

FINITE ELEMENT APPROXIMATION OF THE  $p(\cdot)$ -LAPLACIAN

DOMINIC BREIT, LARS DIENING AND SEBASTIAN SCHWARZACHER

ABSTRACT. We study a priori estimates for the  $p(\cdot)$ -Laplace Dirichlet problem,  $-\operatorname{div}(|\nabla \mathbf{v}|^{p(\cdot)-2} \nabla \mathbf{v}) = \mathbf{f}$ . We show that the gradients of the finite element approximation with zero boundary data converges with rate  $O(h^\alpha)$  if the exponent  $p$  is  $\alpha$ -Hölder continuous. The error of the gradients is measured in the so-called quasi-norm, i.e. we measure the  $L^2$ -error of  $|\nabla \mathbf{v}|^{\frac{p(\cdot)-2}{2}} \nabla \mathbf{v}$ .

variable exponents, convergence analysis, a priori estimates, finite element method, generalized Lebesgue and Sobolev spaces

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## 1. INTRODUCTION

In recent years there has been an extensive interest in the field of variable exponent spaces  $L^{p(\cdot)}$ . Different from the classical Lebesgue spaces  $L^p$ , the exponent is not a constant but a function  $p = p(x)$ . We refer to the recent books [CUF13, DHHR11] for a detailed study of the variable exponent spaces, although the definition of the spaces goes already back to Orlicz [Orl31].

The increasing interest was motivated by the model for electrorheological fluids [RR96, Růž00]. Those are smart materials whose viscosity depends on the applied electric field. Nowadays it is possible to change the viscosity locally by a factor of 1000 in 1ms. This is modeled via a dependence of the viscosity on a variable exponent. Electrorheological fluids can for example be used in the construction of clutches and shock absorbers.

Further applications of the variable exponent spaces can be found in the area of image reconstruction. Here, the change of the exponent is used to model different smoothing properties according the edge detector. This can be seen as a hybrid model of standard diffusion and the TV-model introduced by [CLR06], see also [BCE<sup>+</sup>09] and [HHLN10] for an overview over the related topics.

A model problem for image reconstruction as well as a starting point for the study of electro-rheological fluids is the  $p(\cdot)$ -Laplacian system

$$(1.1) \quad -\operatorname{div}((\kappa + |\nabla \mathbf{v}|)^{p(\cdot)-2} \nabla \mathbf{v}) = \mathbf{f} \quad \text{in } \Omega$$

on a bounded domain  $\Omega$ , where  $\kappa \geq 0$ . Here  $\mathbf{f} : \Omega \rightarrow \mathbb{R}^N$  is a given function and  $\mathbf{v} : \Omega \rightarrow \mathbb{R}^N$  is the vector field we are seeking for. The natural function space for solutions is the generalized Sobolev space  $(W^{1,p(\cdot)}(\Omega))^N$  which is the set of all functions  $\mathbf{u} \in (L^{p(\cdot)}(\Omega))^N$  with distributional gradient  $\nabla \mathbf{u} \in (L^{p(\cdot)}(\Omega))^{N \times n}$ . Here we have

$$(1.2) \quad L^{p(\cdot)}(\Omega) = \left\{ f \in L^1(\Omega) : \int_{\Omega} |f|^{p(\cdot)} dx < \infty \right\}.$$

A major breakthrough in the theory of variable exponent spaces was the fact that the right condition on the exponent was found: the *log-Hölder continuity* (see Section 2). In fact for such exponents  $p(\cdot)$  bounded away from 1 the Hardy-Littlewood maximal operator is bounded in  $L^{p(\cdot)}(\Omega)$ , see [CUFN03, Die04] and [DHHR11, Theorem 4.3.8]. This is a consequence of the so called *key estimate for variable*

*exponent spaces*, which roughly reads

$$(1.3) \quad \left( \int_Q |f| dx \right)^{p(y)} \leq c \int_Q |f|^{p(x)} dx + \text{error}$$

where  $Q$  is a ball or cube,  $y \in Q$  and the “error” denotes an appropriate error term, which is essentially independent of  $f$ ; see Theorem 2.2 for the precise statement. Once the maximal operator is bounded many techniques known for Lebesgue spaces can be extended to the setting of variable exponents.

Under the assumptions on  $p(\cdot)$  mentioned above the existence of a weak solution to (1.1) is quite standard provided  $\mathbf{f} \in (L^{p(\cdot)'(\Omega)})^N$ , i.e. there is a unique function  $\mathbf{v} \in (W_0^{1,p(\cdot)}(\Omega))^N$  such that

$$(1.4) \quad \int_{\Omega} \mathbf{A}(\cdot, \nabla \mathbf{v}) \cdot \nabla \psi dx = \int_{\Omega} \mathbf{f} \cdot \psi dx,$$

for all  $\psi \in (W_0^{1,p(\cdot)}(\Omega))^N$  holds. There is a huge bulk of literature regarding properties of solutions, see for instance [AM01, AM05, Die02] and it seems that the problem is well understood from the analytical point of view.

However, there is only few literature about numerical algorithms for discrete solutions of (1.4). In [LMDPM12] the convergence of discontinuous Galerkin FEM approximations is studied. They prove strong convergence of the gradients but without any explicit rate. The paper [CHP10] is concerned with the corresponding generalized Navier-Stokes equations which are certainly much more sophisticated than (1.1). The convergence of the finite element approximation is shown without convergence rate. To our knowledge no quantitative estimate on the convergence rate for problems with variable exponents is known. To derive such an estimate is the main purpose of the present paper.

Let us give a brief overview about conformal finite element approximation for more classical equations. Let  $\Omega$  be a polyhedron decomposed in simplices with lengths bounded by  $h$ . In case of the Laplace equation  $-\Delta \mathbf{v} = \mathbf{f}$  one can easily show that

$$\|\nabla \mathbf{v} - \nabla \mathbf{v}_h\|_2 \leq ch \|\nabla^2 \mathbf{v}\|_2,$$

where  $\mathbf{v}_h$  is the finite element solution. This is based on best approximation and interpolation results and rather classical.

Let us turn to the  $p$ -Laplacian, i.e. (1.1) with constant  $p$ . As was firstly observed in [BL94] the convergence of nonlinear problems should be measured with the so-called quasi-norm. Indeed, in [EL05, DR07] it was shown that

$$(1.5) \quad \|\mathbf{F}(\nabla \mathbf{v}) - \mathbf{F}(\nabla \mathbf{v}_h)\|_2 \leq ch \|\nabla \mathbf{F}(\nabla \mathbf{v})\|_2,$$

where  $\mathbf{F}(\boldsymbol{\xi}) = (\kappa + |\boldsymbol{\xi}|)^{p-2} \boldsymbol{\xi}$ . The result is based on best approximation theorems and interpolations results in Orlicz spaces. It holds also for more general nonlinear equations [DR07]. Please remark, that  $\mathbf{F} \in W^{1,2}$  is classical see [Uhl77, LB93] for smooth domains. In the case of polyhedral domains with re-entrant corners certainly less regularity is expected (as in the linear case), see [Ebm01, EF01, ELS05]. Let us mention, that for the  $p$ -Laplacian results about rate optimality of the adaptive finite element method has been shown in [DK08, BDK12] followed.

In this work finite element solutions to the  $p(\cdot)$ -Laplacian under the assumption that  $p \in C^{0,\alpha}(\overline{\Omega})$  are studied, see (4.2). To be precise we show in Theorem 4.4 that

$$(1.6) \quad \|\mathbf{F}(\cdot, \nabla \mathbf{v}) - \mathbf{F}(\cdot, \nabla \mathbf{v}_h)\|_2 \leq h^\alpha c (\|\mathbf{F}(\cdot, \nabla \mathbf{v})\|_{1,2}),$$

where  $\mathbf{F}(\cdot, \boldsymbol{\xi}) = (\kappa + |\boldsymbol{\xi}|)^{p(\cdot)-2} \boldsymbol{\xi}$ . If  $p$  is Lipschitz continuous we can recover the result for the constant  $p$ -Laplacian (1.5). As usual the proof of (1.6) is mainly based on interpolation operators and a lemma of Céa type. Therefore we study in Section 3 interpolation operators on Lebesgue spaces of variable exponents. In order to do so we need an extension of the key estimate (1.3) which can be found in

Theorem 2.4. These purely analytic results are improved tools in the variable Lebesgue and Sobolev theory and are therefore of independent interest.

Finally we introduce a numerical scheme, where the exponent  $p(\cdot)$  is locally approximated by constant functions, see (4.10). This seems more appropriate for practical use. Naturally another error term due to the approximation of the exponent occurs. However, we show, that this error does not harm the convergence rate and (1.6) still holds, see Theorem 4.8.

## 2. VARIABLE EXPONENT SPACES

For a measurable set  $E \subset \mathbb{R}^n$  let  $|E|$  be the Lebesgue measure of  $E$  and  $\chi_E$  its characteristic function. For  $0 < |E| < \infty$  and  $f \in L^1(E)$  we define the mean value of  $f$  over  $E$  by

$$\langle f \rangle_E := \int_E f \, dx := \frac{1}{|E|} \int_E f \, dx.$$

For an open set  $\Omega \subset \mathbb{R}^n$  let  $L^0(\Omega)$  denote the set of measurable functions.

Let us introduce the spaces of variable exponents  $L^{p(\cdot)}$ . We use the notation of the recent book [DHHR11]. We define  $\mathcal{P}$  to consist of all  $p \in L^0(\mathbb{R}^n)$  with  $p : \mathbb{R}^n \rightarrow [1, \infty]$  (called variable exponents). For  $p \in \mathcal{P}$  we define  $p_\Omega^- := \text{ess inf}_\Omega p$  and  $p_\Omega^+ := \text{ess sup}_\Omega p$ . Moreover, let  $p^+ := p_{\mathbb{R}^n}^+$  and  $p^- := p_{\mathbb{R}^n}^-$ .

For  $p \in \mathcal{P}$  the generalized Lebesgue space  $L^{p(\cdot)}(\Omega)$  is defined as

$$L^{p(\cdot)}(\Omega) := \{f \in L^0(\Omega) : \|f\|_{L^{p(\cdot)}(\Omega)} < \infty\},$$

where

$$\|f\|_{p(\cdot)} := \|f\|_{L^{p(\cdot)}(\Omega)} := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

We say that a function  $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$  is *log-Hölder continuous* on  $\Omega$  if there exists a constant  $c \geq 0$  and  $\alpha_\infty \in \mathbb{R}$  such that

$$|\alpha(x) - \alpha(y)| \leq \frac{c}{\log(e + 1/|x - y|)} \quad \text{and} \quad |\alpha(x) - \alpha_\infty| \leq \frac{c}{\log(e + |x|)}$$

for all  $x, y \in \mathbb{R}^n$ . The first condition describes the so called local log-Hölder continuity and the second the decay condition. The smallest such constant  $c$  is the log-Hölder constant of  $\alpha$ . We define  $\mathcal{P}^{\log}$  to consist of those exponents  $p \in \mathcal{P}$  for which  $\frac{1}{p} : \mathbb{R}^n \rightarrow [0, 1]$  is log-Hölder continuous. By  $p_\infty$  we denote the limit of  $p$  at infinity, which exists for  $p \in \mathcal{P}^{\log}$ . If  $p \in \mathcal{P}$  is bounded, then  $p \in \mathcal{P}^{\log}$  is equivalent to the log-Hölder continuity of  $p$ . However, working with  $\frac{1}{p}$  gives better control of the constants especially in the context of averages and maximal functions. Therefore, we define  $c_{\log}(p)$  as the log-Hölder constant of  $1/p$ . Expressed in  $p$  we have for all  $x, y \in \mathbb{R}^n$

$$|p(x) - p(y)| \leq \frac{(p^+)^2 c_{\log}(p)}{\log(e + 1/|x - y|)} \quad \text{and} \quad |p(x) - p_\infty| \leq \frac{(p^+)^2 c_{\log}(p)}{\log(e + |x|)}.$$

**Lemma 2.1.** *Let  $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$  with  $p^+ < \infty$  and  $m > 0$ . Then for every  $Q \subset \mathbb{R}^n$  with  $\ell(Q) \leq 1$ ,  $\kappa \in [0, 1]$  and  $t \geq 0$ , that holds  $|Q|^m \leq t \leq |Q|^{-m}$*

$$(\kappa + t)^{p(x) - p(y)} \leq c$$

for all  $x, y \in Q$ . The constants depend on  $c_{\log}(p)$ ,  $m$  and  $p^+$ .

*Proof.* Recall that  $\ell(Q) \leq 1$  and  $|Q| \leq 1$ . Since  $\kappa \leq |Q|^{-m}$  for  $\kappa \in [0, 1]$  we have  $|Q|^m \leq \kappa + t \leq 2|Q|^{-m}$ . Thus the local log-Hölder continuity of  $p$  implies

$$(\kappa + t)^{p(x) - p(y)} \leq 2^{p^+} (|Q|^{-|p(x) - p(y)|})^m \leq c,$$

where the constant depends on  $c_{\log}(p)$ ,  $p^+$  and  $m$ .  $\square$

**2.1. Key estimate of variable exponents.** For every convex function  $\psi$  and every cube  $Q$  we have by Jensen's inequality

$$(2.1) \quad \psi\left(\oint_Q |f(y)| dy\right) \leq \oint_Q \psi(|f(y)|) dy.$$

This simple but crucial estimate allows for example to transfer the  $L^1$ - $L^\infty$  estimates for the interpolation operators to the setting of Orlicz spaces, see [DR07]. Therefore, it is necessary for us to find a suitable substitute for Jensen's inequality in the context of variable exponents.

Our goal is to control  $(\oint_Q |f(y)| dy)^{p(x)}$  in terms of  $\int_Q |f(x)|^{p(x)} dx$ . If  $p$  is constant, this is exactly Jensen's inequality and it holds for all  $f \in L^p(Q)$ . However, for  $p$  variable it is impossible to derive such estimates for all  $f$  and we have to restrict ourselves to a certain set of admissible functions  $f$ . Moreover, an additional error term appears. The following statement is a special case of [DS13, Corollary 1], which is an improvement of [DHR11, Corollary 4.2.5] and related estimates from [DHH<sup>+</sup>09, Sch10].

**Theorem 2.2** (key estimate). *Let  $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$  with  $p^+ < \infty$ . Then for every  $m > 0$  there exists  $c_1 > 0$  only depending on  $m$  and  $c_{\log}(p)$  and  $p^+$  such that*

$$(2.2) \quad \left(\oint_Q |f(y)| dy\right)^{p(x)} \leq c_1 \oint_Q |f(y)|^{p(y)} dy + c_1 |Q|^m.$$

for every cube (or ball)  $Q \subset \mathbb{R}^n$  with  $\ell(Q) \leq 1$ , all  $x \in Q$  and all  $f \in L^1(Q)$  with

$$\oint_Q |f| dy \leq \max\{1, |Q|^{-m}\}.$$

Let us introduce the notation

$$\begin{aligned} \varphi(x, t) &:= t^{p(x)}, \\ (M_Q \varphi)(t) &:= \oint_Q \varphi(x, t) dx, \\ M_Q f &:= \oint_Q |f(x)| dx, \\ \varphi(f) &:= \varphi(\cdot, |f(\cdot)|). \end{aligned}$$

Then we can rewrite (2.2) as

$$(2.3) \quad \varphi(M_Q f) \leq c M_Q(\varphi(f)) + c |Q|^m$$

for all  $f$  with  $M_Q f \leq \max\{1, |Q|^{-m}\}$ .

For our finite element analysis we need this estimate extended to the case of shifted Orlicz functions. For constant  $p$  this has been done in [DR07]. We define the shifted functions  $\varphi_a$  for  $a \geq 0$  by

$$\varphi_a(x, t) := \int_0^t \frac{\varphi'(x, a + \tau)}{a + \tau} \tau d\tau.$$

Then  $\varphi_a(x, \cdot)$  is the shifted N-function of  $t \mapsto t^{p(x)}$ , see (A.2). Note that characteristics and  $\Delta_2$ -constants of  $\varphi_a(x, \cdot)$  are uniformly bounded with respect to  $a \geq 0$ , see Section A.

**Remark 2.3.** *It is easy to see that for all  $a, t \geq 0$  we have*

$$(M_Q \varphi_a)(t) = (M_Q \varphi)_a(t).$$

*If  $p^+ < \infty$ , then  $\Delta_2(\{M_Q \varphi_a\}_{a \geq 0}) < \infty$ .*

In the rest of this section we will prove the following theorem.

**Theorem 2.4** (shifted key estimate). *Let  $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$  with  $p^+ < \infty$ . Then for every  $m > 0$  there exists  $c > 0$  only depending on  $m$ ,  $c_{\log}(p)$  and  $p^+$  such that*

$$\varphi_a(x, M_Q f) \leq c M_Q(\varphi_a(f)) + c |Q|^m$$

*for every cube (or ball)  $Q \subset \mathbb{R}^n$  with  $\ell(Q) \leq 1$ , all  $x \in Q$  and all  $f \in L^1(Q)$  with*

$$a + \int_Q |f| dy \leq \max \{1, |Q|^{-m}\} = |Q|^{-m}.$$

*Proof.* We split the proof into three case:

- (a)  $M_Q f \geq a$ ,
- (b)  $M_Q f \leq a \leq |Q|^m$ ,
- (c)  $M_Q f \leq a$  and  $|Q|^m \leq a$ .

**Case:**  $M_Q f \geq a$

Let  $f_1 := 2f \chi_{\{|f| \geq \frac{a}{2}\}}$ . Then  $M_Q f \geq a$ , implies

$$M_Q f_1 = M_Q(2f \chi_{\{|f| \geq \frac{a}{2}\}}) \geq M_Q f - \frac{a}{2} \geq \frac{1}{2} M_Q f \geq \frac{a}{2}.$$

Thus with the  $\Delta_2$ -condition, Remark 2.3, Lemma A.3 (using  $M_Q f_1 \gtrsim a$ ) we get

$$\varphi_a(x, M_Q f) \leq c \varphi_a(x, M_Q f_1) \sim c \varphi(x, M_Q f_1).$$

Thus, the key estimate of Theorem 2.2 implies

$$\varphi_a(x, M_Q f) \leq c M_Q(\varphi(f_1)) + c |Q|^m.$$

Now by Lemma A.3 and the definition of  $f_1$  we have  $\varphi(f_1) \sim \varphi_a(f_1)$  and  $|f_1| \leq 2|f|$ . Thus

$$\begin{aligned} \varphi_a(x, M_Q f) &\leq c M_Q(\varphi_a(f_1)) + c |Q|^m \\ &\leq c M_Q(\varphi_a(f)) + c |Q|^m. \end{aligned}$$

This proves the claim.

**Case:**  $M_Q f \leq a \leq |Q|^m$

Using Corollary A.6, the key estimate Theorem 2.2, and again Corollary A.6 we estimate

$$\begin{aligned} \varphi_a(x, M_Q f) &\leq c \varphi(x, M_Q f) + c \varphi(x, a) \\ &\leq c \varphi(x, a) \\ &\leq c a \varphi(x, 1) \\ &\leq c |Q|^m. \end{aligned}$$

This proves the claim.

**Case:**  $M_Q f \leq a$  and  $|Q|^m \leq a$

Since  $M_Q f \leq a$ , we have

$$\varphi_a(x, M_Q f) \leq c \varphi''(a)(M_Q f)^2.$$

Recall that  $|Q|^m \leq a \leq |Q|^{-m}$ . We choose  $x_Q^- \in Q$  such that  $p(x_Q^-) = p_Q^-$ . Now we conclude with Lemma 2.1 that

$$\varphi''(a) \leq c \varphi''(x_Q^-, a) \leq c \varphi''(x_Q^-, a + M_Q f)$$

using in the last step that  $M_Q f \leq a$ . This and the previous estimate imply

$$\varphi_a(x, M_Q f) \leq c \varphi''(x_Q^- a + M_Q f)(M_Q f)^2 \leq c \varphi_a(x_Q^-, M_Q f)$$

We can now apply Jensen's inequality for the N-function  $\varphi_a(x_Q^-, \cdot)$  to conclude

$$\varphi_a(x, M_Q f) \leq c M_Q (\varphi_a(x_Q^-, f)).$$

Now, the claim follows if we can prove

$$(2.4) \quad \varphi_a(x_Q^-, f) \leq c \varphi_a(x, f) \quad \text{point wise on } Q.$$

So let  $x \in Q$ . If  $|f(x)| \leq a$ , then with Lemma 2.1 and  $|Q|^m \leq a + |f(x)| \leq 2|Q|^{-m}$  we have

$$\begin{aligned} \varphi_a(x_Q^-, |f(x)|) &\leq c \varphi''_a(x_Q^-, a + |f(x)|) |f(x)|^2 \\ &\leq c \varphi''(x, a + |f(x)|) |f(x)|^2 \\ &\leq c \varphi_a(x, |f(x)|). \end{aligned}$$

It  $|f(x)| \geq a$ , then

$$\begin{aligned} \varphi_a(x_Q^-, |f(x)|) &\leq c \varphi_a(x, |f(x)|) \frac{\varphi''(x_Q^-, a + |f(x)|)}{\varphi''(x, a + |f(x)|)} \\ &\leq c \varphi_a(x, |f(x)|) (\kappa + a + |f(x)|)^{p_Q^- - p(x)} \\ &\leq c \varphi_a(x, |f(x)|) a^{p_Q^- - p(x)}. \\ &\leq c \varphi_a(x, |f(x)|), \end{aligned}$$

where we used Lemma 2.1 in the last step and  $|Q|^m \leq a \leq |Q|^{-m}$ . Therefore (2.4) holds, which concludes the proof.  $\square$

**2.2. Poincaré type estimates.** We will show now that the shifted key estimate of Theorem 2.4 implies the following Poincaré type estimate.

**Theorem 2.5.** *Let  $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$  with  $p^+ < \infty$ . Then for every  $m > 0$  there exists  $c > 0$  only depending on  $m$  and  $c_{\log}(p)$  and  $p^+$  such that*

$$\int_Q \varphi_a\left(x, \frac{|u(x) - \langle u \rangle_Q|}{\ell(Q)}\right) dx \leq c \int_Q \varphi_a(x, |\nabla u(x)|) dx + c |Q|^m$$

for every cube (or ball)  $Q \subset \mathbb{R}^n$  with  $\ell(Q) \leq 1$  and all  $u \in W^{1,p(\cdot)}(Q)$  with

$$a + \int_Q |\nabla u| dy \leq \max\{1, |Q|^{-m}\} = |Q|^{-m}.$$

*Proof.* The proof is similar to the one of Proposition 8.2.8 of [DHHR11]. We can assume  $m \geq 1$ .

By  $\mathcal{W}_k$  we denote the family of dyadic cubes of size  $2^{-k}$ , i.e.  $2^{-k}((0, 1)^n + \bar{k})$  with  $k \in \mathbb{Z}$  and  $\bar{k} \in \mathbb{Z}^n$ . Let us fix  $k_0 \in \mathbb{Z}$  such that  $2^{-k_0-1} \leq \ell(Q) \leq 2^{-k_0} \leq 1$ . Then as in the proof of Proposition 8.2.8 of [DHHR11] we have

$$|u(x) - \langle u \rangle_Q| \leq c \int_Q \frac{|\nabla u(y)|}{|x - y|^{n-1}} dy \leq c \ell(Q) \sum_{k=k_0+2}^{\infty} 2^{-k} T_k(\chi_Q |\nabla u|)(x),$$

where the *averaging operator*  $T_k$  is given by

$$T_k f := \sum_{W \in \mathcal{W}_k} \chi_W M_{2W} f$$

for all  $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ , where  $2W$  is the cube with the same center as  $W$  but twice its diameter.

We estimate by convexity and the locally finiteness of the family  $\mathcal{W}_k$

$$\begin{aligned} I &:= \int_Q \varphi_a \left( x, \frac{|u(x) - \langle u \rangle_Q|}{\ell(Q)} \right) dx \\ &\leq c \sum_{k=k_0}^{\infty} 2^{-k} \int_Q \varphi_a \left( x, \sum_{W \in \mathcal{W}_k} \chi_W(x) M_{2W}(\chi_Q |\nabla u|) \right) dx \\ &\leq c \sum_{k=k_0}^{\infty} 2^{-k} \sum_{W \in \mathcal{W}_k} \int_W \varphi_a(x, M_{2W}(\chi_Q |\nabla u|)) dx \\ &\leq c \sum_{k=k_0}^{\infty} 2^{-k} \sum_{W \in \mathcal{W}_k} \int_{2W} \varphi_a(x, M_{2W}(\chi_Q |\nabla u|)) dx. \end{aligned}$$

Using the assumption on  $u$ ,  $\ell(2W) \leq 2^{-k_0-1} \leq \ell(Q) \leq 1$  and  $m \geq 1$  we estimate

$$a + \int_{2W} \chi_Q |\nabla u| dy \leq a + \frac{|Q|}{|2W|} \int_Q |\nabla u| dy \leq \frac{|Q|}{|2W|} \left( a + \int_Q |\nabla u| dx \right) \leq \max \{1, |2W|^{-m}\}.$$

In particular, this and  $\ell(2W) \leq 1$  allow to apply the shifted key estimate of Theorem 2.4. This yields

$$\begin{aligned} I &\leq c \sum_{k=k_0}^{\infty} 2^{-k} \sum_{W \in \mathcal{W}_k} \int_{2W} \left( \int \chi_Q (\varphi_a(y, |\nabla u|)) dy + |W|^m \right) dx \\ &\leq c \sum_{k=k_0}^{\infty} 2^{-k} \sum_{W \in \mathcal{W}_k} \left( \int_{2W} \chi_Q \varphi_a(y, |\nabla u|) dy + |Q|^m \right) \\ &\leq c \int_Q \varphi_a(y, |\nabla u|) dy + |Q|^m \end{aligned}$$

□

### 3. INTERPOLATION AND VARIABLE EXPONENT SPACES

Let  $\Omega \subset \mathbb{R}^n$  be a connected, open (possibly unbounded) domain with polyhedral boundary. We assume that  $\partial\Omega$  is Lipschitz continuous. For an open, bounded (non-empty) set  $U \subset \mathbb{R}^n$  we denote by  $|U|$  its  $n$ -dimensional Lebesgue measure, by  $h_U$  the diameter of  $U$ , and by  $\rho_U$  the supremum of the diameters of inscribed balls. For  $f \in L^1_{\text{loc}}(\mathbb{R}^n)$  we define

$$\int_U f(x) dx := \frac{1}{|U|} \int_U f(x) dx.$$

For a finite set  $A$  we define  $\#A$  to be the number of elements of  $A$ . We write  $f \sim g$  iff there exist constants  $c, C > 0$ , such that

$$c f \leq g \leq C f,$$

where we always indicate on what the constants may depend. Furthermore, we use  $c$  as a generic constant, i.e. its value may change from line to line but does not depend on the important variables.

Let  $\mathcal{T}_h$  be a simplicial subdivision of  $\Omega$ . By

$$h := \max_{K \in \mathcal{T}_h} h_K$$

we denote the maximum mesh size. We assume that  $\mathcal{T}_h$  is non-degenerate:

$$(3.1) \quad \max_{K \in \mathcal{T}_h} \frac{h_K}{\rho_K} \leq \gamma_0.$$

For  $K \in \mathcal{T}_h$  we define the set of neighbors  $N_K$  and the neighborhood  $S_K$  by

$$\begin{aligned} N_K &:= \{K' \in \mathcal{T}_h : \overline{K'} \cap \overline{K} \neq \emptyset\}, \\ S_K &:= \text{interior} \bigcup_{K' \in N_K} \overline{K'}. \end{aligned}$$

Note that for all  $K, K' \in \mathcal{T}_h$ :  $K' \subset \overline{S_K} \Leftrightarrow K \subset \overline{S_{K'}} \Leftrightarrow \overline{K} \cap \overline{K'} \neq \emptyset$ . Due to our assumption on  $\Omega$  the  $S_K$  are connected, open bounded sets.

It is easy to see that the non-degeneracy (3.1) of  $\mathcal{T}_h$  implies the following properties, where the constants are independent of  $h$ :

- (a)  $|S_K| \sim |K|$  for all  $K \in \mathcal{T}_h$ .
- (b) There exists  $m_1 \in \mathbb{N}$  such that  $\#N_K \leq m_1$ .

For  $G \subset \mathbb{R}^n$  and  $m \in \mathbb{N}_0$  we denote by  $\mathcal{P}_m(G)$  the polynomials on  $G$  of degree less than or equal to  $m$ . Moreover, we set  $\mathcal{P}_{-1}(G) := \{0\}$ . Let us characterize the finite element space  $V_h$  as

$$(3.2) \quad V_h := \{v \in L^1_{\text{loc}}(\Omega) : v|_K \in \mathcal{P}_K\},$$

where

$$(3.3) \quad \mathcal{P}_{r_0}(K) \subset \mathcal{P}_K \subset \mathcal{P}_{r_1}(K)$$

for given  $r_0 \leq r_1 \in \mathbb{N}_0$ . Since  $r_1 \in \mathbb{N}_0$  there exists a constant  $c = c(r_1)$  such that for all  $\mathbf{v}_h \in (V_h)^N$ ,  $K \in \mathcal{T}_h$ ,  $j \in \mathbb{N}_0$ , and  $x \in K$  holds

$$(3.4) \quad h^j |\nabla^j \mathbf{v}_h(x)| \leq c h^j \int_K |\nabla^j \mathbf{v}_h(y)| dy \leq c \int_K |\mathbf{v}_h(y)| dy,$$

where we used (3.1). Here and in the remainder of the paper gradients of functions from  $V_h$  are always understood in a local sense, i.e. on each simplex  $K$  it is the pointwise gradient of the local polynomial.

We will show in this section how the classical results for the interpolation error generalizes to the setting of Sobolev spaces with variable exponents. Instead, of deriving estimates for a specific interpolation operator, we will deduce our results just from some general assumptions on the operator. Note that e.g. the Scott-Zhang operator [SZ90] satisfies all our requirements. More precisely, we assume the following.

**Assumption 3.1.** Let  $l_0 \in \mathbb{N}_0$  and let  $\Pi_h : (W^{l_0,1}(\Omega))^N \rightarrow (V_h)^N$ .

- (a) For some  $l \geq l_0$  and  $m_0 \in \mathbb{N}_0$  holds uniformly in  $K \in \mathcal{T}_h$  and  $\mathbf{v} \in (W^{l,1}(\Omega))^N$

$$(3.5) \quad \sum_{j=0}^{m_0} \int_K |h_K^j \nabla^j \Pi_h \mathbf{v}| dx \leq c(m_0, l) \sum_{k=0}^l h_K^k \int_{S_K} |\nabla^k \mathbf{v}| dx.$$

- (b) For all  $\mathbf{v} \in (\mathcal{P}_{r_0})^N(\Omega)$  holds

$$(3.6) \quad \Pi_h \mathbf{v} = \mathbf{v}.$$

Note that the constant in (3.5) will depend on the non-degeneracy constant  $\gamma_0$  of the mesh. In this way all the results below will depend on  $\gamma_0$ .



**Remark 3.2.** The property (3.5) is called  $W^{l,1}(\Omega)$ -stability of the interpolation operator  $\Pi_h$ . Because of (3.4) we can choose  $m_0$  to be 0 with no loss of generality.

In many cases, e.g. for the Clément and the Scott-Zhang interpolation operator, we have  $\Pi_h \mathbf{v} := (\Pi_h v_1, \dots, \Pi_h v_N)$ . Both operators satisfy Assumption 3.1 (cf. [Clé75, SZ90]). In fact, the scaling invariant formulation (3.5) can be easily derived from the proofs there. Note, that for the Scott-Zhang interpolation operator holds a much stronger property, namely

$$(3.7) \quad \Pi_h \mathbf{v} = \mathbf{v}, \quad \forall \mathbf{v} \in (V_h)^N.$$

**Remark 3.3.** The Scott-Zhang interpolation operator is defined in such a way that it preserves homogeneous boundary conditions, i.e.

$$\Pi_h : W_0^{1,1}(\Omega) \rightarrow V \cap W_0^{1,1}.$$

Thus we have to choose in this case  $l_0 = 1$  in (3.5). However, there is a version of the Scott-Zhang interpolation operator which does not preserve boundary values (cf. remark after (4.6) in [SZ90]). In this case we can choose  $l_0 = 0$  in (3.5).

Now, we will deduce solely from the Assumption 3.1 that the same operator  $\Pi_h$  is also a good interpolation operator for the generalized Sobolev spaces  $(W^{l,p(\cdot)}(\Omega))^N$ . We begin with the stability.

**Lemma 3.4 (Stability).** Let  $\Pi_h$  and  $l$  satisfy Assumption 3.1,  $m > 0$  and let  $p \in \mathcal{P}^{\log}(\Omega)$  with  $p^+ < \infty$ . Then for all  $j \in \mathbb{N}_0$ ,  $K \in \mathcal{T}_h$  and  $\mathbf{v} \in (W^{l,p(\cdot)})^N$  with

$$\max_{k=0,\dots,l} \int_{S_K} h_K^k |\nabla^k \mathbf{v}| dy \leq c_2 \max \{1, |K|^{-m}\}.$$

holds

$$(3.8) \quad \int_K \varphi_a(\cdot, |h_K^j \nabla^j \Pi_h \mathbf{v}|) dx \leq c \sum_{k=0}^l \int_{S_K} \varphi_a(\cdot, |h_K^k \nabla^k \mathbf{v}|) dx + c h_K^m,$$

where  $c = c(k, l, m, c_{\log}, p^+, c_2)$ .

*Proof.* Since  $p^+ < \infty$  the  $\Delta_2$ -constant of  $\varphi_a$  is uniformly bounded with respect to  $a$  and it therefore suffices to consider the case  $c_2 = 1$ . Using 3.1 we gain

$$\begin{aligned} \int_K \varphi_a(\cdot, |h_K^j \nabla^j \Pi_h \mathbf{v}|) dx &\leq c \int_K \varphi_a\left(\cdot, \int_K h_K^j |\nabla^j \Pi_h \mathbf{v}(y)| dy\right) dx \\ &\leq c \int_K \varphi_a\left(\cdot, \sum_{k=0}^l h_K^k \int_{S_K} |\nabla^k \mathbf{v}(y)| dy\right) dx. \end{aligned}$$

Now with Theorem 2.4 as well as the convexity and  $\Delta_2$ -condition of  $\varphi_a$  we find

$$\begin{aligned} \int_K \varphi_a\left(\cdot, \int_{S_K} h_K^j |\nabla^j \Pi_h \mathbf{v}| dy\right) dx &\leq c \sum_{k=0}^l \int_K \varphi_a\left(\cdot, h_K^k \int_{S_K} |\nabla^k \mathbf{v}(y)| dy\right) dx \\ &\leq c \sum_{k=0}^l \int_K \int_{S_K} \varphi_a(\cdot, h_K^j |\nabla^j \mathbf{v}|) dy dx + c h_K^m \\ &\leq c \sum_{k=0}^l \int_K \varphi_a(\cdot, h_K^j |\nabla^j \mathbf{v}|) dx + c h_K^m. \end{aligned}$$

This proves the theorem.  $\square$

**Lemma 3.5 (Approximability).** *Let  $\Pi_h$  and  $l$  satisfy Assumption 3.1 with  $l \leq r_0 + 1$  and let  $p \in \mathcal{P}^{\log}(\Omega)$  with  $p^+ < \infty$  and let  $m > 0$ . Then for all  $j \leq l$ ,  $K \in \mathcal{T}_h$  and  $\mathbf{v} \in (W^{l,p(\cdot)})^N$  with*

$$\int_{S_K} h_K^l |\nabla^l \mathbf{v}| dy \leq c_3 \max \{1, |K|^{-m}\}.$$

holds

$$(3.9) \quad \int_K \varphi_a(\cdot, h_K^j |\nabla^j(\mathbf{v} - \Pi_h \mathbf{v})|) dx \leq c \int_{S_K} \varphi_a(\cdot, h_K^l |\nabla^l \mathbf{v}|) dx + ch_K^m,$$

where  $c = c(k, l, m, c_{\log}, p^+, c_3)$ .

*Proof.* We fix  $\mathbf{p} \in (\mathcal{P}_{l-1})^N$  such that  $\int_{S_K} \nabla^k(\mathbf{v} - \mathbf{p}) dx = 0$  for all  $k = 0, \dots, l-1$ . Then by Poincaré

$$(3.10) \quad \max_{k=0, \dots, l} \int_{S_K} h_K^k |\nabla^k(\mathbf{v} - \mathbf{p})| dx \leq c \int_{S_K} h_K^l |\nabla^l \mathbf{v}| dx \leq c \max \{1, |K|^{-m}\}.$$

Due to (3.6) and  $l-1 \leq r_0$  we have  $\Pi_h \mathbf{p} = \mathbf{p}$ . From this, the convexity of  $\varphi_a$ , and  $\Delta_2(\varphi_a) < \infty$  (uniformly with respect to  $a$ ) we obtain for all  $0 \leq j \leq l$

$$\begin{aligned} I &:= \int_K \varphi_a(\cdot, h_K^j |\nabla^j(\mathbf{v} - \Pi_h \mathbf{v})|) dx \\ &\leq c \int_K \varphi_a(\cdot, h_K^j |\nabla^j(\mathbf{v} - \mathbf{p})|) dx + c \int_K \varphi_a(\cdot, h_K^j |\nabla^j(\Pi_h(\mathbf{v} - \mathbf{p}))|) dx. \end{aligned}$$

Now, we use Lemma 3.4 for the functions  $(\mathbf{v} - \mathbf{p})$  to obtain

$$\begin{aligned} I &\leq c \int_K \varphi_a(\cdot, h_K^j |\nabla^j(\mathbf{v} - \mathbf{p})|) dx + c \sum_{k=0}^l \int_{S_K} \varphi_a(\cdot, h_K^k |\nabla^k(\mathbf{v} - \mathbf{p})|) dx + ch_K^m \\ &\leq c \sum_{k=0}^l \int_{S_K} \varphi_a(\cdot, h_K^k |\nabla^k(\mathbf{v} - \mathbf{p})|) dx + ch_K^m \end{aligned}$$

Due to (3.10) we can apply repeatedly the Poincaré-type inequality from Theorem 2.5 and gain

$$\int_K \varphi_a(\cdot, h_K^j |\nabla^j(\mathbf{v} - \Pi_h \mathbf{v})|) dx \leq c \int_{S_K} \varphi_a(\cdot, h_K^l |\nabla^l \mathbf{v}|) dx + ch_K^m,$$

where we also use the uniform  $\Delta_2$ -constants of  $\varphi_a$  with respect to  $a$ .  $\square$

**Corollary 3.6 (Continuity).** *Under the assumptions of Lemma 3.5 holds*

$$(3.11) \quad \int_K \varphi_a(\cdot, h_K^l |\nabla^l \Pi_h \mathbf{v}|) dx \leq c \int_{S_K} \varphi_a(\cdot, h_K^l |\nabla^l \mathbf{v}|) dx + ch_K^m,$$

with  $c = c(l, m, c_{\log}, p^+, \gamma_0, c_3)$ .

*Proof.* The assertion follows directly from (3.9),  $j = l$ , and the triangle inequality.  $\square$

## 4. CONVERGENCE

Let us now apply the previous results to a finite element approximation of the system (1.4) considered on a bounded Lipschitz domain  $\Omega$  and equipped with zero Dirichlet boundary conditions, where  $p \in \mathcal{P}^{\log}(\Omega)$ . Due to these assumptions it is easy to show, with the help of the theory of monotone operators, that there exists a solution  $\mathbf{v} \in (W_0^{1,p(\cdot)}(\Omega))^N$  such that for all  $\psi \in (W_0^{1,p(\cdot)}(\Omega))^N$  holds

$$(4.1) \quad \int_{\Omega} \mathbf{A}(\cdot, \nabla \mathbf{v}) \cdot \nabla \psi \, dx = \int_{\Omega} \mathbf{f} \cdot \psi \, dx,$$

where  $\mathbf{A}(x, \boldsymbol{\xi}) := (\kappa + |\boldsymbol{\xi}|)^{p(x)-2} \boldsymbol{\xi}$ .

Let  $V_h$  be given as in (3.2) with  $r_0 \geq 1$  such that  $V_h \subseteq W^{1,p(\cdot)}(\Omega)$ . Denote

$$\mathbf{V}_{h,0} := (V_h \cap W_0^{1,p(\cdot)}(\Omega))^N.$$

Thus we now restrict ourselves to conforming finite element spaces containing at least linear polynomials. The finite element approximation of (4.1) reads as follows: find  $\mathbf{v}_h \in \mathbf{V}_{h,0}$  such that for all  $\psi_h \in \mathbf{V}_{h,0}$  holds

$$(4.2) \quad \int_{\Omega} \mathbf{A}(\cdot, \nabla \mathbf{v}_h) \cdot \nabla \psi_h \, dx = \int_{\Omega} \mathbf{f} \cdot \psi_h \, dx.$$

Again it is easy to see that this problem has a unique solution.

In the following we define

$$\mathbf{F}(x, \boldsymbol{\xi}) := (\kappa + |\boldsymbol{\xi}|)^{\frac{p(x)-2}{2}} \boldsymbol{\xi} \text{ for } \kappa \in [0, 1]$$

We begin with a best approximation result.

**Lemma 4.1** (best approximation). *Let  $p \in \mathcal{P}^{\log}(\Omega)$  with  $p^- > 1$  and  $p^+ < \infty$ . Let  $\mathbf{v} \in (W_0^{1,p(\cdot)}(\Omega))^N$  and  $\mathbf{v}_h \in \mathbf{V}_{h,0}$  be solutions of (4.1) and (4.2) respectively. Then there exists  $c > 0$  depending only on  $c_{\log}$ ,  $p^-$ ,  $p^+$  and  $\gamma_0$  such that*

$$(4.3) \quad \|\mathbf{F}(\cdot, \nabla \mathbf{v}) - \mathbf{F}(\cdot, \nabla \mathbf{v}_h)\|_2 \leq c \min_{\psi_h \in \mathbf{V}_{h,0}} \|\mathbf{F}(\cdot, \nabla \mathbf{v}) - \mathbf{F}(\cdot, \nabla \psi_h)\|_2.$$

*Proof.* We use Lemma A.4 and the equations (4.1) and (4.2) to find

$$\begin{aligned} & \int_{\Omega} |\mathbf{F}(\cdot, \nabla \mathbf{v}) - \mathbf{F}(\cdot, \nabla \mathbf{v}_h)|^2 \, dx \\ &= c \int_{\Omega} (\mathbf{A}(\cdot, \nabla \mathbf{v}) - \mathbf{A}(\cdot, \nabla \mathbf{v}_h)) : (\nabla \mathbf{v} - \nabla \mathbf{v}_h) \, dx \\ &= c \int_{\Omega} (\mathbf{A}(\cdot, \nabla \mathbf{v}) - \mathbf{A}(\cdot, \nabla \mathbf{v}_h)) : (\nabla \mathbf{v} - \nabla \psi_h) \, dx \\ &\leq c \int_{\Omega} (\varphi_{|\nabla \mathbf{v}|+\kappa})'(x, |\nabla \mathbf{v} - \nabla \mathbf{v}_h|) |\nabla \mathbf{v} - \nabla \psi_h| \, dx. \end{aligned}$$

for all  $\psi_h \in \mathbf{V}_{h,0}$ . Now, Young's inequality, see Lemma A.1, for  $\varphi_{|\nabla \mathbf{v}|+\kappa}(x, \cdot)$  implies for  $\tau > 0$

$$\begin{aligned} & \int_{\Omega} |\mathbf{F}(\cdot, \nabla \mathbf{v}) - \mathbf{F}(\cdot, \nabla \mathbf{v}_h)|^2 dx \\ & \leq \tau \int_{\Omega} (\varphi_{|\nabla \mathbf{v}|+\kappa})(x, |\nabla \mathbf{v} - \nabla \mathbf{v}_h|) dx \\ & \quad + c(\tau) \int_{\Omega} (\varphi_{|\nabla \mathbf{v}|+\kappa})(x, |\nabla \mathbf{v} - \nabla \psi_h|) dx. \\ & \sim \tau \int_{\Omega} |\mathbf{F}(\cdot, \nabla \mathbf{v}) - \mathbf{F}(\cdot, \nabla \mathbf{v}_h)|^2 dx \\ & \quad + c(\tau) \int_{\Omega} |\mathbf{F}(\cdot, \nabla \mathbf{v}) - \mathbf{F}(\cdot, \nabla \psi_h)|^2 dx, \end{aligned}$$

where we used Lemma A.4 in the last step. Hence, for  $\tau$  small enough we can absorb the  $\tau$ -integral in the l.h.s. This implies

$$\|\mathbf{F}(\cdot, \nabla \mathbf{v}) - \mathbf{F}(\cdot, \nabla \mathbf{v}_h)\|_2^2 \leq c \|\mathbf{F}(\cdot, \nabla \mathbf{v}) - \mathbf{F}(\cdot, \nabla \psi_h)\|_2^2.$$

Taking the minimum over all  $\psi_h \in \mathbf{V}_{h,0}$  proves the claim.  $\square$

**Lemma 4.2.** *Let  $p \in \mathcal{P}^{\log}(\Omega)$  with  $p^- > 1$  and  $p^+ < \infty$  and let  $m > 0$ . Let  $\Pi_h$  satisfy Assumption 3.1 with  $l = 1$  and  $\Pi_h \mathbf{q} = \mathbf{q}$  for all  $\mathbf{q} \in (\mathcal{P}_1)^N(\Omega)$ . (This is e.g. satisfied if  $r_0 \geq 1$ .) Let  $\mathbf{v} \in (W^{1,p(\cdot)}(\Omega))^N$  then for all  $K \in \mathcal{T}_h$  with  $h_K \leq 1$  and all  $\mathbf{Q} \in \mathbb{R}^{N \times n}$  with*

$$|\mathbf{Q}| + \int_{S_K} |\nabla \mathbf{v}| dx \leq c_4 \max\{1, |K|^{-m}\},$$

it holds that

$$(4.4) \quad \int_K |\mathbf{F}(\cdot, \nabla \mathbf{v}) - \mathbf{F}(\cdot, \nabla \Pi_h \mathbf{v})|^2 dx \leq c \int_{S_K} |\mathbf{F}(\cdot, \nabla \mathbf{v}) - \mathbf{F}(\cdot, \mathbf{Q})|^2 dx + ch_K^m,$$

with  $c = c(c_{\log}, p^-, p^+, m, \gamma_0, c_4)$ .

*Proof.* Note that  $\mathbf{v} \in (W^{1,p(\cdot)}(\Omega))^N$  implies  $\mathbf{F}(\cdot, \nabla \mathbf{v}) \in (L^2(\Omega))^{N \times n}$ . For arbitrary  $\mathbf{Q} \in \mathbb{R}^{N \times n}$  we have

$$\begin{aligned} & \int_K |\mathbf{F}(\cdot, \nabla \mathbf{v}) - \mathbf{F}(\cdot, \nabla \Pi_h \mathbf{v})|^2 dx \\ & \leq c \int_K |\mathbf{F}(\cdot, \nabla \mathbf{v}) - \mathbf{F}(\cdot, \mathbf{Q})|^2 dx + c \int_K |\mathbf{F}(\cdot, \nabla \Pi_h \mathbf{v}) - \mathbf{F}(\cdot, \mathbf{Q})|^2 dx. \\ & =: (I) + (II). \end{aligned}$$

Let  $\mathbf{p} \in (\mathcal{P}_1)^N(S_K)$  be such that  $\nabla \mathbf{p} = \mathbf{Q}$ . Due to  $\Pi_h \mathbf{p} = \mathbf{p}$  there holds  $\mathbf{Q} = \nabla \mathbf{p} = \nabla \Pi_h \mathbf{p}$ . We estimate by Lemma A.4 as follows

$$\begin{aligned} (II) & \leq c \int_K \varphi_{|\mathbf{Q}|+\kappa}(\cdot, |\nabla \Pi_h \mathbf{v} - \mathbf{Q}|) dx \\ & = c \int_K \varphi_{|\mathbf{Q}|+\kappa}(\cdot, |\nabla \Pi_h \mathbf{v} - \nabla \Pi_h \mathbf{p}|) dx \\ & = c \int_K \varphi_{|\mathbf{Q}|+\kappa}(\cdot, |\nabla \Pi_h(\mathbf{v} - \mathbf{p})|) dx. \end{aligned}$$

We use the crucial Corollary 3.6 for  $\mathbf{v} - \mathbf{p}$  and  $a = |\mathbf{Q}| + \kappa$  to obtain

$$(II) \leq c \int_{S_K} \varphi_{|\mathbf{Q}|+\kappa}(\cdot, |\nabla(\mathbf{v} - \mathbf{p})|) dx + ch_K^2 = c \int_{S_K} \varphi_{|\mathbf{Q}|+\kappa}(\cdot, |\nabla \mathbf{v} - \mathbf{Q}|) dx + ch_K^m.$$

Now, with Lemma A.4

$$(II) \leq c \int_{S_K} |\mathbf{F}(\cdot, \nabla \mathbf{v}) - \mathbf{F}(\cdot, \mathbf{Q})|^2 dx + ch_K^m.$$

Since,  $|K| \sim |S_K|$  and  $K \subset S_K$  we also have

$$(I) \leq c \int_{S_K} |\mathbf{F}(\cdot, \nabla \mathbf{v}) - \mathbf{F}(\cdot, \mathbf{Q})|^2 dx.$$

Overall,

$$\int_K |\mathbf{F}(\cdot, \nabla \mathbf{v}) - \mathbf{F}(\cdot, \nabla \Pi_h \mathbf{v})|^2 dx \leq c \int_{S_K} |\mathbf{F}(\cdot, \nabla \mathbf{v}) - \mathbf{F}(\cdot, \mathbf{Q})|^2 dx + ch_K^m.$$

Taking the infimum over all  $\mathbf{Q}$  proves (4.4).  $\square$

**Lemma 4.3.** *Let  $p \in C^{0,\alpha}(\overline{\Omega})$  with  $p^- > 1$  and let  $m > 0$ . Let  $\Pi_h$  satisfy Assumption 3.1 with  $l = 1$  and  $\Pi_h \mathbf{q} = \mathbf{q}$  for all  $\mathbf{q} \in (\mathcal{P}_1)^N(\Omega)$ . (This is e.g. satisfied if  $r_0 \geq 1$ .) Let  $\mathbf{v} \in (W^{1,p(\cdot)}(\Omega))^N$  then for all  $K \in \mathcal{T}_h$  with  $h_K \leq 1$  with*

$$\int_{S_K} |\nabla \mathbf{v}| dx \leq c_5 \max \{1, |K|^{-m}\},$$

it holds that

$$(4.5) \quad \begin{aligned} \int_K |\mathbf{F}(\cdot, \nabla \mathbf{v}) - \mathbf{F}(\cdot, \nabla \Pi_h \mathbf{v})|^2 dx &\leq ch_K^{2\alpha} \left( \int_{S_K} \ln(\kappa + |\nabla \mathbf{v}|)^2 (\kappa + |\nabla \mathbf{v}|)^{p(\cdot)} dx + 1 \right) \\ &\quad + ch_K^2 \left( \int_{S_K} |\nabla \mathbf{F}(\cdot, \nabla \mathbf{v})|^2 dx \right) \end{aligned}$$

with  $c = c(c_{\log}, \|p\|_{C^{0,\alpha}(\overline{\Omega})}, \alpha, p^-, m, \gamma_0, c_5)$ .

*Proof.* In order to show (4.5) we need a special choice of  $\mathbf{Q}$  in Lemma 4.2. Precisely we want that

$$(4.6) \quad \int_{S_K} \mathbf{F}(\cdot, \mathbf{Q}) dx = \int_{S_K} \mathbf{F}(\cdot, \nabla \mathbf{v}) dx.$$

This is possible if the function

$$\lambda(\mathbf{Q}) := \int_{S_K} \mathbf{F}(\cdot, \mathbf{Q}) dx = \int_{S_K} (\kappa + |\mathbf{Q}|)^{\frac{p(\cdot)-2}{2}} dx \mathbf{Q}$$

is surjective. Obviously we can choose every direction in  $\mathbb{R}^{N \times n}$  via the direction of  $\mathbf{Q}$ . On the other hand the modulus

$$|\lambda(\mathbf{Q})| = \int_{S_K} (\kappa + |\mathbf{Q}|)^{\frac{p(\cdot)-2}{2}} |\mathbf{Q}| dx$$

is an increasing continuous function of  $|\mathbf{Q}|$  with  $|\lambda(\mathbf{0})| = 0$  and  $|\lambda(\mathbf{Q})| \rightarrow \infty$  for  $|\mathbf{Q}| \rightarrow \infty$ . Hence  $\lambda : \mathbb{R}^{N \times n} \rightarrow \mathbb{R}^{N \times n}$  is surjective. Choosing  $\mathbf{Q}$  via (4.6) we have

$$\begin{aligned} & \int_{S_K} |\mathbf{F}(\cdot, \nabla \mathbf{v}) - \mathbf{F}(\cdot, \mathbf{Q})|^2 dx \\ & \leq c \int_{S_K} |\mathbf{F}(\cdot, \nabla \mathbf{v}) - \langle \mathbf{F}(\cdot, \nabla \mathbf{v}) \rangle_{S_K}|^2 dx + c \int_{S_K} |\mathbf{F}(\cdot, \mathbf{Q}) - \langle \mathbf{F}(\cdot, \mathbf{Q}) \rangle_{S_K}|^2 dx. \end{aligned}$$

By Poincaré's inequality we clearly gain

$$\int_{S_K} |\mathbf{F}(\cdot, \nabla \mathbf{v}) - \langle \mathbf{F}(\cdot, \nabla \mathbf{v}) \rangle_{S_K}|^2 dx \leq c h_K^2 \int_{S_K} |\nabla \mathbf{F}(\cdot, \nabla \mathbf{v})|^2 dx$$

We estimate

$$\int_{S_K} |\mathbf{F}(\cdot, \mathbf{Q}) - \langle \mathbf{F}(\cdot, \mathbf{Q}) \rangle_{S_K}|^2 dx \leq c \int_{S_K} \int_{S_K} |\mathbf{F}(x, \mathbf{Q}) - \mathbf{F}(y, \mathbf{Q})|^2 dx dy.$$

For  $x, y \in S_K$  we estimate

$$\begin{aligned} & |\mathbf{F}(x, \mathbf{Q}) - \mathbf{F}(y, \mathbf{Q})| \\ (4.7) \quad & \leq c |p(x) - p(y)| \ln(\kappa + |\mathbf{Q}|) \left( (\kappa + |\mathbf{Q}|)^{\frac{p(x)-2}{2}} + (\kappa + |\mathbf{Q}|)^{\frac{p(y)-2}{2}} \right) |\mathbf{Q}|. \end{aligned}$$

This,  $p \in C^{0,\alpha}(\bar{\Omega})$  and the previous estimate imply

$$(4.8) \quad \int_{S_K} |\mathbf{F}(\cdot, \mathbf{Q}) - \langle \mathbf{F}(\cdot, \mathbf{Q}) \rangle_{S_K}|^2 dx \leq c h_K^{2\alpha} \left( \int_{S_K} \ln(\kappa + |\mathbf{Q}|)^2 (\kappa + |\mathbf{Q}|)^{p(x)} dx + 1 \right).$$

In order to complete the proof of (4.5) we need to replace  $\mathbf{Q}$  by  $\nabla \mathbf{v}$ . The choice of  $\mathbf{Q}$  in (4.6) yields

$$\begin{aligned} (4.9) \quad |\mathbf{Q}|^{p_{S_K}^-} & \leq c \left( \left| \int_{S_K} \mathbf{F}(\cdot, \mathbf{Q}) dx \right|^2 + 1 \right) = c \left( \left| \int_{S_K} \mathbf{F}(\cdot, \nabla \mathbf{v}) dx \right|^2 + 1 \right) \\ & \leq c \left( \int_{S_K} |\mathbf{F}(\cdot, \nabla \mathbf{v})|^2 dx + 1 \right) \leq c |S_K|^{-1} \leq c h^{-n}. \end{aligned}$$

This allows to apply Lemma 2.1 to get

$$\begin{aligned} & \int_{S_K} \ln(\kappa + |\mathbf{Q}|)^2 (\kappa + |\mathbf{Q}|)^{p(x)} dx \leq c \ln(1 + |\mathbf{Q}|)^2 (\kappa + |\mathbf{Q}|)^{p_{S_K}^-} \\ & \leq c \left( \Psi(|\mathbf{Q}|^{p_{S_K}^-}) + 1 \right), \end{aligned}$$

where  $\Psi(t) := \ln(1+t)t$ . Since  $\Psi$  is convex we have as a consequence of (4.9) and Jensen's inequality

$$\begin{aligned} & \Psi(|\mathbf{Q}|^{p_{S_K}^-}) + 1 \leq c \left( \int_{S_K} \Psi(|\mathbf{F}(\cdot, \nabla \mathbf{v})|) dx + 1 \right) \\ & \leq c \left( \int_{S_K} \ln(\kappa + |\nabla \mathbf{v}|)^2 (\kappa + |\nabla \mathbf{v}|)^{p(x)} dx + 1 \right). \end{aligned}$$

Inserting this in (4.8) proves the claim. □

From Lemma 4.1, Lemma 4.2 and Lemma 4.3 we immediately deduce the following interpolation error estimate.

**Theorem 4.4.** *Let  $p \in C^{0,\alpha}(\overline{\Omega})$  with  $p^- > 1$ . Let  $\mathbf{v} \in (W_0^{1,p(\cdot)}(\Omega))^N$  and  $\mathbf{v}_h \in \mathbf{V}_{h,0}$  be solutions of (4.1) and (4.2) respectively. Then*

$$\begin{aligned} \|\mathbf{F}(\cdot, \nabla \mathbf{v}) - \mathbf{F}(\cdot, \nabla \mathbf{v}_h)\|_2 &\leq c \min_{\psi_h \in \mathbf{V}_{h,0}} \|\mathbf{F}(\cdot, \nabla \mathbf{v}) - \mathbf{F}(\cdot, \nabla \psi_h)\|_2 \\ &\leq c \|\mathbf{F}(\cdot, \nabla \mathbf{v}) - \mathbf{F}(\cdot, \nabla \Pi_h \mathbf{v})\|_2 + c h^\alpha \\ &\leq c h^\alpha, \end{aligned}$$

where  $c$  also depends on  $\|\mathbf{F}(\nabla \mathbf{v})\|_{1,2}$ .

**Remark 4.5.** *In the case of  $p \in C^{0,\alpha}$  with  $\alpha < 1$  one cannot expect that  $\mathbf{F}(\cdot, \nabla \mathbf{v}) \in W^{1,2}(\Omega)$  even locally. However, motivated by the results of [DE08] we believe that  $p \in C^{0,\alpha}$  should imply  $\mathbf{F}(\cdot, \nabla \mathbf{v}) \in \mathcal{N}^{\alpha,2}$ , where  $\mathcal{N}^{\alpha,2}$  is the Nikolskiĭ space with order of differentiability  $\alpha$ . This is enough to ensure that*

$$\sum_K \int_{S_K} |\mathbf{F}(\cdot, \nabla \mathbf{v}) - \langle \mathbf{F}(\cdot, \nabla \mathbf{v}) \rangle_{S_K}|^2 dx \leq c h_K^{2\alpha}.$$

This still implies the  $\mathcal{O}(h^\alpha)$ -convergence of Theorem 4.4.

**4.1. Frozen exponents.** For the application it is convenient to replace the exponent  $p(\cdot)$  by some local approximation.

$$p_{\mathcal{T}} := \sum_{K \in \mathcal{T}} p(x_K) \chi_K,$$

where  $x_K := \operatorname{argmin}_K p$ , i.e.  $p(x_K) = p_K^-$ , and consider

$$\mathbf{A}_{\mathcal{T}}(x, \xi) = \sum_{K \in \mathcal{T}} \chi_K(x) \mathbf{A}(x_K, \xi)$$

instead of  $\mathbf{A}$ . In the following we will show that this does not effect our results about convergence. Using monotone operator theory we can find a function  $\tilde{\mathbf{v}}_h \in \mathbf{V}_{h,0}$

$$(4.10) \quad \int_{\Omega} \mathbf{A}_{\mathcal{T}}(\cdot, \nabla \tilde{\mathbf{v}}_h) \cdot \nabla \psi_h dx = \int_{\Omega} \mathbf{f} \cdot \psi_h dx$$

for all  $\psi \in \mathbf{V}_{h,0}$ . The coercivity  $\mathbf{A}_{\mathcal{T}}$  implies that  $\tilde{\mathbf{v}}_h \in (W^{1,p_{\mathcal{T}}(\cdot)}(\Omega))^N$ . We will show now that also  $\tilde{\mathbf{v}}_h \in (W^{1,p(\cdot)}(\Omega))^N$ . We begin with the local estimate

$$(4.11) \quad \|\nabla \tilde{\mathbf{v}}_h\|_{L^\infty(K)} \leq c \left( \int_K |\nabla \tilde{\mathbf{v}}_h|^{p(x_K)} dx \right)^{\frac{1}{p(x_K)}} \leq c h_K^{-\frac{d}{p(x_K)}}.$$

Thus, we can apply Lemma 2.1 to find

$$\int_{\Omega} |\nabla \tilde{\mathbf{v}}_h|^{p(\cdot)} dx \leq \int_{\Omega} (1 + |\nabla \tilde{\mathbf{v}}_h|)^{p(\cdot)} dx \leq c \int_{\Omega} (1 + |\nabla \tilde{\mathbf{v}}_h|)^{p_{\mathcal{T}}} dx \leq c(\mathbf{f}).$$

**Remark 4.6.** *Indeed, the steps above show, that the norms  $\|\cdot\|_{p_{\mathcal{T}}(\cdot)}$  and  $\|\cdot\|_{p(\cdot)}$  are equivalent on any finite element space consisting locally of polynomials of finite order. Even more, if  $\|g\|_{p(\cdot)} \leq 1$ , then*

$$\int_{\Omega} |g(x)|^{p(x)} dx \leq c \int_{\Omega} |g(x)|^{p_{\mathcal{T}}(x)} dx + c h^m$$

for any fixed  $m$ , where  $c = c(m)$ . The same estimate holds also with  $p$  and  $p_{\mathcal{T}}$  interchanged.

Following the same ideas as before we want to estimate

$$\begin{aligned} \|\mathbf{F}_{\mathcal{T}}(\cdot, \nabla \mathbf{v}) - \mathbf{F}_{\mathcal{T}}(\cdot, \nabla \tilde{\mathbf{v}}_h)\|_2^2 &= \sum_{K \in \mathcal{T}} \int_K |\mathbf{F}(x_K, \nabla \mathbf{v}) - \mathbf{F}(x_K, \nabla \tilde{\mathbf{v}}_h)|^2 dx, \\ \text{where } \mathbf{F}_{\mathcal{T}}(x, \boldsymbol{\xi}) &:= \sum_{K \in \mathcal{T}} \chi_K(x) \mathbf{F}(x_K, \boldsymbol{\xi}). \end{aligned}$$

Now we are confronted with the problem that  $\mathbf{v}$  and  $\tilde{\mathbf{v}}_h$  solve two different equations. Due to this an additional error term occurs. Following the same approach as before we gain for  $\boldsymbol{\psi} \in \mathbf{V}_{0,h}$  arbitrary

$$\begin{aligned} \|\mathbf{F}_{\mathcal{T}}(\cdot, \nabla \mathbf{v}) - \mathbf{F}_{\mathcal{T}}(\cdot, \nabla \tilde{\mathbf{v}}_h)\|_2^2 &\leq c \int_{\Omega} (\mathbf{A}_{\mathcal{T}}(\cdot, \nabla \mathbf{v}) - \mathbf{A}_{\mathcal{T}}(\cdot, \nabla \tilde{\mathbf{v}}_h)) : (\nabla \mathbf{v} - \nabla \tilde{\mathbf{v}}_h) dx \\ &= c \int_{\Omega} (\mathbf{A}(\cdot, \nabla \mathbf{v}) - \mathbf{A}_{\mathcal{T}}(\cdot, \nabla \tilde{\mathbf{v}}_h)) : (\nabla \mathbf{v} - \nabla \tilde{\boldsymbol{\psi}}_h) dx \\ &\quad + c \int_{\Omega} (\mathbf{A}_{\mathcal{T}}(\cdot, \nabla \mathbf{v}) - \mathbf{A}(\cdot, \nabla \mathbf{v})) : (\nabla \mathbf{v} - \nabla \tilde{\mathbf{v}}_h) dx \\ &= c \int_{\Omega} (\mathbf{A}_{\mathcal{T}}(\cdot, \nabla \mathbf{v}) - \mathbf{A}_{\mathcal{T}}(\cdot, \nabla \tilde{\mathbf{v}}_h)) : (\nabla \mathbf{v} - \nabla \tilde{\boldsymbol{\psi}}_h) dx \\ &\quad + c \int_{\Omega} (\mathbf{A}_{\mathcal{T}}(\cdot, \nabla \mathbf{v}) - \mathbf{A}(\cdot, \nabla \mathbf{v})) : (\nabla \mathbf{v} - \nabla \tilde{\mathbf{v}}_h) dx \\ &\quad + c \int_{\Omega} (\mathbf{A}(\cdot, \nabla \mathbf{v}) - \mathbf{A}_{\mathcal{T}}(\cdot, \nabla \mathbf{v})) : (\nabla \mathbf{v} - \nabla \boldsymbol{\psi}_h) dx. \end{aligned}$$

As before this implies

$$\begin{aligned} \|\mathbf{F}_{\mathcal{T}}(\cdot, \nabla \mathbf{v}) - \mathbf{F}_{\mathcal{T}}(\cdot, \nabla \tilde{\mathbf{v}}_h)\|_2^2 &\leq c \|\mathbf{F}_{\mathcal{T}}(\cdot, \nabla \mathbf{v}) - \mathbf{F}_{\mathcal{T}}(\cdot, \nabla \boldsymbol{\psi}_h)\|_2^2 \\ &\quad + c \int_{\Omega} (\mathbf{A}_{\mathcal{T}}(\cdot, \nabla \mathbf{v}) - \mathbf{A}(\cdot, \nabla \mathbf{v})) : (\nabla \mathbf{v} - \nabla \tilde{\mathbf{v}}_h) dx \\ &\quad + c \int_{\Omega} (\mathbf{A}(\cdot, \nabla \mathbf{v}) - \mathbf{A}_{\mathcal{T}}(\cdot, \nabla \mathbf{v})) : (\nabla \mathbf{v} - \nabla \boldsymbol{\psi}_h) dx \\ &=: (I) + (II) + (III). \end{aligned}$$

In particular, we get a best approximation result with the two additional error terms (II) and (III). We begin in the following with (II).

To estimate the difference between  $\mathbf{A}_{\mathcal{T}}$  and  $\mathbf{A}$  we need the estimate

$$\begin{aligned} |\mathbf{A}_{\mathcal{T}}(x, \mathbf{Q}) - \mathbf{A}(x, \mathbf{Q})| &\leq c |p_{\mathcal{T}}(x) - p(x)| |\ln(\kappa + |\mathbf{Q}|)| \left( (\kappa + |\mathbf{Q}|)^{p_{\mathcal{T}}(x)-2} + (\kappa + |\mathbf{Q}|)^{p(x)-2} \right) |\mathbf{Q}| \\ &\leq c h^{\alpha} |\ln(\kappa + |\mathbf{Q}|)| \left( (\kappa + |\mathbf{Q}|)^{p_{\mathcal{T}}(x)-2} + (\kappa + |\mathbf{Q}|)^{p(x)-2} \right) |\mathbf{Q}| \end{aligned}$$

for all  $\mathbf{Q} \in \mathbb{R}^{N \times n}$  using also that  $p \in C^{0,\alpha}$ . Hence, we get

$$\begin{aligned} (II) &:= \int_{\Omega} (\mathbf{A}_{\mathcal{T}}(\cdot, \nabla \mathbf{v}) - \mathbf{A}(\cdot, \nabla \mathbf{v})) : (\nabla \mathbf{v} - \nabla \tilde{\mathbf{v}}_h) dx \\ &\leq c h^{\alpha} \int_{\Omega} |\ln(\kappa + |\nabla \mathbf{v}|)| (\kappa + |\nabla \mathbf{v}|)^{p_{\mathcal{T}}(x)-2} |\nabla \mathbf{v}| |\nabla \mathbf{v} - \nabla \tilde{\mathbf{v}}_h| dx \\ &\quad + c h^{\alpha} \int_{\Omega} |\ln(\kappa + |\nabla \mathbf{v}|)| (\kappa + |\nabla \mathbf{v}|)^{p(x)-2} |\nabla \mathbf{v}| |\nabla \mathbf{v} - \nabla \tilde{\mathbf{v}}_h| dx \\ &=: (II)_1 + (II)_2. \end{aligned}$$



We begin with the estimate for  $(II)_1$  on each  $K \in \mathcal{T}$ . Define the N-function

$$\varphi^K(t) := \int_0^t (\kappa + s)^{p_K-2} s \, ds.$$

Using this definition we estimate

$$(II)_1 \leq c \sum_{K \in \mathcal{T}} \int_K h^\alpha |\ln(\kappa + |\nabla \mathbf{v}|)| (\varphi^K)'(|\nabla \mathbf{v}|) |\nabla \mathbf{v} - \nabla \tilde{\mathbf{v}}_h| \, dx.$$

Using Young's inequality with  $\varphi_{|\nabla \mathbf{u}|}^K$  on  $|\nabla \mathbf{v} - \nabla \tilde{\mathbf{v}}_h|$  and its complementary function on the rest, we get

$$\begin{aligned} (II)_1 &\leq \delta \sum_{K \in \mathcal{T}} \int_K (\varphi^K)_{|\nabla \mathbf{v}|} (|\nabla \mathbf{v} - \nabla \tilde{\mathbf{v}}_h|) \, dx \\ &\quad + c_\delta \sum_{K \in \mathcal{T}} \int_K ((\varphi^K)_{|\nabla \mathbf{v}|})^* \left( h^\alpha |\ln(\kappa + |\nabla \mathbf{v}|)| (\varphi^K)'(|\nabla \mathbf{v}|) \right) \, dx. \end{aligned}$$

Now we use Lemma A.4 for the first line and Lemma A.8 and Lemma A.7 (with  $\lambda = h^\alpha \leq 1$  using  $h \leq 1$ ) for the second line to find

$$\begin{aligned} (II)_1 &\leq \delta c \|\mathbf{F}_{\mathcal{T}}(\cdot, \nabla \mathbf{v}) - \mathbf{F}_{\mathcal{T}}(\cdot, \nabla \tilde{\mathbf{v}}_h)\|_2^2 \\ &\quad + c_\delta \sum_{K \in \mathcal{T}} \int_K (1 + |\ln(\kappa + |\nabla \mathbf{v}|)|)^{\max\{2, p'_K\}} ((\varphi^K)_{|\nabla \mathbf{v}|})^* \left( h^\alpha (\varphi^K)'(|\nabla \mathbf{v}|) \right) \, dx \\ &\leq \delta c \|\mathbf{F}_{\mathcal{T}}(\cdot, \nabla \mathbf{v}) - \mathbf{F}_{\mathcal{T}}(\cdot, \nabla \tilde{\mathbf{v}}_h)\|_2^2 \\ &\quad + c_\delta \sum_{K \in \mathcal{T}} h^{2\alpha} \int_K (1 + |\ln(\kappa + |\nabla \mathbf{v}|)|)^{\max\{2, p'_K\}} (\varphi^K)(|\nabla \mathbf{v}|) \, dx. \end{aligned}$$

The term  $(II)_2$  is estimate similarly taking into account the extra factor  $(\kappa + |\nabla \mathbf{v}|)^{p(x)-p_{\mathcal{T}}(x)}$ . We get

$$\begin{aligned} (II)_2 &\leq \delta c \|\mathbf{F}_{\mathcal{T}}(\cdot, \nabla \mathbf{v}) - \mathbf{F}_{\mathcal{T}}(\cdot, \nabla \tilde{\mathbf{v}}_h)\|_2^2 \\ &\quad + c_\delta \sum_{K \in \mathcal{T}} h^{2\alpha} \int_K (1 + |\ln(\kappa + |\nabla \mathbf{v}|)| (\kappa + |\nabla \mathbf{v}|)^{p(x)-p_{\mathcal{T}}(x)})^{\max\{2, p'_K\}} (\varphi^K)(|\nabla \mathbf{v}|) \, dx. \end{aligned}$$

Overall, this yields

$$(II) \leq \delta c \|\mathbf{F}_{\mathcal{T}}(\cdot, \nabla \mathbf{v}) - \mathbf{F}_{\mathcal{T}}(\cdot, \nabla \tilde{\mathbf{v}}_h)\|_2^2 + c_\delta c_s h^{2\alpha} \int_\Omega (1 + |\nabla \mathbf{v}|^{p(x)s}) \, dx,$$

for some  $s > 1$ . For  $h$  small we can choose  $s$  close to 1.

For  $(III)$  the analogous estimate is

$$(III) \leq c \|\mathbf{F}_{\mathcal{T}}(\cdot, \nabla \mathbf{v}) - \mathbf{F}_{\mathcal{T}}(\cdot, \nabla \psi_h)\|_2^2 + c_s h^{2\alpha} \int_\Omega (1 + |\nabla \mathbf{v}|^{p(x)s}) \, dx.$$

We finally end up with the following result.

**Lemma 4.7** (best approximation). *Let  $p \in C^{0,\alpha}(\overline{\Omega})$  for some  $\alpha \in (0, 1]$  with  $p^- > 1$ . Let  $\mathbf{v} \in (W_0^{1,p(\cdot)}(\Omega))^N$  and  $\tilde{\mathbf{v}}_h \in \mathbf{V}_{h,0}$  be the solutions of (4.1) and (4.10) respectively. Then for some  $s > 1$  (close to 1 for  $h$  small)*

$$\begin{aligned} \|\mathbf{F}_{\mathcal{T}}(\cdot, \nabla \mathbf{v}) - \mathbf{F}_{\mathcal{T}}(\cdot, \nabla \tilde{\mathbf{v}}_h)\|_2 &\leq c \min_{\psi_h \in \mathbf{V}_{h,0}} \|\mathbf{F}_{\mathcal{T}}(\cdot, \nabla \mathbf{v}) - \mathbf{F}_{\mathcal{T}}(\cdot, \nabla \psi_h)\|_2 \\ &\quad + c h^\alpha \left( \int_\Omega (1 + |\nabla \mathbf{v}|^{p(x)s}) \, dx \right)^{\frac{1}{2}}. \end{aligned}$$

We estimate the best approximation error by the projection error using  $\psi_h = \Pi_h \mathbf{v}$ .

$$\begin{aligned} \|\mathbf{F}_{\mathcal{T}}(\cdot, \nabla \mathbf{v}) - \mathbf{F}_{\mathcal{T}}(\cdot, \nabla \Pi_h \mathbf{v})\|_2 &\leq \|\mathbf{F}(\cdot, \nabla \mathbf{v}) - \mathbf{F}(\cdot, \nabla \Pi_h \mathbf{v})\|_2 \\ &\quad + \|\mathbf{F}_{\mathcal{T}}(\cdot, \nabla \mathbf{v}) - \mathbf{F}(\cdot, \nabla \mathbf{v})\|_2 \\ &\quad + \|\mathbf{F}_{\mathcal{T}}(\cdot, \nabla \Pi_h \mathbf{v}) - \mathbf{F}(\cdot, \nabla \Pi_h \mathbf{v})\|_2 \\ &=: [I] + [II] + [III]. \end{aligned}$$

We already know that  $[I] \leq c h^\alpha$  by Theorem 4.4. The estimate for  $[II]$  and  $[III]$  are similar. Analogously to the estimate for  $|\mathbf{A}_{\mathcal{T}}(x, \mathbf{Q}) - \mathbf{A}(x, \mathbf{Q})|$  we have

$$\begin{aligned} &|\mathbf{F}_{\mathcal{T}}(x, \mathbf{Q}) - \mathbf{F}(x, \mathbf{Q})| \\ &\leq c |p_{\mathcal{T}}(x) - p(x)| |\ln(\kappa + |\mathbf{Q}|)| \left( (\kappa + |\mathbf{Q}|)^{\frac{p_{\mathcal{T}}(x)-2}{2}} + (\kappa + |\mathbf{Q}|)^{\frac{p(x)-2}{2}} \right) |\mathbf{Q}|. \end{aligned}$$

This implies as above

$$\begin{aligned} [II] &\leq c h^\alpha \left( \int_{\Omega} (1 + |\nabla \mathbf{v}|)^{sp(x)} dx \right)^{\frac{1}{2}}, \\ [III] &\leq c h^\alpha \left( \int_{\Omega} (1 + |\nabla \Pi_h \mathbf{v}|)^{sp(x)} dx \right)^{\frac{1}{2}}. \end{aligned}$$

We can use the stability of  $\Pi_h$ , see Lemma 3.4 (for the  $a = 0$  and the exponent  $sp(\cdot)$ ) to get

$$[III] \leq c h^\alpha \left( \int_{\Omega} (1 + |\nabla \mathbf{v}|)^{sp(x)} dx \right)^{\frac{1}{2}}.$$

We summarize our calculations in the following theorem.

**Theorem 4.8.** *Let  $p \in C^{0,\alpha}(\overline{\Omega})$  with  $p^- > 1$ . Let  $\mathbf{v} \in (W_0^{1,p(\cdot)}(\Omega))^N$  and  $\tilde{\mathbf{v}}_h \in \mathbf{V}_{h,0}$  be solutions of (4.1) and (4.10) respectively. Then*

$$\begin{aligned} \|\mathbf{F}_{\mathcal{T}}(\cdot, \nabla \mathbf{v}) - \mathbf{F}_{\mathcal{T}}(\cdot, \nabla \tilde{\mathbf{v}}_h)\|_2 &\leq c \min_{\psi_h \in \mathbf{V}_{h,0}} \|\mathbf{F}_{\mathcal{T}}(\cdot, \nabla \mathbf{v}) - \mathbf{F}_{\mathcal{T}}(\cdot, \nabla \psi_h)\|_2 + c h^\alpha \\ &\leq c \|\mathbf{F}_{\mathcal{T}}(\cdot, \nabla \mathbf{v}) - \mathbf{F}_{\mathcal{T}}(\cdot, \nabla \Pi_h \mathbf{v})\|_2 + c h^\alpha \\ &\leq c h^\alpha, \end{aligned}$$

where  $c$  also depends on  $\|\mathbf{F}(\nabla \mathbf{v})\|_{1,2}$ .

#### APPENDIX A. ORLICZ SPACES

The following definitions and results are standard in the theory of Orlicz spaces and can for example be found in [KR61, RR91]. A continuous, convex function  $\rho : [0, \infty) \rightarrow [0, \infty)$  with  $\rho(0) = 0$ , and  $\lim_{t \rightarrow \infty} \rho(t) = \infty$  is called a *continuous, convex  $\varphi$ -function*.

We say that  $\varphi$  satisfies the  $\Delta_2$ -condition, if there exists  $c > 0$  such that for all  $t \geq 0$  holds  $\varphi(2t) \leq c \varphi(t)$ . By  $\Delta_2(\varphi)$  we denote the smallest such constant. Since  $\varphi(t) \leq \varphi(2t)$  the  $\Delta_2$ -condition is equivalent to  $\varphi(2t) \sim \varphi(t)$  uniformly in  $t$ . For a family  $\varphi_\lambda$  of continuous, convex  $\varphi$ -functions we define  $\Delta_2(\{\varphi_\lambda\}) := \sup_\lambda \Delta_2(\varphi_\lambda)$ . Note that if  $\Delta_2(\varphi) < \infty$  then  $\varphi(t) \sim \varphi(ct)$  uniformly in  $t \geq 0$  for any fixed  $c > 0$ . By  $L^\varphi$  and  $W^{k,\varphi}$ ,  $k \in \mathbb{N}_0$ , we denote the classical Orlicz and Orlicz-Sobolev spaces, i.e.  $f \in L^\varphi$  iff  $\int \varphi(|f|) dx < \infty$  and  $f \in W^{k,\varphi}$  iff  $\nabla^j f \in L^\varphi$ ,  $0 \leq j \leq k$ .

A  $\varphi$ -function  $\rho$  is called a  $N$ -function iff it is strictly increasing and convex with

$$\lim_{t \rightarrow 0} \frac{\rho(t)}{t} = \lim_{t \rightarrow \infty} \frac{t}{\rho(t)} = 0.$$

By  $\rho^*$  we denote the conjugate N-function of  $\rho$ , which is given by  $\rho^*(t) = \sup_{s \geq 0} (st - \rho(s))$ . Then  $\rho^{**} = \rho$ .

**Lemma A.1** (Young's inequality). *Let  $\rho$  be an N-function. Then for all  $s, t \geq 0$  we have*

$$st \leq \rho(s) + \rho^*(t).$$

If  $\Delta_2(\rho, \rho^*) < \infty$ , then additionally for all  $\delta > 0$

$$\begin{aligned} st &\leq \delta \rho(s) + c_\delta \rho^*(t) \text{ and } st \leq c_\delta \rho(s) + \delta \rho^*(t), \\ \rho'(s)t &\leq \delta \rho(s) + c_\delta \rho(t) \text{ and } \rho'(s)t \leq \delta \rho(t) + c_\delta \rho(s), \end{aligned}$$

where  $c_\delta = c(\delta, \Delta_2(\{\rho, \rho^*\}))$ .

**Definition A.2.** Let  $\rho$  be an N-function. We say that  $\rho$  is elliptic, if  $\rho$  is  $C^1$  on  $[0, \infty)$  and  $C^2$  on  $(0, \infty)$  and assume that

$$(A.1) \quad \rho'(t) \sim t \rho''(t)$$

uniformly in  $t > 0$ . The constants hidden in  $\sim$  are called the characteristics of  $\rho$ .

Note that (A.1) is stronger than  $\Delta_2(\rho, \rho^*) < \infty$ . In fact, the  $\Delta_2$ -constants can be estimated in terms of the characteristics of  $\rho$ .

Associated to an elliptic N-function  $\rho$  we define the tensors

$$\mathbf{A}^\rho(\xi) := \frac{\rho'(|\xi|)}{|\xi|} \xi, \quad \xi \in \mathbb{R}^{N \times n} \text{ and } \mathbf{F}^\rho(\xi) := \sqrt{\frac{\rho'(|\xi|)}{|\xi|}} \xi, \quad \xi \in \mathbb{R}^{N \times n}.$$

We define the shifted N-function  $\rho_a$  for  $a \geq 0$  by

$$(A.2) \quad \rho_a(t) := \int_0^t \frac{\rho'(a + \tau)}{a + \tau} \tau \, d\tau.$$

The following auxiliary result can be found in [DE08, DK08, DKS12, RD07] and [BDK12, Lemma 11].

**Lemma A.3.** For all  $a, b, t \geq 0$  we have

$$\begin{aligned} \rho_a(t) &\sim \begin{cases} \rho''(a)t^2 & \text{if } t \lesssim a \\ \rho(t) & \text{if } t \gtrsim a, \end{cases} \\ (\rho_a)_b(t) &\sim \rho_{a+b}(t). \end{aligned}$$

**Lemma A.4** ([DE08, Lemma 2.3]). We have

$$\begin{aligned} (\mathbf{A}^\rho(\mathbf{P}) - \mathbf{A}^\rho(\mathbf{Q})) \cdot (\mathbf{P} - \mathbf{Q}) &\sim |\mathbf{F}^\rho(\mathbf{P}) - \mathbf{F}^\rho(\mathbf{Q})|^2 \\ &\sim \rho_{|\mathbf{P}|}(|\mathbf{P} - \mathbf{Q}|) \\ &\sim \rho''(|\mathbf{P}| + |\mathbf{Q}|) |\mathbf{P} - \mathbf{Q}|^2 \end{aligned}$$

uniformly in  $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{n \times N}$ . Moreover, uniformly in  $\mathbf{Q} \in \mathbb{R}^{n \times N}$ ,

$$\begin{aligned} \mathbf{A}^\rho(\mathbf{Q}) \cdot \mathbf{Q} &\sim |\mathbf{F}^\rho(\mathbf{Q})|^2 \sim \rho(|\mathbf{Q}|) \\ |\mathbf{A}^\rho(\mathbf{P}) - \mathbf{A}^\rho(\mathbf{Q})| &\sim (\rho_{|\mathbf{P}|})'(|\mathbf{P} - \mathbf{Q}|). \end{aligned}$$

The constants depend only on the characteristics of  $\rho$ .

**Lemma A.5** (Change of Shift). *Let  $\rho$  be an elliptic  $N$ -function. Then for each  $\delta > 0$  there exists  $C_\delta \geq 1$  (only depending on  $\delta$  and the characteristics of  $\rho$ ) such that*

$$\begin{aligned}\rho_{|\mathbf{a}|}(t) &\leq C_\delta \rho_{|\mathbf{b}|}(t) + \delta \rho_{|\mathbf{a}|}(|\mathbf{a} - \mathbf{b}|), \\ (\rho_{|\mathbf{a}|})^*(t) &\leq C_\delta (\rho_{|\mathbf{b}|})^*(t) + \delta \rho_{|\mathbf{a}|}(|\mathbf{a} - \mathbf{b}|),\end{aligned}$$

for all  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$  and  $t \geq 0$ .

The case  $\mathbf{a} = 0$  or  $\mathbf{b} = 0$  implies the following corollary.

**Corollary A.6** (Removal of Shift). *Let  $\rho$  be an elliptic  $N$ -function. Then for each  $\delta > 0$  there exists  $C_\delta \geq 1$  (only depending on  $\delta$  and the characteristics of  $\rho$ ) such that*

$$\begin{aligned}\rho_{|\mathbf{a}|}(t) &\leq C_\delta \rho(t) + \delta \rho(|\mathbf{a}|), \\ \rho(t) &\leq C_\delta \rho_{|\mathbf{a}|}(t) + \delta \rho(|\mathbf{a}|),\end{aligned}$$

for all  $\mathbf{a} \in \mathbb{R}^d$  and  $t \geq 0$ .

**Lemma A.7.** *Let  $\rho$  be an elliptic  $N$ -function. Then  $(\rho_a)^*(t) \sim (\rho^*)_{\rho'(a)}(t)$  uniformly in  $a, t \geq 0$ . Moreover, for all  $\lambda \in [0, 1]$  we have*

$$\rho_a(\lambda a) \sim \lambda^2 \rho(a) \sim (\rho_a)^*(\lambda \rho'(a)).$$

**Lemma A.8.** *Let  $\rho(t) := \int_0^t (\kappa + s)^{q-2} s \, ds$  with  $q \in (1, \infty)$  and  $t \geq 0$ . Then*

$$\begin{aligned}\rho_a(\lambda t) &\leq c \max\{\lambda^q, \lambda^2\} \rho(t), \\ (\rho_a)^*(\lambda t) &\leq c \max\{\lambda^{q'}, \lambda^2\} \rho(t)\end{aligned}$$

uniformly in  $a, \lambda \geq 0$ .

**Remark A.9.** *Let  $p \in \mathcal{P}(\Omega)$  with  $p^- > 1$  and  $p^+ < \infty$ . The results above extend to the function  $\varphi(x, t) = \int_0^t (\kappa + s)^{p(x)-2} s \, ds$  uniformly in  $x \in \Omega$ , where the constants only depend on  $p^-$  and  $p^+$ .*

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