# BERGMAN COMPLEXES OF LATTICE PATH MATROIDS 

EMANUELE DELUCCHI, MARTIN DLUGOSCH


#### Abstract

We give an explicit description of the poset of cells of Bergman complexes of lattice path matroids and establish a criterion for its simpliciality, in terms of the shape of the bounding paths.


## 1. Introduction

The term Bergman fan has come to denote the polyhedral fan given as the logarithmic limit set of a complex algebraic subvariety $V$ of $\mathbb{C}^{n}$. Logarithmic limit sets were first introduced by George M. Bergman in 22, and their fan structure was described by Bieri and Groves [4]. Sturmfels 19 made the key observation that, if $V$ is defined by linear equations, its Bergman fan can be constructed from the matroid associated to $V$. In this case, the Bergman fan is the tropicalization of the complement variety of the hyperplane arrangement determined in $V$ by the intersection of the standard coordinate hyperplanes of $\mathbb{C}^{n}$. This leads to the very important role played by Bergman fans in tropical geometry (we point the interested reader to the book [13] for more on this subject). As Sturmfels' construction can be carried out also for nonrepresentable matroids, we get a Bergman fan - indeed, a tropical variety - associated to every matroid $M$. There is no loss of information in considering, instead of the whole fan, just its intersection with the unit sphere: the resulting spherical complex $\Gamma_{M}$ is called the Bergman complex of the given matroid $M$ (see Section 2.3 for the precise definitions). The homotopy type of the complex $\Gamma_{M}$ coincides with that of the order complex of $\mathscr{L}(M)$, the lattice of flats of $M$ (see Section 2.1). This has been established by Ardila and Klivans [1], who proved that in fact the order complex of $\mathscr{L}(M)$ subdivides $\Gamma_{M}$. This raises the question of the polyhedral structure of $\Gamma_{M}$. Here, progress was made by Feichtner and Sturmfels [10], who proved that $\Gamma_{M}$ is subdivided by the nested set complex of $M$, a much coarser (simplicial) complex than the order complex of $\mathscr{L}(M)$. As an additional improvement, the second author 9 described a decomposition of the matroid types (see Section 2.2) associated to faces of the Bergman complex into connected direct summands. Recently, Rincón [17 found and implemented an algorithm to efficiently compute a certain simplicial subdivision of the nested set complex, called cyclic Bergman complex.

From a combinatorial point of view, an open problem is to find an explicit description of the face structure of the Bergman complex $\Gamma_{M}$, improving on the aforementioned results which describe simplicial complexes subdividing $\Gamma_{M}$. A more general question, of interest for computations as well as for the structure theory, and which can't be effectively answered in general either, is whether the Bergman complex of a given matroid is itself simplicial. In this paper we address both the characterization of simpliciality as well as the explicit face structure of Bergman complexes for a special class of matroids.

Lattice path matroids were introduced by Bonin, de Mier and Noy 5 as a family of transversal matroids whose bases can be characterized by means of the lattice paths contained in the region of the plane bounded by two given lattice paths. This class of matroids enjoys a host of nice enumerative and structural properties 7],
and can be characterized among all matroids by a list of excluded minors 6, Theorem 3.1]. Their rich structure theory allowed lattice path matroids to attract attention both as a fertile setting for significative stepping stones toward general results (e.g., Stanley's $M$-vector conjecture 18 or the theory hyperplane splits of matroid polytopes [8]) as well as by offering a convenient general framework for the study of some subclasses of independent interest, such as generalized Catalan matroids [7, Section 4]. In the context of the study of Grassmannians, where generalized Catalan matroids are called Schubert matroids, lattice path matroids correspond precisely to 'Richardson matroids', a special case of the positroids used as indices of cells in Postnikov's stratification of the totally nonnegative Grassmannian 16 (see e.g. [11, Introduction] for an overview of this subject) - for these 'special cells' the computation of the corresponding 'Grassmann necklace' is particularly tractable [14, Section 6].

In this paper we determine the polyhedral structure of the Bergman complex of a given lattice path matroid and formulate a necessary and sufficient condition for $\Gamma_{M}$ to be simplicial. In the geometric spirit of lattice path matroids, our characterizations are in terms of the shape of the bounding paths.

The structure of the paper is as follows. First, in Section 2 we review some basic notions and we derive a description of faces and vertices of the Bergman complex of a lattice path matroid in terms of 'bays' and 'land necks' of the bounding paths (Lemma 2.6, see also Remark 8).

Section 3 contains our first main result, Theorem 3.1, where, in terms of bays and land necks, we characterize simpliciality of faces of the Bergman complex and, as a corollary, we characterize those lattice path matroids that have a simplicial Bergman complex (Corollary 3.2).

In Section 4 we introduce a poset (again defined in terms of bays and (non-)land necks, see Definition 4.2) which we prove to be isomorphic to the face poset of the Bergman complex (Theorem4.3). We close with an explicit expression for the polyhedral structure of faces of the Bergman complex (Corollary 4.4).

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## 2. Preliminaries

2.1. Matroids. We sketch some of the basics of matroid theory, in order to provide the basic definitions and to set some notation. For a thorough introduction to the subject and as a standard reference we point to 15 .

Definition 2.1. Let $E$ denote a finite set, $\mathscr{P}(E)$ the set of its subsets. A nonempty family $\mathscr{B} \subseteq \mathscr{P}(E)$ is the set of bases of a matroid on the ground set $E$ if, given $B_{1}, B_{2} \in \mathscr{B}$ and $e \in B_{1} \backslash B_{2}$, there is $f \in B_{2} \backslash B_{1}$ such that $\left(B_{1} \backslash\{e\}\right) \cup\{f\} \in \mathscr{B}$. A matroid can be given as the pair $(E, \mathscr{B})$. Given a matroid $M$, we will denote by $\mathscr{B}(M)$ its set of bases.

A loop of $M$ is any $e \in E$ that is not contained in any $B \in \mathscr{B}(M)$. A coloop of $M$ is any $e \in E$ that is contained in every $B \in \mathscr{B}(M)$.

Remark 1. If $M$ is a matroid on the ground set $E$, then all the elements of $\mathscr{B}(M)$ have the same cardinality, which is called the rank of $M$. More generally, given a subset $A \subseteq E$, define the $r a n k$ of $A$ to be

$$
\operatorname{rk}(A):=\max \{|A \cap B|: B \in \mathscr{B}(M)\} .
$$

Consider the families

$$
\begin{array}{r}
\mathscr{B}(M)[A]:=\{B \cap A: B \in \mathscr{B}(M),|A \cap B|=\operatorname{rk}(A)\}, \\
\mathscr{B}(M) / A:=\{B \backslash A: B \in \mathscr{B}(M),|A \cap B|=\operatorname{rk}(A)\} .
\end{array}
$$

These satisfy Definition 2.1 and thus describe a matroid $M[A]$ on the ground set $A$ and a matroid $M / A$ on the ground set $E \backslash A$, respectively. The matroid $M[A]$ is called the restriction of $M$ to $A$, while $M / A$ is the contraction of $A$ in $M$. A matroid that is obtained from $M$ by a sequence of contractions and restrictions is called a minor of $M$. Notice that for every $A \subseteq E$ the $\operatorname{rank}$ of $M[A]$ is $\operatorname{rk}(A)$. In particular, the rank of $M$ is $\operatorname{rk}(E)$.

Remark 2. Let $M_{1}$ and $M_{2}$ be matroids on disjoint ground sets $E_{1}, E_{2}$. Their direct sum is the matroid $M_{1} \oplus M_{2}$ on the ground set $E_{1} \cup E_{2}$ with set of bases

$$
\mathscr{B}\left(M_{1} \oplus M_{2}\right)=\left\{B_{1} \cup B_{2}: B_{1} \in \mathscr{B}\left(M_{1}\right), B_{2} \in \mathscr{B}\left(M_{2}\right)\right\} .
$$

A matroid $M$ on the ground set $E$ is connected if there is no nontrivial partition $E=E_{1} \uplus E_{2}$ with $M=M\left[E_{1}\right] \oplus M\left[E_{2}\right]$. Let $E=E_{1} \uplus \cdots \uplus E_{c}$ be a partition of $E$ such that $M=M\left[E_{1}\right] \oplus \ldots \oplus M\left[E_{c}\right]$ and $M\left[E_{j}\right]$ is connected for every $j=1, \ldots, c$. Then the $M\left[E_{j}\right]$ are uniquely determined up to renumbering and are called the connected components of $M$.

The closure of $A$ is the union of all $X \subseteq E$ such that $X$ contains $A$ and the ranks of $X$ and $A$ coincide. We denote by $\mathscr{L}(M)$ the set of all closed sets, also called flats of $M$. It is customary to endow $\mathscr{L}(M)$ with the partial order given by inclusion, which makes it a geometric lattice, referred to as the lattice of flats of $M$.

If the rank of $A \subseteq E$ equals its cardinality, we call $A$ independent. Let $M_{1}$ and $M_{2}$ be matroids on the ground set $E$. Then $M_{2}$ is called a weak map image of $M_{1}$ if there is a bijection $\varphi: E \rightarrow E$ for which the preimage of every independent set of $M_{2}$ is an independent set of $M_{1}$.
2.2. Matroid polytopes. Let $M$ be a matroid of rank $d$ on the ground set $[n]:=$ $\{1, \ldots, n\}$. For every $B \in \mathscr{B}(M)$ we consider a vector $v(B) \in \mathbb{R}^{n}$ defined by $v(B)_{i}=1$ if $i \in B, v(B)_{i}=0$ else. The matroid polytope of $M$ is the convex hull

$$
P_{M}:=\operatorname{conv}\{v(B): B \in \mathscr{B}(M)\} .
$$

This is a polytope of dimension $n-c(M)$, where $c(M)$ is the number of connected components of $M$, contained in the hypersimplex $\Delta_{d}=\operatorname{conv}\left\{d e_{i}: i=1, \ldots, n\right\}$. It can be readily seen that for two matroids $M_{1}, M_{2}$ we have $P_{M_{1} \oplus M_{2}}=P_{M_{1}} \times P_{M_{2}}$.

Given a flat $F \in \mathscr{L}(M)$, define the halfspace

$$
H_{F}^{+}:=\left\{x \in \mathbb{R}^{n}: \sum_{i \in F} x_{i} \leqslant \operatorname{rank}(F)\right\}
$$

and let $H_{F}$ be the hyperplane bounding $H_{F}^{+}$. According to 10, we have

$$
P_{M}=\Delta_{d} \cap \bigcap_{F \in \mathscr{L}(M)} H_{F}^{+}
$$

Let us consider the poset $\mathscr{F}(M)$ of faces (closed cells) of $P_{M}$ ordered by inclusion (see 20, Definition 2.6]). Every $f \in \mathscr{F}(M)$ is the matroid polytope of a matroid $M_{f}$ called "the matroid type" of the face $f$, with set of bases

$$
\mathscr{B}\left(M_{f}\right)=\{B \in \mathscr{B}(M): v(B) \in V(f)\},
$$

where $V(f)$ denotes the set of vertices of $f$. Notice that a matroid type's ground set is always the full ground set $E$, while minors have strictly smaller ground sets. In general, a matroid type is a direct sum of minors.

A maximal element $f \in \mathscr{F}(M)$ (a facet of $P_{M}$ ) can be of one of two types:
(i) $f$ lies on the boundary of $\Delta_{d}$,
(ii) $f$ meets the interior of $\Delta_{d}$.

Remark 3.
(i) If $f$ is of type (i), then there is $j$ such that $x_{j}=0$ for all $x \in f$. This means that $j$ is not contained in any basis of $M_{f}$, i.e., $j$ is a loop of $M_{f}$. Conversely, if $j$ is a loop of $M_{f}$, then $f$ is of type (i).
(ii) Following [10, Prop. 2.6], the facet $f$ is of type (ii) if and only if $f \subseteq H_{F}$ for an $F \in \mathscr{L}(M)$ such that $M / F$ and $M[F]$ are both connected. Such an $F$ is called a flacet of the matroid $M$, and is uniquely determined by $f$. We will write $F_{f}$ for the flacet corresponding to the type-(ii) facet $f$. The bases lying on this facet are the bases of the matroid $M / F_{f} \oplus M\left[F_{f}\right]$.
2.3. Bergman complexes. Let $M$ be a connected matroid of rank $d$ on the ground set $[n]$.

We consider the polar dual $P_{M}^{\vee}$ of $P_{M}$ (see, e.g., 20, Definition 2.10]) and let $\mathscr{F} \vee(M)$ denote its poset of faces. Duality determines canonical order-reversing bijections $\mathscr{F}(M) \xrightarrow{\vee} \mathscr{F}^{\vee}(M) \xrightarrow{\vee} \mathscr{F}(M)$. In particular, every face $\alpha \in \mathscr{F}^{\vee}(M)$ has a matroid type $M_{\alpha}:=M_{\alpha^{v}}$.
Definition 2.2. The Bergman complex of $M$ is the polyhedral subcomplex $\Gamma_{M}$ of $P_{M}^{\vee}$ with set of faces

$$
\Gamma_{M}:=\left\{\alpha \in \mathscr{F}^{\vee}(M): M_{\alpha} \text { loopfree }\right\}
$$

which we regard as a downwards closed subposet of $\mathscr{F}^{\vee}(M)$.
Remark 4. Bergman's original definition of this space [3] is set theoretical, while the polyhedral structure was first studied by Bieri and Groves [4]. Our definition follows Feichtner and Sturmfels' approach 10 and describes what Ardila and Klivans 1 call coarse subdivision.

Remark 5. According to Remark 3, the vertices of the Bergman complex are exactly the faces of the form $f^{\vee}$ where $f$ is a facet of type (ii).

Let $\gamma_{1}, \ldots, \gamma_{k}$ be the vertices of a face $\alpha \in \Gamma_{M}$. The face $\alpha^{\vee}$ of $P_{M}$ is the intersection of its adjacent facets $\gamma_{1}^{\vee}, \ldots, \gamma_{k}^{\vee}$. Therefore

$$
\left\{v(B): B \in \mathscr{B}\left(M_{\alpha}\right)\right\}=V\left(\alpha^{\vee}\right)=\bigcap_{i=1}^{k} V\left(\gamma_{i}^{\vee}\right)=\bigcap_{i=1}^{k}\left\{v(B): B \in \mathscr{B}\left(M_{\gamma_{i}}\right)\right\}
$$

and we can write

$$
\mathscr{B}\left(M_{\alpha}\right)=\bigcap_{i=1}^{k} \mathscr{B}\left(M_{\gamma_{i}}\right) .
$$

Remark 6. The face $\alpha \in \Gamma_{M}$ is simplicial if every proper subset of its vertices determines a proper subface. Equivalently, $\alpha$ fails to be simplicial if and only if

$$
\mathscr{B}\left(M_{\alpha}\right)=\bigcap_{\gamma \in U} \mathscr{B}\left(M_{\gamma}\right)
$$

for a proper subset $U \subsetneq V(\alpha)$.
2.4. Lattice path matroids. Let $p, q$ be lattice paths in the plane with common starting point (say at the origin $(0,0)$ ) and ending point (say at a point $(m, r)$ ). We will assume that $p$ never goes below $q$. We will write $p$ and $q$ as words

$$
p=p_{1} \ldots p_{m+r} \quad q=q_{1} \ldots q_{m+r}
$$

where each letter is $N$ or $E$, signaling a step $\operatorname{North}(0,1)$ or East $(1,0)$.

By $\lceil p, q\rfloor$ we will denote the set of lattice paths from $(0,0)$ to $(m, r)$ that never go above $p$ or below $q$.

For any $s \in\lceil p, q\rfloor$ write $s=s_{1} \ldots s_{m+r}$ and define

$$
B(s):=\left\{i: s_{i}=N\right\}
$$

Lemma $2.3(\boxed{5})$. The set $\{B(s): s \in[p, q]\}$ is the set of bases of a matroid $M(p, q)$ on the ground set $[m+r]$.
Definition 2.4. A lattice path matroid is any matroid of the form $M(p, q)$ for two lattice paths $p, q$ as above.

Remark 7 ([5, Theorem 3.6]). A lattice path matroid $M(p, q)$ is connected if and only if the paths $p$ and $q$ never touch except at $(0,0)$ and $(m, r)$.
Assumption. Unless otherwise stated, in the following we will consider only connected lattice path matroids.

In order to understand the faces of the Bergman complex of lattice path matroids, we will often use the fact that contractions and deletions of lattice path matroids are again lattice path matroids whose bounding paths can be constructed directly from the bounding paths of the original matroid (see [7, Section 3.1])
2.5. Bergman complexes of lattice path matroids. Let $M=M(p, q)$ be a connected lattice path matroid. Given $B \in \mathscr{B}(M)$, here and in what follows we will write $p(B)$ for the corresponding lattice path. Let $\alpha$ be a face of the Bergman complex of $M$. The corresponding path $p(B)$ of a basis $B \in M_{\alpha}$ is called path of $M_{\alpha}$ or just $M_{\alpha}$-path. A node of the matroid type $M_{\alpha}$ is any integer point of the lattice that is visited by every path $p(B)$ with $B \in \mathscr{B}\left(M_{\alpha}\right)$.

Definition 2.5 (Fundamental flats, bays, land necks). We will say that a lattice point $\left(y_{1}, y_{2}\right)$ on the upper path $p$ is a bay of $p$ if $p_{y_{1}+y_{2}} p_{y_{1}+y_{2}+1}=E N$. Similarly, a point $\left(z_{1}, z_{2}\right)$ on $q$ is a bay for $q$ if $q_{z_{1}+z_{2}} q_{z_{1}+z_{2}+1}=N E$. Let $U_{p}$, resp. $U_{q}$, be the set of bays of $p$, resp. of $q$. The fundamental flats of $M$ are sets of the form $\left\{1, \ldots, y_{1}+y_{2}\right\}$ for an $\left(y_{1}, y_{2}\right) \in U_{p}$ or of the form $\left\{z_{1}+z_{1}+1, \ldots, m+r\right\}$ for $\left(z_{1}, z_{2}\right) \in U_{q}$. We point the reader to [7, Section 5] for further detail on the properties of these sets - here it is enough to recall that these are flats of the matroid $M$.

We will say that $i \in[m+r]$ is a land neck of $M(p, q)$ if the endpoint of $p_{1} \ldots p_{i+1}$ lies one unit North of the endpoint of $q_{1} \ldots q_{i}$. The set of land necks is $S(p, q)$.


Figure 1. A lattice path matroid of rank 8 on the ground set [16]. The black dots mark the $p$-bays, the black squares mark the $q$-bays and the thick line shows that the singleton $\{4\}$ is a land neck (however, note that the singleton $\{7\}$ is not).

The following lemma identifies vertices of the Bergman complex of a lattice path matroid.

Lemma 2.6. The flacets (vertices of the Bergman complex) of a connected lattice path matroid $M(p, q)$ are
(a) the fundamental flats,
(b) the singletons $\{i\} \subseteq[m+r] \backslash S(p, q)$.

Proof. According to Remark 5 and Remark 3, we need to identify the connected flats of $M(p, q)$ whose contraction is also connected. To this end, recall that contractions and deletions of lattice path matroids are again lattice path matroids. More precisely, e.g. following 7, Section 3], for every $\{i\} \in[m+r]$ a representation of the contraction of $\{i\}$ in $M(p, q)$ is obtained as follows. If $i$ is not a loop nor a coloop, remove the last North step of $p$ before or at $p_{i}$ as well as the first North step of $q$ after or at $q_{i}$. If $i$ is a loop (resp. a coloop), then remove the common East (resp. North) step at $p_{i}=q_{i}$. With this we can already see that part (b) of the claim identifies exactly the one-element connected flats whose contraction is connected (see Figure 1 for an illustration of this definition).

By Theorem 5.7 of [7], the nontrivial connected flats of $M(p, q)$ are
(1) the fundamental flats and
(2) the flats of the form $F_{y} \cap G_{z}$, where $F_{y}$ (resp. $G_{z}$ ) is the fundamental flat corresponding to some $y=\left(y_{1}, y_{2}\right) \in U_{p}$ (resp. some $z=\left(z_{1}, z_{2}\right) \in U_{q}$ ), and for which $y_{1}>z_{1}$ (see e.g. Figure 3).
Moreover, in this case we have $\operatorname{rk}\left(F_{y} \cap G_{z}\right)=y_{2}-z_{2}$.
We are thus left with showing that contractions of fundamental flats are connected, while contractions of the flats described in (2) are not.

First, consider a fundamental flat, say $F_{y}$ for some $y=\left(y_{1}, y_{2}\right) \in U_{p}$. By Remark 1. the bases of the contraction to $F_{y}$ are of the form $B \backslash F_{y}$ where $B$ is a basis of $M$ whose intersection with $F_{y}$ has maximal rank, i.e., $\left|B \cap\left\{1, \ldots, y_{1}+y_{2}\right\}\right|=y_{2}$. These are the bases represented by paths which reach height $y_{2}$ in the first $y_{1}+y_{2}$ steps. It is easy to see that such paths are exactly those passing through $y$. We conclude that a lattice path representation for $M(p, q) / F_{y}$ consists of all lattice paths starting at $y$ and contained in $\lceil p, q\rfloor$. With Remark 77 we see that the bounding paths of this representation never meet except at $y$ and $(m, r)$ (since $M(p, q)$ is connected and $y$ is a $p$-bay), implying connectedness of the contraction. See Figure 2 for an illustration of this situation. The case of fundamental flats corresponding to $q$-bays is treated analogously.

Now we turn to the connected flats of type (2), and we will prove that their contraction cannot be connected. In fact, let $F_{y}$ and $G_{z}$ as in (2) and write $y=\left(y_{1}, y_{2}\right)$, $z=\left(z_{1}, z_{2}\right)$. By the same argument as above we see that bases of $M(p, q) /\left(F_{y} \cap G_{z}\right)$ correspond to paths with $y_{2}-z_{2}$ North steps between $\left\{z_{1}+z_{2}+1, \ldots, y_{1}+y_{2}\right\}$, which are exactly the paths passing through both $y$ and $z$. Thus, bases of the contraction correspond to concatenations of any path in $\lceil p, q\rfloor$ ending in $z$ with any path in $\lceil p, q\rceil$ beginning at $y$ (Figure 3 illustrates this situation). With Remark 2 we see that $M(p, q) /\left(F_{y} \cap G_{z}\right)$ is not connected.

Through Lemma 2.6 we obtain the following geometric interpretation of Remarks 3 and 5 .
Remark 8 (Faces of the Bergman complex in terms of the lattice paths). Let $F$ be a flacet of $M=M(p, q)$ corresponding to a facet $f$ of the matroid polytope $P_{M}$ and consider $B \in \mathscr{B}(M)$. Then $B \in \mathscr{B}\left(M_{f}\right)$ if and only if one of the following holds
(a) $F$ is a fundamental flat and $p(B)$ goes through the corresponding bay,
(b) $F=\{i\}$ and $p(B)_{i}=N$.


Figure 2. Illustration for the proof Lemma 2.6. Left-hand side: a fundamental flat $F_{y}=\left\{1, \ldots, y_{1}+y_{2}\right\}$ corresponding to a $p$-bay $z=\left(y_{1}, y_{2}\right)$. Right-hand side: its contraction $M / F_{y}$.


Figure 3. Illustration for the proof Lemma 2.6. Left-hand side: a proper, non-trivial, connected flat $F_{y} \cap G_{z}=\left\{z_{1}+z_{2}+1, \ldots, y_{1}+\right.$ $\left.y_{2}\right\}$ of type (2). Right-hand side: its contraction $M /\left(F_{y} \cap G_{z}\right)$.

We will then say that the path $p(B)$ satisfies the constraint imposed by $F$. In both cases we can describe the matroid type corresponding to a single vertex $F_{f}$ of the Bergman complex by

$$
M_{f}=M / F_{f} \oplus M\left[F_{f}\right] .
$$

More generally, using this language of 'paths' and 'constraints' we can say that faces of the Bergman complex correspond to (the matroid type defined by paths satisfying the constraints given by) collections of fundamental flats and non-landnecks. Conversely, any such collection corresponds to a face of the Bergman complex provided that the paths satisfying its constraints determine a loopfree matroid.

With Remark 6 we can also say that a face of the Bergman complex of a lattice path matroid is simplicial if and only if none of the contraints associated to it is redundant (i.e., by removing any of those we obtain a strictly smaller face).

Example 2.7. On the left of Figure 4 we have marked some of the flacets of the lattice path matroid of Figure 1. The associated fundamental flats correspond to the $p$-bays $(3,3),(4,6)$ and the $q$-bay $(2,1)$. The paths that pass through these three points and go North at their 8th step define the bases of a matroid $M^{\prime}$, of which we give a lattice path representation on the right hand side (we remark that in order to get such a presentation of the matroid type, one has to change the initial order of the ground set). We see that $M^{\prime}$ is loopfree, thus it is the matroid type of a face $f$ of the Bergman complex, and that $\{8\}$ is the only singleton which is a vertex of $f$. On the other hand, the $p$-bay $(3,3)$ and the $q$-bay $(4,3)$ do not define a face of the Bergman complex, since every path that goes through both will go


Figure 4. Some flacets and a lattice representation of the matroid type $M_{f}$, where $f^{\vee}$ has those flacets as vertices.

East on the 7th step, thus $\{7\}$ will be a loop of the corresponding matroid type. Analogously, the $p$-bay $(3,3)$ and the two singletons $\{5\},\{6\}$ do not define a face of the Bergman complex because every path that goes through $(3,3)$ and goes North at the 5 th and 6 th step must go East (along $q$ ) on the 4 th step, thus $\{4\}$ will be a loop of the corresponding matroid type.

## 3. A SImpliciality criterion

The goal of this section is to give a complete characterization of which lattice path matroids possess a simplicial Bergman complex. After the discussions of Section 2 , it is clear that it will be enough to consider connected (and in particular loopfree) matroids.

Theorem 3.1. Let $\alpha$ be a face of the Bergman complex of a connected lattice path matroid $M(p, q)$. Then $\alpha$ is simplicial unless the flacets corresponding to its vertices include
(1) a fundamental flat $F$ corresponding to $a$ bay $(x, z)$ of $p$ and
(2) a fundamental flat $G$ corresponding to a bay $(x, y)$ of $q$
with $z-y>1$.
Proof. Let the vertices of $\alpha$ include $F$ and $G$ as in (1) and (2) above. Then, for all $B \in \mathscr{B}\left(M_{\alpha}\right)$, the lattice path $p(B)$ passes through $(x, y)$ and $(x, z)$. So it satisfies $p(B)_{j}=N$ for $j=x+y+1, \ldots, x+z$. We conclude that if $F$ and $G$ correspond to vertices of $V(\alpha)$ then all flacets in $\{x+y+1\}, \ldots,\{x+z\}$ are forced by $F$ and $G$ to be vertices in $V(\alpha)$. More precisely, with $U:=V(\alpha) \backslash\{\{x+y+1\}, \ldots,\{x+z\}\}$, we have $\mathscr{B}\left(M_{\alpha}\right)=\bigcap_{\gamma \in U} \mathscr{B}\left(M_{\gamma}\right)$. Since $z-y>1, U$ is a proper subset of $V(\alpha)$ and, in view of Remark 6, we conclude that $\alpha$ is not simplicial.

For the reverse implication, suppose $\alpha$ is not simplicial. Again by Remark 6 there is $\gamma_{0} \in V(\alpha)$ such that

$$
\begin{equation*}
\mathscr{B}\left(M_{\alpha}\right)=\bigcap_{\gamma \in U \backslash\left\{\gamma_{0}\right\}} \mathscr{B}\left(M_{\gamma}\right) . \tag{1}
\end{equation*}
$$

We distinguish two cases.
Case 1: $F_{\gamma_{0}}$ is a fundamental flat. Without loss of generality $F_{\gamma_{0}}$ corresponds to a $p$-bay $(x, z)$. In terms of Remark 8 , Equation (1) means that every path satisfying the constraints of $V(\alpha) \backslash\left\{\gamma_{0}\right\}$ passes through $(x, z)$. In particular $(x, z)$ is a node of $M_{\alpha}$. There may of course be other nodes south of $(x, z)$ : in fact, the next claim proves that the southernmost node on the vertical of $(x, y)$ lies on the lower bounding path $q$.

Claim 1: Let $u$ be minimal such that $(x, u)$ is a node of $M_{\alpha}$. Then $(x, u)$ lies on $q$.
Proof of Claim 1. First remark that any element of $\{(x, u), \ldots,(x, z)\}$ is indeed a node of $M_{\alpha}$ Fix any path representing a basis of $M_{\alpha}$, say

$$
s=s_{1} \ldots s_{x+u-1} N \ldots N E s_{k} \ldots s_{r+m}
$$

where $k>x+z+1$. Now consider the path

$$
s^{\prime}=s_{1} \ldots s_{x+u-1} E N \ldots N s_{k} \ldots s_{r+m} .
$$

If $s^{\prime}=p(B)$ for a basis $B \in \mathscr{B}(M)$, then it would satisfy the constraints given by $F_{\gamma}$ for $\gamma \in V(\alpha) \backslash \gamma_{0}$, since it passes all the same bays as $s$, and the only step where $s$ goes north but $s^{\prime}$ doesn't is at $\{x+u\}$, which is not a vertex of $M_{\alpha}$ by choice of $u$ (otherwise $(x, u-1)$ would also be a node). However, $s^{\prime}$ does not satisfy the constraint given by $F_{\gamma_{0}}$ since it does not pass through the $p$-bay $(x, z)$. Thus, the basis $B$ with $s^{\prime}=p(B)$ would be an element in the right hand side but not in the left hand side of Equation (11): a contradiction.

We conclude that $s^{\prime} \notin[p, q]$, implying that the point $(x+1, u-1)$ is not a lattice point between $q$ and $p$, i.e., $(x, u)$ lies on $q$.

Claim 1 shows in particular that there is a node $(x, u)$ of $M_{\alpha}$ which lies on $q$. If there is no $q$-bay of the form $(x, y)$, all paths of $M_{\alpha}$ go east in the $(x+u)$-th step (the step before reaching $(x, u))$. But this is a contradiction to $M_{\alpha}$ being loopless. So there has to be a $q$-bay $(x, y)$ which is also a node of $M_{\alpha}$. Now, since $M$ is connected, $y<z$; moreover, in order for all paths satisfying the constraints of $V(\alpha) \backslash\left\{\gamma_{0}\right\}$ to pass through $(x, z), V(\alpha)$ must contain all singletons $\{x+y\}, \ldots,\{x+z\}$. In particular, none of these is a land neck, thus $z-y>1$, q.e.d.
Case 2: $F_{\gamma_{0}}$ is a singleton $\left\{i^{\prime}\right\}$. Then every path $s$ of $M_{\alpha}$ has $s_{i^{\prime}}=N$, and we start by proving that if every path of $M_{\alpha}$ must go north at step $i^{\prime}$, then they all do so at a node.

Claim 2: $M_{\alpha}$ has a node $\left(x_{1}, x_{2}\right)$ with $x_{1}+x_{2}=i^{\prime}$.
Proof of Claim 2. To see this, consider two paths $s, s^{\prime}$ of $M_{\alpha}$ such that $s_{1} \ldots s_{i^{\prime}}$ ends at $\left(y_{1}, y_{2}\right)$ and $s_{1}^{\prime} \ldots s_{i^{\prime}}^{\prime}$ ends at $\left(y_{1}^{\prime}, y_{2}^{\prime}\right)$ with $y_{2}^{\prime} \geqslant y_{2}$. We want to prove that $y_{2}^{\prime}=y_{2}$.

By way of contradiction suppose $y_{2}^{\prime}-y_{2}>0$ and let $\left(a_{1}, a_{2}\right),\left(b_{1}, b_{2}\right)$ with $a_{1}+a_{2}<i^{\prime}<b_{1}+b_{2}$ be on both $s, s^{\prime}$ and such that the path $s_{a_{1}+a_{2}+1} \ldots s_{b_{1}+b_{2}}$ is always below $s_{a_{1}+a_{2}+1}^{\prime} \ldots s_{b_{1}+b_{2}}^{\prime}$. In particular, $s_{a_{1}+a_{2}+1}^{\prime}=$ $N, s_{b_{1}+b_{2}}^{\prime}=E$ and there are no nodes of $M_{\alpha}$ between $\left(a_{1}, a_{2}\right)$ and $\left(b_{1}, b_{2}\right)$. The path

$$
s_{1}^{\prime} \ldots s_{a_{1}+a_{2}}^{\prime} N s_{a_{1}+a_{2}+2} \ldots s_{b_{1}+b_{2}-1} E s_{b_{1}+b_{2}+1} \ldots s_{r+m}
$$

is thus a path of $M_{\alpha}$ that after $i^{\prime}$ steps ends at a point $\left(y_{1}^{\prime \prime}, y_{2}^{\prime \prime}\right)$ with $y_{2}^{\prime \prime}=$ $y_{2}+1$.
By repeating this operation we can assume without loss of generality that $y_{2}^{\prime}-y_{2}=1$. Now in this case, since the $i$ th step of $s^{\prime}$ is $N$, the path

$$
s_{1}^{\prime} \ldots s_{i^{\prime}-1}^{\prime} E s_{i^{\prime}+1} \ldots s_{r+m}
$$

represents an element of the right-hand side but not on the left-hand side of (1): a contradiction.
Thus we see that all paths corresponding to bases of $M_{\alpha}$ must pass through a common node, of the form $\left(x, i^{\prime}-x\right)$, for some $x$. Since $F_{\gamma_{0}}=\left\{i^{\prime}\right\}$, before this node all paths must go North. Thus, both $\left(x, i^{\prime}-x-1\right)$ and $\left(x, i^{\prime}-x\right)$ are nodes of $M_{\alpha}$.

Since $\left\{i^{\prime}\right\}$ is not a land neck, we know that the points $(x, z)$ and $(x, y)$ where the vertical line through $\left(x, i^{\prime}-x\right)$ meets $p$ resp. $q$ must lie more than one unit apart, i.e., $z-y>1$.

To prove the theorem it now suffices to prove the following claim.
Claim 3: There is a p-bay $(x, z)$ north and a $q$-bay $(x, y)$ south of $\left(x, i^{\prime}-x\right)$, and both the flacets corresponding to $(x, y)$ and $(x, z)$ are vertices of $\alpha$.
Proof of Claim 3. Let $(x, u)$ be the lowest node of $M_{\alpha}$ south of $\left(x, i^{\prime}-x-1\right)$. Then there is a path $s$ of $M_{\alpha}$ with $s_{x+u}=E$. Similarly let $v$ be maximal such that $(x, v)$ is a node of $M_{\alpha}$ and let $t$ be a path of $M_{\alpha}$ with $t_{x+v+1}=E$. Consider the paths

$$
\begin{aligned}
s^{\prime} & :=s_{1} \ldots s_{x+u-1} N \ldots N E s_{i^{\prime}+1} \ldots s_{r+m} \\
t^{\prime} & :=t_{1} \ldots t_{i^{\prime}-1} E N \ldots N t_{x+v+2} \ldots t_{r+m}
\end{aligned}
$$



Figure 5. The paths $s, t, s^{\prime}$ and $s$ in the proof of Claim 3.
Neither $s^{\prime}$ nor $t^{\prime}$ can be a path of $M_{\alpha}$, because they violate the constraint of $F_{\gamma_{0}}$ by $s_{i^{\prime}}^{\prime}=t_{i^{\prime}}^{\prime}=E$. Therefore, in order for Equation (1) to hold, for each of the paths $s^{\prime}$ and $t^{\prime}$ either
(i) it is not an element of $[p, q\rfloor$ at all, or
(ii) it must violate the constraint given by some $F_{\gamma}$ with $\gamma \in V(\alpha) \backslash\left\{\gamma_{0}\right\}$. Notice that in the second case, since the only step where $s^{\prime}$ goes east but $s$ does not is $i^{\prime}$, the vertex $\gamma$ can not correspond to a singleton, and thus it must correspond to a bay. In our particular setup we obtain the following case analysis.

For $s^{\prime}$ not to be a path of $M_{\alpha}$ one of the following must occur:
(si) the point $\left(x, i^{\prime}-x-1\right)$ lies on $p$ (since $s^{\prime}$ passes through $\left(x-1, i^{\prime}-x\right)$ ), or
(sii) there is $\gamma \in V(\alpha)$ such that $F_{\gamma}$ corresponds to a $q$-bay $(x, y)$ between $\left(x, i^{\prime}-x-1\right)$ and $(x, u)$.
Similarly, for $t^{\prime}$ not to be a path of $M_{\alpha}$, either
$(t \mathrm{i})$ the point $\left(x, i^{\prime}-x\right)$ lies on $q$ (since $t^{\prime}$ passes through $\left(x+1, i^{\prime}-x-1\right)$ ), or
( tii) there is $\gamma \in V(\alpha)$ such that $F_{\gamma}$ corresponds to a $p$-bay $(x, z)$ between $\left(x, i^{\prime}-x\right)$ and $(x, v)$.
Now, if both $(s i)$ and ( $t \mathrm{i}$ ) were true, $M(p, q)$ would not be connected, contradicting the hypothesis; on the other hand, $(s i)$ and ( $t \mathrm{ii}$ ) together imply that there is a point on $p$ south of a $p$-bay, and similarly from ( ti ) and (sii)
follows the existence of a point on $q$ that is north of a $q$-bay. We conclude that both (sii) and (tii) must hold, proving the claim.

Corollary 3.2. The Bergman complex of a connected lattice path matroid $M(p, q)$ is simplicial if and only if every pair of vertically aligned bays determines a land neck (i.e., if $\left(x_{1}, x_{2}\right) \in U_{p}$ and $\left(x_{1}, x_{2}^{\prime}\right) \in U_{q}$, then $x_{2}-x_{2}^{\prime}=1$ ).
Example 3.3. Consider the lattice path matroid given in Figure 6. The face $\alpha$ whose vertices correspond to the bays $(4,3),(4,6)$ and the singletons $8,9,10$ in Figure 6 is a minimal non-simplicial face.


Figure 6. Illustration of flacets of the Bergman complex that define a non-simplicial face as well as a lattice path representation of the matroid type of this face.

## 4. Combinatorial structure

4.1. The poset of faces. Let $M=M(p, q)$ be a lattice path matroid with set of bays $U$ and set of land necks $S(p, q)$. Define a partial order on $U$ by setting

$$
\left(x_{1}, x_{2}\right)<\left(y_{1}, y_{2}\right) \text { if and only if } x_{1} \leqslant y_{1} \text { and } x_{2}<y_{2}
$$

We will denote with $\Delta(U)$ the set of chains (i.e., totally ordered subsets) of $U$, ordered by inclusion. Every chain

$$
\omega:=\left\{\left(a_{1}, b_{1}\right)<\left(a_{2}, b_{2}\right)<\cdots\right\}
$$

of $U$ defines a partition

$$
\pi(\omega)=\pi_{1}(\omega) \uplus \ldots \uplus \pi_{|\omega|+1}(\omega)
$$

of the set $[m+r]$ with $j$-th block $\pi_{j}(\omega):=\left\{a_{j-1}+b_{j-1}+1, \ldots, a_{j}+b_{j}\right\}$ for $j=1, \ldots,|\omega|+1$, where we set $\left(a_{0}, b_{0}\right)=(0,0)$ and $\left(a_{|\omega|+1}, b_{|\omega|+1}\right)=(m, r)$.
Definition 4.1. The chain $\omega$ defines a matroid

$$
M(\omega)=M_{1}(\omega) \oplus \ldots \oplus M_{|\omega|+1}(\omega)
$$

where $M_{j}(\omega)$ is the lattice path matroid represented by the lattice paths from $\left(a_{j-1}, b_{j-1}\right)$ to ( $a_{j}, b_{j}$ ) lying between $p$ and $q$.

Remark 9. The matroid $M(\omega)$ is a weak map image of $M$. Also, notice that for $\omega \in \Delta(U)$ every $M_{j}(\omega)$ is loopfree.

Let the set $\mathcal{I}:=\mathscr{P}([m+r] \backslash S(p, q))$ be partially ordered by inclusion.

$\omega=\{(2,2)<(5,5)\}$

$\omega=\{(2,2)<(4,3)<(4,6)\}$

Figure 7. The direct summands for $M(\omega)$ are the lattice path matroids that appear in the shaded regions.

Definition 4.2. Given a lattice path matroid $M$ recall the above notations and define $\mathcal{Q}(p, q)$ as the subposet of the product poset $\Delta(U) \times \mathcal{I}$ defined by:
$\left\{\begin{array}{lll}(\omega, J) \in \Delta(U) \times \mathcal{I} & \begin{array}{ll}\text { For } 2 \leqslant i \leqslant|\omega|: & \text { if } a_{i}=a_{i-1}, b_{i}-b_{i-1}>1 ; \\ \pi_{i}(\omega) \cap J=\pi_{i}(\omega) & \text { if } a_{i}=a_{i-1}, b_{i}-b_{i-1}=1 ; \\ \pi_{i}(\omega) \cap J=\varnothing & \text { otherwise }\end{array} \\ \pi_{i}(\omega) \cap J \text { is an independent flat of } M_{i}(\omega) \quad .\end{array}\right.$.
Theorem 4.3. For any connected lattice path matroid $M=M(p, q)$ the posets $\Gamma_{M(p, q)}$ and $\mathcal{Q}(p, q)$ are isomorphic.
Proof. Recall that vertices of $\Gamma_{M}$ are the set of bays and non-land necks singletons. Faces of $\Gamma_{M}$ are cells of $P_{M}^{\vee}$, and thus uniquely determined by their vertices.

We consider a face $\alpha$ of the Bergman complex of $M$ and recall that, with Lemma 2.6 , its set $V(\alpha)$ of vertices consists of two types of elements: bays (identifying vertices corresponding to fundamental flats) and singletons. Accordingly, we have a partition $V(\alpha)=V(\alpha) \cap U \uplus V(\alpha) \cap \mathcal{I}$. We first show that the pair $(V(\alpha) \cap$ $U, V(\alpha) \cap \mathcal{I})$ is contained in $\mathcal{Q}(p, q)$.
(a) $V(\alpha) \cap U \in \Delta(U)$.

To prove this, consider two points $(a, b),(c, d) \in V(\alpha) \cap U$ such that $a \leqslant c$ and $b \geqslant d$. Every lattice path $p(B)$ for $B \in M_{\alpha}$ must pass through both points, therefore $a \leqslant c$ implies $b \leqslant d$, so $b=d$. Now we have $p(B)_{j}=E$ for all $B \in \mathscr{B}\left(M_{\alpha}\right)$ and every $j \in\{a+b+1, \ldots, c+b\}$ - but since $M_{\alpha}$ by definition must be loopless, there can't be any such $j$, thus $a=c$ and we are done.

Write $\omega=\left(a_{1}, b_{1}\right)<\cdots<\left(a_{k}, b_{k}\right)$ for the chain $V(\alpha) \cap U$, write $J:=V(\alpha) \cap \mathcal{I}$ and let $i \in[k]$.
(b) If $a_{i-1}=a_{i}$, then either $b_{i}-b_{i-1}=1$ i.e. the two bays define a land neck and $\pi_{i}(\omega)=\left\{a_{i}+b_{i}\right\} \subseteq S[p, q]$ and so $\varnothing=\pi_{i}(\omega) \cap \mathcal{I} \supseteq \pi_{i}(\omega) \cap J$, as required. Otherwise, we are in the non-simplicial situation of the proof of Theorem 3.1, and every element of $\pi_{i}(\omega)=\left\{a_{i-1}+b_{i-1}+1, \ldots, a_{i}+b_{i}\right\}$ is, as a singleton flacet, a vertex of $\alpha$. Therefore, $\pi_{i}(\omega) \cap J=\pi_{i}(\omega)$.
(c) If $a_{i-1}<a_{i}$, then $\pi_{i}(\omega) \cap J$ is an independent flat of $M$.

Namely, in this case, thinking of paths satisfying constraints (according, e.g., to Remark 8 8 , the set of lattice paths $\left\{p(B)_{a_{i-1}+b_{i-1}+1} \cdots p(B)_{a_{i}+b_{i}}\right.$ : $\left.B \in \mathscr{B}\left(M_{\alpha}\right)\right\}$ is the set of bases of a matroid $M_{i}^{\prime}$ isomorphic to

$$
M_{i}(\omega) /\left(J \cap \pi_{i}(\omega)\right) \oplus M_{i}(\omega)\left[J \cap \pi_{i}(\omega)\right] .
$$

Now, $\pi_{i}(\omega) \cap J$ is independent because every basis of $M_{\alpha}$ contains $J$ (the paths representing these bases must satisfy the constraints in $J$, i.e., go north at every element of $J$ ). The loop-freeness of $M_{\alpha}$ gives the loopfreeness of $M_{i}^{\prime}$ (which is a direct summand in the decomposition of $M_{\alpha}$ ). In particular we see that $M_{i}(\omega) /\left(J \cap \pi_{i}(\omega)\right)$ is loopfree. Then, $J \cap \pi_{i}(\omega)$ is a flat of $M_{i}(\omega)$ (e.g., by [15, Exercise 3.1.8]).
In view of (a), (b), (c) above, the following function is well defined and clearly order-preserving.

$$
\psi: \Gamma_{M} \rightarrow \mathcal{Q}(p, q) ; \quad V \mapsto(V \cap U, V \cap \mathcal{I})
$$

Moreover, it admits an order-preserving inverse given by

$$
\mathcal{Q}(p, q) \rightarrow \Gamma_{M} ; \quad(\omega, J) \mapsto \omega \cup J
$$

is easily seen to be well-defined, as it sends $(\omega, J)$ to the vertex set of the face $\alpha$ with

$$
M_{\alpha}=\bigoplus_{a_{i-1} \neq a_{i}} M_{i}(\omega) /\left(J \cap \pi_{i}(\omega)\right) \oplus M_{i}(\omega)\left[J \cap \pi_{i}(\omega)\right] \oplus \bigoplus_{a_{i-1}=a_{i}} M_{i}(\omega)
$$

which is loopfree because all its direct summands are.
4.2. Polyhedral structure of faces. As a face of $P^{\vee}$, every face of $\Gamma_{M}$ is the convex hull of its vertices. We would like to characterize such polyhedra.

Recall Definition 2.5 and, for a subset $X \subseteq U \cup([m+r] \backslash S(p, q))$, let $\gamma(X)$ denote the set of vertices $v$ of $P^{\vee}$ for which $F_{v} \in X$ either is an element of $X \backslash U$ or corresponds to a bay in $X \cap U$.

Given a set of points $A$ in general position let $\Delta_{A}=\operatorname{conv}(A)$ denote the simplex on the vertex set $A$. Moreover, let $\diamond_{A}$ be the polytope obtained as the suspension of $\Delta_{A}$, see [12, Section 2.2] for a formal definition of suspensions.

Remark 10. Recall that the poset of faces of the join $P * Q$ of two polytopes $P$ and $Q$ is obtained from the face posets $\widehat{\mathscr{F}}(P)$ and $\widehat{\mathscr{F}}(Q)$ (these are the face posets of $P$ and $Q$, each with an added unique smallest element $0_{P}$ resp. $0_{Q}$ ) by

$$
\widehat{\mathscr{F}}(P * Q) \simeq \widehat{\mathscr{F}}(P) \times \widehat{\mathscr{F}}(Q)
$$

Corollary 4.4. The face of $\Gamma_{M}$ corresponding to $(\omega, J)$ is a join
where $A:=\left\{\left(a_{i}, b_{i}\right) \in \omega: a_{i-1}=a_{i} \Rightarrow\left|b_{i-1}-b_{i}\right|=0\right\}$ is the set of all bays in $\omega$ that either form a land neck with some other bay of $\omega$ or are not vertically aligned with any other bay of $\omega$ at all.

Remark 11. Notice that if there are no two vertically aligned bays $\left(a_{i-1}, b_{i-1}\right),\left(a_{i}, b_{i}\right)$ that do not form a land neck $\left(a_{i-1}=a_{i}, b_{i}-b_{i-1}>1\right)$, then every term of the above join is a simplex, and thus $P(\omega, J)$ is a simplex, in agreement with the condition found in Theorem 3.1.

Proof. Since the vertex set of $P(\omega, J)$ is precisely $\gamma(\omega \cup J)$, it is enough to prove isomorphism of face posets.

Given a set $X$ and distinct elements $y_{1}, y_{2} \notin X$, write $B_{X}$ for the poset of all subsets of $X, C_{X, y_{1}, y_{2}}$ for the poset of all subsets $Y \subseteq X \cup\left\{y_{1}, y_{2}\right\}$ such that
$\left\{y_{1}, y_{2}\right\} \subseteq Y$ if and only if $X \subseteq Y$. Then $B_{X} \simeq \widehat{\mathscr{F}}\left(\Delta_{X}\right)$ and $C_{X, y_{1}, y_{2}} \simeq \widehat{\mathscr{F}}\left(\diamond_{X}\right)$ and the face poset of $P(\omega, J)$ is isomorphic to

$$
B_{A} \times \prod_{a_{i-1}<a_{i}} B_{J \cap \pi_{i}(\omega)} \times \prod_{\substack{i: a_{i-1}=a_{i}, b_{i}-b_{i-1}>1}} C_{\pi_{i}(\omega),\left(a_{i-1}, b_{i-1}\right),\left(a_{i}, b_{i}\right)}
$$

Now to prove that the map

$$
\begin{gathered}
\mathcal{Q}(p, q)_{\leqslant} \leqslant(\omega, J) \rightarrow \mathscr{F}(P(\omega, J)), \\
\left(\omega^{\prime}, J^{\prime}\right) \mapsto(\omega^{\prime} \cap A, \underbrace{J^{\prime} \cap \pi_{i}(\omega), \ldots}_{a_{i-1}<a_{i}}, \underbrace{\left.\left(\left\{\left(a_{i-1}, b_{i-1}\right),\left(a_{i}, b_{i}\right)\right\} \cap \omega^{\prime}\right) \cup\left(J^{\prime} \cap \pi_{i}(\omega)\right), \ldots\right)}_{a_{i-1}=a_{i}, b_{i}-b_{i-1}>1}
\end{gathered}
$$

is a poset isomorphism amounts to a routine check in view of Theorem 3.1 and of the fact that, for every $(\omega, J) \in \mathcal{Q}(p, q)$, we get that $\left(\omega, J^{\prime}\right)$ is in $\mathcal{Q}(p, q)$ for all $J^{\prime} \subseteq J$. Here, loop-freeness of the quotients $M_{i}(\omega) /\left(J^{\prime} \cap \pi_{i}(\omega)\right)$ is implied by loop-freeness of $M_{i}(\omega) /\left(J \cap \pi_{i}(\omega)\right)$ since subsets of independent flats are again independent flats.

Example 4.5. Consider the lattice path matroid given in Figure 6 on page 11. The face $\alpha$ whose vertices correspond to the bays $(4,3),(4,6)$ and the singletons $8,9,10$ in Figure 6 is a triangular bipyramid obtained by suspending a 2 -simplex (whose vertices correspond to the singletons $8,9,10$ ) between two additional vertices (corresponding to the bays $(4,3)$ and $(4,6))$.

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(Emanuele Delucchi) Departement de Mathématiques, Université de Fribourg, Chemin du Musée 21, 1700 Fribourg, Switzerland.

E-mail address, Emanuele Delucchi: emanuele.delucchi@unifr.ch
(Martin Dlugosch) Fachbereich Mathematik und Informatik, Universität Bremen, Bibliothekstrasse 1, 28359 Bremen, Bremen, Germany.

E-mail address, Martin Dlugosch: mdlug@math.uni-bremen.de

