ARITHMETIC OF MARKED ORDER POLYTOPES, MONOTONE TRIANGLE RECIPROCITY, AND PARTIAL COLORINGS

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ABSTRACT. For a pair of posets $A \subseteq P$ and an order preserving map $\lambda : A \to \mathbb{R}$, the marked order polytope parametrizes the order preserving extensions of λ to P. We show that the function counting integral-valued extensions is a piecewise polynomial in λ and we prove a reciprocity statement in terms of order-reversing maps. We apply our results to give a geometric proof of a combinatorial reciprocity for monotone triangles due to Fischer and Riegler (2011) and we consider the enumerative problem of counting extensions of partial graph colorings of Herzberg and Murty (2007).

1. INTRODUCTION

Partially ordered sets, or **posets** for short, are among the most fundamental objects in combinatorics. For a finite poset P, Stanley [12] considered the problem of counting (strictly) order preserving maps from P into n-chains and showed that many problems in combinatorics can be cast into this form. Here, a map $\lambda : P \to [n]$ into the n-chain is **order preserving** if $\lambda(p) \leq \lambda(q)$ whenever $p \prec_P q$ and the inequality is strict for **strict** order preservation. In [12] it is shown that the number of order preserving maps into a chain of length n is given by a polynomial $\Omega_P(n)$ in the positive integer n and the number of strictly order preserving maps is related to $\Omega_P(n)$ by a combinatorial reciprocity (see Section 2.5).

In this paper we consider the problem of counting the number of order preserving extensions of a map $\lambda : A \to \mathbb{Z}$ from a subposet $A \subseteq P$ to P. Clearly, this number is finite only when Acomprises all minimal and maximal elements of P and we tacitly assume this throughout. It is also obvious that no extension exists unless λ is order preserving for A and we define $\Omega_{P,A}(\lambda)$ as the number of order preserving maps $\hat{\lambda} : P \to \mathbb{Z}$ such that $\hat{\lambda}|_A = \lambda$. By adjoining a minimum and maximum to P it is seen that $\Omega_{P,A}(\lambda)$ generalizes the order polynomial.

The function $\Omega_P(n)$ can be studied from a geometric perspective by relating it to the Ehrhart function of the **order polytope** [15], the set of order preserving maps $P \to [0, 1]$. The finiteness of P asserts that this is indeed a convex polytope in the finite dimensional real vector space \mathbb{R}^P . The order polytope is a lattice polytope whose facial structure is intimately related to the structure of P and which has a canonical unimodular triangulation again described in terms of the combinatorics of P. Standard facts from Ehrhart theory (see for example [2]) then assert that $\Omega_P(n)$ is a polynomial of degree |P|. We pursue this geometric route and study the **marked order polytope**

$$\mathcal{O}_{P,A}(\lambda) = \left\{ \widehat{\lambda} : P \to \mathbb{R} \text{ order preserving } : \widehat{\lambda}(a) = \lambda(a) \text{ for all } a \in A \right\} \subset \mathbb{R}^P.$$

Marked order polytopes were considered (and named) by Ardila, Bliem, and Salazar [1] in connection with representation theory. In the case that A is a chain, the polytopes already appear in [14]; see Section 2.4. The set $\mathcal{O}_{P,A}(\lambda)$ defines a polyhedron for any choice of $A \subseteq P$ but it is

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a polytope precisely when $\min(P) \cup \max(P) \subseteq A$. It follows that $\Omega_{P,A}(\lambda) = \#(\mathcal{O}_{P,A}(\lambda) \cap \mathbb{Z}^P)$. In Section 2 we elaborate on the geometric-combinatorial properties of $\mathcal{O}_{P,A}(\lambda)$ and we show that $\Omega_{P,A}(\lambda)$ is a piecewise polynomial over the space of integral-valued order preserving maps $\lambda : A \to \mathbb{Z}$. We give an explicit description of the polyhedral domains for which $\Omega_{P,A}(\lambda)$ is a polynomial and we give a combinatorial reciprocity for $\Omega_{P,A}(-\lambda)$. We close by "transferring" our results to the marked chain polytopes of [1].

In Section 3, we use our results to give a geometric interpretation of a combinatorial reciprocity for **monotone triangles** that was recently described by Fischer and Riegler [7]. A monotone triangle is a triangular array of numbers such as

with fixed bottom row such that the entries along the directions \searrow and \nearrow are weakly increasing and strictly increasing in direction \rightarrow ; a more formal treatment is deferred to Section 3. Monotone triangles arose initially in connection with alternating sign matrices [11] and a significant amount of work regarding their enumerative behavior was done in [6]. In particular, it was shown that the number of monotone triangles is a polynomial in the strictly increasing bottom row. In [7] a (signed) interpretation is given for the evaluation of this polynomial at weakly decreasing arguments in terms of **decreasing monotone triangles**. In our language, monotone triangles are extensions of order preserving maps over posets know as **Gelfand-Tsetlin patterns** plus some extra conditions. These extra conditions can be interpreted as excluding the lattice points in a natural subcomplex of the boundary of $\mathcal{O}_{P,A}(\lambda)$. We investigate the combinatorics of this subcomplex and give a geometric interpretation for the combinatorial reciprocity of monotone triangles.

Finally, a well-known result of Stanley [13] gives a combinatorial interpretation for the evaluation of the chromatic polynomial $\chi_G(t)$ of a graph G at negative integers in terms of acyclic orientations. We give a combinatorial reciprocity for the situation of counting extensions of partial colorings which was considered by Herzberg and Murty [8].

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2. Marked order polytopes

Marked order polytopes as defined in the introduction naturally arise as sections of a polyhedral cone, the **order cone**, which parametrizes order preserving maps from a finite poset P to \mathbb{R} . The order cone is the "cone-analog" of the order polytope which was thoroughly studied in [15] and whose main geometric results we reproduce before turning to marked order polytopes. We freely make use of concepts from polyhedral geometry as can be found, for example, in [16]. For a finite set S we identify \mathbb{R}^S with the vector space of real-valued functions $S \to \mathbb{R}$.

2.1. Order cones. The order cone is the set $\mathcal{L}(P) \subseteq \mathbb{R}^P$ of order preserving maps from P into \mathbb{R}

 $\mathcal{L}(P) = \{ \phi \in \mathbb{R}^P : \phi(p) \leq \phi(q) \text{ for all } p \preceq_P q \}.$

This is a closed convex cone and the finiteness of P ensures that $\mathcal{L}(P)$ is polyhedral (i.e. bounded by finitely many halfspaces). The cone is not pointed and the lineality space of $\mathcal{L}(P)$ is spanned by the indicator functions of the connected components of P. Said differently, the largest linear subspace contained in $\mathcal{L}(P)$ is spanned by the functions $\chi: P \to \{0, 1\}$ that satisfy $\chi(p) = \chi(q)$ whenever there is a sequence $p = p_0 p_1 \dots p_{k-1} p_k = q$ such that $p_i p_{i+1}$ are comparable in P. The cone $\mathcal{L}(P) \subseteq \mathbb{R}^P$ is of full dimension |P| and its facet defining equations are given by $\phi(p) = \phi(q)$ for every cover relation $p \prec p$. Every face $F \subseteq \mathcal{L}(P)$ gives rise to a subposet G(F) of P whose Hasse diagram is given by those $p \prec p$ for which $\phi(p) = \phi(q)$ for all $\phi \in F$. Such a subposet G(F) arising from a face $F \subseteq \mathcal{L}(P)$ is called a **face partition**. The following characterization of face partitions is taken from [15].

Proposition 2.1. A subposet $G \subseteq P$ is a face partition if and only if for every $p, q \in G$ with $p \preceq_G q$ we have $[p,q]_P \subseteq G$.

Equivalently, the directed graph obtained from the Hasse diagram of P by contracting the cover relations in G is an acyclic graph and, after removing transitive edges, is the Hasse diagram of a poset that we denote by P/G. Note that G is typically not a connected poset. The face corresponding to such a graph G is then

 $F_P(G) = \{ \phi \in \mathcal{L}(P) : \phi \text{ is constant on every connected component of } G \}$

and $F_P(G)$ is isomorphic to $\mathcal{L}(P/G)$ by a linear and lattice preserving map.

The order cone has a canonical subdivision into unimodular cones that stems from refinements of P induced by elements of $\mathcal{L}(P)$. To describe the constituents of the subdivision, recall that $I \subseteq P$ is an **order ideal** if $p \preceq_P q$ and $q \in I$ implies $p \in I$. Let $\phi \in \mathcal{L}(P)$ be an order preserving map with range $\phi(P) = \{t_0 < t_1 < \cdots < t_k\}$. Then ϕ induces a chain of order ideals

$${}^{P}_{\bullet} : I_{0} \stackrel{\frown}{\neq} I_{1} \stackrel{\frown}{\neq} I_{2} \stackrel{\frown}{\neq} \cdots \stackrel{\frown}{\neq} I_{k} = P$$

by setting $I_j = \{p \in P : \phi(p) \leq t_j\}$. If the poset P is clear from the context, we drop the superscript and simply write I_{\bullet} . Conversely, a given chain of order ideals I_{\bullet} is induced by $\phi \in \mathcal{L}(P)$ if and only if ϕ is constant on $I_j \setminus I_{j-1}$ for $j = 0, 1, \ldots, k$ (with $I_{-1} = \emptyset$) and

$$\phi(I_0) < \phi(I_1 \setminus I_0) < \phi(I_2 \setminus I_1) < \cdots < \phi(I_k \setminus I_{k-1}).$$

This defines the relative interior of a (k + 1)-dimensional simplicial cone in $\mathcal{L}(P)$ whose closure we denote by $F(I_{\bullet})$. Chains of order ideals are ordered by refinement and the maximal elements correspond to saturated chains of order ideals or, equivalently, **linear extensions** of P. For a saturated chain I_{\bullet} , we have $I_j \setminus I_{j-1} = \{p_j\}$ for $j = 0, 1, \ldots, m = |P| - 1$ and $p_i \prec_P p_j$ implies i < j. In this case

$$F(I_{\bullet}) = \left\{ \phi \in \mathbb{R}^P : \phi(p_0) \leq \phi(p_1) \leq \cdots \leq \phi(p_{m-1}) \right\}.$$

Modulo lineality space, this is a unimodular simplicial cone spanned by the characteristic functions $\phi^0, \phi^1, \ldots, \phi^{m-1} : P \to \{0, 1\}$ with $\phi^k(p_j) = 1$ iff $j \ge k$. Faces of $F(I_{\bullet})$ correspond to the coarsenings of I_{\bullet} and since every $\phi \in \mathcal{L}(P)$ induces a unique $I_{\bullet} = I_{\bullet}(\phi)$, this proves the following result which was first shown by Stanley [15] for the order polytope $\mathcal{L}(P) \cap [0, 1]^P$.

Proposition 2.2. Let P be a finite poset. Then

 $\mathcal{T}_P = \left\{ F(I^P_{\bullet}) \; : \; I^P_{\bullet} \; chain \; of \; order \; ideals \; in \; P \; \right\}$

is a subdivision of $\mathcal{L}(P)$ into unimodular simplicial cones.

2.2. Marked order polytopes. Now let $A \subseteq P$ be a subposet such that $\min(P) \cup \max(P) \subseteq A$. For an order preserving map $\lambda : A \to \mathbb{R}$, the marked order polytope

$$\mathcal{O}_{P,A}(\lambda) = \left\{ \widehat{\lambda} \in \mathcal{L}(P) : \widehat{\lambda}(a) = \lambda(a) \text{ for all } a \in A \right\} = \mathcal{L}(P) \cap \operatorname{Ext}_{P,A}(\lambda)$$

is the intersection of the order cone with the affine space $\operatorname{Ext}_{P,A}(\lambda) = \{\widehat{\lambda} \in \mathbb{R}^P : \widehat{\lambda}|_A = \lambda\}$. Every face of $\mathcal{O}_{P,A}(\lambda)$ is a section of a face H of $\mathcal{L}(P)$ with $\operatorname{Ext}_{P,A}(\lambda)$ and is itself a marked order polytope. We denote the dependence of H on λ by $H(\lambda)$. We can describe them in terms of face partitions.

Proposition 2.3. Let G be a face partition of P and let $\lambda : A \to \mathbb{R}$ be an order preserving map for $A \subseteq P$. Then $\operatorname{Ext}_{P,A}(\lambda)$ meets $F_P(G)$ in the relative interior if and only if the following holds for all $a, b \in A$: Let $G_a, G_b \subseteq P$ be the connected components of G containing a and b, respectively. i) If $\lambda(a) < \lambda(b)$ then

$$\bigcup_{p \in G_a} P_{\preceq p} \cap \bigcup_{q \in G_b} P_{\succeq q} = \varnothing.$$

ii) If $\lambda(a) = \lambda(b)$ and a and b are comparable, then $G_a = G_b$.

In this case, $F_P(G) \cap \operatorname{Ext}_{P,A}(\lambda)$ is linearly isomorphic to $\mathcal{O}_{P/G,A/G}(\lambda_G)$ where $\lambda_G : A/G \to \mathbb{R}$ is the well-defined map on the quotient.

Proof. Let P/G be the quotient poset associated to the face partition G. The quotient A/G is a subposet of P/G and $\lambda_G : A/G \to \mathbb{R}$ is a well-defined map if condition i) holds. Moreover, the induced map λ_G is order preserving for A/G if condition i) holds, and in fact strictly if ii) holds. Thus $F_P(G) \cap \operatorname{Ext}_{P,A}(\lambda)$ is linearly isomorphic to $\mathcal{O}_{P/G,A/G}(\lambda_G)$ which is of maximal dimension.

We call a face partition **compatible** with λ if it satisfies the conditions above. In particular, taking the intersection of all compatible face partitions of P, we obtain $\mathcal{O}_{P,A}(\lambda)$ as an improper face.

Corollary 2.4. Let $A \subseteq P$ be a pair of posets and $\lambda : A \to \mathbb{R}$ an order preserving map. Then $\mathcal{O}_{P,A}(\lambda)$ is a convex polytope of dimension

$$\dim \mathcal{O}_{P,A}(\lambda) = |P \setminus \{ p \in P : a \leq p \leq b \text{ for } a, b \in A \text{ with } \lambda(a) = \lambda(b) \}|.$$

Proof. The presentation as the affine section of a cone marks $\mathcal{O}_{P,A}(\lambda)$ as a convex polyhedron. As every element of P has by assumption a lower and upper bound in A, it follows that $\mathcal{O}_{P,A}(\lambda)$ is a polytope. The right-hand side is exactly the number of elements of P whose values are not yet determined by λ and $\mathcal{O}_{P,A}(\lambda)$ has at most this dimension. On the other hand, Lemma 2.5 shows the existence of a subpolytope of exactly this dimension.

2.3. Induced subdivisions and arithmetic. Intersecting every cell of the canonical subdivision \mathcal{T}_P of $\mathcal{L}(P)$ with the affine space $\operatorname{Ext}_{P,A}(\lambda)$ induces a subdivision of $\mathcal{O}_{P,A}(\lambda)$ that we can explicitly describe. To describe the cells in the intersection, let I_{\bullet} be a chain of order ideals of P. For $a \in P$ we denote by $i(I_{\bullet}, a)$ the smallest index j for which $a \in I_j$. We call a chain of order ideals I_{\bullet} of P compatible with λ if

$$i(I_{\bullet}, a) < i(I_{\bullet}, b)$$
 if and only if $\lambda(a) < \lambda(b)$

for all $a, b \in A$. The crucial observation is that relint $F(I_{\bullet}) \cap \operatorname{Ext}_{P,A}(\lambda)$ is not empty iff I_{\bullet} is compatible with λ and in this case $F(I_{\bullet}) \cap \operatorname{Ext}_{P,A}(\lambda)$ is of a particularly nice form.

Lemma 2.5. Let $A \subseteq P$ be a pair of posets and $\lambda : A \to \mathbb{R}$ an order preserving map. If I_{\bullet} is a chain of order ideals of P compatible with λ , then the induced cell $F(I_{\bullet}) \cap \operatorname{Ext}_{P,A}(\lambda)$ is a Cartesian product of simplices.

Proof. Let $\lambda(A) = \{t_0 < t_1 < \cdots < t_r\}$ be the range of λ and pick elements $a_0, a_1, \ldots, a_r \in A$ with $\lambda(a_i) = t_i$. Let $i_j = i(I_{\bullet}, a_j)$ for $j = 0, 1, \ldots, r$ and, since I_{\bullet} is compatible with λ , we have $0 = i_0 < i_1 < \cdots < i_r = k$. It follows that $F(I_{\bullet}) \cap \operatorname{Ext}_{P,A}(\lambda)$ is the set of all $\phi \in \mathbb{R}^P$ such that ϕ is constant on $I_h \setminus I_{h-1}$ for $h = 0, 1, \ldots, k$ (with $I_{-1} = \emptyset$) and

Thus, $F(I_{\bullet}) \cap \operatorname{Ext}_{P,A}(\lambda)$ is linearly isomorphic to $F_0 \times F_1 \times \cdots \times F_{r-1}$ where, by setting $s_j = \phi(I_j \setminus I_{j-1}),$

(2)
$$F_j = \{ \lambda(a_j) \leq s_{i_j+1} \leq s_{i_j+2} \leq \cdots \leq s_{i_{j+1}-1} \leq \lambda(a_{j+1}) \}.$$

is a simplex of dimension $d_j = i_{j+1} - i_j - 1.$

Thus the canonical subdivision of $\mathcal{L}(P)$ induces a subdivision of $\mathcal{O}_{P,A}(\lambda)$ into products of simplices indexed by compatible chains of order ideals. This is the key observation for the following result.

Theorem 2.6. Let $A \subseteq P$ be a pair of posets with $\min(P) \cup \max(P) \subseteq A$. For integral-valued order preserving maps $\lambda : A \to \mathbb{Z}$, the function

$$\Omega_{P,A}(\lambda) = |\mathcal{O}_{P,A}(\lambda) \cap \mathbb{Z}^{P}|$$

is a piecewise polynomial over the order cone $\mathcal{L}(A)$. The cells of the canonical subdivision of $\mathcal{L}(A)$ refine the domains of polynomiality of $\Omega_{P,A}(\lambda)$. In other words, $\Omega_{P,A}(\lambda)$ is a polynomial restricted to any cell $F(I_{\bullet}^{A})$ of the subdivision of $\mathcal{L}(A)$.

Proof. Lemma 2.5 shows that for fixed $\lambda : A \to \mathbb{Z}$ every maximal cell in the induced subdivision of $\mathcal{O}_{P,A}(\lambda)$ is a product of simplices and the proof actually shows that, after taking successive differences, the simplices F_j of (2) are lattice isomorphic to

(3)
$$(\lambda(a_{j+1}) - \lambda(a_j)) \cdot \Delta_{d_j} = \{ y \in \mathbb{R}^{d_j}_{\geq 0} : y_1 + y_2 + \dots + y_{d_j} \leq \lambda(a_{j+1}) - \lambda(a_j) \}.$$

Elementary counting then shows that

(4)
$$|F(I_{\bullet}) \cap \operatorname{Ext}_{P,A}(\lambda) \cap \mathbb{Z}^{P}| = \prod_{j=0}^{r-1} |F_{j} \cap \mathbb{Z}^{P}| = \prod_{j=0}^{r-1} \left(\frac{\lambda(a_{j+1}) - \lambda(a_{j}) + d_{j}}{d_{j}} \right)$$

which is a polynomial in λ of degree $d_0 + d_1 + \cdots + d_{r-1} = \dim F(I_{\bullet}) \cap \operatorname{Ext}_{P,A}(\lambda)$. Möbius inversion on the face lattice of the induced subdivision shows that $\Omega_{P,A}$ is the evaluation of a polynomial at the given λ . To complete the proof, note that $\lambda, \lambda' : A \to \mathbb{R}$ have the same collections of compatible chains of order ideals whenever $\lambda, \lambda' \in \operatorname{relint} C$ for some cell C in the canonical subdivision \mathcal{T}_A of $\mathcal{L}(A)$. \Box

A weaker version of Theorem 2.6 can also be derived from the theory of partition functions [4, Ch. 13]. It can be seen that over $\mathcal{L}(A)$, the marked order polytope is of the form

$$\mathcal{O}_{P,A}(\lambda) = \{ x \in \mathbb{R}^n : Bx \leq c(\lambda) \}$$

where $B \in \mathbb{Z}^{M \times n}$ is a fixed matrix with n = |P| and $c : \mathbb{R}^A \to \mathbb{R}^M$ is an affine map. Moreover, B is unimodular. It follows from the theory of partition functions that the function $\Phi_B : \mathbb{Z}^M \to \mathbb{Z}$ given by

$$g \mapsto \#\{x \in \mathbb{Z}^n : Bx \le g\}$$

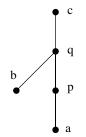
is a piecewise polynomial over the cone $C_B \subset \mathbb{R}^M$ of (real-valued) g such that the polytope above is non-empty. The domains of polynomiality are given by the **type cones** for B; see McMullen [10]. Consequently, we have $\Omega_{P,A}(\lambda) = \Phi_B(c(\lambda))$. It follows that $\mathcal{L}(A)$ is linearly isomorphic to a section of C_B and the canonical subdivision \mathcal{T}_A is a refinement of the induced subdivision by type cones. It is generally difficult to give an explicit description of the subdivision of C_B by type, not to mention the sections of type cones by the image of $c(\lambda)$. So, an additional benefit of the proof presented here is the explicit description of the domains of polynomiality.

In the context of representation theory, the lattice points of certain marked order polytopes bijectively correspond to bases elements of irreducible representations; cf. the discussion in [1, 3]. Bliem [3] used partition functions of *chopped and sliced cones* to show that in the marking λ , the dimension of the corresponding irreducible representation is given by a piecewise quasipolynomial. Theorem 2.6 strengthens his result to piecewise polynomials. Bliem [3, Warning 1] remarks that his 'regions of quasi-polynomiality' might be too fine in the sense that the quasi-polynomials for adjacent regions might coincide. This also happens for the piecewise polynomial described in Theorem 2.6. In the simplest case A = P and $\Omega_{P,A} \equiv 1$.

Question 1. What is the coarsest subdivision of $\mathcal{L}(A)$ for which $\Omega_{P,A}(\lambda)$ is a piecewise polynomial?

For this it is necessary to give a combinatorial condition when two adjacent cells of \mathcal{T}_A carry the same polynomial.

Example 2.7. Consider the following poset *P* given by its Hasse diagram:



Let $A = \{a, b, c\}$ and let $\lambda \colon A \to \mathbb{Z}$ be an order preserving map. If $\lambda(a) < \lambda(b) < \lambda(c)$ then there are two compatible linear extensions of P:

The number of lattice points in the corresponding maximal cells of $\mathcal{O}_{P,A}(\lambda)$ are $\binom{\lambda(c)-\lambda(b)+2}{2}$ and $(\lambda(c)-\lambda(b)+1)(\lambda(b)-\lambda(a)+1)$ respectively. Taking into account overcounting we have to substract the number of order preserving extensions of λ to P for which p and b have the same value. These correspond to lattice points in the cell given by the chain of order ideals

$$\{a\} \subset \{a,b,p\} \subset \{a,b,p,q\} \subset \{a,b,p,q,c\}$$

and their number is $\lambda(c) - \lambda(b) + 1$. In total we have

$$\Omega_{P,A}(\lambda) = \binom{\lambda(c) - \lambda(b) + 2}{2} + (\lambda(c) - \lambda(b) + 1)(\lambda(b) - \lambda(a) + 1) - (\lambda(c) - \lambda(b) + 1).$$

If $\lambda(b) < \lambda(a) < \lambda(c)$ then the only compatible linear extension is

$$b \prec a \prec p \prec q \prec c$$

and

$$\Omega_{P,A}(\lambda) = \binom{\lambda(c) - \lambda(a) + 2}{2}.$$

2.4. Chains and Cayley cones. Let us consider the special case in which $A \subseteq P$ is a chain. It turns out that in this case the relation between $\mathcal{L}(P)$ and $\mathcal{L}(A)$ is very special. A pointed polyhedral cone $K \subset \mathbb{R}^n$ is called a **Cayley cone** over L if there is a linear projection $\pi : K \to L$ onto a pointed simplicial cone L such that every ray of K is injectively mapped to a ray of L. In case K is not pointed, then $K \cong K' \times U$ where K' is pointed and U is a linear space and we require $L \cong L' \times U$ and π is an isomorphism on U. Cayley cones are the "cone-analogs" of Cayley configurations/polytopes [5, Sect. 9.2] which are precisely the preimages under π of bounded hyperplane sections $L \cap H$.

Proposition 2.8. If $A \subseteq P$ is a chain and $\min(P) \cup \max(P) \subseteq A$, then $\mathcal{L}(P)$ is a Cayley cone over $\mathcal{L}(A)$.

Proof. The restriction map $\pi(\phi) = \phi|_A$ for $\phi \in \mathcal{L}(P)$ is a surjective linear projection. Since A is a chain and $\min(P) \cup \max(P) \subseteq A$, A and P are connected posets. The lineality spaces are spanned by 1_A and 1_P , respectively, and π is an isomorphism on lineality spaces. Moreover, $\mathcal{L}_0(A) = \mathcal{L}(A)/(\mathbb{R} \cdot 1_A)$ is linear isomorphic to the cone of order preserving maps $A \to \mathbb{R}_{\geq 0}$ which map $\min(A) = \{a_0\}$ to 0, which shows that $\mathcal{L}_0(A)$ is simplicial.

Thus, we only need to check that $\pi : \mathcal{L}_0(P) \to \mathcal{L}_0(A)$ maps rays to rays. It follows from the description of face partitions (Proposition 2.1) that the rays of $\mathcal{L}_0(P)$ are spanned by indicator functions of proper filters. Let ϕ be such an indicator function. Then also $\phi|_A : A \to \{0,1\}$ is an indicator function of a proper filter of A which proves the claim.

Here is the main property of Cayley cones that make them an indispensable tool in the study of mixed subdivisions and mixed volumes.

Proposition 2.9. Let K be a pointed Cayley cone over L. Let r_1, \ldots, r_k be linearly independent generators of L and let $K_i = \pi^{-1}(r_i)$ be the fiber over the generator r_i . Then for every point $p \in L$ we have

$$\tau^{-1}(p) = \mu_1 K_1 + \mu_2 K_2 + \cdots + \mu_r K_r$$

where $\mu_1, \mu_2, \ldots, \mu_r \geq 0$ are the unique coefficients such that $p = \sum_i \mu_i r_i$.

Proof. Let $\{s_{ij} \in K : 1 \leq i \leq k, 1 \leq j \leq m_i\}$ be a minimal generating set of K such that $\pi(s_{ij}) = r_i$. It follows that $K_i = \operatorname{conv}\{s_{ij} : 1 \leq j \leq m_i\}$. Thus, if $\mu_{ij} \geq 0$ are such that

$$\sum_{i,j} \mu_{ij} s_{ij} \in \pi^{-1}(p)$$

then, by the uniqueness of the μ_i , we have $\sum_j \mu_{ij} = \mu_i$ and $\sum_j \mu_{ij} s_{ij} \in \mu_i K_i$.

If $A = \{a_0 \prec_P a_1 \prec_P \cdots \prec_P a_k\}$ is a chain, recall that $\phi^0, \phi^1, \ldots, \phi^k : A \to \{0, 1\}$ with $\phi^i(a_j) = 1$ iff $j \ge i$ is a minimal generating set of $\mathcal{L}(A)$. If $\lambda : A \to \mathbb{R}$ is an order preserving map, then unique coordinates of $\lambda \in \mathcal{L}(A)$ with respect to $\{\phi^i\}$ are given by $\mu_0 = \lambda(a_0)$ and $\mu_i = \lambda(a_i) - \lambda(a_{i-1})$ for $1 \le i \le r$.

Corollary 2.10. Let P be a poset and $A \subseteq P$ a chain such that $\min(P) \cup \max(P) \subseteq A$. Let $\Phi_i = \mathcal{O}_{P,A}(\phi^i)$ for i = 1, 2, ..., k. Then for any order preserving map $\lambda : A \to \mathbb{R}$ we have

(5)
$$\mathcal{O}_{P,A}(\lambda) = \mu_0 1_P + \mu_1 \Phi_1 + \mu_2 \Phi_2 + \dots + \mu_k \Phi_k.$$

This was already observed by Stanley [14, Thm. 3.2] and used to show that the number of order preserving maps extending a given map on a chain $A \subset P$ satisfy certain log-concavity conditions. This is done by identifying the numbers as mixed volumes which are calculated from the Cayley polytope.

In particular, $\Omega_{P,A}(\lambda)$ counts the number of lattice points in the Minkowski sum (5). It follows from Theorem 2.6 and (4) that over a maximal cell $C \in \mathcal{T}_A$, the function $\Omega_{P,A}(\lambda)$ can be written as a polynomial $f(\mu)$ in the coordinates $\mu = (\mu_1, \ldots, \mu_k)$. The degree of $f(\mu)$ in every variable μ_i is given by

 $\deg_{\mu_i} f(\mu) \ = \ \dim \Phi^i \ = \ |P \setminus (P_{\preceq a_{i-1}} \cup P_{\succeq a_i})|$

The degree in λ_i is more difficult to determine.

Question 2. What is $\deg_{\lambda_i} \Omega_{P,A}(\lambda)$ in terms of the combinatorics of P?

If $A \subseteq P$ is a chain with minimum a_0 and maximum a_k , then the degree of λ_0 and λ_k agrees with μ_1 and μ_k . A related situation is implicitly treated in Fischer [6]: The number $\alpha(n; k_1, k_2, \ldots, k_n)$ of monotone triangles with bottom row $\mathbf{k} = (k_1 \leq k_2 \leq \cdots \leq k_n)$ is a polynomial in \mathbf{k} and is of degree n-1 in every variable k_i . In Section 3, it is shown that $\alpha(n; \mathbf{k})$ is essentially the number of integer-valued order preserving extensions from a particular poset with some extra conditions (i.e. certain faces of the marked order polytope are excluded). However, it appears that these extra condition do not influence the degree.

2.5. Combinatorial reciprocity. For a special choice of A, we recover the classical order polytope.

Example 2.11 (Order polytopes). Let P' be the result of adjoining a minimum $\hat{0}$ and maximum $\hat{1}$ to P. Let $A = \{\hat{0}, \hat{1}\}$ and for n > 0 let $\lambda_n : A \to \mathbb{Z}$ be the order preserving map with $\lambda_n(\hat{0}) = 1$ and $\lambda_n(\hat{1}) = n$. Then $\Omega_{P',A}(\lambda_n) = \Omega_P(n)$ is the order polynomial of P which counts the number of order preserving maps from P to [n]. Equivalently, $\Omega_{P',A}(\lambda_n)$ equals the Ehrhart polynomial

of the order polytope $\mathcal{L}(P) \cap [0,1]^P$ evaluated at n-1. Ehrhart-Macdonald Reciprocity (see for example [2, Thm. 4.1]) then yields that

$$(-1)^{|P|} \Omega_P(-n) = (-1)^{\dim \mathcal{O}_{P',A}(\lambda_n)} \Omega_{P',A}(\lambda_{-n})$$

equals the number of strictly order preserving maps into [n].

We wish to extend this combinatorial reciprocity to our more general setting. For that we say that an extension $\hat{\lambda} : P \to \mathbb{R}$ of λ is **strict** if $\hat{\lambda}(p) = \hat{\lambda}(q)$ and $p \prec q$ implies that $a \preceq p \prec q \preceq b$ for some $a, b \in A$ with $\lambda(a) = \lambda(b)$.

Theorem 2.12. Let $A \subset P$ be a pair of posets with $\min(P) \cup \max(P) \subseteq A$. If $\lambda : A \to \mathbb{Z}$ is an order preserving map, then

 $(-1)^{\dim \mathcal{O}_{P,A}(\lambda)} \Omega_{P,A}(-\lambda)$

equals the number of strict order preserving extensions of λ .

Note that if $F(I_{\bullet}^A)$ is the unique cell of the subdivision of $\mathcal{L}(A)$ that contains λ in the relative interior, then $\Omega_{P,A}(\lambda)$ is the evaluation of a polynomial and it is this polynomial that is evaluated at $-\lambda$ in the course of the theorem above. From the geometric point of view, $(-1)^{\dim \mathcal{O}_{P,A}(\lambda)} \Omega_{P,A}(-\lambda)$ counts the number of lattice points in the relative interior of $\mathcal{O}_{P,A}(\lambda)$. This is reminiscent of Ehrhart-Macdonald reciprocity and in fact follows from it.

Proof. For fixed λ , let I_{\bullet}^{A} such that $\lambda \in \operatorname{relint} F(I_{\bullet}^{A})$. Then $\Omega_{P,A}$ restricted to relint $F(I_{\bullet}^{A})$ is given by some polynomial $P(\mathbf{x}) \in \mathbb{R}[x_{a} : a \in A]$. For $n \in \mathbb{Z}_{>0}$, we have that $n\lambda \in \operatorname{relint} F(I_{\bullet}^{A})$ and thus $\Omega_{P,A}(n\lambda) = P(n\lambda)$. As $\Omega_{P,A}(n\lambda)$ equals the number of lattice points in $n \mathcal{O}_{P,A}(\lambda)$, it follows that $P(n\lambda)$ is the Ehrhart polynomial of $\mathcal{O}_{P,A}(\lambda)$. Now, Ehrhart-Macdonald reciprocity implies that the number of points in the relative interior of $\mathcal{O}_{P,A}(\lambda)$ equals

$$(-1)^{d} \operatorname{Ehr}(\mathcal{O}_{P,A}(\lambda), -1) = (-1)^{d} P(-\lambda) = (-1)^{d} \Omega_{P,A}(-\lambda).$$

m $\mathcal{O}_{P,A}(\lambda).$

where $d = \dim \mathcal{O}_{P,A}(\lambda)$

2.6. Marked chain polytopes. Let us close by transferring our results to the marked chain polytopes of Ardila, Bliem, and Salazar [1]. To that end we write $\phi(C) = \sum \{\phi(c) : c \in C\}$ for a subset $C \subseteq P$ and $\phi : P \to \mathbb{R}$. For a pair of posets $A \subset P$ and an order preserving map $\lambda : A \to \mathbb{R}$, the marked chain polytope is the convex polytope

$$\mathcal{C}_{P,A}(\lambda) = \left\{ \phi \in \mathbb{R}^P_{\geq 0} : \phi(C) \leq \lambda(b) - \lambda(a) \text{ for every chain } C \subseteq [a, b] \text{ and } a, b \in A \right\}$$

The unmarked version of the chain polytope was introduced in [15] to show that certain invariants of P (such as $\Omega_P(n)$) only depend on the comparability graph of P. The marked chain polytopes were introduced in [1] in connection with representation theory. Stanley defined a lattice preserving, piecewise linear map from the order polytope to the chain polytope and this transfer map was extended in [1] to relate the arithmetic of marked order polytope and marked chain polytopes. Thus, appealing to Theorem 3.4 of [1] proves

Corollary 2.13. For a pair of posets
$$A \subset P$$
, $\min(P) \cup \max(P) \subseteq A$, the function

$$\lambda \mapsto |\mathcal{C}_{P,A}(\lambda) \cap \mathbb{Z}^P|$$

is a piecewise polynomial over $\mathcal{L}(A) \cap \mathbb{Z}^A$ and evaluating at $-\lambda$ equals $(-1)^{\dim \mathcal{C}_{P,A}(\lambda)}$ times the number of lattice points in the relative interior of $\mathcal{C}_{P,A}(\lambda)$.

3. MONOTONE TRIANGLE RECIPROCITY

A monotone triangle (MT, for short) of order n, as exemplified in (1), is a triangular array of integers $a = (a_{i,j})_{1 \le j \le i \le n} \in \mathbb{Z}$ such that the entries

- (M1) weakly increase along the north-east direction: $a_{i,j} \leq a_{i-1,j}$ for all $1 \leq j < i \leq n$,
- (M2) weakly increase along the south-east direction: $a_{i,j} \leq a_{i+1,j+1}$ for all $1 \leq j \leq i < n$, and (M3) strictly increase in the rows: $a_{i,j} < a_{i,j+1}$ for all $1 \leq j < i < n$.

The number of monotone triangles with fixed bottom row $\mathbf{k} = (k_1 \leq k_2 \leq \cdots \leq k_n)$ is finite and denoted by $\alpha(n; k_1, k_2, \ldots, k_n)$. Monotone triangles originated in the study of alternating sign matrices [11] where it was shown that alternating sign matrices of order *n* exactly correspond to monotone triangles with bottom row $(1, 2, \ldots, n)$. The study of enumerative properties of monotone triangles with general bottom row was initiated in [6] where it was shown that $\alpha(n; k_1, k_2, \ldots, k_n)$ is a polynomial in the strictly increasing arguments. Note that our definition of a monotone triangle slightly differs from that of Fischer [6] inasmuch that we do not require that the bottom row is strictly increasing.

More precisely, there is a polynomial that agrees with $\alpha(n; \mathbf{k})$ for increasing $\mathbf{k} = (k_1 \leq k_2 \leq \cdots \leq k_n)$ and, by abuse of notation, we identify $\alpha(n; \mathbf{k})$ with this polynomial. As a polynomial, $\alpha(n; \mathbf{k})$ admits evaluations at arbitrary $\mathbf{k} \in \mathbb{Z}^n$ and it is natural to ask if there are domains for which the values $\alpha(n; \mathbf{k})$ have combinatorial significance. An interpretation for the values of α at weakly decreasing arguments was given by Fischer and Riegler [7] in terms of signed enumeration of so called decreasing monotone triangles. A **decreasing monotone triangle** (DMT) is again a triangular array $b = (b_{i,j})_{1 \leq j \leq i \leq n} \in \mathbb{Z}$ such that

(W1) the entries weakly decrease along the north-east direction: $b_{i,j} \ge b_{i-1,j}$ for $1 \le j < i \le n$,

(W2) the entries weakly decrease along the south-east direction: $b_{i,j} \ge b_{i+1,j+1}$ for $1 \le j \le i < n$, (W3) there are no three identical entries per row, and

(W4) two consecutive rows do not contain the same integer exactly once.

An example of a DMT is

(6)
$$\begin{array}{c} & & 3 \\ & 4 \\ & 4 \\ \hline & 3 \\ & 4 \\ \hline & 4 \\ \hline & 4 \\ \hline & 3 \\ & 3 \\ \end{array} \begin{array}{c} 3 \\ & 3 \\ & 3 \\ & 3 \\ & 1 \end{array}$$

The collection of decreasing monotone triangles with bottom row $\mathbf{k} = (k_1 \ge k_2 \ge \cdots \ge k_n) \in \mathbb{Z}^n$ is denoted by $\mathcal{W}_n(\mathbf{k})$. For a DMT *b*, two adjacent and identical elements in a row are called a **duplicate-descendant** if they are either in the last row or the row below contains exactly the same pair. In the example, the duplicate-descendants are underlined. The number of duplicate-descendants of *b* is denoted by dd(*b*). The precise reciprocity statement now is

Theorem 3.1 ([7, Thm. 1]). For weakly decreasing integers $\mathbf{k} = (k_1 \ge k_2 \ge \cdots \ge k_n)$ we have

$$\alpha(n; k_1, k_2, \dots, k_n) = (-1)^{\binom{n}{2}} \sum_{b \in \mathcal{W}_n(\mathbf{k})} (-1)^{\mathrm{dd}(b)}.$$

In this section we give a geometric proof for the result above by relating (decreasing) monotone triangles to special order preserving maps. A **Gelfand-Tsetlin poset** GT_n of order n is the poset on $\{(i, j) \in \mathbb{Z}^2 : 1 \le j \le i \le n\}$ with order relation

$$(i,j) \preceq_{\mathrm{GT}_n} (k,l) :\iff k-i \leq l-j \text{ and } j \leq l.$$

The Hasse diagram for GT_n is given in Figure 1. Throughout, we let $A = \{\kappa_1, \kappa_2, \ldots, \kappa_n\} \subset GT_n$ be the *n*-chain of elements $\kappa_j = (n, j)$ with $1 \leq j \leq n$, depicted by the circled elements in Figure 1. An increasing sequence $k = (k_1 \leq k_2 \leq \cdots \leq k_n)$ corresponds to a order preserving map $k : A \to \mathbb{Z}$ by setting $k(\kappa_i) = k_i$. We call an order preserving map $a : GT_n \to \mathbb{Z}$ a weak monotone triangle (WMT) (also known as **Gelfand-Tsetlin pattern**). Here is the main observation.

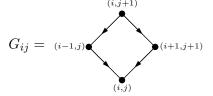
Observation 1. The collection of monotone triangles $a = (a_{ij})_{1 \le j \le i \le n} \in \mathbb{Z}$ for given bottom row $\mathbf{k} = (k_1 \le k_2 \le \cdots \le k_n) \in \mathbb{Z}^n$ bijectively correspond to integral-valued order preserving maps $a : \operatorname{GT}_n \to \mathbb{Z}$ extending $\mathbf{k} : A \to \mathbb{Z}$ and such that $a_{i,j} < a_{i,j+1}$ for all $1 \le j < i < n$.

To put this initial observation to good use, we pass to **real-valued** order preserving maps and we call an order preserving map $a : GT_n \to \mathbb{R}$ extending **k** a **monotone triangle** if it satisfies (M3). Hence, the monotone triangles with bottom row **k** form a special subset of the marked order polytope for GT_n

$$\mathcal{GT}_n(\mathbf{k}) := \mathcal{O}_{\mathrm{GT}_n,A}(\mathbf{k}).$$

Let us denote by $\mathcal{B}_n = \{(i,j) : 1 \le j < i < n\}$ and for $(i,j) \in \mathcal{B}_n$ define
 $Q_{ij} = \{a \in \mathcal{L}(\mathrm{GT}_n) : a_{i,j} = a_{i,j+1}\}$

as the set of real-valued weak monotone triangles which fail (M3) non-exclusively at position (i, j). The Hasse diagram of the face partition $G_{ij} = G(Q_{ij})$ of Q_{ij} is a **diamond** in GT_n :



It is easy to see that G_{ij} is a compatible face partition for strictly increasing bottom row **k** and together with a count of parameters yields the following geometric result.

Proposition 3.2. Let $\mathbf{k} = (k_1 < k_2 < \cdots < k_n)$. For $(i, j) \in \mathcal{B}_n$, the set $Q_{ij}(\mathbf{k}) \subseteq \mathcal{GT}_n(\mathbf{k})$ is a face of codimension 3.

This yields a geometric perspective on monotone triangles.

Corollary 3.3. For $\mathbf{k} = (k_1 \leq k_2 \leq \cdots \leq k_n)$, the set of monotone triangles with bottom row \mathbf{k} are precisely the lattice points in

(7)
$$\mathcal{GT}_n(\mathbf{k}) \setminus \bigcup_{(i,j)\in\mathcal{B}_n} Q_{ij}(\mathbf{k}).$$

Notice that if **k** contains three identical elements $k_j = k_{j+1} = k_{j+2}$, then $\mathcal{GT}_n(\mathbf{k}) \subseteq Q_{n-1,j}(\mathbf{k})$ and the above set is empty. Hence, the number of monotone triangles with bottom row **k** can only be non-zero if **k** contains at most pairs of identical elements.

Corollary 3.3 allows us to write $\alpha(n; \mathbf{k})$ as a polynomial by inclusion-exclusion on the set of faces $\{Q_{ij}(\mathbf{k}) : (i, j) \in \mathcal{B}_n\}$. More refined, we will consider the poset of non-empty intersections of faces of the form Q_{ij} and obtain $\alpha(n; \mathbf{k})$ as a polynomial by Möbius inversion on that poset. This will be relatively easy once we have a characterization of the face partitions of such finite

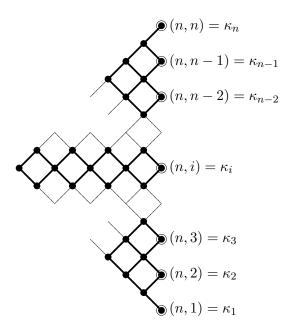
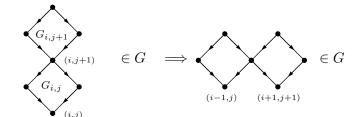


FIGURE 1. Hasse diagram for the Gelfand-Tsetlin poset of order n (in solid black).

intersections. For that we call an subposet $G \subseteq GT_n$ a **diamond poset** if the Hasse diagram of G is a union of graphs $G_{i,j}$. In addition, we call a diamond poset **closed** if $G_{i,j}, G_{i,j+1} \subset G$ implies $G_{i-1,j}, G_{i+1,j+1} \subset G$. That is,



Lemma 3.4. Let $F \subseteq \mathcal{L}(GT_n)$ be a non-empty face. Then

$$F = \bigcap_{(i,j)\in I} Q_{ij}$$

for some $I \subseteq \mathcal{B}_n$ if and only if G(F) is a closed diamond poset.

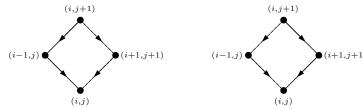
Proof. The face F is exactly the intersection of all facets for which the corresponding cover relation is in G(F). If G(F) is a closed diamond poset, then every cover relation is contained in at least one diamond and hence F is exactly the intersection of all Q_{ij} for which $G_{ij} \subseteq G(F)$.

For the converse, we can assume that G = G(F) is connected and we let $G' = \bigcup \{G_{ij} : F \subseteq Q_{ij}\}$ be the largest diamond poset contained in G. If $G \neq G'$, then by Proposition 2.1 there is a nontrivial directed path $P = p_0 p_1 \dots p_k$ that meets G' only in a connected component containing p_0 and p_k . In particular no edge of P is contained in a diamond of G and, furthermore, P cannot contain vertices (i, j) and (i, j + 1). Indeed, by Proposition 2.1, this would imply that $G_{ij} \subset G'$ which contradicts $P \cap G' = \{p_0, p_k\}$. It follows that $c = p_{i+1} - p_i \in \mathbb{Z}^2$ is a constant direction for all $i = 0, 1, \dots, k - 1$.

Let us assume that c = (+1, 0). Thus, every vertex p_h along P has constant second coordinate $\ell = (p_h)_2$. Let R be an undirected(!) path connecting p_0 and p_k in G' such that

$$\rho(R) = \sum_{r \in R} |r_2 - \ell|$$

is minimal. Such a path exists as p_0 and p_k are in the same connected component of the underlying undirected graph of G', and $\rho(R) > 0$. (Indeed, we have $\rho(R) = 0$ iff R = P after orienting edges). But then R contains a sequence of vertices (i, j), (i - 1, j), (i, j + 1) with j < l or (i, j), (i + 1, j + 1), (i, j + 1) with $j \ge l$ and the value of $\rho(R)$ can be reduced by rerouting along $G_{i,j}$.



Hence, by contradiction, R = P and G = G'.

Let us define Q as the set of all closed diamond subposets of GT_n ordered by reverse inclusion. In light of the above lemma, we have

$$\mathcal{Q} \cong \left\{ \bigcap_{(i,j)\in I} Q_{ij} : I \subseteq \mathcal{B}_n \right\}$$

is a meet-semilattice with greatest element $\hat{1} = \hat{1}_{\mathcal{Q}} := \emptyset$ corresponding to $\mathcal{L}(\mathrm{GT}_n)$. The Möbius function of \mathcal{Q} can now be dealt with in the language of diamond posets. Let us write

$$I(G) = \{(i,j) \in \mathcal{B}_n : G_{ij} \subseteq G\}$$

for $G \in \mathcal{Q}$.

Lemma 3.5. Let $G \in \mathcal{Q}$ and I = I(G). Then

$$\mu_{\mathcal{Q}}(G,\hat{1}) = \begin{cases} 0, & \text{if } (i,j), (i,j+1) \in I \\ (-1)^{|I|}, & \text{otherwise.} \end{cases}$$

Proof. Let \mathcal{A} be the collection of **atoms** of the interval $[G, \hat{1}]_{\mathcal{Q}}$, that is, the elements of \mathcal{Q} covering G. To prove the first claim, we will use the Crosscut Theorem [9, Sec. 3.1.9]

$$\mu_{\mathcal{Q}}(G,\hat{1}) = N_0 - N_1 + \cdots + (-1)^i N_i$$

where N_k is the number of k-element subsets $S \subseteq \mathcal{A}$ such that $\hat{1}$ is the smallest joint upper bound for the elements in S. Now if there is some $Q \prec \hat{1}_Q$ such that every $H \in \mathcal{A}$ is smaller than Q, then this implies $N_k = 0$ for all k and the claim follows.

To that end, let $(i_0, j_0) \in I(G)$ with $(i_0 + 1, j_0), (i_0 + 1, j_0 + 1) \in I(G)$ and i_0 minimal. We claim that $(i_0, j_0) \in I(H)$ for every $H \in \mathcal{A}$. Indeed, assume that $(i_0, j_0) \notin I(H)$. By Lemma 3.4, we have that $H \cup G_{i_0,j_0}$ is a diamond poset but not closed, as $H \in \mathcal{A}$ by assumption. This forces G_{i_0,j_0-1} or G_{i_0,j_0+1} to be in G, and establishing then the closedness condition has to introduce some $G_{i,j} \subseteq G$ with $i < i_0$. However, this contradicts the choice of i_0 and we can take $Q = Q_{i_0j_0}$.

For the other case, observe that the closedness condition for G is vacuous. This stays true for every diamond subposet which are in bijection to the subsets of I(G). Hence $[G, \hat{1}]_{\mathcal{Q}}$ is isomorphic to the boolean lattice on |I(G)| elements.

This yields a partial explanation of condition (W3): A weak monotone triangle $a : \operatorname{GT}_n \to \mathbb{R}$ with strictly increasing bottom row satisfies (W3) and (W4) if and only if $a \in \operatorname{relint} F$ for some face F with $G = G(F) \in \mathcal{Q}$ and $\mu_{\mathcal{Q}}(F, \hat{1}) \neq 0$. For that reason, let us define $\mathcal{Q}_{ess} \subseteq \mathcal{Q}$ as the **essential** subposet of \mathcal{Q} with

$$\mathcal{Q}_{\text{ess}} = \left\{ G \in \mathcal{Q} : \mu_{\mathcal{Q}}(G, \hat{1}) \neq 0 \right\}$$

Hence, we can identify \mathcal{Q}_{ess} with the collection of closed diamond posets G of GT_n such that $G_{i,j} \cup G_{i,j+1} \not\subseteq G$. In particular, $\hat{1} \in \mathcal{Q}_{ess}$ and from the definition of Möbius functions it follows that $\mu_{\mathcal{Q}_{ess}}(G, \hat{1}) = \mu_{\mathcal{Q}}(G, \hat{1})$ for all $G \in \mathcal{Q}_{ess}$.

With that knowledge, we can now write the number of lattice points in (7) as a polynomial in **k**. For the sake of clarity, let us emphasize that the combinatorics of $\mathcal{Q}_{\mathrm{GT}_n,A}(\mathbf{k})$ is independent of the actual choice of a strictly order preserving map $\mathbf{k} : A \to \mathbb{R}$. In this case, every $G \in \mathcal{Q}_{\mathrm{ess}}$ is a compatible face partition of a distinct face of $\mathcal{GT}_n(\mathbf{k})$ which we can identify with the marked order polytope $\mathcal{O}_{\mathrm{GT}_n/G,A/G}(\mathbf{k})$.

Theorem 3.6. For $\mathbf{k} = (k_1 \leq k_2 \leq \cdots \leq k_n)$, the number of monotone triangles with bottom row \mathbf{k} is given by

$$\alpha(n; \mathbf{k}) = \sum_{G \in \mathcal{Q}_{ess}} (-1)^{|I(G)|} \Omega_{\mathrm{GT}_n/G, A/G}(\mathbf{k}),$$

and thus is a polynomial. In particular, $\alpha(n; \mathbf{k}) = 0$ whenever $k_j = k_{j+1} = k_{j+2}$.

Proof. If **k** is strictly order preserving, then the above formula is exactly the Möbius inversion of the function $f_G(\mathbf{k}) = \Omega_{\mathrm{GT}_n/G, A/G}(\mathbf{k})$ for $G \in \mathcal{Q}_{\mathrm{ess}}$ by Corollary 3.3 and Lemmas 3.4 and 3.5.

If **k** has two, but no three identical entries, then $G \in \mathcal{Q}_{ess}$ is not compatible with **k** but can be completed to a compatible face partition \overline{G} . It is easy to see that \overline{G} arises from G by only adding the cover relations $(n, j) \prec_{\operatorname{GT}_n} (n - 1, j)$ and $(n - 1, j) \prec_{\operatorname{GT}_n} (n, j + 1)$ for every $1 \leq j < n$ with $k_j = k_{j+1}$. The map $G \mapsto \overline{G}$ is injective on \mathcal{Q}_{ess} and the image is a poset under reverse inclusion isomorphic to \mathcal{Q}_{ess} . Hence, the above formula counts the number of lattice points in (7).

If **k** has three identical entries, then (7) is the empty set and $\alpha(n; \mathbf{k}) = 0$. Consequently, we have to show that the right hand side is also identically zero for all such **k**. It suffices to assume that **k** has exactly three identical entries as every bottom row with more than three identical elements

belongs to the boundary of some cell for which the interior consists of bottom rows with exactly three identical elements. So, let us assume that $k_j = k_{j+1} = k_{j+2}$ are the only equalities for **k**. Let $G \in \mathcal{Q}_{ess}$ and \overline{G} its completion to a face partition compatible with **k**. Then $\Omega_{\mathrm{GT}_n/\overline{G},A/\overline{G}}(\mathbf{k})$ appears in the sum on the right hand side with multiplicity

$$\sum \left\{ (-1)^{|I(H)|} : H \in \mathcal{Q}_{\text{ess}}, \bar{H} = \bar{G} \right\}.$$

For any such H, let $(i, j) \in \mathcal{B}_n$ be the lexicographic smallest such that $G_{i+1,j} \cup G_{i+1,j+1} \subseteq \overline{H} = \overline{G}$ (existence follows from $k_j = k_{j+1} = k_{j+2}$). Hence $G_{i,j} \subseteq \overline{H}$ by closedness. We distinguish two cases:

- 1. Assume that $G_{ij} \subseteq H$, then the largest diamond subposet $H' \subset H$ not containing G_{ij} is closed as $H \in \mathcal{Q}_{ess}$, and $\bar{H}' = \bar{G}$ as $G_{i+1,j} \cup G_{i+1,j+1} \subseteq \bar{H}$.
- 2. If $G_{ij} \not\subseteq H$, then set $H' = H \cup G_{ij}$. By the minimality of (i, j) we have that H' is closed diamond and $\overline{H'} = \overline{G}$.

This defines a perfect matching on $\{H \in \mathcal{Q}_{ess} : \bar{H} = \bar{G}\}$ and $|I(H)| = |I(H')| \pm 1$ shows that the multiplicity of $\Omega_{\operatorname{GT}_n/\bar{G}, A/\bar{G}}(\mathbf{k})$ is zero.

Coming back to the reciprocity statement for monotone triangles, we note that $b = (b_{ij})_{1 \le j \le i \le n}$ is a decreasing monotone triangle if and only if $-b : \operatorname{GT}_n \to \mathbb{R}$ is a weak monotone triangle satisfying (W3) and (W4).

Proposition 3.7. Let $a = (a_{ij})_{1 \leq j \leq i \leq n} \in \mathbb{Z}$ be a weak monotone triangle with bottom row $\mathbf{k} = (k_1 \leq k_2 \leq \cdots \leq k_n)$ with no three identical elements. Then -a is a DMT with bottom row $-\mathbf{k}$ if and only if there is a unique $G \in \mathcal{Q}_{ess}$ with corresponding face $F \subseteq \mathcal{GT}_n(\mathbf{k})$ such that $a \in \operatorname{relint} F$.

Proof. Let F be the face of $\mathcal{GT}_n(\mathbf{k})$ that has a in the relative interior and let G' = G(F) be its compatible face partition. If \mathbf{k} is not strictly increasing, then G' contains cover relations that reach into A. Let $G \subseteq G'$ be the subposet which arises by deleting those which are not contained in a diamond. Then G is a face partition and $\operatorname{Ext}_{\operatorname{GT}_n,A}(\lambda) \cap F_{\operatorname{GT}_n}(G) = F$.

Now (W4) is equivalent to the condition that every cover relation in G is contained in a diamond. Otherwise there are indices $(i, j), (i + 1, k) \in \mathcal{B}_n$ with $k \in \{j, j + 1\}$ such that $b_{i,j} = b_{i+1,k}$ and $b_{i,j-1} < b_{i,j} < b_{i,j+1}$ and $b_{i,k-1} < b_{i,k} < b_{i,k+1}$ which contradicts (W4). Since **k** does not contain three identical elements, G is the unique diamond poset that gives rise to F. Moreover, $G \in \mathcal{Q}_{ess}$ if and only if every point in the relative interior of F satisfies (W3).

Let us extend the notion of duplicate-descendants to real-valued weak monotone triangles satisfying (W3) and define dd(F) for a non-empty face $F \subseteq \mathcal{GT}_n(\mathbf{k})$ as the number of duplicatedescendants for an arbitrary $a \in \operatorname{relint} F$.

Lemma 3.8. Let $\mathbf{k} = (k_1 \leq k_2 \leq \cdots \leq k_n)$ with no three identical elements and let m be the number of pairs of identical elements. Let $G \in \mathcal{Q}_{ess}$ with corresponding face $F \subseteq \mathcal{GT}_n(\mathbf{k})$. Then

$$|I(G)| + \operatorname{codim} F + m \equiv \operatorname{dd}(F) \mod 2$$

Proof. We induct on l = |I(G)|. For l = 0, we have $F = \mathcal{GT}_n(\mathbf{k})$ which is of codimension 0 and dd(F) = m by definition.

For l > 0 there is a diamond $G_{ij} \subseteq G$ which shares at most one edge with another diamond or a "half-diamond" coming from a pair of equal numbers at the bottom row. Let $G' \subset G$ be the largest diamond poset not containing G_{ij} and let F' be the corresponding face. By induction, the claim holds for G' and |I(G)| = |I(G')| + 1.

If $G_{ij} \cap G(F')$ does not contain an edge, then dd(F) = dd(F') and codim F = codim F' + 3. In the remaining case, G_{ij} shares exactly one edge with G(F') and thus dd(F) = dd(F') + 1. On the other hand, adding G_{ij} to G(F') binds two degrees of freedom and codim F = codim F' + 2. \Box

Proof of Theorem 3.1. By Theorem 3.6, $\alpha \equiv 0$ restricted to the set of order preserving maps $-\mathbf{k} : A \to \mathbb{Z}$ with three identical entries. As α is a polynomial, it follows that this extends to $\alpha(n; \mathbf{k})$. This proves the claim in this case as $\mathcal{W}_n(\mathbf{k}) = \emptyset$.

Let us assume that \mathbf{k} has m pairs of identical elements. Then $\dim \mathcal{GT}_n(-\mathbf{k}) = \binom{n}{2} - m$. For $G \in \mathcal{Q}_{\text{ess}}$ let us denote by $F_G(-\mathbf{k})$ the corresponding non-empty face of $\mathcal{GT}_n(-\mathbf{k})$. By Theorem 3.6 and Theorem 2.12, we have

$$\alpha(n;\mathbf{k}) = (-1)^{\binom{n}{2}} \sum_{G \in \mathcal{Q}_{ess}} (-1)^{|I(G)| + m + \operatorname{codim} F_G(-\mathbf{k})|} \operatorname{relint} F_G(-\mathbf{k}) \cap \mathbb{Z}^{\mathrm{GT}_n}|$$

where we use codim $F_G(-\mathbf{k}) = \binom{n}{2} - m - \dim F_G(-\mathbf{k})$. The claim now follows from Proposition 3.7 and Lemma 3.8.

4. EXTENDING PARTIAL GRAPH COLORINGS

Let G = (V, E) be a graph and k a positive integer. A k-coloring of G is simply a map $c: V \to [k]$. The coloring is called **proper** if $c(u) \neq c(v)$ for every edge $uv \in E$. It is wellknown that the number of proper k-colorings of G is given by a polynomial in k, the **chromatic polynomial** $\chi_G(k)$. Generalizing these notions, Murty and Herzberg [8] considered the problem of counting extensions of partial colorings of G. For a given subset $A \subseteq V$ and a partial coloring $c: A \to [k]$ an extension of c of size m is an m-coloring $\hat{c}: V \to [m]$ such that $\hat{c}(a) = c(a)$ for all $a \in A$. If \hat{c} is moreover a proper coloring, then \hat{c} is called a proper extension. Such extensions only exist for $m \geq k$.

Theorem 4.1 ([8, Thm. 1]). Let G = (V, E) be a graph and $c : A \to [k]$ a partial coloring for $A \subseteq V$. Then either there are no proper extensions or there is a polynomial $\chi_{G,c}(m)$ of degree |V| - |A| such that

 $\chi_{G,c}(m) = \# \{ \widehat{c} : V \to [m] : \widehat{c} \text{ is a proper coloring with } \widehat{c}(a) = c(a) \text{ for all } a \in A \}$

for all $m \geq k$.

We give an alternative proof of their result and a combinatorial interpretation for $\chi_{G,c}(-m)$ extending the combinatorial reciprocity of Stanley [13] for the ordinary chromatic polynomial. Recall that an **orientation** σ of G assigns every edge e a head and a tail. An orientation is **acyclic** if there are no directed cycles. An orientation σ is weakly compatible with a given coloring $c: V \to [m]$ if σ orients an edge e = uv along its color gradient, that is, from u to v whenever c(u) < c(v).

Theorem 4.2. Let G = (V, E) be a graph and let $c : A \to [k]$ be a partial coloring for $A \subseteq V$. Let A_1, A_2, \ldots, A_k be the partition of A into color classes induced by c. For $m \ge k$ we have that $(-1)^{|V\setminus A|} \chi_{G,c}(-m)$ is the number of pairs (\widehat{c}, σ) where $\widehat{c} : V \to [m]$ is a coloring extending c and σ is a weakly compatible acyclic orientation such that there is no directed path with both endpoints in A_i for some $i = 1, 2, \ldots, k$.

In the case that no two vertices of A get the same color, the result simplifies.

Corollary 4.3. Let G = (V, E) be a graph and $A \subseteq V$. If $c : A \to [k]$ is injective and $m \ge k$, then $|\chi_{G,c}(-m)|$ equals the number of pairs (\hat{c}, σ) where \hat{c} is an m-coloring extending c and σ is an acyclic orientation weakly compatible with \hat{c} .

It is also possible to give an interpretation for the evaluations at -m for m < k. Here, we constrain ourselves to one particularly interesting evaluation.

Corollary 4.4. Let G = (V, E) be a graph and $c : A \to [k]$ a partial coloring for $A \subseteq V$. Then $|\chi_{G,c}(-1)|$ equals the number of acyclic orientations of G for which there is no directed path from a to b whenever $a, b \in A$ with $c(a) \ge c(b)$.

Furthermore, choosing $A = \emptyset$, we see that $\chi_{G,c} = \chi_G$ and the above theorem specializes to the classical reciprocity for chromatic polynomials.

Corollary 4.5 ([13, Thm. 1.2]). For a graph G, $|\chi_G(-m)|$ equals the number of pairs (c, σ) for which c is an m-coloring and σ is a weakly compatible acyclic orientation. In particular, $|\chi_G(-1)|$ is the number of acyclic orientations of G.

Proofs. First observe that we may assume that no two vertices of A are assigned the same color by c. Indeed, assume that c(a) = c(b) for some $a, b \in A$. If ab is an edge of G, then no proper coloring can extend c and $\chi_{G,c} \equiv 0$. Moreover, in any orientation of G there is a directed path between a and b. If $ab \notin E$, let G_{ab} be obtained from G by identifying a and b. Then c descends to a partial coloring c_{ab} on G_{ab} and it is easy to see that there is a bijective correspondence between extensions of size m of c and c_{ab} . As for acyclic orientations, note that an acyclic orientation of G yields an acyclic orientation of G_{ab} if and only if there is no directed path between a and b. So, henceforth we assume that $c: A \to [k]$ is injective.

Let G' be the **suspension** of G, that is, the graph G with two additional vertices $\hat{0}, \hat{1}$ that are connected to all vertices of G. For $m \geq k$, let us consider all extensions of c to proper colorings $\hat{c}: V' \to \{0, 1, \ldots, m+1\}$ such that $\hat{c}(\hat{0}) = 0$ and $\hat{c}(\hat{1}) = m+1$. Every such coloring \hat{c} gives rise to a unique compatible acyclic orientation σ by directing every edge along its color gradient. By definition, $\hat{0}$ is a source and $\hat{1}$ is a sink. The acyclicity of σ implies that we can define a partially ordered set P_{σ} on V' by setting $u \preceq_{P_{\sigma}} v$ if there is directed path from u to v. Extending A to $A' = A \cup \{\hat{0}, \hat{1}\}$ and c to c'_m by

$$c'_{m}(a) = \begin{cases} 0, & \text{if } a = \hat{0}, \\ m+1, & \text{if } a = \hat{1}, \text{ and} \\ c(a), & \text{otherwise,} \end{cases}$$

it follows that every proper coloring \hat{c} of G' that extends c'_m and induces σ is a strict order preserving map $\hat{c}: P_{\sigma} \to \{0, 1, \dots, m+1\}$ extending c'_m and vice versa. By Theorem 2.12

(8)
$$\chi_{G,c}(m) = \sum_{\sigma} (-1)^{|V \setminus A|} \Omega_{P_{\sigma},A'}(-c'_m)$$

where the sum is over all acyclic orientations of G' such that for every $a, b \in A'$ there is no directed path from a to b whenever c(a) > c(b). This shows that $\chi_{G,c}(m)$ is a sum of polynomials in mwith positive leading coefficients. For m sufficiently large, there is an extension of c such that every vertex $V \setminus A$ gets a color > k. For the corresponding poset P_{σ} the summand $\Omega_{P_{\sigma},A'}(-c'_m)$ is of degree |V| - |A| in m which completes the proof of Theorem 4.1.

Let $A' = \{\hat{0} = a_0, a_1, \dots, a_{r-1}, a_r = \hat{1}\}$ so that i < j implies $c'_m(a_i) < c'_m(a_j)$. That is, c'_m is a strictly order preserving map for the chain A' with $c'_m(\hat{0}) = 0$ and $c'_m(\hat{1}) = m + 1$. Hence, we can consider the right hand side of (8) as a polynomial in $(0 = c_0 < c_1 < c_2 < \dots < c_r = m)$. However, the number of proper extensions of c is independent of the actual values of $c : A \to [k]$. Indeed, if $d : A \to [k]$ is a different injective partial coloring, then the permutation $\pi : [k] \to [k]$ that takes c to d extends to a relabeling on every extension of c to d. It follows that the right hand side of (8) is a polynomial independent of c_1, \dots, c_{r-1} and

$$(-1)^{|V\setminus A|}\chi_{G,c}(-m) = \sum_{\sigma} \Omega_{P_{\sigma},A'}(-c'_{-m}) = \sum_{\sigma} \Omega_{P_{\sigma},A'}(c'_{m-2}-\chi_A)$$

where $\chi_A : A \to \{0, 1\}$ is the characteristic function on A. Every summand is the number of order preserving maps $P_{\sigma} \to \{0, 1, \ldots, m-1\}$ extending $c'_{m-2} - \chi_A$. Translating back, this is exactly the number of pairs of (not necessarily proper) extensions \hat{c} of c'_m and a weakly compatible acyclic orientation σ which yields Theorem 4.2. As the right hand side of (8) is independent of c_1, \ldots, c_{r-1} we get that

$$(-1)^{|V\setminus A|}\chi_{G,c}(-1) = \sum_{\sigma} \Omega_{P_{\sigma},A'}(-c'_{-1}) = \sum_{\sigma} \Omega_{P_{\sigma},A'}(\mathbf{0})$$

Here every summand is one, so the right hand side counts the number of acyclic orientations such that for every $a, b \in A$ there is no directed path from a to b whenever c(a) > c(b) which proves Corollary 4.4.

Example 4.6. Consider the following graph G with $A = \{a, b\}$:



Let $c: A \to [k]$ be a coloring. If c(a) = c(b) then for all $m \ge k$ the number of extension of c to a proper *m*-coloring of *G* is

$$\chi_{G,c}(m) = (m-1)(m-2)$$

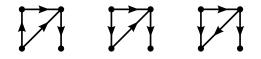
and $(-1)^2 \chi_{G,c}(-1) = 6$ is the number of acyclic orientations of G where there is no directed path between a and b:



If c(a) > c(b) then

 $\chi_{G,c}(m) = (m-2)(m-3) + (m-2) = (m-2)^2$

and $(-1)^2 \chi_{G,c}(-1) = 9$ counts the number of acyclic orientations where there is no directed path from *a* to *b*, i.e. there are three additional acyclic orientations:



The case c(a) < c(b) is clearly analogous.

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