On the uniqueness of unbounded viscosity solutions arising in an optimal terminal wealth problem with transaction costs

Christoph Belak* Olaf Menkens† Jörn Sass‡

October 5, 2015

Abstract

We study the uniqueness of viscosity solutions of a Hamilton-Jacobi-Bellman equation which arises in a portfolio optimization problem in which an investor maximizes expected utility of terminal wealth in the presence of proportional transaction costs. Our main contribution is that the comparison theorem can be applied to prove the uniqueness of the value function in the portfolio optimization problem for logarithmic and power utility.

Keywords: Unbounded viscosity solutions, comparison principle, optimal terminal wealth, transaction costs

AMS Subject Classification (2010): 35D40, 91G80

1 Introduction

The aim of this paper is to establish the uniqueness of viscosity solutions of the following Hamilton-Jacobi-Bellman (HJB) equation

$$0 = \min \left\{ \mathcal{L}^{nt}\mathcal{V}(t,b,s), \mathcal{L}^{buy}\mathcal{V}(t,b,s), \mathcal{L}^{sell}\mathcal{V}(t,b,s) \right\}, \qquad (t,b,s) \in [0,T) \times \mathcal{S}. \tag{1}$$

The differential operators $\mathcal{L}^{nt}$, $\mathcal{L}^{buy}$, and $\mathcal{L}^{sell}$ in (1) are given by

$$\begin{aligned}
\mathcal{L}^{nt} &= -\frac{\partial}{\partial t} - \alpha s \frac{\partial}{\partial s} - \frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2}, \\
\mathcal{L}^{buy} &= (1+\lambda)\frac{\partial}{\partial b} - \frac{\partial}{\partial s}, \\
\mathcal{L}^{sell} &= -(1-\mu)\frac{\partial}{\partial b} + \frac{\partial}{\partial s},
\end{aligned}$$

and the space domain $\mathcal{S}$ is

$$\mathcal{S} = \left\{ (b,s) \in \mathbb{R}^2 : b + (1-\mu)s > 0,\ b + (1+\lambda)s > 0 \right\}.$$

*Fachbereich Mathematik, Technische Universität Kaiserslautern, Erwin-Schrödinger Straße, 67653 Kaiserslautern, Germany (email: belak@mathematik.uni-kl.de). The research of this author was partially supported by Science Foundation Ireland via FMC^2 (Grant No. 08/SRC/FMC1389).

†School of Mathematical Sciences, Dublin City University, Glasnevin, Dublin 9, Ireland (email: olaf.menkens@dcu.ie). The research of this author was partially supported by Science Foundation Ireland via the Edgeworth Centre (Grant No. 07/MI/008) and FMC^2 (Grant No. 08/SRC/FMC1389).

‡Fachbereich Mathematik, Technische Universität Kaiserslautern, Erwin-Schrödinger Straße, 67653 Kaiserslautern, Germany (email: sass@mathematik.uni-kl.de). The research of this author was supported by Deutsche Forschungsgemeinschaft (DFG).

Equation (1) is related to an optimal terminal wealth problem in a market with transaction costs (see Section 2.1 for the model formulation including the definition of the parameters used above). While it is known that uniqueness holds if one imposes a growth condition at infinity and assumes that the viscosity solution vanishes on $\partial\mathcal{S}$, this result is not sufficient to cover some important cases in the corresponding portfolio problem. In particular, if the investor has a utility function of the form

$$U_p(x) = \begin{cases} \frac{1}{p}x^p & \text{if } p < 0, \\ \log x & \text{if } p = 0, \end{cases}$$

the corresponding value function $\mathcal{V}(t, b, s)$ tends to $-\infty$ at the boundary of $\mathcal{S}$ and the existing comparison theorems do not apply. It is therefore our aim to provide a comparison principle which covers these cases as well.

1.1 Portfolio optimization with transaction costs

The problem of optimal investment and consumption in the presence of transaction costs has received considerable attention over the last decades. Despite a wealth of papers, there are still some open problems which have not been addressed in the literature, in particular, in the time-dependent case, when the investor aims to maximize expected utility of terminal wealth without intermediate consumption. While it is known that the corresponding value function is a viscosity solution of (1), and that the HJB equation admits a classical solution, due to the lack of a uniqueness result for a certain subset of model parameters it is not known if this classical solution coincides with the value function in general. Even more, since it is challenging to construct the optimal strategies, one cannot use a classical verification argument to establish the link between these two solutions of the HJB equation. Our aim is therefore to establish a general comparison result for the HJB equation to close this gap. In addition, our uniqueness result ensures the convergence of numerical algorithms to the correct solution, and hence allows us to determine the candidate optimal strategy numerically.

The analysis of optimal investment and consumption in the presence of proportional transaction costs in a continuous-time model was initiated by Magill and Constantinides [21] and insights were gained on the nature of the optimal strategy. A rigorous solution can be found in the seminal article of Davis and Norman [10]. Shreve and Soner [25] obtained similar results under weaker assumptions using a viscosity solution approach. A generalization of this model was studied by Kabanov and Klüppelberg [15] and de Vallière and Kabanov [12]. All five papers considered optimal consumption over an infinite time horizon. The optimal terminal wealth problem with a finite horizon was first treated in Akian et al. [1]. Davis et al. [11] considered the same HJB equation in an option pricing setting and showed that the value function is a viscosity solution thereof. Uniqueness was established in the case of exponential utility. Dai and Yi [9] derived the existence of a regular solution of the HJB equation. Moreover, Kunisch and Sass [19] and Herzog et al. [14] proposed algorithms to approximate the value function and the optimal strategies numerically. Liu and Loewenstein [20] obtained a closed form solution under the assumption that the terminal time is random and Bichuch [4] studied the finite-horizon problem by means of an asymptotic analysis.

The aforementioned papers approach the portfolio problem using PDE methods. In recent years, starting with the seminal paper of Kallsen and Muhle-Karbe [17], the infinite horizon problem has been solved using probabilistic methods. While Kallsen and Muhle-Karbe [17] consider the logarithmic utility case, the results have been extended to power utility in Herczegh and Prokaj [13] and Choi et al. [7]. Finally, Kallsen and Muhle-Karbe [18] and Kallsen and Li [16] use this probabilistic approach to treat the finite-horizon problem for small transaction costs.

1.2 Open problems and our contribution

Despite this wealth of papers, a careful inspection reveals that the finite-horizon terminal wealth problem is still not entirely solved. Open problems include:

(OP1) A rigorous proof of the continuity of the value function in the time variable.

(OP2) Existence of the optimal controls. However, the result is already known to hold in the infinite-horizon case; see, e.g., Shreve and Soner [25, Section 9]. In the finite-horizon setting considered in this paper, the existence result appears to be quite sophisticated, since it requires the existence of an obliquely reflected, two-dimensional process in an unbounded, time-dependent domain.

(OP3) Uniqueness of the value function as the viscosity solution of the HJB equation in the case of logarithmic utility or power utility with a *negative* power. This result is not even known to hold in the infinite-horizon case.

Regarding (OP3), the main problem is to establish a uniqueness result on the unbounded domain $[0,T] \times \overline{\mathcal{S}}$. In the case of logarithmic utility and negative power utility, it is known that the value function on the boundary of the space domain $\overline{\mathcal{S}}$ is equal to negative infinity, which makes a comparison result without growth conditions difficult. We adopt an idea of Vukelja [26] to tackle this problem. Instead of providing a comparison theorem for subsolutions and supersolutions $u(t,b,s)$ and $v(t,b,s)$ directly, we first shift the supersolution v by a parameter $\varepsilon > 0$ in the b direction and prove the comparison result for the shifted supersolution. We obtain the comparison result for supersolutions without shift by sending $\varepsilon \downarrow 0$. This approach has the advantage that the shifted supersolution does not tend to negative infinity at $\partial\mathcal{S}$. For this approach, it seems to be crucial that the HJB equation does not depend explicitly on the variable b.
Although our comparison result is general enough to hold for discontinuous viscosity solutions, we nevertheless prove the continuity of the value function (hence solving (OP1)) in Section 3. This is because the existing proofs of the viscosity property (cf. Proposition 2.4) and the underlying proof of the dynamic programming principle rely on the continuity of $\mathcal{V}$. Although it seems to be possible to prove the corresponding results without the continuity (along the lines of Bouchard and Touzi [6] and Bouchard and Nutz [5]), for the purpose of this paper it seems to be more straightforward to establish the continuity of $\mathcal{V}$ and rely on the existing dynamic programming and viscosity solution results instead. On the other hand, since we believe that our method for proving the comparison principle is of independent interest, we state the result for the general, possibly discontinuous, case.

1.3 Outline of this paper

This paper is organized as follows. In Section 2, we introduce the portfolio optimization problem under transaction costs. Then we recall some known results, and prove some basic properties of the value function and the HJB equation. In Section 3, we rigorously establish the continuity of the value function. Our main result, a comparison principle for the HJB equation (1), is proved in Section 4.

2 The portfolio problem and known results

In this section, we specify the market model and formulate the optimization problem. Then we recall some known results about the properties of the value function.

2.1 Model setup

We consider a market consisting of two assets, namely, a risk-free asset P_0 called *bond* and a risky asset P_1 called *stock*. On the finite time interval $[t, T]$, we assume that the prices of the two assets evolve as

$$\begin{aligned} dP_0(u) &= rP_0(u)du, && P_0(t) = p_0, \\ dP_1(u) &= \alpha P_1(u)du + \sigma P_1(u)dW(u), && P_1(t) = p_1. \end{aligned}$$

We refer to $r \geq 0$ as the *interest rate*, $\alpha \in \mathbb{R}$ as the *drift*, and $\sigma > 0$ as the *volatility* of the stock. Taking P_0 to be the numéraire, we may, without loss of generality, assume that $r = 0$. We assume that W is a standard Brownian motion defined on the canonical Wiener space $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$, where $\Omega = C_0([0, \infty))$ denotes the space of continuous functions $\omega : [0, \infty) \to \mathbb{R}$ satisfying $\omega(0) = 0$ and where $\mathbb{P}$ denotes the Wiener measure. We denote the augmented filtration generated by $(W(u))_{u \geq 0}$ by $\mathbb{F} = \mathbb{F}^0 = (\mathcal{F}(u))_{u \geq 0}$ and, similarly, we let $\mathbb{F}^t = (\mathcal{F}^t(u))_{u \geq t}$ denote the augmented filtration generated by $(W(u) - W(t))_{u \geq t}$.

Remark 2.1. *The assumption $r = 0$ ensures that the differential operator $\mathcal{L}^{nt}$ appearing in the HJB equation* (1) *does not depend on b. This is crucial for our approach to proving the comparison principle. Nevertheless, since we can always choose the bond to be the numéraire in the portfolio optimization problem, this does not pose any restrictions on the market model.*

Denote by b and s the investor's wealth invested in the bond and the stock at time t, respectively. We assume that whenever the investor buys or sells stocks, she has to pay a fee proportional to the size of the transaction. That is, if the investor buys stocks for Δs units of money, she has to pay transaction costs of size $\lambda \Delta s$, where $\lambda \in (0, \infty)$. We assume that the investor pays these costs from the bond account. Therefore, after the transaction, she holds $b - (1 + \lambda)\Delta s$ units of money in the bond and $s + \Delta s$ units of money in the stock. Similarly, if the investor sells stocks for Δs units of money, she has to pay transaction costs of size $\mu \Delta s$, where $\mu \in (0, 1)$.
We can therefore model the investor's trading strategies as follows. Let L and M be two $\mathbb{F}^t$-adapted, nondecreasing, càdlàg processes (i.e. right-continuous paths with left limits) with $L(t-) = M(t-) = 0$. If L and M represent the cumulative units of money used for stock purchases and sales, respectively, the investor's wealth invested in bond and stock, denoted by B and S, respectively, follows

$$dB(u) = -(1 + \lambda)dL(u) + (1 - \mu)dM(u), \qquad u \in [t, T], \tag{2}$$

$$dS(u) = \alpha S(u-)du + \sigma S(u-)dW(u) + dL(u) - dM(u), \qquad u \in [t, T], \tag{3}$$

where we set the initial values to be $B(t-) = b$ and $S(t-) = s$. The *net wealth* X of the investor after liquidation of the stock position at time u is then given by

$$X(u) := \begin{cases} B(u) + (1 - \mu)S(u) & \text{if } S(u) > 0, \\ B(u) + (1 + \lambda)S(u) & \text{if } S(u) \leq 0. \end{cases}$$

It is sometimes necessary to stress the dependence of B, S, and X on the initial values and the trading strategy. To this end, we denote by $X^{L,M}_{t,b,s}(u)$ the net wealth at time u, if $B(t-) = b$, $S(t-) = s$, and if the trading strategy (L, M) is applied. We may drop some of the indices if they are clear from the context. Similarly, this applies to B and S in an obvious analog way.
We require admissible strategies to lead to a nonnegative net wealth. For this purpose, we define the following *solvency cone*

$$\mathcal{S} := \left\{(b, s) \in \mathbb{R}^2 \,\middle|\, b + (1 + \lambda)s > 0, b + (1 - \mu)s > 0\right\}.$$

So, whenever $(B, S) \in \overline{\mathcal{S}}$, the investor can liquidate the stock holdings to end up with nonnegative wealth. Let $t \in [0, T]$. An $\mathbb{F}^t$-adapted trading strategy (L, M) is called *admissible* for initial positions $(b, s) \in \overline{\mathcal{S}}$ if the corresponding pair (B, S) with initial values $B(t-) = b$ and $S(t-) = s$ takes values in $\overline{\mathcal{S}}$ for all $u \in [t, T]$. The set of all admissible trading strategies of this form is denoted by $\mathcal{A}(t, b, s)$.
With this, the objective of the investor is to maximize the expected utility of terminal wealth, i.e., we consider the optimization problem

$$\mathcal{V}(t, b, s) := \sup_{(L,M) \in \mathcal{A}(t,b,s)} \mathbb{E}\left[U_p\left(X_{t,b,s}^{L,M}(T)\right)\right] \tag{4}$$

for a utility function $U_p : (0, \infty) \to \mathbb{R}$ of the form

$$U_p(x) := \begin{cases} \frac{1}{p}x^p & \text{if } p < 1, p \neq 0, \\ \log(x) & \text{if } p = 0. \end{cases} \tag{5}$$

We extend U_p to $[0, \infty)$ by setting $U_p(0) := \lim_{x \downarrow 0} U_p(x)$.

2.2 Properties of the value function and the HJB equation

The aim of this section is to gather and review existing results and establish some preliminary properties of the HJB equation which we require later.
We start by constructing a parametrized family of smooth functions which dominate $\mathcal{V}$. For this, recall that $p < 1$ denotes the parameter associated with the utility function U_p, fix constants $K \geq 1$ and $\gamma \in [1 - \mu, 1 + \lambda]$, and define a function $\varphi_{\gamma,p,K} : [0, T] \times \overline{\mathcal{S}} \to \mathbb{R}$ by

$$\varphi_{\gamma,p,K}(t, b, s) := U_p\left((b + \gamma s) f_{p,K}(t)\right) \tag{6}$$

with $f_{p,K} : [0, T] \to \mathbb{R}_+$ given by

$$f_{p,K}(t) := \exp\left(K \frac{1}{2(1-p)} \frac{\alpha^2}{\sigma^2}(T - t)\right).$$

Note that $\varphi_{1,p,1}(t, b, s)$ is the value function of the portfolio optimization problem in the absence of transaction costs; see Merton [22]. Hence, we can expect $\varphi_{\gamma,p,K} \geq \mathcal{V}$. Indeed, the next lemma shows that $\varphi_{\gamma,p,K}$ is a supersolution of (1) and a classical verification argument shows that $\varphi_{\gamma,p,K} \geq \mathcal{V}$.

Lemma 2.2. *1. The function $\varphi_{\gamma,p,K}$ is a supersolution of* (1) *and a strict supersolution if $\gamma \in (1 - \mu, 1 + \lambda)$ and $K > 1$.*

2. We have $\varphi_{\gamma,p,K} \geq \mathcal{V}$. In particular, $\mathcal{V}(t, b, s) < +\infty$ for all $(t, b, s) \in [0, T] \times \overline{\mathcal{S}}$.

Proof. 1. Direct computations reveal that

$$\begin{aligned}
\mathcal{L}^{nt}\varphi_{\gamma,p,K}(t, b, s) &= \frac{(b + \gamma s)^p}{2(1-p)\sigma^2}(f_{p,K}(t))^p\left[\left(\alpha - \frac{\gamma\sigma^2 s}{b + \gamma s}\right)^2 + (K - 1)\alpha^2\right] \geq 0, \\
\mathcal{L}^{buy}\varphi_{\gamma,p,K}(t, b, s) &= (b + \gamma s)^{p-1}(f_{p,K}(t))^p(1 + \lambda - \gamma) \geq 0, \\
\mathcal{L}^{sell}\varphi_{\gamma,p,K}(t, b, s) &= (b + \gamma s)^{p-1}(f_{p,K}(t))^p(-(1 - \mu) + \gamma) \geq 0,
\end{aligned}$$

where the inequalities are strict if $\gamma \in (1 - \mu, 1 + \lambda)$ and $K > 1$.

2. Fix $(t, b, s) \in [0, T] \times \overline{\mathcal{S}}$, $\varepsilon > 0$, let $(L, M) \in \mathcal{A}(t, b, s)$, and notice that $(L, M) \in \mathcal{A}(t, b+\varepsilon, s)$. Let $(K_j)_{j\in\mathbb{N}}$ be a sequence of compact sets containing (b, s) and $(b + \varepsilon, s)$ such that the K_j increase to $\overline{\mathcal{S}}$ as $j \to \infty$. For each $j \in \mathbb{N}$ we define a stopping time

$$\tau_j := \inf \left\{ u \geq t : \left(B_{t,b}^{L,M}(u) + \varepsilon, S_{t,s}^{L,M}(u) \right) \notin K_j \right\} \wedge T$$

and note that $\tau_j \to T$ as $j \to \infty$.

Note that $B_{t,b+\varepsilon}^{L,M} = B_{t,b}^{L,M} + \varepsilon$ and write $B^\varepsilon := B_{t,b+\varepsilon}^{L,M}$ as well as $S := S_{t,s}^{L,M}$. Itô's Formula for càdlàg semimartingales (see e.g. Protter [24, Theorem II.32]) shows that

$$\begin{aligned}
\varphi_{\gamma,p,K}(\tau_j, B^\varepsilon(\tau_j), S(\tau_j)) = \varphi_{\gamma,p,K}(t, b+\varepsilon, s) &- \int_t^{\tau_j} \mathcal{L}^{nt} \varphi_{\gamma,p,K}(u, B^\varepsilon(u), S(u))\, du \\
&- \int_t^{\tau_j} \mathcal{L}^{buy} \varphi_{\gamma,p,K}(u, B^\varepsilon(u), S(u))\, dL^c(u) \\
&- \int_t^{\tau_j} \mathcal{L}^{sell} \varphi_{\gamma,p,K}(u, B^\varepsilon(u), S(u))\, dM^c(u) \\
&+ \int_t^{\tau_j} \sigma S(u) \frac{\partial}{\partial s} \varphi_{\gamma,p,K}(u, B^\varepsilon(u), S(u))\, dW(u) \\
&+ \sum_{t \leq u \leq \tau_j} \left[\varphi_{\gamma,p,K}(u, B^\varepsilon(u), S(u)) - \varphi_{\gamma,p,K}(u-, B^\varepsilon(u-), S(u-)) \right],
\end{aligned}$$

where L^c and M^c denote the continuous parts of L and M, respectively. Since $\varphi_{\gamma,p,K}$ satisfies $\mathcal{L}^{buy}\varphi_{\gamma,p,K}, \mathcal{L}^{sell}\varphi_{\gamma,p,K} \geq 0$ we see that $\varphi_{\gamma,p,K}$ is non-increasing in the directions of the jumps of $(u, B^\varepsilon(u), S(u))$ by the fundamental theorem of calculus for line integrals and, hence,

$$\sum_{t \leq u \leq \tau_j} \left[\varphi_{\gamma,p,K}(u, B^\varepsilon(u), S(u)) - \varphi_{\gamma,p,K}(u-, B^\varepsilon(u-), S(u-)) \right] \leq 0.$$

Moreover, since $\varphi_{\gamma,p,K}$ is a supersolution of (1) it follows that

$$\begin{aligned}
0 &\leq \int_t^{\tau_j} \mathcal{L}^{nt} \varphi_{\gamma,p,K}(u, B^\varepsilon(u), S(u))\, du, \\
0 &\leq \int_t^{\tau_j} \mathcal{L}^{buy} \varphi_{\gamma,p,K}(u, B^\varepsilon(u), S(u))\, dL^c(u), \\
0 &\leq \int_t^{\tau_j} \mathcal{L}^{sell} \varphi_{\gamma,p,K}(u, B^\varepsilon(u), S(u))\, dM^c(u).
\end{aligned}$$

We therefore obtain

$$\begin{aligned}
&\varphi_{\gamma,p,K}(\tau_j, B^\varepsilon(\tau_j), S(\tau_j)) \\
&\qquad \leq \varphi_{\gamma,p,K}(t, b+\varepsilon, s) + \int_t^{\tau_j} \sigma S(u) \frac{\partial}{\partial s} \varphi_{\gamma,p,K}(u, B^\varepsilon(u), S(u))\, dW(u)
\end{aligned}$$

and by taking expectations on both sides

$$\varphi_{\gamma,p,K}(t, b+\varepsilon, s) \geq \mathbb{E}\left[\varphi_{\gamma,p,K}(\tau_j, B^\varepsilon(\tau_j), S(\tau_j)) \right]$$

for all $j \in \mathbb{N}$. Since

$$\varphi_{\gamma,p,K}(\tau_j, B^\varepsilon(\tau_j), S(\tau_j)) \geq U_p\left(B^\varepsilon(\tau_j) + \gamma S(\tau_j) \right) \geq U_p(\varepsilon)$$

we can send $j \to \infty$ and use Fatou's Lemma to see that

$$\varphi_{\gamma,p,K}(t, b+\varepsilon, s) \geq \mathbb{E}\left[\varphi_{\gamma,p,K}(T, B^{\varepsilon}(T), S(T))\right] = \mathbb{E}\left[U_p(B^{\varepsilon}(T) + \gamma S(T))\right].$$

Next, observe that we have $B^{\varepsilon}(T) + \gamma S(T) \geq X^{L,M}_{t,b+\varepsilon,s}(T)$ since $\gamma \in [1-\mu, 1+\lambda]$. This implies that

$$\varphi_{\gamma,p,K}(t, b+\varepsilon, s) \geq \mathbb{E}\left[U_p\left(X^{L,M}_{t,b+\varepsilon,s}(T)\right)\right] = \mathbb{E}\left[U_p\left(X^{L,M}_{t,b,s}(T) + \varepsilon\right)\right].$$

Now send $\varepsilon \downarrow 0$ and use monotone convergence to obtain

$$\varphi_{\gamma,p,K}(t, b, s) \geq \mathbb{E}\left[U_p\left(X^{L,M}_{t,b,s}(T)\right)\right]$$

and we conclude since (L, M) was chosen arbitrarily. □

The following lemma establishes some further elementary properties of $\mathcal{V}$. We note that these properties have already been observed in Shreve and Soner [25] for the infinite-horizon problem.

Lemma 2.3. *1. The value function is lower bounded, that is*

$$\mathcal{V}(t, b, s) \geq U_p\left(b + \min\{(1-\mu)s, (1+\lambda)s\}\right).$$

2. For every $t \in [0, T]$, the value function $\mathcal{V}(t, b, s)$ is concave in (b, s). In particular, it is locally Lipschitz continuous on $\mathcal{S}$.

Proof. 1. This follows immediately by considering the strategy of closing the stock position at initial time t and no trading afterwards.

2. The concavity is inherited from the utility function U_p. More details can be found in Shreve and Soner [25, Proposition 3.1]. Note that every concave function is locally Lipschitz-continuous in the interior of its domain. □

Note that Lemma 2.3.1 allows us to restrict the set of admissible strategies $\mathcal{A}(t, b, s)$ to those strategies (L, M) which satisfy

$$\mathbb{E}\left[U_p\left(X^{L,M}_{t,b,s}(T)\right)\right] \geq U_p\left(b + \min\{(1-\mu)s, (1+\lambda)s\}\right),$$

which we will assume in what follows. Moreover, combining Lemma 2.2.2 and 2.3.1, we see that

$$\mathcal{V}(t, b, s) = U_p(0) \qquad \text{for all } t \in [0, T], (b, s) \in \partial\mathcal{S}.$$

The following proposition establishes the link between the value function $\mathcal{V}$ and the HJB equation by showing that the value function is a viscosity solution thereof. We refer to Section 4 for the definition of viscosity solutions. The proof of the following proposition can be found in Davis et al. [11] (in a slightly different context), or can be established along the lines of Shreve and Soner [25, Theorem 7.7]. More details can also be found in Belak et al. [3] and Belak [2], who consider a more general setting. Note, however, that the value function needs to be continuous for all these lines of arguments. The proof of the continuity is deferred to Section 3.

Proposition 2.4. *The value function $\mathcal{V}$ is a continuous viscosity solution of*

$$0 = \min\{\mathcal{L}^{nt}\mathcal{V}(t, b, s), \mathcal{L}^{buy}\mathcal{V}(t, b, s), \mathcal{L}^{sell}\mathcal{V}(t, b, s)\}, \qquad (t, b, s) \in [0, T) \times \mathcal{S},$$

with boundary condition

$$\mathcal{V}(t, b, s) = U_p(0), \qquad (b, s) \in \partial\mathcal{S},$$

and terminal condition

$$\mathcal{V}(T, b, s) = U_p\left(b + \min\{(1-\mu)s, (1+\lambda)s\}\right).$$

3 Continuity of the value function

In this section, we prove that $\mathcal{V}$ is uniformly continuous in the time variable t. In addition, we use this result to show that the value function is also jointly continuous in (t, b, s).

3.1 Continuity for $0 < p < 1$

Let us first consider the case $0 < p < 1$. As a preliminary result, we need an estimate on the growth of the net wealth.

Lemma 3.1. *Let $(L, M) \in \mathcal{A}(t, b, s)$.*

1. *There exists a constant $C_0 > 0$ independent of (L, M) such that*

$$\mathbb{E}\left[B(T) + S(T)\right] = \mathbb{E}\left[|B(T) + S(T)|\right] \leq C_0(b + s).$$

2. *There exists a constant $C_1 > 0$ independent of (L, M) such that*

$$\mathbb{E}\left[(B(T) + S(T))^2\right] \leq C_1(1 + b^2 + s^2).$$

Proof. We frequently make use of the fact that we have

$$s \leq |s| \leq C(b + s), \qquad b \leq |b| \leq C(b + s)$$

on $\mathcal{S}$ for $C = 1 + \max\{1/\mu, 1/\lambda\}$. Moreover, the dynamics of B and S imply that we have for every stopping time $\theta \leq T$

$$B(\theta) + S(\theta) \leq b + s + \int_t^\theta \alpha S(u)\, du + \int_t^\theta \sigma S(u)\, dW(u).$$

1. Let $\tau_n := \inf\{u \geq t : |S(u)| \geq n\} \wedge T$. Setting $K := |\alpha| C$, we have

$$\begin{aligned} B(\tau_n) + S(\tau_n) &\leq b + s + \int_t^{\tau_n} \alpha S(u)\, du + \int_t^{\tau_n} \sigma S(u)\, dW(u) \\ &\leq b + s + K \int_t^{\tau_n} B(u) + S(u)\, du + \sigma \int_t^{\tau_n} S(u)\, dW(u). \end{aligned}$$

Taking expectations on both sides implies that

$$\mathbb{E}\left[B(\tau_n) + S(\tau_n)\right] \leq b + s + K\mathbb{E}\left[\int_t^{\tau_n} B(u) + S(u)\, du\right].$$

Since $B(u) + S(u) \geq 0$ we have

$$\mathbb{E}\left[B(\tau_n) + S(\tau_n)\right] \leq b + s + K\mathbb{E}\left[\int_t^{T} B(u) + S(u)\, du\right].$$

Taking the limit $n \to \infty$ together with Fatou's Lemma and using that $\tau_n \to T$ implies that

$$\mathbb{E}\left[B(T) + S(T)\right] \leq b + s + K \int_t^T \mathbb{E}\left[B(u) + S(u)\right]\, du$$

and we conclude by Gronwall's inequality.

2. We have

$$
\begin{aligned}
(B(T)+S(T))^2 &\le (1+B(T)+S(T))^2 \\
&\le \left(1+b+s+K\int_t^T B(u)+S(u)\,du+\sigma\int_t^T S(u)\,dW(u)\right)^2.
\end{aligned}
$$

Using the fact that $(a+b)^2 \le 2a^2+2b^2$ and Hölder's inequality, this implies that there exists a constant $L>0$ such that

$$
\begin{aligned}
&(B(T)+S(T))^2 \\
&\qquad \le L\left(1+b^2+s^2+K^2T\int_t^T (B(u)+S(u))^2\,du+\sigma^2\left[\int_t^T S(u)\,dW(u)\right]^2\right).
\end{aligned}
\tag{7}
$$

Note that

$$
\mathbb{E}\left[\left(\int_t^T S(u)\,dW(u)\right)^2\right]=\mathbb{E}\left[\int_t^T S(u)^2\,du\right]\le \mathbb{E}\left[\int_t^T (B(u)+S(u))^2\,du\right].
$$

Hence, taking expectations in (7) we see that

$$
\begin{aligned}
&\mathbb{E}\left[(B(T)+S(T))^2\right] \\
&\le L\left(1+b^2+s^2+K^2T\int_t^T \mathbb{E}\left[(B(u)+S(u))^2\right]\,du+\sigma^2\int_t^T \mathbb{E}\left[(B(u)+S(u))^2\right]\,du\right)
\end{aligned}
$$

and we can again conclude by Gronwall's inequality. □

The next lemma establishes a crucial time-shifting property of the value function, which will allow us to prove the time continuity by varying the terminal time T instead of the initial time t.

Lemma 3.2. *Denote the value function corresponding to the terminal time T by $\mathcal{V}_T$. Let $t\in[0,T]$ and $h\ge -t$. Then*

$$
\mathcal{V}_T(t,b,s)=\mathcal{V}_{T+h}(t+h,b,s), \qquad (b,s)\in\overline{\mathcal{S}}.
$$

Proof. We denote by $\mathcal{A}_T(t,b,s)$ the set of strategies $(L,M)\in\mathcal{A}(t,b,s)$ with terminal time T. We want to show that from every $(L,M)\in\mathcal{A}_T(t,b,s)$, we can construct admissible $(L_h,M_h)\in\mathcal{A}_{T+h}(t+h,b,s)$ and vice versa.
For this, recall that every $\omega\in\Omega$ is a continuous function $\omega:[0,\infty)\to\mathbb{R}$ with $\omega(0)=0$. Let $h\ge -t$ and $(L,M)\in\mathcal{A}_T(t,b,s)$. Since L and M are $\mathbb{F}^t$-adapted, we see that they can be written as

$$
L(u,\omega)=L\big(u,\omega([t+\cdot]\wedge u)-\omega(t)\big), \qquad M(u,\omega)=M\big(u,\omega([t+\cdot]\wedge u)-\omega(t)\big).
$$

Now, for every $\omega\in\Omega$ and $u\in[t+h,T+h]$, we define

$$
\begin{aligned}
L_h(u,\omega) &:= L\big(u-h,\omega([t+h+\cdot]\wedge u)-\omega(t+h)\big),\\
M_h(u,\omega) &:= M\big(u-h,\omega([t+h+\cdot]\wedge u)-\omega(t+h)\big).
\end{aligned}
$$

Then, clearly, L_h and M_h are $\mathbb{F}^{t+h}$-adapted and $(L_h,M_h)\in\mathcal{A}_{T+h}(t+h,b,s)$.

Since we can similarly construct admissible strategies for terminal time T from strategies with terminal time $T+h$, there is a one-to-one correspondence between the two sets

$$\mathcal{A}_T(t,b,s) \qquad \text{and} \qquad \mathcal{A}_{T+h}(t+h,b,s).$$

Also, it is easy to verify that by the construction of (L_h, M_h) we have

$$\mathbb{E}\left[U_p\left(X_{t,b,s}^{L,M}(T)\right)\right] = \mathbb{E}\left[U_p\left(X_{t+h,b,s}^{L_h,M_h}(T+h)\right)\right],$$

which concludes the proof. ☐

We can now turn to the continuity of $\mathcal{V}$ in the time variable.

Proposition 3.3. *Assume that $p \in (0,1)$ and let $(b,s) \in \overline{\mathcal{S}}$ be fixed. Then $\mathcal{V}(\cdot,b,s)$ is uniformly continuous on $[0,T]$.*

Proof. According to Lemma 3.2, we have

$$|\mathcal{V}_T(t,b,s) - \mathcal{V}_T(t+h,b,s)| = |\mathcal{V}_T(t,b,s) - \mathcal{V}_{T-h}(t,b,s)|$$

for every $h \geq -t$ and, hence, in order to prove continuity in t it suffices to prove continuity in T.

1. We first show that $\mathcal{V}_T$ is increasing in T. For this, let $T_- < T_+$ and fix $t \in [0, T_-]$. Let $(L^-, M^-) \in \mathcal{A}_{T_-}(t,b,s)$ and define (L^+, M^+) such that $(L^+, M^+) = (L^-, M^-)$ on $[t, T_-)$ and such that $S^{L^+,M^+}(u) = 0$ on $[T_-, T_+]$ (i.e., liquidation of the stock position at T_- and no trading afterwards). Then $(L^+, M^+) \in \mathcal{A}_{T_+}(t,b,s)$ and $X_{t,b,s}^{L^+,M^+}(T_+) = X_{t,b,s}^{L^-,M^-}(T_-)$. Since (L^-, M^-) was chosen arbitrarily it follows that $\mathcal{V}_{T_+}(t,b,s) \geq \mathcal{V}_{T_-}(t,b,s)$.

2. Let $\varepsilon > 0$. We are left with showing that

$$\mathcal{V}_{T_+}(t,b,s) - \mathcal{V}_{T_-}(t,b,s) \leq \varepsilon,$$

if $T_+ - T_-$ is sufficiently small. Choose $(L^+, M^+) \in \mathcal{A}_{T_+}(t,b,s)$ to be ε-optimal, i.e.,

$$\mathbb{E}\left[U_p\left(X_{t,b,s}^{L^+,M^+}(T_+)\right)\right] + \varepsilon \geq \mathcal{V}_{T_+}(t,b,s).$$

Let (L^-, M^-) be the restriction of (L^+, M^+) to $[t, T_-]$. We note that clearly $(L^-, M^-) \in \mathcal{A}_{T_-}(t,b,s)$. Define $A := \{X_{t,b,s}^{L^+,M^+}(T_+) - X_{t,b,s}^{L^-,M^-}(T_-) > 0\}$. Then the subadditivity of U_p and Jensen's inequality show that

$$\begin{aligned}
&\mathcal{V}_{T_+}(t,b,s) - \mathcal{V}_{T_-}(t,b,s) \\
&\qquad \leq \mathbb{E}\left[U_p\left(X_{t,b,s}^{L^+,M^+}(T_+)\right) - U_p\left(X_{t,b,s}^{L^-,M^-}(T_-)\right)\right] + \varepsilon \\
&\qquad \leq \mathbb{E}\left[U_p\left(X_{t,b,s}^{L^+,M^+}(T_+)\mathbf{1}_A\right) - U_p\left(X_{t,b,s}^{L^-,M^-}(T_-)\mathbf{1}_A\right)\right] + \varepsilon \\
&\qquad \leq \mathbb{E}\left[U_p\left(\left(X_{t,b,s}^{L^+,M^+}(T_+) - X_{t,b,s}^{L^-,M^-}(T_-)\right)\mathbf{1}_A\right)\right] + \varepsilon \\
&\qquad \leq U_p\left(\mathbb{E}\left[\left(X_{t,b,s}^{L^+,M^+}(T_+) - X_{t,b,s}^{L^-,M^-}(T_-)\right)\mathbf{1}_A\right]\right) + \varepsilon. \qquad (8)
\end{aligned}$$

Write $B := B_{t,b}^{L^+,M^+}$ and $S := S_{t,s}^{L^+,M^+}$. Using that $(L^+, M^+) = (L^-, M^-)$ on $[t, T_-]$, we have $X_{t,b,s}^{L^+,M^+}(T_-) = X_{t,b,s}^{L^-,M^-}(T_-)$ and, hence,

$$
\begin{aligned}
&X_{t,b,s}^{L^+,M^+}(T_+) - X_{t,b,s}^{L^-,M^-}(T_-) \\
&\qquad = X_{t,b,s}^{L^+,M^+}(T_+) - X_{t,b,s}^{L^+,M^+}(T_-) \\
&\qquad = B(T_+) - B(T_-) + S(T_+)[\gamma(T_+) - \gamma(T_-)] + \gamma(T_-)[S(T_+) - S(T_-)],
\end{aligned}
$$

where $\gamma(u) = (1-\mu)\mathbf{1}_{\{S(u)>0\}} + (1+\lambda)\mathbf{1}_{\{S(u)\le 0\}}$. Moreover, note that $S(T_+)[\gamma(T_+) - \gamma(T_-)] \le 0$ and hence

$$
X_{t,b,s}^{L^+,M^+}(T_+) - X_{t,b,s}^{L^-,M^-}(T_-) \le B(T_+) - B(T_-) + \gamma(T_-)[S(T_+) - S(T_-)].
$$

Adhering to the dynamics of (B, S), it follows that

$$
\begin{aligned}
X_{t,b,s}^{L^+,M^+}(T_+) - X_{t,b,s}^{L^-,M^-}(T_-) &\le B(T_+) - B(T_-) + \gamma(T_-)[S(T_+) - S(T_-)] \\
&\le \gamma(T_-) \int_{T_-}^{T^+} \alpha S(u)du + \gamma(T_-) \int_{T_-}^{T^+} \sigma S(u)dW(u).
\end{aligned}
$$

Therefore, we see that

$$
\begin{aligned}
&\left(X_{t,b,s}^{L^+,M^+}(T_+) - X_{t,b,s}^{L^-,M^-}(T_-)\right)\mathbf{1}_A \\
&\qquad \le (1+\lambda)\left|\int_{T_-}^{T_+} \alpha S(u)\,du\right| + (1+\lambda)\left|\int_{T_-}^{T_+} \sigma S(u)\,dW(u)\right|
\end{aligned}
$$

and, hence, there exists a constant $C > 0$ such that

$$
\begin{aligned}
&\mathbb{E}\left[\left(X_{t,b,s}^{L^+,M^+}(T_+) - X_{t,b,s}^{L^-,M^-}(T_-)\right)\mathbf{1}_A\right] \\
&\qquad \le (1+\lambda)\mathbb{E}\left[\left|\int_{T_-}^{T_+} \alpha S(u)\,du\right| + \left|\int_{T_-}^{T_+} \sigma S(u)\,dW(u)\right|\right] \\
&\qquad \le C\mathbb{E}\left[\int_{T_-}^{T_+} |B(u) + S(u)|\,du\right] + C\mathbb{E}\left[\int_{T_-}^{T^+} S(u)^2\,du\right]^{1/2} \\
&\qquad \le C\int_{T_-}^{T_+} \mathbb{E}[B(u) + S(u)]\,du + C\left[\int_{T_-}^{T^+} \mathbb{E}[(B(u) + S(u))^2]\,du\right]^{1/2}.
\end{aligned}
$$

By Lemma 3.1, there exists a constant $K > 0$, independent of (L^+, M^+), such that

$$
\begin{aligned}
&\mathbb{E}\left[\left(X_{t,b,s}^{L^+,M^+}(T_+) - X_{t,b,s}^{L^-,M^-}(T_-)\right)\mathbf{1}_A\right] \\
&\qquad \le K(b+s)(T_+ - T_-) + K(1 + b^2 + s^2)^{1/2}(T_+ - T_-)^{1/2}.
\end{aligned}
$$

Combining this with (8) yields the desired result. □

3.2 Approximation and continuity for $p \le 0$

Note that the only reason why the proof of Proposition 3.3 does not work for $p \le 0$ is because U_p is not subadditive and hence we cannot derive (8). Nevertheless, we can define

$$
U_p^j(x) := U_p(x + 1/j), \qquad \tilde{U}_p^j(x) = U_p^j(x) - U_p^j(0), \qquad x \in [0, \infty),
$$

where $j \in \mathbb{N}$. Note that with this $\tilde{U}_p^j(0) = 0$ and hence $\tilde{U}_p^j$ is subadditive. We denote by $\mathcal{V}^j$ the value function corresponding to $U_p^j(x)$. It can then be verified that $\mathcal{V}^j(\cdot, b, s)$ is also uniformly continuous on $[0, T]$ for all (b, s) fixed. Indeed, in the proof of Proposition 3.3 we only need to replace U_p by $\tilde{U}_p^j$ in (8) (by adding and subtracting $U_p^j(0)$) to make the same proof work.

Lemma 3.4. *Let $p \leq 0$ and fix $(b, s) \in \mathcal{S}$. Then $\lim_{j\to\infty} \mathcal{V}^j(t, b, s) = \mathcal{V}(t, b, s)$ uniformly in t.*

Proof. We consider the case $p < 0$ only. The case $p = 0$ follows similarly. First, note that the family

$$\left\{ U_p\big(X_{t,b,s}^{L,M}(T)\big) \right\}_{t\in[0,T],(L,M)\in\mathcal{A}(t,b,s)} \tag{9}$$

is uniformly integrable. Indeed, choose $q > 1$ arbitrary. Then

$$\mathbb{E}\left[\left| U_p\big(X_{t,b,s}^{L,M}(T)\big) \right|^q \right] = \frac{pq}{|p|^q} \mathbb{E}\left[U_{pq}\big(X_{t,b,s}^{L,M}(T)\big) \right],$$

and since

$$U_{pq}\big(b + \min\{(1-\mu)s, (1+\lambda)s\}\big) \leq \mathbb{E}\left[U_{pq}\big(X_{t,b,s}^{L,M}(T)\big) \right] \leq \varphi_{1,pq,1}(0, b, s)$$

by Lemma 2.3.1 and Lemma 2.2.2 the uniform integrability follows.
Let us now fix $j \in \mathbb{N}$, $(t, b, s) \in [0, T] \times \mathcal{S}$, and $(L, M) \in \mathcal{A}(t, b, s)$ arbitrarily. Let furthermore $\delta > 0$. We calculate

$$\begin{aligned}
0 &\leq \mathbb{E}\left[U_p^j\big(X_{t,b,s}^{L,M}(T)\big) \right] - \mathbb{E}\left[U_p\big(X_{t,b,s}^{L,M}(T)\big) \right] \\
&= \mathbb{E}\left[\Big(U_p^j\big(X_{t,b,s}^{L,M}(T)\big) - U_p\big(X_{t,b,s}^{L,M}(T)\big) \Big) \mathbf{1}_{\{X_{t,b,s}^{L,M}(T) > \delta\}} \right] \\
&\qquad + \mathbb{E}\left[\Big(U_p^j\big(X_{t,b,s}^{L,M}(T)\big) - U_p\big(X_{t,b,s}^{L,M}(T)\big) \Big) \mathbf{1}_{\{X_{t,b,s}^{L,M}(T) \leq \delta\}} \right] \\
&\leq U_p^j(\delta) - U_p(\delta) - \mathbb{E}\left[U_p\big(X_{t,b,s}^{L,M}(T)\big) \mathbf{1}_{\{X_{t,b,s}^{L,M}(T) \leq \delta\}} \right],
\end{aligned}$$

where the last inequality follows from the fact that the difference $U_p^j(x) - U_p(x)$ on $[\delta, \infty)$ is maximal at δ and since $U_p^j \leq 0$. Let now $\varepsilon > 0$. By the uniform integrability of (9) and if δ is small enough, it follows that

$$\mathbb{E}\left[\left| U_p\big(X_{t,b,s}^{L,M}(T)\big) \mathbf{1}_{\{X_{t,b,s}^{L,M}(T) \leq \delta\}} \right| \right] \leq \varepsilon/2,$$

uniformly in t and (L, M). Next, for this choice of δ, there exists $J \in \mathbb{N}$ large enough such that

$$U_p^j(\delta) - U_p(\delta) \leq \varepsilon/2$$

for all $j \geq J$. In total, this implies that

$$\sup_{t\in[0,T]} \sup_{(L,M)\in\mathcal{A}(t,b,s)} \left| \mathbb{E}\left[U_p^j\big(X_{t,b,s}^{L,M}(T)\big) \right] - \mathbb{E}\left[U_p\big(X_{t,b,s}^{L,M}(T)\big) \right] \right| \leq \varepsilon$$

for all $j \geq J$. $\square$

Proposition 3.5. *Let $p \leq 0$ and $(b, s) \in \overline{\mathcal{S}}$. Then $\mathcal{V}(\cdot, b, s)$ is uniformly continuous on $[0, T]$.*

Proof. The uniform continuity for $(b,s) \in \partial\mathcal{S}$ is clear, so let us assume that $(b,s) \in \mathcal{S}$. Let $\varepsilon > 0$, $t \in [0,T]$, and let $(t_n)_{n\in\mathbb{N}}$ be a sequence in $[0,T]$ converging to t. By Lemma 3.4 there exists $j \in \mathbb{N}$ such that

$$\sup_{u\in[0,T]} |\mathcal{V}(u,b,s) - \mathcal{V}^j(u,b,s)| \leq \varepsilon/3$$

and by the continuity of $\mathcal{V}^j$ there exists some $N \in \mathbb{N}$ such that

$$|\mathcal{V}^j(t_n,b,s) - \mathcal{V}^j(t,b,s)| \leq \varepsilon/3$$

for all $n \geq N$. Hence

$$\begin{aligned}
&|\mathcal{V}(t_n,b,s) - \mathcal{V}(t,b,s)| \\
&\leq |\mathcal{V}(t_n,b,s) - \mathcal{V}^j(t_n,b,s)| + |\mathcal{V}^j(t_n,b,s) - \mathcal{V}^j(t,b,s)| + |\mathcal{V}^j(t,b,s) - \mathcal{V}(t,b,s)| \leq \varepsilon
\end{aligned}$$

for all $n \geq N$. □

So, we have in total that $\mathcal{V}$ is locally Lipschitz continuous in (b,s) on $\mathcal{S}$ and uniformly continuous in t. In particular, this implies that $\mathcal{V}$ is jointly continuous in (t,b,s) as we will see in the following corollary.

Corollary 3.6. *The value function $\mathcal{V}$ is continuous on $[0,T] \times \overline{\mathcal{S}}$.*

Proof. Since $\mathcal{V}(t,b,s)$ is locally bounded in a small neighborhood of (b,s) uniformly in t, the local Lipschitz continuity (Lemma 2.3.2) of $\mathcal{V}$ holds uniformly in t. With this, it is easy to prove the joint continuity on $[0,T] \times \mathcal{S}$. Indeed, fix $t \in [0,T]$ and $(b,s) \in \mathcal{S}$ arbitrarily and let (t_n,b_n,s_n) be a sequence converging to (t,b,s). Note that (b_n,s_n) is eventually contained in a compact subset K of $\mathcal{S}$. By the local Lipschitz continuity of $\mathcal{V}$ there exists a constant $L > 0$ such that

$$|\mathcal{V}(u,b_n,s_n) - \mathcal{V}(u,b,s)| \leq L(|b_n - b| + |s_n - s|)$$

for all $u \in [0,T]$ and all n. Hence

$$\begin{aligned}
&\lim_{n\to\infty} |\mathcal{V}(t_n,b_n,s_n) - \mathcal{V}(t,b,s)| \\
&\leq \lim_{n\to\infty} |\mathcal{V}(t_n,b_n,s_n) - \mathcal{V}(t_n,b,s)| + |\mathcal{V}(t_n,b,s) - \mathcal{V}(t,b,s)| \\
&= \lim_{n\to\infty} L(|b_n - b| + |s_n - s|) + |\mathcal{V}(t_n,b,s) - \mathcal{V}(t,b,s)| = 0.
\end{aligned}$$

In order to show that the continuity of $\mathcal{V}$ extends to the boundary of $\mathcal{S}$, choose $(t,b,s) \in [0,T] \times \partial\mathcal{S}$ and let $(t_n,b_n,s_n)_{n\in\mathbb{N}}$ be a sequence converging to (t,b,s). If $s \leq 0$ we have

$$\lim_{n\to\infty} \mathcal{V}(t_n,b_n,s_n) \leq \lim_{n\to\infty} \varphi_{1+\lambda,p,1}(t_n,b_n,s_n) = U_p(0),$$

and if $s > 0$ we have

$$\lim_{n\to\infty} \mathcal{V}(t_n,b_n,s_n) \leq \lim_{n\to\infty} \varphi_{1-\mu,p,1}(t_n,b_n,s_n) = U_p(0).$$

Since $\mathcal{V}(t,b,s) = U_p(0)$ on the boundary this concludes the proof. □

4 Uniqueness of viscosity solutions

The notion of viscosity solutions of partial differential equations can be defined in many equivalent ways. We recall the definition which we will use in what follows. An overview of viscosity solutions and their properties can be found, e.g., in Crandall et al. [8].

4.1 The notion of viscosity solutions

Denote by $\mathbb{S}^2$ the set of symmetric 2×2 matrices with entries in $\mathbb{R}$ and define

$$\begin{aligned} F^{nt}(s,q,r_s,X) &:= -q - \alpha s r_s - \frac{1}{2}\sigma^2 s^2 X_{22}, \\ F^{buy}(r_b,r_s) &:= (1+\lambda) r_b - r_s, \\ F^{sell}(r_b,r_s) &:= -(1-\mu) r_b + r_s, \end{aligned}$$

where $(b,s) \in \mathcal{S}$, $q \in \mathbb{R}$, $r = (r_b, r_s) \in \mathbb{R}^2$, and $X = (X_{ij})_{i,j=1,2} \in \mathbb{S}^2$. Moreover, set

$$F(s,q,r,X) := \min\Big\{F^{nt}(s,q,r_s,X), F^{buy}(r_b,r_s), F^{sell}(r_b,r_s)\Big\}.$$

Then

$$F(s, D_t\mathcal{V}(t,b,s), D_{(b,s)}\mathcal{V}(t,b,s), D^2_{(b,s)}\mathcal{V}(t,b,s)) = 0 \tag{10}$$

corresponds to (1).

Now, let w^* be an upper semicontinuous function on $[0,T] \times \overline{\mathcal{S}}$. For $(t,x) \in [0,T) \times \mathcal{S}$, we define the *superjet* $J^{2,+}w^*(t,x)$ of $w^*(t,x)$ to be the set of all $(q,r,X) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{S}^2$ such that

$$\begin{aligned} \limsup_{\bar{t}\to t,\, \bar{x}\to x} \frac{1}{|t-\bar{t}| + |x-\bar{x}|^2} \Big[& w^*(t,x) - w^*(\bar{t},\bar{x}) - q(t-\bar{t}) \\ & - \langle r, x-\bar{x}\rangle - \frac{1}{2}\langle X(x-\bar{x}), x-\bar{x}\rangle \Big] \le 0, \end{aligned}$$

where we assume that $(\bar{t},\bar{x}) \in [0,T) \times \mathcal{S}$ and where $\langle \cdot,\cdot\rangle$ denotes the inner product on $\mathbb{R}^2$. We define the *subjet* $J^{2,-}w_*(t,x)$ of a lower semicontinuous function $w_*(t,x)$ by setting

$$J^{2,-}w_*(t,x) := -J^{2,+}(-w_*)(t,x).$$

The closure $\overline{J}^{2,+}w^*(t,x)$ of the superjet $J^{2,+}w^*(t,x)$ is defined to be the set of all $(q,r,X) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{S}^2$ for which we can find a sequence $\{(t_j,x_j,q_j,r_j,X_j)\}_{j\in\mathbb{N}}$ with $(t_j,x_j,q_j,r_j,X_j) \in [0,T) \times \mathcal{S} \times J^{2,+}w(t_j,x_j)$ such that

$$\lim_{j\to\infty} (t_j, x_j, w(t_j,x_j), q_j, r_j, X_j) = (t, x, w^*(t,x), q, r, X).$$

The closure $\overline{J}^{2,-}w_*(t,x)$ of the subjet $J^{2,-}w_*(t,x)$ is defined analogously. In terms of the subjets and superjets, a viscosity solution can be defined as follows.

Definition 4.1. *Let $w : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be locally bounded. We denote the upper semicontinuous envelope of w by w^*, and the lower semicontinuous envelope of w by w_*.*

(1) w is called a viscosity subsolution *of* (10) *if, for each $(t,b,s) \in [0,T) \times \mathcal{S}$ and all $(q,r,X) \in J^{2,+}w^*(t,b,s)$, we have*

$$F\big(s,q,r,X\big) \le 0.$$

(2) w is called a viscosity supersolution *of* (10) *if, for each $(t,b,s) \in [0,T) \times \mathcal{S}$ and all $(q,r,X) \in J^{2,-}w_*(t,b,s)$, we have*

$$F\big(s,q,r,X\big) \ge 0.$$

(3) w is called a viscosity solution *of* (10) *if it is both a viscosity subsolution and supersolution.*

Remark 4.2. *Clearly, if w is continuous, then $w = w_* = w^*$ and the definition of a viscosity solution simplifies correspondingly. Hence, in the context of Proposition 2.4, we do not need to rely on the upper and lower semicontinuous envelopes, but can work with the continuous value function $\mathcal{V}$ right away. However, since it is our aim to prove the comparison result (Theorem 4.4) in a more general setting, we resort to the more general notion of a discontinuous viscosity solution in what follows.*

The standard tool in proving the uniqueness of viscosity solutions is Ishii's Lemma, a generalization of the maximum principle for viscosity solutions. We state a specialized version of Ishii's Lemma suitable for our purposes. More general versions can, e.g. be found in Crandall et al. [8, Theorem 3.2] or Pham [23, Lemma 4.4.6 and Remark 4.4.9].

Lemma 4.3 (Ishii's Lemma)**.** *Let u be upper semicontinuous and v be lower semicontinuous on $[0,T]\times\overline{\mathcal{S}}$. Set*

$$\phi_n\big((t,x),(\bar t,\bar x)\big) := u(t,x) - v(\bar t,\bar x) - \frac{n}{2}\big(|t-\bar t|^2 + |x-\bar x|^2\big),$$

where $n\in\mathbb{N}$ and $(t,x),(\bar t,\bar x)\in[0,T)\times\mathcal{S}$. If $((t_0,x_0),(\bar t_0,\bar x_0))$ is a local maximum of ϕ_n, then there exist $X,Y\in\mathbb{S}^2$ such that

$$\begin{aligned}\big(n(t_0-\bar t_0), n(x_0-\bar x_0), X\big) &\in \overline{\mathcal{J}}^{2,+}u(t_0,x_0),\\ \big(n(t_0-\bar t_0), n(x_0-\bar x_0), Y\big) &\in \overline{\mathcal{J}}^{2,-}v(\bar t_0,\bar x_0)\end{aligned}$$

and

$$\begin{pmatrix} X & 0\\ 0 & -Y\end{pmatrix} \le 3n\begin{pmatrix} I & -I\\ -I & I\end{pmatrix},$$

where I denotes the unit matrix in $\mathbb{S}^2$.

4.2 The comparison principle

We are now ready to present the main result of this paper.

Theorem 4.4. *Let $u,v:[0,T]\times\overline{\mathcal{S}}\to\mathbb{R}$ and fix $\varepsilon>0$. Assume that u is an upper semicontinuous viscosity subsolution of (10) and v is a lower semicontinuous viscosity supersolution of (10) such that*

$$U_p(b+\min\{(1-\mu)s,(1+\lambda)s\}) \le u(t,b,s), v(t,b,s) \le \varphi_{\gamma,p,K}(t,b,s) \tag{11}$$

for some $p<1$, $\gamma\in(1-\mu,1+\lambda)$, and $K>1$. If $u(T,b,s)\le v(T,b+\varepsilon,s)$ and $u(t,b,s)\le U_p(0)$ for every $(b,s)\in\partial\mathcal{S}$, then $u(t,b,s)\le v(t,b+\varepsilon,s)$ on $[0,T]\times\overline{\mathcal{S}}$.

Proof. Step 1: Suppose that there exists some $(t^*,b^*,s^*)\in[0,T)\times\overline{\mathcal{S}}$ such that

$$u(t,b^*,s^*) - v(t^*,b^*+\varepsilon,s^*) > 0.$$

Let us note that we have by the growth condition (11)

$$-v(t,b+\varepsilon,s) \le -U_p(b+\varepsilon+\min\{(1-\mu)s,(1+\lambda)s\}) \le -U_p(\varepsilon) < \infty.$$

We therefore have $(b^*,s^*)\notin\partial\mathcal{S}$ since otherwise

$$u(t^*,b^*,s^*) - v(t^*,b^*+\varepsilon,s^*) \le U_p(0) - U_p(\varepsilon) < 0$$

is a contradiction.
Step 2: Define the set

$$\mathcal{D}_\varepsilon := \left\{(t,b,s,\bar t,\bar b,\bar s) : (t,b,s) \in [0,T) \times \mathcal{S}, (\bar t, \bar b - \varepsilon, \bar s) \in [0,T) \times \mathcal{S}\right\}.$$

Now, for some $p' \in (p,1)$ with $p' > 0$, for some $\delta_0 > 0$ to be fixed later, and for every $n \in \mathbb{N}$, we consider the upper semicontinuous functions $\phi_n : \overline{\mathcal{D}}_\varepsilon \to \mathbb{R}$ defined as

$$\begin{aligned}\phi_n(t,b,s,\bar t,\bar b,\bar s) := u(t,b,s) - v(\bar t,\bar b,\bar s) - \delta_0 \varphi_{\gamma,p',K}(t,b,s)\\ - \frac{n}{2}\left(|t-\bar t|^2 + |b - \bar b + \varepsilon|^2 + |s-\bar s|^2\right)\end{aligned}$$

as well as $\phi_\infty : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ given by

$$\phi_\infty(t,b,s) := u(t,b,s) - v(t,b+\varepsilon,s) - \delta_0\varphi_{\gamma,p',K}(t,b,s).$$

Note that if $(t,b,s,\bar t,\bar b,\bar s) \in \overline{\mathcal{D}}_\varepsilon$, then $\bar b + \min\{(1-\mu)\bar s, (1+\lambda)\bar s\} \geq \varepsilon$ and hence $-v(\bar t,\bar b,\bar s) \leq -U_p(\varepsilon) < \infty$. Moreover, since $u \leq \varphi_{\gamma,p,K} \leq \varphi_{\gamma,p',K}$ we have

$$\lim_{|b|,|s|\to\infty} u(t,b,s) - \delta_0\varphi_{\gamma,p',K}(t,b,s) = -\infty$$

which implies that the supremum in

$$M_n := \sup_{\overline{\mathcal{D}}_\varepsilon} \phi_n(t,b,s,\bar t,\bar b,\bar s)$$

is attained at some point $(t_n,b_n,s_n,\bar t_n,\bar b_n,\bar s_n) \in \overline{\mathcal{D}}_\varepsilon$. Also, both $(t_n,b_n,s_n,\bar t_n,\bar b_n,\bar s_n)_{n\in\mathbb{N}}$ and M_n are finite. Similarly, we have

$$M_\infty := \sup_{[0,T]\times\overline{\mathcal{S}}} \phi_\infty(t,b,s) < +\infty$$

and the supremum is attained at some point $(t_\infty,b_\infty,s_\infty) \in [0,T]\times\overline{\mathcal{S}}$. Let us now choose

$$\delta_0 < \frac{u(t^*,b^*,s^*) - v(t^*,b^*+\varepsilon,s^*)}{\varphi_{\gamma,p',K}(t^*,b^*,s^*)}$$

so that we have

$$M_n \geq M_\infty \geq u(t^*,b^*,s^*) - v(t^*,b^*+\varepsilon,s^*) - \delta_0\varphi_{\gamma,p',K}(t^*,b^*,s^*) > 0.$$

Step 3: We want to show that (up to a subsequence)

$$(t_n,b_n,s_n,\bar t_n,\bar b_n,\bar s_n) \to (t_\infty,b_\infty,s_\infty,t_\infty,b_\infty+\varepsilon,s_\infty), \qquad M_n \to M_\infty \tag{12}$$

and

$$n\left(|t_n-\bar t_n|^2 + |b_n-\bar b_n+\varepsilon|^2 + |s_n-\bar s_n|^2\right) \to 0. \tag{13}$$

First, let us recall that the sequence $(t_n,b_n,s_n,\bar t_n,\bar b_n,\bar s_n)_{n\in\mathbb{N}}$ is bounded and hence so is the sequence

$$\left(u(t_n,b_n,s_n) - v(\bar t_n,\bar b_n,\bar s_n) - \delta_0\varphi_{\gamma,p',K}(t_n,b_n,s_n)\right)_{n\in\mathbb{N}}$$

because $u - v - \delta_0\varphi_{\gamma,p',K}$ is upper semicontinuous. Now, since $M_n \geq M_\infty$, we have

$$\begin{aligned}0 &\leq \frac{n}{2}\left(|t_n-\bar t_n|^2 + |b_n-\bar b_n+\varepsilon|^2 + |s_n-\bar s_n|^2\right)\\ &= u(t_n,b_n,s_n) - v(\bar t_n,\bar b_n,\bar s_n) - \delta_0\varphi_{\gamma,p',K}(t_n,b_n,s_n) - M_n\\ &\leq u(t_n,b_n,s_n) - v(\bar t_n,\bar b_n,\bar s_n) - \delta_0\varphi_{\gamma,p',K}(t_n,b_n,s_n) - M_\infty,\end{aligned} \tag{14}$$

which implies that the sequence

$$\left(\frac{n}{2}\left(|t_n-\bar{t}_n|^2+|b_n-\bar{b}_n+\varepsilon|^2+|s_n-\bar{s}_n|^2\right)\right)_{n\in\mathbb{N}}$$

is bounded. We can hence find a subsequence of $(t_n,b_n,s_n,\bar{t}_n,\bar{b}_n,\bar{s}_n)_{n\in\mathbb{N}}$ (which we again denote by $(t_n,b_n,s_n,\bar{t}_n,\bar{b}_n,\bar{s}_n)_{n\in\mathbb{N}}$ for simplicity) such that

$$(t_n,b_n,s_n,\bar{t}_n,\bar{b}_n,\bar{s}_n)\to(\hat{t},\hat{b},\hat{s},\hat{t},\hat{b}+\varepsilon,\hat{s})\in\overline{\mathcal{D}}_\varepsilon.$$

Passing to the limit in (14) now implies that

$$\begin{aligned}
0&\le\limsup_{n\to\infty}\frac{n}{2}\left(|t_n-\bar{t}_n|^2+|b_n-\bar{b}_n+\varepsilon|^2+|s_n-\bar{s}_n|^2\right)\\
&\le\limsup_{n\to\infty}u(t_n,b_n,s_n)-v(\bar{t}_n,\bar{b}_n,\bar{s}_n)-\delta_0\varphi_{\gamma,p',K}(t_n,b_n,s_n)-M_\infty\\
&\le u(\hat{t},\hat{b},\hat{s})-v(\hat{t},\hat{b}+\varepsilon,\hat{s})-\delta_0\varphi_{\gamma,p',K}(\hat{t},\hat{b},\hat{s})-M_\infty\le 0,
\end{aligned}$$

which proves (12) and (13).
Step 4: Next we show that $t_\infty\neq T$ and $(b_\infty,s_\infty)\notin\partial\mathcal{S}$. Suppose on the contrary that we have $t_\infty=T$. Then

$$\begin{aligned}
0<M_\infty&=u(T,b_\infty,s_\infty)-v(T,b_\infty+\varepsilon,s_\infty)-\delta_0\varphi_{\gamma,p',K}(T,b_\infty,s_\infty)\\
&\le u(T,b_\infty,s_\infty)-v(T,b_\infty+\varepsilon,s_\infty)\le 0
\end{aligned}$$

which is a contradiction. Similarly, assuming that $(b_\infty,s_\infty)\in\partial\mathcal{S}$ leads to a contradiction since

$$\begin{aligned}
0<M_\infty&=u(t_\infty,b_\infty,s_\infty)-v(t_\infty,b_\infty+\varepsilon,s_\infty)-\delta_0\varphi_{\gamma,p',K}(t_\infty,b_\infty,s_\infty)\\
&\le U_p(0)-U_p(\varepsilon)<0.
\end{aligned}$$

Hence, $t_\infty\neq T$ and $(b_\infty,s_\infty)\notin\partial\mathcal{S}$. Additionally, since $t_n,\bar{t}_n\to t_\infty$, $b_n\to b_\infty$, $\bar{b}_n\to b_\infty+\varepsilon$, and $s_n,\bar{s}_n\to s_\infty$, we have $(t_n,b_n,s_n,\bar{t}_n,\bar{b}_n,\bar{s}_n)\in\mathcal{D}_\varepsilon$ for n sufficiently large.
Step 5: Let n be large enough such that $(t_n,b_n,s_n,\bar{t}_n,\bar{b}_n,\bar{s}_n)\in\mathcal{D}_\varepsilon$. Then we can apply Theorem 4.3 (Ishii's Lemma) to the upper semicontinuous function $u-\delta_0\varphi_{\gamma,p',K}$ and the lower semicontinuous function v to obtain the existence of $X,Y\in\mathbb{S}^2$ such that

$$\begin{aligned}
&(n(t_n-\bar{t}_n),(n(b_n-\bar{b}_n+\varepsilon),n(s_n-\bar{s}_n)),X)\in\overline{J}^{2,+}[u-\delta_0\varphi_{\gamma,p',K}](t_n,b_n,s_n),\\
&(n(t_n-\bar{t}_n),(n(b_n-\bar{b}_n+\varepsilon),n(s_n-\bar{s}_n)),Y)\in\overline{J}^{2,-}v(\bar{t}_n,\bar{b}_n,\bar{s}_n),
\end{aligned}$$

and such that

$$\begin{pmatrix}X&0\\0&-Y\end{pmatrix}\le 3n\begin{pmatrix}I&-I\\-I&I\end{pmatrix}. \tag{15}$$

Since $\varphi_{\gamma,p',K}$ is smooth, it follows that

$$\begin{aligned}
\Big(&n(t_n-\bar{t}_n)+\delta_0\frac{\partial}{\partial t}\varphi_{\gamma,p',K}(t_n,b_n,s_n),\\
&\Big(n(b_n-\bar{b}_n+\varepsilon)+\delta_0\frac{\partial}{\partial b}\varphi_{\gamma,p',K}(t_n,b_n,s_n),\\
&\qquad n(s_n-\bar{s}_n)+\delta_0\frac{\partial}{\partial s}\varphi_{\gamma,p',K}(t_n,b_n,s_n)\Big),\\
&\qquad\qquad X+\delta_0D^2_{(b,s)}\varphi_{\gamma,p',K}(t_n,b_n,s_n)\Big)\in\overline{J}^{2,+}u(t_n,b_n,s_n).
\end{aligned} \tag{16}$$

For ease of notation, let us define

$$r_t^n := n(t_n - \bar{t}_n), \quad r_b^n := n(b_n - \bar{b}_n + \varepsilon), \quad r_s^n := n(s_n - \bar{s}_n).$$

Step 6: Since u is a viscosity subsolution of (10), we can use (16) and the linearity of the operators $\mathcal{L}^{nt}$, $\mathcal{L}^{buy}$, and $\mathcal{L}^{sell}$ to obtain

$$\begin{aligned}\min\Big\{F^{nt}(s_n, r_t^n, r_s^n, X) + \delta_0 \mathcal{L}^{nt}\varphi_{\gamma,p',K}(t_n, b_n, s_n),\\ F^{buy}(r_b^n, r_s^n) + \delta_0 \mathcal{L}^{buy}\varphi_{\gamma,p',K}(t_n, b_n, s_n),\\ F^{sell}(r_b^n, r_s^n) + \delta_0 \mathcal{L}^{sell}\varphi_{\gamma,p',K}(t_n, b_n, s_n)\Big\} \leq 0.\end{aligned} \tag{17}$$

Similarly, since v is a viscosity supersolution we have

$$\min\Big\{F^{nt}(\bar{s}_n, r_t^n, r_s^n, Y), F^{buy}(r_b^n, r_s^n), F^{sell}(r_b^n, r_s^n)\Big\} \geq 0. \tag{18}$$

Our aim is to show that (17) and (18) lead to a contradiction.
Suppose first that we have in (17)

$$F^{buy}(r_b^n, r_s^n) + \delta_0 \mathcal{L}^{buy}\varphi_{\gamma,p',K}(t_n, b_n, s_n) \leq 0.$$

But since by (18) we have $F^{buy}(r_b^n, r_s^n) \geq 0$ it follows that

$$\delta_0 \mathcal{L}^{buy}\varphi_{\gamma,p',K}(t_n, b_n, s_n) \leq 0$$

which is a contradiction since $\varphi_{\gamma,p',K}$ is a strict supersolution of the HJB equation by Lemma 2.2.1 and since $\gamma \in (1-\mu, 1+\lambda)$ and $K > 1$. In a similar fashion, assuming that

$$F^{sell}(r_b^n, r_s^n) + \delta_0 \mathcal{L}^{sell}\varphi_{\gamma,p',K}(t_n, b_n, s_n) \leq 0$$

leads to a contradiction. We must therefore have

$$F^{nt}(s_n, r_t^n, r_s^n, X) + \delta_0 \mathcal{L}^{nt}\varphi_{\gamma,p',K}(t_n, b_n, s_n) \leq 0.$$

Thus (18) implies that

$$F^{nt}(s_n, r_t^n, r_s^n, X) - F^{nt}(\bar{s}_n, r_t^n, r_s^n, Y) + \delta_0 \mathcal{L}^{nt}\varphi_{\gamma,p',K}(t_n, b_n, s_n) \leq 0.$$

Direct computations show that

$$\begin{aligned}F^{nt}(s_n, r_t^n, r_s^n, X) - F^{nt}(\bar{s}_n, r_t^n, r_s^n, Y) &= -r_t^n - \alpha s_n r_s^n - \frac{1}{2}\sigma^2 s_n^2 X_{22} + r_t^n + \alpha \bar{s}_n r_s^n + \frac{1}{2}\sigma^2 \bar{s}_n^2 Y_{22}\\ &= -\alpha n|s_n - \bar{s}_n|^2 - \frac{1}{2}\sigma^2[s_n^2 X_{22} - \bar{s}_n^2 Y_{22}].\end{aligned}$$

By (15) we have $s_n^2 X_{22} - \bar{s}_n^2 Y_{22} \leq 3n|s_n - \bar{s}_n|^2$ and, therefore,

$$\begin{aligned}F^{nt}(s_n, r_t^n, r_s^n, X) - F^{nt}(\bar{s}_n, r_t^n, r_s^n, Y) &= -\alpha n|s_n - \bar{s}_n|^2 - \frac{1}{2}\sigma^2[s_n^2 X_{22} - \bar{s}_n^2 Y_{22}]\\ &\geq -\alpha n|s_n - \bar{s}_n|^2 - \frac{3}{2}\sigma^2 n|s_n - \bar{s}_n|^2\\ &\geq -2\max\{|\alpha|, \frac{3}{2}\sigma^2\} n|s_n - \bar{s}_n|^2.\end{aligned}$$

We therefore have

$$\begin{aligned}0 &\geq F^{nt}(s_n, r_t^n, r_s^n, X) - F^{nt}(\bar{s}_n, r_t^n, r_s^n, Y) + \delta_0 \mathcal{L}^{nt} \varphi_{\gamma,p',K}(t_n, b_n, s_n) \\ &\geq -2\max\{\alpha, \frac{3}{2}\sigma^2\} n|s_n - \bar{s}_n|^2 + \delta_0 \mathcal{L}^{nt} \varphi_{\gamma,p',K}(t_n, b_n, s_n)\end{aligned}$$

and since $n|s_n - \bar{s}_n|^2 \to 0$ as $n \to \infty$ we obtain

$$0 \geq \delta_0 \mathcal{L}^{nt} \varphi_{\gamma,p',K}(t_\infty, b_\infty, s_\infty) > 0$$

which is again a contradiction and hence finishes the proof. □

The comparison theorem implies the following uniqueness result. In particular, the value function $\mathcal{V}$ is the unique viscosity solution of the HJB equation.

Corollary 4.5. *Let u, v be upper semicontinuous viscosity solutions of the HJB equation satisfying*

$$U_p(b + \min\{(1-\mu)s, (1+\lambda)s\}) \leq u(t,b,s), v(t,b,s) \leq \varphi_{\gamma,p,K}(t,b,s)$$

with $u(t,b,s) = v(t,b,s) = U_p(0)$ on $\partial\mathcal{S}$ and

$$u^*(T,b,s) = u_*(T,b,s) = \mathcal{V}(T,b,s) = v^*(T,b,s) = v_*(T,b,s). \tag{19}$$

Then $u = v$.

Proof. Let $\varepsilon > 0$ be arbitrary. Since u and v are viscosity solutions, u_* is a viscosity supersolution and $v^* = v$ is a viscosity subsolution. Moreover, by (19),

$$\begin{aligned}v(T,b,s) &= U_p(b + \min\{(1-\mu)s, (1+\lambda)s\}) \\ &\leq U_p(b + \varepsilon + \min\{(1-\mu)s, (1+\lambda)s\}) = u(T, b+\varepsilon, s) = u_*(T, b+\varepsilon, s).\end{aligned}$$

Hence $v(t,b,s) \leq u_*(t, b+\varepsilon, s) \leq u(t, b+\varepsilon, s)$ everywhere by Theorem 4.4. Sending ε to zero shows that $v \leq u$ by the upper semicontinuity of u. Switching the roles of v and u shows the reverse inequality. □

5 Acknowledgments

We would like to thank two anonymous referees for their fruitful comments and suggestions. This paper was finalized while the second author was on sabbatical at HU Berlin (visiting Prof. Kurt Helmes) and TU Kaiserslautern (visiting Prof. Ralf Korn), respectively. The second author gratefully acknowledges the hospitality of both hosts and thanks his colleagues at DCU for making this sabbatical possible.

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