Embedding partial Steiner triple systems with few triples

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Abstract

It was proved in 2009 that any partial Steiner triple system of order u has an embedding of order v for each admissible integer $v \geq 2u+1$. This result is best-possible in the sense that, for each $u \geq 9$, there exists a partial Steiner triple system of order u that does not have an embedding of order v for any v < 2u+1. Many partial Steiner triple systems do have embeddings of orders smaller than 2u+1, but little has been proved about when these embeddings exist. In this paper we construct embeddings of orders less than 2u+1 for partial Steiner triple systems with few triples. In particular, we show that a partial Steiner triple system of order $u \geq 62$ with at most $\frac{u^2}{50} - \frac{11u}{100} - \frac{116}{75}$ triples has an embedding of order v for each admissible integer $v \geq \frac{8u+17}{5}$.

1 Introduction

A Steiner triple system of order v is a pair (V, \mathcal{B}) where V is a set with v elements and \mathcal{B} is a set of (unordered) triples chosen from V such that every (unordered) pair of elements of V is a subset of exactly one triple. In 1847, Kirkman [16] proved that there exists a Steiner triple system of order v if and only if $v \equiv 1, 3 \pmod{6}$. Such integers are called admissible. A partial Steiner triple system of order v is a pair v where v is a set with v elements and v is a set of triples chosen from v such that every pair of elements of v is a subset of at most one triple. The leave of a partial Steiner triple system v is the graph v with vertex set v and edge set given by v if and only if the pair v occurs in no triple in v. An embedding of a partial Steiner triple system v is a (complete) Steiner triple system v such that v is a v and v is a v if v is a completion. The embedding spectrum of a partial Steiner triple system is the set of all orders for which it has an embedding. A triangle

decomposition of a graph G is a set of triangles whose edge sets partition the edge set of G. A Steiner triple system of order v can be considered as a triangle decomposition of a complete graph of order v.

In 1977 Lindner [19] conjectured that any partial Steiner triple system of order u has an embedding of order v for each $v \ge 2u + 1$ such that $v \equiv 1, 3 \pmod{6}$. This is best-possible in the sense that, for each $u \ge 9$, there exists a partial Steiner triple system of order u that does not have an embedding of order v for any v < 2u + 1 [10]. Most existing results on embedding partial Steiner triple systems concern embeddings of order at least 2u + 1. In particular, a series of results [23, 18, 1, 3] guaranteed the existence of progressively smaller embeddings, culminating in a complete proof of Lindner's conjecture in [4].

Theorem 1 ([4]). Any partial Steiner triple system of order u has an embedding of order v for each $v \ge 2u + 1$ such that $v \equiv 1, 3 \pmod{6}$.

Of course, many partial Steiner triple systems do have embeddings of order less than 2u+1. Much less is known about when these embeddings exist and they have only been constructed for limited classes of partial Steiner triple systems. Colbourn, Colbourn and Rosa [9] showed that there is a completion of every partial Steiner triple system of order u which has exactly u-2 triples and the property that every triple contains at least one of two specified elements. Bryant [2] gave necessary and sufficient conditions for a partial Steiner triple system of order u every vertex of whose leave has degree d or 0 to have an embedding of order u+d, and found the complete embedding spectrum in the case d=2. Bryant, Maenhaut, Quinn and Webb [7] found the embedding spectra of all partial Steiner triple systems of order 10 with 3-regular leaves. The main result of [6] determines the embedding spectra of partial Steiner triple systems whose leaves are complete bipartite graphs. Emphasising the difficulty of this problem in general, Colbourn [8] has shown that it is NP-complete to decide whether a partial Steiner triple system has an embedding of order less than 2u+1. Recently, numerous embeddings of order less than 2u+1 have been constructed for Steiner triple systems whose leaves have few edges and low maximum degree [15].

This paper focusses on embeddings of partial Steiner triple systems with few triples. We show that every partial Steiner triple system of order at least 62 which satisfies an upper bound on its number of triples has an embedding of order v for each admissible integer $v \ge \frac{8u+17}{5}$.

Theorem 2. Any partial Steiner triple system of order $u \ge 62$ with at most $\frac{u^2}{50} - \frac{11u}{100} - \frac{116}{75}$ triples has an embedding of order v for each integer $v \ge \frac{8u+17}{5}$ such that $v \equiv 1, 3 \pmod{6}$.

The bound of $\frac{8u+17}{5}$ is an artefact of the techniques applied in constructing the embeddings. The lower bound on u could be reduced, but at the expense of further complicating the proof. As a complement to this result, it follows from Corollary 17 that for sufficiently large values of u there is a partial Steiner triple system of order u with at most $\frac{u^2}{50} - \frac{11u}{100} - \frac{116}{75}$ triples that does not have an embedding of order v for any v < (1.346)u.

In fact, we will prove a result stronger than Theorem 2 concerning triangle decompositions of graphs of order v having many vertices of degree v-1.

Theorem 3. Let G be a graph such that every vertex of G has even degree and $|E(G)| \equiv 0 \pmod{3}$. If G has order $v \geq 103$, $|E(G)| \geq {v \choose 2} - (\frac{3v^2}{128} - \frac{27v}{64} - \frac{409}{128})$ and at least $\frac{3v+17}{8}$ vertices of G have degree v-1, then there is a triangle decomposition of G.

It is worth noting that this result settles the following conjecture, due to Nash-Williams [20], in the case of graphs of this very specific form (it also includes some cases not covered by the conjecture).

Conjecture ([20]). Let G be a graph such that every vertex of G has even degree and $|E(G)| \equiv 0 \pmod{3}$. If G has minimum degree at least $\frac{3}{4}|V(G)|$, then there is a triangle decomposition of G.

2 Background and Notation

In this section we introduce some terminology and concepts which we will need in later sections.

All graphs considered in this paper are simple and loopless. For a nonnegative integer v, a complete graph of order v will be denoted by K_v and, for a given set V, the complete graph on vertex set V will be denoted by K_V . For nonnegative integers u and w, a complete bipartite graph with parts of size u and w will be denoted by $K_{u,w}$ and, for disjoint sets U and W, the complete bipartite graph with parts U and W will be denoted by $K_{U,W}$. For graphs G and H, we denote by $G \cup H$ the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$, we denote by $G \cap H$ the graph with vertex set $V(G) \cap V(H)$ and edge set $E(G) \cap E(H)$, we denote by $G \cap H$ the graph with vertex set $V(G) \cap V(H)$ and edge set $E(G) \cap E(H)$, and, if V(G) and V(H) are disjoint, we denote by $G \vee H$ the graph $G \cup H \cup K_{V(G),V(H)}$ (our use of this last notation will imply that V(G) and V(H) are disjoint).

For a graph G and a vertex x of G, the set of vertices adjacent in G to x is denoted by $Nbd_G(x)$, the degree in G of x is denoted by $deg_G(x)$, and the maximum degree of G is denoted by $\Delta(G)$. A graph G is even if $deg_G(x)$ is even for each $x \in V(G)$, and is r-regular if $deg_G(x) = r$

for each $x \in V(G)$. A matching is a 1-regular graph, a cycle is a connected 2-regular graph, and a path is a connected graph with two vertices of degree 1 and every other vertex of degree 2 (we allow trivial matchings with no vertices or edges but demand that cycles and paths have at least two vertices). The length of a cycle or path is the number of edges it has. We let (x_1, \ldots, x_p) denote the cycle with vertex set $\{x_1, \ldots, x_p\}$ and edge set $\{x_1x_2, x_2x_3, \ldots, x_{p-1}x_p, x_px_1\}$. A cycle of length 3 is called a triangle.

A decomposition of a graph G is a set of subgraphs of G whose edge sets partition the edge set of G. If every graph in a decomposition is a cycle we call the decomposition a cycle decomposition and if every graph in a decomposition is a triangle we call the decomposition a triangle decomposition. A packing of a graph G is a decomposition of a subgraph H of G, and the graph G - H is the leave of the packing. Note that if L is the leave of a cycle packing of a graph G, then $\deg_L(x) \equiv \deg_G(x) \pmod{2}$ for each $x \in V(G)$. It is clear that a Steiner triple system of order u can be considered as a triangle decomposition of K_v and that a partial Steiner triple system of order u can be considered as a triangle packing of K_u . If (U, A) is a partial Steiner triple system of order u and u is the leave of u, then an embedding of u of order u can be considered as a triangle decomposition of u.

A proper edge colouring of a graph G with c colours can be considered as a decomposition of G into c matchings. The minimum number of matchings in any such decomposition of G is called the *chromatic index* of G and is denoted by $\chi'(G)$. Vizing [24] proved that $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$ for any graph G.

Definition. If L is a graph of order u and w is a nonnegative integer we will say that (L, w) is admissible if

- u + w is odd;
- $\deg_L(x) \equiv u + 1 \pmod{2}$ for each $x \in V(L)$; and
- $|E(L)| + uw + \frac{w(w-1)}{2} \equiv 0 \pmod{3}$.

When L is the leave of a partial Steiner triple system, (L, w) is admissible if and only if $u + w \equiv 1, 3 \pmod{6}$. The following lemma gives some well-known necessary conditions for the existence of an embedding of a partial Steiner triple system.

Lemma 4 ([2]). Let L be a graph and let w be a nonnegative integer such that there is a triangle decomposition of $L \vee K_w$. Then (L, w) is admissible and there is a subgraph L' of L such that

• there is a triangle decomposition of L - L';

- $\chi'(L') \leq w$; and
- $|E(L')| \ge \frac{w(u-w+1)}{2}$.

Definition. Let \mathcal{P} and \mathcal{P}' be cycle packings of a graph G and let S be a subset of V(G). We say that \mathcal{P} and \mathcal{P}' are equivalent on S if we can write $\mathcal{P} = \{C_1, \ldots, C_t\}$ and $\mathcal{P}' = \{C'_1, \ldots, C'_t\}$ such that

- for $i \in \{1, ..., t\}, |E(C_i)| = |E(C_i')|$;
- for each $x \in S$ and for $i \in \{1, ..., t\}$, $x \in V(C_i)$ if and only if $x \in V(C'_i)$; and
- for each $xy \in E(G)$ such that $x, y \in S$ and for $i \in \{1, ..., t\}$, $xy \in E(C_i)$ if and only if $xy \in E(C_i')$.

3 Preliminary Results

Most of our effort will go into proving Lemma 15 which shows that, for any sufficiently dense graph L of order $u \geq 62$, there is a triangle decomposition of $L \vee K_w$ if (L, w) is admissible and $w = \frac{3u+k}{5}$ for some $k \in \{17, 19, 21, 23\}$ (see Section 4). Once this is established the general result follows relatively easily (see Section 5). The crucial point is that any odd integer v can be written as u'+w' for some integers u' and w' such that $w' = \frac{3u'+k'}{5}$ for some $k' \in \{17, 19, 21, 23\}$.

Note that, if u and w are integers such that $w = \frac{3u+k}{5}$ for some $k \in \{17, 19, 21, 23\}$, then $u \not\equiv 0 \pmod{5}$ and

$$k = \begin{cases} 17, & \text{if } u \equiv 1 \pmod{5}; \\ 19, & \text{if } u \equiv 2 \pmod{5}; \\ 21, & \text{if } u \equiv 3 \pmod{5}; \\ 23, & \text{if } u \equiv 4 \pmod{5}. \end{cases}$$

It is a routine matter to verify inequalities in u and w under these conditions, and we will often leave such tasks to the reader in what follows. Note in particular that, under these conditions, $w \equiv u + 1 \pmod{2}$ and if $u \geq 11$ then $10 \leq w \leq u - 1$.

In this section of the paper we establish a number of preliminary results that will be required for our main construction in Section 4. We first require two results, namely Lemmas 6 and 7, that, given a graph, find a subgraph of it whose components are even-length paths and even-length cycles. The proofs of both of these lemmas require Lemma 5, which is a well-known and easy consequence of the max-flow min-cut theorem [11, 13].

Lemma 5. Let H be a bipartite graph with parts A and B. Then H contains two edge-disjoint matchings M_1 and M_2 with $|E(M_1)| + |E(M_2)| = 2|A| - d$ if and only if $2|S| - \sum_{x \in B} \max(2, |\operatorname{Nbd}_H(x) \cap S|) \leq d$ for each subset S of A.

Proof. Let a and b be vertices not in V(H). Let N be a flow network on node set $V(H) \cup \{a, b\}$ with source a and sink b, such that the underlying graph of N is $K_{\{a\},A} \cup H \cup K_{B,\{b\}}$, each edge in $K_{\{a\},A}$ has capacity 2 and is directed from a to a vertex in A, each edge in H has capacity 1 and is directed from a vertex in A to a vertex in B, and each edge in $K_{B,\{b\}}$ has capacity 2 and is directed from a vertex in B to b. There is an obvious correspondence between k-flows in N and subgraphs of H with maximum degree 2 and size k. Further, any such subgraph of H can be decomposed into two edge-disjoint matchings. So it follows from the max-flow min-cut theorem that H contains two edge-disjoint matchings M_1 and M_2 such that $|E(M_1)| + |E(M_2)| = 2|A| - d$ if and only if the capacity over every a-b cut in N is at least 2|A| - d. It is not difficult to check that this is the case if and only if $2|S| - \sum_{x \in B} \max(2, |\mathrm{Nbd}_H(x) \cap S|) \leq d$ for each subset S of A.

Lemma 6. Let L^* be a graph of order $u \geq 22$, and let w be an integer such that $w = \frac{3u+k}{5}$ for some $k \in \{17, 19, 21, 23\}$. If $|E(L^*)| \geq \frac{w(u-w+1)}{2}$ and $\Delta(L^*) \leq w-8$, then L^* contains a subgraph F such that |E(F)| = u - w + 1 and each component of F is an even-length path or an even-length cycle.

Proof. Among all partitions of $V(L^*)$ into one part of size $\frac{u-w+1}{2}$ and one part of size $\frac{u+w-1}{2}$, let $\{A,B\}$, where $|A|=\frac{u-w+1}{2}$ and $|B|=\frac{u+w-1}{2}$, be one such that $|E(L^*)\cap E(K_B)|$ is minimised. Let R be the largest subset of A for which $L^*\cap K_{R,B}$ contains two edge-disjoint matchings M_1 and M_2 such that $|E(M_1)|=|E(M_2)|=|R|$. If R=A it can be seen that $M_1\cup M_2$ is a subgraph of L^* with the required properties, so we can assume that $R\neq A$. It is not difficult to see from the definition of R that the sizes of any two edge-disjoint matchings in $L^*\cap K_{A,B}$ add to at most |A|+|R|. It follows by Lemma 5 that there is a subset S of A such that $d\geq |A|-|R|$, where $d=2|S|-\sum_{x\in B}\max(2,|\mathrm{Nbd}_{L^*}(x)\cap S|)$.

Let s = |S|, $B' = \{x \in B : |\operatorname{Nbd}_{L^*}(x) \cap S| = 1\}$ and $B'' = \{x \in B : |\operatorname{Nbd}_{L^*}(x) \cap S| \ge 2\}$. In the case $s \ge 2$, it can be seen that

$$\sum_{x \in S} |\text{Nbd}_{L^*}(x) \cap B| \le s|B''| + |B'| \le s(s - \frac{d}{2}),$$

where the second inequality follows from 2|B''| + |B'| = 2s - d. Hence, in this case, $|\text{Nbd}_{L^*}(y) \cap B| \le s - \frac{d}{2}$ for some $y \in S$. In the case s = 1, it is easy to see that $|\text{Nbd}_{L^*}(y) \cap B| \le 1$ for the unique vertex $y \in S$. In either case, if there were a vertex z in B such that $|\text{Nbd}_{L^*}(z) \cap B| \le 1$

 $B| > |\operatorname{Nbd}_{L^*}(y) \cap B|$ then the partition $\{(A \setminus \{y\}) \cup \{z\}, (B \setminus \{z\}) \cup \{y\}\}\}$ would contradict our definition of $\{A, B\}$. If $|\operatorname{Nbd}_{L^*}(y) \cap B| \leq 1$, this implies that $|E(L^*)| \leq \Delta(L^*)|A| + \frac{1}{2}|B| = (w-8)\frac{u-w+1}{2} + \frac{u+w-1}{4}$, which can be seen to contradict $|E(L^*)| \geq \frac{w(u-w+1)}{2}$ for $u \geq 11$. Thus we can assume that $s \geq 2$, that $|\operatorname{Nbd}_{L^*}(x) \cap B| \leq s - \frac{d}{2}$ for each $x \in B$, and, since $\sum_{x \in B} \max(2, |\operatorname{Nbd}_{L^*}(x) \cap S|) \geq |\operatorname{Nbd}_{L^*}(y) \cap B| \geq 2$, that $d \leq 2s - 2$.

Let F be a subgraph of L^* such that $M_1 \cup M_2$ is a subgraph of F, each component of F is an even-length path or an even-length cycle, and subject to these conditions F is of maximum size. If $|E(F)| \ge u - w + 1$ then we are finished, so we can suppose for a contradiction that $|E(F)| \le u - w - 1$ (note that |E(F)| and u - w + 1 are even). Let $\{B_0, B_1, B_2\}$ be the partition of B such that $B_i = \{x \in B : \deg_F(x) = i\}$ for $i \in \{0, 1, 2\}$. Define a partition $\{E_1, E_2, E_3, E_4, E_5\}$ of $E(L^*)$ as follows.

$$E_{1} = \{xy \in E(L^{*}) : x \in A \setminus S\}$$

$$E_{2} = \{xy \in E(L^{*}) : x \in S, y \notin A \setminus S\}$$

$$E_{3} = \{xy \in E(L^{*}) : x \in B_{2}, y \notin A\}$$

$$E_{4} = \{xy \in E(L^{*}) : x \in B_{0}, y \notin A \cup B_{2}\}$$

$$E_{5} = \{xy \in E(L^{*}) : x, y \in B_{1}\}$$

We will find an upper bound on the size of each part of this partition and use these to obtain a contradiction to $|E(L^*)| \ge \frac{w(u-w+1)}{2}$.

Because $\Delta(L^*) \leq w - 8$, we have $|E_1| \leq (\frac{u-w+1}{2} - s)(w - 8)$. We saw earlier that $\sum_{x \in S} |\operatorname{Nbd}_{L^*}(x) \cap B| \leq s(s - \frac{d}{2})$, and thus $|E_2| \leq \binom{s}{2} + s(s - \frac{d}{2})$. We also saw that $|\operatorname{Nbd}_{L^*}(x) \cap B| \leq s - \frac{d}{2}$ for each $x \in B$, and so we have that $|E_3| \leq |B_2|(s - \frac{d}{2})$ and $|E_5| \leq \frac{1}{2}|B_1|(s - \frac{d}{2})$. If there were a vertex z_1 in B_0 which was incident with two edges $z_1 z_2$ and $z_1 z_3$ such that $z_2, z_3 \in B_0 \cup B_1$, then the graph with edge set $E(F) \cup \{z_1 z_2, z_1 z_3\}$ would contradict our definition of F, and it follows that $|E_4| \leq |B_0|$. Thus, because $|E_1| + |E_2| + |E_3| + |E_4| + |E_5| = |E(L^*)| = \frac{w(u-w+1)}{2}$,

$$\theta = \frac{w(u-w+1)}{2} - \left(\left(\frac{u-w+1}{2} - s \right)(w-8) + \left(\frac{s}{2} \right) + s(s-\frac{d}{2}) \right) + |B_2|(s-\frac{d}{2}) + |B_0| + \frac{1}{2}|B_1|(s-\frac{d}{2}))$$

is nonpositive. We will obtain a contradiction by showing that θ is positive.

Because $|B_0| + |B_1| + |B_2| = |B| = \frac{u+w-1}{2}$, we can substitute $|B_0| = \frac{u+w-1}{2} - |B_1| - |B_2|$ and simplify to obtain

$$\theta = \frac{1}{2}(s(2w+d-3s-15)-(2|B_2|+|B_1|)(s-\frac{d}{2}-1)+|B_1|+7u-9w+9).$$

Note that $2|R| + 2|B_2| + |B_1| = 2|E(F)| \le 2(u - w - 1)$ and thus, because $d \ge |A| - |R| = \frac{u - w + 1}{2} - |R|$, we have $2|B_2| + |B_1| \le u - w - 3 + 2d$. Using this fact, along with $|B_1| \ge 0$ and

simplifying we see

$$\theta \ge \frac{1}{4}d(u - w - 2s + 2d + 1) + \frac{1}{2}s(3w - u - 3s - 12) + 4u - 5w + 3,$$

(note that $s - \frac{d}{2} - 1 \ge 0$ because $d \le 2s - 2$). Since $s \le \frac{u - w + 1}{2}$ and $d \ge 1$, we have that $\frac{1}{4}d(u - w - 2s + 2d + 1) \ge \frac{1}{4}(u - w - 2s + 3)$, and consequently we have

$$\theta \ge \frac{1}{2}s(3w - u - 3s - 13) + \frac{1}{4}(17u - 21w + 15).$$

Since $2 \le s \le \frac{u-w+1}{2}$ and $u \ge 22$, it can be seen that $6 \le 3s \le 3w - u - 19$, and consequently that $\frac{1}{2}s(3w - u - 3s - 13) \ge 3w - u - 19$. Thus,

$$\theta \ge \frac{1}{4}(13u - 9w - 61)$$

and so, since $u \ge 14$, θ is positive as required.

Lemma 7. Let L^* be a graph of order $u \geq 16$, and let w be an integer such that $w = \frac{3u+k}{5}$ for some $k \in \{17, 19, 21, 23\}$. If $|E(L^*)| \geq \frac{w(u-w+1)}{2}$, $\Delta(L^*) \leq w-8$, and A is a set of $\frac{u-w+1}{2}$ vertices of L^* such that $\deg_{L^*}(x) \geq u-w$ for each $x \in A$, then L^* contains a subgraph F such that |E(F)| = u-w+1, $A \subseteq V(F)$, and each component of F is an even-length path or an even-length cycle.

Proof. Let $B = V(L^*) \setminus A$ and consider the graph $L^* \cap K_{A,B}$. Since $\deg_{L^*}(x) \geq u - w$ for each $x \in A$, we have that $|\operatorname{Nbd}_{L^*}(x) \cap B| \geq u - w - (|A| - 1) \geq \frac{u - w + 1}{2}$ for each $x \in A$. Clearly then, $\sum_{x \in B} \max(2, |\operatorname{Nbd}_{L^*}(x) \cap S|) \geq u - w + 1 \geq 2|S|$ for each $S \subseteq A$ with $|S| \geq 2$, and $\sum_{x \in B} \max(2, |\operatorname{Nbd}_{L^*}(x) \cap S|) \geq \frac{u - w + 1}{2} \geq 2|S|$ for each $S \subseteq A$ with |S| = 1 (note that $u \geq 16$ implies $\frac{u - w + 1}{2} \geq 2$). Thus, by Lemma 5, there are two edge-disjoint matchings M_1 and M_2 in L^* such that $|E(M_1)| + |E(M_2)| = 2|A| = u - w + 1$. It can be seen that $M_1 \cup M_2$ is a subgraph of L^* with the required properties.

We will also require Lemma 10 which is similar to Lemma 5.1 of [4], except that it concerns general cycle decompositions of graphs rather than simply partial Steiner triple systems. Lemma 9 is used only in the proof of Lemma 10. These two lemmas are proved using path switching techniques as encapsulated below in Lemma 8 which appears as Lemma 9 in [15].

Lemma 8 ([15]). Let G be a graph, let \mathcal{P} be a cycle packing of G, let L be the leave of \mathcal{P} , and let a and b be vertices in G such that $\operatorname{Nbd}_G(a) = \operatorname{Nbd}_G(b)$. Then there exists a partition of the set $(\operatorname{Nbd}_L(a) \cup \operatorname{Nbd}_L(b)) \setminus ((\operatorname{Nbd}_L(a) \cap \operatorname{Nbd}_L(b)) \cup \{a,b\})$ into pairs such that for each pair $\{x,y\}$ of the partition, there exists a cycle packing $\mathcal{P}_{\{x,y\}}$ of G which is equivalent to \mathcal{P} on $V(G) \setminus \{a,b\}$ and whose leave $L_{\{x,y\}}$ differs from L only in that ax, ay, bx and by are edges in $L_{\{x,y\}}$ if and only if they are not edges in L.

Lemma 9. Let L be a graph, let T be a set of vertices which is disjoint from V(L), let \mathcal{P} be a cycle packing of $L \vee K_T$, let G be the leave of \mathcal{P} , and let $a, b, c \in T$ and $d \in V(L)$ be distinct vertices such that $ab, cd \in E(G)$. Let $\mathcal{B} = \{C \in \mathcal{P} : C = (b, y, z) \text{ for some } y \in V(L), z \in T\}$ and suppose that each edge in $E(K_T) \setminus E(G)$ which is incident with b is in a triangle in \mathcal{B} and that $|\operatorname{Nbd}_G(x) \cap T| \geq 1$ for each vertex x in V(L) which is in a triangle in \mathcal{B} . Then there is a cycle packing of $L \vee K_T$ which is equivalent to \mathcal{P} on $V(L) \cup \{b\}$ and whose leave contains the edges bc and cd.

Proof. If $bc \in E(G)$ then we are finished immediately, so we may assume that $bc \notin E(G)$. Let $y_0 = d$ and $z_1 = c$. The hypotheses of the lemma guarantee that we can create a sequence $y_0, z_1, y_1, \ldots, z_t, y_t, z_{t+1}$ of vertices in $V(L) \cup T$ such that

- $y_i z_{i+1} \in E(G)$ for $i \in \{0, ..., t\}$;
- $(b, z_i, y_i) \in \mathcal{P}$ for $i \in \{1, ..., t\}$; and
- z_1, \ldots, z_t, b are distinct vertices in T, y_0, \ldots, y_t are distinct vertices in U, and either $bz_{t+1} \in E(G)$ or $z_{t+1} \in \{z_1, \ldots, z_t\}$.

Case 1. Suppose that $bz_{t+1} \in E(G)$ (this includes the case where $z_{t+1} = a$). Then let

$$\mathcal{P}' = (\mathcal{P} \setminus \{(b, y_i, z_i) : i \in \{1, \dots, t\}\}) \cup \{(b, y_i, z_{i+1}) : i \in \{1, \dots, t\}\}.$$

It is routine to check that \mathcal{P}' is a cycle packing of $L \vee K_T$ with the required conditions.

Case 2. Suppose that $z_{t+1} = z_s$ for some $s \in \{1, \ldots, t\}$. If the (a, z_s) -switch in $\mathcal{P} \setminus \{(b, y_s, z_s)\}$ with origin y_{s-1} does not have terminus y_s , then let σ be this switch. Otherwise, let σ be the (a, z_s) -switch in $\mathcal{P} \setminus \{(b, y_s, z_s)\}$ with origin y_t . It follows from its definition that σ does not terminate at y_{s-1} . Note also that σ does not terminate at b because b is adjacent to both a and z_s in the leave of $\mathcal{P} \setminus \{(b, y_s, z_s)\}$. Unless s = 1 and σ has origin y_0 , let \mathcal{P}^{\dagger} be the cycle packing of $L \vee K_T$ obtained by applying σ to $\mathcal{P} \setminus \{(b, y_s, z_s)\}$ (we will deal with the exceptional case shortly). Consider the cycle packing $\mathcal{P}^{\dagger} \cup \{(b, y_s, z_s)\}$ of $L \vee K_T$. We are now in a situation similar to Case 1, with the relevant vertex sequence being $y_0, z_1, y_1, \ldots, z_{s-1}, y_{s-1}, a$ if σ has origin y_{s-1} and being $y_0, z_1, y_1, \ldots, z_t, y_t, a$ otherwise. Thus, we can complete the proof as in Case 1 (note that $\mathcal{P}^{\dagger} \cup \{(b, y_s, z_s)\}$ is equivalent to \mathcal{P} on $V(L) \cup (T \setminus \{a, z_s\})$ by Lemma 8). In the case where s = 1 and σ has origin y_0 , let \mathcal{P}^{\dagger} be the cycle packing of $L \vee K_T$ obtained by applying the (a, z_1) -switch with origin y_1 to $\mathcal{P} \setminus \{(b, y_1, z_1)\}$ (this switch does not terminate at y_0 , because it is not σ) and observe that $\mathcal{P}^{\dagger} \cup \{(a, b, y_1)\}$ has the required properties (note that $\mathcal{P}^{\dagger} \cup \{(a, b, y_1)\}$ is equivalent to \mathcal{P} on $V(L) \cup (T \setminus \{a, z_1\})$ by Lemma 8).

Lemma 10. Let L be a graph, let T be a set of vertices which is disjoint from V(L), let \mathcal{P} be a cycle packing of $L \vee K_T$, and let G be the leave of \mathcal{P} . Suppose that

- $|E(G) \cap E(K_T)| \ge 1$;
- every cycle in \mathcal{P} that contains an edge in $E(K_T)$ is a triangle with one vertex in V(L); and
- $|\mathrm{Nbd}_G(x) \cap T| \ge 1$ for each $x \in V(L)$, and $|\mathrm{Nbd}_G(y) \cap T| \ge 2$ for some $y \in V(L)$.

Then there is a cycle packing $\mathcal{P}' \cup \{C\}$ of $L \vee K_T$ such that C is a triangle with $V(C) \cap V(L) = \{y\}$ and \mathcal{P}' is equivalent to \mathcal{P} on V(L).

Proof. Case 1. Suppose there are vertices q and r in T such that $ry, qr \in E(G)$. Let s be a vertex in $(\operatorname{Nbd}_G(y) \cap T) \setminus \{r\}$. If s = q then $\mathcal{P} \cup \{(q, r, y)\}$ is a cycle packing of $L \vee K_T$ with the required properties, so we may assume that $s \neq q$. Apply Lemma 9 to \mathcal{P} , taking a = q, b = r, c = s and d = y, to obtain a cycle packing \mathcal{P}' of $L \vee K_T$ which is equivalent to \mathcal{P} on $V(L) \cup \{r\}$ and whose leave contains the edges rs and sy. Note that the leave of \mathcal{P}' also contains the edge ry because \mathcal{P}' is equivalent to \mathcal{P}' on $V(L) \cup \{r\}$. Then $\mathcal{P}' \cup \{(r, s, y)\}$ is a cycle packing of $L \vee K_T$ with the required properties.

Case 2. Suppose that no vertex in $\operatorname{Nbd}_G(y) \cap T$ is adjacent in G to another vertex in T. Let r be a vertex in $\operatorname{Nbd}_G(y) \cap T$ and let pq be an edge in $E(G) \cap E(T)$. Apply Lemma 9 to \mathcal{P} with a = p, b = q, c = r and d = y, to obtain a cycle packing \mathcal{P}' of $L \vee K_T$ which is equivalent to \mathcal{P} on $V(L) \cup \{q\}$ and whose leave contains the edges qr and ry. We are now in a situation covered by Case 1 and can complete the proof as we did there.

Finally we will require Lemma 18 from [15]. The phrasing has been altered substantially so as to avoid the need to introduce technical definitions from that paper. This lemma allows us to obtain a triangle decomposition of a graph $L \vee K_w$ from a cycle packing of $L \vee K_{w-5}$ possessing very specific properties.

Lemma 11 ([15]). Let L be a graph of order u, let w be an integer such that $\frac{3u+17}{5} \leq w \leq u-1$, and let T be a set of w-5 vertices which is disjoint from V(L). Suppose that (L,w) is admissible and there exists a cycle packing \mathcal{P} of $L \vee K_T$ with a leave $F \cup G$ such that the following hold.

- F is a subgraph of L such that every component of F is an even-length path or an even-length cycle, and G is a subgraph of $K_{V(L),T}$.
- Each cycle of length at least 4 in \mathcal{P} is a subgraph of $K_{D,T}$, for some proper subset D of $V(L) \setminus V(F)$ such that each vertex in D is in exactly one such cycle and $|D| = \frac{5w-3u-13}{2}$.

• For some distinct vertices $a_1, a_2 \in D$, we have

$$\deg_G(x) = \begin{cases} \deg_F(x) + 1, & \text{if } x \in V(F); \\ 3, & \text{if } x \in D \setminus \{a_1, a_2\}; \\ 1, & \text{if } x \in (U \setminus (V(F) \cup D)) \cup \{a_1, a_2\}. \end{cases}$$

Then there exists a triangle decomposition of $L \vee K_w$.

4 Construction

In this section we will prove that there is a triangle decomposition of $L \vee K_w$ for any sufficiently dense graph L of order $u \geq 62$ and any integer w such that (L, w) is admissible and $w = \frac{3u+k}{5}$ for some $k \in \{17, 19, 21, 23\}$. Our approach is first to find a triangle packing of L whose leave L^* has exactly $\frac{w(u-w+1)}{2}$ edges and has maximum degree at most w-8 (see Lemma 12). We then construct a triangle decomposition of $L^* \vee K_w$ (see Lemma 14). The union of these sets of triangles is a triangle decomposition of $L \vee K_w$. Choosing $|E(L^*)| = \frac{w(u-w+1)}{2}$ means that a triangle decomposition of $L^* \vee K_w$ that has no triangle with all three vertices in $V(L^*)$ also has no triangle with no vertices in $V(L^*)$. This makes possible a proof of Lemma 14 based on the repeated application of Lemma 10.

We prove Lemma 12 by first taking a maximum triangle packing of a complete graph on vertex set V(L) and deleting from this packing any triangles which are not subgraphs of L. We then adjust the resulting packing to ensure that it has the required properties.

Lemma 12. Let L be a graph of order $u \ge 62$, let w be an integer such that $w = \frac{3u+k}{5}$ for some $k \in \{17, 19, 21, 23\}$, and suppose that (L, w) is admissible. If $|E(L)| \ge {u \choose 2} - \frac{w(u-w+1)-u-2}{4}$, then there is a triangle packing of L with a leave L^* such that $|E(L^*)| = \frac{w(u-w+1)}{2}$ and $\Delta(L^*) \le w-8$.

Proof. Let U = V(L), let $L^c = K_U - L$, and note that $|E(L^c)| \leq \frac{w(u-w+1)-u-2}{4}$. By the main result of [21] there is a triangle packing \mathcal{M} of K_U whose leave G contains at most $\frac{u+2}{2}$ edges (note that $u \geq 6$). Let

$$\mathcal{P}^{\dagger} = \{ C \in \mathcal{M} : E(C) \subseteq E(L) \}$$

be a triangle packing of L, and let H^{\dagger} be the leave of \mathcal{P}^{\dagger} . For each $C \in \mathcal{M} \setminus \mathcal{P}$, we have that $|E(C) \cap E(L)| \leq 2$, $|E(C) \cap E(L^c)| \geq 1$, and no other triangle in \mathcal{M} contains any edge in $E(C) \cap E(L^c)$. Thus,

$$|E(H^{\dagger})| \le 2|E(L^c)| + |E(G)| \le \frac{w(u-w+1)}{2}$$

using $|E(L^c)| \leq \frac{w(u-w+1)-u-2}{4}$ and $|E(G)| \leq \frac{u+2}{2}$. So, because (L,w) being admissible implies that $|E(L)| \equiv \frac{w(u-w+1)}{2} \pmod{3}$, we can delete triangles from \mathcal{P}^{\dagger} to obtain a triangle packing \mathcal{P} of L such that the leave H of \mathcal{P} has exactly $\frac{w(u-w+1)}{2}$ edges. Note that $\deg_H(x) \equiv \deg_L(x) \equiv w \pmod{2}$ for each $x \in U$, using the fact that (L,w) is admissible. Because $|\mathcal{P}| = \frac{1}{3}(|E(L)| - |E(H)|)$ it can be seen to follow from our hypothesis $|E(L)| \geq \binom{u}{2} - \frac{w(u-w+1)-u-2}{4}$ that

$$|\mathcal{P}| \ge \frac{1}{12}(u(2u - 3w - 1) + 3w(w - 1) + 2).$$

If $\Delta(H) \leq w - 8$ then \mathcal{P} is a triangle packing of L with the required properties, so we may assume $\Delta(H) \geq w - 6$. It suffices to show that we can find a triangle packing \mathcal{P}' of L with a leave H' such that $|E(H')| = \frac{w(u-w+1)}{2}$, $\Delta(H') \leq \Delta(H)$ and H' has fewer vertices of degree $\Delta(H)$ than H, because then, by repeating this procedure, we will eventually obtain a triangle packing of L with the required properties.

Let a be a vertex such that $\deg_H(a) = \Delta(H)$, let $A = \operatorname{Nbd}_H(a)$, let $S = \{x \in U : \deg_H(x) \ge w - 8\}$ and let $\mathcal{Q} = \{C \in \mathcal{P} : |V(C) \cap S| \ge 1\}$. Clearly $|S| \le \frac{w(u-w+1)}{w-8}$. Also, note that each vertex in S occurs in at most $\frac{1}{2}((u-1)-(w-8)) = \frac{u-w+7}{2}$ triangles in \mathcal{P} . The proof now splits into two cases according to whether $E(H) \cap E(K_A) = \emptyset$.

Case 1. Suppose that there is an edge xy in $E(H) \cap E(K_A)$. Because $|S| \leq \frac{w(u-w+1)}{w-8}$ and each vertex in S occurs in at most $\frac{u-w+7}{2}$ triangles in \mathcal{P} , we have that $|\mathcal{Q}| \leq \frac{w(u-w+1)(u-w+7)}{2(w-8)}$. Thus, using our lower bound on $|\mathcal{P}|$ it is routine to check that $|\mathcal{P}| > |\mathcal{Q}|$ for $u \geq 52$. It follows that there is a triangle C in $\mathcal{P} \setminus \mathcal{Q}$. It is easy to see that

$$\mathcal{P}' = (\mathcal{P} \setminus \{C\}) \cup \{(x, y, a)\}$$

is a triangle packing of L with the required properties.

Case 2. Suppose that $E(H) \cap E(K_A) = \emptyset$. It follows that each vertex in A has degree in H at most $u - |A| \le u - w + 6$, which for $u \ge 42$ implies degree at most w - 10. Thus, A and S are disjoint sets, and $|S| \le u - |A| \le u - w + 6$. Because $|S \setminus \{a\}| \le u - w + 5$ and each vertex in S occurs in at most $\frac{u-w+7}{2}$ triangles in \mathcal{P} , we have that at most $\frac{(u-w+5)(u-w+7)}{2}$ triangles in \mathcal{P} contain a vertex in $S \setminus \{a\}$ and hence that at most $\frac{(u-w+5)(u-w+7)}{2}$ edges of K_A are in triangles in \mathcal{Q} . Obviously, at most $|E(L^c)| \le \frac{w(u-w+1)-u-2}{4}$ edges of K_A are in $E(L^c)$. Thus, because each edge of K_A is either in $E(L^c)$ or in a triangle in \mathcal{P} and because it is routine to check that, for $u \ge 62$,

$$|E(K_A)| \ge {w-6 \choose 2} > \frac{(u-w+5)(u-w+7)}{2} + \frac{w(u-w+1)-u-2}{4}$$

we have that some edge $xy \in E(K_A)$ is in a triangle (x, y, z) in $\mathcal{P} \setminus \mathcal{Q}$. It is easy to see that

$$\mathcal{P}' = (\mathcal{P} \setminus \{(x, y, z)\}) \cup \{(x, y, a)\}$$

is a triangle packing of L with the required properties.

In order to prove Lemma 14 we first use a proper edge colouring of a subgraph L^* to obtain a cycle packing of $L^* \vee K_{w-5}$ consisting of triangles, each of which contains exactly two vertices in $V(L^*)$, and one or two longer cycles. We then repeatedly apply Lemma 10 until we obtain a cycle packing of $L^* \vee K_{w-5}$ to which we can apply Lemma 11 and thus complete the proof. Lemma 13 is a technical result that is used only in the proof of Lemma 14.

Lemma 13. Let L^* be a graph of order $u \geq 32$ and let w be an integer such that $w = \frac{3u+k}{5}$ for some $k \in \{17, 19, 21, 23\}$. If $|E(L^*)| = \frac{w(u-w+1)}{2}$, $\Delta(L^*) \leq w-8$ and $\deg_{L^*}(x) \equiv u+1 \pmod 2$ for each $x \in V(L^*)$, then there is a decomposition $\{F, F_1, \ldots, F_{w-7}\}$ of L^* such that

- |E(F)| = u w + 1 and each component of F is an even-length path or an even-length cycle;
- F_i is a matching for $i \in \{1, \ldots, w-7\}$; and
- there is a proper subset D of $U \setminus V(F)$ such that $|D| = \frac{5w-3u-13}{2}$, $|D \setminus V(F_{w-7})| \ge 2$, and $\deg_{L^*}(x) \le w 10$ for each $x \in D$.

Proof. Let $U = V(L^*)$ and let M be a subset of U such that $|M| = \frac{u-w+1}{2}$ and $\deg_{L^*}(x) \ge \deg_{L^*}(y)$ for each $x \in M$ and $y \in U \setminus M$. By Lemma 6, there is a subgraph F of L^* such that |E(F)| = u - w + 1 and each component of F is an even-length path or an even-length cycle. Further, by Lemma 7, we can assume that $M \subseteq V(F)$ if $\deg_{L^*}(x) \ge u - w$ for each $x \in M$. Since $\Delta(L^* - F) \le \Delta(L^*) \le w - 8$ there is a decomposition $\{F_1, \ldots, F_{w-7}\}$ of $L^* - F$ into w - 7 matchings by Vizing's theorem [24].

It is easy to check that $u - |V(F)| \ge 6$ using $|V(F)| \le \frac{3}{2}|E(F)|$. Let D' be a subset of $U \setminus V(F)$ such that |D'| = 5 and $\deg_{L^*}(x) \le \deg_{L^*}(y)$ for each $x \in D'$ and $y \in U \setminus (V(F) \cup D')$. Note that D' is a proper subset of $U \setminus V(F)$. It suffices to show that $\deg_{L^*}(x) \le w - 10$ for each $x \in D'$ and that $|D' \setminus V(F_i)| \ge 2$ for some $i \in \{1, \ldots, w - 7\}$. This is because we can reindex the matchings in $\{F_1, \ldots, F_{w-7}\}$, if necessary, so that $|D' \setminus V(F_{w-7})| \ge 2$, and take D to be a subset of D' such that $|D| = \frac{5w - 3u - 13}{2}$ and $|D \setminus V(F_{w-7})| \ge 2$ (note that $\frac{5w - 3u - 13}{2} \in \{2, 3, 4, 5\}$). The proof splits into two cases according to whether $\deg_{L^*}(x) \ge u - w$ for each $x \in M$.

Case 1. Suppose that $\deg_{L^*}(x) \leq u - w - 1$ for some $x \in M$. It is easy to check that $u - |V(F)| \geq |M| + |D'|$ using $|V(F)| \leq \frac{3}{2}|E(F)|$, and this implies that $M \cap D' = \emptyset$ and hence that $\deg_{L^*}(x) \leq u - w - 1$ for each $x \in D'$. Since $u - w - 1 \leq w - 10$ for $u \geq 11$, we have $\deg_{L^*}(x) \leq w - 10$ for each $x \in D'$. Furthermore, using the fact that $\deg_{L^*}(x) \leq u - w - 1$ for

each $x \in D'$, it can be shown that

$$\sum_{x \in D'} |\{i : i \in \{1, \dots, w - 7\}, x \notin V(F_i)\}| \ge 5(w - 7) - 5(u - w - 1) > w - 7,$$

which implies that $|D' \setminus V(F_i)| \ge 2$ for some $i \in \{1, ..., w-7\}$. (Informally, we are evaluating the sum over the vertices in D' of the number of matchings "missing" at a vertex and then concluding that some matching must be missing at two vertices in D'.)

Case 2. Suppose that $\deg_{L^*}(x) \geq u - w$ for each $x \in M$. Then $M \subseteq V(F)$. Consider the set $S = (U \setminus V(F)) \cup M$ and note that $|S| \geq w - 1$ using $|V(F)| \leq \frac{3}{2}|E(F)|$. The definitions of D' and M imply that $\deg_{L^*}(x) \leq \deg_{L^*}(y)$ for each $x \in D'$ and $y \in S \setminus D'$. Also note that $\sum_{x \in S} \deg_{L^*}(x) \leq (w - 1)(u - w + 1)$, since $|E(L^*)| = \frac{w(u - w + 1)}{2}$ and $\sum_{x \in V(F) \setminus M} \deg_F(x) \geq 2|E(F)| - 2|M| \geq u - w + 1$. So, because $|S| \geq w - 1$ and |D'| = 5, it can be seen that $\max(\{\deg_{L^*}(x) : x \in D'\}) \leq \frac{(w - 1)(u - w + 1)}{w - 5}$ and $\sum_{x \in D'} \deg_{L^*}(x) \leq 5(u - w + 1)$. As in Case 1, it is routine to use these facts to show that, for $u \geq 32$, $\deg_{L^*}(x) \leq w - 10$ for each $x \in D$ and that $|D' \setminus V(F_i)| \geq 2$ for some $i \in \{1, \ldots, w - 7\}$.

Lemma 14. Let L^* be a graph of order $u \geq 32$ and let w be an integer such that $w = \frac{3u+k}{5}$ for some $k \in \{17, 19, 21, 23\}$. If $|E(L^*)| = \frac{w(u-w+1)}{2}$, $\Delta(L^*) \leq w-8$ and $\deg_{L^*}(x) \equiv u+1 \pmod 2$ for each $x \in V(L^*)$, then there is a triangle decomposition of $L^* \vee K_w$.

Proof. Let $U = V(L^*)$ and let $T = \{z_1, \ldots, z_{w-5}\}$ be a set of vertices which is disjoint from U. By Lemma 13, there is a decomposition $\{F, F_1, \ldots, F_{w-7}\}$ of L^* such that

- |E(F)| = u w + 1 and each component of F is an even-length path or an even-length cycle;
- F_i is a matching for $i \in \{1, \dots, w-7\}$; and
- there is a proper subset D of $U \setminus V(F)$ such that $|D| = \frac{5w 3u 13}{2}$, $|D \setminus V(F_{w-7})| \ge 2$, and $\deg_{L^*}(x) \le w 10$ for each $x \in D$.

Let $D = \{d_1, \dots, d_s\}$, where $s = \frac{5w - 3u - 13}{2}$, and let

$$\mathcal{C}_{0} = \begin{cases}
\{(d_{1}, z_{w-5}, d_{2}, z_{w-6})\}, & \text{if } |D| = 2; \\
\{(d_{1}, z_{w-7}, d_{2}, z_{w-6}, d_{3}, z_{w-5})\}, & \text{if } |D| = 3; \\
\{(d_{1}, z_{w-5}, d_{2}, z_{w-6}), (d_{3}, z_{w-5}, d_{4}, z_{w-6})\}, & \text{if } |D| = 4; \\
\{(d_{1}, z_{w-7}, d_{2}, z_{w-6}, d_{3}, z_{w-5}), (d_{4}, z_{w-5}, d_{5}, z_{w-6})\}, & \text{if } |D| = 5.
\end{cases}$$

Note that $\bigcup_{C \in \mathcal{C}_0} E(C) \subseteq E(K_{D,\{z_{w-6},z_{w-5}\}}) \cup \{d_1 z_{w-7}, d_2 z_{w-7}\}$ and that each vertex in D is in exactly one cycle in \mathcal{C}_0 . Let

$$\mathcal{T}_0 = \{(x, y, z_i) : i \in \{1, \dots, w - 7\}, xy \in E(F_i)\}.$$

It is routine to verify that $\mathcal{T}_0 \cup \mathcal{C}_0$ is a cycle packing of $L^* \vee K_T$ whose leave is $F \cup G_0$ for some subgraph G_0 of $K_{U,T} \cup K_T$.

Choose two distinct vertices $a_1, a_2 \in D$ (note that $|D| \geq 2$). Define a function $\tau : U \rightarrow \{1, 2, 3\}$ by

$$\tau(x) = \begin{cases} \deg_F(x) + 1, & \text{if } x \in V(F); \\ 3, & \text{if } x \in D \setminus \{a_1, a_2\}; \\ 1, & \text{otherwise.} \end{cases}$$

Note that we have $\deg_{G_0}(x) = (w-5) - \deg_{L^*-F}(x)$ for each $x \in U \setminus D$ and $\deg_{G_0}(x) = (w-7) - \deg_{L^*-F}(x)$ for each $x \in D$. Thus, because $\deg_{L^*}(x) \equiv u+1 \pmod 2$ for each $x \in U$ and $w \equiv u+1 \pmod 2$, it can be seen that $\deg_{G_0}(x) \equiv \tau(x) \pmod 2$ for each $x \in U$. Further, using the fact that $\deg_{L^*}(x) \leq w-8$ for each $x \in U \setminus D$ and $\deg_{L^*}(x) \leq w-10$ for each $x \in D$, we have that $\deg_{G_0}(x) \geq 3 \geq \tau(x)$ for each $x \in U$. Also, using the definition of τ and our expressions for |D|, $|E(L^*)|$ and |E(F)|, we obtain

$$\sum_{x \in U} (\deg_{G_0}(x) - \tau(x)) = u(w - 5) - 2|D| - 2(|E(L^*)| - |E(F)|) - \sum_{x \in U} \tau(x) = 2\binom{w - 5}{2}.$$

Let $\mathcal{P}_0 = \mathcal{T}_0 \cup \mathcal{C}_0$ and $r = {w-5 \choose 2}$. We claim that for each $i \in \{0, \dots, r\}$ there is a packing \mathcal{P}_i of $L \vee K_T$ with leave $F \cup G_i$ for some subgraph G_i of $K_{U,T} \cup K_T$ such that the following hold.

- $|E(G_i \cap K_T)| = {w-5 \choose 2} i$.
- $\deg_{G_i}(x) \equiv \tau(x) \pmod{2}$ and $\deg_{G_i}(x) \geq \tau(x)$ for each $x \in U$.
- Every cycle in \mathcal{P}_i that contains an edge in K_T is a triangle with one vertex in V(L).
- $\sum_{x \in U} (\deg_{G_i}(x) \tau(x)) = 2({w-5 \choose 2} i).$
- Each cycle of length at least 4 in \mathcal{P}_i is a subgraph of $K_{V(L),T}$, each vertex in D is in exactly one such cycle, and each vertex in $V(L) \setminus D$ is not in any such cycle.

To see that these cycle packings exist suppose inductively that a cycle packing \mathcal{P}_k with the required properties exists for some $k \in \{0, \ldots, r-1\}$. We will show that we can construct a packing \mathcal{P}_{k+1} with the required properties. The properties of \mathcal{P}_k imply that there is a vertex $y_k \in U$ such that $\deg_{G_i}(y_k) \geq \tau(y_k) + 2$. Let \mathcal{P}_{k+1} be the packing of $L \vee K_T$ obtained by applying Lemma 10 to \mathcal{P}_k , taking $y = y_k$, and note that by Lemma 10 there is a triangle $C_{k+1} \in \mathcal{P}_{k+1}$ such that $V(C_{k+1}) \cap U = \{y_k\}$ and $\mathcal{P}_{k+1} \setminus \{C_{k+1}\}$ is equivalent to \mathcal{P}_k on U. From this it can be seen that \mathcal{P}_{k+1} has the required properties (note that $\deg_{G_{k+1}}(y_k) = \deg_{G_k}(y_k) - 2$ and $\deg_{G_{k+1}}(x) = \deg_{G_k}(x)$ for each $x \in U \setminus \{y_k\}$).

Consider the cycle packing \mathcal{P}_r . From its properties we have that $E(G_r \cap K_T) = \emptyset$, that $\deg_{G_r}(x) = \tau(x)$ for each $x \in U$, that each cycle of length at least 4 in \mathcal{P}_i is a subgraph of $K_{D,T}$ and that each vertex in D is in exactly one such cycle. Thus we can apply Lemma 11 to \mathcal{P}_r to obtain a triangle decomposition of $L^* \vee K_w$.

Combining Lemmas 12 and 14, we obtain the following result.

Lemma 15. Let L be a graph of order $u \ge 62$, let w be an integer such that $w = \frac{3u+k}{5}$ for some $k \in \{17, 19, 21, 23\}$, and suppose that (L, w) is admissible. If $|E(L)| \ge {u \choose 2} - \frac{w(u-w+1)-u-2}{4}$, there is a triangle decomposition of $L \lor K_w$.

Proof. We can apply Lemma 12 to obtain a triangle packing \mathcal{P} of L with a leave L^* such that $|E(L^*)| = \frac{w(u-w+1)}{2}$ and $\Delta(L^*) \leq w-8$. Because (L,w) is admissible, we obviously also have $\deg_{L^*}(x) \equiv u+1$ for each $x \in V(L^*)$. So we can apply Lemma 14 to obtain a triangle decomposition \mathcal{D} of $L^* \vee K_w$. Clearly $\mathcal{P} \cup \mathcal{D}$ is a triangle decomposition of $L \vee K_w$.

5 Main Result

By applying Lemma 15 with L chosen judiciously, we can obtain our main results without too much further effort.

Proof of Theorem 2. Let (U, \mathcal{A}) be a partial Steiner triple system of order $u \geq 62$ such that $|\mathcal{A}| \leq \frac{u^2}{50} - \frac{11u}{100} - \frac{116}{75}$, and let v be an integer such that $v \geq \frac{8u+17}{5}$ and $v \equiv 1, 3 \pmod{6}$. Let L be the leave of (U, \mathcal{A}) . Let $u' = \frac{5v-k}{8}$ and $w' = \frac{3v+k}{8}$ where k = 21, 23, 17, 19 when $v \equiv 1, 3, 5, 7 \pmod{8}$ respectively. It is easy to see that u' and w' are integers such that u' + w' = v, $u' \geq u \geq 62$ and $w' = \frac{3u'+k}{5}$. Consider the graph $L' = L \vee K_{u'-u}$ (L' = L if u' = u) and note that (L', w') is admissible because (L, v - u) is admissible. It is routine to check that $|\mathcal{A}| \leq \frac{u^2}{50} - \frac{11u}{100} - \frac{116}{75}$ implies that $|\mathcal{E}(L')| \geq \binom{u'}{2} - \frac{w'(u'-w'+1)-u'-2}{4}$. Thus we can apply Lemma 15 to produce a triangle decomposition of $L' \vee K_{w'}$. Because $L' \vee K_{w'}$ is isomorphic to $L \vee K_{v-u}$, the proof is complete.

Proof of Theorem 3. Note that v is odd because G is even and contains vertices of degree v-1. Let $u'=\frac{5v-k}{8}$ and $w'=\frac{3v+k}{8}$ where k=21,23,17,19 when $v\equiv 1,3,5,7 \pmod 8$ respectively. It is easy to see that u' and w' are integers such that $u'+w'=v, u'\geq 62$ and $w'=\frac{3u'+k}{5}$. Our hypotheses imply that G has at least w' vertices of degree v-1, so G is isomorphic to $L'\vee K_{w'}$ for some graph L' of order u'. It follows from the facts that G is even and $|E(G)|\equiv 0 \pmod 3$ that (L',w') is admissible. It is routine to check that

 $|E(G)| \ge {v \choose 2} - (\frac{3v^2}{128} - \frac{27v}{64} - \frac{409}{128})$ implies that $|E(L')| \ge {u' \choose 2} - \frac{w'(u'-w'+1)-u'-2}{4}$. So we can apply Lemma 15 to obtain a triangle decomposition of $L' \lor K_{w'}$. Because $L' \lor K_{w'}$ is isomorphic to G, the proof is complete.

We conclude by establishing the existence of partial Steiner triple systems with specified numbers of triples which do not have any embeddings of order close to u.

Lemma 16. Let u and w be positive integers such that u + w is odd and $w \le u - 5$. There is a partial Steiner triple system (U, A) of order u such that every embedding of (U, A) has order at least u + w and

$$|\mathcal{A}| = \begin{cases} \frac{1}{6}(3u + w^2 - 4w - 3), & \text{if } w \equiv 1, 3 \pmod{6}; \\ \frac{1}{6}(3u + w^2 - 4w + 13), & \text{if } w \equiv 5 \pmod{6}; \\ \frac{1}{6}(3u + w^2 - 2w - 3), & \text{if } w \equiv 0, 2 \pmod{6}; \\ \frac{1}{6}(3u + w^2 - 2w + 1), & \text{if } w \equiv 4 \pmod{6}. \end{cases}$$

Proof. Let U be a set of u vertices, let a be a vertex in U and let S be a subset of $U \setminus \{a\}$ with |S| = w. If w is odd then let S' = S, and if w is even then let S' be a subset of $U \setminus \{a\}$ such that $S \subseteq S'$ and |S'| = w + 1. Let \mathcal{A}_1 be a triangle decomposition of $K_{\{a\}} \vee M$, where M is a matching with vertex set $U \setminus (S \cup \{a\})$ (note that $|U \setminus (S \cup \{a\})| = u - w - 1$ is even because u + w is odd). Observe that $|S'| \equiv 1, 3 \pmod{6}$ if $w \equiv 0, 1, 2, 3 \pmod{6}$ and $|S'| \equiv 5 \pmod{6}$ if $w \equiv 4, 5 \pmod{6}$. Thus, using the main result of [21], there is a triangle packing \mathcal{A}_2 of $K_{S'}$, such that if $w \equiv 0, 1, 2, 3 \pmod{6}$ then the leave of \mathcal{A}_2 is empty, and if $w \equiv 4, 5 \pmod{6}$ then the leave of \mathcal{A}_2 is a cycle $C = (c_1, c_2, c_3, c_4)$ of length 4 such that $S' \setminus S = \{c_1\}$ when $w \equiv 4 \pmod{6}$. It follows from $w \leq u - 5$ that $|U \setminus (S' \cup \{a\})| \geq 3$. Let z_1 and z_2 be distinct vertices in $U \setminus (S' \cup \{a\})$ and let

$$\mathcal{A}_{3} = \begin{cases} \emptyset, & \text{if } w \equiv 0, 1, 2, 3 \pmod{6}; \\ \{(z_{1}, c_{1}, c_{2}), (z_{1}, c_{3}, c_{4}), (z_{2}, c_{2}, c_{3}), (z_{2}, c_{1}, c_{4}))\}, & \text{if } w \equiv 5 \pmod{6}; \\ \{(z_{1}, c_{3}, c_{4}), (z_{2}, c_{2}, c_{3}))\}, & \text{if } w \equiv 4 \pmod{6}. \end{cases}$$

Let $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$ and observe that (U, \mathcal{A}) is a partial Steiner triple system with a leave L such that $|\mathrm{Nbd}_L(a)| = w$ and the subgraph of L induced by $\mathrm{Nbd}_L(a)$ is empty. Thus, it can be deduced from Lemma 4 that every embedding of (U, \mathcal{A}) has order at least u + w. In each case, simple counting shows that \mathcal{A} contains the required number of triples.

Corollary 17. For any positive integers u and t such that $\frac{u+1}{2} \le t < \frac{1}{6}(u^2 - 5u + 16)$, there is a partial Steiner triple system of order u with at most t triples that does not have an embedding of order $u + \sqrt{6t - 3u} - 1$ or smaller.

Proof. Let w be the smallest integer such that $w > \sqrt{6t - 3u} - 1$ and u + w is odd. Note that $\frac{u+1}{2} \le t < \frac{1}{6}(u^2 - 5u + 16)$ implies $0 < \sqrt{6t - 3u} - 1 < u - 5$, and hence that $1 \le w \le u - 5$. By Lemma 16 there is a partial Steiner triple system (U, \mathcal{A}) of order u such that every embedding of (U, \mathcal{A}) has order at least u + w and

$$|\mathcal{A}| = \begin{cases} \frac{1}{6}(3u + w^2 - 4w - 3), & \text{if } w \equiv 1, 3 \pmod{6}; \\ \frac{1}{6}(3u + w^2 - 4w + 13), & \text{if } w \equiv 5 \pmod{6}; \\ \frac{1}{6}(3u + w^2 - 2w - 3), & \text{if } w \equiv 0, 2 \pmod{6}; \\ \frac{1}{6}(3u + w^2 - 2w + 1), & \text{if } w \equiv 4 \pmod{6}. \end{cases}$$

It only remains to show that $|A| \leq t$.

When $w \neq 5$ it is easy to check that $|\mathcal{A}| \leq \frac{1}{6}(3u+w^2-2w+1)$. Thus, since $w \leq \sqrt{6t-3u}+1$, it can be seen that $|\mathcal{A}| \leq t$. When w = 5, we have $|\mathcal{A}| = \frac{u+6}{2}$. Further, it follows from the definition of w that $\sqrt{6t-3u}-1 \geq 3$, and hence that $t \geq \frac{u+6}{2}$ (note that t is an integer). \square

As mentioned in the introduction, it follows from Corollary 17 that for sufficiently large values of u there is a partial Steiner triple system of order u with at most $\frac{u^2}{50} - \frac{11u}{100} - \frac{116}{75}$ triples that does not have an embedding of order v for any v < (1.346)u.

Applying Lemma 16 with w = 2 demonstrates that, for each admissible integer $u \geq 7$, there is a partial Steiner triple system of order u with $\frac{u-1}{2}$ triples that has no completion. This raises the following question which bears some resemblance to Evans' conjecture for partial latin squares [12] (proved in [22]).

Question. Does every partial Steiner triple system of admissible order u with fewer than $\frac{u-1}{2}$ triples have a completion?

References

- [1] L.D. Andersen, A.J.W Hilton and E. Mendelsohn, Embedding partial Steiner triple systems, Proc. London Math. Soc. **41** (1980), 557–576.
- [2] D. Bryant, A conjecture on small embeddings of partial Steiner triple systems, J. Combin.
 Des. 10 (2002), 313–321.
- [3] D. Bryant, Embeddings of partial Steiner triple systems, J. Combin. Theory Ser. A **106** (2004), 77–108.

- [4] D. Bryant and D. Horsley, A proof of Lindner's conjecture on embeddings of partial Steiner triple systems, J. Combin. Designs 17 (2009), 63–89.
- [5] D. Bryant, D. Horsley and B. Maenhaut, Decompositions into 2-regular subgraphs and equitable partial cycle decompositions, J. Combin. Theory Ser. B **93** (2005), 67–72.
- [6] D. Bryant and D. Horsley, Steiner triple systems with two disjoint subsystems, J. Combin. Des. 14 (2006), no. 1, 14–24.
- [7] D. Bryant, B. Maenhaut, K. Quinn and B. S. Webb, Existence and embeddings of partial Steiner triple systems of order ten with cubic leaves, Discrete Math. **284** (2004), 83–95.
- [8] C.J. Colbourn, Embedding partial Steiner triple systems is NP-complete, J. Combin. Theory Ser. A **35** (1983), 100–105.
- [9] C.J. Colbourn, M.J. Colbourn and A. Rosa, Completing small partial triple systems, Discrete Math. **45** (1983), 165–179.
- [10] C.J. Colbourn and A. Rosa, Triple Systems, Clarendon Press, Oxford (1999).
- [11] P. Elias, A. Feinstein, and C. Shannon, A note on the maximum flow through a network, IRE T. Inform. Theor. 2 (1956), 117-119.
- [12] T. Evans, Embedding incomplete latin squares, Amer. Math. Monthly 67 (1960), 958–961.
- [13] L.R. Ford and D.R. Fulkerson, Maximal flow through a network, Canad. J. Math. 8 (1956), 399404.
- [14] P. Hall, On Representatives of Subsets, J. London Math. Soc. 10 (1935), 26–30.
- [15] D. Horsley, Small embeddings of partial Steiner triple systems, J. Combin. Des. (to appear).
- [16] T.P. Kirkman, On a problem in combinations, Cambridge and Dublin Math. J. 2 (1847), 191–204.
- [17] D. König, Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre, Math. Ann. 77 (1916), 453–465.
- [18] C.C. Lindner, A partial Steiner triple system of order n can be embedded in a Steiner triple system of order 6n + 3, J. Combin. Theory Ser. A 18 (1975), 349–351.

- [19] C.C. Lindner and T. Evans, Finite embedding theorems for partial designs and algebras, SMS 56, Les Presses de l'Université de Montréal, (1977).
- [20] C.St.J.A. Nash-Williams, An unsolved problem concerning decomposition of graphs into triangles, Combinatorial Theory and its Applications III, Colloquia Math. Soc. J. Bolyai 4 (1970), 1179–1181.
- [21] J. Schönheim, On maximal systems of k-tuples, Studia Sci. Math. Hungar. 1 (1966), 363–368.
- [22] B. Smetaniuk, A new construction on Latin squares. I. A proof of the Evans conjecture, Ars Combin. 11 (1981), 155–172.
- [23] C. Treash, The completion of finite incomplete Steiner triple systems with applications to loop theory. J. Combin. Theory Ser. A **10** (1971), 259–265.
- [24] V.G. Vizing, On an estimate of the chromatic class of a *p*-graph (in Russian), Diskret Analiz **3** (1964), 25–30.