# A priori estimates and analytical construction of radially symmetric solutions in the gas dynamics 

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#### Abstract

In this article we derive $\mathscr{C}^{1}$-a priori estimates on the Riemann invariants of the Euler compressible equations in the case of cylindrical or spherical symmetry. These estimates allow to construct shock waves with a time of existence proportional to the distance to the origin at the initial time. 2000 Mathematics Subject Classification: 35L60, 35Q31, 76N10.


Keywords: Euler compressible equations, shock wave solution, long time of existence.

## 1 Introduction

We are interested in the Euler compressible system in the isentropic case:

$$
\left\{\begin{array}{l}
\partial_{t} \rho+\operatorname{div}(\rho u)=0,  \tag{1.1}\\
\partial_{t} u+(u \cdot \nabla) u+\frac{p^{\prime}(\rho)}{\rho} \nabla \rho=0 .
\end{array}\right.
$$

In the case of cylindrical $(d=2)$ or spherical $(d=3)$ symmetry, the system (1.1) can be written

$$
\left\{\begin{array}{l}
\partial_{t} \rho+\partial_{r}(\rho u)=\frac{-(d-1) \rho u}{r}  \tag{1.2}\\
\partial_{t} u+u \partial_{r} u+\frac{p^{\prime}(\rho)}{\rho} \partial_{r} \rho=0
\end{array}\right.
$$

The case $d=1$ corresponds to the one-dimensional case. Our goal here is to construct shock wave solutions in the isentropic spherical or cylindrical case with a reasonable lower bound on the time of existence.

Very often only the case of a perfect polytropic gas satisfying the sate law $p v=\mathfrak{R} T$ is considered. However, in numerous phenomenon such as cavitation or sonoluminescence [2, 4] or considering a dusty gas [7, 9, 19, 24, 25], it seems more adapted to consider at least a Van der Waals gas satisfying $p(v-b)=\mathfrak{R} T$. We prove results in this more general framework.

The Euler compressible equations have already been widely studied. Concerning regular solutions, general classical criteria on hyperbolic systems (Leray [12], Gårding [5], Kato [10]) provide us local in time existence of smooth solutions for the Cauchy problem. However, the time of existence can be very small: several results by T. C. Sideris [22, 23], T. Makino, S. Ukai \& S. Kawashima [16], J.-Y. Chemin [3] provide explosion's criteria. We also know that some regular solutions can be global in time: for example stationary solutions, or
under some expansivity hypothesis (see T. T. Li [13], D. Serre [21] or M. Grassin [6, M. Lécureux-Mercier [11]).

Concerning piecewise regular solutions, a result by A. Majda [15] states that we can associate a piecewise regular solution to a given piecewise initial data satisfying some compatibility conditions. But the time of existence of these solutions can be once again very small.

In this paper, we construct single propagating shock waves with a long time of existence. We consider only 2-shocks. The general framework is the one of a Bethe-Weyl gas (for example Van der Waals gas or perfect gas, the definition of a Bethe-Weyl fluid is given in definition [2.2. ) in the particular case of spherical or cylindrical symmetry. More precisely : we consider the cylindrical or spherical case for a Bethe-Weyl fluid. We assume that at initial time we have a piecewise regular solution with a discontinuity jump at radius $R_{0}$ satisfying the Lax compatibility conditions. We assume furthermore that, prolongating each regular piece of this initial condition into a regular global in space function, we can find a regular solution of system (1.2) with this initial condition admitting a long time of existence. Then we add some expansivity hypotheses on the Rimann-invariant. The definition of the Riemann invariants is given in (3.1). We obtain that the time of existence of the $\mathrm{CS} / \mathrm{SS}$ solution is proportional to the radius $R_{0}$ of the initial discontinuity. The statement of the main result is more precisely given in Theorem 5.6.

The strategy of the proof is as follows: we use a method that Li Ta Tsien [13] employed in order to construct 1D shock waves for an isentropic gas. This method is inspired from the scalar one-dimensional case $\partial_{t} u+\partial_{x}(f(u))=0$, in which it is possible to obtain a shock wave solution just gluing two regular solutions along a line of discontinuity satisfying the Rankine-Hugoniot shock condition

$$
U=\frac{f\left(u^{+}\right)-f\left(u^{-}\right)}{u^{+}-u^{-}}
$$

where $u^{+}$is the limit of $u$ at the discontinuity from the right and $u^{-}$is the limit of $u$ at the discontinuity from the left. This provides us an ODE that the line of discontinuity has to satisfy, being defined as $\frac{\mathrm{d} x}{\mathrm{~d} t}=U$. Then, we have just to check that under suitable conditions the Lax entropy conditions meaning the characteristic curves are entering the shock (see Figure 1) are satisfied.

We want to apply this strategy to the isentropic spherical Euler equation. However, in the case of a system of two equations in one space dimension, the strategy above is no longer possible since now the Rankine-Hugoniot conditions provide us two equations. Consequently, we have not only an equation for the behavior of the shock, but also a compatibility condition along the shock. Graphically, we see that, for a 2 -shock, if the 2 characteristics are still entering the shock, the 1-characteristics are entering from the right and exiting from the left (see Figure 1), which brings us an uncertainty zone between the shock and the 1 -characteristic exiting from the foot initial position of the shock to the left. In order to construct a shock wave solution from two regular solutions, we have to study the existence of a smooth solution in this angular domain between the shock and the 1-characteristic. Li Ta Tsien [13] already studied this subject in the one-dimensional case. In particular, he obtained local in time existence of a smooth solution for the angular problem.

More precisely, we first have to estimate the time of existence of a smooth solutions in an angular domain whose boundaries are chosen in a way to be a 1-characteristic on the


Figure 1: Shock curve and characteristics in the scalar case (left) and in the case of a system of two equations (right). the domain of uncertainty is in gray on the right picture.
left and a shock on the right (see Figure (1) ; the time of existence is then obtained by deriving $\mathscr{C}^{1}$ estimates on the solution of this boundary problem. Finally, thanks to a priori estimates on the Riemann invariant, we obtain a lower bound on the time of existence of these solutions, proportional to the position at initial time of the discontinuity.

This paper is structured as followed: in Section 2, we describe the thermodynamical quantities and their properties. In Section 3, we compute a priori estimates in $\mathscr{C}^{0}$ along the characteristics. In Section 4, we compute a priori estimates in $\mathscr{C}{ }^{1}$ along the characteristics. In Section 5, we introduce the shock conditions, the angular problem and we prove the main results of this article: in Proposition 5.5 we give an estimate of the time of existence of a smooth solution in an angular domain and in Theorem 5.6 we finally construct a shock wave. In Sections $\mathbb{A}$ and $B$, we finally give some details about a useful lemma on ODEs and about explicit computations in the cases of a perfect gas and of a Van der Waals gas.

## 2 Thermodynamics

### 2.1 Fundamental relationships

Definition 2.1. We consider a fluid, whose internal energy is a regular function of its specific volume ${ }^{1} v=1 / \rho$ and of its specific entropy $s$. We say that the gas is entitled with a complete state law, or energy law $e=e(v, s)$.

For a gas entitled with a complete state law, the fundamental thermodynamic principle is then

$$
\begin{equation*}
\mathrm{d} e=-p \mathrm{~d} v+T \mathrm{~d} s \tag{2.1}
\end{equation*}
$$

where $p$ is the pressure and $T$ the temperature of the gas. Consequently, the pressure $p$

[^0]and the temperature $T$ can be defined as
\[

$$
\begin{equation*}
p=-\left.\frac{\partial e}{\partial v}\right|_{s}, \quad T=\left.\frac{\partial e}{\partial s}\right|_{v}, \tag{2.2}
\end{equation*}
$$

\]

where the notation | precises the variable maintained constant in the partial derivation.
The higher order derivatives of $e$ have also an important role; we introduce the following adimensional quantities:

$$
\begin{equation*}
\gamma=-\left.\frac{v}{p} \frac{\partial p}{\partial v}\right|_{s}, \quad \Gamma=-\left.\frac{v}{T} \frac{\partial T}{\partial v}\right|_{s}, \quad \delta=\left.\frac{p v}{T^{2}} \frac{\partial T}{\partial s}\right|_{v}, \quad \mathscr{G}=-\frac{v}{2} \frac{\left.\frac{\partial^{3} e}{2 v^{3}}\right|_{s}}{\left.\frac{\partial^{2} e}{\partial v^{2}}\right|_{s}} . \tag{2.3}
\end{equation*}
$$

The coefficient $\gamma$ is called the adiabatic exponent, and $\Gamma$ is the Grüneisen coefficient. The quantities $\gamma, \delta, \Gamma$ and $\mathscr{G}$ characterise the geometrical properties of the isentropic curves in the ( $v, p$ ) plane (see [17]). They can be expressed in function of $e$ through the relationships:

$$
\gamma=\frac{v}{p} \frac{\partial^{2} e}{\partial v^{2}}, \quad \Gamma=-\frac{v}{T} \frac{\partial^{2} e}{\partial s \partial v}, \quad \delta=\frac{p v}{T^{2}} \frac{\partial^{2} e}{\partial s^{2}}
$$

We also introduce the calorific capacity at constant volume $c_{v}$ and the calorific capacity at constant pressure $c_{p}$ by

$$
\begin{equation*}
c_{v}=\left.\frac{\partial e}{\partial T}\right|_{v}=\frac{T}{\left.\frac{\partial^{2} e}{\partial s^{2}}\right|_{v}}, \quad \quad c_{p}=\left.T \frac{\partial s}{\partial T}\right|_{p} \tag{2.4}
\end{equation*}
$$

These two quantities are linked with $\frac{p v}{T}$ and with $\gamma, \delta, \Gamma$ through

$$
\begin{equation*}
\delta c_{v}=\frac{p v}{T}, \quad c_{p}=\frac{p v}{T} \frac{\gamma}{\gamma \delta-\Gamma^{2}} . \tag{2.5}
\end{equation*}
$$

The quantity $\gamma_{*}=\frac{c_{p}}{c_{v}}$ can besides be expressed as $\gamma_{*}=\frac{\gamma \delta}{\gamma \delta-\Gamma^{2}}$. It is not equal to $\gamma$ in the general case, but for an ideal gas we have $\delta=\Gamma=\gamma-1$, and consequently $\gamma_{*}=\gamma$.

### 2.2 Thermodynamical constraints.

It is very natural to assume that the massic volume $v$ is positive. We assume furthermore that the pressure $p$ and the temperature $T$ are positive, which imposes that $e$ is a function increasing in $T$ and decreasing in $v$.

A classical thermodynamical hypothesis requires furthermore $e$ to be a convex function of $s$ and $v$, which means:

$$
\gamma \delta-\Gamma^{2} \geqslant 0, \quad \delta \geqslant 0, \quad \gamma \geqslant 0
$$

Furthermore, we require usually $\Gamma>0$ and $\mathscr{G}>0$. The condition $\Gamma>0$ is not thermodynamically required but is satisfied for many gases and ensures that the isentropes do not cross each other in the $(v, p)$ plan. The condition $\mathscr{G}>0$ means that the isentropes are strictly convex in the ( $v, p$ ) plan.

Definition 2.2. We call Bethe-Weyl fluid any fluid endowed with a complete state law $e$ bounded below such that

- the pressure and the temperature defined by (2.2) are positive,
- the coefficients $\gamma, \delta, \Gamma$ and $\mathscr{G}$ defined by (2.3) satisfy :

$$
\begin{equation*}
\gamma>0, \quad \gamma \delta \geqslant \Gamma^{2}, \quad \Gamma>0, \quad \mathscr{G}>0 \tag{2.6}
\end{equation*}
$$

- there exists a maximal density $\left.\left.\rho_{\max } \in\right] 0,+\infty\right]$ such that $\lim _{\rho \rightarrow \rho_{\max }} p(\rho, s)=+\infty$.

The condition $\gamma \geqslant 0$ means that $p$ increases with the density $\rho=1 / v$, which allows us to define the adiabatic sound speed by

$$
\begin{equation*}
c=\sqrt{\left.\frac{\partial p}{\partial \rho}\right|_{s}}=\sqrt{\gamma \frac{p}{\rho}} . \tag{2.7}
\end{equation*}
$$

Then, we can check that $\mathscr{G}$ can be expressed in function of $\rho$ and $c$ through the expression

$$
\mathscr{G}=\left.\frac{1}{c} \frac{\partial(\rho c)}{\partial \rho}\right|_{s} .
$$

### 2.3 Van der Waals Gas

Definition 2.3. A gas is said to follow the Van der Waals law, if there exists a constant $\mathfrak{R}$ such that it satisfies the following pressure law:

$$
\begin{equation*}
p(v-b)=\Re T, \tag{2.8}
\end{equation*}
$$

where $v$ is the massic volume and $b$ is the covolume, representing the compressibility limit of the fluid, due to the volume of the molecules. The constant $\mathfrak{R}=8.314 \mathrm{~J} . \mathrm{K}^{-1} \cdot \mathrm{~mol}^{-1}$ is called the perfect gas constant.

In the case $b=0$, we obtain the perfect gas law.
The state law (2.8) is a particular case of the state law $p=\frac{\mathfrak{R} T}{V-b}-\frac{a}{V^{2}}$. This last law is not considered here as it authorises the change of phase when $T$ goes under a threshold $T_{c}=\frac{8 a}{276{ }^{2}}$ (see [18]).

In order to obtain expressions for the quantities $e, p, \gamma, \Gamma \ldots$ in function of $v$ and $s$, we use the fundamental relationship (2.1). This equation gives us the PDE: $\partial_{v} e+\frac{\Re}{v-b} \partial_{s} e=0$. Thus, we introduce new variables $w=(v-b)^{-\Re}, \sigma=(v-b)^{-\Re} \exp (s)$ and $\hat{e}(w, \sigma)=e(v, s)$. We obtain $\partial_{w} \hat{e}=0$, so that $e=\mathcal{E}\left((v-b)^{-\mathfrak{R}} \exp (s)\right)$ for any regular function $\mathcal{E}$.

If we assume furthermore that $c_{v}$ is constant, thanks to the definition of $c_{v}$ and (2.1), we get that $\left.\frac{\partial^{2} e}{\partial s^{2}}\right|_{v}=\left.\frac{1}{c_{v}} \frac{\partial e}{\partial s}\right|_{v}$, hence $\sigma \mathcal{E}^{\prime \prime}=\left(\frac{1}{c_{v}}-1\right) \mathcal{E}^{\prime}$ and $\mathcal{E}(\sigma)=C \sigma^{1 / c_{v}}$ which leads to:

$$
e=(v-b)^{-\frac{\mathfrak{\Re}}{c_{v}}} \exp \left(\frac{s}{c_{v}}\right), \quad \quad p=\frac{\mathfrak{R}}{c_{v}} \frac{e}{v-b} .
$$

After some computations we finally obtain

$$
\gamma=\gamma_{0} \frac{v}{v-b}, \quad \Gamma=\delta=\left(\gamma_{0}-1\right) \frac{v}{v-b}, \quad \mathscr{G}=\frac{\gamma_{0}+1}{2} \frac{v}{v-b},
$$

where

$$
\begin{equation*}
\gamma_{0}:=\frac{\Re}{c_{v}}+1 . \tag{2.9}
\end{equation*}
$$

The conditions of Section 2.2 are then satisfied for $\gamma_{0}>1$.
In the following, we consider a general Bethe-Weyl fluid satisfying $1<\mathscr{G}<2$. Another general assumption is that the application $\left(\rho \mapsto \frac{c(\rho)}{\rho}\right)$ is integrable in 0 .

In particular, we can consider a Van der Waals fluid with constant and strictly positive calorific capacity $c_{v}$ :

$$
\begin{equation*}
c_{v}>0 \tag{2.10}
\end{equation*}
$$

which implies $\gamma_{0}>1$ and $\mathscr{G}>1$.
We can also consider the standard $p$-system : $p=\rho^{\gamma}$ with $\gamma>1$.

## 3 A priori $\mathscr{C}^{0}$ estimates along the characteristics

Let us remind that in this paper we consider cylindrical or spherical Euler equation in the isentropical case.

First we want to obtain $\mathscr{C}^{0}$ estimates on regular solutions of (1.2), $\rho$ and $u$. To do that, we use the Riemann invariants of the system and we make computations along the characteristics.

### 3.1 Change of variables

Lemma 3.1. We assume that $\left(\rho \mapsto \frac{c(\rho)}{\rho}\right)$ is integrable in 0. Let us introduce the Riemann invariants

$$
\begin{equation*}
w_{1}=u-H(\rho), \quad w_{2}=u+H(\rho) \tag{3.1}
\end{equation*}
$$

where $H(\rho)$ is a primitive of $\rho \mapsto \frac{c(\rho)}{\rho}$ vanishing in 0 . Then, the system (1.2) reduces to

$$
\left\{\begin{array}{l}
\partial_{t} w_{1}+\lambda_{1}(w) \partial_{r} w_{1}=f(r, w)  \tag{3.2}\\
\partial_{t} w_{2}+\lambda_{2}(w) \partial_{r} w_{2}=-f(r, w)
\end{array}\right.
$$

where $f(r, w)=\frac{(d-1) u c}{r}, \lambda_{1}=u-c(\rho), \lambda_{2}=u+c(\rho)$.
Proof. Direct computation.
We prove below some properties of the new unknown $H, w_{1}, w_{2}$, which will be useful in the proof of the main theorem.

Lemma 3.2. Let us consider a Bethe-Weyl gas. We assume that $1<\mathscr{G}<2$ and that $\left(\rho \mapsto \frac{c(\rho)}{\rho}\right)$ is integrable in 0 . Then we have $H \geqslant c$ and in particular $u-H \leqslant u-c$, that is to say $w_{1} \leqslant \lambda_{1}$.

In particular, $w_{1}=u-H \geqslant 0$ implies $\lambda_{1}=u-c \geqslant 0$.
Remark 3.3. In the case of a perfect polytropic gas, we have $\mathscr{G}=\frac{\gamma_{0}+1}{2}$ and the condition $1<\mathscr{G}<2$ is equivalent to $1<\gamma_{0}<3$, which is a natural hypothesis on $\gamma_{0}$.
Proof. Note first that $c^{\prime}(\rho)=H^{\prime}(\rho)(\mathscr{G}-1) \geqslant 0$. Then $H^{\prime}(\rho)-c^{\prime}(\rho)=H^{\prime}(\rho)(2-\mathscr{G})=$ $\frac{c(\rho)}{\rho}(2-\mathscr{G}) \geqslant 0$. Hence, integrating on $[0, \rho]$, we obtain $H(\rho) \geqslant c(\rho)$.

Lemma 3.4. Let $\rho^{+}>0$. Let us define, for $\rho \geqslant \rho^{+}$,

$$
\begin{equation*}
F\left(\rho, \rho^{+}\right):=\left(p-p^{+}\right)\left(\frac{1}{\rho^{+}}-\frac{1}{\rho}\right) \tag{3.3}
\end{equation*}
$$

Then, for all $\rho \geqslant \rho^{+}$, we have $H(\rho)-H\left(\rho^{+}\right) \leqslant \sqrt{F\left(\rho, \rho^{+}\right)}$.
Proof. Let $\rho \geqslant \rho^{+}$. Let us derivate $\sqrt{F\left(\rho, \rho^{+}\right)}-H(\rho)$ with respect to $\rho$. We obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} \rho}\left(\sqrt{F\left(\rho, \rho^{+}\right)}-H(\rho)\right) & =\frac{1}{2 \sqrt{F\left(\rho, \rho^{+}\right)}}\left(c^{2}\left(\frac{1}{\rho^{+}}-\frac{1}{\rho}\right)+\frac{1}{\rho^{2}}\left(p-p^{+}\right)\right)-\frac{c}{\rho} \\
& =\frac{c}{2 \rho \sqrt{F\left(\rho, \rho^{+}\right)}}\left(c \rho\left(\frac{1}{\rho^{+}}-\frac{1}{\rho}\right)+\frac{1}{c \rho}\left(p-p^{+}\right)-2 \sqrt{F\left(\rho, \rho^{+}\right)}\right) \\
& =\frac{c}{2 \rho \sqrt{F\left(\rho, \rho^{+}\right)}}\left(\sqrt{c \rho\left(\frac{1}{\rho^{+}}-\frac{1}{\rho}\right)}-\sqrt{\frac{1}{c \rho}\left(p-p^{+}\right)}\right)^{2} \\
& \geqslant 0 .
\end{aligned}
$$

Noting that $F\left(\rho^{+}, \rho^{+}\right)=0$ and integrating on $\left[\rho^{+}, \rho\right]$, we obtain $\sqrt{F\left(\rho, \rho^{+}\right)}-H(\rho) \geqslant$ $-H\left(\rho^{+}\right)$, which is the expected result.

Lemma 3.5. Let $u^{+}, \rho^{-}, \rho^{+} \in \mathbb{R}$. Let us assume $\rho^{-} \geqslant \rho^{+}>0$ and define $u^{-}:=u^{+}+$ $\sqrt{F\left(\rho^{-}, \rho^{+}\right)}$, with $F$ defined as in (3.3). Then $u^{-}-H\left(\rho^{-}\right) \geqslant u^{+}-H\left(\rho^{+}\right)$, that is to say $w_{1}^{-} \geqslant w_{1}^{+}$.

In particular, $w_{1}^{+}=u^{+}-H\left(\rho^{+}\right) \geqslant 0$ implies $w_{1}^{-}=u^{-}-H\left(\rho^{-}\right) \geqslant 0$.
Proof. This is a direct consequence of Lemma 3.4. Indeed, the inequality $\sqrt{F\left(\rho^{-}, \rho^{+}\right)} \geqslant$ $H^{-}-H^{+}$implies
$u^{-}-H\left(\rho^{-}\right)=u^{+}+\sqrt{F\left(\rho^{-}, \rho^{+}\right)}-H\left(\rho^{-}\right) \geqslant u^{+}+H\left(\rho^{-}\right)-H\left(\rho^{+}\right)-H\left(\rho^{-}\right)=u^{+}-H\left(\rho^{+}\right)$.

## $3.2 \mathscr{C}^{0}$ estimate on $w_{1}$ and $w_{2}$

Relying on computations along the characteristics, we now obtain estimates in $\mathbf{L}^{\infty}$ for $w_{1}$ and $w_{2}$, the Riemann invariants associated to a regular solution of (1.2).

Lemma 3.6. Let us consider a Bethe-Weyl gas, satisfying $1<\mathscr{G}<2$ and such that $\left(\rho \mapsto \frac{c(\rho)}{\rho}\right)$ is integrable in 0 . Let $w=\left(w_{1}, w_{2}\right)$ be a regular solution of (3.2) with a time of existence $\mathcal{T}>0$. Let $X_{1}$ and $X_{2}$ be the characteristics defined by

$$
\begin{align*}
\frac{\mathrm{d} X_{1}}{\mathrm{~d} t} & =\lambda_{1}\left(w\left(t, X_{1}(t)\right)\right), & \frac{\mathrm{d} X_{2}}{\mathrm{~d} t} & =\lambda_{2}\left(w\left(t, X_{1}(t)\right)\right)  \tag{3.4}\\
X_{1}(0) & =r_{1}, & X_{2}(0) & =r_{2}
\end{align*}
$$

Let us assume that for all $r>0,\left(w_{2}-w_{1}\right)(0, r) \geqslant 0$ and $w_{1}(0, r)>0$. Then, we get: $X_{1}^{\prime}>0, X_{2}^{\prime}>0$ and, for all $t \in[0, \mathcal{T}[$ we have the estimates:

$$
\begin{gather*}
w_{1}\left(0, r_{1}\right) \leqslant w_{1}\left(t, X_{1}(t)\right) \leqslant w_{1}\left(0, r_{1}\right)+\left\|w_{2}(0, \cdot)\right\|_{\mathbf{L}^{\infty}}^{2} \int_{0}^{t} \frac{d-1}{4 X_{1}(\tau)} \mathrm{d} \tau  \tag{3.5}\\
\frac{w_{2}\left(0, r_{2}\right)}{1+w_{2}\left(0, r_{2}\right) \int_{0}^{t} \frac{(d-1)}{4 X_{2}(\tau)} \mathrm{d} \tau} \leqslant w_{2}\left(t, X_{2}(t)\right) \leqslant w_{2}\left(0, r_{2}\right) \tag{3.6}
\end{gather*}
$$

Furthermore, for all $t \in[0, \mathcal{T}[$ :

$$
\begin{equation*}
w_{1}\left(0, X_{1}(0 ; t, r)\right) \leqslant w_{1}(t, r) \leqslant w_{2}(t, r) \leqslant w_{2}\left(0, X_{2}(0 ; t, r)\right) \tag{3.7}
\end{equation*}
$$

where, for $i \in\{1,2\}, X_{i}(0 ; t, r)$ designates the position at time 0 of the characteristic that satifies the initial condition $X_{i}(t)=r$.

Proof. First, let $t_{0}>0$. We introduce $\chi$ be the solution of the ODE:

$$
\frac{\mathrm{d} \chi}{\mathrm{~d} t}=u(t, \chi), \quad \chi\left(t_{0}\right)=r_{0}
$$

Note that the solution of this ODE exists at least in finite time. Some trouble could appear only if $\chi$ meets the line of equation $r=0$. In the same way, the characteristics $X_{1}$ and $X_{2}$ satisfying (3.4) are defined at least in finite time. Let $r_{0}, r_{1}, r_{2}>0$ be fixed. Let us denote $T \in] 0, \mathcal{T}$ [ a time such that $\chi, X_{1}, X_{2}$ are defined on $[0, T]$.
Positivity of the density: Let us prove first that $\rho$ remains non-negative because of the first equation of (1.2). By the definition of $\chi$, for any $t \in[0, T]$.

$$
\frac{\mathrm{d}}{\mathrm{~d} t}\left(\chi^{d-1} \rho(t, \chi) \exp \left(\int_{t_{0}}^{t} \partial_{r} u(s, \chi(s)) \mathrm{d} s\right)\right)=0
$$

Consequently, thet density being non-negative at time $t_{0}=0$, the densitity remains nonnegative along the lines $(t, \chi(t))$; that is to say: the inequality $w_{2} \geqslant w_{1}$ is satisfied for all time $t \in[0, T]$ since it is true at time $t=0$.
$w_{1}$ is increasing along the characteristic: At least on a small time interval containing 0 , since $w_{1}\left(0, r_{1}\right)>0$ we have $w_{1}\left(t, X_{1}(t)\right) \geqslant 0$. We want to prove that $w_{1}$ remains nonnegative along the characteristic. Assume there is a time at which $w_{1}\left(t, X_{1}(t)\right)=0$. Let us denote $t_{0} \leqslant T$ the first time at which $w_{1}\left(t_{0}, X_{1}\left(t_{0}\right)\right)=0$. Thanks to the previous result on the positivity of the density, we obtain $w_{2}\left(t, X_{1}(t)\right) \geqslant w_{1}\left(t, X_{1}(t)\right) \geqslant 0$ on $\left[0, t_{0}\right]$. Then on $\left[0, t_{0}\right]$ we have $u=\frac{w_{2}+w_{1}}{2} \geqslant 0$ and $f(r, w)=\frac{(d-1) u c}{r} \geqslant 0$ which implies

$$
\frac{\mathrm{d} w_{1}\left(t, X_{1}(t)\right)}{\mathrm{d} t} \geqslant 0
$$

Integrating, we get $w_{1}\left(t_{0}, X_{1}\left(t_{0}\right)\right) \geqslant w_{1}\left(0, r_{1}\right)>0$, which is in contradiction with the hypothesis. Finally, $w_{1}$ is strictly positive for all $t \in[0, T]$ and increasing along the first characteristic. In particular, we also have $u>0$ for all $t \in[0, T]$.

Upper bound on $w_{2}$ : Along the second characteristic we get, since $u \geqslant w_{1} \geqslant 0$,

$$
\frac{\mathrm{d} w_{2}\left(t, X_{2}(t)\right)}{\mathrm{d} t}=-f\left(X_{2}, w\left(t, X_{2}(t)\right)\right) \leqslant 0
$$

Integrating we obtain $w_{2}\left(t, X_{2}(t)\right) \leqslant w_{2}\left(0, r_{2}\right)$, which provides us with an upper bound.
Lower bound on $w_{2}$ : Thanks to Lemma 3.2 we know that $c \leqslant H$. Hence, we obtain

$$
\frac{\mathrm{d} w_{2}\left(t, X_{2}(t)\right)}{\mathrm{d} t}=\frac{-(d-1)}{X_{2}(t)} u c \geqslant \frac{-(d-1)}{X_{2}(t)} u H=\frac{-(d-1)}{4 X_{2}(t)}\left(w_{2}^{2}-w_{1}^{2}\right) \geqslant \frac{-(d-1)}{4 X_{2}(t)} w_{2}^{2}
$$

Consequently, we have $-\frac{1}{w_{2}^{2}\left(t, X_{2}(t)\right)} \frac{\mathrm{d} w_{2}\left(t, X_{2}(t)\right)}{\mathrm{d} t} \leqslant \frac{d-1}{4 X_{2}(t)}$ and integrating we finally obtain

$$
\frac{1}{w_{2}\left(t, X_{2}(t)\right)} \leqslant \frac{1}{w_{2}\left(0, r_{2}\right)}+\int_{0}^{t} \frac{d-1}{4 X_{2}(\tau)} \mathrm{d} \tau
$$

Since $w_{2} \geqslant 0$, we can invert this relation, and obtain the desired lower bound on $w_{2}$.
Upper bound on $w_{1}$ : Similarly for $w_{1}$ we get

$$
\frac{\mathrm{d} w_{1}\left(t, X_{1}(t)\right)}{\mathrm{d} t}=\frac{(d-1)}{X_{1}(t)} u c \leqslant \frac{d-1}{X_{1}(t)} u H=\frac{(d-1)}{4 X_{1}(t)}\left(w_{2}^{2}-w_{1}^{2}\right) \leqslant \frac{(d-1)}{4 X_{1}(t)}\left\|w_{2}\right\|_{\mathbf{L}^{\infty}([0, T])}^{2}
$$

Hence, as announced,

$$
w_{1}\left(t, X_{1}(t)\right) \leqslant w_{1}\left(0, r_{1}\right)+\int_{0}^{t} \frac{(d-1)}{4 X_{1}(\tau)}\left\|w_{2}\right\|_{\mathbf{L}^{\infty}([0, T])}^{2} \mathrm{~d} \tau
$$

Time of existence: Note now that $\frac{\mathrm{d} \chi}{\mathrm{d} t}=u(t, \chi(t)) \geqslant 0$ implies that this curve never meets the origin and is defined on $\mathbb{R}^{+}$. Similarly, $\frac{\mathrm{d} X_{2}}{\mathrm{~d} t}=(u+c)\left(t, X_{2}(t)\right) \geqslant 0$ implies the 2 -characteristics are going away from the origin and are defined for all $t \in \mathbb{R}^{+}$.

Concerning the first characteristic, Lemma 3.2 gives us that $\frac{\mathrm{d} X_{1}}{\mathrm{~d} t}=(u-c)\left(t, X_{1}(t)\right) \geqslant$ $(u-H)\left(t, X_{1}(t)\right)=w_{1}\left(t, X_{1}(t)\right) \geqslant 0$. Consequently, the 1-characteristics are going away from the origin and are defined for all $t \in[0, \mathcal{T}[$.

Let us modify slightly the hypotheses in order to obtain a better result: we take off the hypothesis $w_{1,0}>0$. We obtain a similar result, but now the time of validity is finite:

Proposition 3.7. Let us consider a Bethe-Weyl gas, satisfying $1<\mathscr{G}<2$ and such that $\left(\rho \mapsto \frac{c(\rho)}{\rho}\right)$ is integrable in 0 . Let $T>0, R>0$ and let $w=\left(w_{1}, w_{2}\right)$ be a regular solution of (3.2) with a time of existence $\mathcal{T}>0$. We denote $w_{1} 1,0=w_{1}(0, \cdot)$ and $w_{2,0}=w_{2}(0, \cdot)$, the initial condition of this regular solution. Let $T, R$ be two stricty positive real numbers and $X_{1}$ and $X_{2}$ be the characteristics defined by (3.4), crossing in $(T, R)$.

We assume that for all $r>0,\left(w_{2,0}-w_{1,0}\right)(r) \geqslant 0$ and $\min \left(w_{1,0}+w_{2,0}\right)>0$. Then there exists a time $T_{0}>0$ such that :

- for all $t \in\left[0, T_{0}\right]$,

$$
\begin{equation*}
w_{1,0}\left(r_{1}\right) \leqslant w_{1}(t, R) \leqslant w_{2}(t, R) \leqslant w_{2,0}\left(r_{2}\right) \tag{3.8}
\end{equation*}
$$

- for all $t \in\left[0, T_{0}\right]$,

$$
\begin{gather*}
w_{1,0}\left(r_{1}\right) \leqslant w_{1}\left(t, X_{1}(t)\right) \leqslant w_{1,0}\left(r_{1}\right)+\left\|w_{2,0}\right\|_{\mathbf{L} \infty}^{2} \int_{0}^{t} \frac{d-1}{4 X_{1}(\tau)} \mathrm{d} \tau  \tag{3.9}\\
 \tag{3.10}\\
\frac{w_{2,0}\left(r_{2}\right)}{1+\frac{w_{2,0}\left(r_{2}\right)}{r_{2}} \frac{(d-1) t}{4}} \leqslant w_{2}\left(t, X_{2}(t)\right) \leqslant w_{2,0}\left(r_{2}\right)
\end{gather*}
$$

Furthermore, the time of existence $T_{0}$ is defined by $T_{0}=\min (\mathcal{T},+\infty)$ if $\min \left(w_{1,0}\right) \geqslant 0$ and, in the case $\min \left(w_{1,0}\right)<0$, by
$T_{0}=\min \left(\mathcal{T}, \frac{-4 r_{2}}{(d-1) \min _{r_{1}>0}\left(w_{1,0}\left(r_{1}\right)\right) \min _{r_{2}>0}\left(w_{2,0}\left(r_{2}\right)\right)}\left(\min _{r_{1}>0}\left(w_{1,0}\left(r_{1}\right)\right)+\min _{r_{2}>0}\left(w_{2,0}\left(r_{2}\right)\right)\right)\right)$,
Proof. As in the preceding proposition, we know that if the density if non-negative at the initial time, then it is non-negative for any positive time. Hence, $w_{2}(t, r) \geqslant w_{1}(t, r)$ for all $(t, r) \in\left[0, \mathcal{T}\left[\times \mathbb{R}_{+}\right.\right.$.

By hypothesis, $\inf _{r>0} u(0, r) \geqslant \frac{1}{2}\left(\inf \left(w_{1,0}+w_{2,0}\right)\right)>0$. Consequently, by semi-continuity of $\left(t \mapsto \inf _{r>0} u(t, r)\right)$, there exists $\left.t_{0} \in\right] 0, \mathcal{T}$ [ such that on a small time interval $\left[0, t_{0}\right]$ we have $u \geqslant 0$. Hence for all $t \in\left[0, t_{0}\right], \frac{\mathrm{d} w_{1}\left(t, X_{1}(t)\right)}{\mathrm{d} t}>0$ and $\frac{\mathrm{d} w_{2}\left(t, X_{2}(t)\right)}{\mathrm{d} t}<0$. Hence for $t \in\left[0, t_{0}\right], w\left(t, X_{1}(t)\right) \geqslant w_{1}\left(0, r_{1}\right)$ and $w_{2}\left(t, X_{2}(t)\right) \leqslant w_{2}\left(0, r_{2}\right)$.

Assume furthermore that $u(0, r)>0$ on $\left[0, t_{0}\right]$, then: $\frac{\mathrm{d} w_{2}\left(t, X_{2}(t)\right)}{\mathrm{d} t} \geqslant \frac{-(d-1)}{4 X_{2}(t)} w_{2}^{2}$ and $\frac{\mathrm{d} w_{1}\left(t, X_{1}(t)\right)}{\mathrm{d} t} \leqslant \frac{(d-1)}{4 X_{1}(t)} w_{2}^{2}$. Hence, the same estimates as before hold true:

$$
\frac{1}{w_{2}\left(t, X_{2}(t)\right)} \leqslant \frac{1}{w_{2}\left(0, r_{2}\right)}+\int_{0}^{t} \frac{(d-1)}{4 X_{2}(\tau)} \mathrm{d} \tau
$$

Since, for $t \in\left[0, T_{0}\right], \lambda_{2}=u+c \geqslant 0, X_{2}$ is increasing and as $w_{2} \geqslant u$ implies $w_{2} \geqslant 0$, we can invert the relation, obtaining

$$
w_{2}\left(t, X_{2}(t)\right) \geqslant \frac{w_{2}\left(0, r_{2}\right)}{1+w_{2}\left(0, r_{2}\right) \frac{(d-1) t}{4 r_{2}}}
$$

Let us use these estimate to find a lower bound for $u$. If $t \leqslant T_{0}$ then

$$
u(t, R)=\frac{1}{2}\left(w_{1}+w_{2}\right)(t, R) \geqslant \frac{1}{2}\left(w_{1}\left(0, r_{1}\right)+\frac{w_{2}\left(0, r_{2}\right)}{1+w_{2}\left(0, r_{2}\right) \frac{(d-1) t}{4 r_{2}}}\right)
$$

If $w_{1}\left(0, r_{1}\right) \geqslant 0$, then $u$ is non-negative for all time and we recover the result of Lemma 3.6. If $w_{1}\left(0, r_{1}\right)<0$ and $w_{1}\left(0, r_{1}\right)+w_{2}\left(0, r_{2}\right)>0$, then $u$ is non-negative if

$$
t \leqslant \frac{-4 r_{2}}{(d-1) w_{1}\left(0, r_{1}\right) w_{2}\left(0, r_{2}\right)}\left(w_{1}\left(0, r_{1}\right)+w_{2}\left(0, r_{2}\right)\right),
$$

which provides us a lower bound for $T_{0}$.

Remark 3.8. In Lemma 3.6 and Proposition 3.7, we are integrating along the characteristics on the time interval $[0, t]$. Note that we could obtain similar result integrating on any time interval $[\beta, t]$ with $\beta \in[0, t]$.

## 4 A priori $\mathscr{C}^{1}$ estimates along the characteristics

We want now to obtain estimates in $\mathbf{L}^{\infty}$ on $\partial_{r} w_{1}$ and $\partial_{r} w_{2}$ where $w=\left(w_{1}, w_{2}\right)$ is a regular solution of (3.2). We apply the same strategy as in the previous section. That is to say, we want to have a diagonal form for the system of equation of variables $\partial_{r} w_{1}$ and $\partial_{r} w_{2}$, obtained by derivating with respect to $r$ the system (3.2), and then make computations along the characteristics. As the obtained system in $\partial_{r} w_{1}$ and $\partial_{r} w_{2}$ is not diagonal, we have to introduce new variables $v_{1}$ and $v_{2}$ as described below.

### 4.1 Change of variable

Note that in the following section, $e$ stands for $\exp$ and not for the internal energy. We introduce below fuctions depending on the radius $r$ and on the riemann invariant $w_{1}$ and $w_{2}$; we denote $\partial_{1}$ for $\frac{\partial}{\partial w_{1}}, \partial_{2}$ for $\frac{\partial}{\partial w_{2}}$.
Lemma 4.1. Let us define

$$
\begin{equation*}
v_{1}=e^{h}\left(\partial_{r} w_{1}+\Phi\right), \quad v_{2}=e^{k}\left(\partial_{r} w_{2}+\Psi\right), \tag{4.1}
\end{equation*}
$$

where $h, k, \Phi$ and $\Psi$ are such that:

$$
\begin{align*}
\partial_{2} h & =\frac{\partial_{2} \lambda_{1}}{\lambda_{1}-\lambda_{2}}, & \partial_{1} k & =\frac{\partial_{1} \lambda_{2}}{\lambda_{2}-\lambda_{1}},  \tag{4.2}\\
\partial_{2}\left(e^{h} \Phi\right) & =\frac{-e^{h} \partial_{2} f}{\lambda_{1}-\lambda_{2}}, & \partial_{1}\left(e^{k} \Psi\right) & =\frac{e^{k} \partial_{1} f}{\lambda_{2}-\lambda_{1}} . \tag{4.3}
\end{align*}
$$

Then $v_{1}$ and $v_{2}$ satisfy the equations

$$
\left\{\begin{align*}
\partial_{t} v_{1}+\lambda_{1}(w) \partial_{r} v_{1} & =a_{0}(r, w) v_{1}^{2}+a_{1}(r, w) v_{1}+a_{2}(r, w)  \tag{4.4}\\
\partial_{t} v_{2}+\lambda_{2}(w) \partial_{r} v_{2} & =b_{0}(r, w) v_{2}^{2}+b_{1}(r, w) v_{2}+b_{2}(r, w)
\end{align*}\right.
$$

where

$$
\begin{aligned}
& a_{0}(r, w)=-e^{-h} \partial_{1} \lambda_{1}, \\
& a_{1}(r, w)=\partial_{1} f+2 \Phi \partial_{1} \lambda_{1}+\left(\partial_{1} h-\partial_{2} h\right) f, \\
& a_{2}(r, w)=e^{h}\left(\partial_{r} f-\Phi \partial_{1} f-\Phi^{2} \partial_{1} \lambda_{1}+\left(\partial_{1} \Phi-\partial_{2} \Phi\right) f+\lambda_{1} \partial_{r} \Phi\right), \\
& b_{0}(r, w)=-e^{-k} \partial_{2} \lambda_{2}, \\
& b_{1}(r, w)=-\partial_{2} f+2 \partial_{2} \lambda_{2} \Psi+\left(\partial_{1} k-\partial_{2} k\right) f, \\
& b_{2}(r, w)=e^{k}\left(-\partial_{r} f+\partial_{2} f \Psi-\partial_{2} \lambda_{2} \Psi^{2}+\lambda_{2} \partial_{r} \Psi+\left(\partial_{1} \Psi-\partial_{2} \Psi\right) f\right) .
\end{aligned}
$$

Remark 4.2. Note that, for any gas law, we have $\partial_{1} \lambda_{1}=\partial_{2} \lambda_{2}=\frac{\mathscr{Y}}{2} \geqslant 0$. Hence, we have $a_{0} \leqslant 0$ and $b_{0} \leqslant 0$.
Proof. On the first hand, derivating $v_{1}$ with respect to $t$ and $r$, we get

$$
\begin{aligned}
& \partial_{t} v_{1}=e^{h}\left(\partial_{t} \partial_{r} w_{1}+\partial_{1} \Phi \partial_{t} w_{1}+\partial_{2} \Phi \partial_{t} w_{2}\right)+e^{h}\left(\partial_{1} h \partial_{t} w_{1}+\partial_{2} h \partial_{t} w_{2}\right)\left(\partial_{r} w_{1}+\Phi\right) \\
& \partial_{r} v_{1}=e^{h}\left(\partial_{r}^{2} w_{1}+\partial_{1} \Phi \partial_{r} w_{1}+\partial_{2} \Phi \partial_{r} w_{2}+\partial_{r} \Phi\right)+e^{h}\left(\partial_{1} h \partial_{r} w_{1}+\partial_{2} h \partial_{r} w_{2}\right)\left(\partial_{r} w_{1}+\Phi\right) .
\end{aligned}
$$

Hence

$$
\begin{aligned}
& e^{-h}\left(\partial_{t} v_{1}+\lambda_{1} \partial_{r} v_{1}\right) \\
& =\partial_{t} \partial_{r} w_{1}+\lambda_{1} \partial_{r}^{2} w_{1}+\lambda_{1} \partial_{r} \Phi+\partial_{1} \Phi\left(\partial_{t} w_{1}+\lambda_{1} \partial_{r} w_{1}\right)+\partial_{2} \Phi\left(\partial_{t} w_{2}+\lambda_{1} \partial_{r} w_{2}\right) \\
& \quad+\left(\partial_{r} w_{1}+\Phi\right)\left(\partial_{1} h\left(\partial_{t} w_{1}+\lambda_{1} \partial_{r} w_{1}\right)+\partial_{2} h\left(\partial_{t} w_{2}+\lambda_{1} \partial_{r} w_{2}\right)\right)
\end{aligned}
$$

Note that $\partial_{t} w_{1}+\lambda_{1} \partial_{r} w_{1}=f$, and $\partial_{t} w_{2}+\lambda_{1} \partial_{r} w_{2}=-f+\left(\lambda_{1}-\lambda_{2}\right) \partial_{r} w_{2}$ so that

$$
\begin{align*}
& e^{-h}\left(\partial_{t} v_{1}+\lambda_{1} \partial_{r} v_{1}\right) \\
& =\partial_{t} \partial_{r} w_{1}+\lambda_{1} \partial_{r}^{2} w_{1}+\lambda_{1} \partial_{r} \Phi+f \partial_{1} \Phi+\partial_{2} \Phi\left(-f+\left(\lambda_{1}-\lambda_{2}\right) \partial_{r} w_{2}\right)  \tag{4.5}\\
& \quad+\left(\partial_{r} w_{1}+\Phi\right)\left(f \partial_{1} h+\partial_{2} h\left(-f+\left(\lambda_{1}-\lambda_{2}\right) \partial_{r} w_{2}\right)\right)
\end{align*}
$$

On the other hand, derivating with respect to time the equation in $w_{1}$, we obtain:

$$
\begin{aligned}
& \partial_{t} \partial_{r} w_{1}+\lambda_{1} \partial_{r}^{2} w_{1} \\
& =\partial_{r} f+\partial_{1} f \partial_{r} w_{1}+\partial_{2} f \partial_{r} w_{2}-\partial_{1} \lambda_{1}\left(\partial_{r} w_{1}\right)^{2}-\partial_{2} \lambda_{1} \partial_{r} w_{2} \partial_{r} w_{1} \\
& =\partial_{r} f+\partial_{1} f\left(e^{-h} v_{1}-\Phi\right)+\partial_{2} f \partial_{r} w_{2}-\partial_{1} \lambda_{1}\left(e^{-h} v_{1}-\Phi\right)^{2}-\partial_{2} \lambda_{1} \partial_{r} w_{2}\left(e^{-h} v_{1}-\Phi\right) \\
& =-e^{-2 h} v_{1}^{2} \partial_{1} \lambda_{1}+e^{-h} v_{1}\left(\partial_{1} f+2 \Phi \partial_{1} \lambda_{1}\right)+\left(\partial_{r} f-\Phi \partial_{1} f-\Phi^{2} \partial_{1} \lambda_{1}\right) \\
& \quad+\partial_{r} w_{2}\left(\partial_{2} f+\Phi \partial_{2} \lambda_{1}\right)-\partial_{2} \lambda_{1} e^{-h} v_{1} \partial_{r} w_{2}
\end{aligned}
$$

Replacing $\partial_{t} \partial r w_{1}+\lambda_{1} \partial_{1}^{2} w_{1}$ by its expression in (4.5), we get

$$
\begin{aligned}
e^{-h} & \left(\partial_{t} v_{1}+\lambda_{1} \partial_{r} v_{1}\right) \\
= & -e^{-2 h} v_{1}^{2} \partial_{1} \lambda_{1}+e^{-h} v_{1}\left(\partial_{1} f+2 \Phi \partial_{1} \lambda_{1}\right)+\left(\partial_{r} f-\Phi \partial_{1} f-\Phi^{2} \partial_{1} \lambda_{1}\right) \\
& +\partial_{r} w_{2}\left(\partial_{2} f+\Phi \partial_{2} \lambda_{1}\right)-\partial_{2} \lambda_{1} e^{-h} v_{1} \partial_{r} w_{2}+\lambda_{1} \partial_{r} \Phi+f\left(\partial_{1} \Phi-\partial_{2} \Phi\right)+\partial_{2} \Phi\left(\lambda_{1}-\lambda_{2}\right) \partial_{r} w_{2} \\
& +e^{-h} v_{1} f\left(\partial_{1} h-\partial_{2} h\right)+\partial_{2} h\left(\lambda_{1}-\lambda_{2}\right) e^{-h} v_{1} \partial_{r} w_{2} \\
= & -e^{-2 h} v_{1}^{2} \partial_{1} \lambda_{1}+e^{-h} v_{1}\left(\partial_{1} f+2 \Phi \partial_{1} \lambda_{1}+f\left(\partial_{1} h-\partial_{2} h\right)\right) \\
& +\left(\partial_{r} f-\Phi \partial_{1} f-\Phi^{2} \partial_{1} \lambda_{1}+f\left(\partial_{1} \Phi-\partial_{2} \Phi\right)+\lambda_{1} \partial_{r} \Phi\right) \\
& +\partial_{r} w_{2}\left(\partial_{2} f+\Phi \partial_{2} \lambda_{1}+\partial_{2} \Phi\left(\lambda_{1}-\lambda_{2}\right)\right)+\left(\partial_{2} h\left(\lambda_{1}-\lambda_{2}\right)-\partial_{2} \lambda_{1}\right) e^{-h} v_{1} \partial_{r} w_{2} .
\end{aligned}
$$

With our choice for $h$ and $\Phi$, the last line above vanishes.
In the same way, we have for $v_{2}$

$$
\begin{aligned}
& \partial_{t} v_{2}=e^{k}\left(\partial_{t} \partial_{r} w_{2}+\partial_{1} \Psi \partial_{t} w_{1}+\partial_{2} \Psi \partial_{t} w_{2}\right)+e^{k}\left(\partial_{1} k \partial_{t} w_{1}+\partial_{2} k \partial_{t} w_{2}\right)\left(\partial_{r} w_{2}+\Psi\right) \\
& \partial_{r} v_{2}=e^{k}\left(\partial_{r}^{2} w_{2}+\partial_{1} \Psi \partial_{r} w_{1}+\partial_{2} \Psi \partial_{r} w_{2}+\partial_{r} \Psi\right)+e^{k}\left(\partial_{1} k \partial_{r} w_{1}+\partial_{2} k \partial_{r} w_{2}\right)\left(\partial_{r} w_{2}+\Psi\right)
\end{aligned}
$$

Hence

$$
\begin{aligned}
& e^{-k}\left(\partial_{t} v_{2}+\lambda_{2} \partial_{r} v_{2}\right) \\
& =\partial_{t} \partial_{r} w_{2}+\lambda_{2} \partial_{r}^{2} w_{2}+\lambda_{2} \partial_{r} \Psi+\partial_{1} \Psi\left(\partial_{t} w_{1}+\lambda_{2} \partial_{r} w_{1}\right)+\partial_{2} \Psi\left(\partial_{t} w_{2}+\lambda_{2} \partial_{r} w_{2}\right) \\
& \quad+\left(\partial_{r} w_{2}+\Psi\right)\left(\partial_{1} k\left(\partial_{t} w_{1}+\lambda_{2} \partial_{r} w_{1}\right)+\partial_{2} k\left(\partial_{t} w_{2}+\lambda_{2} \partial_{r} w_{2}\right)\right)
\end{aligned}
$$

Note that $\partial_{t} w_{1}+\lambda_{2} \partial_{r} w_{1}=f+\left(\lambda_{2}-\lambda_{1}\right) \partial_{r} w_{1}$, and $\partial_{t} w_{2}+\lambda_{2} \partial_{r} w_{2}=-f$ so that

$$
\begin{aligned}
& e^{-k}\left(\partial_{t} v_{2}+\lambda_{2} \partial_{r} v_{2}\right) \\
& =\partial_{t} \partial_{r} w_{2}+\lambda_{2} \partial_{r}^{2} w_{2}+\lambda_{2} \partial_{r} \Psi+\left(f+\left(\lambda_{2}-\lambda_{1}\right) \partial_{r} w_{1}\right) \partial_{1} \Psi-f \partial_{2} \Psi \\
& \quad+\left(\partial_{r} w_{2}+\Psi\right)\left(\left(f+\left(\lambda_{2}-\lambda_{1}\right) \partial_{r} w_{1}\right) \partial_{1} k-f \partial_{2} k\right)
\end{aligned}
$$

Derivating with respect to time the equation in $w_{2}$, we obtain:

$$
\begin{aligned}
& \partial_{t} \partial_{r} w_{2}+\lambda_{2} \partial_{r}^{2} w_{2} \\
& =-\partial_{r} f-\partial_{1} f \partial_{r} w_{1}-\partial_{2} f \partial_{r} w_{2}-\partial_{1} \lambda_{2} \partial_{r} w_{1} \partial_{r} w_{2}-\partial_{2} \lambda_{2}\left(\partial_{r} w_{2}\right)^{2} \\
& =-\partial_{r} f-\partial_{1} f \partial_{r} w_{1}-\partial_{2} f\left(e^{-k} v_{2}-\Psi\right)-\partial_{1} \lambda_{2}\left(e^{-k} v_{2}-\Psi\right) \partial_{r} w_{1}-\partial_{2} \lambda_{2}\left(e^{-k} v_{2}-\Psi\right)^{2} \\
& =-e^{-2 k} v_{2}^{2} \partial_{2} \lambda_{2}+e^{-k} v_{2}\left(-\partial_{2} f+2 \Psi \partial_{2} \lambda_{2}\right)+\left(-\partial_{r} f+\Psi \partial_{2} f-\Psi^{2} \partial_{2} \lambda_{2}\right) \\
& \quad+\partial_{r} w_{1}\left(-\partial_{1} f+\Psi \partial_{1} \lambda_{2}\right)-\partial_{1} \lambda_{2} e^{-k} v_{2} \partial_{r} w_{1} .
\end{aligned}
$$

Replacing, we get

$$
\begin{aligned}
e^{-k} & \left(\partial_{t} v_{2}+\lambda_{2} \partial_{r} v_{2}\right) \\
= & -e^{-2 k} v_{2}^{2} \partial_{2} \lambda_{2}+e^{-k} v_{2}\left(-\partial_{2} f+2 \Psi \partial_{2} \lambda_{2}\right)+\left(-\partial_{r} f+\Psi \partial_{2} f-\Psi^{2} \partial_{2} \lambda_{2}\right) \\
& +\partial_{r} w_{1}\left(-\partial_{1} f+\Psi \partial_{1} \lambda_{2}\right)-\partial_{1} \lambda_{2} e^{-k} v_{2} \partial_{r} w_{1}+\lambda_{2} \partial_{r} \Psi+\left(f+\left(\lambda_{2}-\lambda_{1}\right) \partial_{r} w_{1}\right) \partial_{1} \Psi-f \partial_{2} \Psi \\
& +e^{-k} v_{2}\left(\left(f+\left(\lambda_{2}-\lambda_{1}\right) \partial_{r} w_{1}\right) \partial_{1} k-f \partial_{2} k\right) \\
= & -e^{-2 k} v_{2}^{2} \partial_{2} \lambda_{2}+e^{-k} v_{2}\left(-\partial_{2} f+2 \Psi \partial_{2} \lambda_{2}+f\left(\partial_{1} k-\partial_{2} k\right)\right) \\
& +\left(-\partial_{r} f+\Psi \partial_{2} f-\Psi^{2} \partial_{2} \lambda_{2}+\lambda_{2} \partial_{r} \Psi+f\left(\partial_{1} \Psi-\partial_{2} \Psi\right)\right) \\
& +\partial_{r} w_{1}\left(-\partial_{1} f+\Psi \partial_{1} \lambda_{2}+\left(\lambda_{2}-\lambda_{1}\right) \partial_{1} \Psi\right)+e^{-k} v_{2} \partial_{r} w_{1}\left(\left(\lambda_{2}-\lambda_{1}\right) \partial_{1} k-\partial_{1} \lambda_{2}\right)
\end{aligned}
$$

With our choice for $k$ and $\Psi$, the last line above vanishes.
Similarly as above, we want to derive a diagonal system on $r v_{1}$ and $r v_{2}$.
Corollary 4.3. With the notation of Lemma 4.1, we have for all $\ell \in \mathbb{N}$,

$$
\left\{\begin{array}{l}
\partial_{t}\left(r^{\ell} v_{1}\right)+\lambda_{1}(w) \partial_{r}\left(r^{\ell} v_{1}\right)=\frac{a_{0}(r, w)}{r^{\ell}}\left(r^{\ell} v_{1}\right)^{2}+\left(a_{1}(r, w)+\frac{\ell \lambda_{1}}{r}\right)\left(r^{\ell} v_{1}\right)+r^{\ell} a_{2}(r, w),  \tag{4.6}\\
\partial_{t}\left(r^{\ell} v_{2}\right)+\lambda_{2}(w) \partial_{r}\left(r^{\ell} v_{2}\right)=\frac{b_{0}(r, w)}{r^{\ell}}\left(r^{\ell} v_{2}\right)^{2}+\left(b_{1}(r, w)+\frac{\ell \lambda_{2}}{r}\right)\left(r^{\ell} v_{2}\right)+r^{\ell} b_{2}(r, w) .
\end{array}\right.
$$

In the following we give some properties of the coefficients $a_{0}, a_{1}, a_{2}, b_{0}, b_{1}, b_{2}$ :
Lemma 4.4. Let us consider a Bethe-Weyl gas, satisfying $1<\mathscr{G}<2$ and such that $\left(\rho \mapsto \frac{c(\rho)}{\rho}\right)$ is integrable in 0 . There exist smooth functions $\bar{a}_{i}: \mathbb{R}^{2} \rightarrow \mathbb{R}$ and $\bar{b}_{i}: \mathbb{R}^{2} \rightarrow \mathbb{R}$ such that, for $i \in\{1,2\}$ the coefficients $a_{i}$ and $b_{i}$ defined as in Lemma 4.1 can be written $a_{i}(r, w)=\frac{\bar{a}_{i}(w)}{r^{i}}$ and $b_{i}(r, w)=\frac{\bar{b}_{i}(w)}{r^{i}}$. In particular, $\bar{a}_{i}$ and $\bar{b}_{i}$ are not depending on $r$.

Similarly, there exist smooth functions $\varphi: \mathbb{R}^{2} \rightarrow \mathbb{R}$ and $\psi: \mathbb{R}^{2} \rightarrow \mathbb{R}$ such that the coefficients $\Phi$ and $\Psi$ can be written $\Phi(r, w)=\frac{\varphi(w)}{r}$ and $\Psi(r, w)=\frac{\psi(w)}{r^{i}}$. In particular $\varphi$ and $\psi$ are not depending on $r$.

Proof. We have $\lambda_{1}=u-c$, and $u=\frac{w_{1}+w_{2}}{2}, H=\frac{w_{2}-w_{1}}{2}$. Hence $\partial_{2} \lambda_{1}=\frac{1}{2}-\frac{c^{\prime}}{2 H^{\prime}}$ where $\partial_{2} \lambda_{1}$ stands for $\frac{\partial \lambda_{1}\left(w_{1}, w_{2}\right)}{\partial w_{2}}$, and

$$
\partial_{2} h=\frac{H^{\prime \prime}}{4\left(H^{\prime}\right)^{2}}
$$

Finally $h=\frac{1}{2} \ln \left(H^{\prime}\right)$ satisfies the equation. In the same way, we obtain $k=h=\frac{1}{2} \ln \left(H^{\prime}\right)$.
Using $H^{\prime}(\rho)=\frac{c(\rho)}{\rho}$ and $\mathscr{G}=\frac{1}{c(\rho)}(c(\rho) \rho)^{\prime}$, we note that $c^{\prime}=H^{\prime}(\mathscr{G}-1)$ and that $H^{\prime \prime}=\frac{H^{\prime}}{\rho}(\mathscr{G}-2)$. Then

$$
\partial_{2}\left(e^{h} \Phi\right)=\frac{\sqrt{H^{\prime}}}{2 c} \frac{d-1}{r}\left(\frac{c}{2}+u \frac{c^{\prime}}{2 H^{\prime}}\right)=\frac{d-1}{2 r}\left(\frac{1}{2} \sqrt{H^{\prime}}+u \frac{c^{\prime}}{2 c \sqrt{H^{\prime}}}\right) .
$$

Let us define $g, A$ and $B$ such that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} \rho}\left(2 \sqrt{H^{\prime}(\rho)} g(\rho)\right) & =\frac{1}{\rho} \sqrt{H^{\prime}(\rho)} \\
A & =1+g \\
\frac{\mathrm{~d}}{\mathrm{~d} \rho}\left(2 \sqrt{H^{\prime}(\rho)} B(\rho)\right) & =H^{\prime}(\rho) \sqrt{H^{\prime}(\rho)}(1+2 g(\rho))
\end{aligned}
$$

Then, noting that, for any function $f$ of $u$ and $\rho$ we have

$$
\partial_{2} f=\partial_{2} u \partial_{u} f+\partial_{2} \rho \partial_{\rho} f=\frac{1}{2} \partial_{u} f+\frac{1}{2 H^{\prime}(\rho)} \partial_{\rho} f
$$

Hence, we can check that $\partial_{2}\left(e^{h} \Phi\right)=\frac{d-1}{2 r} \partial_{2}\left(2 \sqrt{H^{\prime}} u A-2 \sqrt{H^{\prime}} B\right)$ and we can choose

$$
\Phi=\frac{d-1}{r}(u A-B)
$$

In the same way, we have

$$
\Psi=\frac{d-1}{r}(u A+B)
$$

Note that

$$
\partial_{\rho} A=\frac{1}{2 \rho}(1-g(\mathscr{G}-2)), \quad \partial_{\rho} B=\frac{1}{2 \rho}(c(1+2 g)-B(\mathscr{G}-2))
$$

Let us now compute the expression of $a_{0}, a_{1}, a_{2}$ :

$$
\begin{aligned}
a_{0} & =-e^{-h} \partial_{1} \lambda_{1}=-\frac{1}{\sqrt{H^{\prime}}}\left(\frac{1}{2}+\frac{c^{\prime}}{2 H^{\prime}}\right)=\frac{-1}{\sqrt{H^{\prime}}} \frac{\mathscr{G}}{2} \leqslant 0, \\
a_{1} & =\frac{d-1}{2 r}(c-u(\mathscr{G}-1))+\mathscr{G} \frac{d-1}{r}(u A-B)-\frac{d-1}{2 r} u(\mathscr{G}-2) \\
& =\frac{d-1}{r}\left[\frac{c}{2}-B \mathscr{G}+\frac{u}{2}(3+2 \mathscr{G} g)\right],
\end{aligned}
$$

and

$$
\begin{aligned}
a_{2}= & \frac{(d-1) \sqrt{H^{\prime}}}{r^{2}}\left[-u c-\frac{d-1}{2}\left((u A-B)(c-u(\mathscr{G}-1))+\mathscr{G}(u A-B)^{2}\right)\right. \\
& -\frac{d-1}{2} u(u(1-(\mathscr{G}-2)(A-1))+(B(\mathscr{G}-2)-c(2 A-1))) \\
& -(u-c)(u A-B)] \\
= & \frac{(d-1) \sqrt{H^{\prime}}}{r^{2}}\left[-u c-\left(u^{2} A-u(B+c A)+c B\right)\right. \\
& -\frac{d-1}{2}\left(u(A c+B(\mathscr{G}-1))-B c-u^{2} A(\mathscr{G}-1)+\mathscr{G}\left(u^{2} A^{2}-2 u A B+B^{2}\right)\right. \\
& \left.\left.\left.+u^{2}(1-(\mathscr{G}-2)(A-1))+u(B(\mathscr{G}-2)-c(2 A-1))\right)\right)\right]
\end{aligned}
$$

Let us now consider the expression of $b_{0}, b_{1}, b_{2}$ as given in (4.8).
$b_{0}=a_{0}=\frac{-\mathscr{G}}{2 \sqrt{H^{\prime}}}$,
$b_{1}=\frac{d-1}{2 r}(u(2 \mathscr{G}(A-1)+3)+2 \mathscr{G} B-c)$,
$b_{2}=\frac{(d-1) \sqrt{H^{\prime}}}{r^{2}}\left[-u^{2} A-u(c(A-1)+B)-c B+\frac{d-1}{2}\left(u^{2}\left(-\mathscr{G} A^{2}+(2 \mathscr{G}-3) A+1-\mathscr{G}\right)\right.\right.$

$$
\left.\left.+u(B(2 \mathscr{G}-3-2 \mathscr{G} A)-c(A-1))+B c-\mathscr{G} B^{2}\right)\right]
$$

## 4.2 $\mathscr{C}^{1}$ estimate on $w_{1}$ and $w_{2}$

We derive now $\mathscr{C}^{1}$-estimates for $w_{1}$ and $w_{2}$. First we derive an upper bound on $\partial_{r} w_{1}$ and $\partial_{r} w_{2}$.

Lemma 4.5. Let us consider a Bethe-Weyl fluid and let $w=\left(w_{1}, w_{2}\right)$ be a regular solution of (3.2) with a time of existence $\mathcal{T}>0$. Let $\alpha, R>0$ with $\alpha<\mathcal{T}$. Let $t \mapsto X_{1}(t)$ and $t \mapsto X_{2}(t)$ be the characteristics defined as in (3.4) and passing through ( $\alpha, R$ ). Let $T$ be such that $T>\alpha$ and $T<\mathcal{T}$. We assume that $X_{1}(t), X_{2}(t)$ are well-defined on $[\alpha, T]$. Let furthermore $v_{1}, v_{2}$ be defined as in (4.1). Then, for all $t \in[0, \mathcal{T}[$, we have

$$
\begin{aligned}
v_{1}\left(t, X_{1}(t)\right) e^{-\int_{\alpha}^{t} a_{1}\left(s, X_{1}(s)\right) \mathrm{d} s} & \leqslant v_{1}\left(\alpha, X_{1}(\alpha)\right)+\int_{\alpha}^{t} a_{2}\left(s, X_{1}(s)\right) e^{\int_{s}^{t} a_{1}\left(\tau, X_{1}(\tau)\right) \mathrm{d} \tau} \\
v_{2}\left(t, X_{2}(t)\right) e^{-\int_{\alpha}^{t} b_{1}\left(s, X_{2}(s) \mathrm{d} s\right.} & \leqslant v_{2}\left(\alpha, X_{2}(\alpha)\right)+\int_{\alpha}^{t} b_{2}\left(s, X_{2}(s)\right) e^{\int_{s}^{t} b_{1}\left(\tau, X_{2}(\tau)\right) \mathrm{d} \tau}
\end{aligned}
$$

More generally, for any $\ell \in \mathbb{N}$,

$$
\begin{aligned}
& X_{1}(t)^{\ell} v_{1}\left(t, X_{1}(t)\right) \\
\leqslant & X_{1}(\alpha)^{\ell} v_{1}\left(\alpha, X_{1}(\alpha)\right) e^{\int_{\alpha}^{t}\left(\frac{\ell \lambda_{1}}{X_{1}(s)}+a_{1}\left(s, X_{1}(s)\right)\right) \mathrm{d} s}+\int_{\alpha}^{t} X_{1}(s)^{\ell} a_{2}\left(s, X_{1}(s)\right) e^{\int_{s}^{t}\left(\frac{\ell \lambda_{1}}{X_{1}(\tau)}+a_{1}\left(\tau, X_{1}(\tau)\right)\right) \mathrm{d} \tau}, \\
& X_{2}(t)^{\ell} v_{2}\left(t, X_{2}(t)\right) \\
\leqslant & X_{2}(\alpha)^{\ell} v_{2}\left(\alpha, X_{2}(\alpha)\right) e^{\int_{\alpha}^{t}\left(\frac{\ell \lambda_{2}}{X_{2}(s)}+b_{1}\left(s, X_{2}(s)\right)\right) \mathrm{d} s}+\int_{\alpha}^{t} X_{2}^{\ell}(s) b_{2}\left(s, X_{2}(s)\right) e^{f_{s}^{t}\left(\frac{\ell \lambda_{2}}{X_{2}(\tau)}+b_{1}\left(\tau, X_{2}(\tau)\right)\right) \mathrm{d} \tau} .
\end{aligned}
$$

Proof. Let us consider $v_{1}$, the argument being similar for $v_{2}$. Let us denote $y_{1}(t)=$ $v_{1}\left(t, X_{1}(t)\right)$. The coefficient $a_{0}$ being non-positive, we have by Lemma 4.1 $y_{1}^{\prime}(t) \leqslant a_{1} y_{1}+a_{2}$. Hence,

$$
\left(y_{1} e^{-\int_{\alpha}^{t} a_{1}\left(s, X_{1}(s)\right) \mathrm{d} s}\right)^{\prime} \leqslant a_{2}\left(t, X_{1}(t)\right) e^{-\int_{\alpha}^{t} a_{1}\left(s, X_{1}(s)\right) \mathrm{d} s}
$$

Integrating we obtain the desired estimate. The second set of estimates is obtained in the same way considering the system (4.6) instead of (4.4).

We now derive a lower bound on $v_{1}$ and $v_{2}$. First we consider the case in which the initial condition $v_{1}(\alpha, R)$ is positive.

Lemma 4.6. Let us consider a Bethe-Weyl fluid and let $w=\left(w_{1}, w_{2}\right)$ be a regular solution of (3.2) with a time of existence $\mathcal{T}>0$. Let $\alpha, R>0$ with $\alpha<\mathcal{T}$. Let $t \mapsto X_{1}(t)$ and $t \mapsto X_{2}(t)$ be the characteristics defined as in (3.4) and passing through $(\alpha, R)$. Let $T$ be such that $T>\alpha$ and $T<\mathcal{T}$. We assume that $X_{1}(t), X_{2}(t)$ are well-defined on $[\alpha, T]$.

Let us denote $\bar{a}_{1}=r a_{1}, \bar{a}_{2}=r^{2} a_{2}, \bar{b}_{1}=r b_{1}, \bar{b}_{2}=r^{2} b_{2}$ as in Lemma 4.4, and

$$
\begin{array}{lll}
A_{0}=\max _{[0, T]}\left|a_{0}\left(t, X_{1}(t)\right)\right|, & A_{1}=\max _{[0, T]}\left|\bar{a}_{1}\left(t, X_{1}(t)\right)\right|, & A_{2}=\max _{[0, T]}\left|\bar{a}_{2}\left(t, X_{1}(t)\right)\right|, \\
B_{0}=\max _{[0, T]}\left|b_{0}\left(t, X_{2}(t)\right)\right|, & B_{1}=\max _{[0, T]}\left|\bar{b}_{1}\left(t, X_{2}(t)\right)\right|, & B_{2}=\max _{[0, T]}\left|\bar{b}_{2}\left(t, X_{2}(t)\right)\right| . \tag{4.8}
\end{array}
$$

We also denote, for all $\zeta_{0}, \zeta_{1}, \zeta_{2}$ positive, $x\left(\zeta_{0}, \zeta_{1}, \zeta_{2}\right)$ the unique positive solution of

$$
\begin{equation*}
Q(r)=\frac{\zeta_{1}}{\sqrt{\zeta_{0} \zeta_{2}}}, \quad \text { where } Q(r):=r e^{r} \tag{4.9}
\end{equation*}
$$

Furthermore, we denote

$$
\begin{align*}
K_{a}(\theta) & =\left(\int_{\alpha}^{\theta}\left|a_{2}\left(t, X_{1}(t)\right)\right| \mathrm{d} t\right) \cdot \exp \left(\int_{\alpha}^{\theta}\left|a_{1}\left(t, X_{1}(t)\right)\right| \mathrm{d} t\right),  \tag{4.10}\\
K_{b}(\theta) & =\left(\int_{\alpha}^{\theta}\left|b_{2}\left(t, X_{2}(t)\right)\right| \mathrm{d} t\right) \cdot \exp \left(\int_{\alpha}^{\theta}\left|b_{1}\left(t, X_{2}(t)\right)\right| \mathrm{d} t\right) . \tag{4.11}
\end{align*}
$$

Then:

1. If $v_{1}(\alpha, R) \geqslant 0$, then for all $t \in\left[\alpha, \alpha+\frac{x\left(A_{0}, A_{1}, A_{2}\right)}{A_{1}} \min _{[0, T]} X_{1}(t)\right]$, we have

$$
\begin{aligned}
\frac{-K_{a}(t)}{1-K_{a}(t) \int_{\alpha}^{t}\left|a_{0}\left(\tau, X_{1}(\tau)\right)\right| e^{\int_{\alpha}^{\tau}\left|a_{1}\left(u, X_{1}(u)\right)\right| \mathrm{d} u} \mathrm{~d} \tau} & \leqslant v_{1}\left(t, X_{1}(t)\right) e^{-\int_{\alpha}^{t} a_{1}\left(s, X_{1}(s)\right) \mathrm{d} s} \\
& \leqslant v_{1}\left(\alpha, X_{1}(\alpha)\right)+K_{a}(t) .
\end{aligned}
$$

2. If $v_{2}(\alpha, R) \geqslant 0$, then for all $t \in\left[\alpha, \alpha+\frac{x\left(B_{0}, B_{1}, B_{2}\right)}{B_{1}} \min _{[0, T]} X_{2}(t)\right]$, we have

$$
\begin{aligned}
\frac{-K_{b}(t)}{1-K_{b}(t) \int_{\alpha}^{t}\left|b_{0}\left(\tau, X_{2}(\tau)\right)\right| e^{\int_{\alpha}^{\tau}\left|b_{1}\left(u, X_{2}(u)\right)\right| \mathrm{d} u} \mathrm{~d} \tau} & \leqslant v_{2}\left(t, X_{2}(t)\right) e^{-\int_{\alpha}^{t} b_{1}\left(s, X_{2}(s)\right) \mathrm{d} s} \\
& \leqslant v_{2}\left(\alpha, X_{2}(\alpha)\right)+K_{b}(t) .
\end{aligned}
$$

Remark 4.7. The quantities $A_{i}$ and $B_{i}$, for $i \in\{0,1,2\}$, are not depending directly on $r$ but only on $\left\|w_{1}\right\|_{\mathbf{L}^{\infty}}$ and $\left\|w_{2}\right\|_{\mathbf{L}^{\infty}}$. In the case $\frac{\mathrm{d} X_{2}}{\mathrm{~d} t}>0$ and $\alpha=0$, we obtain a lower bound for the time of existence of $\partial_{r} w_{2}$

$$
T_{e x}\left(\partial_{r} w_{2}\right) \geqslant \frac{x\left(B_{0}, B_{1}, B_{2}\right)}{B_{1}} R .
$$

Remark 4.8. The same lemma can be applied to (4.6) with

$$
\begin{array}{lll}
\tilde{a}_{0}=\frac{a_{0}}{r^{\ell}}, & \tilde{a}_{1}=a_{1}+\frac{\ell \lambda_{1}}{r}, & \tilde{a}_{2}=r^{\ell} a_{2}, \\
\tilde{b}_{0}=\frac{b_{0}}{r^{\ell}}, & \tilde{b}_{1}=b_{1}+\frac{\ell \lambda_{2}}{r}, & \tilde{b}_{2}=r^{\ell} b_{2} .
\end{array}
$$

The time of validity of the estimates remains similar, replacing $A_{1}$ by $\tilde{A}_{1}=\max _{[0, T]}\left(\bar{a}_{1}+\right.$ $\left.\left.\lambda_{1}\right)\left(t, X_{1}(t)\right)\right)$ and $B_{1}$ by $\left.\tilde{B}_{1}=\max _{[0, T]}\left(\bar{b}_{1}+\lambda_{2}\right)\left(t, X_{2}(t)\right)\right)$, with $A_{0}, B_{0}, A_{2}, B_{2}$ unchanged.

Proof. First note that the upper bounds come directly from Lemma 4.5.
Let us consider the equation on $v_{2}$. Without a lot of changes we can adapt the following to $v_{1}$. Let us denote $y_{2}(t)=v_{2}\left(t, X_{2}(t)\right)$ where $v_{2}$ is defined in in (4.1). The equation on $v_{2}$ in (4.4) writes, with $y_{2}(t)=v_{2}\left(t, X_{2}(t)\right)$,

$$
\begin{equation*}
y_{2}^{\prime}=b_{0} y_{2}^{2}+b_{1} y_{2}+b_{2}, \tag{4.12}
\end{equation*}
$$

where $b_{0} \leqslant 0$.

According to a Lemma A. 2 (see also Hörmander [8), if we have $y_{2}(\alpha) \geqslant 0$ and, for $\theta>\alpha$

$$
\begin{equation*}
\int_{\alpha}^{\theta}\left|b_{2}\left(t, X_{2}(t)\right)\right| \mathrm{d} t \int_{\alpha}^{\theta}\left|b_{0}\left(t, X_{2}(t)\right)\right| \mathrm{d} t \exp \left(2 \int_{\alpha}^{\theta}\left|b_{1}\left(t, X_{2}(t)\right)\right| \mathrm{d} t\right)<1 \tag{4.13}
\end{equation*}
$$

then the equation (4.12) with initial condition $y_{2}(\alpha)=v_{2}\left(\alpha, X_{2}(\alpha)\right)$ in $t=\alpha$ admits a solution on $[\alpha, \theta]$ and, with $K_{b}$ as in (4.11), we have the estimate

$$
\begin{aligned}
& \frac{-K_{b}(\theta)}{1-K_{b}(\theta) \int_{\alpha}^{\theta}\left|b_{0}\left(t, X_{2}(t)\right)\right| \mathrm{d} t \exp \left(\int_{\alpha}^{\theta}\left|b_{1}\left(t, X_{2}(t)\right)\right| \mathrm{d} t\right)} \\
& \leqslant y_{2}(\theta) e^{-\int_{\alpha}^{\theta} b_{1}\left(s, X_{1}(s)\right) \mathrm{d} s} \leqslant v_{2}\left(\alpha, X_{2}(\alpha)\right)+K_{b}(\theta) .
\end{aligned}
$$

Note, thanks to Lemma 4.4 that we have

$$
b_{0}(r, w)=\bar{b}_{0}(w), \quad b_{1}(r, w)=\frac{\bar{b}_{1}(w)}{r}, \quad b_{2}(r, w)=\frac{\bar{b}_{2}(w)}{r^{2}}
$$

As we are not considering the 1-D case, $b_{0}, b_{2}$ are not constantly zero, see the expression of $b_{0}, b_{2}$ in Lemma 4.1,

We have:

$$
\begin{aligned}
& \int_{\alpha}^{\theta}\left|b_{2}\left(t, X_{2}(t)\right)\right| \mathrm{d} t \int_{\alpha}^{\theta}\left|b_{0}\left(t, X_{2}(t)\right)\right| \mathrm{d} t \exp \left(2 \int_{\alpha}^{\theta}\left|b_{1}\left(t, X_{2}(t)\right)\right| \mathrm{d} t\right) \\
\leqslant & \frac{B_{0} B_{2}}{B_{1}^{2}}\left[\frac{B_{1}}{\min _{[\alpha, \theta]}\left(X_{2}(t)\right)}(\theta-\alpha) \exp \left[\frac{B_{1}}{\min _{[\alpha, \theta]}\left(X_{2}(t)\right)}(\theta-\alpha)\right]\right]^{2} \\
= & \frac{B_{0} B_{2}}{B_{1}^{2}} Q\left(\frac{B_{1}}{\min _{[\alpha, \theta]}\left(X_{2}(t)\right)}(\theta-\alpha)\right)^{2} .
\end{aligned}
$$

Hence, the condition (4.13) is satisfied for all $\theta \leqslant T$ such that $Q\left(\frac{B_{1}}{\min _{[\alpha, \theta]}\left(X_{2}(t)\right)}(\theta-\alpha)\right)<$ $\frac{B_{1}}{\sqrt{B_{0} B_{2}}}$. So, it is sufficient to ask that $\theta$ satisfies:

$$
\theta \leqslant \alpha+\min _{t \in[\alpha, \theta]}\left(X_{2}(t)\right) \frac{x\left(B_{0}, B_{1}, B_{2}\right)}{B_{1}}
$$

Let us now derive a lower bound on $v_{1}$ and $v_{2}$ in the case the initial condition is negative.
Lemma 4.9. Let $\alpha, R>0$. With the same notations as introduced in Lemma 4.6. we obtain:

1. If $v_{1}(\alpha, R) \leqslant 0$, then $v_{1}$ is well-defined on every interval $[\alpha, \theta]$ such that

$$
\theta \leqslant \alpha+\frac{\min _{t \in[\alpha, \theta]} X_{1}(t)}{A_{1}} Q^{-1}\left(\Theta_{+}\right)
$$

where

$$
\begin{equation*}
\Theta_{+}=\frac{A_{1}}{\sqrt{A_{0} A_{2}}} \frac{1}{\left(1+\frac{\sqrt{A_{0}}}{\sqrt{A_{2}}}\left(\left|v_{1}(\alpha, R)\right| \min _{t \in[\alpha, \theta]} X_{1}(t)\right)\right)} \tag{4.14}
\end{equation*}
$$

and we have the estimate

$$
\begin{array}{r}
\frac{-\left(\left|v_{1}(\alpha, R)\right|+K_{a}\right)}{1-\left(\left|v_{1}(\alpha, R)\right|+K_{a}\right) \int_{\alpha}^{\theta}\left|a_{0}\left(t, X_{1}(t)\right)\right| e^{\int_{\alpha}^{t}\left|a_{1}\left(u, X_{1}(u)\right)\right| \mathrm{d} u} \mathrm{~d} t} \\
\leqslant v_{1}\left(t, X_{1}(t)\right) e^{\int_{\alpha}^{t} a_{1}\left(s, X_{1}(s)\right) \mathrm{d} s} \leqslant K_{a}+v_{1}(\alpha, R)
\end{array}
$$

2. If $v_{2}(\alpha, R) \leqslant 0$, then $v_{2}$ is well-defined on every interval $[\alpha, \theta]$ such that

$$
\theta \leqslant \alpha+\frac{\min _{t \in[\alpha, \theta]} X_{2}(t)}{B_{1}} Q^{-1}\left(\Xi_{+}\right)
$$

where

$$
\begin{equation*}
\Xi_{+}=\frac{B_{1}}{\sqrt{B_{0} B_{2}}} \frac{1}{\left(1+\frac{\sqrt{B_{0}}}{\sqrt{B_{2}}}\left(\left|v_{2}(\alpha)\right| \min _{t \in[\alpha, \theta]} X_{2}(t)\right)\right)} \tag{4.15}
\end{equation*}
$$

and we have the estimate

$$
\begin{aligned}
& \frac{-\left(\left|v_{2}(\alpha, R)\right|+K_{b}\right)}{1-\left(\left|v_{2}(\alpha, R)\right|+K_{b}\right) \int_{\alpha}^{\theta}\left|b_{0}\left(t, X_{2}(t)\right)\right| e^{\int_{\alpha}^{t}\left|b_{1}\left(u, X_{2}(u)\right)\right| \mathrm{d} u} \mathrm{~d} t} \\
& \leqslant v_{2}\left(t, X_{2}(t)\right) e^{\int_{\alpha}^{t} b_{1}\left(s, X_{2}(s)\right) \mathrm{d} s} \leqslant K_{b}+v_{2}(\alpha, R)
\end{aligned}
$$

Proof. First note that the upper bounds come directly from Lemma 4.5
Let us consider $v_{1}$. The same computations apply to $v_{2}$ after small changes. Let us denote $z_{1}(t)=-v_{1}\left(t, X_{1}(t)\right)$. Then, by hypothesis $z_{1}(\alpha) \geqslant 0$. According to Lemma 4.1, $z_{1}$ satisfies the ODE:

$$
z_{1}^{\prime}=-a_{0} z_{1}^{2}+a_{1} z_{1}-a_{2}
$$

where $-a_{0} \geqslant 0$ (see Lemma 4.4). Let us introduce $K_{a}$ as in (4.10). To apply Hörmander Lemma (see Lemma A.2), conditions (A.2) and (A.3) have to be satisfied. Note that, since $-a_{0} \geqslant 0$, condition (A.2) implies condition (A.3) so it is sufficient to see what is a sufficient condition allowing (A.2) to be satisfied. Condition (A.2) is equivalent to

$$
\left(z_{1}(\alpha)+K_{a}\right) \int_{\alpha}^{\theta}\left|a_{0}\left(t, X_{1}(t)\right)\right| \mathrm{d} t e^{\int_{\alpha}^{\theta}\left|a_{1}\left(t, X_{1}(t)\right)\right| \mathrm{d} t}-1<0
$$

It is sufficient to have

$$
z_{1}(\alpha) A_{0}(\theta-\alpha) e^{\frac{A_{1}}{\min _{t} X_{1}(t)}(\theta-\alpha)}+\frac{A_{0} A_{2}}{\min _{t} X_{1}(t)^{2}}(\theta-\alpha)^{2} e^{\frac{2 A_{1}}{\min _{t} X_{1}(t)}(\theta-\alpha)}-1<0
$$

Let us denote $\Theta=\frac{A_{1}}{\min _{t \in[\alpha, \theta]} X_{1}(t)}(\theta-\alpha) e^{\frac{A_{1}}{\min _{t \in[\alpha, \theta]} X_{1}(t)}(\theta-\alpha)}=Q\left(\frac{A_{1}}{\min _{t \in[\alpha, \theta]} X_{1}(t)}(\theta-\alpha)\right)$, where $Q$ is defined as in (4.9). Then to satisfy condition (A.2) it is sufficient to have

$$
z_{1}(\alpha) \frac{A_{0}}{A_{1}} \min _{t \in[\alpha, \theta]}\left\{X_{1}(t)\right\} \Theta+\frac{A_{0} A_{2}}{A_{1}^{2}} \Theta^{2}-1<0
$$

Let $\Delta=\left(z_{1}(\alpha) \frac{A_{0}}{A_{1}} \min _{t}\left\{X_{1}(t)\right\}\right)^{2}+4 \frac{A_{0} A_{2}}{A_{1}^{2}}$ be the discriminant of this equation in $\Theta$. This equation admits two distinct roots, one positive and one negative. Let us denote $\xi_{+}$the positive root. Then we have

$$
\begin{aligned}
\xi_{+} & =\frac{A_{1} z_{1}(\alpha) \min _{t \in[\alpha, \theta]} X_{1}(t)}{2 A_{2}}\left(-1+\sqrt{1+\frac{4 A_{2}}{z_{1}(\alpha)^{2} A_{0} \min _{t \in[\alpha, \theta]} X_{1}(t)^{2}}}\right) \\
& =\frac{A_{1}}{\sqrt{A_{0} A_{2}}} \frac{1}{\sqrt{1+\frac{A_{0}}{4 A_{2}}\left(z_{1}(\alpha) \min _{t \in[\alpha, \theta]} X_{1}(t)\right)^{2}}+\frac{\sqrt{A_{0}}}{2 \sqrt{A_{2}}}\left(z_{1}(\alpha) \min _{t \in[\alpha, \theta]} X_{1}(t)\right)} \\
& \geqslant \frac{A_{1}}{\sqrt{A_{0} A_{2}}} \frac{1}{1+\frac{\sqrt{A_{0}}}{\sqrt{A_{2}}}\left(z_{1}(\alpha) \min _{t \in[\alpha, \theta]} X_{1}(t)\right)}=: \Theta_{+} .
\end{aligned}
$$

Then (A.2) is satisfied if $0 \leqslant \Theta \leqslant \Theta_{+}$. Hence, the application $Q$ defined in (4.9) being strictly increasing on $\mathbb{R}_{+}$, it is sufficient to have

$$
\theta \leqslant \alpha+\frac{\min _{t \in[\alpha, \theta]} X_{1}(t)}{A_{1}} Q^{-1}\left(\Theta_{+}\right)
$$

## 5 Construction of a shock wave

### 5.1 Rankine-Hugoniot conditions

The Rankine-Hugoniot conditions (cf. [1, p. 312]) appear when we consider weak and piecewise smooth solution for first order systems. For the Euler equations (1.1), these conditions are written:

$$
\left\{\begin{array}{l}
-U[\rho]+[\rho(u \cdot \nu)]=0 \\
-U[\rho u]+[\rho(u \cdot \nu) u+p \nu]=0
\end{array}\right.
$$

through a discontinuity with normal vector $\nu$ and with normal speed $U$. The usual notation [.] stands for the jump between the two limit values at the both sides of the discontinuity. We denote $u^{+}$the limit of $u$ from the right and $u^{-}$the limit of $u$ from the left. In the spherical case, these conditions become

$$
\left\{\begin{array}{l}
-U[\rho]+[\rho u]=0  \tag{5.1}\\
-U[\rho u]+\left[\rho u^{2}+p\right]=0
\end{array}\right.
$$

A weak solution with a discontinuity is called shock when $U$ differs from the speed of the fluid on the both sides of the discontinuity. Note that the first condition of (5.1) gives us $\rho^{+}\left(U-u^{+}\right)=\rho^{-}\left(U-u^{-}\right)$so that $U-u^{+}$and $U-u^{-}$have the same sign. If we assume $U-u^{ \pm} \geqslant 0$ then the shock moves from the left to the right.

Let $W^{ \pm}=U-u^{ \pm}$. With some classical computations, we get that (5.1) is equivalent to

$$
\begin{equation*}
j:=\rho^{+}\left(U-u^{+}\right)=\rho^{-}\left(U-u^{-}\right), \quad\left[p+\rho W^{2}\right]=0 \tag{5.2}
\end{equation*}
$$

Hence, we get

$$
u^{+}-u^{-}=j\left(\frac{1}{\rho^{-}}-\frac{1}{\rho^{+}}\right), \quad p^{+}-p^{-}=j^{2}\left(\frac{1}{\rho^{-}}-\frac{1}{\rho^{+}}\right) .
$$

Finally, we obtain $\left(u^{+}-u^{-}\right)^{2}=\left(p^{+}-p^{-}\right)\left(\frac{1}{\rho^{-}-} \frac{1}{\rho^{+}}\right) \geqslant 0$. Let us recall furthermore the Lax entropy conditions for a 2 -shock (see 20]):

$$
\begin{array}{cc}
\rho^{-}>\rho^{+}, & p^{-}>p^{+}, \\
\lambda_{1}\left(w^{-}\right)<U\left(w^{+}, w^{-}\right)<\lambda_{2}\left(w^{-}\right), & U\left(w^{+}\right) \geqslant u^{+},  \tag{5.4}\\
\geqslant \lambda_{2}\left(w^{+}\right) .
\end{array}
$$

Finally, the jump conditions at the shock are

$$
\begin{equation*}
U=\frac{\rho^{+} u^{+}-\rho^{-} u^{-}}{\rho^{+}-\rho^{-}}, \quad u^{-}-u^{+}=\sqrt{\left(p^{+}-p^{-}\right)\left(\frac{1}{\rho^{-}}-\frac{1}{\rho^{+}}\right)}, \quad \rho^{-}>\rho^{+} . \tag{5.5}
\end{equation*}
$$

Proposition 5.1. For a Bethe-Weyl gas, the Rankine-Hugoniot and Lax shock conditions can be reduced to (5.5).

Proof. By (5.5), we have immediatly $u^{-}>u^{+}$and $p^{-}>p^{+}$since $\frac{\partial p}{\partial \rho}=c^{2} \geqslant 0$. Hence (5.3) is satisfied.

Let us prove (5.4). First, note that

$$
U-u^{-}=\frac{\rho^{+} \sqrt{F\left(\rho^{-}, \rho^{+}\right)}}{\rho^{-}-\rho^{+}},
$$

where $F$ is defined as in (3.3). Hence $U-u^{-} \geqslant 0 \geqslant-c^{-}$and $U \geqslant u^{-}-c^{-}=\lambda_{1}^{-}$.
Let us compute now $U-u^{-}-c^{-}$:

$$
\begin{aligned}
U-u^{-}-c^{-} & =\frac{\rho^{+}}{\rho^{-}-\rho^{+}} \sqrt{\rho^{-} c^{-}\left(\frac{1}{\rho^{+}}-\frac{1}{\rho^{-}}\right)}\left(\sqrt{\frac{1}{\rho^{-} c^{-}}\left(p^{-}-p^{+}\right)}-\sqrt{\rho^{-} c^{-}\left(\frac{1}{\rho^{+}}-\frac{1}{\rho^{-}}\right)}\right) \\
& =\frac{\rho^{+}}{\rho^{-}-\rho^{+}} \sqrt{\rho^{-} c^{-}\left(\frac{1}{\rho^{+}}-\frac{1}{\rho^{-}}\right)} \frac{\frac{1}{\rho^{-} c^{-}}\left(p^{-}-p^{+}-\left(\rho^{-} c^{-}\right)^{2}\left(\frac{1}{\rho^{+}}-\frac{1}{\rho^{-}}\right)\right)}{\left(\sqrt{\frac{1}{\rho^{-} c^{-}}\left(p^{-}-p^{+}\right)}+\sqrt{\rho^{-} c^{-}\left(\frac{1}{\rho^{+}}-\frac{1}{\rho^{-}}\right)}\right)}
\end{aligned}
$$

Let us denote $\xi\left(\rho^{-}, \rho^{+}\right)=p^{-}-p^{+}-\left(\rho^{-} c^{-}\right)^{2}\left(\frac{1}{\rho^{+}}-\frac{1}{\rho^{-}}\right)$. We have, for $\rho \geqslant \rho^{+}$

$$
\frac{\partial \xi\left(\rho, \rho^{+}\right)}{\partial \rho}=-2 \rho c^{2} \mathscr{G}\left(\frac{1}{\rho^{+}}-\frac{1}{\rho}\right) \leqslant 0 .
$$

Integrating on $\left[\rho^{+}, \rho^{-}\right]$, we obtain $\xi\left(\rho^{-}, \rho^{+}\right) \leqslant \xi\left(\rho^{+}, \rho^{+}\right)=0$. Finally, we have $U-u^{-}-c^{-} \leqslant$ 0.

A similar computation prove that $U-u^{+}-c^{+} \geqslant 0$.

Let us invert the second of the jump conditions (5.5), in order to express it as a condition on $w_{1}^{-}$, depending on $w_{2}^{-}, w_{1}^{+}, w_{2}^{+}$.
Lemma 5.2. We assume that there exists $\left(w_{1,0}^{-}, w_{2,0}^{-}\right)$and $\left(w_{1,0}^{+}, w_{2,0}^{+}\right)$such that $\mathcal{F}\left(w_{0}^{-}, w_{0}^{+}\right)=$ 0 . Then, there exists $g \in \mathscr{C}^{1}\left(\mathbb{R}^{3}, \mathbb{R}\right)$ such that the second condition of (5.5) is equivalent to the compatibility condition

$$
\begin{equation*}
w_{1}^{-}=g\left(w_{2}^{-}, w_{1}^{+}, w_{2}^{+}\right) \tag{5.6}
\end{equation*}
$$

as long as the condition $\rho^{-}>\rho^{+}$is satisfied. That is to say, as long as $g\left(w_{2}^{-}, w^{+}\right)<$ $w_{2}^{-}-w_{2}^{+}+w_{1}^{+}$.

Proof. We can write (5.5) as the following:

$$
\begin{equation*}
\mathcal{F}\left(w^{-}, w^{+}\right):=u^{-}-u^{+}-\sqrt{F\left(\rho^{-}, \rho^{+}\right)}=0 \tag{5.7}
\end{equation*}
$$

with $F\left(\rho, \rho^{+}\right)=\left(\frac{1}{\rho}-\frac{1}{\rho^{+}}\right)\left(p\left(\rho^{+}\right)-p(\rho)\right)$ as in (3.3). Since by hypothesis $\rho^{+}<\rho^{-}$, we have

$$
\begin{align*}
\frac{\partial \mathcal{F}}{\partial w_{1}^{-}} & =\frac{1}{4 \sqrt{F}}\left(\sqrt{\rho^{-} c^{-}\left(\frac{1}{\rho^{+}}-\frac{1}{\rho^{-}}\right)}+\sqrt{\frac{1}{\rho^{-} c^{-}}\left(p^{-}-p^{+}\right)}\right)^{2} \\
& >0 \tag{5.8}
\end{align*}
$$

According to the hypotheses $\mathcal{F}\left(w^{-}\left(0, R_{0}\right), w^{+}\left(0, R_{0}\right)\right)=0$, that is to say that the jump condition given by the second equation of (5.5) is satisfied at time $t=0$. By the implicit function Theorem, there exist (locally) a unique function $g\left(w_{2}^{-}, w^{+}\right)$such that

$$
\mathcal{F}\left(g\left(w_{2}^{-}, w^{+}\right), w_{2}^{-}, w^{+}\right)=0
$$

Furthermore

$$
\begin{aligned}
\frac{\partial \mathcal{F}}{\partial w_{2}^{-}} & =\frac{-1}{4 \sqrt{F}}\left(\sqrt{\rho^{-} c^{-}\left(\frac{1}{\rho^{+}}-\frac{1}{\rho^{-}}\right)}-\sqrt{\frac{1}{\rho^{-} c^{-}}\left(p^{-}-p^{+}\right)}\right)^{2} \leqslant 0 \\
\frac{\partial \mathcal{F}}{\partial w_{1}^{+}} & =\frac{-1}{4 \sqrt{F}}\left(\sqrt{\rho^{+} c^{+}\left(\frac{1}{\rho^{+}}-\frac{1}{\rho^{-}}\right)}+\sqrt{\frac{1}{\rho^{+} c^{+}}\left(p^{-}-p^{+}\right)}\right)^{2} \leqslant 0 \\
\frac{\partial \mathcal{F}}{\partial w_{2}^{+}} & =\frac{1}{4 \sqrt{F}}\left(\sqrt{\rho^{+} c^{+}\left(\frac{1}{\rho^{+}}-\frac{1}{\rho^{-}}\right)}-\sqrt{\frac{1}{\rho^{+} c^{+}}\left(p^{-}-p^{+}\right)}\right)^{2} \geqslant 0
\end{aligned}
$$

and we can see that the sign of the partial derivatives remain constant. As $\left(\frac{\partial g}{\partial w_{2}^{-}}, \frac{\partial g}{\partial w_{1}^{+}}, \frac{\partial g}{\partial w_{2}^{+}}\right)=$ $\frac{-1}{\partial_{w_{1}^{-}} \mathcal{F}}\left(\partial_{w_{2}^{-}} \mathcal{F}, \partial_{1^{+}} \mathcal{F}, \partial_{2^{+}} \mathcal{F}\right)$ is well defined as long as $\rho^{-}>\rho^{+}$. Hence, the implicit function is defined as long as the condition $\rho^{-}>\rho^{+}$is satisfied.

Remark 5.3. Note that the condition $\rho^{-}>\rho^{+}$along the discontinuity line $\mathcal{K}$ is equivalent to $w_{2}^{-}>w_{2}^{+}$on $\mathcal{K}$. Indeed, as $H$ is a strictly increasing function of $\rho, \rho^{-}>\rho^{+}$is equivalent to $H^{-}>H^{+}$, that is to say $w_{2}^{-}-w_{1}^{-}>w_{2}^{+}-w_{1}^{+}$. On the discontinuity line $\mathcal{K}, w_{1}^{-}=$ $g\left(w_{2}^{-}, w^{+}\right)$. Hence, $\rho^{-}>\rho^{+}$is equivalent to $g\left(w_{2}^{-}, w_{1}^{+}, w_{2}^{+}\right)<w_{2}^{-}-w_{2}^{+}+w_{1}^{+}$.

Let us denote $h\left(w_{2}^{-}, w_{1}^{+}, w_{2}^{+}\right)=w_{2}^{-}-w_{2}^{+}+w_{1}^{+}$. As $\partial_{w_{1}^{-}} \mathcal{F}>0$, then $g\left(w_{2}^{-}, w^{+}\right)<$ $h\left(w_{2}^{-}, w^{+}\right)$is equivalent to $\mathcal{F}\left(g\left(w_{2}^{-}, w^{+}\right), w_{2}^{-}, w^{+}\right)<\mathcal{F}\left(h\left(w_{2}^{-}, w^{+}\right), w_{2}^{-}, w^{+}\right)$. We note that

$$
u\left(h, w_{2}^{-}\right)=\frac{1}{2}\left(h+w_{2}^{-}\right)=\frac{1}{2}\left(2 w_{2}^{-}-\left(w_{2}^{+}-w_{1}^{+}\right)\right)=w_{2}^{-}-H\left(w_{1}^{+}, w_{2}^{+}\right)
$$

and

$$
\rho\left(h, w_{2}^{-}\right)=H^{-1}\left(\frac{1}{2}\left(w_{2}^{-}-h\right)\right)=H^{-1}\left(\frac{1}{2}\left(w_{2}^{+}-w_{1}^{+}\right)\right)=\rho\left(w_{1}^{+}, w_{2}^{+}\right)
$$

Hence we obtain that $\mathcal{F}\left(g, w_{2}^{-}, w^{+}\right)<\mathcal{F}\left(h, w_{2}^{-}, w^{+}\right)$is equivalent to

$$
0<\left(w_{2}^{-}-H^{+}\right)-u^{+}=w_{2}^{-}-\left(u^{+}+H^{+}\right)=w_{2}^{-}-w_{2}^{+}
$$

### 5.2 Angular domain

We want now to construct a solution by solving three problems : two classical problems with initial conditions obtained by prolongating the initial conditions on the right and on the left of $R_{0}$; and an angular problem with boundary conditions given in (5.6)-(5.9). According to T. T. Li \& W. C. Yu [14, Chap. 3], this last problem admits a local in time solution. In order to obtain an estimate on the time of existence we make a priori estimates on the solution.

Let us denote $D_{-}$the domain in the $(r, t)$-plan which is bounded on the right by the curve $\mathcal{C}_{1}$ defined in (5.9) and by the lines of equations $t=0, t=T_{*}$ (see Figure 2). In the same way, we denote $D_{+}$the domain in the $(r, t)$-plan which is bounded on the left by the curve $\mathcal{K}$ defined in (5.9) and by the lines of equations $t=0, t=T_{*}$. The domain $D_{0}$ is the domain in the $(r, t)$-plan which is bounded on the left by the curve $\mathcal{C}_{1}$ defined in (5.9) and on the right by the curve $\mathcal{K}$ defined in (5.9).


Figure 2: Angular Domain and some related curves.

More precisely, let $R_{0}>0$. Let us define the free boundary domain $D_{0}$ (see Figure 2) where the boundaries $\mathcal{C}_{1}=\left\{\left(t, x_{1}(t)\right), t \geqslant 0\right\}$ and $\mathcal{K}=\left\{\left(t, x_{2}(t)\right), t \geqslant 0\right\}$ are respectively the curves defined by

$$
\begin{equation*}
\frac{\mathrm{d} x_{1}}{\mathrm{~d} t}=\lambda_{1}\left(w\left(t, x_{1}(t)\right)\right), \quad x_{1}(0)=R_{0}, \quad \frac{\mathrm{~d} x_{2}}{\mathrm{~d} t}=U\left(t, w\left(t, x_{2}(t)\right)\right), \quad x_{2}(0)=R_{0} \tag{5.9}
\end{equation*}
$$

where $U$ is the speed of a 2-shock linking $w^{-}\left(t, x_{2}(t)\right)$ to $w^{+}\left(t, x_{2}(t)\right)$ and is defined by the first equation of (5.5).

So that the angular problem is well-posed, we need to add some boundary conditions on $\mathcal{C}_{1}$ and $\mathcal{K}$. Defining $g$ thanks to Lemma 5.2, we have the following boundary conditions:

$$
\begin{equation*}
w_{2}\left(t, x_{1}(t)\right)=\omega(t), \quad w_{1}\left(t, x_{2}(t)\right)=g\left(t, x_{2}(t), w_{2}\left(t, x_{2}(t)\right)\right) \tag{5.10}
\end{equation*}
$$

where $\omega$ is chosen so that $w_{2}$ is $\mathscr{C}^{1}$ through $\mathcal{C}_{1}$. The definition of $g$ ensures that the compatibility condition given by the second equation of (5.5) is satisfied along the discontinuity line $\mathcal{K}$.

Lemma 5.4. Let us assume that, along the shock $\mathcal{K}$, the conditions (5.5) are satisfied. Then along the shock $\mathcal{K}$, we have

$$
\begin{aligned}
\partial_{r} w_{1}^{-}\left(t, x_{2}(t)\right)= & \frac{1}{U-\lambda_{1}^{-}}\left[\left(U-\lambda_{2}^{-}\right) \partial_{w_{2}^{-}} g \partial_{r} w_{2}^{-}-\left(\partial_{w_{2}^{-}} g+1\right) f^{-}\right. \\
& \left.+\left(U-\lambda_{1}^{+}\right) \partial_{w_{1}^{+}} g \partial_{r} w_{1}^{+}+\left(U-\lambda_{2}^{+}\right) \partial_{w_{2}^{+}} g \partial_{r} w_{2}^{+}+\left(\partial_{w_{1}^{+}} g-\partial_{w_{2}^{+}} g\right) f^{+}\right] .
\end{aligned}
$$

Proof. Derivating $w_{1}^{-}\left(t, x_{2}(t)\right)$ we obtain

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{~d} t} w_{1}^{-}\left(t, x_{2}(t)\right) \\
= & \partial_{w_{2}^{-}} g\left(\left(U-\lambda_{2}^{-}\right) \partial_{r} w_{2}^{-}-f^{-}\right)+\partial_{w_{1}^{+}} g\left(\left(U-\lambda_{1}^{+}\right) \partial_{r} w_{1}^{+}+f^{+}\right)+\partial_{w_{2}^{+}} g\left(\left(U-\lambda_{2}^{+}\right) \partial_{r} w_{2}^{+}-f^{+}\right)
\end{aligned}
$$

but also $\frac{\mathrm{d}}{\mathrm{d} t} w_{1}^{-}\left(t, x_{2}(t)\right)=\partial_{t} w_{1}^{-}+U \partial_{r} w_{1}^{-}=\left(U-\lambda_{1}^{-}\right) \partial_{r} w_{1}^{-}+f^{-}$.

Proposition 5.5. Let $R_{0}>0$. Let us consider a Bethe-Weyl gas, satisfying $1<\mathscr{G}<2$ and such that $\left(\rho \mapsto \frac{c(\rho)}{\rho}\right)$ is integrable in 0 .

Let us consider the free boundary problem consisting in the system (1.2) in $D_{0}$ with the following boundary conditions:

$$
w_{2}\left(t, x_{1}(t)\right)=\omega(t) \geqslant 0, \quad \text { on } \mathcal{C}_{1} ; \quad w_{1}\left(t, x_{2}(t)\right)=G\left(t, w_{2}\right)>0, \quad \text { on } \mathcal{K}
$$

with $\mathcal{C}_{1}, \mathcal{K}$, defined as above.
Assume that the compatibility conditions (5.5) are satisfied at time $t=0$ and that there exists $C_{0}>0$ such that, for any $M, T>0$ :

$$
\begin{array}{ll}
\text { along } \mathcal{C}_{1}, & \left|\partial_{r} w_{2}\left(t, x_{1}(t)\right)\right|=\left|\frac{\omega^{\prime}+f}{\lambda_{1}-\lambda_{2}}\left(t, x_{1}(t)\right)\right| \leqslant \frac{C_{0}}{x_{1}(t)} \\
\text { along } \mathcal{K}, & \partial_{t} G\left(t, x_{2}(t)\right) \geqslant \frac{-C_{0}}{x_{2}(t)}, \quad \sup _{t \in[0, T]}\left|\partial_{2} G\left(t, x_{2}(t)\right)\right| \leqslant C_{0}
\end{array}
$$

Then, assuming that there exists $\delta>0$ such that $\rho^{-}-\rho^{+} \geqslant \delta>0$ along $\mathcal{K}$, there exists a $\mathscr{C}^{1}$ solution of (1.2) in $D_{0}$ whose time of existence is bounded below in the following way

$$
T_{e x} \geqslant C R_{0}
$$

where $C$ depends on the boundary conditions.

Proof. Let $(t, r) \in D_{0}$. The 2-characteristic going through $(t, r)$ originates in $D_{-}$and crosses $\mathcal{C}_{1}$ in $\left(\beta, x_{1}(\beta)\right)$; meanwhile the 1-characteristic going through $(t, r)$ originates in $\left(\alpha, x_{2}(\alpha)\right) \in \mathcal{K}$.

As $w_{1}>0$ along $\mathcal{K}$, we can apply Lemma 3.6 on $D_{0}$ to obtain a $\mathscr{C}^{0}$ estimate of $w_{1}$ and $w_{2}$. We obtain

$$
0 \leqslant w_{1}\left(\alpha, x_{2}(\alpha)\right) \leqslant w_{1}(t, r) \leqslant w_{2}(t, r) \leqslant w_{2}\left(\beta, x_{1}(\beta)\right) \leqslant\|\omega\|_{\mathbf{L}^{\infty}}
$$

Next, we want to obtain $\mathscr{C}^{1}$ estimates for $w_{1}$ and $w_{2}$. First, we derive an $\mathbf{L}^{\infty}$ estimate for $\partial_{r} w_{2}$. Let us remind that $v_{2}=e^{h}\left(\partial_{r} w_{2}+\Psi\right)$ with $\Psi(r, w)=\frac{\psi(w)}{r}$.

- In the case $\left(\partial_{r} w_{2}+\Psi\right)\left(\beta, x_{1}(\beta)\right) \geqslant 0$, then we have $v_{2}\left(\beta, x_{1}(\beta)\right) \geqslant 0$ and we can apply Lemma 4.6 on $D_{0}$ to obtain an $\mathbf{L}^{\infty}$-estimate of $\partial_{r} w_{2}$ in $D_{0}$. Since $X_{2}^{\prime} \geqslant 0$, we obtain a lower bound for the time of existence of $\partial_{r} w_{2}$ :

$$
T_{e x} \geqslant x_{1}(\beta) \frac{x\left(B_{0}, B_{1}, B_{2}\right)}{B_{1}} \geqslant R_{0} \frac{x\left(B_{0}, B_{1}, B_{2}\right)}{B_{1}}
$$

where $B_{0}, B_{1}, B_{2}$ are defined as in (4.8).

- In the case $v_{2}\left(\beta, x_{1}(\beta)\right) \leqslant 0$, as $X_{2}^{\prime} \geqslant 0$, by Lemma 4.9 we obtain the estimate

$$
T_{e x} \geqslant \frac{R_{0}}{B_{1}} Q^{-1}\left(\Xi_{+}\right)
$$

where $B_{1}$ is defined as in (4.8) and $\Xi_{+}=\frac{B_{1}}{\sqrt{B_{0} B_{2}}} \frac{1}{1+\frac{\sqrt{B_{0}}}{\sqrt{B_{2}}}\left(\left|v_{2}\left(\beta, x_{1}(\beta)\right)\right| x_{1}(\beta)\right)}$.
Since $\Psi(w, r)=\frac{\psi(w)}{r} \leqslant \frac{\|\psi\|_{\mathbf{L}^{\infty}}}{r}$ and, by hypothesis $\left|\partial_{r} w_{2}\left(\beta, x_{1}(\beta)\right)\right| \leqslant \frac{C_{0}}{x_{1}(\beta)}$, we have

$$
\left|v_{2}\left(\beta, x_{1}(\beta)\right)\right| x_{1}(\beta) \leqslant\|\psi\|_{\mathbf{L}^{\infty}}+C_{0}:=C_{1}
$$

and we finally have

$$
T_{e x} \geqslant \frac{R_{0}}{B_{1}} Q^{-1}\left(\frac{B_{1}}{\sqrt{B_{0} B_{2}}} \frac{1}{1+\frac{\sqrt{B_{0}}}{\sqrt{B_{2}}} C_{1}}\right)
$$

Let us now find an estimate in $\mathbf{L}^{\infty}$ for $\partial_{r} w_{1}$. We want to proceed in the same way. First we prove that there exists $C$ such that $\partial_{r} w_{1}+\Phi \geqslant \frac{-C}{r}$ along $\mathcal{K}$. derivating the boundary condition, we get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} t} w_{1}\left(t, x_{2}(t)\right) & =\partial_{t} w_{1}+U \partial_{r} w_{1} \\
& =f+\left(U-\lambda_{1}\right) \partial_{r} w_{1} \\
& =\partial_{t} G+\partial_{2} G\left(\left(U-\lambda_{2}\right) \partial_{r} w_{2}-f\right)
\end{aligned}
$$

Hence $\partial_{r} w_{1}=\frac{1}{U-\lambda_{1}}\left[\partial_{t} G-f+\partial_{2} G\left(\left(U-\lambda_{2}\right) \partial_{r} w_{2}-f\right)\right]$. By hypothesis we have a lower bound on $\partial_{t} G$. We can find a similar lower bound on $f$, whose expression is known. By hypothesis $U-\lambda_{2}^{-} \leqslant 0, U-\lambda_{1}^{-}>0$ and they are depending only on $\left(w_{1}, w_{2}\right)$. Thanks to

Lemma 4.5, we obtain the following upper bound on $v_{2}=e^{h}\left(\partial_{r} w_{2}+\Psi\right)$, reminding that, by Lemma 4.4, $b_{i}(r, w)=\bar{b}_{i}(w) / r^{i}$ for any $i \in\{0,1,2\}$ :
$v_{2}\left(t, X_{2}(t)\right) e^{-\int_{\alpha}^{t} \frac{\left(\lambda_{2}+\bar{b}_{1}\right)\left(s, X_{2}(s)\right)}{X_{2}(s)} \mathrm{d} s} \leqslant \frac{X_{2}(\alpha) v_{2}\left(\alpha, X_{1}(\alpha)\right)}{X_{2}(t)}+\frac{1}{X_{2}(t)} \int_{\alpha}^{t} \frac{\bar{b}_{2}}{X_{2}(s)} e^{-\int_{\alpha}^{s} \frac{\left(\lambda_{2}+\bar{b}_{1}\right)\left(\tau, X_{2}(\tau)\right)}{X_{2}(\tau)} \mathrm{d} \tau} \mathrm{d} s$.
Reminding that $X_{1}^{\prime} \geqslant 0$ and $X_{2}^{\prime} \geqslant 0$, if $\left(\alpha, X_{2}(\alpha)\right) \in \mathcal{C}_{1}$, we have, for any $s \geqslant \alpha, 1 / X_{2}(s) \leqslant$ $1 / X_{2}(\alpha) \leqslant 1 / R_{0}$. Using furthermore the hypothesis $\left|r\left(\partial_{r} w_{2}+\Psi\right)\right| \leqslant C_{0}$ along $\mathcal{C}_{1}$, we obtain

$$
v_{2}\left(t, X_{2}(t)\right) \leqslant\left(\frac{C_{0}}{X_{2}(t)}+\frac{1}{X_{2}(t)}(t-\alpha) \frac{B_{2}}{R_{0}}\right) e^{(t-\alpha)\left(\sup \left(\lambda_{2}\right)+B_{1}\right) / R_{0}}
$$

which provide us the estimate $v_{2}\left(t, x_{2}(t)\right) \leqslant \frac{1}{x_{2}(t)} C e^{\frac{C(t-\alpha)}{R_{0}}}$. We obtain the desired estimate taking a time $t$ bounded.

- If $\left(\partial_{r} w_{1}+\Phi\right)\left(\alpha, x_{2}(\alpha)\right) \geqslant 0$, then we have $v_{1}\left(\alpha, x_{2}(\alpha)\right) \geqslant 0$ and we can apply Lemma 4.6 on $D_{0}$ to obtain estimate of $\partial_{r} w_{1}$ in $\mathbf{L}^{\infty}$ in $D_{0}$. Since $X_{1}^{\prime} \geqslant 0$, we obtain a lower bound for the time of existence of $\partial_{r} w_{1}$ :

$$
T_{e x} \geqslant x_{2}(\alpha) \frac{x\left(A_{0}, A_{1}, A_{2}\right)}{A_{1}} \geqslant R_{0} \frac{x\left(A_{0}, A_{1}, A_{2}\right)}{A_{1}},
$$

where $A_{0}, A_{1}, A_{2}$ are defined as in (4.7) and $x\left(A_{0}, A_{1}, A_{2}\right)$ is defined in Lemma 4.6.

- If $v_{1}\left(\alpha, x_{2}(\alpha)\right) \leqslant 0$, as $X_{1}^{\prime} \geqslant 0$, by Lemma 4.9 we obtain the estimate

$$
T_{e x} \geqslant \frac{R_{0}}{A_{1}} Q^{-1}\left(\Theta_{+}\right)
$$

where $A_{1}$ is defined as in (4.7) and $\Theta_{+}=\frac{A_{1}}{\sqrt{A_{0} A_{2}}} \frac{1}{1+\frac{\sqrt{A_{0}}}{\sqrt{A_{2}}}\left(\left|v_{1}\left(\alpha, x_{2}(\alpha)\right)\right| x_{2}(\alpha)\right)}$. Since $\Phi(w, r)=\frac{\varphi(w)}{r} \leqslant \frac{C_{1}}{r}$ by Lemma 4.4 and $\left|\partial_{r} w_{1}\left(\alpha, x_{2}(\alpha)\right)\right| \leqslant \frac{C_{0}}{x_{2}(\alpha)}$, we have

$$
\left|v_{1}\left(\alpha, x_{2}(\alpha)\right)\right| x_{2}(\alpha) \leqslant C_{1}+C_{0}:=C_{2},
$$

and we finally have

$$
T_{e x} \geqslant \frac{R_{0}}{A_{1}} Q^{-1}\left(\frac{A_{1}}{\sqrt{A_{0} A_{2}}} \frac{1}{1+\frac{\sqrt{A_{0}}}{\sqrt{A_{2}}} C_{2}}\right) .
$$

We are now able to construct a piecewise solution:
Theorem 5.6. Let us consider a Bethe-Weyl gas satisfying $1<\mathscr{G}<2$ and such that ( $\left.\rho \mapsto \frac{c(\rho)}{\rho}\right)$ is integrable in 0 .

Let $R_{0}>0$. Assume there exists two regular solutions of (1.2) $\left(\rho^{-}, u^{-}\right)$and $\left(\rho^{+}, u^{+}\right)$ with a time of existence $\mathcal{T}>0,\left(\rho^{-}, u^{-}\right)$admitting an initial condition at time $t=0$ defined on $\left[0, R_{0}\right]$ and $\left(\rho^{+}, u^{+}\right)$admitting an initial condition at time $t=0$ on $\left[R_{0},+\infty[\right.$. We assume that the compatibility condition (5.5) are satisfied at time $t=0$, in $r=R_{0}$. We
denote $w=\left(w_{1}, w_{2}\right)$ the associated Riemann invariant defined as in (3.1). The superspcript $\pm$ means the quantity asoociated to $\left(\rho^{ \pm}, u^{ \pm}\right)$definied in the domains $D^{ \pm}$. The subscript 0 indicates the initial condition at time $t=0$.

Let us assume that

$$
\min _{r \leqslant R_{0}} w_{1,0}^{-}(r)>0, \quad w_{1,0}^{-}\left(R_{0}\right)+\inf _{r \geqslant R_{0}} w_{1,0}^{+}(r)>0
$$

and that there exists $C_{0}>0$ such that, for all $t \in\left[0, \mathcal{T}\left[\right.\right.$ and any $r$ where $w^{-}$(respectively $\left.w^{+}\right)$is defined:

$$
\left|\partial_{r} w_{2}^{-}(t, r)\right| \leqslant \frac{C_{0}}{r}, \partial_{r} w_{1}^{+}(t, r) \quad \geqslant \frac{-C_{0}}{r}, \partial_{r} w_{2}^{+}(t, r) \quad \leqslant \frac{C_{0}}{r}
$$

Then, as long as $\rho^{-}>\rho^{+}$along $\mathcal{K}$, there exists a shock wave solution of (1.2) with initial conditions

$$
\begin{array}{lll}
\text { for } r \leqslant R_{0}, & u(0, r)=u_{0}^{-}(r), & \rho(0, r)=\rho_{0}^{-}(r), \\
\text { for } r \geqslant R_{0}, & u(0, r)=u_{0}^{+}(r), & \rho(0, r)=\rho_{0}^{+}(r),
\end{array}
$$

and its time of existence is bounded below in the following way

$$
T_{e x} \geqslant \min \left(\mathcal{T}, C R_{0}\right),
$$

where $C$ depends on the initial conditions.
For example, we can use the result obtained in 11 in order to have $\mathcal{T}=+\infty$. For $\left(\rho^{+}, u^{+}\right)$, a stationary solution could also work.
Proof. Let $D_{-}, D_{0}, D_{+}$be defined as before. Let us define $G\left(t, w_{2}\right):=g\left(w_{2}, w^{+}\right)$, where $g$ has been defined in Lemma 5.2. Using the estimate of Lemma 3.5 in $D_{+}$, we obtain along the discontinuity line $\mathcal{K}: w_{1}^{-}=g\left(w_{2}^{-}, w^{+}\right) \geqslant w_{1}^{+} \geqslant w_{1,0}^{+} \geqslant \inf _{\left\{r \geqslant R_{0}\right\}} w_{1,0}^{+}$. Besides, $u>0$ in $D_{-}$, in particular $u^{-}\left(0, R_{0}\right)>0$, and by continuity of $u^{-}$in $D_{-} \cup D_{0}$, at least for a small time, $u^{-}$is non-negative in $D_{0}$. Note $\left[0, t_{0}\right]$ the time interval in which $u^{-}$is non-negative in $D_{0}$. For $(t, r) \in D_{0}$ such that $t \leqslant t_{0}$. Let us denote $\beta$ the time at which the 2-characteristic going through $(t, r) \operatorname{cross} \mathcal{C}_{1}$.

$$
u^{-}(t, r) \geqslant \frac{1}{2}\left(w_{1}^{-}+w_{2}^{-}\right)\left(t, x_{2}(t)\right) \geqslant \frac{1}{2}\left(\min _{r \geqslant R_{0}}\left(w_{1,0}^{+}(r)\right)+\frac{w_{2}\left(\beta, x_{1}(\beta)\right)}{\left.1+w_{2}\left(\beta, x_{1}(\beta)\right) \frac{(d-1)(t-\beta)}{4 x_{1}(\beta)}\right)}\right) .
$$

Besides, as $\left(\beta, x_{1}(\beta)\right) \in D_{-}$, we have $w_{2}\left(\beta, x_{1}(\beta)\right) \geqslant w_{1}\left(\beta, x_{1}(\beta)\right) \geqslant w_{1}\left(0, R_{0}\right)$. Furthermore, $x_{1}^{\prime} \geqslant 0$ then $x_{1}(\beta) \geqslant R_{0}$. Then

$$
u(t, r) \geqslant \frac{1}{2}\left(w_{1}^{-}+w_{2}^{-}\right)\left(t, x_{2}(t)\right) \geqslant \frac{1}{2}\left(\min _{r \geqslant R_{0}}\left(w_{1,0}^{+}\right)+\frac{1}{\left.\frac{1}{w_{1}^{-}\left(0, R_{0}\right)}+\frac{(d-1)(t-\beta)}{4 R_{0}}\right)}\right)
$$

If $w_{1,0}^{+} \geqslant 0, u^{-} \geqslant 0$ in all $D_{0}$. Otherwise, if $w_{1,0}^{+}<0$ then $u^{-}$is non-negative as long as

$$
t-\beta \leqslant \frac{4 R_{0}}{(d-1)}\left(\frac{1}{\min \left(w_{1,0}^{+}\right)}+\frac{1}{w_{2}\left(\beta, x_{1}(\beta)\right)}\right)
$$

Then we obtain that $u^{-}$is non-negative if

$$
t \leqslant \frac{4 R_{0}}{(d-1)}\left(\frac{1}{\min \left(w_{1,0}^{+}\right)}+\frac{1}{w_{1}^{-}\left(0, R_{0}\right)}\right)
$$

Hence, we can apply Proposition 3.7 to $(t, r) \in D_{0}$ to find a time of validity for the $\mathscr{C}^{0}{ }_{-}$ estimates proportional to $R_{0}$.

Besides, the expression of $\partial_{r} w_{1}^{-}$along $\mathcal{K}$ in Lemma 5.4 and the hypotheses on $\partial_{r} w_{1}^{+}$, $\partial_{r} w_{2}^{+}$allow us to check that the hypotheses of Proposition 5.5 are satisfied. Hence, we have a regular solution in the angular domain satisfying the boundary condition $w_{2}\left(t, x_{1}(t)\right)=$ $w_{2}^{-}\left(t, x_{1}(t)\right)$ and $w_{1}\left(t, x_{2}(t)\right)=g\left(w_{2}, w^{+}\right)$.

### 5.3 Checking hypotheses

We determine now a set of hypotheses on the initial conditions so that the hypothesis " $\rho^{-}>\rho^{+}$" of the previous theorem is satisfied. The condition $\left|\partial_{r} w_{2}\right| \leqslant \frac{C}{r}$ along $\mathcal{C}_{1}$ can not be computed with this method: the construction of a regular solution in $D_{-}$has to be obtained by another result on regular solution (see for example D. Serre [21] or M. Grassin [6] or M. Lécureux-Mercier [11]).

Proposition 5.7. In the same context as in Theorem 5.6, we assume furthermore that $w_{1,0}^{-}\left(R_{0}\right)+\min _{\left\{r \geqslant R_{0}\right\}} w_{1,0}^{+}>0$ and $w_{1,0}^{-}\left(R_{0}\right)>\max _{\left\{r \geqslant R_{0}\right\}} w_{2,0}^{+}$. Then, the estimates of Proposition 3.7 are available in $D_{0}$ for a time proportional to $R_{0}$ and furthermore, along $\mathcal{K}, \rho^{+}<\rho^{-}$.

Proof. Let $t \in \mathbb{R}_{+}$we denote $\beta$ the time at which the 2 -characteristic going through $\left(t, x_{2}(t)\right)$ crosses $\mathcal{C}_{1}$. By Proposition 3.7, we have

$$
\begin{aligned}
w_{2}^{-}\left(t, x_{2}(t)\right) & \geqslant \frac{w_{2}\left(\beta, x_{1}(\beta)\right)}{1+(t-\beta) \frac{(d-1)}{4 x_{1}(\beta)} w_{2}\left(\beta, x_{1}(\beta)\right)} \\
& \geqslant \frac{1}{\frac{1}{w_{1,0}^{-}\left(R_{0}\right)}+(t-\beta) \frac{(d-1)}{4 R_{0}}}
\end{aligned}
$$

Besides, $w_{2}^{+}\left(t, x_{2}(t)\right) \leqslant \max \left(w_{2,0}^{+}\right)$. By remark5.3, the condition $\rho^{+}<\rho^{-}$on $\mathcal{K}$ is equivalent to $w_{2}^{-}>w_{2}^{+}$. Hence it is sufficient to have

$$
\frac{1}{\frac{1}{w_{1,0}^{-}\left(R_{0}\right)}+(t-\beta) \frac{(d-1)}{4 R_{0}}}>\max \left(w_{2,0}^{+}\right)
$$

Finally it is sufficient to have $w_{1,0}^{-}\left(R_{0}\right)>\max \left(w_{2,0}^{+}\right)$and $t<\frac{4 R_{0}}{(d-1)} \frac{w_{1}^{-}\left(0, R_{0}\right)-\max w_{2,0}^{+}}{w_{1}^{-}\left(0, R_{0}\right) \max w_{2,0}^{+}}$.

## A Time of existence for ODE

Lemma A. 1 (Maximum principle for ODE.). Let $a, b:[0, T] \rightarrow \mathbb{R}$ be continuous applications. Assume that $w, z:[0, T] \rightarrow \mathbb{R}$ are continuous applications such that $z(0) \geqslant w(0)$ and $w^{\prime}=a w^{2}+b, z^{\prime} \geqslant a z^{2}+b$ in $[0, T]$. Then $z(t) \geqslant w(t)$ for all $t \in[0, T]$.

Proof. By hypothesis,

$$
(z-w)^{\prime}(t)=z^{\prime}-\left(a(t) w(t)^{2}+b(t)\right) \geqslant a(t)\left(z(t)^{2}-w(t)^{2}\right)
$$

that is to say $(z-w)^{\prime}(t) \geqslant a(t)(z(t)-w(t))(z(t)+w(t))$. It follows that,

$$
\frac{\mathrm{d}}{\mathrm{~d} t}\left((z-w)(t) e^{-\int_{0}^{t} a(s)(z+w)(s) \mathrm{d} s}\right) \geqslant 0
$$

Hence $(z-w)(t) e^{-\int_{0}^{t} a(s)(z+w)(s) \mathrm{d} s} \geqslant(z-w)(0) \geqslant 0$, for all $t \in[0, T]$.
Lemma A.2. Let $T>0$ and $a_{0}, a_{1}, a_{2} \in \mathscr{C}^{0}([0, T] ; \mathbb{R})$. Let $a_{0}^{+}=\max \left(a_{0}, 0\right)$ and $K$ be defined by

$$
\begin{equation*}
K=\int_{0}^{T}\left|a_{2}(t)\right| \mathrm{d} t \exp \left(\int_{0}^{T}\left|a_{1}(t)\right| \mathrm{d} t\right) . \tag{A.1}
\end{equation*}
$$

If $y_{0} \geqslant 0$ and

$$
\begin{align*}
\frac{1}{y_{0}+K} & >\int_{0}^{T} a_{0}^{+}(t) \mathrm{d} t \exp \left(\int_{0}^{T}\left|a_{1}(t)\right| \mathrm{d} t\right)  \tag{A.2}\\
\frac{1}{K} & >\int_{0}^{T}\left|a_{0}(t)\right| \mathrm{d} t \exp \left(\int_{0}^{T}\left|a_{1}(t)\right| \mathrm{d} t\right) \tag{A.3}
\end{align*}
$$

then the maximal solution of the Cauchy problem

$$
\begin{equation*}
y^{\prime}=a_{0}(t) y^{2}+a_{1}(t) y+a_{2}(t), \quad y(0)=y_{0} \tag{A.4}
\end{equation*}
$$

is defined at least on $[0, T]$ and satisfies

$$
\begin{array}{ll}
\frac{1}{y(T)}>\frac{1}{y_{0}+K}-\int_{0}^{T}\left|a_{0}^{+}(t)\right| \mathrm{d} t \exp \left(\int_{0}^{T}\left|a_{1}(t)\right| \mathrm{d} t\right), & \text { if } y(T) \geqslant 0 \\
\frac{1}{|y(T)|}>\frac{1}{K}-\int_{0}^{T}\left|a_{0}(t)\right| \mathrm{d} t \exp \left(\int_{0}^{T}\left|a_{1}(t)\right| \mathrm{d} t\right), & \text { if } y(T)<0 \tag{A.6}
\end{array}
$$

Proof. First, denoting $\tilde{y}=\exp \left(-\int_{0}^{t} a_{1}(s) \mathrm{d} s\right) y, \tilde{a}_{0}=\exp \left(\int_{0}^{t} a_{1}(s) \mathrm{d} s\right) a_{0}$ and $\tilde{a}_{2}=$ $\exp \left(-\int_{0}^{t} a_{1}(s) \mathrm{d} s\right) a_{2}$, we see that the equation (A.4) becomes the ordinary differential equation

$$
\begin{equation*}
\tilde{y}^{\prime}=\tilde{a}_{0}(t) \tilde{y}^{2}+\tilde{a}_{2}(t) . \tag{A.7}
\end{equation*}
$$

and $\tilde{y}(0)=y(0)$. We can thus assume without loss of generality that $a_{1} \equiv 0$.

Let us introduce the increasing function $v$ defined by $v(t)=\int_{0}^{t}\left|\tilde{a}_{2}(s)\right| \mathrm{d} s$. Let $z$ be the maximal solution of the Cauchy problem

$$
z^{\prime}=\tilde{a}_{0}^{+}(t)(z+K)^{2}, \quad z(0)=y_{0}
$$

Then $z$ is increasing and since $y_{0}+K>0$, we have

$$
\frac{1}{z(t)+K}=\frac{1}{y_{0}+K}-\int_{0}^{t} \tilde{a}_{0}^{+}(s) \mathrm{d} s .
$$

Note that the right hand side does not vanish for all $t \in[0, T]$, thanks to (A.2). Besides, we have $(z+v)(0)=y_{0}$ and

$$
\begin{aligned}
(z+v)^{\prime} & =\tilde{a}_{0}^{+}(t)(z+K)^{2}+\left|\tilde{a}_{2}\right| \\
& \geqslant \tilde{a}_{0}^{+}(t)(z+v)^{2}+\tilde{a}_{2} \\
& \geqslant \tilde{a}_{0}(t)(z+v)^{2}+\tilde{a}_{2} .
\end{aligned}
$$

Consequently, according to Lemma A.1 we have $y(t) \leqslant(z+v)(t) \leqslant z(t)+K$ for all $t \in[0, T]$ if $y$ exists. In particular, as long as $y(t)>0$, we have

$$
\frac{1}{y(t)} \geqslant \frac{1}{z(t)+K}=\frac{1}{y_{0}+K}-\int_{0}^{t} \tilde{a}_{0}^{+}(s) \mathrm{d} s=\frac{1}{y_{0}+K}-\int_{0}^{t} a_{0}^{+}(\tau) e^{\int_{0}^{\tau} a_{1}(s) \mathrm{d} s} \mathrm{~d} \tau,
$$

hence

$$
\begin{equation*}
\frac{1}{y(t)} \geqslant \frac{1}{y_{0}+K}-\int_{0}^{t} a_{0}^{+}(\tau) \mathrm{d} \tau e^{\int_{0}^{t}\left|a_{1}(s)\right| \mathrm{d} s} . \tag{A.8}
\end{equation*}
$$

Assume now that $y$ vanishes and changes its sign in $t_{0} \in[0, T]$. We can apply the same procedure as above to $Y=-y$, replacing $y_{0}$ by $Y\left(t_{0}\right)=Y_{0}=0$ and beginning at time $t_{0}$. The application $Y$ is then solution of $Y^{\prime}=A_{0} Y^{2}+a_{1} Y+A_{2}$, where $A_{0}=-a_{0}, A_{2}=-a_{2}$. Denoting $A_{0}^{+}=\max \left(-a_{0}, 0\right)$, we get that, for all $t \geqslant t_{0}$ such that $Y(t)>0$

$$
\frac{1}{Y(t)} \geqslant \frac{1}{K}-\int_{t_{0}}^{t} A_{0}^{+}(\tau) \mathrm{d} \tau e^{\int_{0}^{t}\left|a_{1}(s)\right| \mathrm{d} s} .
$$

Consequently, for all $t$ such that $y(t)<0$

$$
\begin{equation*}
\frac{1}{|y(t)|} \geqslant \frac{1}{K}-\int_{0}^{t}\left|a_{0}(\tau)\right| \mathrm{d} \tau e^{\int_{0}^{t}\left|a_{1}(s)\right| \mathrm{d} s} . \tag{A.9}
\end{equation*}
$$

Finally, the inequalities (A.8)-(A.9) give us some bounds on $y$ for all time, $y$ being positive or negative. Hence, $y$ can not tend to $\pm \infty$ and exists up to time $T$. Indeed, we proved above that there exists a function $\varphi \in \mathscr{C}^{0}([0, T], \mathbb{R})$ such that, if $y$ is solution of (A.4), then $|y(t)| \leqslant \varphi(t)$ for all $t \in[0, T]$. Let us denote $T_{*}$ the maximal time of existence of $y$. If we assume $T_{*}<T$, then we obtain that $y$ is bounded on $\left[0, T_{*}\left[\operatorname{by} \max _{[0, T]} \varphi\right.\right.$, which contradicts the fact that $y$ has to go out of all compact set when $t \rightarrow T_{*}<\infty$.

## B Explicit expression of the coefficients

We use here the same notation as in Lemma 4.4.
Remark B.1. Note that, in the case $\rho \mapsto \frac{\sqrt{H^{\prime}(\rho)}}{\rho}$ is integrable at $+\infty$, we have

$$
A=1+g=\frac{1}{2 \sqrt{H^{\prime}}} \int_{+\infty}^{\rho} \frac{1}{u} \sqrt{H^{\prime}(u)}(\mathscr{G}-1) \mathrm{d} u \leqslant 0
$$

Hence $1+2 g \leqslant 0$ and we have also $B \leqslant 0$. Besides,

$$
(1+g) H-B=\frac{1}{2 \sqrt{H^{\prime}}} \int_{0}^{\rho}\left(H^{\prime} \sqrt{H^{\prime}}+\frac{H \sqrt{H^{\prime}}}{\rho}(\mathscr{G}-1)\right) \geqslant 0 .
$$

Hence, if $w_{2}=u+H \geqslant 0$, then $\Psi \leqslant \frac{(d-1)}{r}(1+g)(u+H) \leqslant 0$
Remark B.2. In the case $d=3$, we obtain

$$
\begin{aligned}
a_{2} & =\frac{2 \sqrt{H^{\prime}}}{r^{2}}\left[-u^{2}\left(\mathscr{G} A^{2}-2 A(\mathscr{G}-2)+\mathscr{G}-1\right)+u(2(A-1)(\mathscr{G} B+c)+4 B)-\mathscr{G} B^{2}\right] \\
& =\frac{2 \sqrt{H^{\prime}}}{r^{2}}\left[-\mathscr{G}\left(u\left(A-1+\frac{2}{\mathscr{G}}\right)-B\right)^{2}-4 u^{2}\left(\frac{3}{4}-\frac{1}{\mathscr{G}}\right)+2 u c(A-1)\right],
\end{aligned}
$$

and
$b_{2}=\frac{2 \sqrt{H^{\prime}}}{r^{2}}\left[-u^{2}\left(\mathscr{G} A^{2}-2 A(\mathscr{G}-2)+\mathscr{G}-1\right)+u(2 B(\mathscr{G}-2-\mathscr{G} A)-2 c(A-1))-\mathscr{G} B^{2}\right]$
In the case $d=2$, we obtain

$$
a_{2}=\frac{\sqrt{H^{\prime}}}{2 r^{2}}\left[-\mathscr{G}\left(u\left(A-1+\frac{5}{2 \mathscr{G}}\right)-B\right)^{2}-\frac{25}{4} u^{2}\left(\frac{16}{25}-\frac{1}{\mathscr{G}}\right)+3 u c(A-1)\right] .
$$

## B. 1 Perfect gas.

For a perfect gas, we have: $c(\rho)=\sqrt{\gamma_{0}\left(\gamma_{0}-1\right)}(\rho)^{\frac{\gamma_{0}-1}{2}}$, thus

$$
H=2 \sqrt{\frac{\gamma_{0}}{\gamma_{0}-1}}(\rho)^{\frac{\gamma_{0}-1}{2}} .
$$

This implies, denoting $\nu=\frac{\gamma_{0}+1}{\gamma_{0}-1}>1$

$$
\begin{aligned}
\rho & =\left(\frac{\gamma_{0}-1}{4 \gamma_{0}}\right)^{\frac{1}{\gamma_{0}-1}} H^{\nu-1}, \\
c(\rho) & =\frac{\gamma_{0}-1}{2} H, \\
H^{\prime} & =\frac{\gamma_{0}-1}{2}\left(\frac{4 \gamma_{0}}{\gamma_{0}-1}\right)^{\frac{1}{\gamma_{0}-1}} H^{2-\nu}
\end{aligned}
$$

Consequently, noting that $\frac{\gamma_{0}-1}{2}=\frac{1}{\nu-1}, u=\frac{w_{1}+w_{2}}{2}$ and $H=\frac{w_{2}-w_{1}}{2}$, we get

$$
\begin{aligned}
\lambda_{1} & =\frac{w_{1}+w_{2}}{2}-\frac{1}{\nu-1}\left(\frac{w_{2}-w_{1}}{2}\right), \\
\lambda_{2} & =\frac{w_{1}+w_{2}}{2}+\frac{1}{\nu-1}\left(\frac{w_{2}-w_{1}}{2}\right), \\
\partial_{1} \lambda_{1} & =\partial_{2} \lambda_{2}=\frac{\mathscr{G}}{2}=\frac{\nu}{2(\nu-1)}>0, \\
\partial_{2} \lambda_{1} & =\partial_{1} \lambda_{2}=\frac{(\nu-2)}{2(\nu-1)} \geqslant 0 .
\end{aligned}
$$

Then $h=k=\ln \left(H^{(2-\nu) / 2}\right)$ and

$$
\begin{aligned}
g= & \frac{\nu-1}{(2-\nu)} \leqslant 0 \\
B= & \frac{\nu}{(2-\nu)(4-\nu)} H, \\
a_{0}= & \frac{-\nu}{2(\nu-1)^{1 / 2}(2 \nu+2)^{\frac{\nu-1}{2}}} H^{\nu-1}, \\
a_{1}= & \frac{d-1}{(2-\nu) r}\left[3 u+\frac{2(4-3 \nu) H}{(\nu-1)(4-\nu)}\right], \\
a_{2}= & \frac{d-1}{(2-\nu) r^{2}}\left[-u^{2}+\frac{4 u H}{(4-\nu)}-\frac{\nu H^{2}}{(\nu-1)(4-\nu)}\right. \\
& \left.+\frac{d-1}{2}\left(\frac{-2 u^{2}}{(2-\nu)}-\frac{8 u H}{(2-\nu)(4-\nu)}+\left(\frac{\nu H^{2}}{(\nu-1)(4-\nu)}-\frac{\nu^{3}}{(\nu-1)(2-\nu)(4-\nu)^{2}}\right) H^{2}\right)\right]
\end{aligned}
$$

## B. 2 Van der Waals gas.

For a Van der Waals gas, we have: $c(\rho)=\frac{1}{1-b \rho} \sqrt{\gamma_{0}\left(\gamma_{0}-1\right)}\left(\frac{\rho}{1-b \rho}\right)^{\frac{\gamma_{0}-1}{2}}$, thus

$$
H=2 \sqrt{\frac{\gamma_{0}}{\gamma_{0}-1}}\left(\frac{\rho}{1-b \rho}\right)^{\frac{\gamma_{0}-1}{2}} .
$$

This implies, denoting $\tilde{b}=b\left(\frac{\gamma_{0}-1}{4 \gamma_{0}}\right)^{\frac{1}{\gamma_{0}-1}}$ and $\nu=\frac{\gamma_{0}+1}{\gamma_{0}-1}>1$

$$
\begin{aligned}
\rho & =\frac{\left(\frac{\gamma_{0}-1}{4 \gamma_{0}}\right)^{\frac{1}{\gamma_{0}-1}} H^{\nu-1}}{1+\tilde{b} H^{\nu-1}}, \\
c(\rho) & =\frac{\gamma_{0}-1}{2}\left(1+\tilde{b} H^{\nu-1}\right) H \\
H^{\prime} & =\frac{\gamma_{0}-1}{2}\left(\frac{4 \gamma_{0}}{\gamma_{0}-1}\right)^{\frac{1}{\gamma_{0}-1}}\left(1+\tilde{b} H^{\nu-1}\right)^{2} H^{2-\nu}
\end{aligned}
$$

Consequently, noting that $\frac{\gamma_{0}-1}{2}=\frac{1}{\nu-1}, u=\frac{w_{1}+w_{2}}{2}$ and $H=\frac{w_{2}-w_{1}}{2}$, we get

$$
\begin{aligned}
\lambda_{1} & =\frac{w_{1}+w_{2}}{2}-\frac{1}{\nu-1}\left(\frac{w_{2}-w_{1}}{2}+\tilde{b}\left(\frac{w_{2}-w_{1}}{2}\right)^{\nu}\right), \\
\lambda_{2} & =\frac{w_{1}+w_{2}}{2}+\frac{1}{\nu-1}\left(\frac{w_{2}-w_{1}}{2}+\tilde{b}\left(\frac{w_{2}-w_{1}}{2}\right)^{\nu}\right), \\
\partial_{1} \lambda_{1} & =\partial_{2} \lambda_{2}=\frac{G}{2}=\frac{\nu}{2(\nu-1)}\left(1+\tilde{b} H^{\nu-1}\right)>0, \\
\partial_{2} \lambda_{1} & =\partial_{1} \lambda_{2}=\frac{1}{2(\nu-1)}\left((\nu-2)-\tilde{b} \nu H^{\nu-1}\right) .
\end{aligned}
$$

Then $h=k=\ln \left(H^{(2-\nu) / 2}\left(1+\tilde{b} H^{\nu-1}\right)\right)$ and

$$
\begin{aligned}
g & =\frac{\nu-1}{(2-\nu)\left(1+\tilde{b} H^{\nu-1}\right)} \\
B & =\frac{1}{(2-\nu)(4-\nu)} \frac{H}{1+\tilde{b} H^{\nu-1}}\left(1+\frac{(2-\nu)(4-\nu)}{2+\nu} \tilde{b} H^{\nu-1}\right) \\
a_{0} & =\frac{-\nu}{2(\nu-1)^{1 / 2}(2 \nu+2)^{\frac{\nu-1}{2}}} H^{\nu-1}
\end{aligned}
$$

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[^0]:    ${ }^{1}$ specific is a synonym of massic

