s-Inversion Sequences and P-Partitions of Type B

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Abstract

Given a sequence $s = (s_1, s_2, ...)$ of positive integers, the inversion sequences with respect to s, or s-inversion sequences, were introduced by Savage and Schuster in their study of lecture hall polytopes. A sequence $(e_1, e_2, ..., e_n)$ of nonnegative integers is called an s-inversion sequence of length n if $0 \le e_i < s_i$ for $1 \le i \le n$. Let I(n) be the set of s-inversion sequences of length n for s = (1, 4, 3, 8, 5, 12, ...), that is, $s_{2i} = 4i$ and $s_{2i-1} = 2i - 1$ for $i \ge 1$, and let P_n be the set of signed permutations on $\{1^2, 2^2, ..., n^2\}$. Savage and Visontai conjectured that when n =2k, the ascent number over I_n is equidistributed with the descent number over P_k . For a positive integer n, we use type B P-partitions to give a characterization of signed permutations over which the descent number is equidistributed with the ascent number over I_n . When n is even, this confirms the conjecture of Savage and Visontai. Moreover, let I'_n be the set of s-inversion sequences of length n for s = (2, 2, 6, 4, 10, 6, ...), that is, $s_{2i} = 2i$ and $s_{2i-1} = 4i - 2$ for $i \ge 1$. We find a set of signed permutations over which the descent number is equidistributed with the ascent number over I'_n .

Keywords: inversion sequence, signed permutation, type B P-partition, equidistribution

AMS Subject Classifications: 05A05, 05A15

1 Introduction

The notion of s-inversion sequences was introduced by Savage and Schuster [8] in their study of lecture hall polytopes. Let $s = (s_1, s_2, ...)$ be a sequence of positive integers. An inversion sequence of length n with respect to s, or an s-inversion sequence of length n, is a sequence $e = (e_1, e_2, ..., e_n)$ of nonnegative integers such that $0 \le e_i < s_i$ for

 $1 \leq i \leq n$. An ascent of an s-inversion sequence $e = (e_1, e_2, \ldots, e_n)$ is defined to be an integer $i \in \{0, 1, \ldots, n-1\}$ such that

$$\frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}},$$

under the assumption that $e_0 = 0$ and $s_0 = 1$. The ascent number $\operatorname{asc}(e)$ of e is meant to be the number of ascents of e.

The generating function of the ascent number over s-inversion sequences can be viewed as a generalization of the Eulerian polynomial for permutations, since the ascent number over the s-inversion sequences of length n for s = (1, 2, 3, ...) is equidistributed with the descent number over the permutations on $\{1, 2, ..., n\}$, see Savage and Schuster [8]. For an inversion sequence $e = (e_1, e_2, ..., e_n)$ with respect to $s = (s_1, s_2, ...,)$, let

$$\operatorname{amaj}(e) = \sum_{i \in \operatorname{Asc}(e)} (n-i),$$

and

$$lhp(e) = -|e| + \sum_{i \in Asc(e)} (s_{i+1} + \dots + s_n),$$

where Asc(e) is the set of ascents of e, and $|e| = e_1 + e_2 + \cdots + e_n$ is the weight of e. Savage and Schuster [8] showed that the multivariate generating function for the ascent number ase(e), the major index amaj(e), the lecture hall statistic lhp(e) and the weight |e| is related to the Ehrhart series of *s*-lecture hall polytopes and the generating function of *s*-lecture hall partitions.

Savage and Visontai [9] found a connection between the generating function of the ascent number over s-inversion sequences of length n and a conjecture of Brenti [3] on the real-rootedness of Eulerian polynomials of finite Coxeter groups. The real-rootedness of the Eulerian polynomial of type A was known to Frobenius [6], see also [2, 7]. Brenti [3] proved the real-rootedness of the Eulerian polynomials of Coxeter groups of type B and exceptional Coxeter groups. For the sequence s = (2, 4, 6, ...) and an s-inversion sequence $e = (e_1, e_2, ..., e_n)$, Savage and Visontai [9] defined the type D ascent set of e as given by

Asc_D(e) =
$$\left\{ i \left| \frac{e_i}{i} < \frac{e_{i+1}}{i+1}, 1 \le i \le n-1 \right\} \cup \{0 \mid \text{if } 2e_1 + e_2 \ge 3\}. \right.$$

Let $T_n(x)$ be the generating function of the type D ascent number over s-inversion sequence of length n for s = (2, 4, 6, ...). For example, $T_3(x) = 2(x^3 + 11x^2 + 11x + 1)$. Let $D_n(x)$ be the n-th Eulerian polynomial of type D. Recall that the type D Coxeter group of rank n, denoted D_n , is the group of even-signed permutations on $\{1, 2, ..., n\}$, see Björner and Brenti [1]. The descent set $\text{Des}_D(\sigma)$ of an even-signed permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in D_n$ is defined by

$$Des_D(\sigma) = \{ i \mid \sigma_i > \sigma_{i+1}, 1 \le i \le n-1 \} \cup \{ 0 \mid \text{if } \sigma_1 + \sigma_2 < 0 \}$$

The descent number of σ is meant to be the number of descents in $\text{Des}_D(\sigma)$. Let $D_n(x)$ denote the generating function of the descent number over D_n . Savage and Visontai [9] showed that $T_n(x) = 2D_n(x)$. By proving that $T_n(x)$ has only real roots for $n \ge 1$, they deduced the real-rootedness of the Eulerian polynomials of type D and settled the last unsolved case of the conjecture of Brenti [3].

Savage and Visontai [9] proved that for any sequence s of positive integers and any positive integer n, the generating function of the ascent number over s-inversion sequences of length n has only real roots. Let I_n denote the set of s-inversion sequences of length n for the specific sequence s = (1, 4, 3, 8, 5, 12, ...), that is, for $i \ge 1$, $s_{2i} = 4i$ and $s_{2i-1} = 2i - 1$. Let P_n denote the set of signed permutations on the multiset $\{1^2, 2^2, ..., n^2\}$. Savage and Visontai [9] posed the following conjecture, which implies the real-rootedness of the generating function of the descent number over P_n .

Conjecture 1.1 ([9, Conjecture 3.27]) For $n \ge 1$, the descent number over P_n is equidistributed with the ascent number over I_{2n} .

In this paper, we give a proof of Conjecture 1.1. Let $P_n(x)$ denote the generating function of the descent number over the set P_n of signed permutations on $\{1^2, 2^2, \ldots, n^2\}$, and let $I_n(x)$ denote the generating function of the ascent number over I_n . Savage and Schuster [8] found a relation for $I_n(x)$. We show that the generating function $P_n(x)$ equals the generating function of the descent number over linear extensions of certain signed labeled forests. By using *P*-partitions of type *B* introduced by Chow [5], we show that the generating function for the descent number over linear extensions satisfies the same relation as $I_{2n}(x)$. Thus the generating function $P_n(x)$ satisfies the same relation as $I_{2n}(x)$. This proves Conjecture 1.1.

We also find characterizations of signed permutations such that the descent number is equidistributed with the ascent number over three other classes of s-inversion sequences. To be specific, we show that the descent number over the set of signed permutations on the multiset $\{1^2, 2^2, \ldots, (n-1)^2, n\}$ such that n is assigned a minus sign is equidistributed with the ascent number over I_{2n-1} . For $s = (2, 2, 6, 4, 10, 6, \ldots)$, that is, for $i \ge 1$, $s_{2i} = 2i$ and $s_{2i-1} = 4i - 2$, let I'_n denote the set of s-inversion sequences of length n. We show that the descent number over P_n is equidistributed with the ascent number over I'_{2n} and the descent number over the set of signed permutations on $\{1^2, 2^2, \ldots, (n-1)^2, n\}$ is equidistributed with the ascent number over I'_{2n-1} .

2 Proof of Conjecture 1.1

In this section, we present a proof of Conjecture 1.1 by establishing a connection between the generating function $P_n(x)$ of the descent number over P_n and the generating function of the descent number over linear extensions of certain signed labeled forests with 2n vertices. Let F_n be the plane forest with n trees containing exactly two vertices, and let $F_n(x)$ denote the generating function of the descent number over linear extensions of (F_n, w) , where w ranges over certain signed labelings of F_n . Keep in mind that a plane forest means a set of plane trees that are arranged in linear order. We shall show that $F_n(x) = P_n(x)$. On the other hand, by using P-partitions of type B introduced by Chow [5], we obtain a relation for $F_n(x)$. Savage and Schuster [8] have shown that the same relation is satisfied by the generating function $I_{2n}(x)$, so we get $I_{2n}(x) = F_n(x)$. This confirms Conjecture 1.1, that is, $P_n(x) = I_{2n}(x)$.

Let us give an overview of linear extensions of a signed labeled forest. Let F be a plane forest with n vertices, and let S be a set of n distinct positive integers. A labeling of F on S is an assignment of the elements in S to the vertices of F such that each element in S is assigned to only one vertex. A signed labeling of F on S is a labeling of F on S with each label possibly associated with a minus sign. For example, Figure 2.1 illustrates a signed labeled forest on $\{1, 2, \ldots, 9\}$. We use (F, w) to stand for a plane

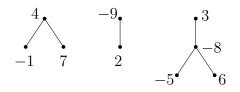


Figure 2.1: A signed labeled forest on $\{1, 2, \ldots, 9\}$

forest F associated with a signed labeling w.

Linear extensions of a signed labeled forest (F, w) are defined based on linear extensions of F. For a plane forest F with n vertices, say x_1, x_2, \ldots, x_n , a linear extension of F is a permutation $x_{i_1}x_{i_2}\cdots x_{i_n}$ of the vertices of F such that $x_{i_j} <_F x_{i_k}$ implies j < k, where $<_F$ is the order relation of F. Let $\mathcal{L}(F)$ denote the set of linear extensions of F. Then the set of linear extensions of (F, w) is defined as

$$\mathcal{L}(F,w) = \left\{ w(x_{i_1})w(x_{i_2})\cdots w(x_{i_n}) \mid x_{i_1}x_{i_2}\cdots x_{i_n} \in \mathcal{L}(F) \right\},\$$

where w(x) denotes the label of a vertex x of F.

Notice that a linear extension of (F, w) is a signed permutation on the labeling set S of F. We next define the generating function $F_n(x)$ of the descent number over linear extensions of the plane forest F_n associated with certain signed labelings. Let us recall the descent number of a signed permutation on a multiset. A signed permutation on a multiset M is a permutation on M for which each element is possibly assigned a minus sign. For example, $\overline{3}1\overline{2}13$ is a signed permutation on $\{1^2, 2, 3^2\}$, where we use a bar to indicate that an element is assigned a minus sign. The descent set of a signed permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ is defined as

$$\{i \mid \sigma_i > \sigma_{i+1}, \ 1 \le i \le n-1\} \cup \{0 \mid \text{if } \sigma_1 < 0\},\tag{2.1}$$

see Savage and Visontai [9]. However, for the purpose of this paper, we use the following reformation of the descent set of a signed permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$

$$Des_B(\sigma) = \{ i \mid \sigma_i > \sigma_{i+1}, \ 1 \le i \le n-1 \} \cup \{ n \mid \text{if } \sigma_n > 0 \}.$$
(2.2)

The number of descents in $\text{Des}_B(\sigma)$ is referred to as the descent number of σ , denoted $\text{des}_B(\sigma)$. In fact, via the bijection

$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \longmapsto \sigma' = (-\sigma_n)(-\sigma_{n-1}) \cdots (-\sigma_1),$$

we see that the descent numbers defined by (2.1) and $\text{Des}_B(\sigma)$ are equidistributed over signed permutations on any multiset.

To define the generating function $F_n(x)$, we introduce some specific single labelings for the plane forest F_n . We write T_1, T_2, \ldots, T_n for the *n* trees of F_n , which are listed from left to right. For $1 \leq i \leq n$, let u_i denote the root of T_i and let v_i denote the child of u_i . Let *w* be a signed labeling of F_n on $\{1, 2, \ldots, 2n\}$ such that for $1 \leq i \leq n$,

$$\{|w(u_i)|, |w(v_i)|\} = \{2i - 1, 2i\}.$$

Let w_i $(1 \le i \le n)$ be the signed labeling of T_i induced by w. There are eight possibilities for each w_i . However, for the purpose of establishing the following equidistribution theorem, we need only four cases as given below:

Case 1:
$$w_i(u_i) = 2i$$
 and $w_i(v_i) = 2i - 1$;
Case 2: $w_i(u_i) = 2i$ and $w_i(v_i) = \overline{2i - 1}$;
Case 3: $w_i(u_i) = \overline{2i}$ and $w_i(v_i) = 2i - 1$;
Case 4: $w_i(u_i) = \overline{2i - 1}$ and $w_i(v_i) = \overline{2i}$.

For the *j*-th case, we say that w_i is of type *j*. Let $L(F_n)$ denote the set of signed labelings of F_n such that the induced labeling w_i of T_i is one of the above four types. We shall show that the descent number over P_n is equidistributed with the descent number over the set of linear extensions of (F_n, w) , where *w* ranges over the set $L(F_n)$. Specifically, define

$$F_n(x) = \sum_{w \in L(F_n)} \sum_{\sigma \in \mathcal{L}(F_n, w)} x^{\operatorname{des}_B(\sigma)}$$

We have the following equidistribution theorem.

Theorem 2.1 For $n \ge 1$, we have

$$F_n(x) = P_n(x).$$

Proof. We proceed to construct a descent preserving bijection ϕ from the set

$$\{\sigma \in \mathcal{L}(F_n, w) \mid w \in L(F_n)\}$$

to the set P_n of signed permutations on $\{1^2, 2^2, \ldots, n^2\}$. Let $\sigma = \sigma_1 \sigma_2 \cdots \sigma_{2n}$ be a linear extension in $\mathcal{L}(F_n, w)$, where $w \in L(F_n)$. Define $\phi(\sigma) = \tau = \tau_1 \tau_2 \cdots \tau_{2n}$ as follows. For $1 \leq j \leq 2n, \tau_j$ has the same sign as σ_j , and $|\tau_j| = i$ if $|\sigma_j| = 2i-1$ or $|\sigma_j| = 2i$. It is routine to check that ϕ is a bijection. Moreover, it is readily verified that $j \in \{1, 2, \ldots, 2n\}$ is a descent of σ if and only if it is a descent of τ . This completes the proof.

The next theorem gives an expression for the generating function $F_n(x)$.

Theorem 2.2 For $n \ge 1$, we have

$$\frac{F_n(x)}{(1-x)^{2n+1}} = \sum_{t \ge 0} (t+1)^n (2t+1)^n x^t.$$
(2.3)

To prove Theorem 2.2, we need a decomposition of P-partitions of type B into σ compatible maps due to Chow [5], where σ is a linear extension of P. When the poset P is associated with an ordinary labeling, type B P-partitions reduce to ordinary Ppartitions introduced by Stanley [10]. To make a connection to Theorem 2.2, it is
sufficient to consider the case when P is a plane forest. In this case, we do not need
the structure of P-partitions of type B in full generality as given by Chow. For the case
when P is a plane forest, a P-partition of type B was described by Chen, Gao and Guo
[4].

Let F be a plane forest, and w be a signed labeling of F. Let \mathbb{N} be the set of nonnegative integers. A (F, w)-partition of type B is a map f from the set of vertices of F to \mathbb{N} that satisfies the following conditions:

- (1) $f(x) \leq f(y)$ if $x \geq_F y$;
- (2) f(x) < f(y) if $x >_F y$ and w(x) < w(y);
- (3) $f(x) \ge 1$ if x is a root of F with w(x) > 0.

Analogous to the decomposition of ordinary *P*-partitions given by Stanley [10], Chow [5] showed that type *B* (*F*, *w*)-partitions can be decomposed into σ -compatible maps, where σ is a linear extension of (*F*, *w*). For a signed permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$, a σ -compatible map *g* is a map from { $\sigma_1, \sigma_2, \ldots, \sigma_n$ } to \mathbb{N} that satisfies the following conditions:

- (1) $g(\sigma_1) \ge g(\sigma_2) \ge \cdots \ge g(\sigma_n);$
- (2) For $i \in \{1, 2, \dots, n-1\}$, $g(\sigma_i) > g(\sigma_{i+1})$ if $\sigma_i > \sigma_{i+1}$;

(3) $g(\sigma_n) \ge 1$ if $\sigma_n > 0$.

Let A(F, w) denote the set of type B(F, w)-partitions, and let A_{σ} denote the set of σ -compatible maps. The following decomposition is due to Chow [5].

Theorem 2.3 ([5, Theorem 2.1.4]) Let F be a plane forest associated with a signed labeling w. Then

$$A(F,w) = \bigcup_{\sigma \in \mathcal{L}(F,w)} A_{\sigma}.$$
 (2.4)

For a nonnegative integer t, let $\Omega_F(w,t)$ denote the number of type B(F,w)partitions f such that $f(x) \leq t$ for any $x \in F$. When w is an ordinary labeling, Stanley [10] has established a relation between the generating function of the descent number over linear extensions of (F,w) and the generating function of $\Omega_F(w,t)$. For signed labeled forests, we have the following relation.

Theorem 2.4 Let F be a plane forest with n vertices, and w be a signed labeling of F on $\{1, 2, ..., n\}$. Then

$$\frac{\sum_{\sigma \in \mathcal{L}(F,w)} x^{\operatorname{des}_B(\sigma)}}{(1-x)^{n+1}} = \sum_{t \ge 0} \Omega_F(w,t) x^t.$$
(2.5)

Proof. We essentially follow the proof of Stanley [10] for ordinary *P*-partitions. For a signed permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ on $\{1, 2, \ldots, n\}$, let $\Omega_{\sigma}(t)$ denote the number of σ -compatible maps g with $g(\sigma_1) \leq t$. For any linear extension σ in $\mathcal{L}(F, w)$, in view of the decomposition (2.4), relation (2.5) can be deduced from the following relation

$$\sum_{t \ge 0} \Omega_{\sigma}(t) x^{t} = \frac{x^{\text{des}_{B}(\sigma)}}{(1-x)^{n+1}}.$$
(2.6)

For $1 \leq i \leq n$, let d_i denote the number of descents of σ that are greater than or equal to i, that is,

$$d_i = |\{j \mid \sigma_j \ge \sigma_{j+1}, \, i \le j \le n-1\} \cup \{n \mid \text{if } \sigma_n > 0\}|.$$

Setting $\lambda_i = g(\sigma_i) - d_i$, we are led to a one-to-one correspondence between the set of σ -compatible maps g with $g(\sigma_1) \leq t$ and the set of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \leq t - \text{des}_B(\sigma)$, where the latter is counted by

$$\binom{n+t-\mathrm{des}_B(\sigma)}{n},$$

see, for example, Stanley [10]. Thus,

$$\sum_{t \ge 0} \Omega_{\sigma}(t) x^{t} = \sum_{t \ge 0} \binom{n + t - \operatorname{des}_{B}(\sigma)}{n} x^{t}$$
$$= \frac{x^{\operatorname{des}_{B}(\sigma)}}{(1 - x)^{n+1}},$$

which agrees with (2.6). This completes the proof.

We are now ready to prove Theorem 2.2.

Proof of Theorem 2.2. By Theorem 2.4, the assertion (2.3) is equivalent to the following relation

$$\sum_{w \in L(F_n)} \Omega_{F_n}(w, t) = \left((t+1)(2t+1) \right)^n.$$
(2.7)

Let T_1, T_2, \ldots, T_n be the *n* trees of F_n listed from left to right. Thus, for any signed labeled forest (F_n, w) , we have

$$\Omega_{F_n}(w,t) = \prod_{i=1}^{n} \Omega_{T_i}(w_i,t),$$
(2.8)

where w_i is the signed labeling of T_i induced by w. Recall that for a signed labeling w in $L(F_n)$, each induced labeling w_i has four choices. For $1 \le j \le 4$, let $w_i^{(j)}$ be the signed labeling of T_i that is of type j, so that the left-hand side of (2.7) can be rewritten as

$$\sum_{w \in L(F_n)} \Omega_{F_n}(w, t) = \prod_{i=1}^n \left(\Omega_{T_i}(w_i^{(1)}, t) + \Omega_{T_i}(w_i^{(2)}, t) + \Omega_{T_i}(w_i^{(3)}, t) + \Omega_{T_i}(w_i^{(4)}, t) \right).$$
(2.9)

We claim that for any $1 \le i \le n$,

$$\Omega_{T_i}(w_i^{(1)}, t) + \Omega_{T_i}(w_i^{(2)}, t) + \Omega_{T_i}(w_i^{(3)}, t) + \Omega_{T_i}(w_i^{(4)}, t) = (t+1)(2t+1).$$
(2.10)

Assume that u_i is the root of T_i and v_i is the child of u_i . Let f be a type $B(T_i, w_i^{(j)})$ -partition such that $f(v_i) \leq t$. Then we have

$$0 < f(u_i) \le f(v_i) \le t, \quad \text{if } j = 1; \\ 0 < f(u_i) \le f(v_i) \le t, \quad \text{if } j = 2; \\ 0 \le f(u_i) < f(v_i) \le t, \quad \text{if } j = 3; \\ 0 \le f(u_i) \le f(v_i) \le t, \quad \text{if } j = 4. \end{cases}$$

It follows that

$$\Omega_{T_i}(w_i^{(1)}, t) = \Omega_{T_i}(w_i^{(2)}, t) = \Omega_{T_i}(w_i^{(3)}, t) = \binom{t+1}{2}$$
(2.11)

and

$$\Omega_{T_i}(w_i^{(4)}, t) = \binom{t+2}{2}.$$
(2.12)

Hence we obtain (2.10), completing the proof.

In addition to Theorem 2.1 and Theorem 2.2, a formula of Savage and Schuster [8] is needed to prove Conjecture 1.1. Recall that I_n is the set of *s*-inversion sequences of length *n* for the specific sequence s = (1, 4, 3, 8, 5, 12, ...), and $I_n(x)$ is the generating function of the ascent number over I_n . Savage and Schuster [8, Theorem 13] showed that for $n \ge 1$,

$$\frac{I_n(x)}{(1-x)^{n+1}} = \sum_{t \ge 0} (t+1)^{\lceil \frac{n}{2} \rceil} (2t+1)^{\lfloor \frac{n}{2} \rfloor} x^t.$$
(2.13)

Replacing n by 2n in (2.13), we get

$$\frac{I_{2n}(x)}{(1-x)^{2n+1}} = \sum_{t \ge 0} (t+1)^n (2t+1)^n x^t.$$
(2.14)

Comparing (2.14) with (2.3), we obtain $F_n(x) = I_{2n}(x)$. By Theorem 2.1 we arrive at $P_n(x) = I_{2n}(x)$, completing the proof of Conjecture 1.1.

3 Further equidistributions

In this section, we give characterizations of three sets of signed permutations over which the descent number is equidistributed with the ascent number over the sets I_{2n-1} , I'_{2n-1} and I'_{2n} respectively.

Recall that I_{2n-1} stands for the set of *s*-inversion sequences of length 2n - 1 for s = (1, 4, 3, 8, 5, 12, ...), and I'_n stands for the set of *s*-inversion sequences of length *n* for s = (2, 2, 6, 4, 10, 6, ...). Let U_n be the set of signed permutations on the multiset $\{1^2, 2^2, \ldots, (n-1)^2, n\}$, and let V_n be the subset of U_n consisting of signed permutations such that the element *n* carries a minus sign. We show that the descent number over V_n is equidistributed with the ascent number over I_{2n-1} , the descent number over U_n is equidistributed with the ascent number over I'_{2n-1} , and the descent number over P_n is equidistributed with the ascent number over I'_{2n} .

Theorem 3.1 For $n \ge 1$, we have

$$\sum_{\sigma \in V_n} x^{\operatorname{des}_B(\sigma)} = \sum_{e \in I_{2n-1}} x^{\operatorname{asc}(e)}.$$
(3.1)

Proof. Let $V_n(x)$ denote the sum on the left-hand side of (3.1). Since a relation on the generating function $I_{2n-1}(x)$ is given by (2.13), it suffices to show that $V_n(x)$ satisfies the

same relation as $I_{2n-1}(x)$, that is, for $n \ge 1$,

$$\frac{V_n(x)}{(1-x)^{2n}} = \sum_{t \ge 0} (t+1)^n (2t+1)^{n-1} x^t.$$
(3.2)

We claim that $V_n(x)$ equals the generating function of the descent number over linear extensions of certain signed labeled forests. To this end, let F'_n be the plane forest which is obtained from F_{n-1} by adding a single vertex as the rightmost component. Let $T_1, T_2, \ldots, T_{n-1}$ denote n-1 trees of F_{n-1} , and let T_n denote a single vertex. Hence F'_n consists of plane trees $T_1, \ldots, T_{n-1}, T_n$. Write $L(F'_n)$ for the set of signed labelings w of F'_n such that $w(T_n) = -(2n-1)$ and the induced signed labeling of w on F_{n-1} belongs to $L(F_{n-1})$. Set

$$F'_n(x) = \sum_{w \in L(F'_n)} \sum_{\sigma \in \mathcal{L}(F'_n, w)} x^{\operatorname{des}_B(\sigma)}.$$
(3.3)

Using the same reasoning as in the proof of Theorem 2.1, one can construct a descent preserving bijection between the set

$$\{\sigma \in \mathcal{L}(F'_n, w) \, | \, w \in L(F'_n)\}$$

and the set V_n . Hence we get

$$F'_{n}(x) = V_{n}(x),$$
 (3.4)

so that (3.2) can be rewritten as

$$\frac{F'_n(x)}{(1-x)^{2n}} = \sum_{t \ge 0} (t+1)^n (2t+1)^{n-1} x^t.$$
(3.5)

Applying Theorem 2.4 to the set of signed labeled forests (F'_n, w) with $w \in L(F'_n)$, we obtain

$$\frac{F'_n(x)}{(1-x)^{2n}} = \sum_{t \ge 0} \sum_{w \in L(F'_n)} \Omega_{F'_n}(w,t) \, x^t.$$

Thus, (3.5) can be deduced from the following relation

$$\sum_{w \in L(F'_n)} \Omega_{F'_n}(w, t) = (t+1)^n (2t+1)^{n-1}.$$
(3.6)

Notice that for a signed labeling w in $L(F'_n)$, each induced labeling w_i of T_i for $1 \leq i \leq n-1$ has four types, and the induced labeling w_n of T_n satisfies $w_n(T_n) = -(2n-1)$. For $1 \leq i \leq n-1$ and $1 \leq j \leq 4$, let $w_i^{(j)}$ be the signed labeling of T_i that is of type j. In the proof of Theorem 2.2, it has been shown that for $1 \leq i \leq n-1$,

$$\Omega_{T_i}(w_i^{(1)}, t) + \Omega_{T_i}(w_i^{(2)}, t) + \Omega_{T_i}(w_i^{(3)}, t) + \Omega_{T_i}(w_i^{(4)}, t) = (t+1)(2t+1).$$
(3.7)

Moreover, it is clear that $\Omega_{T_n}(w_n, t) = t + 1$. Hence we deduce that

$$\sum_{w \in L(F'_n)} \Omega_{F'_n}(w,t) = \Omega_{T_n}(w_n,t) \prod_{i=1}^{n-1} \left(\Omega_{T_i}(w_i^{(1)},t) + \Omega_{T_i}(w_i^{(2)},t) + \Omega_{T_i}(w_i^{(3)},t) + \Omega_{T_i}(w_i^{(4)},t) \right)$$
$$= (t+1)^n (2t+1)^{n-1},$$

as required.

We now show that the descent number over P_n and U_n is equidistributed with the ascent number over I'_{2n} and I'_{2n-1} , respectively.

Theorem 3.2 For $n \ge 1$, we have

$$\sum_{\sigma \in P_n} x^{\operatorname{des}_B(\sigma)} = \sum_{e \in I'_{2n}} x^{\operatorname{asc}(e)}$$
(3.8)

and

$$\sum_{\sigma \in U_n} x^{\operatorname{des}_B(\sigma)} = \sum_{e \in I'_{2n-1}} x^{\operatorname{asc}(e)}.$$
(3.9)

To prove Theorem 3.2, we need the following formulas (3.11) and (3.12) of Savage and Schuster [8] for the generating function of the ascent number over *s*-inversion sequences of length *n*. For a sequence $s = (s_1, s_2, ...)$ of positive integers, let $f_n^{(s)}(t)$ denote the number of sequences $(\lambda_1, \lambda_2, ..., \lambda_n)$ of nonnegative integers such that

$$0 \le \frac{\lambda_1}{s_1} \le \frac{\lambda_2}{s_2} \le \dots \le \frac{\lambda_n}{s_n} \le t.$$
(3.10)

Theorem 3.3 ([8, Theorem 5]) Let $s = (s_1, s_2, ...)$ be a sequence of positive integers. Then

$$\sum_{t \ge 0} f_n^{(s)}(t) x^t = \frac{1}{(1-x)^{n+1}} \sum_e x^{\operatorname{asc}(e)},$$
(3.11)

where e ranges over s-inversion sequences of length n.

In particular, for s = (2, 2, 6, 4, 10, 6, ...) and

$$s' = s/2 = (1, 1, 3, 2, 5, 3, \ldots),$$

Savage and Schuster [8, Theorem 14] showed that

$$f_n^{(s')}(t) = (t+1)^{\lceil \frac{n}{2} \rceil} \left(\frac{t+2}{2}\right)^{\lfloor \frac{n}{2} \rfloor}.$$
 (3.12)

We now turn to the proof of Theorem 3.2.

Proof of Theorem 3.2. It follows from (3.12) that

$$f_n^{(s)}(t) = f_n^{(s')}(2t) = (t+1)^{\lfloor \frac{n}{2} \rfloor} (2t+1)^{\lceil \frac{n}{2} \rceil}.$$

By Theorem 3.3, we obtain that

$$\frac{\sum\limits_{e \in I'_n} x^{\operatorname{asc}(e)}}{(1-x)^{n+1}} = \sum_{t \ge 0} (t+1)^{\lfloor \frac{n}{2} \rfloor} (2t+1)^{\lceil \frac{n}{2} \rceil} x^t.$$
(3.13)

To prove (3.8), we replace n by 2n in (3.13) to get

$$\frac{\sum\limits_{e \in I'_{2n}} x^{\operatorname{asc}(e)}}{(1-x)^{2n+1}} = \sum_{t \ge 0} (t+1)^n (2t+1)^n x^t.$$
(3.14)

Combining the above relation and the formula (2.14) for $I_{2n}(x)$, we obtain

$$\sum_{e \in I'_{2n}} x^{\operatorname{asc}(e)} = I_{2n}(x).$$

Thus (3.8) follows from the fact that $I_{2n}(x) = P_n(x)$.

Next we prove (3.9). Replacing n by 2n - 1 in (3.13), we get

$$\frac{\sum\limits_{e \in I'_{2n-1}} x^{\operatorname{asc}(e)}}{(1-x)^{2n}} = \sum_{t \ge 0} (t+1)^{n-1} (2t+1)^n x^t.$$

Hence (3.9) can be deduced from the following relation

$$\frac{\sum\limits_{\sigma \in U_n} x^{\deg_B(\sigma)}}{(1-x)^{2n}} = \sum_{t \ge 0} (t+1)^{n-1} (2t+1)^n x^t.$$
(3.15)

To prove (3.15), let

$$U_n(x) = \sum_{\sigma \in U_n} x^{\operatorname{des}_B(\sigma)}.$$

We claim that $U_n(x)$ coincides with the generating function of the descent number over linear extensions of certain signed labeled forests. In the notation F'_n, T_1, \ldots, T_n as defined in the proof of Theorem 3.1, we use $\overline{L}(F'_n)$ to denote the set of signed labelings w of F'_n such that $w(T_n) = 2n - 1$ or -(2n - 1) and the induced labeling of w on F_{n-1} belongs to $L(F_{n-1})$. Let

$$G_n(x) = \sum_{w \in \overline{L}(F'_n)} \sum_{\sigma \in \mathcal{L}(F'_n, w)} x^{\operatorname{des}_B(\sigma)}.$$

Again, using the argument in the proof of Theorem 2.1, one can construct a descent preserving bijection between the set

$$\left\{\sigma\in\mathcal{L}(F'_n,w)\,|\,w\in\overline{L}(F'_n)\right\}$$

and the set U_n . It then follows that

$$G_n(x) = U_n(x). \tag{3.16}$$

Therefore, (3.15) is equivalent to

$$\frac{G_n(x)}{(1-x)^{2n}} = \sum_{t \ge 0} (t+1)^{n-1} (2t+1)^n x^t.$$
(3.17)

Applying Theorem 2.4 to the set of signed labeled forests (F'_n, w) with $w \in \overline{L}(F'_n)$, we find that

$$\frac{G_n(x)}{(1-x)^{2n}} = \sum_{t \ge 0} \sum_{w \in \overline{L}(F'_n)} \Omega_{F'_n}(w,t) x^t.$$

Hence (3.17) can be deduced from the following relation

$$\sum_{w \in \overline{L}(F'_n)} \Omega_{F'_n}(w, t) = (t+1)^{n-1} (2t+1)^n.$$
(3.18)

To prove (3.18), for $1 \le i \le n-1$ and $1 \le j \le 4$, let $w_i^{(j)}$ be the signed labeling of T_i that is of type j, and let w'_n and w''_n be the signed labelings of T_n such that $w'_n(T_n) = 2n-1$ and $w''_n(T_n) = -(2n-1)$. As shown in the proof of Theorem 2.2, for $1 \le i \le n-1$,

$$\Omega_{T_i}(w_i^{(1)}, t) + \Omega_{T_i}(w_i^{(2)}, t) + \Omega_{T_i}(w_i^{(3)}, t) + \Omega_{T_i}(w_i^{(4)}, t) = (t+1)(2t+1).$$

Evidently, $\Omega_{T_n}(w'_n, t) = t$ and $\Omega_{T_n}(w''_n, t) = t + 1$. Hence the sum on the left-hand side of (3.18) equals

$$(\Omega_{T_n}(w'_n, t) + \Omega_{T_n}(w''_n, t)) \prod_{i=1}^{n-1} \left(\Omega_{T_i}(w_i^{(1)}, t) + \Omega_{T_i}(w_i^{(2)}, t) + \Omega_{T_i}(w_i^{(3)}, t) + \Omega_{T_i}(w_i^{(4)}, t) \right)$$

= $(t+1)^{n-1}(2t+1)^n$,

as required.

Acknowledgments. This work was supported by the 973 Project and the National Science Foundation of China.

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