{B}^{A}(v)$ completes a ranking of $T_{B}(v)$, we may assume that $T(v) \neq T_{B}(v)$ and $P$ contains some vertex outside $T_{B}(v)$. By hypothesis all such vertices having a neighbor in $T_{B}(v)$ are in the same component of $T(v)-v$, so we may assume $x \in V(T(v))-V\left(T_{B}(v)\right)$ and $y \in V\left(T_{B}(v)\right)$.

Since $v$ is labeled from $A$ and $T_{B}(v)-v$ is labeled from $B$ with $y \in V\left(T_{B}(v)\right)$, we have $\ell \in A \cup B$. By hypothesis all vertices of $T(v)-V\left(T_{B}(v)\right)$ having neighbors in $T_{B}(v)$ have labels larger than $\max (A \cup B)$, so $x$ has no neighbor in $T_{B}(v)$. Hence $P$ contains some internal vertex $z$ outside $T_{B}(v)$ with a neighbor in $T_{B}(v)$. By hypothesis, $f(z)>\max (A \cup B) \geq \ell$.

Set $j=\lfloor d / 3\rfloor$. Break the labels from 1 to $3\left|V\left(T_{k-1, j}^{*}\right)\right|$ into three segments, with $X$ consisting of the lowest $\left|V\left(T_{k-1, j-1}^{*}\right)\right|$ labels, $Y$ the next $\left|V\left(T_{k-1, j}^{*}\right)\right|-\left|V\left(T_{k-1, j-1}^{*}\right)\right|$ labels, and $Z$ the remaining high labels. For $k \geq 3$, we give Ranker a strategy in the on-line ranking game on $T_{k, d}$ that uses labels from $X \cup Y \cup Z$. Since $\stackrel{\circ}{\rho}\left(T_{k, d}\right) \leq 3\left|V\left(T_{k-1, j}^{*}\right)\right|=3\left((k-1)^{j}+\sum_{i=0}^{j-1}(k-1)^{i}\right)<6(k-1)^{j}$, this establishes the following.
Theorem 2.7. If $d \geq 0$ and $k \geq 3$, then $\stackrel{\circ}{\rho}\left(T_{k, d}\right) \leq 6(k-1)^{\lfloor d / 3\rfloor}$.
The goal of our strategy for Ranker is to label from $X \cup Y$ many vertices that lie within distance $j-1$ of a leaf, reserving $Z$ for a small number of middle vertices.

Algorithm 2.8. Compute $f(v)$ according to the following table.

| Value of $\mathbf{f}(\mathrm{v})$ | Conditions |
| :---: | :--- |
| (I) $f_{X}^{X}(v)$ | (1) $T_{X}(v)$ is isomorphic to a subgraph of $T_{k-1, j-1}^{*}$, and <br> $(2)$ either $T_{X}(v)=T(v)$ or there exists a vertex $u$ in $T(v)$ labeled from $Y$ such <br> that $T_{X}(v)$ is the component of $T(v)-u$ containing $v$. |
| (II) $f_{X \cup Y}^{X}(v)$ | (1) The eccentricity of $v$ in $T(v)$ is at least $d-j$, and <br> (2) there exists no vertex $u$ in $T(v)$ labeled from $Y$ such that $T_{X}(v)$ is the <br> component of $T(v)-u$ containing $v$. |
| (III) $f_{X \cup Y \cup Z}^{Z}(v)$ | (1) The eccentricity of $v$ in $T(v)$ is less than $d-j$, and <br> $(2)$ either $T_{X}(v)$ is not isomorphic to a subgraph of $T_{k-1, j-1}^{*}$ or $T_{X}(v) \neq T(v)$. |

Before we go any further, we need to show that Algorithm 2.8 is, in fact, an algorithm. Note that $d-j \geq 2 j$.

Proposition 2.9. When playing the on-line ranking game on $T_{k, d}$, each presented vertex $v$ satisfies the conditions of exactly one of the three cases.

Proof. If the eccentricity of $v$ in $T(v)$ is less than $d-j$, then Case II does not apply. If furthermore $T_{X}(v)$ is isomorphic to a subgraph of $T_{k-1, j-1}^{*}$ and $T_{X}(v)=T(v)$, then Case I applies but Case III does not. Otherwise, Case III applies, but Case I does not since if $T_{X}(v)$ is isomorphic to a subgraph of $T_{k-1, j-1}^{*}$, then $T_{X}(v) \neq T(v)$ and a vertex $u$ in $T(v)$ such that $T_{X}(v)$ is the component of $T(v)-u$ containing $v$ would have eccentricity at most $\max \{d-j-2,2 j-1\}$, which is less than $d-j$, precluding $u$ from being labeled from $Y$.

If the eccentricity of $v$ in $T(v)$ is at least $d-j$, then Case III does not apply. If furthermore $T_{X}(v)$ is isomorphic to a subgraph of $T_{k-1, j-1}^{*}$, then the eccentricity of $v$ in $T_{X}(v)$ is at most $2 j-2$, so $T_{X}(v) \neq T(v)$ since $2 j-2<d-j$. Thus Case I only applies if $T_{X}(v)$ is isomorphic to a subgraph of $T_{k-1, j-1}^{*}$ and there exists a vertex $u$ in $T(v)$ labeled from $Y$ such that $T_{X}(v)$ is the component of $T(v)-u$ containing $v$.

If there does exist a vertex $u$ in $T(v)$ labeled from $Y$ such that $T_{X}(v)$ is the component of $T(v)-u$ containing $v$, then $u$ had eccentricity at least $d-j$ in $T(u)$, so $T_{X}(v)$ is isomorphic to a subgraph of $T_{k-1, j-1}^{*}$ since $T_{k, d}$ has diameter $d$. Hence Case I applies. If there exists no vertex $u$ in $T(v)$ labeled from $Y$ such that $T_{X}(v)$ is the component of $T(v)-u$ containing $v$, then Case II applies.

We now show that Algorithm 2.8 produces a valid label in each of the three cases. Assume that the algorithm has assigned valid labels before the presentation of $v$. Note that for $(A, B) \in\{(X, X),(Y, X \cup$ $Y),(Z, X \cup Y \cup Z)\}$, each vertex $u \in V\left(T_{B}(v)\right)$ labeled from $A$ was given label $f_{B}^{A}(u)$ when it arrived, and $\min A>\max (B-A)$. Hence by Lemma 2.5 $f_{B}^{A}(v)$ exists if every component of $T_{B}(v)-v$ lacks some label in $A$.

Proposition 2.10. In Case $I, f_{X}^{X}(v)$ exists, and setting $f(v)=f_{X}^{X}(v)$ is a valid move for Ranker.
Proof. Note that $f_{X}^{X}(v)$ exists by Lemma 2.5 because $\left|V\left(T_{X}(v)\right)\right| \leq|X|$. Furthermore, $f_{X}^{X}(v)$ provides a valid label for $v$ by Lemma 2.6 because either $T_{X}(v)=T(v)$ or there exists a vertex $u$ in $T(v)$ such that $f(u)>\max X$ and $T_{X}(v)$ is a component of $T(v)-u$, making $u$ the only vertex outside $T_{X}(v)$ neighboring a vertex inside $T_{X}(v)$.

If $y$ satisfies the conditions of Case II, then let $H(y)$ be the component of $T(y)-y$ having greatest diameter.

Lemma 2.11. If $y$ is labeled from $Y$, then each vertex separated from $H(y)$ by $y$ (at any point in the game) is labeled from $X$.

Proof. The eccentricity of $y$ in $T(y)$ is at least $d-j$, so $H(y)$ has diameter at least $d-j-1$. This forces each other component of $T(y)-y$ to be isomorphic to a subtree of $T_{k-1, j-1}^{*}$. Any vertex $r$ of such a component is labeled from $X$, since $T(r)$ was isomorphic to a subgraph of $T_{k-1, j-1}^{*}$, implying $T_{X}(r)=T(r)$. Furthermore, any subsequently presented vertex $s$ satisfying $y \in V(T(s))$ that is separated from $H(y)$ by $y$ is labeled from $X$, since $T_{X}(s)$ is isomorphic to a subgraph of $T_{k-1, j-1}^{*}$ and is the component of $T(s)-y$ containing $s$.

Lemma 2.12. Every path in $T_{X \cup Y}(v)$ contains at most two vertices labeled from $Y$ (including possibly $v)$.

Proof. Let $y, y^{\prime}$, and $y^{\prime \prime}$ be distinct vertices in $T_{X \cup Y}(v)$ labeled from $Y$ (one could possibly be $v$ ). Since $y^{\prime}$ and $y^{\prime \prime}$ are labeled from $Y$, neither is separated from $H(y)$ by $y$, by Lemma 2.11. If $u$ is the neighbor of $y$ in $H(y)$, then the edge $u y$ must be part of any path containing $y$ and at least one of $y^{\prime}$ or $y^{\prime \prime}$. Hence edge-disjoint $y^{\prime}, y$ - and $y, y^{\prime \prime}$-paths do not exist, so no path contains $y$ between $y^{\prime}$ and $y^{\prime \prime}$. By symmetry, no path contains each of $y, y^{\prime}$, and $y^{\prime \prime}$.

Lemma 2.13. If $T(v)$ contains a vertex labeled from $Y$ (possibly $v$ ), then $T(v)$ contains a vertex labeled from $Z$, and no path in $T(v)$ contains a vertex labeled from $Z$ and multiple vertices of $T_{X \cup Y}(v)$ labeled from $Y$.

Proof. For the first claim, let $y$ be the first vertex in $T(v)$ labeled from $Y$. The diameter of $H(y)$ is greater than the diameter of $T_{k-1, j-1}^{*}$ because $d-j-1>2 j-2$, so some vertex $r \in V(H(y))$ violated the first condition of Case I when presented and was thus not labeled from $X$. Since $r$ was presented before $y$, it is labeled from $Z$.

For the second claim, let $z$ be a vertex of $T(v)$ labeled from $Z$, and $y^{\prime}$ and $y^{\prime \prime}$ be distinct vertices of $T_{X \cup Y}(v)$ labeled from $Y$. If $u$ is the neighbor of $z$ in the direction of $v$, then the edge $u z$ must be part of any path containing $z$ and at least one of $y^{\prime}$ or $y^{\prime \prime}$. Hence edge-disjoint $y^{\prime}, z$ - and $z, y^{\prime \prime}$-paths do not exist, so no path can contain $z$ between $y^{\prime}$ and $y^{\prime \prime}$.

By Lemma 2.11, any vertex separated from $H\left(y^{\prime}\right)$ by $y^{\prime}$ is labeled from $X$, so $y^{\prime \prime}$ is not separated from $z$ by $y^{\prime}$. Similarly, $y^{\prime}$ is not separated from $z$ by $y^{\prime \prime}$. Thus no path can contain each of $z, y^{\prime}$, and $y^{\prime \prime}$.


Figure 2. A labeling of $T_{k, d}$ for vertices with high eccentricity $\left(x_{i} \in X, y_{i} \in Y, z_{i} \in Z\right)$.

Proposition 2.14. In Case II, $f_{X \cup Y}^{Y}(v)$ exists, and setting $f(v)=f_{X \cup Y}^{Y}(v)$ is a valid move for Ranker.
Proof. Let $S$ be the set consisting of $v$ and every vertex in $T_{X \cup Y}(v)$ labeled from $Y$. By Lemma 2.12, the elements of $S$ are only separated by vertices labeled from $X$, so the smallest subtree $T$ of $T_{X \cup Y}(v)$ containing all of $S$ has all its internal vertices labeled from $X$. Therefore the set of internal vertices of $T$ induces a tree $T^{\prime}$ isomorphic to a subtree of $T_{k-1, j-1}^{*}$. By Lemma 2.13, some vertex not labeled from $Y$ neighbors a vertex in $T^{\prime}$ if $T^{\prime} \neq \emptyset$, or else some path in $T(v)$ contains a vertex labeled from $Z$ and multiple vertices of $T_{X \cup Y}(v)$ labeled from $Y$. Thus $|S|=|V(T)|-\left|V\left(T^{\prime}\right)\right| \leq\left|V\left(T_{k-1, j}^{*}\right)\right|-\left|V\left(T_{k-1, j-1}^{*}\right)\right|=|Y|$, so $f_{X \cup Y}^{Y}(v)$ exists by Lemma 2.5.

Finally, the only vertices outside $T_{X \cup Y}(v)$ that neighbor a vertex inside $T_{X \cup Y}(v)$ are in $H(y)$ and labeled from $Z$. Hence $f_{X \cup Y}^{Y}(v)$ provides a valid label for $v$, by Lemma 2.6.

Lemma 2.15. If $v$ is assigned a label $m \in Z$ previously unused in $T(v)$, then $v$ is a leaf of some subtree of $T_{X \cup Z}(v)$ containing every label in $Z$ smaller than $m$.
Proof. By Lemma 2.11 two vertices labeled from $Z$ are never separated by a vertex labeled from $Y$, so all vertices in $T(v)-v$ labeled from $Z$ lie in $T_{X \cup Z}(v)$. We use induction on $m$, with the base case $m=\min Z$ being trivial. If $m>\min Z$, let $u$ be the first vertex in $T_{X \cup Z}(v)$ labeled with $m-1$. Since $u$ arrived as a leaf of some subtree containing every label in $Z$ smaller than $m-1$, adding to that tree the $u, v$-path through $T_{X \cup Z}(v)$ yields the desired tree.

Lemma 2.16. The largest subtree $T$ of $T_{k, d}$ having diameter $d-j-1$ has at most $2\left|V\left(T_{k-1, j}^{*}\right)\right|$ vertices.
Proof. Let $u_{1} u_{2}$ be the central edge of $T$ if $d-j-1$ is odd and any edge containing the central vertex of $T$ if $d-j-1$ is even. Deleting $u_{1} u_{2}$ from $T$ then leaves two trees $T_{1}$ and $T_{2}$ containing $u_{1}$ and $u_{2}$, respectively, with $u_{i}$ having degree at most $k-1$ and eccentricity at most $\lfloor(d-j-1) / 2\rfloor$ in $T_{i}$. Thus each $T_{i}$ is isomorphic to a subtree of $T_{k-1, j}^{*}$, since $\lfloor(d-j-1) / 2\rfloor \leq j$ for $j=\lfloor d / 3\rfloor$. Hence $|V(T)|=\left|V\left(T_{1}\right)\right|+\left|V\left(T_{2}\right)\right| \leq 2\left|V\left(T_{k-1, j}^{*}\right)\right|$.
Proposition 2.17. In Case III, $f_{X \cup Y \cup Z}^{Z}(v)$ exists, and setting $f(v)=f_{X \cup Y \cup Z}^{Z}(v)$ is a valid move for Ranker.
Proof. Note that $T_{X \cup Y \cup Z}(v)=T(v)$, so if $f_{X \cup Y \cup Z}^{Z}(v)$ exists, then by Lemma 2.6 it is a valid label for $v$. If $T(v)$ uses at most $2\left|V\left(T_{k-1, j}^{*}\right)\right|$ labels from $Z$, then by Lemma $2.5 f_{X \cup Y \cup Z}^{Z}(v)$ exists, since $|Z|=2\left|V\left(T_{k-1, j}^{*}\right)\right|$. By Lemma 2.15 and the first condition of Case III, the number of labels from $Z$ used in $T(v)$ is at most the number of times a vertex $u$ in $T_{X \cup Z}(v)$ was presented as a leaf of $T_{X \cup Y}(u)$ having
eccentricity less than $d-j$ in $T_{X \cup Y}(u)$. Since any leaf added adjacent to a vertex having eccentricity at least $d-j$ will itself have eccentricity at least $d-j$, it suffices to show that growing a subtree of $T_{k, d}$ by iteratively adding one leaf $2\left|V\left(T_{k-1, j}^{*}\right)\right|$ times eventually forces some new leaf to have eccentricity at least $d-j$ at the time of its insertion. Since any leaf whose insertion raises the diameter of the tree has eccentricity equal to the higher diameter, this statement follows from Lemma 2.16

## 3. Trees with few internal vertices

Recall that $\mathcal{T}^{p, q}$ is the family of trees having at most $p$ internal vertices and diameter at most $q$. We first exhibit a strategy for Ranker on $\mathcal{T}^{p, q}$ that uses no label larger than $p+q+1$. We can improve this bound for the class of double stars by proving $\stackrel{\circ}{\rho}\left(\mathcal{T}^{2,3}\right)=4$ (since every tree with diameter 3 has exactly two internal vertices, $\mathcal{T}^{2,3}$ is the family of trees with diameter 3 ). This extends the work of Schiermeyer, Tuza, and Voigt 13, who characterized the families of graphs with on-line ranking number 1, 2, and 3 .
3.1. Upper bound on $\mathcal{T}^{p, q}$. During the on-line ranking game on $\mathcal{T}^{p, q}$, let $S$ be the component of the current graph containing the unlabeled presented vertex $v$. We give Ranker a procedure for ranking $v$ based solely on $S$ and the labels given to the other vertices of $S$.

Algorithm 3.1. If $v$ is the only vertex in $S$, let $f(v)=q+1$. If $v$ is not the only vertex in $S$, then let $m$ denote the largest label already used on $S$. If there exists a label smaller than $m$ that completes $a$ ranking when assigned to $v$, give $v$ the largest such label. Otherwise, let $f(v)=m+1$.

Lemma 3.2. If $v$ arrives as a leaf of a nontrivial component $S$ whose highest ranked vertex has label m, then Algorithm 3.1 will assign $v$ a label smaller than $m$.
Proof. Suppose that Algorithm 3.1 assigns $f(v)=m+1$. Let $v_{0}=v$. We now select vertices $v_{1}, \ldots, v_{j}$ from $S$ such that $v_{0}, v_{1}, \ldots, v_{j}$ in order form a path $P$ and $v_{j}$ arrived as an isolated vertex. For $i \geq 0$, let $v_{i+1}$ be a vertex with the least label among all vertices that were adjacent to $v_{i}$ when $v_{i}$ was presented, unless $v_{i}$ arrived as an isolated vertex, in which case set $j=i$. Since $S$ is finite, the process must end with some vertex $v_{j}$. Since $v_{i}$ was presented as a neighbor of $v_{i+1}, P$ is a path.

Note that Algorithm 3.1 assigns $f(u)=a \neq q+1$ only if $u$ arrives as a neighbor of a vertex $w$ such that $f(w) \leq a+1$. Since $f\left(v_{1}\right)=1$ (otherwise $f\left(v_{0}\right)=f\left(v_{1}\right)-1<m$ ), we must have $f\left(v_{i}\right) \leq i$ for $1 \leq i<j$. Also, $f\left(v_{j}\right)=q+1$ because $v_{j}$ arrived as an isolated vertex. Since $v_{j}$ was chosen as the neighbor with the least label when $v_{j-1}$ arrived, $f(u)>q$ for any such neighbor $u$. Hence $f\left(v_{j-1}\right) \geq q$. Therefore $j-1 \geq q$, which gives $P$ length $q+1$, contradicting $S$ having diameter at most $q$.
Theorem 3.3. Algorithm 3.1 uses no label larger than $p+q+1$.
Proof. By Lemma 3.2, the only way for a new largest label greater than $q+1$ to be used on $S$ is for the unlabeled vertex to arrive as an internal vertex. Only the $p$ internal vertices of an element of $\mathcal{T}^{p, q}$ can be presented as such, and each time a new largest label is used it increases the largest used value by 1 , so the largest label that could be used on one of them would be $p+q+1$.
3.2. Double stars. For any forest $F$, Schiermeyer, Tuza, and Voigt [13] proved $\rho(F)=1$ if and only if $F$ has no edges, $\stackrel{\circ}{\rho}(F)=2$ if and only if $F$ has an edge but no component with more than one edge, and $\stackrel{\rho}{\rho}(F)=3$ if and only if $F$ is a star forest with maximum degree at least 2 or $F$ is a linear forest whose largest component is $P_{4}$. Since $P_{4}$ is the only member of $\mathcal{T}^{2,3}$ having on-line ranking number less than 4 , proving $\stackrel{\circ}{\rho}\left(\mathcal{T}^{2,3}\right)=4$ only requires a strategy for Ranker, and our result implies $\stackrel{\circ}{\rho}(T)=4$ for any $T \in \mathcal{T}^{2,3}-\left\{P_{4}\right\}$. We now make some observations about the on-line ranking game on $\mathcal{T}^{2,3}$ before giving a strategy for Ranker.

When a vertex $u$ is presented, let $G(u)$ be the graph at that time, and let $T(u)$ be the component of $G(u)$ containing $u$. When the first edge(s) appear, the presented vertex $v$ is the center of a star; thus $T(v)$ is a star, while $G(v)$ may include isolated vertices in addition to $T(v)$. Let $v^{\prime}$ be the first vertex to complete a path of length 3 . The graph $G\left(v^{\prime}\right)$ is connected and has two internal vertices, properties that remain true as subsequent vertices are presented. Let $T$ be the final tree.

Consider the round when a vertex $u$ is presented. If $u$ is presented after $v^{\prime}$, or $u=v^{\prime}$ and $u$ is a leaf of $T(u)$, then $G(u)=T(u)$, and $u$ must be a leaf in $T$. If $u$ is presented after $v$ but before $v^{\prime}$, then either $T(u)=u$ or $T(u)$ is a star not centered at $u$. If additionally $G(u)$ is disconnected, then $u$ must wind up as a leaf in $T$, since $T$ has diameter 3. Call $u$ a forced leaf in this case, the case that $u$ is presented after $v^{\prime}$, or the case that $u=v^{\prime}$ and $u$ is presented as a leaf of $T(u)$. Otherwise, if $u$ is presented after $v$ but before $v^{\prime}$, then $u$ is a leaf of $T(u)$, and say that $u$ is undetermined (since $u$ may or may not wind up as a leaf in $T$ ). Also call $v$ undetermined, as well as $v^{\prime}$ if $v^{\prime}$ is not a forced leaf.

Algorithm 3.4. Give label 3 to the first vertex presented, label 2 to any subsequent vertex presented before $v$, and label 1 to any forced leaf. The rest of the algorithm specifies how to rank the undetermined vertices in terms of the labeling of $G(v)$.

If $G(v)=P_{2}$, then give label 4 to $v$ and label 2 to any subsequent undetermined vertex. If $G(v)$ has more than one edge (disconnected or not), and $v$ is adjacent to the vertex labeled 3, then give label 4 to $v$ and label 3 to any subsequent undetermined vertex.

If neither of the previous cases hold, then $G(v)$ is disconnected, and $v$ and $v^{\prime}$ are the only undetermined vertices. If $G(v)$ has exactly one edge, and $v$ is adjacent to the vertex labeled 3, then give label 2 to $v$ and label 4 to $v^{\prime}$. In the remaining case, $v$ is not adjacent to the vertex labeled 3; give label 3 to $v$ and label 4 to $v^{\prime}$.




Figure 3. Possibilities for $G(v)$.

Proposition 3.5. $\stackrel{\circ}{\rho}\left(\mathcal{T}^{2,3}\right)=4$.
Proof. Because $P_{4}$ is the only tree with exactly two internal vertices having on-line ranking number at most 3, we need only to verify that Algorithm 3.4 is a valid strategy for Ranker.

If $G(v)=P_{2}$, then every vertex labeled 1 is a leaf, and the only label besides 1 that can be used more than once is 2 . Any two vertices labeled 2 must be separated by one of the first two vertices presented, each of which receives a higher label.

If $G(v)$ has more than one edge, and $v$ is added adjacent to the vertex labeled 3 , then every vertex labeled 1 is a leaf, and the only vertex labeled 4 is $v$, which is an internal vertex. If the other internal vertex is labeled 3 , then each leaf adjacent to it is labeled 1 or 2 . Any two vertices labeled 3 must be separated from each other by $v$, which is labeled 4 , and any two vertices labeled 2 must be separated from each other by an internal vertex, which is labeled either 3 or 4 . If the internal vertex besides $v$ is labeled 2 , then each adjacent leaf must be labeled 1 . Any two vertices with the same label of 2 or 3 would have to be separated from each other by $v$, which is labeled 4 .

If $G(v)$ has exactly one edge but more than two vertices, and $v$ is adjacent to the vertex labeled 3 , then any vertex labeled 1 will be a leaf, only the first vertex presented will be labeled 3 , and any two vertices labeled 2 will be separated from each other by $v^{\prime}$, which is the only vertex labeled 4 .

If $G(v)$ has more than two vertices, and $v$ is not adjacent to the vertex labeled 3 , then any vertex labeled 1 will be a leaf, and any two vertices with the same label of 2 or 3 will be separated from each other by $v^{\prime}$, which is the only vertex labeled 4 .

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