UNIVERSALITY OF RANDOM GRAPHS FOR GRAPHS OF MAXIMUM DEGREE TWO

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Abstract. For a family $\mathcal{F}$ of graphs, a graph G is called $\mathcal{F}$-*universal* if G contains every graph in $\mathcal{F}$ as a subgraph. Let $\mathcal{F}_n(d)$ be the family of all graphs on n vertices with maximum degree at most d. Dellamonica, Kohayakawa, Rödl and Ruciński [17] showed that, for $d \geq 3$, the random graph $G(n,p)$ is $\mathcal{F}_n(d)$-universal with high probability provided $p \geq C\big(\frac{\log n}{n}\big)^{1/d}$ for a sufficiently large constant $C = C(d)$. In this paper we prove the missing part of the result, that is, the random graph $G(n,p)$ is $\mathcal{F}_n(2)$-universal with high probability provided $p \geq C\big(\frac{\log n}{n}\big)^{1/2}$ for a sufficiently large constant C.

1. Introduction

For a positive integer n and a real number p in the range $0 \leq p \leq 1$, the random graph $G(n,p)$ on a set V of n elements may be obtained from the complete graph on V by choosing each edge with probability p, independently of all other edges.

After the random graph $G(n,p)$ was first introduced by Erdős [22] in 1947, the theory of the random graph has become an active area of research. One of the most interesting problems is the containment problem, in which one tries to obtain conditions on p for the property that $G(n,p)$ contains a given graph H as a subgraph with high probability. For example, when n is even and the given graph H is a perfect matching on V, then it is easy to see that $np - \log n \to \infty$ is a necessary condition, as there is an isolated vertex with substantial probability if $np - \log n$ is bounded. Erdős and Rényi [23] showed the condition is also sufficient. In the case that H is a Hamiltonian cycle, Komlós and Szemerédi [29] and Korshunov [30] discovered that the easy necessary condition $np - \log n - \log\log n \to \infty$ is also sufficient. More generally, if H is a factor of a strictly balanced graph, including a triangle, a cycle or a complete graph, Johansson, Kahn and Vu [28] determined a necessary and sufficient condition for the containment problem with respect to H. For more information, the reader is referred to Bollobás [11] and Janson, Łuczak and Ruciński [26] and the references therein.

One may also consider a family $\mathcal{F}$ of graphs rather than a single graph H. For a family $\mathcal{F}$ of graphs, a graph G is called $\mathcal{F}$-*universal* if G contains every graph in $\mathcal{F}$ as a subgraph. There is extensive research on $\mathcal{F}$-universal graphs when $\mathcal{F}$ are families of trees [13, 16], spanning trees [10,

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14, 15, 24], planar graphs of bounded degree [10], graphs of bounded size [7, 32], graphs of bounded degree [2, 3, 4, 5, 12], and spanning graphs of bounded degree [1, 27], etc.

Since an $\mathcal{F}$-universal graph G must have the maximum degree greater than or equal to the maximum degrees of all graphs in $\mathcal{F}$, a family $\mathcal{F}$ of graphs of bounded degree may be considered. For example, one may consider the family $\mathcal{T}_n(d)$ of all trees on n vertices with maximum degree at most d. Bhatt, Chung, Leighton and Rosenberg [10] showed that there is a $\mathcal{T}_n(d)$-universal graph on n vertices with maximum degree depending only on d. For $d \geq \log n$, Johannsen, Krivelevich and Samotij [27] proved that there is a positive constant c such that $G(n,p)$ with $p \geq cdn^{-1/3} \log n$ is asymptotically almost surely (a.a.s.) $\mathcal{T}_n(d)$-universal; where, in general, a property holds *asymptotically almost surely*, or simply *a.a.s.*, if it holds with probability tending to 1 as $n \to \infty$. In particular, we have that $G(n,p)$ with $p \geq cn^{-1/3}(\log n)^2$ is a.a.s. $\mathcal{T}_n(\log n)$-universal, and hence, $\mathcal{T}_n(d)$-universal for a constant d. For the family $\mathcal{T}_{(1-\varepsilon)n}(d)$ of all trees on $(1-\varepsilon)n$ vertices with maximum degree at most d, Alon, Krivelevich and Sudakov [6] showed that for every positive constant ε and positive integer d, there exists a constant $c = c(\varepsilon, d)$ such that $G(n,p)$ with $p = c/n$ is a.a.s. $\mathcal{T}_{(1-\varepsilon)n}(d)$-universal. For more related results, [19, 8, 9] may be referred.

In this paper, we consider the family $\mathcal{F}_n(d)$ of all graphs on n vertices with maximum degree at most d. For an even n and $d = 1$, the $\mathcal{F}_n(1)$-universality is equivalent to the containment problem for a perfect matching. Provided that n is divisible by $d+1$, one may easily see that $p \geq \left(\frac{(\log n)^{1/d}}{n}\right)^{2/(d+1)}$ is a necessary condition for $G(n,p)$ being a.a.s. $\mathcal{F}_n(d)$-universal, since $\mathcal{F}_n(d)$ contains a K_{d+1}-factor, and hence, every vertex must be contained in a copy of K_{d+1}. On the other hand, Dellamonica, Kohayakawa, Rödl and Ruciński [20, 21] proved that $p \geq C\left(\frac{\log n}{n}\right)^{1/(2d)}$ is sufficient, for a sufficiently large constant C.

Theorem 1 ([20, 21])**.** *For every integer $d \geq 2$, there exists a positive constant $C = C(d)$ such that if $p \geq C\left(\frac{(\log n)^2}{n}\right)^{1/(2d)}$, then the random graph $G(n,p)$ is a.a.s. $\mathcal{F}_n(d)$-universal.*

Recently the above result was notably improved for $d \geq 3$.

Theorem 2 ([17, 18])**.** *For every integer $d \geq 3$, there exists a positive constant $C = C(d)$ such that if $p \geq C\left(\frac{\log n}{n}\right)^{1/d}$, then the random graph $G(n,p)$ is a.a.s. $\mathcal{F}_n(d)$-universal.*

In this paper, we show that the statement of Theorem 2 holds for $d = 2$.

Theorem 3. *There exists a positive constant C such that if $p \geq C\left(\frac{\log n}{n}\right)^{1/2}$, then the random graph $G(n,p)$ is a.a.s. $\mathcal{F}_n(2)$-universal.*

The rest of this paper is organized as follows. In the next section we define a notion of a '*good*' graph and introduce two main lemmas, which imply Theorem 3. Sections 3 and 4 are for the proofs of the two lemmas.

In this paper, we will use the following notation and convention.

Notation and convention: For a graph G and $v \in V(G)$, the set $N_G(v)$ denotes the set of neighbors of v in G. Similarly, for $U \subset V(G)$, the set $N_G(U)$ denotes the set of vertices which are adjacent to a vertex in U. The graph $G[U]$ denotes the subgraph of G induced on U. For simplicity, we omit floor and ceiling symbols when they are not essential.

2. Good graph and two lemmas

In order to show Theorem 3, by monotonicity, it suffices to show the statement of Theorem 3 with $p = C\left(\frac{\log n}{n}\right)^{1/2}$ for a sufficiently large constant C. Hence, from now on, we fix p as $p = C\left(\frac{\log n}{n}\right)^{1/2}$. Throughout the paper, we let

$$\delta = 0.01 \quad \text{and} \quad \varepsilon = 0.001.$$

Now we provide the definition of a '*good*' graph. Let V be a vertex set on n vertices. We fix a partition $V = V_0 \cup V_1 \cup \cdots \cup V_6$ such that

$$|V_1| = \cdots = |V_6| = \varepsilon n \quad \text{and} \quad |V_0| = (1-6\varepsilon)n \geq (3/4)n.$$

For a graph G on V and $k = 1$ or 2, let $U \subset V$ and $\mathcal{L}$ be a collection of pairwise disjoint k-subsets of $V \setminus U$. We consider a bipartite graph $B(\mathcal{L}, U)$ between $\mathcal{L}$ and U, in which $L \in \mathcal{L}$ and $u \in U$ are adjacent if and only if $L \subset N_G(u)$.

Now we are ready to define a good graph.

Definition 4. *A graph G on V is called '(n, C)-good' if the following properties hold.*

$(P1)$ *There exists a matching $\mathcal{M}$ of $G[V_0]$ with $|\mathcal{M}| = 2\varepsilon n$ such that for all $U \subset V \setminus V(\mathcal{M})$ with $|U| \leq \dfrac{\delta n}{C^2 \log n}$, we have*

$$\left|\left\{\{a,b\} \in \mathcal{M} \;\middle|\; a \sim u, b \sim u \text{ for some } u \in U\right\}\right| \geq \frac{C^2 \log n}{16n}|\mathcal{M}||U|.$$

$(P2)$ *Let $k = 1$ or 2, and $\mathcal{L}$ be a collection of pairwise disjoint k-subsets of V.*

If $|\mathcal{L}| \leq \dfrac{\delta}{C^k}\left(\dfrac{n}{\log n}\right)^{k/2}$, then, for V_i with $V_i \cap \left(\bigcup_{L\in\mathcal{L}} L\right) = \emptyset$, $i = 1, ..., 6$, we have that

$$|N_{B(\mathcal{L},V_i)}(\mathcal{L})| \geq (1-\delta)C^k\left(\frac{\log n}{n}\right)^{k/2}|\mathcal{L}||V_i|. \tag{2.1}$$

If $|\mathcal{L}| \geq \dfrac{\log n}{C^{k-1}}\left(\dfrac{n}{\log n}\right)^{k/2}$, then, for all U with $|U| \geq \dfrac{\log n}{C^{k-1}}\left(\dfrac{n}{\log n}\right)^{k/2}$ and $U \cap \left(\bigcup_{L\in\mathcal{L}} L\right) = \emptyset$, the graph $B(\mathcal{L}, U)$ has at least one edge.

$$\left(\text{No requirement is needed when } \frac{\delta}{C^k}\left(\frac{n}{\log n}\right)^{k/2} < |\mathcal{L}| < \frac{\log n}{C^{k-1}}\left(\frac{n}{\log n}\right)^{k/2}\right).$$

Remark 5. *For $p = C\left(\frac{\log n}{n}\right)^{1/2}$, the above inequality (2.1) may be written as*

$$|N_{B(\mathcal{L},V_i)}(\mathcal{L})| \geq (1-\delta)p^k|\mathcal{L}||V_i|.$$

Notice that $p^k|\mathcal{L}||V_i|$ is the expected number of edges in $B(\mathcal{L}, V_i)$ if G were $G(n,p)$. It is easy to see that only few vertices of V_i are of degree 2 or more in $B(\mathcal{L}, V_i)$. Hence, $|N_{B(\mathcal{L},V_i)}(\mathcal{L})|$ is almost the same as the number of edges in $B(\mathcal{L}, V_i)$.

We will show the following two lemmas.

Lemma 6. *There exists a positive constant C such that an (n, C)-good graph is $\mathcal{F}_n(2)$-universal provided that n is sufficiently large.*

Lemma 7. *There exists a positive constant C such that the random graph $G(n, p)$ on V with $p = C\big(\frac{\log n}{n}\big)^{1/2}$ is a.a.s. (n, C)-good.*

Our proof of Lemmas 6 and 7 will be given in Sections 3 and 4, respectively. Theorem 3 clearly follows from Lemmas 6 and 7.

3. Universality of good graph

For the proof of Lemma 6, we may assume that H is a maximal graph in $\mathcal{F}_n(2)$, in the sense that no edge may be added to H to be a graph in $\mathcal{F}_n(2)$. Then, it is easy to see that all but at most one vertex of H have degree 2. We will show that there exists a positive constant C such that, for a sufficiently large n, an (n, C)-good graph G contains a copy of H as a subgraph. To this end, a partition of $W := V(H)$ will be used, and each part will be embedded at a time. A subset of W is called *k-independent* in H if the distance between every distinct pair of vertices in the subset is greater than k.

Lemma 8. *Let H be a maximal graph in $\mathcal{F}_n(2)$. Then there is a partition $W := V(H) = W_0 \cup W_1 \cup \cdots \cup W_6$ with*

$$|W_0| = 4\varepsilon n, \quad |W_6| = 2\varepsilon n, \quad |W_i| \geq 2\varepsilon n, \;\; i = 1, 2, ..., 5, \tag{3.1}$$

such that

(1) $W_1, ..., W_5$ are 2-independent.
(2) W_6 is 3-independent, and all vertices of W_6 are of degree 2.
(3) $W_0 = N_H(W_6)$.

Proof. We first construct W_6 and W_0. Since the maximum degree of H is 2, for each vertex v in H, there are at most 6 vertices that are of distance 3 or less from v, excluding v itself. By the greedy algorithm, it is easy to see that there is a 3-independent set of size at least $n/7$. Hence, we may choose W_6 satisfying $|W_6| = 2\varepsilon n$ and (2) as there is at most one vertex of degree less than 2. Let $W_0 := \bigcup_{w \in V_6} N_H(w)$. Clearly, $|W_0| = 4\varepsilon n$ as W_6 is 3-independent.

Next, we consider W_i for $1 \leq i \leq 5$. Let H^2 be the graph on the vertex set W in which two vertices are adjacent if and only if two vertices are of distance at most 2 in H. Since H has the maximum degree 2, the maximum degree of H^2 is at most 4. Using Hajnal–Szemerédi Theorem [25], we may partition W into 5 independent sets of H^2 so that each part is of size at least $n/5 - 1$. By removing all vertices in $W_0 \cup W_6$ from each part, $W_1, W_2, ..., W_5$ may be obtained. Then, it is clear that each W_i is 2-independent in H and $|W_i|$ is at least $n/5 - 1 - 6\varepsilon n \geq 2\varepsilon n$, for $i = 1, ..., 5$. □

A bijection from W to $V = V(G)$ is called an *embedding* of H to G if it maps each edge of H to an edge of G. We now find an embedding of H to G using an algorithm modified from the embedding algorithm in [17, 18]: Take a partition $W_0, ..., W_6$ of W as described in Lemma 8. We will embed W_i into $V_0 \cup \cdots \cup V_i$.

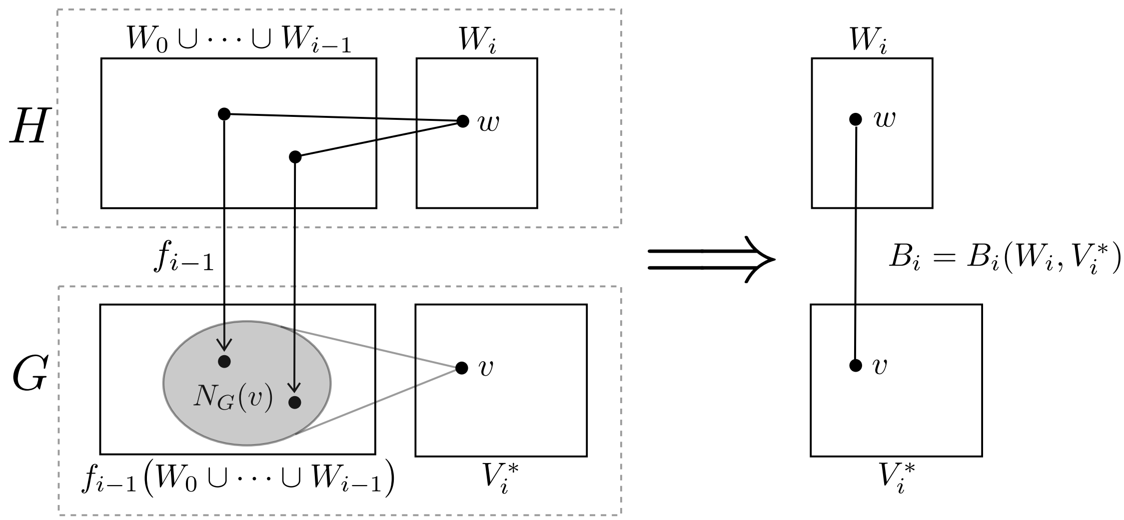

FIGURE 1. The bipartite graph $B_i = B_i(W_i, V_i^*)$

To map W_0 into V_0, recall that $W_0 = N_H(W_6)$ and $|N_H(w)| = 2$ for all $w \in W_6$. For a matching $\mathcal{M} = \{e_1, ..., e_{2\varepsilon n}\}$ in $G[V_0]$ with (P1) and $W_6 = \{w_1, ..., w_{2\varepsilon n}\}$, it is enough for us to take a bijection, say f_0, from W_0 to $V(\mathcal{M})$ such that $N_H(w_i)$ is mapped to e_i, where $V(\mathcal{M})$ is the set of end vertices of all edges in $\mathcal{M}$.

The mapping f_0 is an embedding of W_0 to V_0: Since V_6 is a 3-independent set in H, the sets $N_H(w)$, $w \in V_6$, are pairwise disjoint and there is no edge between them. Hence, every edge e in W_0 belongs to $N_H(w)$ for some $w \in W_6$. As every $N_H(w)$ is mapped to an edge in $\mathcal{M} \subset E(G[V_0])$ under f_0, the edge e is mapped to an edge in $G[V_0]$.

Assuming an embedding

$$f_{i-1} : W_0 \cup W_1 \cup \cdots \cup W_{i-1} \to V_0 \cup V_1 \cup \cdots \cup V_{i-1}$$

is defined, $i = 1, 2, ..., 6$, we will embed W_i into $V_i^* := V_i \cup (V_0 \cup V_1 \cup \cdots \cup V_{i-1}) \setminus \mathrm{Image}(f_{i-1})$ to extend f_{i-1} to an embedding

$$f_i : W_0 \cup W_1 \cup \cdots \cup W_i \to V_0 \cup V_1 \cup \cdots \cup V_i.$$

Let $B_i(W_i, V_i^*)$, or just B_i, be the bipartite graph in which $w \in W_i$ and $v \in V_i^*$ are adjacent if and only if

$$f_{i-1}\Big(N_H(w) \cap (W_0 \cup \cdots \cup W_{i-1})\Big) \subset N_G(v).$$

(See Figure 1). Or equivalently, for $L_i(w) := f_{i-1}\Big(N_H(w) \cap (W_0 \cup \cdots \cup W_{i-1})\Big)$, $\mathcal{L}_i := \Big\{L_i(w) : w \in W_i\Big\}$ and the bipartite graph $B(\mathcal{L}_i, V_i^*)$ defined just before Definition 4,

$$w \sim v \text{ in } B_i(W_i, V_i^*) \quad \text{if and only if} \quad L_i(w) \sim v \text{ in } B(\mathcal{L}_i, V_i^*). \tag{3.2}$$

If possible, take a W_i-matching of B_i, i.e., a matching in B_i that covers all vertices in W_i. (Later, we will show that this is possible). The image of $w \in W_i$ under the mapping f_i is defined to be the vertex in V_i^* that is matched to w in the W_i-matching. For $w \notin W_i$, $f_i(w) = f_{i-1}(w)$.

The mapping f_i is an embedding of W_i to V_i^*: For an edge e in $W_0 \cup \cdots \cup W_i$, at most one end point of e is in W_i since W_i is 2-independent, especially independent. If both ends of e are in $W_0 \cup \cdots \cup W_{i-1}$, then $f_i(e) = f_{i-1}(e)$ is an edge in G. If one end, say w, of e is in W_i, then the other end, say w', is in $N_H(w) \cap (W_0 \cup \cdots \cup W_{i-1})$. Hence, $w \sim f_i(w)$ in $B_i(W_i, V_i^*)$ implies that

$$f_{i-1}\Big(N_H(w) \cap (W_0 \cup \cdots \cup W_{i-1})\Big) \subset N_G(f_i(w)),$$

in particular, $f_i(w') = f_{i-1}(w') \in N_G(f_i(w))$, i.e., $\{f_i(w'), f_i(w)\}$ is an edge.

It remains to show that there exists a W_i-matching in $B_i = B_i(W_i, V_i^*)$ for $i = 1, ..., 6$. We first show the following, which guarantees Hall's condition for a subset U of W_i satisfying some condition.

Lemma 9. *Let $i = 1, ..., 6$. If $U \subset W_i$ satisfying $|U| \leq |V_i^*| - n/C$, then*

$$|N_{B_i}(U)| \geq |U|.$$

Proof. Let $U = U_0 \cup U_1 \cup U_2$, where

$$U_j := \Big\{w \in W_i : |N_H(w) \cap (W_0 \cup \cdots \cup W_{i-1})| = j\Big\}.$$

If $U_0 \neq \emptyset$, then $N_{B_i}(U) = V_i^*$ and hence $|N_{B_i}(U)| = |V_i^*| > |U|$ as $|U| \leq |V_i^*| - n/C$. We now assume that $U_0 = \emptyset$. Take U_k such that $|U_k| \geq |U|/2$.

Case 1: the case when $|U_k| \leq \frac{\delta}{C^k}\big(\frac{n}{\log n}\big)^{k/2}$. For $L_i(u) := f_{i-1}\Big(N_H(u) \cap (W_1 \cup \cdots \cup W_{i-1})\Big)$ and $\mathcal{L}(U_k) = \Big\{L_i(u) : u \in U_k\Big\}$, We have that

$$N_{B(\mathcal{L}_i, V_i^*)}\big(\mathcal{L}(U_k)\big) \cap V_i \subset N_{B_i}(U_k) :$$

For $v \in V_i$ with $L(u) \sim v$ in $B(\mathcal{L}_i, V_i^*)$ for some $u \in U_k$, it follows from (3.2) that $u \sim v$ in B_i for $u \in U_k$, or equivalently, $v \in N_{B_i}(U_k)$. Notice that

$$N_{B(\mathcal{L}_i, V_i^*)}\big(\mathcal{L}(U_k)\big) \cap V_i = N_{B(\mathcal{L}(U_k), V_i)}\Big(\mathcal{L}(U_k)\Big).$$

Property (P2) implies that

$$\begin{aligned} |N_{B_i}(U_k)| &\geq |N_{B(\mathcal{L}(U_k), V_i)}(\mathcal{L}(U_k))| \geq (1-\delta)C^k \varepsilon n \Big(\frac{\log n}{n}\Big)^{k/2} |U_k| \\ &\geq \frac{\varepsilon C^2 (\log n)}{2} |U_k|, \end{aligned} \tag{3.3}$$

as $k = 1$ or 2. In particular, $|N_{B_i}(U_k)| \geq 2|U_k| \geq |U|$.

Case 2: the case when $\frac{\delta}{C^k}\big(\frac{n}{\log n}\big)^{k/2} < |U_k| \leq \frac{\log n}{C^{k-1}}\big(\frac{n}{\log n}\big)^{k/2}$. Taking a subset U_k' of U_k of size $\frac{\delta}{C^k}\big(\frac{n}{\log n}\big)^{k/2}$, it follows from (3.3) that

$$\begin{aligned} |N_{B_i}(U_k)| &\geq |N_{B_i}(U_k')| \geq \frac{\varepsilon \delta C^{2-k} (\log n)}{2} \Big(\frac{n}{\log n}\Big)^{k/2} \\ &= \frac{\varepsilon \delta C}{2} \cdot \frac{\log n}{C^{k-1}} \Big(\frac{n}{\log n}\Big)^{k/2} \geq 2|U_k| \geq |U| \end{aligned}$$

as C is sufficiently large and ε and δ are absolute constants.

Case 3: the case when $|U_k| > \frac{\log n}{C^{k-1}}\left(\frac{n}{\log n}\right)^{k/2}$. We will show that

$$|N_{B_i}(U_k)| \geq |V_i^*| - n/C \ \ (\geq |U|).$$

We first observe that there is no edge of $B_i = B_i(W_i, V_i^*)$ between U_k and $V_i^* \setminus N_{B_i}(U_k)$. Hence, for $\mathcal{L}(U_k)$ defined as in Case 1, it follows from (3.2) that there is no edge of $B(\mathcal{L}_i, V_i^*)$ between $\mathcal{L}(U_k)$ and $V_i^* \setminus N_{B_i}(U_k)$. This means that the induced subgraph $B\Big(\mathcal{L}(U_k), V_i^* \setminus N_{B_i}(U_k)\Big)$ of $B(\mathcal{L}_i, V_i^*)$ has no edge. Since $|\mathcal{L}(U_k)| = |U_k| > \frac{\log n}{C^{k-1}}\left(\frac{n}{\log n}\right)^{k/2}$, the property (P2) yields that

$$|V_i^* \setminus N_{B_i}(U_k)| < \frac{\log n}{C^{k-1}}\left(\frac{n}{\log n}\right)^{k/2} \leq \frac{n}{C},$$

which is equivalent to $|N_{B_i}(U_k)| \geq |V_i^*| - n/C$ as desired. □

Corollary 10. *For $i = 1, ..., 5$, there exists a W_i-matching in $B_i(W_i, V_i^*)$.*

Proof. One can easily show that $|W_i| < |V_i^*| - n/C$ for $i = 1, ..., 5$. Indeed, we have

$$\begin{aligned} |V_i^*| &= |V_0 \cup \cdots \cup V_i| - |W_0 \cup \cdots \cup W_{i-1}| \\ &= |W_i \cup \cdots \cup W_6| - |V_{i+1} \cup \cdots \cup V_6|, \end{aligned}$$

and

$$\begin{aligned} |V_i^*| - |W_i| &= |W_{i+1} \cup \cdots \cup W_6| - |V_{i+1} \cup \cdots \cup V_6| \\ &\geq (6-i)2\varepsilon n - (6-i)\varepsilon n = (6-i)\varepsilon n \geq \varepsilon n > \frac{n}{C} \end{aligned}$$

where the first inequality follows from (3.1) and the last inequality holds for a sufficiently large constant C. Clearly, for all $U \subset W_i$, we have $|U| < |V_i^*| - n/C$ for $1 \leq i \leq 5$. Hence, Lemma 9 yields that for every $U \subset W_i$, we have $|N_{B_i}(U)| \geq |U|$. Consequently, Hall's theorem implies Corollary 10. □

Next, we consider the case when $i = 6$.

Lemma 11. *There exists a W_6-matching in $B_6 = B_6(W_6, V_6^*)$.*

Proof. It suffices to check Hall's condition, that is, for every $U \subset W_6$,

$$|N_{B_6}(U)| \geq |U|. \tag{3.4}$$

If $|U| \leq |V_6^*| - n/C = 2\varepsilon n - n/C$, then Lemma 9 implies (3.4). Hence, we assume that

$$|U| > 2\varepsilon n - n/C.$$

Notice that

$$|U| + \left|N_{B_6}\Big(V_6^* \setminus N_{B_6}(U)\Big)\right| \leq |W_6| = 2\varepsilon n$$

since U and $N_{B_6}\Big(V_6^* \setminus N_{B_6}(U)\Big)$ are disjoint. If $|V_6^* \setminus N_{B_6}(U)| \geq \frac{\delta n}{C^2 \log n}$, take a subset Y of $V_6^* \setminus N_{B_6}(U)$ with $|Y| = \frac{\delta n}{C^2 \log n}$. Then, by equation (3.2) and Property (P1), we infer

$$\Big|N_{B_6}\Big(V_6^* \setminus N_{B_6}(U)\Big)\Big| \geq |N_{B_6}(Y)| \geq \frac{\varepsilon\delta}{8} n > \frac{n}{C}$$

and $|U| + \Big|N_{B_6}\Big(V_6^* \setminus N_{B_6}(U)\Big)\Big| > 2\varepsilon n$, which is a contradiction. Therefore, $|V_6^* \setminus N_{B_6}(U)| < \frac{\delta n}{C^2 \log n}$. Then, Property (P1) together with (3.2) implies that

$$\Big|N_{B_6}\Big(V_6^* \setminus N_{B_6}(U)\Big)\Big| \geq \frac{C^2 \varepsilon \log n}{8} \big|V_6^* \setminus N_{B_6}(U)\big| > \big|V_6^* \setminus N_{B_6}(U)\big|.$$

Since $|N_{B_6}(U)| + |V_6^* \setminus N_{B_6}(U)| = |V_6^*| = 2\varepsilon n$ and $|U| + \Big|N_{B_6}\Big(V_6^* \setminus N_{B_6}(U)\Big)\Big| \leq |W_6| = 2\varepsilon n$, we have

$$\begin{aligned} |N_{B_6}(U)| + |V_6^* \setminus N_{B_6}(U)| &\geq |U| + \Big|N_{B_6}\Big(V_6^* \setminus N_{B_6}(U)\Big)\Big| \\ &> |U| + \big|V_6^* \setminus N_{B_6}(U)\big|, \end{aligned}$$

that is, $|N_{B_6}(U)| > |U|$. □

4. Random graph is good.

In order to show Lemma 7, we need to show that there exists a positive constant C such that the random graph $G(n,p)$ with $p = C\big(\frac{\log n}{n}\big)^{1/2}$ a.a.s. satisfies Properties (P1) and (P2) in Definition 4. Our proof of Properties (P1) and (P2) of $G(n,p)$ will be given in Sections 4.1 and 4.2, respectively. In the proofs, we will use the following version of Chernoff's bound.

Lemma 12 (**Chernoff's bound**, Corollary 4.6 in [31])**.** *Let X_i be independent random variables such that $\Pr[X_i = 1] = p_i$ and $\Pr[X_i = 0] = 1 - p_i$, and let $X = \sum_{i=1}^n X_i$. For $0 < \lambda < 1$,*

$$\Pr\Big[|X - \mathbb{E}(X)| \geq \lambda \mathbb{E}(X)\Big] \leq 2\exp\Big(-\frac{\lambda^2}{3}\mathbb{E}(X)\Big).$$

4.1. **Property (P1).** In order to show that $G(n,p)$ with $p = C\big(\frac{\log n}{n}\big)^{1/2}$ a.a.s. satisfies Property (P1), it suffices to show the following lemma.

Lemma 13. *There exists a positive constant C such that $G(n,p)$ with $p = C\big(\frac{\log n}{n}\big)^{1/2}$ a.a.s. satisfies the following: There exists a matching $\mathcal{M}$ with $|\mathcal{M}| = 2\varepsilon n$ in the subgraph of $G(n,p)$ induced on V_0 such that for all $U \subset V \setminus V(\mathcal{M})$ with $|U| \leq \frac{\delta n}{C^2 \log n}$, we have*

$$\Big|\Big\{\{a,b\} \in \mathcal{M} \;\Big|\; a \sim u, b \sim u \text{ for some } u \in U\Big\}\Big| \geq \frac{C^2 \log n}{16n}|\mathcal{M}||U|.$$

Proof. Let $G_1 = G\big(n - 6\varepsilon n, \frac{2\log n}{n}\big)$ on the vertex set V_0 and $G_2 = G(n, p/2)$ on the vertex set V. It is easy to see that $G(n,p)$ on V stochastically contains $G_1 \cup G_2$. Hence, it is enough to show that $G_1 \cup G_2$ a.a.s. contains a matching $\mathcal{M}$ described in Lemma 13.

The result of Erdős and Rényi [23] implies that G_1 a.a.s. contains a matching in V_0 covering all but at most one vertex. Hence, G_1 on V_0 a.a.s. contains a matching of size $2\varepsilon n$. Take such a matching $\mathcal{M}$ in G_1.

Let

$$X(U) := \Big|\Big\{e \in \mathcal{M} \;\Big|\; e \subset N_{G_2}(u) \text{ for some } u \in U\Big\}\Big|.$$

Notice that $X(U)$ is the sum of independent and identically distributed (i.i.d.) random variables X_e, $e \in \mathcal{M}$, where

$$X_e = \begin{cases} 1 & \text{if } e \subset N_{G_2}(u) \text{ for some } u \in U \\ 0 & \text{otherwise.} \end{cases}$$

Since $|U| \le \frac{\delta n}{C^2 \log n}$, we have that for each $e \in \mathcal{M}$,

$$\begin{aligned} \Pr\Big[X_e = 0\Big] &= \Big(1 - \Big(\frac{p}{2}\Big)^2\Big)^{|U|} \le 1 - \frac{|U|p^2}{4} + \frac{|U|^2 p^4}{32} \\ &\le 1 - \Big(1 - \frac{\delta}{8}\Big)\frac{|U|p^2}{4} \le 1 - \frac{|U|p^2}{8}, \end{aligned}$$

or equivalently, $\Pr\big[X_e = 1\big] \ge \frac{1}{8}|U|p^2$. Hence,

$$\mathbb{E}\big(X(U)\big) \ge \frac{p^2}{8}|\mathcal{M}||U| = \frac{C^2 \log n}{8n}|\mathcal{M}||U|.$$

Chernoff's bound (Lemma 12) yields that

$$\begin{aligned} \Pr\Big[X(U) < \frac{C^2 \log n}{16n}|\mathcal{M}||U|\Big] &\le \Pr\Big[|X(U) - \mathbb{E}\big(X(U)\big)| \ge \frac{1}{2}\mathbb{E}\big(X(U)\big)\Big] \\ &\le 2\exp\Big(-0.01 p^2 |\mathcal{M}||U|\Big) \\ &= 2\exp\Big(-0.01 \cdot 2\varepsilon C^2 |U| \log n\Big) \\ &\le 2\exp\big(-3|U|\log n\big) \le \frac{2}{n^{3|U|}}. \end{aligned}$$

Therefore, we infer

$$\begin{aligned} &\Pr\Big[\exists\, U \in V \setminus V(\mathcal{M}) \text{ with } |U| \le \frac{\delta n}{C^2 \log n} \text{ such that } X(U) < \frac{C^2 \log n}{16n}|\mathcal{M}||U|\Big] \\ &\le \sum_{\ell=1}^{n} n^\ell \frac{2}{n^{3\ell}} \le n \cdot \frac{2}{n^2} = o(1), \end{aligned}$$

which completes the proof of Lemma 13. □

4.2. **Property (P2).** We now show that $G(n,p)$ with $p = C\big(\frac{\log n}{n}\big)^{1/2}$ a.a.s. satisfies Property (P2). First, recall the following definition which was given just before Definition 4: For a graph G on V and $k = 1$ or 2, let $U \subset V$ and $\mathcal{L}$ be a collection of pairwise disjoint k-subsets of $V \setminus U$. We consider a bipartite graph $B(\mathcal{L}, U)$ between $\mathcal{L}$ and U, in which $L \in \mathcal{L}$ and $u \in U$ are adjacent if and only if $L \subset N_G(u)$.

In order to show that $G(n,p)$ with $p = C\big(\frac{\log n}{n}\big)^{1/2}$ a.a.s. satisfies Property (P2), it suffices to show the following lemma.

Lemma 14. *There exists a positive constant C such that $G(n,p)$ with $p = C\big(\frac{\log n}{n}\big)^{1/2}$ a.a.s. satisfies the following: Let $k = 1$ or 2, and $\mathcal{L}$ be a collection of pairwise disjoint k-subsets of V.*

(a) If $|\mathcal{L}| \le \frac{\delta}{C^k}\Big(\frac{n}{\log n}\Big)^{k/2}$*, then, for* V_i *with* $V_i \cap \Big(\bigcup_{L\in\mathcal{L}} L\Big) = \emptyset$*,* $i = 1, ..., 6$*, we have that*

$$|N_{B(\mathcal{L},V_i)}(\mathcal{L})| \ge (1-\delta)C^k\Big(\frac{\log n}{n}\Big)^{k/2}|\mathcal{L}||V_i|.$$

(b) If $|\mathcal{L}| \ge \frac{\log n}{C^{k-1}}\Big(\frac{n}{\log n}\Big)^{k/2}$*, then, for all* U *with* $|U| \ge \frac{\log n}{C^{k-1}}\Big(\frac{n}{\log n}\Big)^{k/2}$ *and* $U \cap \Big(\bigcup_{L\in\mathcal{L}} L\Big) = \emptyset$*, the graph* $B(\mathcal{L}, U)$ *has at least one edge.*

Proof. For a proof of (a) of Lemma 14, we observe that $X(\mathcal{L}, V_i) := |N_{B(\mathcal{L},V_i)}(\mathcal{L})|$ is the sum of i.i.d. random variables X_v, $v \in V_i$, where

$$X_v = \begin{cases} 1 & \text{if } L \subset N_G(v) \text{ for some } L \in \mathcal{L} \\ 0 & \text{otherwise.} \end{cases}$$

Since $|L| = k$ for all $L \in \mathcal{L}$ and $p^k|\mathcal{L}| \le \delta = 0.01$, we have

$$\mathbb{E}\Big(X(\mathcal{L}, V_i)\Big) = |V_i|\Big(1-(1-p^k)^{|\mathcal{L}|}\Big) \ge (1-\delta/2)\, p^k|\mathcal{L}||V_i| = (1-\delta/2)\, C^k\Big(\frac{\log n}{n}\Big)^{k/2}|\mathcal{L}||V_i|.$$

Chernoff's bound (Lemma 12) implies that

$$\begin{aligned}\Pr\Big[X(\mathcal{L}, V_i) < (1-\delta)C^k\Big(\frac{\log n}{n}\Big)^{k/2}|\mathcal{L}||V_i|\Big] &\le \Pr\Big[|X(\mathcal{L}, V_i) - \mathbb{E}\Big(X(\mathcal{L}, V_i)\Big)| \ge \frac{\delta}{2}\mathbb{E}\Big(X(\mathcal{L}, V_i)\Big)\Big] \\ &\le 2\exp\Big(-\frac{\delta^2}{12}\mathbb{E}\Big(X(\mathcal{L}, V_i)\Big)\Big),\end{aligned}$$

and by $p^k|V_i| \ge p^2\varepsilon n = C^2\varepsilon\log n$,

$$\Pr\Big[X(\mathcal{L}, V_i) < (1-\delta)C^k\Big(\frac{\log n}{n}\Big)^{k/2}|\mathcal{L}||V_i|\Big] \le 2\exp\big(-3|\mathcal{L}|\log n\big) = \frac{2}{n^{3|\mathcal{L}|}}.$$

Therefore, we infer that

$$\begin{aligned}&\Pr\Big[\exists\, V_i, \mathcal{L} \text{ with } 1 \le |\mathcal{L}| \le \tfrac{\delta}{C^k}\big(\tfrac{n}{\log n}\big)^{k/2} \text{ such that } X(\mathcal{L}, V_i) < (1-\delta)C^k\Big(\frac{\log n}{n}\Big)^{k/2}|\mathcal{L}||V_i|\Big] \\ &\le\ 6\sum_{\ell=1}^{n} n^{\ell}\frac{2}{n^{3\ell}} \le 6n\frac{2}{n^2} = o(1),\end{aligned}$$

which completes the proof of (a) of Lemma 14.

For a proof of (b) of Lemma 14, we observe that the number $Y(\mathcal{L}, U)$ of edges in $B(\mathcal{L}, U)$ is the sum of i.i.d. random variables $Y_{L,u}$ for $L \in \mathcal{L}$ and $u \in U$, where

$$Y_{L,u} = \begin{cases} 1 & \text{if } L \subset N_G(u) \\ 0 & \text{otherwise.} \end{cases}$$

Since $|L| = k$ for all $L \in \mathcal{L}$, we have $\mathbb{E}\Big(Y(\mathcal{L}, U)\Big) = p^k|\mathcal{L}||U|$. Chernoff's bound (Lemma 12) yields that

$$\begin{aligned}\Pr\Big[Y(\mathcal{L}, U) = 0\Big] &\le \Pr\Big[\big|Y(\mathcal{L}, U) - \mathbb{E}\big(Y(\mathcal{L}, U)\big)\big| \ge \frac{1}{2}\mathbb{E}\big(Y(\mathcal{L}, U)\big)\Big] \\ &\le 2\exp\Big(-\frac{1}{12}\mathbb{E}\big(Y(\mathcal{L}, U)\big)\Big) \le 2\exp\Big(-\frac{1}{12}p^k|\mathcal{L}||U|\Big).\end{aligned}$$

For $\ell \geq \frac{\log n}{C^{k-1}}\left(\frac{n}{\log n}\right)^{k/2}$ and $r \geq \frac{\log n}{C^{k-1}}\left(\frac{n}{\log n}\right)^{k/2}$, the number of $\mathcal{L}$ with $|\mathcal{L}| = \ell$ is at most $\binom{n}{k}^{\ell} \leq n^{k\ell}$ and the number of U with $|U| = r$ is at most $\binom{n}{r}$, and we have

$$\Pr\Big[\exists\, \mathcal{L}, U \text{ with } |\mathcal{L}| = \ell, |U| = r \text{ such that } Y(\mathcal{L}, U) = 0\Big]$$
$$\leq n^{k\ell} n^{r} \cdot 2\exp\Big(-\frac{1}{12}p^{k}\ell r\Big) \leq 2\exp\Big(\big(k\ell + r\big)\log n - \frac{1}{12}p^{k}\ell r\Big).$$

Since $p^{k}\ell = C^{k}\left(\frac{\log n}{n}\right)^{k/2}\ell \geq C\log n$ and $p^{k}r = C^{k}\left(\frac{\log n}{n}\right)^{k/2} r \geq C\log n$, we have that

$$(k\ell + r)\log n \leq 0.01C(\ell + r)\log n \leq 0.01\big(p^{k}\ell r + p^{k}\ell r\big) \leq 0.02p^{k}\ell r,$$

and hence,

$$n^{k\ell}n^{r} \cdot 2\exp\Big(-\frac{1}{12}p^{k}\ell r\Big) \leq 2\exp\Big(-\frac{1}{24}p^{k}\ell r\Big) \leq 2\exp\Big(-\tfrac{C^{2-k}}{24}\left(\tfrac{n}{\log n}\right)^{k/2}(\log n)^{2}\Big) \leq 2\exp\big(-n^{1/2}\big).$$

Therefore, we infer that

$$\Pr\Big[\exists\, \mathcal{L}, U \text{ with } |\mathcal{L}|, |U| \geq \tfrac{\log n}{C^{k-1}}\left(\tfrac{n}{\log n}\right)^{k/2} \text{ such that } Y(\mathcal{L}, U) = 0\Big]$$
$$\leq \quad \frac{n}{k} \cdot n \cdot 2\exp\big(-n^{1/2}\big) = o(1),$$

which completes the proof of (b) of Lemma 14. □

5. Concluding remarks

One may ask about how the approach of this paper can be used for the case that $d \geq 3$. We believe that our approach for finding a suitable matching given in Lemma 13 can be also applied in order to find a suitable family of vertex disjoint d-cliques when $d \geq 3$ and $p \geq C\left(\frac{\log n}{n}\right)^{1/d}$. This approach together with an embedding algorithm modified from the algorithm in Dellamonica, Kohayakawa, Rödl and Ruciński [17, 18] may provide a simpler proof of Theorem 2.

As a further research direction, we are interested in resolving the following problem.

Problem 15. *For an integer $d \geq 2$, determine the largest constant $a = a(d)$ with $0 \leq a \leq 1$ such that if $p \geq n^{-a+o(1)}$, then $G(n,p)$ is a.a.s. $\mathcal{F}_n(d)$-universal.*

An easy observation mentioned in the introduction gives an upper bound $\frac{2}{d+1}$ for a. The current best lower bound is $\frac{1}{d}$ based on the result in Dellamonica, Kohayakawa, Rödl and Ruciński [17, 18] and this paper.

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