On the Robust Optimal Stopping Problem *†

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Abstract

We study a robust optimal stopping problem with respect to a set $\mathcal{P}$ of mutually singular probabilities. This can be interpreted as a zero-sum controller-stopper game in which the stopper is trying to maximize its pay-off while an adverse player wants to minimize this payoff by choosing an evaluation criteria from $\mathcal{P}$. We show that the *upper Snell envelope* $\overline{Z}$ of the reward process Y is a supermartingale with respect to an appropriately defined nonlinear expectation $\underline{\mathscr{E}}$, and $\overline{Z}$ is further an $\underline{\mathscr{E}}-$martingale up to the first time τ^* when $\overline{Z}$ meets Y. Consequently, τ^* is the optimal stopping time for the robust optimal stopping problem and the corresponding zero-sum game has a value. Although the result seems similar to the one obtained in the classical optimal stopping theory, the mutual singularity of probabilities and the game aspect of the problem give rise to major technical hurdles, which we circumvent using some new methods.

Keywords: robust optimal stopping, zero-sum game of control and stopping, volatility uncertainty, dynamic programming principle, Snell envelope, nonlinear expectation, weak stability under pasting, path-dependent stochastic differential equations with controls.

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1 Introduction

We solve a continuous-time *robust* optimal stopping problem with respect to a non-dominated set $\mathcal{P}$ of mutually singular probabilities on the canonical space Ω of continuous paths. This optimal stopping problem can also be interpreted as a zero-sum controller-stopper game in which the stopper is trying to maximize its pay-off while an adverse player wants to minimize this payoff by choosing an evaluation criteria from $\mathcal{P}$. In our main result, Theorem 5.1, we construct an optimal stopping time and show that the corresponding game has a value. More precisely, we obtain that

$$\sup_{\tau\in\mathcal{T}}\inf_{\mathbb{P}\in\mathcal{P}}\mathbb{E}_{\mathbb{P}}\big[Y_\tau\big]=\inf_{\mathbb{P}\in\mathcal{P}}\mathbb{E}_{\mathbb{P}}\big[Y_{\tau^*}\big]=\inf_{\mathbb{P}\in\mathcal{P}}\sup_{\tau\in\mathcal{T}}\mathbb{E}_{\mathbb{P}}\big[Y_\tau\big]. \tag{1.1}$$

Here $\mathcal{T}$ denotes the set of all stopping times with respect to the natural filtration $\mathbf{F}$ of the canonical process B, Y is an $\mathbf{F}-$adapted RCLL (càdlàg) process satisfying an one-sided uniform continuity condition (see (3.1)), and τ^* is the first time Y meets its *upper Snell envelope* $\overline{Z}_t(\omega):=\inf\limits_{\mathbb{P}\in\mathcal{P}(t,\omega)}\sup\limits_{\tau\in\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}\big[Y^{t,\omega}_\tau\big]$, $(t,\omega)\in[0,T]\times\Omega$. (Please refer to Section 2 for the definition of the shifted process $Y^{t,\omega}$.)

The proof of this result turns out to be quite technical for three reasons. First, since the probability set $\mathcal{P}$ does not admit a dominating probability, there is no dominated convergence theorem for the nonlinear expectation $\underline{\mathscr{E}}_t[\cdot](\omega):=\inf\limits_{\mathbb{P}\in\mathcal{P}(t,\omega)}\mathbb{E}_{\mathbb{P}}[\cdot]$, $(t,\omega)\in[0,T]\times\Omega$. So we can not follow techniques similar to the ones used in the classical theory of optimal stopping due to El Karoui [14] to obtain the martingale property of the upper Snell envelope $\overline{Z}$. Second, we do not have a measurable selection theorem for stopping strategies, which complicates the proof of the dynamic programming principle. Moreover, the local approach that used comparison principle of viscosity solutions to show the existence of game value (see e.g. [15] and [1]) does not work for our path-dependent set-up.

In Theorem 5.1, we demonstrate that $\overline{Z}$ is an $\underline{\mathscr{E}}-$supermartingale, and an $\underline{\mathscr{E}}-$martingale up to τ^*, the first time $\overline{Z}$ meets Y, from which (1.1) immediately follows. To prove this theorem, we use a more global approach rather than the local approach. We start with a dynamic programming principle (DPP), see Proposition 4.1, whose "super-solution" part is technically difficult due to the lack of measurable selection for stopping times. We overcome this issue by using a countable dense subset of $\mathcal{T}^t$ to construct a suitable approximation. This dynamic programming result is used to show the continuity of the upper Snell envelope, which plays an important role in the main theorem as our results heavily rely on construction of approximating stopping times for τ^*. However the dynamic programming principle directly enters the proof of Theorem 5.1 to show the supermartingale property of $\overline{Z}$ only after we upgrade the super side of the DPP for random transit horizons in Proposition 4.3. We would like to emphasize that the submartingale property of the upper Snell envelope $\overline{Z}$ until τ^* does not directly follow from the dynamic programming principle. Instead, we build a delicate approximation scheme that involves carefully pasting probabilities and leveraging the martingale property of the single-probability Snell envelopes until they meet Y.

Let us say a few words about our assumptions. It should not come us a surprise that as a function of (t,ω), the probability set $\mathcal{P}(t,\omega)$ needs to be adapted. The most important assumption on the probability class

$$\{\mathcal{P}(t,\omega)\}_{(t,\omega)\in[0,T]\times\Omega}$$

is the weak stability under pasting, see (P2) in Section 3. It is hard to envision that a dynamic programming result could hold without a stability under *pasting* assumption. This assumption along with the aforementioned continuity assumption (3.1) on Y (the regularity assumptions on the reward are common and can be verified for example of pay-offs of all financial derivatives) allows us to construct approximate strategies for the controller by appropriately choosing its conditional distributions. Our stability assumption is weaker than its counterpart in Ekren, Touzi and Zhang [13]; see for example our Remark 3.4 for a further discussion. We show in Section 6 that this assumption (along with other assumptions we make on the probability class) are satisfied for some path-dependent SDEs with controls, which represents a large class of models on simultaneous drift and volatility uncertainty. (A stronger stability assumption as in [13] leads to results which is applicable only for volatility uncertainty.) We see Section 6 as one of the main contributions of our paper, which we dedicate almost half our paper to. Another assumption we make on the probability class is that the augmentation of the filtration generated by the canonical process with respect to each probability in the class is right-continuous. This is because, as mentioned above, we exploit the results from the classic optimal stopping theory on the martingale property of the Snell envelopes for a given probability. Again the example in Section 6 is shown to satisfy this assumption.

Relevant Literature. Since the seminal work [35], the martingale approach was extensively used in optimal stopping theory (see e.g. [26], [14], Appendix D of [20]) and has been applied to various problems stemming from mathematical finance, the most important example of which is the computation of the super hedging price of the American contingent claims [6, 17, 18, 22]. Optimal stopping under Knightian uncertainty/nonlinear expectations/risk measures or the closely related controller-stopper-games have attracted a lot of attention in the recent years: [23, 24, 16, 8, 9, 32, 2, 3, 4, 5, 7, 25]. In this literature, the set of probabilities is assumed to be dominated by a single probability or the controller is only allowed to influence the drift.

When the set of probabilities contain mutually singular probabilities or the controller can influence not only the drift but also the volatility, results are available only in some particular cases. Karazas and Sudderth [21] considered the controller-stopper-game in which the controller is allowed to control the volatility as well as the drift and resolved the saddle point problem for case of one-dimensional state variable using the characterization of the value function in terms of the scale function of the state variable. In the multi-dimensional case [1] showed the existence of the value of a game using a comparison principle for viscosity solutions.

Our technical set-up follows closely that of [13] which analyzed a control problem with discretionary stopping (i.e., $\sup\limits_{\tau\in\mathcal{T}}\sup\limits_{\mathbb{P}\in\mathcal{P}}\mathbb{E}_{\mathbb{P}}[Y_\tau]$) in a non-Markovian framework with mutually singular probability priors. (The solution of this problem was an important technical step in extending the notion of viscosity solutions to the fully nonlinear path-dependent PDEs in [11] and [12].) Nutz and Zhang [29] independently and around the same time addressed the problem we are considering by using a different (and an elegant) approach: They exploited the "tower property" of the nonlinear expectation $\underline{\mathscr{E}}$ developed in [28] to derive the $\underline{\mathscr{E}}$-martingale property of the *discrete time version* of the *lower Snell envelope* $\underline{Z}_t(\omega) := \sup\limits_{\tau\in\mathcal{T}^t}\inf\limits_{\mathbb{P}\in\mathcal{P}(t,\omega)}\mathbb{E}_{\mathbb{P}}\big[Y^{t,\omega}_\tau\big]$, $(t,\omega)\in[0,T]\times\Omega$. In contrast, we take an approach we consider to be very natural: We work with the upper Snell envelope and build our approximations *directly* in continuous time leveraging the known results from the classical optimal stopping theory. In their introduction, [29] states that they can not work on upper Snell envelope due to the measurability selection issue; see paragraph 3 on page 3 of their paper. Our paper overcomes this issue. A major benefit of our approach is that we do not have to assume that the reward process is bounded since we do not have to rely on the approximation from discrete to continuous time. Another benefit is the weaker continuity assumption we impose on the value function in the path; compare Assumptions 4.1 in our paper and Assumption 3.2 in [29]. The latter requires the value of any stopping strategy to be continuous with the same modulus of continuity, which is an assumption that is not easily verifiable. One strong suit of [29] is the saddle point analysis.

The rest of the paper is organized as follows: In Section 2 we will introduce notations and some preliminary results such as the regular conditional probability distribution. In Section 3, we set-up the stage for our main result by imposing some assumptions on the reward process and the classes of mutually singular probabilities. Then Section 4 studies properties of the upper Snell envelope of the reward process such as path regularity and dynamic programming principles. They are the essence to resolve our main result on the robust optimal stopping problem stated in Section 5. In Section 6, we give an example of path-dependent SDEs with controls that satisfies all our assumptions. The proofs of our results are deferred to Section 7, and the Appendix contains some technical lemmata needed for the proofs of the main results.

2 Notation and Preliminaries

Let $(\mathbb{M},\varrho_{\mathbb{M}})$ be a generic metric space and let $\mathscr{B}(\mathbb{M})$ be the Borel $\sigma-$field of $\mathbb{M}$. For any $x\in\mathbb{M}$ and $\delta>0$, $O_\delta(x) := \{x'\in\mathbb{M} : \varrho_{\mathbb{M}}(x,x')<\delta\}$ and $\overline{O}_\delta(x) := \{x'\in\mathbb{M} : \varrho_{\mathbb{M}}(x,x')\le\delta\}$ respectively denote the open and closed ball centered at x with radius δ. Fix $d\in\mathbb{N}$. Let $\mathcal{S}_d^{>0}$ stand for all $\mathbb{R}^{d\times d}-$valued positively definite matrices. We denote by $\mathscr{B}(\mathcal{S}_d^{>0})$ the Borel $\sigma-$field of $\mathcal{S}_d^{>0}$ under the relative Euclidean topology.

Given $0\le t\le T<\infty$, let $\Omega^{t,T} := \big\{\omega\in\mathbb{C}\big([t,T];\mathbb{R}^d\big) : \omega(t)=0\big\}$ be the canonical space over the period $[t,T]$, whose null path $\omega(\cdot)\equiv 0$ will be denoted by $\mathbf{0}^{t,T}$. For any $t\le s\le S\le T$, we introduce a semi-norm $\|\cdot\|_{s,S}$ on $\Omega^{t,T}$: $\|\omega\|_{s,S}:=\sup\limits_{r\in[s,S]}|\omega(r)|$, $\forall\,\omega\in\Omega^{t,T}$. In particular, $\|\cdot\|_{t,T}$ is a norm on $\Omega^{t,T}$, called uniform norm, under which $\Omega^{t,T}$ is a separable complete metric space. Also, the *truncation* mapping $\Pi^{t,T}_{s,S}$ from $\Omega^{t,T}$ to $\Omega^{s,S}$ is defined by

$$\big(\Pi^{t,T}_{s,S}(\omega)\big)(r):=\omega(r)-\omega(s),\quad \forall\,\omega\in\Omega^{t,T},\ \forall\,r\in[s,S].$$

The canonical process $B^{t,T}$ on $\Omega^{t,T}$ is a $d-$dimensional Brownian motion under the Wiener measure $\mathbb{P}^{t,T}_0$ on $\big(\Omega^{t,T},\mathscr{B}(\Omega^{t,T})\big)$. Let $\mathbf{F}^{t,T}=\Big\{\mathcal{F}^{t,T}_s:=\sigma\big(B^{t,T}_r;r\in[t,s]\big)\Big\}_{s\in[t,T]}$ be the natural filtration of $B^{t,T}$ and let $\mathcal{C}^{t,T}$ collect all *cylinder* sets in $\mathcal{F}^{t,T}_T$: $\mathcal{C}^{t,T}:=\Big\{\bigcap\limits_{i=1}^{m}\big(B^{t,T}_{t_i}\big)^{-1}(\mathcal{E}_i): m\in\mathbb{N},\ t<t_1<\cdots<t_m\le T,\ \{\mathcal{E}_i\}_{i=1}^m\subset\mathscr{B}(\mathbb{R}^d)\Big\}$. It is well-known that

$$\mathscr{B}(\Omega^{t,T})=\sigma(\mathcal{C}^{t,T})=\sigma\Big\{\big(B^{t,T}_r\big)^{-1}(\mathcal{E}):r\in[t,T],\mathcal{E}\in\mathscr{B}(\mathbb{R}^d)\Big\}=\mathcal{F}^{t,T}_T.$$

Let $\mathscr{P}^{t,T}$ denote the $\mathbf{F}^{t,T}-$progressively measurable $\sigma-$field of $[t,T]\times\Omega^{t,T}$ and let $\mathcal{T}^{t,T}$ collect all $\mathbf{F}^{t,T}-$stopping times. We set $\mathcal{T}^{t,T}_s:=\{\tau\in\mathcal{T}^{t,T}:\tau\ge s\}$ for each $s\in[t,T]$ and will use the convention $\inf\emptyset:=\infty$.

From now on, we shall fix a time horizon $T\in(0,\infty)$ and drop it from the above notations, i.e., $\big(\Omega^{t,T},\mathbf{0}^{t,T},\ B^{t,T},\mathbb{P}^{t,T}_0,\mathbf{F}^{t,T},\mathscr{P}^{t,T},\mathcal{T}^{t,T}_s\big)\longrightarrow\big(\Omega^t,\mathbf{0}^t,\ B^t,\mathbb{P}^t_0,\mathbf{F}^t,\mathscr{P}^t,\mathcal{T}^t_s\big)$. When $S=T$, $\Pi^{t,T}_{s,T}$ will be simply denoted by Π^t_s. For any $0\le t\le s\le T$, $\omega\in\Omega^t$ and $\delta>0$, define $O^s_\delta(\omega):=\big\{\omega'\in\Omega^t:\|\omega'-\omega\|_{t,s}<\delta\big\}$ $\big($In particular, $O^T_\delta(\omega)=O_\delta(\omega)=\big\{\omega'\in\Omega^t:\|\omega'-\omega\|_{t,T}<\delta\big\}\big)$. Since Ω^t is the set of $\mathbb{R}^d-$valued continuous functions on $[t,T]$ starting from 0,

$$\begin{aligned}O^s_\delta(\omega)&=\bigcup_{n\in\mathbb{N}}\big\{\omega'\in\Omega^t:\|\omega'-\omega\|_{t,s}\le\delta-\delta/n\big\}=\bigcup_{n\in\mathbb{N}}\bigcap_{r\in(t,s)\cap\mathbb{Q}}\big\{\omega'\in\Omega^t:|\omega'(r)-\omega(r)|\le\delta-\delta/n\big\}\\&=\bigcup_{n\in\mathbb{N}}\bigcap_{r\in(t,s)\cap\mathbb{Q}}\big\{\omega'\in\Omega^t:B^t_r(\omega')\in\overline{O}_{\delta-\delta/n}\big(\omega(r)\big)\big\}\in\mathcal{F}^t_s.\end{aligned}\tag{2.1}$$

We fix a countable dense subset $\big\{\widehat{\omega}^t_j\big\}_{j\in\mathbb{N}}$ of Ω^t under $\|\cdot\|_{t,T}$, and set $\Theta^t_s:=\big\{O^s_\delta(\widehat{\omega}^t_j):\delta\in\mathbb{Q}_+,\ j\in\mathbb{N}\big\}\subset\mathcal{F}^t_s$.

Given $t\in[0,T]$ and a probability $\mathbb{P}$ on $\big(\Omega^t,\mathscr{B}(\Omega^t)\big)=\big(\Omega^t,\mathcal{F}^t_T\big)$, let us set $\mathscr{N}^{\mathbb{P}}:=\big\{\mathcal{N}\subset\Omega^t:\mathcal{N}\subset A$ for some $A\in\mathcal{F}^t_T$ with $\mathbb{P}(A)=0\big\}$. The $\mathbb{P}-$augmentation $\mathbf{F}^{\mathbb{P}}$ of $\mathbf{F}^t$ consists of $\mathcal{F}^{\mathbb{P}}_s:=\sigma\big(\mathcal{F}^t_s\cup\mathscr{N}^{\mathbb{P}}\big)$, $s\in[t,T]$. We denote by $\mathcal{T}^{\mathbb{P}}$ the collection of all $\mathbf{F}^{\mathbb{P}}-$stopping times and set $\mathcal{T}^{\mathbb{P}}_s:=\{\tau\in\mathcal{T}^{\mathbb{P}}:\tau\ge s\}$ for each $s\in[t,T]$. In particular, we will write $\big(\overline{\mathscr{N}}^t,\overline{\mathcal{T}}^t,\overline{\mathcal{T}}^t_s\big)$ for $\big(\mathscr{N}^{\mathbb{P}^t_0},\mathcal{T}^{\mathbb{P}^t_0},\mathcal{T}^{\mathbb{P}^t_0}_s\big)$ and $\overline{\mathbf{F}}^t=\big\{\overline{\mathcal{F}}^t_s\big\}_{s\in[t,T]}$ for $\mathbf{F}^{\mathbb{P}^t_0}=\Big\{\mathcal{F}^{\mathbb{P}^t_0}_s\Big\}_{s\in[t,T]}$.

The completion of $\big(\Omega^t,\mathcal{F}^t_T,\mathbb{P}\big)$ is the probability space $\big(\Omega^t,\mathcal{F}^{\mathbb{P}}_T,\overline{\mathbb{P}}\big)$ with $\overline{\mathbb{P}}\big|_{\mathcal{F}^t_T}=\mathbb{P}$, we still write $\mathbb{P}$ for $\overline{\mathbb{P}}$ for convenience. In particular, the expectation on $\big(\Omega^t,\overline{\mathcal{F}}^t_T,\mathbb{P}^t_0\big)$ will be simply denoted by $\mathbb{E}_t$. A probability space $\big(\Omega^t,\mathcal{F}',\mathbb{P}'\big)$ is called an extension of $\big(\Omega^t,\mathcal{F}^t_T,\mathbb{P}\big)$ if $\mathcal{F}^t_T\subset\mathcal{F}'$ and $\mathbb{P}'\big|_{\mathcal{F}^t_T}=\mathbb{P}$.

For any metric space $\mathbb{M}$ and any $\mathbb{M}-$valued process $X=\{X_s\}_{s\in[t,T]}$, we set $\mathbf{F}^X=\Big\{\mathcal{F}^X_s:=\sigma\big(X_r;r\in[t,s]\big)\Big\}_{s\in[t,T]}$ as the natural filtration of X and let $\mathbf{F}^{X,\mathbb{P}}=\Big\{\mathcal{F}^{X,\mathbb{P}}_s:=\sigma\big(\mathcal{F}^X_s\cup\mathscr{N}^{\mathbb{P}}\big)\Big\}_{s\in[t,T]}$. $\big($In particular, $\mathbf{F}^{\mathbb{P}}=\mathbf{F}^{B^t,\mathbb{P}}$.$\big)$ If X is $\mathbf{F}^{\mathbb{P}}-$adapted, it holds for any $s\in[t,T]$ that $\mathcal{F}^X_s\subset\mathcal{F}^{\mathbb{P}}_s$ and thus $\mathcal{F}^{X,\mathbb{P}}_s\subset\mathcal{F}^{\mathbb{P}}_s$.

The following spaces about $\mathbb{P}$ will be frequently used in the sequel.

1) For any sub$-\sigma-$field $\mathcal{G}$ of $\mathcal{F}^t_T$, let $L^1(\mathcal{G},\mathbb{P})$ be the space of all real-valued, $\mathcal{G}-$measurable random variables ξ with $\|\xi\|_{L^1(\mathcal{G},\mathbb{P})}:=\mathbb{E}_{\mathbb{P}}\big[|\xi|\big]<\infty$.

2) Let $\mathbb{D}(\mathbf{F}^t,\mathbb{P})$ be the space of all real$-$valued, $\mathbf{F}^t-$adapted processes $\{X_s\}_{s\in[t,T]}$ whose paths are all right-continuous and satisfy $\mathbb{E}_{\mathbb{P}}[X_*]<\infty$, where $X_*:=\|X\|_{t,T}=\sup\limits_{s\in[t,T]}|X_s|$.

If the superscript $t=0$, we will drop them from the above notations. For example, $\mathbf{0}=\mathbf{0}^{0,T}$ and $\mathcal{T}=\mathcal{T}^{0,T}$.

2.1 Concatenation of Sample Paths

In the rest of this section, let us fix $0\le t\le s\le T$. We concatenate an $\omega\in\Omega^t$ and an $\widetilde{\omega}\in\Omega^s$ at time s by:

$$\big(\omega\otimes_s\widetilde{\omega}\big)(r):=\omega(r)\,\mathbf{1}_{\{r\in[t,s)\}}+\big(\omega(s)+\widetilde{\omega}(r)\big)\,\mathbf{1}_{\{r\in[s,T]\}},\quad\forall\,r\in[t,T],$$

which is still of Ω^t. For any non-empty $\widetilde{A}\subset\Omega^s$, we set $\omega\otimes_s\emptyset=\emptyset$ and $\omega\otimes_s\widetilde{A}:=\big\{\omega\otimes_s\widetilde{\omega}:\widetilde{\omega}\in\widetilde{A}\big\}$.

The next result shows that $A\in\mathcal{F}^t_s$ consists of elements $\omega\otimes_s\Omega^s$ with $\omega\in A$.

Lemma 2.1. *Let $A \in \mathcal{F}^t_s$. If $\omega \in A$, then $\omega \otimes_s \Omega^s \subset A$. Otherwise, if $\omega \notin A$, then $\omega \otimes_s \Omega^s \subset A^c$.*

For any $\mathcal{F}^t_s-$measurable random variable η, since $\{\omega' \in \Omega^t : \eta(\omega') = \eta(\omega)\} \in \mathcal{F}^t_s$, Lemma 2.1 shows that

$$\omega \otimes_s \Omega^s \subset \{\omega' \in \Omega^t : \eta(\omega') = \eta(\omega)\} \quad \text{i.e.,} \quad \eta(\omega \otimes_s \widetilde{\omega}) = \eta(\omega), \quad \forall\, \widetilde{\omega} \in \Omega^s. \tag{2.2}$$

To wit, the value $\eta(\omega)$ depends only on $\omega|_{[t,s]}$.

On the other hand, for any $A \subset \Omega^t$ we set $A^{s,\omega} := \{\widetilde{\omega} \in \Omega^s : \omega \otimes_s \widetilde{\omega} \in A\}$ as the projection of A on Ω^s along ω. In particular, $\emptyset^{s,\omega} = \emptyset$.

For any $r \in [s,T]$, the operation $(\ )^{s,\omega}$ projects an $\mathcal{F}^t_r-$measurable set to an $\mathcal{F}^s_r-$measurable set while the operation $\omega \otimes_s \cdot$ takes an $\mathcal{F}^s_r-$measurable set as input and returns an $\mathcal{F}^t_r-$measurable set:

Lemma 2.2. *Given $\omega \in \Omega^t$ and $r \in [s,T]$, we have $A^{s,\omega} \in \mathcal{F}^s_r$ for any $A \in \mathcal{F}^t_r$, and $\omega \otimes_s \widetilde{A} \in \mathcal{F}^t_r$ for any $\widetilde{A} \in \mathcal{F}^s_r$.*

Corollary 2.1. *Given $\tau \in \mathcal{T}^t$ and $\omega \in \Omega^t$, if $\tau(\omega \otimes_s \Omega^s) \subset [r,T]$ for some $r \in [s,T]$, then $\tau^{s,\omega} \in \mathcal{T}^s_r$.*

For any $\mathcal{D} \subset [t,T] \times \Omega^t$, we accordingly set $\mathcal{D}^{s,\omega} := \big\{(r,\widetilde{\omega}) \in [s,T] \times \Omega^s : \big(r, \omega \otimes_s \widetilde{\omega}\big) \in \mathcal{D}\big\}$.

Lemma 2.3. *Given $\omega \in \Omega^t$ and $T_0 \in [s,T]$, we have $\mathcal{D}^{s,\omega} \in \mathscr{B}\big([s,T_0]\big) \otimes \mathcal{F}^s_{T_0}$ for any $\mathcal{D} \in \mathscr{B}([t,T_0]) \otimes \mathcal{F}^t_{T_0}$.*

2.2 Regular Conditional Probability Distributions

Let $\mathbb{P}$ be a probability on $\big(\Omega^t, \mathcal{F}^t_T\big)$. In virtue of Theorem 1.3.4 and (1.3.15) of [37], there exists a family $\{\mathbb{P}^\omega_s\}_{\omega \in \Omega^t}$ of probabilities on $\big(\Omega^t, \mathcal{F}^t_T\big)$, called the *regular conditional probability distribution* (r.c.p.d.) of $\mathbb{P}$ with respect to $\mathcal{F}^t_s$, such that

(i) For any $A \in \mathcal{F}^t_T$, the mapping $\omega \to \mathbb{P}^\omega_s(A)$ is $\mathcal{F}^t_s-$measurable;

(ii) For any $\xi \in L^1\big(\mathcal{F}^t_T, \mathbb{P}\big)$, $\mathbb{E}_{\mathbb{P}^\omega_s}[\xi] = \mathbb{E}_{\mathbb{P}}\big[\xi\big|\mathcal{F}^t_s\big](\omega)$ for $\mathbb{P}-$a.s. $\omega \in \Omega^t$; (2.3)

(iii) For any $\omega \in \Omega^t$, $\mathbb{P}^\omega_s\big(\omega \otimes_s \Omega^s\big) = 1$. (2.4)

Given $\omega \in \Omega^t$, by Lemma 2.2, $\omega \otimes_s \widetilde{A} \in \mathcal{F}^t_T$ for any $\widetilde{A} \in \mathcal{F}^s_T$. So we can deduce from (2.4) that

$$\mathbb{P}^{s,\omega}\big(\widetilde{A}\big) := \mathbb{P}^\omega_s\big(\omega \otimes_s \widetilde{A}\big), \quad \forall\, \widetilde{A} \in \mathcal{F}^s_T \tag{2.5}$$

defines a probability on $\big(\Omega^s, \mathcal{F}^s_T\big)$. The Wiener measures, however, are invariant under path shift:

Lemma 2.4. *Let $0 \le t \le s \le T$. It holds for $\mathbb{P}^t_0-$a.s. $\omega \in \Omega^t$ that $\big(\mathbb{P}^t_0\big)^{s,\omega} = \mathbb{P}^s_0$.*

Thanks to the existence of r.c.p.d. we can define conditional distributions using (2.5). Then by introducing path regularity for the reward process Y, one can treat *path-dependent* problems in ways similar to *state-dependent* problems. This can be seen as the general idea behind a dynamic programming in the path-dependent setting and the path-dependent PDEs introduced in [10].

2.3 Shifted Random Variables and Shifted Processes

Given a random variable ξ and a process $X = \{X_r\}_{r \in [t,T]}$ on Ω^t, for any $\omega \in \Omega^t$ we define the shifted random variable $\xi^{s,\omega}$ by $\xi^{s,\omega}(\widetilde{\omega}) := \xi(\omega \otimes_s \widetilde{\omega})$, $\forall\, \widetilde{\omega} \in \Omega^s$ and the shifted process $X^{s,\omega}$ by $X^{s,\omega}_r(\widetilde{\omega}) = X(r, \omega \otimes_s \widetilde{\omega})$, $(r,\widetilde{\omega}) \in [s,T] \times \Omega^s$.

In light of Lemma 2.2 and the regular conditional probability distribution, shifted random variables/processes "inherit" measurability and integrability as follows:

Proposition 2.1. *Let $\mathbb{M}$ be a generic metric space and let $\omega \in \Omega^t$.*
(1) If an $\mathbb{M}-$valued random variable ξ on Ω^t is $\mathcal{F}^t_r-$measurable for some $r \in [s,T]$, then $\xi^{s,\omega}$ is $\mathcal{F}^s_r-$measurable.
(2) If an $\mathbb{M}-$valued process $\{X_r\}_{r \in [t,T]}$ is $\mathbf{F}^t-$adapted (resp. $\mathbf{F}^t-$progressively measurable), then the shifted process $\big\{X^{s,\omega}_r\big\}_{r \in [s,T]}$ is $\mathbf{F}^s-$adapted (resp. $\mathbf{F}^s-$progressively measurable).
(3) For any $\mathcal{D} \in \mathscr{P}^t$, we have $\mathcal{D}^{s,\omega} \in \mathscr{P}^s$.

Proposition 2.2. *(1) If $\xi \in L^1\big(\mathcal{F}^t_T,\mathbb{P}\big)$ for some probability $\mathbb{P}$ on $\big(\Omega^t,\mathscr{B}(\Omega^t)\big)$, then it holds for $\mathbb{P}-$a.s. $\omega\in\Omega^t$ that the shifted random variable $\xi^{s,\omega}\in L^1\big(\mathcal{F}^s_T,\mathbb{P}^{s,\omega}\big)$ and*

$$\mathbb{E}_{\mathbb{P}^{s,\omega}}\big[\xi^{s,\omega}\big]=\mathbb{E}_{\mathbb{P}}\big[\xi\big|\mathcal{F}^t_s\big](\omega)\in\mathbb{R}. \tag{2.6}$$

(2) If $X\in\mathbb{D}\big(\mathbf{F}^t,\mathbb{P}\big)$ for some probability $\mathbb{P}$ on $\big(\Omega^t,\mathscr{B}(\Omega^t)\big)$, then it holds for $\mathbb{P}-$a.s. $\omega\in\Omega^t$ that the shifted process $X^{s,\omega}\in\mathbb{D}\big(\mathbf{F}^s,\mathbb{P}^{s,\omega}\big)$.

As a consequence of (2.6), a shifted $\mathbb{P}-$null set also has zero measure.

Lemma 2.5. *(1) Let $\mathbb{P}$ be a probability on $\big(\Omega^t,\mathscr{B}(\Omega^t)\big)$. For any $\mathcal{N}\in\mathscr{N}^{\mathbb{P}}$, it holds for $\mathbb{P}-$a.s. $\omega\in\Omega^t$ that $\mathcal{N}^{s,\omega}\in\mathscr{N}^{\mathbb{P}^{s,\omega}}$. In particular, for any $\mathcal{N}\in\overline{\mathscr{N}}^t$, it holds for $\mathbb{P}^t_0-$a.s. $\omega\in\Omega^t$ that $\mathcal{N}^{s,\omega}\in\overline{\mathscr{N}}^s$.*

(2) For any $\mathcal{D}\in\mathscr{B}\big([t,T]\big)\otimes\mathcal{F}^t_T$ with $(dr\times d\mathbb{P}^t_0)\big(\mathcal{D}\cap([s,T]\times\Omega^t)\big)=0$, it holds for $\mathbb{P}^t_0-$a.s. $\omega\in\Omega^t$ that $\big(dr\times d\mathbb{P}^s_0\big)\big(\mathcal{D}^{s,\omega}\big)=0$.

(3) For any $\tau\in\overline{\mathcal{T}}^t_s$, it holds for $\mathbb{P}^t_0-$a.s. $\omega\in\Omega^t$ that $\tau^{s,\omega}\in\overline{\mathcal{T}}^s$.

Based on Lemma 2.5 (1), we have the following extension of Proposition 2.2 (1).

Proposition 2.3. *Let $\mathbb{P}$ be a probability on $\big(\Omega^t,\mathscr{B}(\Omega^t)\big)$. For any $\xi\in L^1\big(\mathcal{F}^{\mathbb{P}}_T,\mathbb{P}\big)$, it holds for $\mathbb{P}-$a.s. $\omega\in\Omega^t$ that the shifted random variable $\xi^{s,\omega}\in L^1\big(\mathcal{F}^{\mathbb{P}^{s,\omega}}_T,\mathbb{P}^{s,\omega}\big)$ and (2.6) holds.*

In the next three sections, we will gradually provide the technical set-up and preparation for our main result (Theorem 5.1) on the robust optimal stopping problem.

3 Weak Stability under Pasting

In the proof of Theorem 5.1, we will use an approximation scheme which exploits results from the classic optimal stopping theory for a given probability. For this purpose, we consider the following probability set.

Definition 3.1. *For any $t\in[0,T]$, let $\mathfrak{P}_t$ collect all probabilities $\mathbb{P}$ on $\big(\Omega^t,\mathscr{B}(\Omega^t)\big)$ such that $\mathbf{F}^{\mathbb{P}}$ is right-continuous.*

We will also need some regularity assumption on the reward process.

Standing assumptions on reward process Y.

($\mathbf{Y}$) Y is an $\mathbf{F}-$adapted process that satisfies an one-sided continuity condition in (t,ω) with respect to some modulus of continuity function ρ_0 in the following sense

$$Y_{t_1}(\omega_1)-Y_{t_2}(\omega_2)\le\rho_0\Big(\mathbf{d}_\infty\big((t_1,\omega_1),(t_2,\omega_2)\big)\Big),\quad\forall\,0\le t_1\le t_2\le T,\ \forall\,\omega_1,\omega_2\in\Omega, \tag{3.1}$$

where $\mathbf{d}_\infty\big((t_1,\omega_1),(t_2,\omega_2)\big):=(t_2-t_1)+\|\omega_1(\cdot\wedge t_1)-\omega_2(\cdot\wedge t_2)\|_{0,T}$.

Remark 3.1. *(1) As pointed out in Remark 3.2 of [13],* (3.1) *implies that each path of Y is RCLL with positive jumps. (2) Also, one can deduce from* (3.1) *that the process Y is left upper semi-continuous (left u.s.c.): i.e., for any $(t,\omega)\in(0,T]\times\Omega$, $Y_t(\omega)\ge\varlimsup_{s\nearrow t}Y_s(\omega)$. It follows that the shifted process $Y^{t,\omega}$ is also left u.s.c. Then we can apply the classical optimal stopping theory to $Y^{t,\omega}$ under each $\mathbb{P}\in\mathfrak{P}_t$. Actually, the proof of Theorem 5.1 relies on the comparison of $\overline{Z}^{t,\omega}$ with the Snell envelope of $Y^{t,\omega}$ under each $\mathbb{P}\in\mathfrak{P}_t$.*

The next result show that the integrability of the shifted reward process is independent of the given path history:

Lemma 3.1. *Assume (Y). For any $t\in[0,T]$ and any probability $\mathbb{P}$ on $\big(\Omega^t,\mathscr{B}(\Omega^t)\big)$, if $Y^{t,\omega}\in\mathbb{D}(\mathbf{F}^t,\mathbb{P})$ for some $\omega\in\Omega$, then $Y^{t,\omega'}\in\mathbb{D}(\mathbf{F}^t,\mathbb{P})$ for all $\omega'\in\Omega$.*

We shall focus on the following subset of $\mathfrak{P}_t$ that makes the shifted reward process integrable.

Assumption 3.1. *For any $t\in[0,T]$, the set $\mathfrak{P}^Y_t:=\big\{\mathbb{P}\in\mathfrak{P}_t:Y^{t,\mathbf{0}}\in\mathbb{D}(\mathbf{F}^t,\mathbb{P})\big\}$ is not empty.*

Remark 3.2. *(1) If $Y \in \mathbb{D}(\mathbf{F},\mathbb{P}_0)$, then $\mathbb{P}_0^t \in \mathfrak{P}_t^Y$ for any $t\in[0,T]$. (2) As we will see in Lemma 6.2, when the modulus of continuity ρ_0 has polynomial growth, the laws of solutions to the controlled SDEs* (6.2) *over period $[t,T]$ belong to $\mathfrak{P}_t^Y$.*

Under (Y) and Assumption 3.1, we see from Lemma 3.1 that for any $t\in[0,T]$ and $\mathbb{P}\in\mathfrak{P}_t^Y$,

$$Y^{t,\omega}\in\mathbb{D}\big(\mathbf{F}^t,\mathbb{P}\big), \quad \forall\,\omega\in\Omega. \tag{3.2}$$

Next, we need the probability classes to be adapted and weakly stable under pasting in the following sense:

Standing assumptions on probability class.

(**P0**) For any $t\in[0,T]$, let us consider a family $\{\mathcal{P}(t,\omega)=\mathcal{P}_Y(t,\omega)\}_{\omega\in\Omega}$ of subsets of $\mathfrak{P}_t^Y$ such that

$$\mathcal{P}(t,\omega_1)=\mathcal{P}(t,\omega_2) \;\text{ if }\; \omega_1|_{[0,t]}=\omega_2|_{[0,t]}. \tag{3.3}$$

We further assume that the probability class $\{\mathcal{P}(t,\omega)\}_{(t,\omega)\in[0,T]\times\Omega}$ satisfy the following two conditions for some modulus of continuity function $\widehat{\rho}_0$: for any $0\le t<s\le T$, $\omega\in\Omega$ and $\mathbb{P}\in\mathcal{P}(t,\omega)$

(**P1**) There exist an extension $(\Omega^t,\mathcal{F}',\mathbb{P}')$ of $(\Omega^t,\mathcal{F}_T^t,\mathbb{P})$ and $\Omega'\in\mathcal{F}'$ with $\mathbb{P}'(\Omega')=1$ such that for any $\widetilde{\omega}\in\Omega'$, $\mathbb{P}^{s,\widetilde{\omega}}\in\mathcal{P}(s,\omega\otimes_t\widetilde{\omega})$;

(**P2**) For any $\delta\in\mathbb{Q}_+$ and $\lambda\in\mathbb{N}$, let $\{\mathcal{A}_j\}_{j=0}^{\lambda}$ be a $\mathcal{F}_s^t-$partition of Ω^t such that for $j=1,\cdots,\lambda$, $\mathcal{A}_j\subset O_{\delta_j}^s(\widetilde{\omega}_j)$ for some $\delta_j\in\big((0,\delta]\cap\mathbb{Q}\big)\cup\{\delta\}$ and $\widetilde{\omega}_j\in\Omega^t$. Then for any $\mathbb{P}_j\in\mathcal{P}(s,\omega\otimes_t\widetilde{\omega}_j)$, $j=1,\cdots,\lambda$, there exists a $\widehat{\mathbb{P}}\in\mathcal{P}(t,\omega)$ such that

(i) $\widehat{\mathbb{P}}(A\cap\mathcal{A}_0)=\mathbb{P}(A\cap\mathcal{A}_0)$, $\forall\,A\in\mathcal{F}_T^t$;

(ii) For any $j=1,\cdots,\lambda$ and $A\in\mathcal{F}_s^t$, $\widehat{\mathbb{P}}(A\cap\mathcal{A}_j)=\mathbb{P}(A\cap\mathcal{A}_j)$ and

$$\sup_{\tau\in\mathcal{T}_s^t}\mathbb{E}_{\widehat{\mathbb{P}}}\big[\mathbf{1}_{A\cap\mathcal{A}_j}Y_\tau^{t,\omega}\big]\le\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\widetilde{\omega}\in A\cap\mathcal{A}_j\}}\Big(\sup_{\zeta\in\mathcal{T}^s}\mathbb{E}_{\mathbb{P}_j}\big[Y_\zeta^{s,\omega\otimes_t\widetilde{\omega}}\big]+\widehat{\rho}_0(\delta)\Big)\Big]. \tag{3.4}$$

From now on, when writing $Y_\tau^{t,\omega}$, we mean $(Y^{t,\omega})_\tau$ not $(Y_\tau)^{t,\omega}$.

Remark 3.3. *(1) By* (3.3)*, one can regard $\mathcal{P}(t,\omega)$ as a path-dependent subset of $\mathfrak{P}_t$. In particular, $\mathcal{P}:=\mathcal{P}(0,\mathbf{0})=\mathcal{P}(0,\omega)$, $\forall\,\omega\in\Omega$.*

(2) As we will show in Section 7, both sides of (3.4) *are finite. In particular, the expectation on right-hand-side is well-defined since the mapping $\widetilde{\omega}\to\sup_{\zeta\in\mathcal{T}^s}\mathbb{E}_{\widetilde{\mathbb{P}}}\big[Y_\zeta^{s,\omega\otimes_t\widetilde{\omega}}\big]$ is continuous under norm $\|\ \|_{t,T}$ for any $\widetilde{\mathbb{P}}\in\mathfrak{P}_s^Y$.*

(3) The condition (P2) can be viewed as a weak stability under pasting since it is implied by the stability under finite pasting $\big($see e.g. (4.18) of [36]$\big)$: for any $0\le t<s\le T$, $\omega\in\Omega$, $\mathbb{P}\in\mathcal{P}(t,\omega)$, $\delta\in\mathbb{Q}_+$ and $\lambda\in\mathbb{N}$, let $\{\mathcal{A}_j\}_{j=0}^{\lambda}$ be a $\mathcal{F}_s^t-$partition of Ω^t such that for $j=1,\cdots,\lambda$, $\mathcal{A}_j\subset O_{\delta_j}^s(\widetilde{\omega}_j)$ for some $\delta_j\in\big((0,\delta]\cap\mathbb{Q}\big)\cup\{\delta\}$ and $\widetilde{\omega}_j\in\Omega^t$. Then for any $\mathbb{P}_j\in\mathcal{P}(s,\omega\otimes_t\widetilde{\omega}_j)$, $j=1,\cdots,\lambda$, the probability defined by

$$\widehat{\mathbb{P}}(A)=\mathbb{P}(A\cap\mathcal{A}_0)+\sum_{j=1}^{\lambda}\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\widetilde{\omega}\in\mathcal{A}_j\}}\mathbb{P}_j\big(A^{s,\widetilde{\omega}}\big)\Big], \quad \forall\,A\in\mathcal{F}_T^t \tag{3.5}$$

is in $\mathcal{P}(t,\omega)$.

Remark 3.4. *The reason we assume (P2) rather than the stability of finite pasting* (3.5) *lies in the fact that the latter does not hold for our example of path-dependent SDEs with controls (Section 6) as pointed out in Remark 3.6 of [27], while the former is sufficient for our approximation methods in proving the main results.*

4 The Dynamic Programming Principle

The key to solving problem (1.1) is the following upper *Snell* envelope of the reward processes:

$$\overline{Z}_t(\omega):=\inf_{\mathbb{P}\in\mathcal{P}(t,\omega)}\sup_{\tau\in\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}\big[Y_\tau^{t,\omega}\big], \quad \forall\,(t,\omega)\in[0,T]\times\Omega. \tag{4.1}$$

In this section, we derive some basic properties of $\overline{Z}$ and the dynamic programming principles it satisfies. These results will provide an important technical step for the proof of Theorem 5.1. Let (Y), (P0), (P1), (P2) and Assumption 3.1 hold throughout the section.

Given $(t,\omega)\in[0,T]\times\Omega$, since Y_t is $\mathcal{F}_t-$measurable, (2.2) implies that $Y_t^{t,\omega}=Y_t(\omega)$. it then follows from (4.1) that

$$Y_t(\omega) = \inf_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}_\mathbb{P}\big[Y_t^{t,\omega}\big] \le \overline{Z}_t(\omega) \le \inf_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}_\mathbb{P}\big[Y_*^{t,\omega}\big] < \infty, \quad \forall\,(t,\omega)\in[0,T]\times\Omega. \tag{4.2}$$

We need two assumptions on $\overline{Z}$ before discussing its path regularity properties and dynamic programming principle.

Assumption 4.1. *There exists a modulus of continuity function $\rho_1 \ge \rho_0$ such that for any $t\in[0,T]$*

$$\big|\overline{Z}_t(\omega_1) - \overline{Z}_t(\omega_2)\big| \le \rho_1\big(\|\omega_1-\omega_2\|_{0,t}\big), \quad \forall\,\omega_1,\omega_2\in\Omega. \tag{4.3}$$

Remark 4.1. *If $\mathcal{P}(t,\omega)$ does not depend on ω for all $t\in[0,T]$, then* (3.1) *implies Assumption 4.1.*

Remark 4.2. *Assumption 4.1 implies that $\overline{Z}$ is $\mathbf{F}-$adapted.*

Assumption 4.2. *For any $\alpha>0$, there exists a modulus of continuity function ρ_α such that for any $t\in[0,T)$*

$$\sup_{\omega\in O^t_\alpha(\mathbf{0})}\ \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}_\mathbb{P}\Big[\rho_1\Big(\delta + 2\sup_{r\in[t,(t+\delta)\wedge T]}|B^t_r|\Big)\Big] \le \rho_\alpha(\delta), \quad \forall\,\delta\in(0,T]. \tag{4.4}$$

Similar to (3.2), one has the following integrability result of shifted processes of $\overline{Z}$.

Lemma 4.1. *Given $(t,\omega)\in[0,T]\times\Omega$, it holds for any $\mathbb{P}\in\mathcal{P}(t,\omega)$ and $s\in[t,T]$ that $\mathbb{E}_\mathbb{P}\Big[\big|\overline{Z}^{t,\omega}_s\big|\Big]<\infty$.*

As to the dynamic programming principle, we present first a basic version in which the transit horizon is deterministic:

Proposition 4.1. *For any $0\le t\le s\le T$ and $\omega\in\Omega$,*

$$\overline{Z}_t(\omega) = \inf_{\mathbb{P}\in\mathcal{P}(t,\omega)}\ \sup_{\tau\in\mathcal{T}^t} \mathbb{E}_\mathbb{P}\Big[\mathbf{1}_{\{\tau<s\}}Y^{t,\omega}_\tau + \mathbf{1}_{\{\tau\ge s\}}\overline{Z}^{t,\omega}_s\Big]. \tag{4.5}$$

Consequently, all paths of $\overline{Z}$ are continuous:

Proposition 4.2. *For any $(t,\omega)\in[0,T]\times\Omega$ and $\mathbb{P}\in\mathcal{P}(t,\omega)$, $\overline{Z}^{t,\omega}$ is an $\mathbf{F}^t-$adapted process with all continuous paths and $\big\{\overline{Z}^{t,\omega}_\tau\big\}_{\tau\in\mathcal{T}^\mathbb{P}}$ is $\mathbb{P}-$uniformly integrable.*

The continuity of $\overline{Z}$ allows us to derive the super side of a general dynamic programming principle with random transit horizons.

Proposition 4.3. *For any $(t,\omega)\in[0,T]\times\Omega$ and $\nu\in\mathcal{T}^t$,*

$$\overline{Z}_t(\omega) \ge \inf_{\mathbb{P}\in\mathcal{P}(t,\omega)}\ \sup_{\tau\in\mathcal{T}^t} \mathbb{E}_\mathbb{P}\Big[\mathbf{1}_{\{\tau<\nu\}}Y^{t,\omega}_\tau + \mathbf{1}_{\{\tau\ge \nu\}}\overline{Z}^{t,\omega}_\nu\Big]. \tag{4.6}$$

5 Robust Optimal Stopping

In this section, we state our main result on robust optimal stopping problem. Let (Y), (P0), (P1), (P2) and Assumption 3.1−4.2 hold throughout the section.

For any $t\in[0,T]$, we set $\mathscr{L}_t := \{\text{random variable } \xi \text{ on } \Omega: \xi^{t,\omega}\in L^1(\mathcal{F}^t_T,\mathbb{P}),\ \forall\,\omega\in\Omega,\ \mathbb{P}\in\mathcal{P}(t,\omega)\}$ and define on $\mathscr{L}_t$ a nonlinear expectation: $\underline{\mathscr{E}}_t[\xi](\omega) := \inf_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}_\mathbb{P}[\xi^{t,\omega}]$, $\forall\,\omega\in\Omega,\ \xi\in\mathscr{L}_t$.

Remark 5.1. *Given $\tau\in\mathcal{T}$, $Y_\tau, \overline{Z}_\tau \in \mathscr{L}_t$ for any $t\in[0,T]$, thanks to* (3.2) *and Proposition 4.2.*

Similar to the classic optimal stopping theory, we will show that the first time $\overline{Z}$ meets Y

$$\tau^* := \inf\{t \in [0,T] : \overline{Z}_t = Y_t\} \tag{5.1}$$

is an optimal stopping time for (1.1), and the upper Snell envelope $\overline{Z}$ has a martingale characterization with respect to the nonlinear expectation $\underline{\mathscr{E}} := \{\underline{\mathscr{E}}_t\}_{t\in[0,T]}$:

Theorem 5.1. *Let (Y), $(P0)$, $(P1)$, $(P2)$ and Assumption 3.1-Assumption 4.2 hold. If $\sup\limits_{(t,\omega)\in[0,T]\times\Omega} Y_t(\omega) = \infty$, we further assume that for some $L>0$*

$$Y_{t_2}(\omega) - Y_{t_1}(\omega) \le L + \sup_{r\in[0,t_1]} |Y_r(\omega)| + \rho_1\Big(\sup_{r\in[t_1,t_2]} \big|\omega(r) - \omega(t_1)\big|\Big), \quad \forall\, 0 \le t_1 \le t_2 \le T,\ \ \forall\, \omega \in \Omega. \tag{5.2}$$

Then $\overline{Z}$ is an $\underline{\mathscr{E}}-$supermartingale and is even an $\underline{\mathscr{E}}-$martingale up to time τ^* in sense that

$$\big(\overline{Z}_{\gamma\wedge t}\big)(\omega) \ge \underline{\mathscr{E}}_t\big[\,\overline{Z}_\gamma\big](\omega) \quad \text{and} \quad \big(\overline{Z}_{\tau^*\wedge\gamma\wedge t}\big)(\omega) = \underline{\mathscr{E}}_t\big[\,\overline{Z}_{\tau^*\wedge\gamma}\big](\omega), \quad \forall\, (t,\omega) \in [0,T]\times\Omega,\ \ \forall\, \gamma \in \mathcal{T}. \tag{5.3}$$

In particular, the $\mathbf{F}-$stopping time τ^* satisfies (1.1).

A few remarks are in order:

Remark 5.2. (*1*) *Similar to [29], we can apply* (1.1) *to subhedging of American options in a financial market with volatility uncertainty.*
(*2*) *As to a worst-case risk measure $\mathfrak{R}(\xi) := \sup\limits_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_\mathbb{P}[-\xi]$ defined for any bounded financial position ξ, applying* (1.1) *to a given bounded reward process Y yields that*

$$\inf_{\tau\in\mathcal{T}} \mathfrak{R}(Y_\tau) = -\sup_{\tau\in\mathcal{T}} \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_\mathbb{P}\big[Y_\tau\big] = -\inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_\mathbb{P}\big[Y_{\tau^*}\big] = \mathfrak{R}\big(Y_{\tau^*}\big).$$

So τ^* is also an optimal stopping time for the optimal stopping problem of $\mathfrak{R}$.
(*3*) *From the perspective of a zero-sum controller-stopper game in which the stopper chooses the termination time while the controller selects the distribution law from $\mathcal{P}$,* (1.1) *shows that such a game has a value $\underline{\mathscr{E}}_0[Y_{\tau^*}] = \inf\limits_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_\mathbb{P}\big[Y_{\tau^*}\big]$ as its lower value $\sup\limits_{\tau\in\mathcal{S}} \inf\limits_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_\mathbb{P}\big[Y_\tau\big]$ coincides with the upper one $\inf\limits_{\mathbb{P}\in\mathcal{P}} \sup\limits_{\tau\in\mathcal{S}} \mathbb{E}_\mathbb{P}\big[Y_\tau\big]$.*

6 Example: Path-dependent Controlled SDEs

In this section we will present an example of the probability class $\{\mathcal{P}(t,\omega)\}_{(t,\omega)\in[0,T]\times\Omega}$ in case of path-dependent stochastic differential equations with controls.

Let $\kappa>0$ and let $b\colon [0,T]\times\Omega\times\mathbb{R}^{d\times d} \to \mathbb{R}^d$ be a $\mathscr{P}\otimes\mathscr{B}(\mathbb{R}^{d\times d})/\mathscr{B}(\mathbb{R}^d)-$measurable function such that

$$|b(t,\omega,u)-b(t,\omega',u)| \le \kappa\|\omega-\omega'\|_{0,t} \quad \text{and} \quad |b(t,\mathbf{0},u)| \le \kappa(1+|u|), \quad \forall\, \omega,\omega' \in \Omega,\ (t,u) \in [0,T]\times\mathbb{R}^{d\times d}. \tag{6.1}$$

Lemma 6.1. *Given $(t,\omega) \in [0,T]\times\Omega$, the mapping $b^{t,\omega}(r,\widetilde{\omega},u) := b(r,\omega\otimes_t\widetilde{\omega},u)$, $\forall\,(r,\widetilde{\omega},u) \in [t,T]\times\Omega^t\times\mathbb{R}^{d\times d}$ is $\mathscr{P}^t\otimes\mathscr{B}(\mathbb{R}^{d\times d})/\mathscr{B}(\mathbb{R}^d)-$measurable.*

Given $(t,\omega)\in[0,T]\times\Omega$, by (6.1) and Lemma 6.1, $b^{t,\omega}$ is a $\mathscr{P}^t\otimes\mathscr{B}(\mathbb{R}^{d\times d})/\mathscr{B}(\mathbb{R}^d)-$measurable function that satisfies

$$|b^{t,\omega}(r,\widetilde{\omega},u)-b^{t,\omega}(r,\widetilde{\omega}',u)| \le \kappa\|\widetilde{\omega}-\widetilde{\omega}'\|_{t,r} \text{ and } |b^{t,\omega}(r,\mathbf{0}^t,u)| \le \kappa\big(1+\|\omega\|_{0,t}+|u|\big),\ \ \forall\,\widetilde{\omega},\widetilde{\omega}' \in \Omega^t,\ (r,u) \in [t,T]\times\mathbb{R}^{d\times d}.$$

Let $t \in [0,T]$ and let $\mathcal{U}_t$ collect all $\mathcal{S}_d^{>0}-$valued, $\mathbf{F}^t-$progressively measurable processes $\{\mu_s\}_{s\in[t,T]}$ such that $|\mu_s| \le \kappa$, $ds\times d\mathbb{P}_0^t-$a.s. Given $\mu \in \mathcal{U}_t$, a slight extension of Theorem V.12.1 of [33] shows that the following stochastic differential equation (SDE) on the probability space $\big(\Omega^t,\mathcal{F}_T^t,\mathbb{P}_0^t\big)$:

$$X_s = \int_t^s b^{t,\omega}(r,X,\mu_r)dr + \int_t^s \mu_r\, dB_r^t, \quad s \in [t,T], \tag{6.2}$$

admits a unique solution $X^{t,\omega,\mu}$, which is an $\overline{\mathbf{F}}^t-$adapted continuous process satisfying $E_t\big[\big(X_*^{t,\omega,\mu}\big)^p\big]<\infty$ for any $p\ge 1$. Note that the SDE (6.2) depends on $\omega\big|_{[0,t]}$ via the generator $b^{t,\omega}$.

Without loss of generality, we may assume that all paths of $X^{t,\omega,\mu}$ are continuous and starting from 0. $\big($Otherwise, by setting $\mathcal{N} := \{\omega \in \Omega^t : X_t^{t,\omega,\mu}(\omega) \neq \mathbf{0}$ or the path $X_\cdot^{t,\omega,\mu}(\omega)$ is not continuous$\} \in \overline{\mathscr{N}}^t$, one can take $\widetilde{X}_s^{t,\omega,\mu} := \mathbf{1}_{\mathcal{N}^c} X_s^{t,\omega,\mu}$, $s\in[t,T]$. It is an $\overline{\mathbf{F}}^t-$adapted process that satisfies (6.2) and whose paths are all continuous and starting from 0.$\big)$

Applying the Burkholder-Davis-Gundy inequality, Gronwall's inequality and using the Lipschitz continuity of b in $\omega-$variable, one can easily derive the following estimates for $X^{t,\omega,\mu}$: for any $p \ge 1$

$$\mathbb{E}_t\bigg[\sup_{r\in[t,s]}\big|X_r^{t,\omega,\mu}-X_r^{t,\omega',\mu}\big|^p\bigg]\le C_p\|\omega-\omega'\|_{0,t}^p\,(s-t)^p, \quad \forall\,\omega'\in\Omega,\ \forall\, s\in[t,T], \tag{6.3}$$

and
$$\mathbb{E}_t\bigg[\sup_{r\in[\zeta,(\zeta+\delta)\wedge T]}\big|X_r^{t,\omega,\mu}-X_\zeta^{t,\omega,\mu}\big|^p\bigg]\le\varphi_p(\|\omega\|_{0,t})\,\delta^{\,p/2}, \quad \text{for any } \overline{\mathbf{F}}^t\text{−stopping time } \zeta \text{ and } \delta>0, \tag{6.4}$$

where C_p is a constant depending on p,κ,T and $\varphi_p:\mathbb{R}_+\to\mathbb{R}_+$ is a continuous function depending on p,κ,T.

Similar to Lemma 3.3 of [29], the following result shows that the shift of $X^{t,\omega,\mu}$ is exactly the solution of SDE (6.2) with shifted drift coefficient and shifted control.

Proposition 6.1. *Given $0\le t\le s\le T$, $\omega\in\Omega$ and $\mu\in\mathcal{U}_t$, let $\mathcal{X}:=X^{t,\omega,\mu}$. It holds for $\mathbb{P}_0^t-$a.s. $\widetilde{\omega}\in\Omega^t$ that $\mu^{s,\widetilde{\omega}}\in\mathcal{U}_s$ and that $\mathcal{X}^{s,\widetilde{\omega}}=X^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega}),\mu^{s,\widetilde{\omega}}}+\mathcal{X}_s(\widetilde{\omega})$.*

As a mapping from Ω^t to Ω^t, $X^{t,\omega,\mu}$ is $\overline{\mathcal{F}}_s^t/\mathcal{F}_s^t-$measurable for any $s\in[t,T]$: To see this, let us pick up an arbitrary $\mathcal{E}\in\mathscr{B}(\mathbb{R}^d)$. The $\overline{\mathbf{F}}^t-$adaptness of $X^{t,\omega,\mu}$ shows that for any $r\in[t,s]$

$$\big(X^{t,\omega,\mu}\big)^{-1}\Big(\big(B_r^t\big)^{-1}(\mathcal{E})\Big)=\big\{\widetilde{\omega}\in\Omega^t: X^{t,\omega,\mu}(\widetilde{\omega})\in\big(B_r^t\big)^{-1}(\mathcal{E})\big\}=\big\{\widetilde{\omega}\in\Omega^t: X_r^{t,\omega,\mu}(\widetilde{\omega})\in\mathcal{E}\big\}\in\overline{\mathcal{F}}_s^t. \tag{6.5}$$

Thus $\big(B_r^t\big)^{-1}(\mathcal{E})\in\mathcal{G}_s^{X^{t,\omega,\mu}}:=\Big\{A\subset\Omega^t:\big(X^{t,\omega,\mu}\big)^{-1}(A)\in\overline{\mathcal{F}}_s^t\Big\}$, which is clearly a $\sigma-$field of Ω^t. It follows that $\mathcal{F}_s^t\subset\mathcal{G}_s^{X^{t,\omega,\mu}}$, i.e.,

$$\big(X^{t,\omega,\mu}\big)^{-1}(A)\in\overline{\mathcal{F}}_s^t, \quad \forall\,A\in\mathcal{F}_s^t, \tag{6.6}$$

proving the measurability of the mapping $X^{t,\omega,\mu}$. We define the law of $X^{t,\omega,\mu}$ under $\mathbb{P}_0^t$ by

$$\mathfrak{p}^{t,\omega,\mu}(A):=\mathbb{P}_0^t\circ\big(X^{t,\omega,\mu}\big)^{-1}(A), \quad \forall\,A\in\mathcal{G}_T^{X^{t,\omega,\mu}},$$

and denote by $\mathbb{P}^{t,\omega,\mu}$ the restriction of $\mathfrak{p}^{t,\omega,\mu}$ on $\big(\Omega^t,\mathcal{F}_T^t\big)$.

The filtrations $\mathbf{F}^{\mathbb{P}^{t,\omega,\mu}}$ are all right-continuous:

Proposition 6.2. *For any $(t,\omega)\in[0,T]\times\Omega$ and $\mu\in\mathcal{U}_t$, $\mathbb{P}^{t,\omega,\mu}$ belongs to $\mathfrak{P}_t$.*

Remark 6.1. *The reason we consider the law of $X^{t,\omega,\mu}$ under $\mathbb{P}_0^t$ over $\mathcal{G}_T^{X^{t,\omega,\mu}}$ $\big($the largest $\sigma-$field to induce $\mathbb{P}_0^t$ under the mapping $X^{t,\omega,\mu}\big)$ rather than $\mathcal{F}_T^t$ is as follows. Our proofs for Proposition 6.2 and Proposition 6.3 rely heavily on the inverse mapping $W^{t,\omega,\mu}$ of $X^{t,\omega,\mu}$, which is an $\mathbf{F}^t-$progessively measurable processes having only $\mathfrak{p}^{t,\omega,\mu}-$a.s. continuous paths. Consequently, as we will show in the proof of the following Proposition 6.3, it holds for $\mathfrak{p}^{t,\omega,\mu}-$a.s. $\widetilde{\omega}\in\Omega^t$ that the shifted probability $\big(\mathbb{P}^{t,\omega,\mu}\big)^{s,\widetilde{\omega}}$ is the law of the solution to the shifted SDE and thus belongs to $\mathcal{P}(s,\omega\otimes_t\widetilde{\omega})$. This explains why our assumption (P1) needs an extension $(\Omega^t,\mathcal{F}',\mathbb{P}')$ of the probability space $(\Omega^t,\mathcal{F}_T^t,\mathbb{P})$.*

Now, we set $\mathcal{P}(t,\omega):=\big\{\mathbb{P}^{t,\omega,\mu}:\mu\in\mathcal{U}_t\big\}$. Given $\varpi\ge 1$, let ρ_0 be a modulus of continuity function such that

$$\rho_0(\delta)\le\kappa(1+\delta^{\varpi}), \quad \forall\,\delta>0, \tag{6.7}$$

and let Y satisfy (Y) with ρ_0.

Lemma 6.2. *Assume* (Y) *and* (6.7)*. For any* $(t,\omega)\in[0,T]\times\Omega$*, we have* $\mathcal{P}(t,\omega)\subset\mathfrak{P}_t^Y$*.*

For any $\omega_1,\omega_2\in\Omega$ with $\omega_1|_{[0,t]}=\omega_2|_{[0,t]}$, since (6.2) depends only on $\omega|_{[0,t]}$ for a given path $\omega\in\Omega$, we see that $X^{t,\omega_1,\mu}=X^{t,\omega_2,\mu}$ and thus $\mathbb{P}^{t,\omega_1,\mu}=\mathbb{P}^{t,\omega_2,\mu}$ for any $\mu\in\mathcal{U}_t$. It follows that $\mathcal{P}(t,\omega_1)=\mathcal{P}(t,\omega_2)$. So assumption (P0) is satisfied.

Proposition 6.3. *Assume* (Y) *and* (6.7)*. Then the probability class* $\{\mathcal{P}(t,\omega)\}_{(t,\omega)\in[0,T]\times\Omega}$ *satisfies (P1), (P2), Assumptions 4.1 and 4.2.*

7 Proofs

7.1 Proofs of the results in Section 2

Proof of Lemma 2.1: Set $\Lambda:=\Big\{A\subset\Omega^t: A=\underset{\omega\in A}{\cup}\big(\omega\otimes_s\Omega^s\big)\Big\}$. For any $A\in\Lambda$, we claim that

$$\omega\otimes_s\Omega^s\subset A^c \text{ for any } \omega\in A^c. \tag{7.1}$$

Assume not, there is an $\omega\in A^c$ and an $\widetilde{\omega}\in\Omega^s$ such that $\omega\otimes_s\widetilde{\omega}\in A$, thus $\big(\omega\otimes_s\widetilde{\omega}\big)\otimes_s\Omega^s\subset A$. Then $\omega\in\omega\otimes_s\Omega^s=\big(\omega\otimes_s\widetilde{\omega}\big)\otimes_s\Omega^s\subset A$. A contradiction appear.

For any $r\in[t,s]$ and $\mathcal{E}\in\mathscr{B}(\mathbb{R}^d)$, if $\omega\in\big(B_r^t\big)^{-1}\big(\mathcal{E}\big)$, then for any $\widetilde{\omega}\in\Omega^s$, $\big(\omega\otimes_s\widetilde{\omega}\big)(r)=\omega(r)\in\mathcal{E}$, i.e., $\omega\otimes_s\widetilde{\omega}\in\big(B_r^t\big)^{-1}\big(\mathcal{E}\big)$. Thus $\omega\otimes_s\Omega^s\subset\big(B_r^t\big)^{-1}\big(\mathcal{E}\big)$, which implies that $\big(B_r^t\big)^{-1}\big(\mathcal{E}\big)\in\Lambda$. In particular, $\emptyset\in\Lambda$ and $\Omega^t\in\Lambda$. For any $A\in\Lambda$, (7.1) implies that $A^c\in\Lambda$. For any $\{A_n\}_{n\in\mathbb{N}}\subset\Lambda$, $\underset{n\in\mathbb{N}}{\cup}A_n=\underset{n\in\mathbb{N}}{\cup}\Big(\underset{\omega\in A_n}{\cup}\big(\omega\otimes_s\Omega^s\big)\Big)=\underset{\omega\in\underset{n\in\mathbb{N}}{\cup}A_n}{\cup}\big(\omega\otimes_s\Omega^s\big)$, namely, $\underset{n\in\mathbb{N}}{\cup}A_n\in\Lambda$. Thus, Λ is a $\sigma-$field of Ω^t containing all generating sets of $\mathcal{F}_s^t$. It then follows that $\mathcal{F}_s^t\subset\Lambda$, proving the lemma. □

Proof of Lemma 2.2: If we regard $\omega\otimes_s\cdot$ as a mapping Ψ from Ω^s to Ω^t, i.e., $\Psi(\widetilde{\omega}):=\omega\otimes_s\widetilde{\omega}$, $\forall\,\widetilde{\omega}\in\Omega^s$, then $A^{s,\omega}=\Psi^{-1}(A)$ for any $A\subset\Omega^t$. Given $t'\in[t,r]$ and $\mathcal{E}\in\mathscr{B}(\mathbb{R}^d)$, we can deduce that

$$\Big(\big(B_{t'}^t\big)^{-1}(\mathcal{E})\Big)^{s,\omega}=\begin{cases}\Omega^s, & \text{if } t'\in[t,s) \text{ and } \omega(t')\in\mathcal{E};\\ \emptyset, & \text{if } t'\in[t,s) \text{ and } \omega(t')\notin\mathcal{E};\\ \big\{\widetilde{\omega}\in\Omega^s:\omega(s)+\widetilde{\omega}(t')\in\mathcal{E}\big\}=\big(B_{t'}^s\big)^{-1}(\mathcal{E}')\in\mathcal{F}_r^s, & \text{if } t'\in[s,r],\end{cases}$$

where $\mathcal{E}'=\{x-\omega(s):x\in\mathcal{E}\}\in\mathscr{B}(\mathbb{R}^d)$. So $\big(B_{t'}^t\big)^{-1}(\mathcal{E})\in\Lambda:=\Big\{A\subset\Omega^t: A^{s,\omega}=\Psi^{-1}(A)\in\mathcal{F}_r^s\Big\}$, which is clearly a $\sigma-$field of Ω^t. It follows that $\mathcal{F}_r^t\subset\Lambda$, i.e., $A^{s,\omega}\in\mathcal{F}_r^s$ for any $A\in\mathcal{F}_r^t$. On the other hand, the continuity of paths in Ω^t shows that

$$\omega\otimes_s\Omega^s=\Big\{\omega'\in\Omega^t:\omega'(t')=\omega(t'),\,\forall\,t'\in(t,s)\cap\mathbb{Q}\Big\}=\underset{t'\in(t,s)\cap\mathbb{Q}}{\cap}\big(B_{t'}^t\big)^{-1}\big(\omega(t')\big)\in\mathcal{F}_s^t. \tag{7.2}$$

For any $\widetilde{A}\in\mathcal{F}_r^s$, applying Lemma A.1 with $S=T$ gives that $(\Pi_s^t)^{-1}\big(\widetilde{A}\big)\in\mathcal{F}_r^t$, which together with (7.2) shows that $\omega\otimes_s\widetilde{A}=(\Pi_s^t)^{-1}\big(\widetilde{A}\big)\cap\big(\omega\otimes_s\Omega^s\big)\in\mathcal{F}_r^t$. □

Proof of Corollary 2.1: Let $\tau\in\mathcal{T}^t$, $\omega\in\Omega^t$ and assume that $\tau(\omega\otimes_s\Omega^s)\subset[r,T]$ for some $r\in[s,T]$. Given $\widetilde{r}\in[r,T]$, we set $A:=\{\omega'\in\Omega^t:\tau(\omega')\le\widetilde{r}\}\in\mathcal{F}_{\widetilde{r}}^t$ and can deduce from Lemma 2.2 that

$$\{\widetilde{\omega}\in\Omega^s:\tau^{s,\omega}(\widetilde{\omega})\le\widetilde{r}\}=\{\widetilde{\omega}\in\Omega^s:\tau(\omega\otimes_s\widetilde{\omega})\le\widetilde{r}\}=\{\widetilde{\omega}\in\Omega^s:\omega\otimes_s\widetilde{\omega}\in A\}=A^{s,\omega}\in\mathcal{F}_{\widetilde{r}}^s.$$

So $\tau^{s,\omega}\in\mathcal{T}_r^s$. □

Proof of Lemma 2.3: Define a mapping $\widetilde{\Psi}:[s,T_0]\times\Omega^s\to[s,T_0]\times\Omega^t$ by $\widetilde{\Psi}(r,\widetilde{\omega}):=\big(r,\omega\otimes_s\widetilde{\omega}\big)$, $\forall\,(r,\widetilde{\omega})\in[s,T_0]\times\Omega^s$. In particular, $\mathcal{D}^{s,\omega}=\widetilde{\Psi}^{-1}(\mathcal{D})$ for any $\mathcal{D}\subset[t,T_0]\times\Omega^t$. For any $\mathcal{E}\in\mathscr{B}\big([t,T_0]\big)$ and $A\in\mathcal{F}_{T_0}^t$, Lemma 2.2 shows that

$$\widetilde{\Psi}^{-1}\big(\mathcal{E}\times A\big)=\big\{(r,\widetilde{\omega})\in[s,T_0]\times\Omega^s:\big(r,\omega\otimes_s\widetilde{\omega}\big)\in\mathcal{E}\times A\big\}=\big(\mathcal{E}\cap[s,T_0]\big)\times A^{s,\omega}\in\mathscr{B}\big([s,T_0]\big)\otimes\mathcal{F}_{T_0}^s.$$

Hence, the rectangular measurable set $\mathcal{E}\times A\in\Lambda_{T_0}:=\big\{\mathcal{D}\subset[t,T_0]\times\Omega^t:\widetilde{\Psi}^{-1}(\mathcal{D})\in\mathscr{B}\big([s,T_0]\big)\otimes\mathcal{F}^s_{T_0}\big\}$, which is clearly a $\sigma-$field of $[t,T_0]\times\Omega^t$. It follows that $\mathscr{B}([t,T_0])\otimes\mathcal{F}^t_{T_0}\subset\Lambda_{T_0}$, i.e., $\mathcal{D}^{s,\omega}=\widetilde{\Psi}^{-1}(\mathcal{D})\in\mathscr{B}\big([s,T_0]\big)\otimes\mathcal{F}^s_{T_0}$ for any $\mathcal{D}\in\mathscr{B}([t,T_0])\otimes\mathcal{F}^t_{T_0}$. □

Proof of Lemma 2.4: Given $\widetilde{A}\in\mathcal{F}^s_T$, since $(\Pi^t_s)^{-1}(\widetilde{A})\in\mathcal{F}^t_T$ by Lemma A.1, (2.4) and (2.3) imply that for $\mathbb{P}^t_0-$a.s. $\omega\in\Omega^t$

$$\big(\mathbb{P}^t_0\big)^{s,\omega}\big(\widetilde{A}\big)=\big(\mathbb{P}^t_0\big)^{\omega}_s\big(\omega\otimes_s\widetilde{A}\big)=\big(\mathbb{P}^t_0\big)^{\omega}_s\big((\omega\otimes_s\Omega^s)\cap(\Pi^t_s)^{-1}(\widetilde{A})\big)=\big(\mathbb{P}^t_0\big)^{\omega}_s\big((\Pi^t_s)^{-1}(\widetilde{A})\big)=\mathbb{E}_t\Big[\mathbf{1}_{(\Pi^t_s)^{-1}(\widetilde{A})}\big|\mathcal{F}^t_s\Big](\omega).$$

It is easy to see that $(\Pi^t_s)^{-1}(\mathcal{F}^s_T)=\sigma\big(B^t_r-B^t_s;r\in[s,T]\big)$. Thus $(\Pi^t_s)^{-1}(\widetilde{A})$ is independent of $\mathcal{F}^t_s$ under $\mathbb{P}^t_0$. Applying (A.1) with $S=T$ yield that for $\mathbb{P}^t_0-$a.s. $\omega\in\Omega^t$,

$$\big(\mathbb{P}^t_0\big)^{s,\omega}\big(\widetilde{A}\big)=\mathbb{E}_t\Big[\mathbf{1}_{(\Pi^t_s)^{-1}(\widetilde{A})}\big|\mathcal{F}^t_s\Big](\omega)=\mathbb{E}_t\Big[\mathbf{1}_{(\Pi^t_s)^{-1}(\widetilde{A})}\Big]=\mathbb{P}^t_0\big((\Pi^t_s)^{-1}(\widetilde{A})\big)=\mathbb{P}^s_0(\widetilde{A}).$$

Since $\mathscr{C}^s_T$ is a countable set by Lemma A.2, we can find a $\mathcal{N}\in\overline{\mathscr{N}}^t$ such that for any $\omega\in\mathcal{N}^c$, $\big(\mathbb{P}^t_0\big)^{s,\omega}\big(\widetilde{A}\big)=\mathbb{P}^s_0(\widetilde{A})$ holds for each $\widetilde{A}\in\mathscr{C}^s_T$. To wit, $\mathscr{C}^s_T\subset\Lambda:=\Big\{\widetilde{A}\in\mathcal{F}^s_T:\big(\mathbb{P}^t_0\big)^{s,\omega}\big(\widetilde{A}\big)=\mathbb{P}^s_0(\widetilde{A})\text{ for any }\omega\in\mathcal{N}^c\Big\}$. It is easy to see that Λ is a Dynkin system. As $\mathscr{C}^s_T$ is closed under intersection, Lemma A.2 and Dynkin System Theorem show that $\mathcal{F}^s_T=\sigma\big(\mathscr{C}^s_T\big)\subset\Lambda$. Namely, it holds for any $\omega\in\mathcal{N}^c$ that $\big(\mathbb{P}^t_0\big)^{s,\omega}\big(\widetilde{A}\big)=\mathbb{P}^s_0(\widetilde{A})$, $\forall\,\widetilde{A}\in\mathcal{F}^s_T$. □

Proof of Proposition 2.1: 1) Let ξ be an $\mathbb{M}-$valued, $\mathcal{F}^t_r-$measurable random variable for some $r\in[s,T]$. For any $\mathcal{M}\in\mathscr{B}(\mathbb{M})$, since $\xi^{-1}(\mathcal{M})\in\mathcal{F}^t_r$, Lemma 2.2 shows that

$$\big(\xi^{s,\omega}\big)^{-1}(\mathcal{M})=\big\{\widetilde{\omega}\in\Omega^s:\xi(\omega\otimes_s\widetilde{\omega})\in\mathcal{M}\big\}=\big\{\widetilde{\omega}\in\Omega^s:\omega\otimes_s\widetilde{\omega}\in\xi^{-1}(\mathcal{M})\big\}=\big(\xi^{-1}(\mathcal{M})\big)^{s,\omega}\in\mathcal{F}^s_r.\tag{7.3}$$

Thus $\xi^{s,\omega}$ is $\mathcal{F}^s_r-$measurable.

2) Let $\{X_r\}_{r\in[t,T]}$ be an $\mathbb{M}-$valued, $\mathbf{F}^t-$adapted process. For any $r\in[s,T]$ and $\mathcal{M}\in\mathscr{B}(\mathbb{M})$, similar to (7.3), one can deduce that $\big(X^{s,\omega}_r\big)^{-1}(\mathcal{M})=\big(X^{-1}_r(\mathcal{M})\big)^{s,\omega}\in\mathcal{F}^s_r$, which shows that $\big\{X^{s,\omega}_r\big\}_{r\in[s,T]}$ is $\mathbf{F}^s-$adapted.

Next, let $\{X_r\}_{r\in[t,T]}$ be an $\mathbb{M}-$valued, $\mathbf{F}^t-$progressively measurable process. Given $T_0\in[s,T]$ and $\mathcal{M}\in\mathscr{B}(\mathbb{M})$, since $\mathcal{D}_0:=\big\{(r,\omega')\in[t,T_0]\times\Omega^t:X_r(\omega')\in\mathcal{M}\big\}\in\mathscr{B}\big([t,T_0]\big)\otimes\mathcal{F}^t_{T_0}$, we can deduce from Lemma 2.3 that

$$\big\{(r,\widetilde{\omega})\in[s,T_0]\times\Omega^s:\ X^{s,\omega}_r(\widetilde{\omega})\in\mathcal{M}\big\}=\big\{(r,\widetilde{\omega})\in[s,T_0]\times\Omega^s:\ (r,\omega\otimes_s\widetilde{\omega})\in\mathcal{D}_0\big\}=\mathcal{D}^{s,\omega}_0\in\mathscr{B}\big([s,T_0]\big)\otimes\mathcal{F}^s_{T_0},$$

which shows the $\mathbf{F}^s-$progressive measurability of $\big\{X^{s,\omega}_r\big\}_{r\in[s,T]}$.

3) Let $\mathcal{D}\in\mathscr{P}^t$. Since $\mathbf{1}_{\mathcal{D}}=\big\{\mathbf{1}_{\mathcal{D}}(r,\omega')\big\}_{(r,\omega')\in[t,T]\times\Omega^t}$ is an $\mathbf{F}^t-$progressively measurable process, part (2) shows that

$$\mathbf{1}_{\mathcal{D}^{s,\omega}}\big(r,\widetilde{\omega}\big)=\mathbf{1}_{\mathcal{D}}\big(r,\omega\otimes_s\widetilde{\omega}\big)=\big(\mathbf{1}_{\mathcal{D}}\big)^{s,\omega}\big(r,\widetilde{\omega}\big),\quad\forall\,(r,\widetilde{\omega})\in[s,T]\times\Omega^s$$

is an $\mathbf{F}^s-$progressively measurable process. Thus, $\mathcal{D}^{s,\omega}\in\mathscr{P}^s$. □

Proof of Proposition 2.2: 1) Given $\omega\in\Omega^t$, we see from Proposition 2.1 (1) that $\xi^{s,\omega}$ is $\mathcal{F}^s_T-$measurable. Also, we can deduce from (2.5), (2.4) and (2.3) that for $\mathbb{P}-$a.s. $\omega\in\Omega^t$

$$\begin{aligned}\mathbb{E}_{\mathbb{P}^{s,\omega}}\big[\xi^{s,\omega}\big]&=\int_{\Omega^s}\xi^{s,\omega}(\widetilde{\omega})d\,\mathbb{P}^{s,\omega}(\widetilde{\omega})=\int_{\Omega^s}\xi\big(\omega\otimes_s\widetilde{\omega}\big)d\,\mathbb{P}^{\omega}_s\big(\omega\otimes_s\widetilde{\omega}\big)=\int_{\omega\otimes_s\Omega^s}\xi(\omega')d\mathbb{P}^{\omega}_s(\omega')\\&=\int_{\Omega^t}\xi(\omega')d\mathbb{P}^{\omega}_s(\omega')=\mathbb{E}_{\mathbb{P}^{\omega}_s}\big[\xi\big]=\mathbb{E}_{\mathbb{P}}\big[\xi\big|\mathcal{F}^t_s\big](\omega)<\infty,\end{aligned}$$

which leads to (2.6).

2) Let $\omega\in\Omega^t$. Proposition 2.1 (2) shows that $\big\{X^{s,\omega}_r\big\}_{r\in[s,T]}$ is $\mathbf{F}^s-$adapted. Clearly, the shifted process $X^{s,\omega}$ also inherits the right continuity of process X. If $\mathbb{E}_{\mathbb{P}}[X_*]<\infty$, since

$$(X_*)^{s,\omega}(\widetilde{\omega})=\sup_{r\in[t,T]}|X_r|(\omega\otimes_s\widetilde{\omega})\ge\sup_{r\in[s,T]}|X_r|(\omega\otimes_s\widetilde{\omega})=\sup_{r\in[s,T]}|X^{s,\omega}_r|(\widetilde{\omega})=(X^{s,\omega})_*(\widetilde{\omega}),\quad\forall\,\widetilde{\omega}\in\Omega^s,$$

(2.6) implies that for $\mathbb{P}-$a.s. $\omega\in\Omega^t$, $\mathbb{E}_{\mathbb{P}^{s,\omega}}\big[(X^{s,\omega})_*\big]\le\mathbb{E}_{\mathbb{P}^{s,\omega}}\big[(X_*)^{s,\omega}\big]=\mathbb{E}_{\mathbb{P}}\big[X_*\big|\mathcal{F}^t_s\big](\omega)<\infty$. □

Proof of Lemma 2.5: 1) Let $\mathcal{N} \in \mathscr{N}^{\mathbb{P}}$. There exists an $A \in \mathcal{F}^t_T$ with $\mathbb{P}(A)=0$ such that $\mathcal{N} \subset A$. For any $\omega \in \Omega^t$, Lemma 2.2 shows that $\mathcal{N}^{s,\omega} \subset A^{s,\omega} \in \mathcal{F}^s_T$ and one can deduce that $(\mathbf{1}_A)^{s,\omega}(\widetilde{\omega}) = \mathbf{1}_{\{\omega \otimes_s \widetilde{\omega} \in A\}} = \mathbf{1}_{\{\widetilde{\omega} \in A^{s,\omega}\}} = \mathbf{1}_{A^{s,\omega}}(\widetilde{\omega})$, $\forall\, \widetilde{\omega} \in \Omega^s$. Then (2.6) implies that for $\mathbb{P}-$a.s. $\omega \in \Omega^t$

$$\mathbb{P}^{s,\omega}\big(A^{s,\omega}\big) = \mathbb{E}_{\mathbb{P}^{s,\omega}}\big[\mathbf{1}_{A^{s,\omega}}\big] = \mathbb{E}_{\mathbb{P}^{s,\omega}}\big[(\mathbf{1}_A)^{s,\omega}\big] = \mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_A \big| \mathcal{F}^t_s\big](\omega) = 0.$$

Thus, $\mathcal{N}^{s,\omega} \in \mathscr{N}^{\mathbb{P}^{s,\omega}}$. In particular, if $\mathcal{N} \in \overline{\mathscr{N}}^t$, one can deduce from Lemma 2.4 that for $\mathbb{P}^t_0-$a.s. $\omega \in \Omega^t$, $\mathcal{N}^{s,\omega} \in \overline{\mathscr{N}}^s$.
2) Let $\mathcal{D} \in \mathscr{B}\big([t,T]\big) \otimes \mathcal{F}^t_T$ with $(dr \times d\mathbb{P}^t_0)\big(\mathcal{D} \cap ([s,T] \times \Omega^t)\big) = 0$. We set $\mathcal{D}_r := \{\omega \in \Omega^t : (r,\omega) \in \mathcal{D}\}$, $\forall\, r \in [t,T]$. Fubini Theorem shows that

$$0 = (dr \times d\mathbb{P}^t_0)\,\big(\mathcal{D} \cap ([s,T] \times \Omega^t)\big) = \int_s^T \bigg(\int_{\Omega^t} \mathbf{1}_{\mathcal{D}_r}(\omega)\, d\mathbb{P}^t_0(\omega)\bigg) dr = \int_{\Omega^t} \bigg(\int_s^T \mathbf{1}_{\mathcal{D}_r}(\omega)\, dr\bigg) d\mathbb{P}^t_0(\omega) = \mathbb{E}_t\bigg[\int_s^T \mathbf{1}_{\mathcal{D}_r} dr\bigg].$$

Thus $\int_s^T \mathbf{1}_{\mathcal{D}_r} dr \in L^1(\mathcal{F}^t_T, \mathbb{P}^t_0)$ is equal to 0, $\mathbb{P}^t_0-$a.s., which together with (2.6) and Lemma 2.4 implies that

$$\mathbb{E}_s\bigg[\Big(\int_s^T \mathbf{1}_{\mathcal{D}_r} dr\Big)^{s,\omega}\bigg] = \mathbb{E}_t\bigg[\int_s^T \mathbf{1}_{\mathcal{D}_r} dr \Big| \mathcal{F}^t_s\bigg](\omega) = 0 \tag{7.4}$$

holds for any $\omega \in \Omega^t$ except on a $\mathcal{N} \in \overline{\mathscr{N}}^t$.

Given $\omega \in \mathcal{N}^c$, applying Lemma 2.3 with $T_0 = T$ shows that $\mathcal{D}^{s,\omega} \in \mathscr{B}\big([s,T]\big) \otimes \mathcal{F}^s_T$. Since

$$\big\{\widetilde{\omega} \in \Omega^s : (r,\widetilde{\omega}) \in \mathcal{D}^{s,\omega}\big\} = \big\{\widetilde{\omega} \in \Omega^s : \big(r, \omega \otimes_s \widetilde{\omega}\big) \in \mathcal{D}\big\} = \big\{\widetilde{\omega} \in \Omega^s : \omega \otimes_s \widetilde{\omega} \in \mathcal{D}_r\big\}, \quad \forall\, r \in [s,T],$$

we can deduce from Fubini Theorem and (7.4) that

$$\begin{aligned}\big(dr \times d\mathbb{P}^s_0\big)\big(\mathcal{D}^{s,\omega}\big) &= \int_s^T \Big(\int_{\Omega^s} \mathbf{1}_{\mathcal{D}^{s,\omega}}(r,\widetilde{\omega}) d\mathbb{P}^s_0(\widetilde{\omega})\Big) dr = \int_{\Omega^s} \Big(\int_s^T \mathbf{1}_{\mathcal{D}_r}\big(\omega \otimes_s \widetilde{\omega}\big) dr\Big) d\mathbb{P}^s_0(\widetilde{\omega}) \\ &= \int_{\Omega^s} \Big(\int_s^T \mathbf{1}_{\mathcal{D}_r} dr\Big)^{s,\omega}(\widetilde{\omega})\, d\mathbb{P}^s_0(\widetilde{\omega}) = \mathbb{E}_s\bigg[\Big(\int_s^T \mathbf{1}_{\mathcal{D}_r} dr\Big)^{s,\omega}\bigg] = 0.\end{aligned}$$

3) Let $\tau \in \overline{\mathcal{T}}^t_s$ and $r \in [s,T]$. As $A_r := \{\tau \le r\} \in \overline{\mathcal{F}}^t_r$, there exists an $\widetilde{A}_r \in \mathcal{F}^t_r$ such that $\mathcal{N}_r := A_r \,\Delta\, \widetilde{A}_r \in \overline{\mathscr{N}}^t$ (see e.g. Problem 2.7.3 of [19]). By part (1), it holds for all $\omega \in \Omega^t$ except on a $\mathbb{P}^t_0-$null set $\widehat{\mathcal{N}}(r)$ that $\mathcal{N}^{s,\omega}_r \in \overline{\mathscr{N}}^s$. Given $\omega \in \big(\widehat{\mathcal{N}}(r)\big)^c$, since $A^{s,\omega}_r \,\Delta\, \widetilde{A}^{s,\omega}_r = \big(A_r \,\Delta\, \widetilde{A}_r\big)^{s,\omega} = \mathcal{N}^{s,\omega}_r \in \overline{\mathscr{N}}^s$ and since $\widetilde{A}^{s,\omega}_r \in \mathcal{F}^s_r$ by Lemma 2.2, we can deduce that $A^{s,\omega}_r \in \overline{\mathcal{F}}^s_r$ and it follows that

$$\{\tau^{s,\omega} \le r\} = \{\widetilde{\omega} \in \Omega^s : \tau^{s,\omega}(\widetilde{\omega}) \le r\} = \{\widetilde{\omega} \in \Omega^s : \tau(\omega \otimes_s \widetilde{\omega}) \le r\} = \{\widetilde{\omega} \in \Omega^s : \omega \otimes_s \widetilde{\omega} \in A_r\} = A^{s,\omega}_r \in \overline{\mathcal{F}}^s_r. \tag{7.5}$$

Set $\widehat{\mathcal{N}} := \underset{r \in (s,T) \cap \mathbb{Q}}{\cup} \widehat{\mathcal{N}}(r)$ and let $\omega \in \widehat{\mathcal{N}}^c$. For any $r \in [s,T)$, there exists a sequence $\{r_n\}_{n \in \mathbb{N}}$ in $(s,T) \cap \mathbb{Q}$ such that $\lim_{n \to \infty} \downarrow r_n = r$. Then (7.5) and the right-continuity of Brownian filtration $\overline{\mathbf{F}}^s$ (under $\mathbb{P}^s_0$) imply that $\{\tau^{s,\omega} \le r\} = \underset{n \in \mathbb{N}}{\cap} \{\tau^{s,\omega} \le r_n\} \in \overline{\mathcal{F}}^s_{r+} = \overline{\mathcal{F}}^s_r$. Hence $\tau^{s,\omega} \in \overline{\mathcal{T}}^s$. □

Proof of Proposition 2.3: Let $\xi \in L^1\big(\mathcal{F}^{\mathbb{P}}_T, \mathbb{P}\big)$. One can approximate ξ^+ from below by a sequence of positive simple $\mathcal{F}^{\mathbb{P}}_T-$measurable random variables: $\xi^+ = \lim_{n \to \infty} \uparrow \xi_n$, where $\xi_n := \sum_{i=1}^{4^n-1} \frac{i}{2^n} \mathbf{1}_{A^n_i}$ and $A^n_i := \big\{\xi^+ \in \big[\frac{i}{2^n}, \frac{i+1}{2^n}\big)\big\} \in \mathcal{F}^{\mathbb{P}}_T$.

Let $n \in \mathbb{N}$. For $i = 1, \cdots, 4^n - 1$, by e.g. Problem 2.7.3 of [19], there exists an $\widetilde{A}^n_i \in \mathcal{F}^t_T$ such that $A^n_i \,\Delta\, \widetilde{A}^n_i \in \mathscr{N}^{\mathbb{P}}$. Setting $\mathcal{A}^n_i := \widetilde{A}^n_i \big\backslash \underset{j<i}{\cup} \widetilde{A}^n_j \in \mathcal{F}^t_T$, one can deduce that

$$\begin{aligned}A^n_i \backslash \mathcal{A}^n_i \;&=\; A^n_i \cap \Big[\big(\widetilde{A}^n_i\big)^c \cup \Big(\underset{j<i}{\cup} \widetilde{A}^n_j\Big)\Big] = \big(A^n_i \backslash \widetilde{A}^n_i\big) \cup \Big(\underset{j<i}{\cup} \big(\widetilde{A}^n_j \cap A^n_i\big)\Big) \\ &\subset\; \big(A^n_i \Delta \widetilde{A}^n_i\big) \cup \Big(\underset{j<i}{\cup} \big(\widetilde{A}^n_j \cap (A^n_j)^c\big)\Big) \subset \underset{j \le i}{\cup} \big(A^n_j \Delta \widetilde{A}^n_j\big) \in \mathscr{N}^{\mathbb{P}}.\end{aligned} \tag{7.6}$$

Define $\eta_n := \sum_{i=1}^{4^n-1} \frac{i}{2^n} \mathbf{1}_{\mathcal{A}^n_i}$, which is an $\mathcal{F}^t_T-$measurable bounded random variable. By Proposition 2.2 (1), it holds for all $\omega \in \Omega^t$ except on a $\mathcal{N}_n \in \mathscr{N}^{\mathbb{P}}$ that

$$\eta^{s,\omega}_n \in L^1\big(\mathcal{F}^s_T, \mathbb{P}^{s,\omega}\big) \quad \text{and} \quad \mathbb{E}_{\mathbb{P}^{s,\omega}}\big[\eta^{s,\omega}_n\big] = \mathbb{E}_{\mathbb{P}}\big[\eta_n \big| \mathcal{F}^t_s\big](\omega). \tag{7.7}$$

Clearly, η_n coincides with ξ_n over $\mathcal{Q}_n := \underset{i=1}{\overset{4^n-1}{\cup}}\big(A^n_i \cap \mathcal{A}^n_i\big) \cup \big(A^n_0 \cap \mathcal{A}^n_0\big)$, where $A^n_0 := \Big(\underset{i=1}{\overset{4^n-1}{\cup}} A^n_i\Big)^c$ and $\mathcal{A}^n_0 := \Big(\underset{i=1}{\overset{4^n-1}{\cup}} \mathcal{A}^n_i\Big)^c$. Since $\big\{A^n_i\big\}_{i=0}^{4^n-1}$ is a disjoint union of Ω^t and since $A^n_0 \backslash \mathcal{A}^n_0 = A^n_0 \cap \Big(\underset{i=1}{\overset{4^n-1}{\cup}} \mathcal{A}^n_i\Big) = \underset{i=1}{\overset{4^n-1}{\cup}}\big(\mathcal{A}^n_i \cap A^n_0\big) \subset \underset{i=1}{\overset{4^n-1}{\cup}}\big(\widetilde{A}^n_i \cap (A^n_i)^c\big) \subset \underset{i=1}{\overset{4^n-1}{\cup}}\big(A^n_i \Delta \widetilde{A}^n_i\big) \in \mathscr{N}^{\mathbb{P}}$, we see from (7.6) that $\mathcal{Q}^c_n = \underset{i=1}{\overset{4^n-1}{\cup}}\big(A^n_i \backslash \mathcal{A}^n_i\big) \cup \big(A^n_0 \backslash \mathcal{A}^n_0\big) \in \mathscr{N}^{\mathbb{P}}$.

Set $\mathfrak{N}_0 := \underset{n\in\mathbb{N}}{\cup} \mathcal{Q}^c_n \in \mathscr{N}^{\mathbb{P}}$. As

$$\xi^+ = \lim_{n\to\infty}\uparrow \eta_n \text{ over } \underset{n\in\mathbb{N}}{\cap} \mathcal{Q}_n = \mathfrak{N}^c_0, \tag{7.8}$$

applying the conditional version of monotone convergence theorem yields that

$$\lim_{n\to\infty}\uparrow \mathbb{E}_{\mathbb{P}}\big[\eta_n\big|\mathcal{F}^t_s\big](\omega) = \mathbb{E}_{\mathbb{P}}\big[\xi^+\big|\mathcal{F}^t_s\big](\omega) \in \mathbb{R}_+ \tag{7.9}$$

holds for all $\omega \in \Omega^t$ except on a $\mathbb{P}-$null set $\mathfrak{N}_1$. By Lemma 2.5 (1), there exists another $\mathbb{P}-$null set $\mathfrak{N}_2$ such that for any $\omega \in \mathfrak{N}^c_2$, $\mathfrak{N}^{s,\omega}_0 \in \mathscr{N}^{\mathbb{P}^{s,\omega}}$.

Now, let $\mathfrak{N} := \mathfrak{N}_1 \cup \mathfrak{N}_2 \cup \Big(\underset{n\in\mathbb{N}}{\cup} \mathcal{N}_n\Big) \in \mathscr{N}^{\mathbb{P}}$. Given $\omega \in \mathfrak{N}^c$, $\mathfrak{N}^{s,\omega}_0$ is a $\mathbb{P}^{s,\omega}-$null set. For any $\widetilde{\omega} \in \big(\mathfrak{N}^{s,\omega}_0\big)^c = (\mathfrak{N}^c_0)^{s,\omega}$, (7.8) shows that

$$(\xi^+)^{s,\omega}(\widetilde{\omega}) = \xi^+(\omega \otimes_s \widetilde{\omega}) = \lim_{n\to\infty}\uparrow \eta_n(\omega \otimes_s \widetilde{\omega}) = \lim_{n\to\infty}\uparrow \eta^{s,\omega}_n(\widetilde{\omega}). \tag{7.10}$$

So over $\big(\mathfrak{N}^{s,\omega}_0\big)^c$, $(\xi^+)^{s,\omega}$ coincides with $\overline{\lim_{n\to\infty}}\, \eta^{s,\omega}_n$, which is $\mathcal{F}^s_T-$measurable by (7.7). It follows that $(\xi^+)^{s,\omega}$ is $\mathcal{F}^{\mathbb{P}^{s,\omega}}_T-$measurable.

Moreover, applying the monotone convergence theorem to (7.10), we see from (7.7) and (7.9) that

$$\mathbb{E}_{\mathbb{P}^{s,\omega}}\big[(\xi^+)^{s,\omega}\big] = \lim_{n\to\infty}\uparrow \mathbb{E}_{\mathbb{P}^{s,\omega}}\big[\eta^{s,\omega}_n\big] = \lim_{n\to\infty}\uparrow \mathbb{E}_{\mathbb{P}}\big[\eta_n\big|\mathcal{F}^t_s\big](\omega) = \mathbb{E}_{\mathbb{P}}\big[\xi^+\big|\mathcal{F}^t_s\big](\omega) \in \mathbb{R}_+.$$

The similar result also holds for ξ^-, then the conclusion follows. □

7.2 Proofs of the results in Section 3

Proof of Lemma 3.1: Let $t \in [0,T]$ and $\mathbb{P}$ be a probability on $\big(\Omega^t, \mathscr{B}(\Omega^t)\big)$. Suppose that $Y^{t,\omega} \in \mathbb{D}(\mathbf{F}^t, \mathbb{P})$ for some $\omega \in \Omega$ and fix $\omega' \in \Omega$. The $\mathbf{F}-$adaptness of Y, Proposition 2.1 (2) and Remark 3.1 (1) show that $Y^{t,\omega'}$ is an $\mathbf{F}^t-$adapted process with all RCLL paths. Given $\widetilde{\omega} \in \Omega^t$, (3.1) implies that for any $s \in [t,T]$

$$\big|Y^{t,\omega'}_s(\widetilde{\omega}) - Y^{t,\omega}_s(\widetilde{\omega})\big| = \big|Y_s(\omega' \otimes_t \widetilde{\omega}) - Y_s(\omega \otimes_t \widetilde{\omega})\big| \le \rho_0\big(\|\omega' \otimes_t \widetilde{\omega} - \omega \otimes_t \widetilde{\omega}\|_{0,s}\big) = \rho_0\big(\|\omega' - \omega\|_{0,t}\big). \tag{7.11}$$

It follows that $\mathbb{E}_{\mathbb{P}}\big[Y^{t,\omega'}_*\big] = \mathbb{E}_{\mathbb{P}}\Big[\sup_{s\in[t,T]} |Y^{t,\omega'}_s|\Big] \le \mathbb{E}_{\mathbb{P}}\Big[\sup_{s\in[t,T]} |Y^{t,\omega}_s|\Big] + \rho_0\big(\|\omega' - \omega\|_{0,t}\big) = \mathbb{E}_{\mathbb{P}}\big[Y^{t,\omega}_*\big] + \rho_0\big(\|\omega' - \omega\|_{0,t}\big)$. So $Y^{t,\omega'} \in \mathbb{D}\big(\mathbf{F}^t, \mathbb{P}\big)$. □

Proof of Remark 3.2 (1): Given $t \in [0,T]$, Proposition 2.2 (2) and Lemma 2.4 imply that for $\mathbb{P}_0-$a.s. $\omega \in \Omega$, $Y^{t,\omega} \in \mathbb{D}\big(\mathbf{F}^t, (\mathbb{P}_0)^{t,\omega}\big) = \mathbb{D}\big(\mathbf{F}^t, \mathbb{P}^t_0\big)$. Then by Lemma 3.1, $Y^{t,\mathbf{0}} \in \mathbb{D}\big(\mathbf{F}^t, \mathbb{P}^t_0\big)$, which together with the right-continuity of $\overline{\mathbf{F}}^t$ show that $\mathbb{P}^t_0 \in \mathfrak{P}^Y_t$. □

Proof of Remark 3.3: 2) Let $\widetilde{\mathbb{P}} \in \mathfrak{P}^Y_s$. Given $\widetilde{\omega}_1, \widetilde{\omega}_2 \in \Omega^t$ and $\zeta \in \mathcal{T}^s$, similar to (7.11), we can deduce that

$$\begin{aligned}\big|Y^{s,\omega\otimes_t\widetilde{\omega}_1}_\zeta(\widehat{\omega}) - Y^{s,\omega\otimes_t\widetilde{\omega}_2}_\zeta(\widehat{\omega})\big| &= \big|Y\big(\zeta(\widehat{\omega}), (\omega \otimes_t \widetilde{\omega}_1) \otimes_s \widehat{\omega}\big) - Y\big(\zeta(\widehat{\omega}), (\omega \otimes_t \widetilde{\omega}_2) \otimes_s \widehat{\omega}\big)\big| \\ &\le \rho_0\big(\|(\omega\otimes_t\widetilde{\omega}_1)\otimes_s\widehat{\omega} - (\omega\otimes_t\widetilde{\omega}_2)\otimes_s\widehat{\omega}\|_{0,\zeta(\widehat{\omega})}\big) = \rho_0\big(\|\widetilde{\omega}_1 - \widetilde{\omega}_2\|_{t,s}\big), \quad \forall\, \widehat{\omega} \in \Omega^s.\end{aligned}$$

It follows that

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\big[Y^{s,\omega\otimes_t\widetilde{\omega}_1}_\zeta\big] \le \mathbb{E}_{\widetilde{\mathbb{P}}}\big[Y^{s,\omega\otimes_t\widetilde{\omega}_2}_\zeta\big] + \rho_0\big(\|\widetilde{\omega}_1 - \widetilde{\omega}_2\|_{t,s}\big). \tag{7.12}$$

Taking supremum over $\zeta \in \mathcal{T}^s$ yields that $\sup\limits_{\zeta\in\mathcal{T}^s} \mathbb{E}_{\widetilde{\mathbb{P}}}\big[Y_\zeta^{s,\omega\otimes_t\widetilde{\omega}_1}\big] \le \sup\limits_{\zeta\in\mathcal{T}^s} \mathbb{E}_{\widetilde{\mathbb{P}}}\big[Y_\zeta^{s,\omega\otimes_t\widetilde{\omega}_2}\big] + \rho_0\big(\|\widetilde{\omega}_1-\widetilde{\omega}_2\|_{t,T}\big)$. Exchanging the roles of $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ shows that the mapping $\widetilde{\omega} \to \sup\limits_{\zeta\in\mathcal{T}^s} \mathbb{E}_{\widetilde{\mathbb{P}}}\big[Y_\zeta^{s,\omega\otimes_t\widetilde{\omega}}\big]$ is continuous under norm $\|\ \|_{t,T}$ and thus $\mathcal{F}_T^t-$measurable.

Next, let us show that both sides of (3.4) are finite: Let $j=1,\cdots,\lambda$ and $A\in\mathcal{F}_s^t$. For any $\tau\in\mathcal{T}_s^t$, (3.2) shows that $\big|\mathbb{E}_{\widehat{\mathbb{P}}}\big[\mathbf{1}_{A\cap\mathcal{A}_j}Y_\tau^{t,\omega}\big]\big| \le \mathbb{E}_{\widehat{\mathbb{P}}}\big[\big|Y_\tau^{t,\omega}\big|\big] \le \mathbb{E}_{\widehat{\mathbb{P}}}\big[Y_*^{t,\omega}\big] < \infty$, which leads to that

$$-\infty < -\mathbb{E}_{\widehat{\mathbb{P}}}\big[Y_*^{t,\omega}\big] \le \sup_{\tau\in\mathcal{T}_s^t} \mathbb{E}_{\widehat{\mathbb{P}}}\big[\mathbf{1}_{A\cap\mathcal{A}_j}Y_\tau^{t,\omega}\big] \le \mathbb{E}_{\widehat{\mathbb{P}}}\big[Y_*^{t,\omega}\big] < \infty.$$

On the other hand, given $\widetilde{\omega}\in A\cap\mathcal{A}_j$ and $\zeta\in\mathcal{T}^s$, applying (7.12) with $(\widetilde{\omega}_1,\widetilde{\omega}_2)=(\widetilde{\omega},\widetilde{\omega}_j)$ and $(\widetilde{\omega}_1,\widetilde{\omega}_2)=(\widetilde{\omega}_j,\widetilde{\omega})$ respectively yields that

$$\Big|\mathbb{E}_{\mathbb{P}_j}\big[Y_\zeta^{s,\omega\otimes_t\widetilde{\omega}}\big]\Big| \le \Big|\mathbb{E}_{\mathbb{P}_j}\big[Y_\zeta^{s,\omega\otimes_t\widetilde{\omega}_j}\big]\Big| + \Big|\mathbb{E}_{\mathbb{P}_j}\big[Y_\zeta^{s,\omega\otimes_t\widetilde{\omega}} - Y_\zeta^{s,\omega\otimes_t\widetilde{\omega}_j}\big]\Big| \le \mathbb{E}_{\mathbb{P}_j}\big[Y_*^{s,\omega\otimes_t\widetilde{\omega}_j}\big] + \rho_0\big(\|\widetilde{\omega}-\widetilde{\omega}_j\|_{t,s}\big) \le \mathbb{E}_{\mathbb{P}_j}\big[Y_*^{s,\omega\otimes_t\widetilde{\omega}_j}\big] + \rho_0(\delta).$$

It then follows from (3.2) that

$$\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\widetilde{\omega}\in A\cap\mathcal{A}_j\}}\Big(\sup_{\zeta\in\mathcal{T}^s}\mathbb{E}_{\mathbb{P}_j}\big[Y_\zeta^{s,\omega\otimes_t\widetilde{\omega}}\big] + \widehat{\rho}_0(\delta)\Big)\Big] \le \Big(\mathbb{E}_{\mathbb{P}_j}\big[Y_*^{s,\omega\otimes_t\widetilde{\omega}_j}\big] + \rho_0(\delta) + \widehat{\rho}_0(\delta)\Big)\,\mathbb{P}(A\cap\mathcal{A}_j) < \infty$$

as well as that

$$\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\widetilde{\omega}\in A\cap\mathcal{A}_j\}}\Big(\sup_{\zeta\in\mathcal{T}^s}\mathbb{E}_{\mathbb{P}_j}\big[Y_\zeta^{s,\omega\otimes_t\widetilde{\omega}}\big] + \widehat{\rho}_0(\delta)\Big)\Big] \ge \Big(-\mathbb{E}_{\mathbb{P}_j}\big[Y_*^{s,\omega\otimes_t\widetilde{\omega}_j}\big] - \rho_0(\delta) + \widehat{\rho}_0(\delta)\Big)\mathbb{P}(A\cap\mathcal{A}_j) > -\infty.$$

3) Given $A\in\mathcal{F}_T^t$, for any $j=1,\cdots,\lambda$ and $\widetilde{\omega}\in\mathcal{A}_j$, since $\mathcal{A}_j\in\mathcal{F}_s^t$, Lemma 2.1 shows that $(\mathcal{A}_j)^{s,\widetilde{\omega}}=\Omega^s$ (or $(\mathbf{1}_{\mathcal{A}_j})^{s,\widetilde{\omega}}\equiv 1$), which implies that $(A\cap\mathcal{A}_0)^{s,\widetilde{\omega}}=\emptyset$. So it is easy to calculate that $\widehat{\mathbb{P}}(A\cap\mathcal{A}_0)=\mathbb{P}(A\cap\mathcal{A}_0)$.

Next, let $j=1,\cdots,\lambda$ and $A\in\mathcal{F}_s^t$. We see from Lemma 2.1 again that

$$\text{if } \widetilde{\omega}\in A\cap\mathcal{A}_j \text{ (resp. } \notin A\cap\mathcal{A}_j\text{), then } (A\cap\mathcal{A}_j)^{s,\widetilde{\omega}}=\Omega^s \text{ (resp. } =\emptyset\text{)}. \tag{7.13}$$

It follows that

$$\widehat{\mathbb{P}}(A\cap\mathcal{A}_j) = \sum_{j'=1}^{\lambda}\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\widetilde{\omega}\in\mathcal{A}_{j'}\}}\mathbb{P}_{j'}\big((A\cap\mathcal{A}_j)^{s,\widetilde{\omega}}\big)\Big] = \sum_{j'=1}^{\lambda}\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\widetilde{\omega}\in A\cap\mathcal{A}_j\}}\mathbf{1}_{\{\widetilde{\omega}\in\mathcal{A}_{j'}\}}\mathbb{P}_{j'}\big(\Omega^s\big)\Big] = \mathbb{P}(A\cap\mathcal{A}_j).$$

Given $\tau\in\mathcal{T}_s^t$, since $\tau^{s,\widetilde{\omega}}\in\mathcal{T}^s$ by Corollary 2.1, we can deduce from (7.13) again that

$$\begin{aligned}
\mathbb{E}_{\widehat{\mathbb{P}}}\Big[\mathbf{1}_{A\cap\mathcal{A}_j}Y_\tau^{t,\omega}\Big] &= \sum_{j'=1}^{\lambda}\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\widetilde{\omega}\in\mathcal{A}_{j'}\}}\mathbb{E}_{\mathbb{P}_{j'}}\Big[\big(\mathbf{1}_{A\cap\mathcal{A}_j}Y_\tau^{t,\omega}\big)^{s,\widetilde{\omega}}\Big]\Big] = \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\widetilde{\omega}\in A\cap\mathcal{A}_j\}}\mathbb{E}_{\mathbb{P}_j}\Big[(Y_\tau^{t,\omega})^{s,\widetilde{\omega}}\Big]\Big] \\
&= \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\widetilde{\omega}\in A\cap\mathcal{A}_j\}}\mathbb{E}_{\mathbb{P}_j}\Big[Y_{\tau^{s,\widetilde{\omega}}}^{s,\omega\otimes_t\widetilde{\omega}}\Big]\Big] \le \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\widetilde{\omega}\in A\cap\mathcal{A}_j\}}\sup_{\zeta\in\mathcal{T}^s}\mathbb{E}_{\mathbb{P}_j}\Big[Y_\zeta^{s,\omega\otimes_t\widetilde{\omega}}\Big]\Big],
\end{aligned}$$

where we used the fact that

$$\begin{aligned}
(Y_\tau^{t,\omega})^{s,\widetilde{\omega}}(\widehat{\omega}) &= Y_\tau^{t,\omega}(\widetilde{\omega}\otimes_s\widehat{\omega}) = Y\big(\tau(\widetilde{\omega}\otimes_s\widehat{\omega}),\omega\otimes_t(\widetilde{\omega}\otimes_s\widehat{\omega})\big) = Y\big(\tau^{s,\widetilde{\omega}}(\widehat{\omega}),(\omega\otimes_t\widetilde{\omega})\otimes_s\widehat{\omega}\big) \\
&= Y^{s,\omega\otimes_t\widetilde{\omega}}\big(\tau^{s,\widetilde{\omega}}(\widehat{\omega}),\widehat{\omega}\big) = Y_{\tau^{s,\widetilde{\omega}}}^{s,\omega\otimes_t\widetilde{\omega}}(\widehat{\omega}), \quad \forall\,\widehat{\omega}\in\Omega^s.
\end{aligned}$$

□

7.3 Proofs of the results in Section 4

Proof of Remark 4.1: Let $t\in[0,T]$ and $\omega_1,\omega_2\in\Omega$. For any $\mathbb{P}\in\mathcal{P}_t$, $\tau\in\mathcal{T}^t$ and $\widetilde{\omega}\in\Omega^t$, (7.11) shows that $\big|Y_s^{t,\omega_1}(\widetilde{\omega})-Y_s^{t,\omega_2}(\widetilde{\omega})\big|\le\rho_0\big(\|\omega_1-\omega_2\|_{0,t}\big)$, $\forall\,s\in[t,T]$. In particular, $\big|Y^{t,\omega_1}\big(\tau(\widetilde{\omega}),\widetilde{\omega}\big)-Y^{t,\omega_2}\big(\tau(\widetilde{\omega}),\widetilde{\omega}\big)\big|\le\rho_0\big(\|\omega_1-\omega_2\|_{0,t}\big)$. It then follows that

$$\mathbb{E}_{\mathbb{P}}\big[Y_\tau^{t,\omega_1}\big] \le \mathbb{E}_{\mathbb{P}}\big[Y_\tau^{t,\omega_2}\big] + \rho_0\big(\|\omega_1-\omega_2\|_{0,t}\big). \tag{7.14}$$

Taking supremum over $\tau\in\mathcal{T}^t$ and then taking infimum over $\mathbb{P}\in\mathcal{P}_t$ yield that $\overline{Z}_t(\omega_1)\le\overline{Z}_t(\omega_2)+\rho_0\big(\|\omega_1-\omega_2\|_{0,t}\big)$. Exchanging the role of ω_1 and ω_2, we obtain (4.3) with $\rho_1=\rho_0$. □

Proof of Lemma 4.1: Let $0\le t\le s\le T$, $\omega\in\Omega$ and $\mathbb{P}\in\mathcal{P}(t,\omega)$. If $t=s$, as $\overline{Z}_t$ is $\mathcal{F}_t-$measurable by Remark 4.2, (2.2) shows that $\mathbb{E}_\mathbb{P}\Big[\big|\overline{Z}^{t,\omega}_t\big|\Big]=\mathbb{E}_\mathbb{P}\big[|\overline{Z}_t(\omega)|\big]=|\overline{Z}_t(\omega)|<\infty$. So let us assume $t<s$. For any $\widetilde{\omega}\in\Omega^t$, one can deduce that

$$\begin{aligned}Y^{s,\omega\otimes_t\widetilde{\omega}}_*(\widehat{\omega})&=\sup_{r\in[s,T]}\big|Y^{s,\omega\otimes_t\widetilde{\omega}}_r(\widehat{\omega})\big|=\sup_{r\in[s,T]}\big|Y\big(r,(\omega\otimes_t\widetilde{\omega})\otimes_s\widehat{\omega}\big)\big|\le\sup_{r\in[t,T]}\big|Y\big(r,\omega\otimes_t(\widetilde{\omega}\otimes_s\widehat{\omega})\big)\big|\\&=\sup_{r\in[t,T]}\big|Y^{t,\omega}_r(\widetilde{\omega}\otimes_s\widehat{\omega})\big|=Y^{t,\omega}_*(\widetilde{\omega}\otimes_s\widehat{\omega})=\big(Y^{t,\omega}_*\big)^{s,\widetilde{\omega}}(\widehat{\omega}),\quad\forall\,\widehat{\omega}\in\Omega^s.\end{aligned}\tag{7.15}$$

By (P1), there exist an extension $(\Omega^t,\mathcal{F}',\mathbb{P}')$ of $(\Omega^t,\mathcal{F}^t_T,\mathbb{P})$ and $\Omega'\in\mathcal{F}'$ with $\mathbb{P}'(\Omega')=1$ such that for any $\widetilde{\omega}\in\Omega'$, $\mathbb{P}^{s,\widetilde{\omega}}\in\mathcal{P}(s,\omega\otimes_t\widetilde{\omega})$. Since $Y^{t,\omega}\in\mathbb{D}(\mathbf{F}^t,\mathbb{P})$ by (3.2), we see from (2.6) that for all $\widetilde{\omega}\in\Omega^t$ except on some $\mathcal{N}\in\mathscr{N}^\mathbb{P}$, $\mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\Big[\big(Y^{t,\omega}_*\big)^{s,\widetilde{\omega}}\Big]=\mathbb{E}_\mathbb{P}\big[Y^{t,\omega}_*\big|\mathcal{F}^t_s\big](\widetilde{\omega})$. Let A be the $\mathcal{F}^t_T-$measurable set containing $\mathcal{N}$ and with $\mathbb{P}(A)=0$. For any $\widetilde{\omega}\in\Omega'\cap A^c\in\mathcal{F}'$, (4.2) and (7.15) imply that

$$Y_s(\omega\otimes_t\widetilde{\omega})\le\overline{Z}_s(\omega\otimes_t\widetilde{\omega})\le\sup_{\tau\in\mathcal{T}^s}\mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\big[Y^{s,\omega\otimes_t\widetilde{\omega}}_\tau\big]\le\mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\big[Y^{s,\omega\otimes_t\widetilde{\omega}}_*\big]\le\mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\Big[\big(Y^{t,\omega}_*\big)^{s,\widetilde{\omega}}\Big]=\mathbb{E}_\mathbb{P}\big[Y^{t,\omega}_*\big|\mathcal{F}^t_s\big](\widetilde{\omega}),$$

so $\Omega'\cap A^c\subset\widetilde{\mathcal{A}}:=\big\{Y^{t,\omega}_s\le\overline{Z}^{t,\omega}_s\le\mathbb{E}_\mathbb{P}\big[Y^{t,\omega}_*\big|\mathcal{F}^t_s\big]\big\}$. Remark 4.2 and Proposition 2.1 (2) show that $\widetilde{\mathcal{A}}\in\mathcal{F}^t_s$, it then follows that $\mathbb{P}(\widetilde{\mathcal{A}})=\mathbb{P}'(\widetilde{\mathcal{A}})\ge\mathbb{P}'(\Omega'\cap A^c)=1$. To wit,

$$Y^{t,\omega}_s\le\overline{Z}^{t,\omega}_s\le\mathbb{E}_\mathbb{P}\big[Y^{t,\omega}_*\big|\mathcal{F}^t_s\big],\quad\mathbb{P}-\text{a.s.},\tag{7.16}$$

which leads to that $\mathbb{E}_\mathbb{P}\Big[\big|\overline{Z}^{t,\omega}_s\big|\Big]\le\mathbb{E}_\mathbb{P}\Big[\big|Y^{t,\omega}_s\big|+\mathbb{E}_\mathbb{P}\big[Y^{t,\omega}_*\big|\mathcal{F}^t_s\big]\Big]=\mathbb{E}_\mathbb{P}\Big[\big|Y^{t,\omega}_s\big|\Big]+\mathbb{E}_\mathbb{P}\big[Y^{t,\omega}_*\big]\le2\mathbb{E}_\mathbb{P}\big[Y^{t,\omega}_*\big]<\infty.$ □

Proof of Proposition 4.1: Fix $0\le t\le s\le T$ and $\omega\in\Omega$. If $t=s$, Remark 4.2 and (2.2) imply that $\overline{Z}^{t,\omega}_t=\overline{Z}_t(\omega)$. Then (4.5) clearly holds. So we just assume $t<s$ and define

$$\mathcal{Y}_r:=Y^{t,\omega}_r\quad\text{and}\quad\mathcal{Z}_r:=\overline{Z}^{t,\omega}_r,\quad\forall\,r\in[t,T].\tag{7.17}$$

1) *To show*

$$\overline{Z}_t(\omega)\le\inf_{\mathbb{P}\in\mathcal{P}(t,\omega)}\sup_{\tau\in\mathcal{T}^t}\mathbb{E}_\mathbb{P}\Big[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_\tau+\mathbf{1}_{\{\tau\ge s\}}\mathcal{Z}_s\Big],\tag{7.18}$$

we shall paste the local approximating minimizers $\mathbb{P}_{\widetilde{\omega}}$ of $\overline{Z}^{t,\omega}_s(\widetilde{\omega})$ according to (P2) and then make some estimations.

Fix $\varepsilon>0$ and let $\delta\in\mathbb{Q}_+$ such that $\rho_0(\delta)\vee\widehat{\rho}_0(\delta)\vee\rho_1(\delta)<\varepsilon/4$. Given $\widetilde{\omega}\in\Omega^t$, we can find a $\mathbb{P}_{\widetilde{\omega}}\in\mathcal{P}(s,\omega\otimes_t\widetilde{\omega})$ such that

$$\overline{Z}_s(\omega\otimes_t\widetilde{\omega})\ge\sup_{\tau\in\mathcal{T}^s}\mathbb{E}_{\mathbb{P}_{\widetilde{\omega}}}\big[Y^{s,\omega\otimes_t\widetilde{\omega}}_\tau\big]-\varepsilon/4.\tag{7.19}$$

Similarly to (A.5), $O^s_\delta(\widetilde{\omega})$ is an open set of Ω^t. For any $\widetilde{\omega}'\in O^s_\delta(\widetilde{\omega})$, an analogy to (7.14) shows that

$$\mathbb{E}_{\mathbb{P}_{\widetilde{\omega}}}\big[Y^{s,\omega\otimes_t\widetilde{\omega}'}_\tau\big]\le\mathbb{E}_{\mathbb{P}_{\widetilde{\omega}}}\big[Y^{s,\omega\otimes_t\widetilde{\omega}}_\tau\big]+\rho_0\big(\|\omega\otimes_t\widetilde{\omega}'-\omega\otimes_t\widetilde{\omega}\|_{0,s}\big)=\mathbb{E}_{\mathbb{P}_{\widetilde{\omega}}}\big[Y^{s,\omega\otimes_t\widetilde{\omega}}_\tau\big]+\rho_0\big(\|\widetilde{\omega}'-\widetilde{\omega}\|_{t,s}\big),\quad\forall\,\tau\in\mathcal{T}^s.$$

Taking supremum over $\tau\in\mathcal{T}^s$, we can deduce from (4.3) and (7.19) that

$$\begin{aligned}\sup_{\tau\in\mathcal{T}^s}\mathbb{E}_{\mathbb{P}_{\widetilde{\omega}}}\big[Y^{s,\omega\otimes_t\widetilde{\omega}'}_\tau\big]&\le\sup_{\tau\in\mathcal{T}^s}\mathbb{E}_{\mathbb{P}_{\widetilde{\omega}}}\big[Y^{s,\omega\otimes_t\widetilde{\omega}}_\tau\big]+\rho_0\big(\|\widetilde{\omega}'-\widetilde{\omega}\|_{t,s}\big)\le\overline{Z}_s\big(\omega\otimes_t\widetilde{\omega}\big)+\frac{1}{2}\varepsilon\\&\le\overline{Z}_s(\omega\otimes_t\widetilde{\omega}')+\rho_1\big(\|\widetilde{\omega}'-\widetilde{\omega}\|_{t,s}\big)+\frac{1}{2}\varepsilon\le\mathcal{Z}_s(\widetilde{\omega}')+\frac{3}{4}\varepsilon,\quad\forall\,\widetilde{\omega}'\in O^s_\delta(\widetilde{\omega}).\end{aligned}\tag{7.20}$$

Next, fix $\mathbb{P}\in\mathcal{P}(t,\omega)$ and $\lambda\in\mathbb{N}$. For $j=1,\cdots,\lambda$, we set $\mathcal{A}_j:=\Big(O^s_\delta(\widehat{\omega}^t_j)\backslash\big(\underset{j'<j}{\cup}O^s_\delta(\widehat{\omega}^t_{j'})\big)\Big)\in\mathcal{F}^t_s$ by (2.1) and set $\mathbb{P}_j:=\mathbb{P}_{\widehat{\omega}^t_j}$ (where $\widehat{\omega}^t_j$ is defined right after (2.1)). Let $\widehat{\mathbb{P}}_\lambda$ be the probability of $\mathcal{P}(t,\omega)$ in (P2) for $\big\{(\mathcal{A}_j,\delta_j,\widetilde{\omega}_j,\mathbb{P}_j)\big\}_{j=1}^\lambda=\big\{(\mathcal{A}_j,\delta,\widehat{\omega}^t_j,\mathbb{P}_j)\big\}_{j=1}^\lambda$ and $\mathcal{A}_0:=\Big(\overset{\lambda}{\underset{j=1}{\cup}}\mathcal{A}_j\Big)^c\in\mathcal{F}^t_s$. So

$$\mathbb{E}_{\widehat{\mathbb{P}}_\lambda}[\xi]=\mathbb{E}_\mathbb{P}[\xi],\ \ \forall\,\xi\in L^1(\mathcal{F}^t_s,\widehat{\mathbb{P}}_\lambda)\cap L^1\big(\mathcal{F}^t_s,\mathbb{P}\big)\quad\text{and}\quad\mathbb{E}_{\widehat{\mathbb{P}}_\lambda}[\mathbf{1}_{\mathcal{A}_0}\xi]=\mathbb{E}_\mathbb{P}[\mathbf{1}_{\mathcal{A}_0}\xi],\ \ \forall\,\xi\in L^1(\mathcal{F}^t_T,\widehat{\mathbb{P}}_\lambda)\cap L^1\big(\mathcal{F}^t_T,\mathbb{P}\big).\tag{7.21}$$

Given $\tau \in \mathcal{T}^t$, one can deduce from (3.2), (7.21), (3.4), (7.20) and Lemma 4.1 that

$$\begin{aligned}
\mathbb{E}_{\widehat{\mathbb{P}}_\lambda}\big[\mathcal{Y}_\tau\big] &= \mathbb{E}_{\widehat{\mathbb{P}}_\lambda}\big[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_{\tau\wedge s}+\mathbf{1}_{\{\tau\ge s\}\cap\mathcal{A}_0}\mathcal{Y}_{\tau\vee s}\big]+\sum_{j=1}^{\lambda}\mathbb{E}_{\widehat{\mathbb{P}}_\lambda}\big[\mathbf{1}_{\{\tau\ge s\}\cap\mathcal{A}_j}Y^{t,\omega}_{\tau\vee s}\big] \\
&\le \mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_{\tau\wedge s}+\mathbf{1}_{\{\tau\ge s\}\cap\mathcal{A}_0}\mathcal{Y}_{\tau\vee s}\big]+\sum_{j=1}^{\lambda}\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau(\widetilde{\omega})\ge s\}\cap\{\widetilde{\omega}\in\mathcal{A}_j\}}\Big(\sup_{\zeta\in\mathcal{T}^s}\mathbb{E}_{\mathbb{P}_j}\big[Y^{s,\omega\otimes_t\widetilde{\omega}}_\zeta\big]+\widehat{\rho}_0(\delta)\Big)\Big] \\
&\le \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_\tau+\mathbf{1}_{\{\tau\ge s\}\cap\mathcal{A}_0}\mathcal{Y}_\tau+\mathbf{1}_{\{\tau\ge s\}\cap\mathcal{A}_0^c}\mathcal{Z}_s\Big]+\varepsilon \\
&= \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_\tau+\mathbf{1}_{\{\tau\ge s\}}\mathcal{Z}_s\Big]+\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau\ge s\}\cap\mathcal{A}_0}\big(\mathcal{Y}_\tau-\mathcal{Z}_s\big)\Big]+\varepsilon \\
&\le \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_\tau+\mathbf{1}_{\{\tau\ge s\}}\mathcal{Z}_s\Big]+\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\mathcal{A}_0}\big(\mathcal{Y}_*+|\mathcal{Z}_s|\big)\Big]+\varepsilon.
\end{aligned}$$

Taking supremum over $\tau\in\mathcal{T}^t$ yields that

$$\overline{Z}_t(\omega)\le\sup_{\tau\in\mathcal{T}^t}\mathbb{E}_{\widehat{\mathbb{P}}_\lambda}\big[\mathcal{Y}_\tau\big]\le\sup_{\tau\in\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_\tau+\mathbf{1}_{\{\tau\ge s\}}\mathcal{Z}_s\Big]+\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\big(\underset{j=1}{\overset{\lambda}{\cup}}\mathcal{A}_j\big)^c}\big(\mathcal{Y}_*+|\mathcal{Z}_s|\big)\Big]+\varepsilon. \tag{7.22}$$

Since $\underset{j\in\mathbb{N}}{\cup}\mathcal{A}_j=\underset{j\in\mathbb{N}}{\cup}O^s_\delta(\widehat{\omega}^t_j)=\Omega^t$ and since $\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_*+\big|\mathcal{Z}_s\big|\big]<\infty$ by (3.2) and Lemma 4.1, letting $\lambda\to\infty$ in (7.22), we can deduce from the dominated convergence theorem that $\overline{Z}_t(\omega)\le\sup_{\tau\in\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_\tau+\mathbf{1}_{\{\tau\ge s\}}\mathcal{Z}_s\Big]+\varepsilon$. Eventually, taking infimum over $\mathbb{P}\in\mathcal{P}(t,\omega)$ on the right-hand-side and then letting $\varepsilon\to0$, we obtain (7.18).

2) *As to the reverse of* (7.18), *it suffices to show for a given* $\mathbb{P}\in\mathcal{P}(t,\omega)$ *that*

$$\sup_{\tau\in\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_\tau+\mathbf{1}_{\{\tau\ge s\}}\mathcal{Z}_s\Big]\le\sup_{\tau\in\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_\tau\big]. \tag{7.23}$$

Let us start with the main idea of proving (7.23)*: Contrary to* (7.19)*, we need upper bounds for* $\overline{Z}^{t,\omega}_s$ *this time. First note that* $\overline{Z}^{t,\omega}_s(\widetilde{\omega})\le\sup_{\zeta\in\mathcal{T}^s}\mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\Big[Y^{s,\omega\otimes_t\widetilde{\omega}}_\zeta\Big]$, $\forall\,\widetilde{\omega}\in\Omega^t$. *Given* $\zeta\in\mathcal{T}^s$, (2.6) *implies that*

$$\mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\Big[Y^{s,\omega\otimes_t\widetilde{\omega}}_\zeta\Big]=\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\zeta(\Pi^t_s)}\big|\mathcal{F}^t_s\big](\widetilde{\omega})\le\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\widehat{\tau}}\big|\mathcal{F}^t_s\big](\widetilde{\omega}) \tag{7.24}$$

holds for any $\widetilde{\omega}\in\Omega^t$ *except on a* $\mathbb{P}-$*null set* $\mathcal{N}_\zeta$, *where* $\widehat{\tau}$ *is an optimal stopping time. Since* $\mathcal{T}^s$ *is an uncountable set, we can not take supremum over* $\zeta\in\mathcal{T}^s$ *for* $\mathbb{P}-a.s.$ $\widetilde{\omega}\in\Omega^t$ *in* (7.24) *to obtain*

$$\mathcal{Z}_s\le\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\widehat{\tau}}\big|\mathcal{F}^{\mathbb{P}}_s\big],\quad\mathbb{P}-a.s. \tag{7.25}$$

To overcome this difficulty, we shall consider a "dense" countable subset Γ *of* $\mathcal{T}^s$ *in sense of* (7.26).

2a) *Construction of* Γ*:* For any $n\in\mathbb{N}$, we set $\mathscr{D}_n:=\big((s,T)\cap\{i2^{-n}\}_{i\in\mathbb{N}}\big)\cup\{T\}$ and $\mathscr{D}:=\underset{n\in\mathbb{N}}{\cup}\mathscr{D}_n$. Given $q\in\mathscr{D}$, we simply denote the countable subset Θ^s_q of $\mathcal{F}^s_q$ by $\{O^q_j\}_{j\in\mathbb{N}}$ and define $\Upsilon^q_k:=\Big\{q\mathbf{1}_{\underset{j\in I}{\cup}O^q_j}+T\mathbf{1}_{\underset{j\in I}{\cap}(O^q_j)^c}:I\subset\{1,\cdots,k\}\Big\}\subset\mathcal{T}^s$, $\forall\,k\in\mathbb{N}$. For any $n,k\in\mathbb{N}$, we set $\Gamma_{n,k}:=\Big\{\underset{q\in\mathscr{D}_n}{\wedge}\tau_q:\tau_q\in\Upsilon^q_k\Big\}\subset\mathcal{T}^s$. Then $\Gamma:=\underset{n,k\in\mathbb{N}}{\cup}\Gamma_{n,k}$ is clearly a countable subset of $\mathcal{T}^s$.

Since the filtration $\mathbf{F}^{\mathbb{P}}$ is right-continuous, and since the process $\mathcal{Y}$ is right-continuous and left upper semi-continuous by Remark 3.1 (2), the classic optimal stopping theory shows that $\underset{\tau\in\mathcal{T}^{\mathbb{P}}_s}{\operatorname{esssup}}\,\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_\tau\big|\mathcal{F}^{\mathbb{P}}_s\big]$ admits an optimal stopping time $\widehat{\tau}\in\mathcal{T}^{\mathbb{P}}_s$, which is the first time after s the process $\mathcal{Y}$ meets the RCLL modification of its Snell envelope $\Big\{\underset{\tau\in\mathcal{T}^{\mathbb{P}}_r}{\operatorname{esssup}}\,\mathbb{E}_{\mathbb{P}}[\mathcal{Y}_\tau|\mathcal{F}^{\mathbb{P}}_r]\Big\}_{r\in[t,T]}$.

Fix $\varepsilon>0$. We claim that there exists a $\widehat{\tau}'\in\mathcal{T}^t_s$ such that

$$\mathbb{E}_{\mathbb{P}}\big[\big|\mathcal{Y}_{\widehat{\tau}'}-\mathcal{Y}_{\widehat{\tau}}\big|\big]<\varepsilon/4. \tag{7.26}$$

To see this, let n be an integer ≥2. Given $i=1,\cdots,n$, we set $s^n_i:=s+\frac{i}{n}(T-s)$ and $A^n_i:=\{s^n_{i-1}<\widehat{\tau}\le s^n_i\}\in\mathcal{F}^{\mathbb{P}}_{s^n_i}$ with $s^n_0=-1$. By e.g. Problem 2.7.3 of [19], there exists an $(A')^n_i\in\mathcal{F}^t_{s^n_i}$ such that $A^n_i\,\Delta\,(A')^n_i\in\mathscr{N}^{\mathbb{P}}$. Define

$(\mathcal{A}')^n_i := (A')^n_i \backslash \underset{i'<i}{\cup} (A')^n_{i'} \in \mathcal{F}^t_{s^n_i}$ and $\mathcal{A}'_n := \overset{n}{\underset{i=1}{\cup}} (\mathcal{A}')^n_i = \overset{n}{\underset{i=1}{\cup}} (A')^n_i \in \mathcal{F}^t_T$. Then $\tau_n := \sum_{i=1}^n \mathbf{1}_{A^n_i}\, s^n_i$ is a $\mathcal{T}^{\mathbb{P}}_s$−stopping time while $\tau'_n := \sum_{i=1}^n \mathbf{1}_{(\mathcal{A}')^n_i}\, s^n_i + \mathbf{1}_{(\mathcal{A}'_n)^c} T$ defines an $\mathcal{T}^t_s$−stopping time. Clearly, τ_n coincides with τ'_n over $\overset{n}{\underset{i=1}{\cup}} \big(A^n_i \cap (\mathcal{A}')^n_i\big)$, whose complement $\overset{n}{\underset{i=1}{\cup}} \big(A^n_i \backslash (\mathcal{A}')^n_i\big)$ belongs to $\mathscr{N}^{\mathbb{P}}$ by a similar argument to (7.6). To wit, $\tau_n = \tau'_n$, $\mathbb{P}$−a.s. Since $\lim\limits_{n\to\infty} \tau_n = \widehat{\tau}$ and since $\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_*\big] < \infty$ by (3.2), we can deduce from the right-continuity of the shifted process $\mathcal{Y}$ and the dominated convergence theorem that

$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}\big[\big|\mathcal{Y}_{\tau'_n} - \mathcal{Y}_{\widehat{\tau}}\big|\big] = \lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}\big[\big|\mathcal{Y}_{\tau_n} - \mathcal{Y}_{\widehat{\tau}}\big|\big] = 0. \tag{7.27}$$

So there exists an $N \in \mathbb{N}$ such that $\mathbb{E}_{\mathbb{P}}\big[\big|\mathcal{Y}_{\tau'_N} - \mathcal{Y}_{\widehat{\tau}}\big|\big] < \varepsilon/4$, i.e., (7.26) holds for $\widehat{\tau}' = \tau'_N$.

2b) *In the next two steps, we will gradually demonstrate* (7.25).

Since $\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_*\big] < \infty$ and since $\zeta(\Pi^t_s) \in \mathcal{T}^t_s \subset \mathcal{T}^{\mathbb{P}}_s$ for any $\zeta \in \mathcal{T}^s$ by Lemma A.1, applying Lemma A.4 (1) with $X = B^t$ show that except on an $\mathcal{N} \in \mathscr{N}^{\mathbb{P}}$

$$\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\zeta(\Pi^t_s)}\big|\mathcal{F}^t_s\big] = \mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\zeta(\Pi^t_s)}\big|\mathcal{F}^{\mathbb{P}}_s\big] \le \underset{\tau\in\mathcal{T}^{\mathbb{P}}_s}{\text{esssup}}\, \mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_\tau\big|\mathcal{F}^{\mathbb{P}}_s\big] = \mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\widehat{\tau}}\big|\mathcal{F}^{\mathbb{P}}_s\big] = \mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\widehat{\tau}}\big|\mathcal{F}^t_s\big], \quad \forall\, \zeta \in \Gamma. \tag{7.28}$$

Also in light of (2.6), there exists another $\widetilde{\mathcal{N}} \in \mathscr{N}^{\mathbb{P}}$ such that for any $\widetilde{\omega} \in \widetilde{\mathcal{N}}^c$,

$$\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\zeta(\Pi^t_s)}\big|\mathcal{F}^t_s\big](\widetilde{\omega}) = \mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\Big[\big(\mathcal{Y}_{\zeta(\Pi^t_s)}\big)^{s,\widetilde{\omega}}\Big] = \mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\Big[Y^{s,\omega\otimes_t\widetilde{\omega}}_\zeta\Big], \quad \forall\, \zeta \in \Gamma, \tag{7.29}$$

where we used the fact that for any $\widehat{\omega} \in \Omega^s$

$$\big(\mathcal{Y}_{\zeta(\Pi^t_s)}\big)^{s,\widetilde{\omega}}(\widehat{\omega}) = \mathcal{Y}_{\zeta(\Pi^t_s)}(\widetilde{\omega} \otimes_s \widehat{\omega}) = Y\Big(\zeta\big(\Pi^t_s(\widetilde{\omega} \otimes_s \widehat{\omega})\big), \omega \otimes_t (\widetilde{\omega} \otimes_s \widehat{\omega})\Big) = Y\big(\zeta(\widehat{\omega}), (\omega \otimes_t \widetilde{\omega}) \otimes_s \widehat{\omega}\big) = Y^{s,\omega\otimes_t\widetilde{\omega}}_\zeta(\widehat{\omega}).$$

By (P1), there exist an extension $(\Omega^t, \mathcal{F}', \mathbb{P}')$ of $(\Omega^t, \mathcal{F}^t_T, \mathbb{P})$ and $\Omega' \in \mathcal{F}'$ with $\mathbb{P}'(\Omega') = 1$ such that for any $\widetilde{\omega} \in \Omega'$, $\mathbb{P}^{s,\widetilde{\omega}} \in \mathcal{P}(s, \omega \otimes_t \widetilde{\omega})$. Let $\widehat{A}$ be the $\mathcal{F}^t_T$−measurable set containing $\mathcal{N} \cup \widetilde{\mathcal{N}}$ and with $\mathbb{P}(\widehat{A}) = 0$.

Now, fix $\widetilde{\omega} \in \Omega' \cap \widehat{A}^c \in \mathcal{F}'$. There exists a $\zeta_{\widetilde{\omega}} \in \mathcal{T}^s$ such that

$$\sup_{\zeta\in\mathcal{T}^s} \mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\big[Y^{s,\omega\otimes_t\widetilde{\omega}}_\zeta\big] \le \mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\big[Y^{s,\omega\otimes_t\widetilde{\omega}}_{\zeta_{\widetilde{\omega}}}\big] + \varepsilon/4. \tag{7.30}$$

2c) *Next, we will approximate $\zeta_{\widetilde{\omega}}$ by a sequence $\{\zeta^n\}_{n\in\mathbb{N}}$ in Γ:* As $\mathbb{P}^{s,\widetilde{\omega}} \in \mathcal{P}(s, \omega\otimes_t\widetilde{\omega})$, (3.2) shows that $\mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\big[Y^{s,\omega\otimes_t\widetilde{\omega}}_*\big] < \infty$. So there exists a $\delta = \delta(\widetilde{\omega}) > 0$ such that

$$\mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\big[\mathbf{1}_A Y^{s,\omega\otimes_t\widetilde{\omega}}_*\big] < \varepsilon/4 \quad \text{for any } A \in \mathcal{F}^s_T \text{ with } \mathbb{P}^{s,\widetilde{\omega}}(A) < \delta. \tag{7.31}$$

Given $n \in \mathbb{N}$ and $i \in \big\{\lfloor 2^n s\rfloor, \cdots, \lfloor 2^n T\rfloor\big\}$, we set $q^n_i := \frac{i+1}{2^n} \wedge T \in \mathscr{D}_n$ and $\widetilde{A}^n_i := \{\frac{i}{2^n} \le \zeta_{\widetilde{\omega}} < \frac{i+1}{2^n}\} \in \mathcal{F}^s_{q^n_i}$. Lemma A.8 shows that for some sequence $\big\{O^{n,i}_\ell\big\}_{\ell\in\mathbb{N}}$ in $\Theta^s_{q^n_i} = \big\{O^{q^n_i}_j\big\}_{j\in\mathbb{N}}$

$$\widetilde{A}^n_i \subset \underset{\ell\in\mathbb{N}}{\cup} O^{n,i}_\ell \quad \text{and} \quad \mathbb{P}^{s,\widetilde{\omega}}(\widetilde{A}^n_i) > \mathbb{P}^{s,\widetilde{\omega}}\Big(\underset{\ell\in\mathbb{N}}{\cup} O^{n,i}_\ell\Big) - \frac{\delta}{\lfloor 2^n T\rfloor^2}. \tag{7.32}$$

Moreover, there exists an $\ell^n_i \in \mathbb{N}$ such that

$$\mathbb{P}^{s,\widetilde{\omega}}\big(\mathcal{O}^n_i\big) > \mathbb{P}^{s,\widetilde{\omega}}\Big(\underset{\ell\in\mathbb{N}}{\cup} O^{n,i}_\ell\Big) - \frac{\delta}{\lfloor 2^n T\rfloor^2} \tag{7.33}$$

with $\mathcal{O}^n_i := \overset{\ell^n_i}{\underset{\ell=1}{\cup}} O^{n,i}_\ell \in \mathcal{F}^s_{q^n_i}$. Clearly, $\zeta^n_i := q^n_i \mathbf{1}_{\mathcal{O}^n_i} + T\mathbf{1}_{(\mathcal{O}^n_i)^c} \in \Upsilon^{q^n_i}_{k^n_i}$ for some $k^n_i \in \mathbb{N}$. Setting $\widehat{\mathcal{O}}^n_i := \mathcal{O}^n_i \backslash \overset{i-1}{\underset{i'=\lfloor 2^n s\rfloor}{\cup}} \mathcal{O}^n_{i'} \in \mathcal{F}^s_{q^n_i}$, similar to (7.6) we can deduce that

$$\widetilde{A}^n_i \backslash \widehat{\mathcal{O}}^n_i = \widetilde{A}^n_i \cap \Big[(\mathcal{O}^n_i)^c \cup \big(\overset{i-1}{\underset{i'=\lfloor 2^n s\rfloor}{\cup}} \mathcal{O}^n_{i'}\big)\Big] \subset \Big(\big(\underset{\ell\in\mathbb{N}}{\cup} O^{n,i}_\ell\big) \backslash \mathcal{O}^n_i\Big) \cup \Big(\overset{i-1}{\underset{i'=\lfloor 2^n s\rfloor}{\cup}} \big(\mathcal{O}^n_{i'} \cap (\widetilde{A}^n_{i'})^c\big)\Big).$$

It then follows from (7.32) and (7.33) that

$$\mathbb{P}^{s,\widetilde{\omega}}(\widetilde{A}_i^n \backslash \widehat{\mathcal{O}}_i^n) \le \mathbb{P}^{s,\widetilde{\omega}}\Big(\big(\underset{\ell\in\mathbb{N}}{\cup} O_\ell^{n,i}\big)\backslash \mathcal{O}_i^n\Big) + \sum_{i'=\lfloor 2^n s\rfloor}^{i-1} \mathbb{P}^{s,\widetilde{\omega}}\Big(\big(\underset{\ell\in\mathbb{N}}{\cup} O_\ell^{n,i'}\big)\backslash \widetilde{A}_{i'}^n\Big) < \frac{i\delta}{\lfloor 2^n T\rfloor^2} \le \frac{\delta}{\lfloor 2^n T\rfloor}. \tag{7.34}$$

Set $\widehat{\mathcal{O}}_n := \underset{i=\lfloor 2^n s\rfloor}{\overset{\lfloor 2^n T\rfloor}{\cup}} \widehat{\mathcal{O}}_i^n = \underset{i=\lfloor 2^n s\rfloor}{\overset{\lfloor 2^n T\rfloor}{\cup}} \mathcal{O}_i^n$ and $k_n := \max\{k_i^n : i = \lfloor 2^n s\rfloor, \cdots, \lfloor 2^n T\rfloor\}$, we see that $\widehat{\zeta}^n := \underset{i=\lfloor 2^n s\rfloor}{\overset{\lfloor 2^n T\rfloor}{\wedge}} \widehat{\zeta}_i^n = \sum_{i=\lfloor 2^n s\rfloor}^{\lfloor 2^n T\rfloor} q_i^n \mathbf{1}_{\widehat{\mathcal{O}}_i^n} + \mathbf{1}_{\widehat{\mathcal{O}}_n^c} T$ is a stopping time of Γ_{n,k_n}, which equals to $\zeta^n := \sum_{i=\lfloor 2^n s\rfloor}^{\lfloor 2^n T\rfloor} q_i^n \mathbf{1}_{\widetilde{A}_i^n} \in \mathcal{T}^s$ over $\mathcal{A}_n := \underset{i=\lfloor 2^n s\rfloor}{\overset{\lfloor 2^n T\rfloor}{\cup}} \big(\widetilde{A}_i^n \cap \widehat{\mathcal{O}}_i^n\big) \in \mathcal{F}_T^s$. As $\underset{i=\lfloor 2^n s\rfloor}{\overset{\lfloor 2^n T\rfloor}{\cup}} \widetilde{A}_i^n = \Omega^s$, (7.34) implies that

$$\mathbb{P}^{s,\widetilde{\omega}}(\mathcal{A}_n^c) = \mathbb{P}^{s,\widetilde{\omega}}\Big(\underset{i=\lfloor 2^n s\rfloor}{\overset{\lfloor 2^n T\rfloor}{\cup}} \big(\widetilde{A}_i^n \backslash \widehat{\mathcal{O}}_i^n\big)\Big) = \sum_{i=\lfloor 2^n s\rfloor}^{\lfloor 2^n T\rfloor} \mathbb{P}^{s,\widetilde{\omega}}\big(\widetilde{A}_i^n \backslash \widehat{\mathcal{O}}_i^n\big) < \delta. \tag{7.35}$$

It then follows from (7.31) that

$$\mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\Big[\big|Y_{\zeta^n}^{s,\omega\otimes_t\widetilde{\omega}} - Y_{\widehat{\zeta}^n}^{s,\omega\otimes_t\widetilde{\omega}}\big|\Big] = \mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\Big[\mathbf{1}_{\mathcal{A}_n^c}\big|Y_{\zeta^n}^{s,\omega\otimes_t\widetilde{\omega}} - Y_{\widehat{\zeta}^n}^{s,\omega\otimes_t\widetilde{\omega}}\big|\Big] \le 2\mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\Big[\mathbf{1}_{\mathcal{A}_n^c} Y_*^{s,\omega\otimes_t\widetilde{\omega}}\Big] < \varepsilon/2,$$

which together with (7.28) and (7.29) shows that

$$\mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\big[Y_{\zeta^n}^{s,\omega\otimes_t\widetilde{\omega}}\big] < \mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\big[Y_{\widehat{\zeta}^n}^{s,\omega\otimes_t\widetilde{\omega}}\big] + \varepsilon/2 \le \mathbb{E}_{\mathbb{P}}[\mathcal{Y}_{\widehat{\tau}}|\mathcal{F}_s^t](\widetilde{\omega}) + \varepsilon/2.$$

Since $\lim_{n\to\infty}\downarrow \zeta^n = \zeta_{\widetilde{\omega}}$ and since $\mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\big[Y_*^{s,\omega\otimes_t\widetilde{\omega}}\big] < \infty$, letting $n\to\infty$, we can deduce from (7.30), the right-continuity of the shifted process $Y^{s,\omega\otimes_t\widetilde{\omega}}$ and the dominated convergence theorem that for any $\widetilde{\omega} \in \Omega' \cap \widehat{A}^c$

$$\mathcal{Z}_s(\widetilde{\omega}) = \overline{Z}_s(\omega\otimes_t\widetilde{\omega}) \le \sup_{\zeta\in\mathcal{T}^s} \mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\big[Y_\zeta^{s,\omega\otimes_t\widetilde{\omega}}\big] \le \mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\big[Y_{\zeta_{\widetilde{\omega}}}^{s,\omega\otimes_t\widetilde{\omega}}\big] + \varepsilon/4 = \lim_{n\to\infty} \mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\big[Y_{\zeta^n}^{s,\omega\otimes_t\widetilde{\omega}}\big] + \varepsilon/4 \le \mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\widehat{\tau}}\big|\mathcal{F}_s^t\big](\widetilde{\omega}) + \frac{3}{4}\varepsilon.$$

Since $\mathcal{Z}_s \in \mathcal{F}_s^t$ by Remark 4.2 and Proposition 2.1 (2), an analogy to (7.16) yields that

$$\mathcal{Z}_s \le \mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\widehat{\tau}}\big|\mathcal{F}_s^t\big] + \frac{3}{4}\varepsilon, \quad \mathbb{P}-\text{a.s.} \tag{7.36}$$

If sending ε to 0 and applying Lemma A.4 (1) with $X = B^t$ now, we will immediately obtain (7.25).

2d) Given $\tau \in \mathcal{T}^t$, we set $\overline{\tau} := \mathbf{1}_{\{\tau<s\}}\tau + \mathbf{1}_{\{\tau\ge s\}}\widehat{\tau}'$. For any $r \in [t,s)$, as $\widehat{\tau}' \in \mathcal{T}_s^t$, one can deduce that $\{\overline{\tau} \le r\} = \{\tau < s\} \cap \{\tau \le r\} = \{\tau \le r\} \in \mathcal{F}_r^t$. On the other hand, for any $r \in [s,T]$, $\{\overline{\tau} \le r\} = \big(\{\tau < s\} \cap \{\tau \le r\}\big) \cup \big(\{\tau \ge s\} \cap \{\widehat{\tau}' \le r\}\big) = \{\tau < s\} \cup \big(\{\tau \ge s\} \cap \{\widehat{\tau}' \le r\}\big) \in \mathcal{F}_r^t$. So $\overline{\tau} \in \mathcal{T}^t$ and it follows from (7.36) and (7.26) that

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_\tau + \mathbf{1}_{\{\tau\ge s\}}\mathcal{Z}_s\Big] &\le \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_{\tau\wedge s} + \mathbf{1}_{\{\tau\ge s\}}\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\widehat{\tau}}\big|\mathcal{F}_s^t\big]\Big] + \frac{3}{4}\varepsilon = \mathbb{E}_{\mathbb{P}}\Big[\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_{\tau\wedge s} + \mathbf{1}_{\{\tau\ge s\}}\mathcal{Y}_{\widehat{\tau}}\big|\mathcal{F}_s^t\big]\Big] + \frac{3}{4}\varepsilon \\ &= \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_\tau + \mathbf{1}_{\{\tau\ge s\}}\mathcal{Y}_{\widehat{\tau}}\Big] + \frac{3}{4}\varepsilon \le \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_\tau + \mathbf{1}_{\{\tau\ge s\}}\mathcal{Y}_{\widehat{\tau}'}\Big] + \varepsilon = \mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\overline{\tau}}\big] + \varepsilon \le \sup_{\tau\in\mathcal{T}^t} \mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_\tau\big] + \varepsilon.\end{aligned}$$

Taking supremum over $\tau \in \mathcal{T}^t$ on the left-hand-side then letting $\varepsilon \to 0$ yield (7.23). So we proved the proposition. □

Proof of Proposition 4.2: 1) Fix $\omega \in \Omega$. Letting $0 \le t < s \le T$ such that $\sup_{t\le r<r'\le s} \big|\omega(r') - \omega(r)\big| \le T$. we shall show

$$\big|\overline{Z}_s(\omega) - \overline{Z}_t(\omega)\big| \le 2\rho_\alpha(\delta_{t,s}), \tag{7.37}$$

where $\alpha := 1 + \|\omega\|_{0,T}$ and $\delta_{t,s} := (s-t) \vee \sup_{t\le r<r'\le s} \big|\omega(r') - \omega(r)\big| \le T$.

Given $\varepsilon > 0$, there exists a $\mathbb{P} = \mathbb{P}(t,\omega,\varepsilon) \in \mathcal{P}(t,\omega)$ such that

$$\overline{Z}_t(\omega) \ge \sup_{\tau\in\mathcal{T}^t} \mathbb{E}_{\mathbb{P}}\big[Y_\tau^{t,\omega}\big] - \varepsilon \ge \sup_{\tau\in\mathcal{T}^t} \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau<s\}}Y_\tau^{t,\omega} + \mathbf{1}_{\{\tau\ge s\}}\overline{Z}_s^{t,\omega}\Big] - \varepsilon \ge \mathbb{E}_{\mathbb{P}}\Big[\overline{Z}_s^{t,\omega}\Big] - \varepsilon, \tag{7.38}$$

where we used (7.23) in the second inequality and took $\tau = s$ in the last inequality. In light of (4.3)

$$\begin{aligned}\big|\overline{Z}_s(\omega)-\overline{Z}^{t,\omega}_s(\widetilde{\omega})\big| &= \big|\overline{Z}_s(\omega)-\overline{Z}(s,\omega\otimes_t\widetilde{\omega})\big| \le \rho_1\big(\|\omega-\omega\otimes_t\widetilde{\omega}\|_{0,s}\big) = \rho_1\Big(\sup_{r\in[t,s]}\big|\widetilde{\omega}(r)+\omega(t)-\omega(r)\big|\Big)\\ &\le \rho_1\Big(\sup_{r\in[t,s]}\big|\widetilde{\omega}(r)\big|+\sup_{r\in[t,s]}\big|\omega(r)-\omega(t)\big|\Big) \le \rho_1\Big(\sup_{r\in[t,(t+\delta_{t,s})\wedge T]}\big|B^t_r(\widetilde{\omega})\big|+\delta_{t,s}\Big), \quad \forall\,\widetilde{\omega}\in\Omega^t. \qquad (7.39)\end{aligned}$$

Since $\|\omega\|_{0,t} \le \|\omega\|_{0,T} < \alpha$, (7.38) and (4.4) imply that

$$\overline{Z}_s(\omega)-\overline{Z}_t(\omega) \le \mathbb{E}_{\mathbb{P}}\Big[\overline{Z}_s(\omega)-\overline{Z}^{t,\omega}_s\Big]+\varepsilon \le \mathbb{E}_{\mathbb{P}}\Big[\rho_1\Big(\delta_{t,s}+\sup_{r\in[t,(t+\delta_{t,s})\wedge T]}\big|B^t_r\big|\Big)\Big]+\varepsilon \le \rho_\alpha(\delta_{t,s})+\varepsilon.$$

Letting $\varepsilon \to 0$ yields that

$$\overline{Z}_s(\omega)-\overline{Z}_t(\omega) \le \rho_\alpha(\delta_{t,s}). \qquad (7.40)$$

On the other hand, let $\widehat{\mathbb{P}}$ be an arbitrary probability in $\mathcal{P}(t,\omega)$. Applying Proposition 4.1 yields that

$$\overline{Z}_t(\omega)-\overline{Z}_s(\omega) \le \sup_{\tau\in\mathcal{T}^t}\mathbb{E}_{\widehat{\mathbb{P}}}\Big[\mathbf{1}_{\{\tau<s\}}Y^{t,\omega}_\tau+\mathbf{1}_{\{\tau\ge s\}}\overline{Z}^{t,\omega}_s\Big]-\overline{Z}_s(\omega). \qquad (7.41)$$

For any $\tau \in \mathcal{T}^t$ and $\widetilde{\omega} \in \{\tau < s\}$, (3.1) shows that

$$\begin{aligned}Y^{t,\omega}_\tau(\widetilde{\omega})-Y^{t,\omega}_s(\widetilde{\omega}) &= Y\big(\tau(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)-Y\big(s,\omega\otimes_t\widetilde{\omega}\big) \le \rho_0\Big(\mathbf{d}_\infty\big((\tau(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}),(s,\omega\otimes_t\widetilde{\omega})\big)\Big)\\ &\le \rho_0\Big((s-t)+\sup_{r\in[t,T]}\big|\widetilde{\omega}\big(r\wedge\tau(\widetilde{\omega})\big)-\widetilde{\omega}\big(r\wedge s\big)\big|\Big) \le \rho_1\Big((s-t)+2\sup_{r\in[t,s]}\big|B^t_r(\widetilde{\omega})\big|\Big).\end{aligned}$$

Plugging this into (7.41), we can deduce from (4.4), (4.2) and (7.39) that

$$\begin{aligned}\overline{Z}_t(\omega)-\overline{Z}_s(\omega) &\le \sup_{\tau\in\mathcal{T}^t}\mathbb{E}_{\widehat{\mathbb{P}}}\Big[\mathbf{1}_{\{\tau<s\}}\rho_1\Big((s-t)+2\sup_{r\in[t,s]}\big|B^t_r\big|\Big)+\mathbf{1}_{\{\tau<s\}}Y^{t,\omega}_s+\mathbf{1}_{\{\tau\ge s\}}\overline{Z}^{t,\omega}_s-\overline{Z}_s(\omega)\Big]\\ &\le \rho_\alpha(s-t)+\mathbb{E}_{\widehat{\mathbb{P}}}\Big[\overline{Z}^{t,\omega}_s-\overline{Z}_s(\omega)\Big] \le 2\rho_\alpha(\delta_{t,s}),\end{aligned}$$

which together with (7.40) proves (7.37). As $\lim\limits_{t\nearrow s}\!\downarrow \delta_{t,s} = \lim\limits_{s\searrow t}\!\downarrow \delta_{t,s} = 0$, the continuity of $\overline{Z}$ easily follows.

2) Let $(t,\omega) \in [0,T]\times\Omega$ and $\mathbb{P} \in \mathcal{P}(t,\omega)$. Remark 4.2, Proposition 2.1 (2) and part (1) show that $\overline{Z}^{t,\omega}$ is an $\mathbf{F}^t-$adapted process with all continuous paths.

As $\mathbb{E}_{\mathbb{P}}[Y^{t,\omega}_*] < \infty$ by (3.2), using (7.16) and applying Lemma A.4 (1) with $X = B^t$ show that for any $s \in [t,T]$

$$Y^{t,\omega}_s \le \overline{Z}^{t,\omega}_s \le \mathbb{E}_{\mathbb{P}}[Y^{t,\omega}_*|\mathcal{F}^t_s] = \mathbb{E}_{\mathbb{P}}\big[Y^{t,\omega}_*\big|\mathcal{F}^{\mathbb{P}}_s\big], \quad \mathbb{P}-\text{a.s.}$$

Then by the continuity of process $\overline{Z}$ and the right continuity of processes Y, $\big\{\mathbb{E}_{\mathbb{P}}\big[Y^{t,\omega}_*\big|\mathcal{F}^{\mathbb{P}}_s\big]\big\}_{s\in[t,T]}$, it holds $\mathbb{P}-$a.s. that $Y^{t,\omega}_s \le \overline{Z}^{t,\omega}_s \le \mathbb{E}_{\mathbb{P}}\big[Y^{t,\omega}_*\big|\mathcal{F}^{\mathbb{P}}_s\big]$ for any $s \in [t,T]$. It follows that for any $\tau \in \mathcal{T}^{\mathbb{P}}$, $\big|\overline{Z}^{t,\omega}_\tau\big| \le Y^{t,\omega}_* + \mathbb{E}_{\mathbb{P}}\big[Y^{t,\omega}_*\big|\mathcal{F}^{\mathbb{P}}_\tau\big]$, $\mathbb{P}-$a.s. Hence, $\big\{\overline{Z}^{t,\omega}_\tau\big\}_{\tau\in\mathcal{T}^{\mathbb{P}}}$ is $\mathbb{P}-$uniformly integrable. □

Proof of Proposition 4.3: When $t = T$, (4.6) trivially holds as an equality. So let us fix $(t,\omega) \in [0,T)\times\Omega$ and $\nu \in \mathcal{T}^t$. We still define $\mathcal{Y}$ and $\mathcal{Z}$ as in (7.17). To obtain (4.6), it suffices to show for a given $\mathbb{P} \in \mathcal{P}(t,\omega)$ that

$$\sup_{\tau\in\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau<\nu\}}\mathcal{Y}_\tau+\mathbf{1}_{\{\tau\ge\nu\}}\mathcal{Z}_\nu\Big] \le \sup_{\tau\in\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_\tau\big]. \qquad (7.42)$$

Define the Snell envelope $Z^{\mathbb{P}}$ of $\mathcal{Y}$ under $\mathbb{P}$: $Z^{\mathbb{P}}_s := \underset{\tau\in\mathcal{T}^{\mathbb{P}}_s}{\operatorname{esssup}}\,\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_\tau\big|\mathcal{F}^{\mathbb{P}}_s\big]$, $s \in [t,T]$. Since the filtration $\mathbf{F}^{\mathbb{P}}$ is right-continuous, and since the process $\mathcal{Y}$ is right-continuous and left upper semi-continuous by Remark 3.1 (2), the classic optimal stopping theory shows that $Z^{\mathbb{P}}$ admits an RCLL modification $\big\{\mathscr{Z}^{\mathbb{P}}_s\big\}_{s\in[t,T]}$ such that for any $\varsigma \in \mathcal{T}^{\mathbb{P}}$, $\tau^\varsigma_{\mathbb{P}} := \inf\big\{r \in [\varsigma,T] : \mathscr{Z}^{\mathbb{P}}_r = \mathcal{Y}_r\big\} \in \mathcal{T}^{\mathbb{P}}_\varsigma$ is an optimal stopping time for $\underset{\tau\in\mathcal{T}^{\mathbb{P}}_\varsigma}{\operatorname{esssup}}\,\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_\tau\big|\mathcal{F}^{\mathbb{P}}_\varsigma\big]$.

For any $s \in [t,T]$, we know from (7.25) that $\mathcal{Z}_s \le \mathbb{E}_\mathbb{P}\big[\mathcal{Y}_{\tau^s_\mathbb{P}}\big|\mathcal{F}^\mathbb{P}_s\big] = Z^\mathbb{P}_s = \mathscr{Z}^\mathbb{P}_s$, $\mathbb{P}-$a.s. The continuity of $\overline{Z}$ (by Proposition 4.2) and the right-continuity of $\mathscr{Z}^\mathbb{P}$ then imply that

$$\mathbb{P}\big\{\mathcal{Z}_s \le \mathscr{Z}^\mathbb{P}_s,\ \ \forall\, s \in [t,T]\big\} = 1. \tag{7.43}$$

It follows that

$$\mathcal{Z}_\nu \le \mathscr{Z}^\mathbb{P}_\nu = \underset{\tau\in\mathcal{T}^\mathbb{P}_\nu}{\operatorname{esssup}}\,\mathbb{E}_\mathbb{P}\big[\mathcal{Y}_\tau\big|\mathcal{F}^\mathbb{P}_\nu\big] = \mathbb{E}_\mathbb{P}\big[\mathcal{Y}_{\tau^\nu_\mathbb{P}}\big|\mathcal{F}^\mathbb{P}_\nu\big], \quad \mathbb{P}-\text{a.s.}, \tag{7.44}$$

where the first equality is due to a well-known result in the optimal stopping theorem, see e.g. Theorem D.7 of [20].

Let $\tau \in \mathcal{T}^t$ and Set $\overline{\tau} := \mathbf{1}_{\{\tau<\nu\}}\tau + \mathbf{1}_{\{\tau\ge\nu\}}\tau^\nu_\mathbb{P}$. Given $r \in [t,T]$, since $\{\tau<\nu\} \in \mathcal{F}^t_{\tau\wedge\nu}$ and $\tau^\nu_\mathbb{P} \in \mathcal{T}^\mathbb{P}_\nu$, we see that $\{\tau\ge\nu\}\in\mathcal{F}^t_{\tau\wedge\nu}\subset\mathcal{F}^\mathbb{P}_\nu\subset\mathcal{F}^\mathbb{P}_{\tau^\nu_\mathbb{P}}$. It follows that $\{\tau<\nu\}\cap\{\tau\le r\}\in\mathcal{F}^t_r\subset\mathcal{F}^\mathbb{P}_r$ and $\{\tau\ge\nu\}\cap\{\tau^\nu_\mathbb{P}\le r\}\in\mathcal{F}^\mathbb{P}_r$, which together show

$$\{\overline{\tau} \le r\} = \big(\{\tau<\nu\}\cap\{\tau\le r\}\big) \cup \big(\{\tau\ge\nu\}\cap\{\tau^\nu_\mathbb{P}\le r\}\big) \in \mathcal{F}^\mathbb{P}_r.$$

Thus $\overline{\tau} \in \mathcal{T}^\mathbb{P}$. For any $\varepsilon > 0$, similar to (7.26), there exists a $\overline{\tau}_\varepsilon \in \mathcal{T}^t$ such that $\mathbb{E}_\mathbb{P}\big[\big|\mathcal{Y}_{\overline{\tau}_\varepsilon} - \mathcal{Y}_{\overline{\tau}}\big|\big] < \varepsilon$. Then we can deduce from (7.44) that

$$\begin{aligned}
\mathbb{E}_\mathbb{P}\Big[\mathbf{1}_{\{\tau<\nu\}}\mathcal{Y}_\tau + \mathbf{1}_{\{\tau\ge\nu\}}\mathcal{Z}_\nu\Big] &\le \mathbb{E}_\mathbb{P}\big[\mathbf{1}_{\{\tau<\nu\}}\mathcal{Y}_\tau\big] + \mathbb{E}_\mathbb{P}\Big[\mathbf{1}_{\{\tau\ge\nu\}}\mathbb{E}_\mathbb{P}\big[\mathcal{Y}_{\tau^\nu_\mathbb{P}}\big|\mathcal{F}^\mathbb{P}_\nu\big]\Big] = \mathbb{E}_\mathbb{P}\big[\mathbf{1}_{\{\tau<\nu\}}\mathcal{Y}_\tau\big] + \mathbb{E}_\mathbb{P}\Big[\mathbb{E}_\mathbb{P}\big[\mathbf{1}_{\{\tau\ge\nu\}}\mathcal{Y}_{\tau^\nu_\mathbb{P}}\big|\mathcal{F}^\mathbb{P}_\nu\big]\Big] \\
&= \mathbb{E}_\mathbb{P}\Big[\mathbf{1}_{\{\tau<\nu\}}\mathcal{Y}_\tau + \mathbf{1}_{\{\tau\ge\nu\}}\mathcal{Y}_{\tau^\nu_\mathbb{P}}\Big] = \mathbb{E}_\mathbb{P}\big[\mathcal{Y}_{\overline{\tau}}\big] \le \mathbb{E}_\mathbb{P}\big[\mathcal{Y}_{\overline{\tau}_\varepsilon}\big] + \varepsilon \le \sup_{\zeta\in\mathcal{T}^t}\mathbb{E}_\mathbb{P}\big[\mathcal{Y}_\zeta\big] + \varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$ and then taking supremum over $\tau \in \mathcal{T}^t$ on the left-hand-side yield (7.42).

7.4 Proofs of the results in Section 5

Proof of Remark 5.1: Let $\tau \in \mathcal{T}$ and $(t,\omega) \in [0,T]\times\Omega$. As Y_τ and $\overline{Z}_\tau$ are $\mathcal{F}_T-$measurable by Remark 4.2, Proposition 2.1 (1) shows that $(Y_\tau)^{t,\omega}$ and $(\overline{Z}_\tau)^{t,\omega}$ are in turn $\mathcal{F}^t_T-$measurable. Since $Y_{\tau\wedge t}, \overline{Z}_{\tau\wedge t} \in \mathcal{F}_t$, one can deduce from (2.2) that

$$\begin{aligned}
\big|(\overline{Z}_\tau)^{t,\omega}(\widetilde{\omega})\big| &= \mathbf{1}_{\{\tau(\omega\otimes_t\widetilde{\omega})<t\}}\big|\overline{Z}(\tau(\omega\otimes_t\widetilde{\omega})\wedge t, \omega\otimes_t\widetilde{\omega})\big| + \mathbf{1}_{\{\tau(\omega\otimes_t\widetilde{\omega})\ge t\}}\big|\overline{Z}(\tau(\omega\otimes_t\widetilde{\omega})\vee t, \omega\otimes_t\widetilde{\omega})\big| \\
&= \mathbf{1}_{\{\tau(\omega\otimes_t\widetilde{\omega})<t\}}\big|\overline{Z}_{\tau\wedge t}(\omega\otimes_t\widetilde{\omega})\big| + \mathbf{1}_{\{\tau(\omega\otimes_t\widetilde{\omega})\ge t\}}\big|\overline{Z}^{t,\omega}\big((\tau\vee t)^{t,\omega}(\widetilde{\omega}),\widetilde{\omega}\big)\big| \\
&= \mathbf{1}_{\{\tau(\omega\otimes_t\widetilde{\omega})<t\}}\big|\overline{Z}_{\tau\wedge t}(\omega)\big| + \mathbf{1}_{\{\tau(\omega\otimes_t\widetilde{\omega})\ge t\}}\big|\overline{Z}^{t,\omega}_{(\tau\vee t)^{t,\omega}}(\widetilde{\omega})\big|,
\end{aligned}$$

and similarly $\quad \big|(Y_\tau)^{t,\omega}(\widetilde{\omega})\big| \le \mathbf{1}_{\{\tau(\omega\otimes_t\widetilde{\omega})<t\}}\big|Y_{\tau\wedge t}(\omega)\big| + \mathbf{1}_{\{\tau(\omega\otimes_t\widetilde{\omega})\ge t\}}Y^{t,\omega}_*(\widetilde{\omega}), \quad \forall\,\widetilde{\omega}\in\Omega^t.$

For any $\mathbb{P}\in\mathcal{P}(t,\omega)$, as $(\tau\vee t)^{t,\omega} \in \mathcal{T}^t$ by Corollary 2.1, we see from (3.2), (4.2) and Proposition 4.2 that

$$\mathbb{E}_\mathbb{P}\big[\big|(Y_\tau)^{t,\omega}\big| + \big|(\overline{Z}_\tau)^{t,\omega}\big|\big] \le \big|Y_{\tau\wedge t}(\omega)\big| + \big|\overline{Z}_{\tau\wedge t}(\omega)\big| + \mathbb{E}_\mathbb{P}\big[Y^{t,\omega}_*\big] + \mathbb{E}_\mathbb{P}\Big[\big|\overline{Z}^{t,\omega}_{(\tau\vee t)^{t,\omega}}\big|\Big] < \infty.$$

Thus, $Y_\tau, \overline{Z}_\tau \in \mathscr{L}_t$. □

Proof of Theorem 5.1:

1) We first show that the random time τ^* defined in (5.1) is an $\mathbf{F}-$stopping time: Given $\delta \ge 0$, we define $\tau_\delta := \inf\big\{t\in[0,T] : \overline{Z}_t \le Y_t + \delta\big\}$. Since

$$\overline{Z}_T(\omega) = \inf_{\mathbb{P}\in\mathcal{P}(T,\omega)}\mathbb{E}_\mathbb{P}\big[Y^{T,\omega}_T\big] = \inf_{\mathbb{P}\in\mathcal{P}(T,\omega)}\mathbb{E}_\mathbb{P}\big[Y(T,\omega)\big] = Y(T,\omega), \quad \forall\,\omega\in\Omega, \tag{7.45}$$

it follows that $\overline{Z}_T = Y_T \le Y_T + \delta$. So $\tau_\delta \le T$. For any $s \in [0,T)$, Remark 3.1 (1), the continuity of process $\overline{Z}$ (by Proposition 4.2) as well as the $\mathbf{F}-$adaptness of Y and $\overline{Z}$ by Remark 4.2 imply that

$$\begin{aligned}
\{\tau_\delta > s\} &= \{\omega\in\Omega\colon \overline{Z}_t(\omega) - Y_t(\omega) > \delta,\ \forall\, t\in[0,s]\} = \bigcup_{i\in\mathbb{N}}\{\omega\in\Omega\colon \overline{Z}_t(\omega) - Y_t(\omega) \ge \delta + 1/i,\ \forall\, t\in[0,s]\} \\
&= \bigcup_{i\in\mathbb{N}}\{\omega\in\Omega\colon \overline{Z}_t(\omega) - Y_t(\omega) \ge \delta + 1/i,\ \forall\, t\in\mathbb{Q}_s\} = \bigcup_{i\in\mathbb{N}}\bigcap_{t\in\mathbb{Q}_s}\{\omega\in\Omega\colon \overline{Z}_t(\omega) - Y_t(\omega) \ge \delta + 1/i\} \in \mathcal{F}_s,
\end{aligned}$$

where $\mathbb{Q}_s := \big([0,s]\cap\mathbb{Q}\big)\cup\{s\}$. So τ_δ is an $\mathbf{F}-$stopping time. In particular, we see from (4.2) that

$$\tau^* := \inf\big\{t\in[0,T]:\overline{Z}_t = Y_t\big\} = \inf\big\{t\in[0,T]:\overline{Z}_t \le Y_t\big\}$$

is an $\mathbf{F}-$stopping time.

2) When $t=T$, (5.3) clearly holds. So let us fix $(t,\omega)\in[0,T)\times\Omega$ and $\gamma\in\mathcal{T}$. We still define $\mathcal{Y}$ and $\mathcal{Z}$ as in (7.17). If $\widehat{t}:=\gamma(\omega)\le t$, i.e., $\omega\in\big\{\gamma=\widehat{t}\,\big\}\in\mathcal{F}_{\widehat{t}}\subset\mathcal{F}_t$, Lemma 2.1 implies that $\omega\otimes_t\Omega^t\subset\big\{\gamma=\widehat{t}\,\big\}$. Then applying (2.2) to $\overline{Z}_{\widehat{t}}\in\mathcal{F}_{\widehat{t}}\subset\mathcal{F}_t$ yields that $\big(\overline{Z}_\gamma\big)^{t,\omega}(\widetilde{\omega})=\big(\overline{Z}_\gamma\big)(\omega\otimes_t\widetilde{\omega})=\overline{Z}\big(\gamma(\omega\otimes_t\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)=\overline{Z}\big(\widehat{t},\omega\otimes_t\widetilde{\omega}\big)=\overline{Z}\big(\widehat{t},\omega\big)$. It follows that

$$\underline{\mathscr{E}}_t\big[\,\overline{Z}_\gamma\big](\omega)=\inf_{\mathbb{P}\in\mathcal{P}(t,\omega)}\mathbb{E}_{\mathbb{P}}\Big[\big(\overline{Z}_\gamma\big)^{t,\omega}\Big]=\inf_{\mathbb{P}\in\mathcal{P}(t,\omega)}\mathbb{E}_{\mathbb{P}}\big[\,\overline{Z}(\widehat{t},\omega)\big]=\overline{Z}\big(\widehat{t},\omega\big)=\overline{Z}\big(\gamma(\omega)\wedge t,\omega\big)=\big(\overline{Z}_{\gamma\wedge t}\big)(\omega). \tag{7.46}$$

On the other hand, if $\gamma(\omega)>t$, i.e., $\omega\in\{\gamma>t\}\in\mathcal{F}_t$. Lemma 2.1 again shows that $\omega\otimes_t\Omega^t\subset\{\gamma>t\}$. Applying Corollary 2.1 with $(\tau,s,r)=(\gamma,t,t)$ shows that $\gamma^{t,\omega}\in\mathcal{T}^t$, then taking $\tau=\nu=\gamma^{t,\omega}$ in (4.6) yields that

$$\big(\overline{Z}_{\gamma\wedge t}\big)(\omega)=\overline{Z}_t(\omega)\ge\inf_{\mathbb{P}\in\mathcal{P}(t,\omega)}\sup_{\tau\in\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau<\gamma^{t,\omega}\}}\mathcal{Y}_\tau+\mathbf{1}_{\{\tau\ge\gamma^{t,\omega}\}}\mathcal{Z}_{\gamma^{t,\omega}}\Big]\ge\inf_{\mathbb{P}\in\mathcal{P}(t,\omega)}\mathbb{E}_{\mathbb{P}}\big[\mathcal{Z}_{\gamma^{t,\omega}}\big]=\underline{\mathscr{E}}_t\big[\,\overline{Z}_\gamma\big](\omega), \tag{7.47}$$

which together with (7.46) shows that $\overline{Z}$ is an $\underline{\mathscr{E}}-$supermartingale.

Next, let us show the $\underline{\mathscr{E}}-$submartingality of $\big\{\overline{Z}_{\tau^*\wedge t}\big\}_{t\in[0,T]}$: If $\tau^*(\omega)\wedge\gamma(\omega)\le t$, an analogy to (7.46) shows that

$$\underline{\mathscr{E}}_t\big[\,\overline{Z}_{\tau^*\wedge\gamma}\big](\omega)=\big(\overline{Z}_{\tau^*\wedge\gamma\wedge t}\big)(\omega). \tag{7.48}$$

Suppose $\tau^*(\omega)\wedge\gamma(\omega)>t$, i.e., $\omega\in\{\tau^*\wedge\gamma>t\}\in\mathcal{F}_t$. By Lemma 2.1,

$$\omega\otimes_t\Omega^t\subset\{\tau^*\wedge\gamma>t\}. \tag{7.49}$$

The demonstration of

$$\big(\overline{Z}_{\tau^*\wedge\gamma\wedge t}\big)(\omega)\le\underline{\mathscr{E}}_t\big[\,\overline{Z}_{\tau^*\wedge\gamma}\big](\omega) \tag{7.50}$$

in case of $\tau^*(\omega)\wedge\gamma(\omega)>t$ is relatively lengthy. We split it into several steps. The main idea is: We approximate τ^* by the hitting time $\tau^n:=\inf\big\{s\in[0,T]:\overline{Z}_s\le Y_s+1/n\big\}$ and then approximate the corresponding shifted stopping time $\zeta^n:=\big(\gamma\wedge(\tau^n\vee t)\big)^{t,\omega}$ by stopping time ζ^n_k that takes finite values $t^k_i:=t+\frac{i}{k}(T-t)$, $i=1,\cdots,k$. We will paste in accordance with (P2) the local approximating minimizers $\mathbb{P}^i_{\widetilde{\omega}}$ of $\mathcal{Z}_{t^k_i}(\widetilde{\omega})$ over the set $\{\zeta^n_k=t^k_i\}$ backwardly to get a probability $\mathbb{P}_1\in\mathcal{P}(t,\omega)$ that satisfies $\mathbb{E}_{\mathbb{P}_1}\Big[\mathcal{Y}_\tau\big|\mathcal{F}^{\mathbb{P}_1}_{\zeta^n_k}\Big]\le\mathcal{Z}_{\zeta^n_k}+\varepsilon$ for all stopping times τ. Taking essential supremum over τ's shows that

$$\mathscr{Z}^{\mathbb{P}_1}_{\zeta^n_k}\le\mathcal{Z}_{\zeta^n_k}+\varepsilon, \tag{7.51}$$

where $\mathscr{Z}^{\mathbb{P}_1}$ denotes the Snell envelope of $\mathcal{Y}$ under the single probability $\mathbb{P}_1$. By the martingale property of $\mathscr{Z}^{\mathbb{P}_1}$,

$$\overline{Z}_t(\omega)\le\mathscr{Z}^{\mathbb{P}_1}_t\le\mathbb{E}_{\mathbb{P}_1}\Big[\mathscr{Z}^{\mathbb{P}_1}_{\zeta^n_k\wedge\tau_{\mathbb{P}_1}}\Big], \tag{7.52}$$

where $\tau_{\mathbb{P}_1}$ is the optimal stopping time for $\mathscr{Z}^{\mathbb{P}_1}$. As the first time $\mathscr{Z}^{\mathbb{P}_1}$ meets $\mathcal{Y}$, $\tau_{\mathbb{P}_1}\ge(\tau^*)^{t,\omega}$. Since $\tau^*=\lim_{n\to\infty}\uparrow\tau^n$ and $\lim_{k\to\infty}\zeta^n_k=\zeta^n$, for n,k large enough we have $\tau_{\mathbb{P}_1}\ge\zeta^n_k$ except for a tiny probability. Then combining (7.52) with (7.51) and applying a series of estimations yield that $\overline{Z}_t(\omega)\le\mathbb{E}_{\mathbb{P}_1}\big[\mathcal{Z}_{\zeta^n_k}\big]+\varepsilon\le\mathbb{E}_{\mathbb{P}}\big[\mathcal{Z}_{\zeta^n_k}\big]+\varepsilon$. Finally, letting $k,n\to\infty$, $\varepsilon\to0$ and taking infimum over $\mathbb{P}\in\mathcal{P}(t,\omega)$ lead to (7.50).

2a) *In the first step, we paste the local approximating minimizers $\mathbb{P}^i_{\widetilde{\omega}}$ of $\mathcal{Z}_{t^k_i}(\widetilde{\omega})$ over the set $\{\zeta^n_k=t^k_i\}$ backwardly.*

Fix $\mathbb{P}\in\mathcal{P}(t,\omega)$, $\varepsilon\in(0,1)$ and $\alpha,n,k,\lambda\in\mathbb{N}$ with $k\ge2$. We let $\{\omega^\alpha_j\}_{j\in\mathbb{N}}$ be a subsequence of $\{\widehat{\omega}^t_j\}_{j\in\mathbb{N}}$ in $O_\alpha(\mathbf{0}^t)$, and have seen from part (1) that $\tau^n:=\inf\big\{s\in[0,T]:\overline{Z}_s\le Y_s+1/n\big\}$ is an $\mathbf{F}-$stopping time. Since $\gamma(\omega\otimes_t\Omega^t)\subset(t,T]$ and $\tau^*(\omega\otimes_t\Omega^t)\subset(t,T]$ by (7.49), Corollary 2.1 shows that both $\zeta^n:=\big(\gamma\wedge(\tau^n\vee t)\big)^{t,\omega}$ and $\zeta^*:=(\tau^*)^{t,\omega}$ are $\mathcal{T}^t-$stopping times. We set $t_i=t^k_i:=t+\frac{i}{k}(T-t)$ for $i=1,\cdots,k$ and define $\zeta^n_k:=\mathbf{1}_{\{\zeta^n\le t_1\}}t_1+\sum_{i=2}^k\mathbf{1}_{\{t_{i-1}<\zeta^n\le t_i\}}t_i\in\mathcal{T}^t$.

There exists a $\delta\in\mathbb{Q}_+$ such that $\rho_0(\delta)\vee\widehat{\rho}_0(\delta)\vee\rho_1(\delta)<\varepsilon/4$. Given $(i,j)\in\{1,\cdots,k\}\times\{1,\cdots,\lambda\}$, we set $\mathcal{A}^i_j:=\{\zeta^n_k=t_i\}\cap\Big(O^{t_i}_\delta(\omega^\alpha_j)\backslash\underset{j'<j}{\cup}O^{t_i}_\delta(\omega^\alpha_{j'})\Big)\in\mathcal{F}^t_{t_i}$ by (2.1). There exists a $\mathbb{P}^i_j\in\mathcal{P}\big(t_i,\omega\otimes_t\omega^\alpha_j\big)$ such that $\overline{Z}_{t_i}(\omega\otimes_t\omega^\alpha_j)\ge\sup_{\tau\in\mathcal{T}^{t_i}}\mathbb{E}_{\mathbb{P}^i_j}\Big[Y^{t_i,\omega\otimes_t\omega^\alpha_j}_\tau\Big]-\varepsilon/4$. For any $\widetilde{\omega}\in\mathcal{A}^i_j$ with $\mathcal{A}^i_j\neq\emptyset$, similar to (7.20), one can deduce from (3.1) and (4.3) that

$$\begin{aligned}\sup_{\tau\in\mathcal{T}^{t_i}}\mathbb{E}_{\mathbb{P}^i_j}\Big[\mathcal{Y}^{t_i,\widetilde{\omega}}_\tau\Big]&=\sup_{\tau\in\mathcal{T}^{t_i}}\mathbb{E}_{\mathbb{P}^i_j}\Big[Y^{t_i,\omega\otimes_t\widetilde{\omega}}_\tau\Big]\le\sup_{\tau\in\mathcal{T}^{t_i}}\mathbb{E}_{\mathbb{P}^i_j}\Big[Y^{t_i,\omega\otimes_t\omega^\alpha_j}_\tau\Big]+\rho_0\big(\|\widetilde{\omega}-\omega^\alpha_j\|_{t,t_i}\big)\le\overline{Z}_{t_i}(\omega\otimes_t\omega^\alpha_j)+\frac{\varepsilon}{4}+\rho_0\big(\|\widetilde{\omega}-\omega^\alpha_j\|_{t,t_i}\big)\\&<\overline{Z}_{t_i}(\omega\otimes_t\widetilde{\omega})+\rho_1\big(\|\widetilde{\omega}-\omega^\alpha_j\|_{t,t_i}\big)+\frac{1}{2}\varepsilon<\overline{Z}_{t_i}(\omega\otimes_t\widetilde{\omega})+\frac{3}{4}\varepsilon=\mathcal{Z}_{t_i}(\widetilde{\omega})+\frac{3}{4}\varepsilon.\end{aligned}\tag{7.53}$$

Setting $\mathbb{P}^\lambda_k:=\mathbb{P}$, we recursively pick up $\mathbb{P}^\lambda_i$, $i=k-1,\cdots,1$ from $\mathcal{P}(t,\omega)$ such that (P2) holds for for $\Big(s,\widehat{\mathbb{P}},\mathbb{P},\big\{(\mathcal{A}_j,\delta_j,\widetilde{\omega}_j,\mathbb{P}_j)\big\}_{j=1}^\lambda\Big)=\Big(t_i,\mathbb{P}^\lambda_i,\mathbb{P}^\lambda_{i+1},\big\{(\mathcal{A}^i_j,\delta,\omega^\alpha_j,\mathbb{P}^i_j)\big\}_{j=1}^\lambda\Big)$ and $\mathcal{A}_0=\mathcal{A}^i_0:=\Big(\underset{j=1}{\overset{\lambda}{\cup}}\mathcal{A}^i_j\Big)^c\in\mathcal{F}^t_{t_i}$. Then

$$\sup_{\tau\in\mathcal{T}^t_{t_i}}\mathbb{E}_{\mathbb{P}^\lambda_i}\big[\mathbf{1}_{A\cap\mathcal{A}^i_j}Y^{t,\omega}_\tau\big]\le\mathbb{E}_{\mathbb{P}^\lambda_{i+1}}\Big[\mathbf{1}_{\{\widetilde{\omega}\in A\cap\mathcal{A}^i_j\}}\Big(\sup_{\zeta\in\mathcal{T}^{t_i}}\mathbb{E}_{\mathbb{P}^i_j}\big[Y^{t_i,\omega\otimes_t\widetilde{\omega}}_\zeta\big]+\widehat{\rho}_0(\delta)\Big)\Big],\quad\forall\,j=1,\cdots,\lambda,\ \ \forall\,A\in\mathcal{F}^t_{t_i}.\tag{7.54}$$

And similar to (7.21), we have

$$\mathbb{E}_{\mathbb{P}^\lambda_i}[\xi]\;=\;\mathbb{E}_{\mathbb{P}^\lambda_{i+1}}[\xi],\qquad\forall\,\xi\in L^1(\mathcal{F}^t_{t_i},\mathbb{P}^\lambda_i)\cap L^1\big(\mathcal{F}^t_{t_i},\mathbb{P}^\lambda_{i+1}\big),\tag{7.55}$$

$$\text{and}\quad\mathbb{E}_{\mathbb{P}^\lambda_i}[\mathbf{1}_{\mathcal{A}^i_0}\xi]\;=\;\mathbb{E}_{\mathbb{P}^\lambda_{i+1}}[\mathbf{1}_{\mathcal{A}^i_0}\xi],\qquad\forall\,\xi\in L^1(\mathcal{F}^t_T,\mathbb{P}^\lambda_i)\cap L^1\big(\mathcal{F}^t_T,\mathbb{P}^\lambda_{i+1}\big).\tag{7.56}$$

2b) *Now, let us consider the Snell envelope $Z^{\mathbb{P}^\lambda_1}$ of $\mathcal{Y}$ under $\mathbb{P}^\lambda_1$, i.e., $Z^{\mathbb{P}^\lambda_1}_s:=\underset{\tau\in\mathcal{T}^{\mathbb{P}^\lambda_1}_s}{\mathrm{esssup}}\,\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathcal{Y}_\tau\Big|\mathcal{F}^{\mathbb{P}^\lambda_1}_s\Big]$, $s\in[t,T]$.*

As mentioned in the proof of Proposition 4.3, $Z^{\mathbb{P}^\lambda_1}$ admits an RCLL modification $\Big\{\mathscr{Z}^{\mathbb{P}^\lambda_1}_s\Big\}_{s\in[t,T]}$ such that for any $\varsigma\in\mathcal{T}^{\mathbb{P}^\lambda_1}$, $\tau^\varsigma_{\mathbb{P}^\lambda_1}:=\inf\Big\{r\in[\varsigma,T]:\mathscr{Z}^{\mathbb{P}^\lambda_1}_r=\mathcal{Y}_r\Big\}\in\mathcal{T}^{\mathbb{P}^\lambda_1}_\varsigma$ is an optimal stopping time for $\underset{\tau\in\mathcal{T}^{\mathbb{P}^\lambda_1}_\varsigma}{\mathrm{esssup}}\,\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathcal{Y}_\tau\Big|\mathcal{F}^{\mathbb{P}^\lambda_1}_\varsigma\Big]$. Simply denoting $\tau^t_{\mathbb{P}^\lambda_1}$ by τ_λ, we also know that $\mathscr{Z}^{\mathbb{P}^\lambda_1}$ $\bigg($resp. $\Big\{\mathscr{Z}^{\mathbb{P}^\lambda_1}_{\tau_\lambda\wedge s}\Big\}_{s\in[t,T]}\bigg)$ is a supermartingale (resp. martingale) with respect to $\big(\mathbf{F}^{\mathbb{P}^\lambda_1},\mathbb{P}^\lambda_1\big)$. It follows from Optional Sampling Theorem that

$$\overline{Z}_t(\omega)=\inf_{\mathbb{P}\in\mathcal{P}(t,\omega)}\sup_{\tau\in\mathcal{T}^t}\mathbb{E}_\mathbb{P}\big[\mathcal{Y}_\tau\big]\le\sup_{\tau\in\mathcal{T}^t}\mathbb{E}_{\mathbb{P}^\lambda_1}\big[\mathcal{Y}_\tau\big]\le\sup_{\tau\in\mathcal{T}^{\mathbb{P}^\lambda_1}}\mathbb{E}_{\mathbb{P}^\lambda_1}\big[\mathcal{Y}_\tau\big]=Z^{\mathbb{P}^\lambda_1}_t=\mathscr{Z}^{\mathbb{P}^\lambda_1}_t=\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathscr{Z}^{\mathbb{P}^\lambda_1}_{\zeta^n_k\wedge\tau_\lambda}\Big].\tag{7.57}$$

Applying (7.43) with $\mathbb{P}=\mathbb{P}^\lambda_1$ shows that $\mathbb{P}^\lambda_1\Big\{\mathcal{Z}_s\le\mathscr{Z}^{\mathbb{P}^\lambda_1}_s,\ \ \forall\,s\in[t,T]\Big\}=1$. By the continuity of $\overline{Z}$ and the right continuity of $\mathscr{Z}^{\mathbb{P}^\lambda_1}$, it holds for $\mathbb{P}^\lambda_1-$a.s. $\widetilde{\omega}\in\Omega^t$ that $\mathcal{Z}_s(\widetilde{\omega})\le\mathscr{Z}^{\mathbb{P}^\lambda_1}_s(\widetilde{\omega})$ for any $s\in[t,T]$. Since $\tau^*(\omega\otimes_t\widetilde{\omega})>t$ by (7.49), one can deduce that

$$\begin{aligned}\zeta^*(\widetilde{\omega})=\tau^*(\omega\otimes_t\widetilde{\omega})&=\inf\{s\in[0,T]:\overline{Z}_s(\omega\otimes_t\widetilde{\omega})=Y_s(\omega\otimes_t\widetilde{\omega})\}=\inf\{s\in[t,T]:\overline{Z}_s(\omega\otimes_t\widetilde{\omega})=Y_s(\omega\otimes_t\widetilde{\omega})\}\\&=\inf\{s\in[t,T]:\mathcal{Z}_s(\widetilde{\omega})=\mathcal{Y}_s(\widetilde{\omega})\}\le\inf\{s\in[t,T]:\mathscr{Z}^{\mathbb{P}^\lambda_1}_s(\widetilde{\omega})=\mathcal{Y}_s(\widetilde{\omega})\}=\tau_\lambda(\widetilde{\omega}).\end{aligned}\tag{7.58}$$

Next, let us use (7.53)−(7.56) *to show that*

$$\mathbf{1}_{\underset{i=1}{\overset{k-1}{\cup}}(\mathcal{A}^i_0)^c}\mathscr{Z}^{\mathbb{P}^\lambda_1}_{\zeta^n_k}\le\mathbf{1}_{\underset{i=1}{\overset{k-1}{\cup}}(\mathcal{A}^i_0)^c}\big(\mathcal{Z}_{\zeta^n_k}+\varepsilon\big),\quad\mathbb{P}^\lambda_1-a.s.\tag{7.59}$$

To see this, we let $(i,j)\in\{1,\cdots,k-1\}\times\{1,\cdots,\lambda\}$, $\tau\in\mathcal{T}^t_{t_i}$ and $A\in\mathcal{F}^t_{t_i}$. Since $\mathcal{A}^i_j\subset\mathcal{A}^{i'}_0$ for $i'\in\{1,\cdots,k-1\}\backslash\{i\}$, we can deduce from (7.56), (3.2), (7.54), (7.53), (7.55) and Proposition 4.2 that

$$\begin{aligned}\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{A\cap\mathcal{A}^i_j}\mathcal{Y}_\tau\Big]&=\cdots=\mathbb{E}_{\mathbb{P}^\lambda_{i-1}}\Big[\mathbf{1}_{A\cap\mathcal{A}^i_j}\mathcal{Y}_\tau\Big]=\mathbb{E}_{\mathbb{P}^\lambda_i}\Big[\mathbf{1}_{A\cap\mathcal{A}^i_j}\mathcal{Y}_\tau\Big]\le\mathbb{E}_{\mathbb{P}^\lambda_{i+1}}\Big[\mathbf{1}_{\{\widetilde{\omega}\in A\cap\mathcal{A}^i_j\}}\Big(\sup_{\zeta\in\mathcal{T}^{t_i}}\mathbb{E}_{\mathbb{P}^i_j}\big[Y^{t_i,\omega\otimes_t\widetilde{\omega}}_\zeta\big]+\widehat{\rho}_0(\delta)\Big)\Big]\\&\le\mathbb{E}_{\mathbb{P}^\lambda_{i+1}}\Big[\mathbf{1}_{A\cap\mathcal{A}^i_j}\big(\mathcal{Z}_{t_i}+\varepsilon\big)\Big]=\mathbb{E}_{\mathbb{P}^\lambda_i}\Big[\mathbf{1}_{A\cap\mathcal{A}^i_j}\big(\mathcal{Z}_{t_i}+\varepsilon\big)\Big]=\cdots=\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{A\cap\mathcal{A}^i_j}\big(\mathcal{Z}_{t_i}+\varepsilon\big)\Big],\end{aligned}$$

where we used the fact that $\mathcal{Z}_{t_i} \in \mathcal{F}^t_{t_i}$ by Remark 4.2 and Proposition 2.1 (2). Letting A vary over $\mathcal{F}^t_{t_i}$ and applying Lemma A.4 (1) with $(\mathbb{P}, X) = \big(\mathbb{P}^\lambda_1, B^t\big)$ yield that

$$\mathbf{1}_{\mathcal{A}^i_j}\big(\mathcal{Z}_{t_i} + \varepsilon\big) \ge \mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{\mathcal{A}^i_j}\mathcal{Y}_\tau\Big|\mathcal{F}^t_{t_i}\Big] = \mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{\mathcal{A}^i_j}\mathcal{Y}_\tau\Big|\mathcal{F}^{\mathbb{P}^\lambda_1}_{t_i}\Big], \quad \mathbb{P}^\lambda_1 - a.s. \tag{7.60}$$

For any $\tau \in \mathcal{T}^{\mathbb{P}^\lambda_1}_{t_i}$, similar to (7.27), one can find a sequence $\{\tau^i_\ell\}_{\ell\in\mathbb{N}}$ of $\mathcal{T}^t_{t_i}$ such that $\lim\limits_{\ell\to\infty}\mathbb{E}_{\mathbb{P}^\lambda_1}\big[\big|\mathcal{Y}_{\tau^i_\ell} - \mathcal{Y}_\tau\big|\big] = 0$. Then $\{\tau^i_\ell\}_{\ell\in\mathbb{N}}$ in turn has a subsequence (we still denote it by $\{\tau^i_\ell\}_{\ell\in\mathbb{N}}$) such that $\lim\limits_{\ell\to\infty}\mathcal{Y}_{\tau^i_\ell} = \mathcal{Y}_\tau$, $\mathbb{P}^\lambda_1$−a.s. As $\mathbb{E}_{\mathbb{P}^\lambda_1}[\mathcal{Y}_*] < \infty$ by (3.2), a conditional-expectation version of the dominated convergence theorem and (7.60) imply that

$$\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{\mathcal{A}^i_j}\mathcal{Y}_\tau\Big|\mathcal{F}^{\mathbb{P}^\lambda_1}_{t_i}\Big] = \lim_{\ell\to\infty}\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{\mathcal{A}^i_j}\mathcal{Y}_{\tau^i_\ell}\Big|\mathcal{F}^{\mathbb{P}^\lambda_1}_{t_i}\Big] \le \mathbf{1}_{\mathcal{A}^i_j}\big(\mathcal{Z}_{t_i} + \varepsilon\big), \quad \mathbb{P}^\lambda_1 - a.s.$$

Since $\mathcal{A}^i_j \in \mathcal{F}^t_{t_i}$, it follows that

$$\begin{aligned}
\mathbf{1}_{\mathcal{A}^i_j}\mathscr{Z}^{\mathbb{P}^\lambda_1}_{\zeta^n_k} &= \mathbf{1}_{\mathcal{A}^i_j}\mathscr{Z}^{\mathbb{P}^\lambda_1}_{t_i} = \mathbf{1}_{\mathcal{A}^i_j}Z^{\mathbb{P}^\lambda_1}_{t_i} = \mathbf{1}_{\mathcal{A}^i_j}\operatorname*{esssup}_{\tau\in\mathcal{T}^{\mathbb{P}^\lambda_1}_{t_i}}\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathcal{Y}_\tau\Big|\mathcal{F}^{\mathbb{P}^\lambda_1}_{t_i}\Big] = \operatorname*{esssup}_{\tau\in\mathcal{T}^{\mathbb{P}^\lambda_1}_{t_i}}\mathbf{1}_{\mathcal{A}^i_j}\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathcal{Y}_\tau\Big|\mathcal{F}^{\mathbb{P}^\lambda_1}_{t_i}\Big] \\
&= \operatorname*{esssup}_{\tau\in\mathcal{T}^{\mathbb{P}^\lambda_1}_{t_i}}\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{\mathcal{A}^i_j}\mathcal{Y}_\tau\Big|\mathcal{F}^{\mathbb{P}^\lambda_1}_{t_i}\Big] \le \mathbf{1}_{\mathcal{A}^i_j}\big(\mathcal{Z}_{t_i} + \varepsilon\big) = \mathbf{1}_{\mathcal{A}^i_j}\big(\mathcal{Z}_{\zeta^n_k} + \varepsilon\big), \quad \mathbb{P}^\lambda_1 - a.s.
\end{aligned}$$

Summing them up over $j \in \{1, \cdots, \lambda\}$ and then over $i \in \{1, \cdots, k-1\}$ yields (7.59).

2c) *In this step, we will use* (7.57) *and* (7.59) *to show*

$$\overline{Z}_t(\omega) \le \mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{\mathscr{A}_\lambda}\mathcal{Z}_{\zeta^n_k} + \mathbf{1}_{\mathscr{A}^c_\lambda}\mathcal{Y}_{\tau_\lambda}\Big] + \varepsilon, \tag{7.61}$$

where $\mathscr{A}_\lambda := \{\zeta^n_k \le \zeta^*\} \cap \Big(\bigcup\limits_{i=1}^{k-1}(\mathcal{A}^i_0)^c\Big) = \{\zeta^n_k \le \zeta^*\} \cap \Big(\bigcup\limits_{i=1}^{k-1}\bigcup\limits_{j=1}^{\lambda}\mathcal{A}^i_j\Big)$.

We first claim that $\mathscr{A}_\lambda \in \mathcal{F}^t_{\zeta^n_k\wedge\zeta^*} \cap \mathcal{F}^{\mathbb{P}^\lambda_1}_{\zeta^n_k\wedge\tau_\lambda}$. To see this claim, we set an auxiliary set $\widehat{\mathscr{A}_\lambda} := \{\zeta^n_k \le \tau_\lambda\} \cap \Big(\bigcup\limits_{i=1}^{k-1}(\mathcal{A}^i_0)^c\Big)$. Given $s \in [t, T]$, if $s < t_1$, then $\mathscr{A}_\lambda \cap \{\zeta^n_k \wedge \zeta^* \le s\} = \mathscr{A}_\lambda \cap \{\zeta^n_k \le s\} = \emptyset$ and $\widehat{\mathscr{A}_\lambda} \cap \{\zeta^n_k \wedge \tau_\lambda \le s\} = \widehat{\mathscr{A}_\lambda} \cap \{\zeta^n_k \le s\} = \emptyset$. Otherwise, let k' be the largest integer from $\{1, \cdots, k-1\}$ such that $t_{k'} \le s$. Since $(\mathcal{A}^i_0)^c = \bigcup\limits_{j=1}^{\lambda}\mathcal{A}^i_j \subset \{\zeta^n_k = t_i\}$ for $i = 1, \cdots, k-1$,

$$\begin{aligned}
\mathscr{A}_\lambda \cap \{\zeta^n_k \wedge \zeta^* \le s\} &= \mathscr{A}_\lambda \cap \{\zeta^n_k \le s\} = \{\zeta^n_k \le \zeta^*\} \cap \Big(\bigcup_{i=1}^{k'}(\mathcal{A}^i_0)^c\Big) \cap \{\zeta^n_k \le s\} \\
\text{and} \quad \widehat{\mathscr{A}_\lambda} \cap \{\zeta^n_k \wedge \tau_\lambda \le s\} &= \widehat{\mathscr{A}_\lambda} \cap \{\zeta^n_k \le s\} = \{\zeta^n_k \le \tau_\lambda\} \cap \Big(\bigcup_{i=1}^{k'}(\mathcal{A}^i_0)^c\Big) \cap \{\zeta^n_k \le s\}.
\end{aligned}$$

Clearly, $\bigcup\limits_{i=1}^{k'}(\mathcal{A}^i_0)^c \in \mathcal{F}^t_{t_{k'}} \subset \mathcal{F}^t_s \subset \mathcal{F}^{\mathbb{P}^\lambda_1}_s$. As $\{\zeta^n_k \le \zeta^*\} \in \mathcal{F}^t_{\zeta^n_k\wedge\zeta^*} \subset \mathcal{F}^t_{\zeta^n_k}$ and $\{\zeta^n_k \le \tau_\lambda\} \in \mathcal{F}^{\mathbb{P}^\lambda_1}_{\zeta^n_k\wedge\tau_\lambda} \subset \mathcal{F}^{\mathbb{P}^\lambda_1}_{\zeta^n_k}$, we also have $\{\zeta^n_k \le \zeta^*\} \cap \{\zeta^n_k \le s\} \in \mathcal{F}^t_s$ and $\{\zeta^n_k \le \tau_\lambda\} \cap \{\zeta^n_k \le s\} \in \mathcal{F}^{\mathbb{P}^\lambda_1}_s$. It follows that $\mathscr{A}_\lambda \cap \{\zeta^n_k \wedge \zeta^* \le s\} \in \mathcal{F}^t_s$ and $\widehat{\mathscr{A}_\lambda} \cap \{\zeta^n_k \wedge \tau_\lambda \le s\} \in \mathcal{F}^{\mathbb{P}^\lambda_1}_s$. Hence $\mathscr{A}_\lambda \in \mathcal{F}^t_{\zeta^n_k\wedge\zeta^*}$ and $\widehat{\mathscr{A}_\lambda} \in \mathcal{F}^{\mathbb{P}^\lambda_1}_{\zeta^n_k\wedge\tau_\lambda}$.

By (7.58), $\mathcal{N} := \{\zeta^* > \tau_\lambda\} \in \mathscr{N}^{\mathbb{P}^\lambda_1}$. Since $\mathscr{A}_\lambda \cap \mathcal{N}^c \subset \{\zeta^n_k \le \tau_\lambda\}$ and since $\{\zeta^n_k \le \zeta^* \wedge \tau_\lambda\} \in \mathcal{F}^{\mathbb{P}^\lambda_1}_{\zeta^n_k\wedge\zeta^*\wedge\tau_\lambda} \subset \mathcal{F}^{\mathbb{P}^\lambda_1}_{\zeta^n_k\wedge\tau_\lambda}$, one can deduce that

$$\mathscr{A}_\lambda \cap \mathcal{N}^c = \mathscr{A}_\lambda \cap \{\zeta^n_k \le \tau_\lambda\} \cap \mathcal{N}^c = \{\zeta^n_k \le \zeta^* \wedge \tau_\lambda\} \cap \Big(\bigcup_{i=1}^{k-1}(\mathcal{A}^i_0)^c\Big) \cap \mathcal{N}^c = \{\zeta^n_k \le \zeta^* \wedge \tau_\lambda\} \cap \widehat{\mathscr{A}_\lambda} \cap \mathcal{N}^c \in \mathcal{F}^{\mathbb{P}^\lambda_1}_{\zeta^n_k\wedge\tau_\lambda}.$$

As $\mathscr{A}_\lambda \cap \mathcal{N} \in \mathscr{N}^{\mathbb{P}^\lambda_1}$, we see that $\mathscr{A}_\lambda \in \mathcal{F}^{\mathbb{P}^\lambda_1}_{\zeta^n_k\wedge\tau_\lambda}$.

Since $\Big\{\mathscr{Z}^{\mathbb{P}^\lambda_1}_{\tau_\lambda\wedge s}\Big\}_{s\in[t,T]}$ is a martingale with respect to $\big(\mathbf{F}^{\mathbb{P}^\lambda_1}, \mathbb{P}^\lambda_1\big)$, it follows from Optional Sampling Theorem that

$$\mathbf{1}_{\mathscr{A}^c_\lambda}\mathscr{Z}^{\mathbb{P}^\lambda_1}_{\zeta^n_k\wedge\tau_\lambda} = \mathbf{1}_{\mathscr{A}^c_\lambda}\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathscr{Z}^{\mathbb{P}^\lambda_1}_{\tau_\lambda}\Big|\mathcal{F}^{\mathbb{P}^\lambda_1}_{\zeta^n_k\wedge\tau_\lambda}\Big] = \mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{\mathscr{A}^c_\lambda}\mathscr{Z}^{\mathbb{P}^\lambda_1}_{\tau_\lambda}\Big|\mathcal{F}^{\mathbb{P}^\lambda_1}_{\zeta^n_k\wedge\tau_\lambda}\Big], \quad \mathbb{P}^\lambda_1 - a.s.$$

Taking expectation $\mathbb{E}_{\mathbb{P}^\lambda_1}$ yields that

$$\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{\mathscr{A}^c_\lambda}\mathscr{Z}^{\mathbb{P}^\lambda_1}_{\zeta^n_k\wedge\tau_\lambda}\Big]=\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{\mathscr{A}^c_\lambda}\mathscr{Z}^{\mathbb{P}^\lambda_1}_{\tau_\lambda}\Big]=\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{\mathscr{A}^c_\lambda}\mathcal{Y}_{\tau_\lambda}\Big]. \tag{7.62}$$

Since $\zeta^n_k\le\tau_\lambda$ holds $\mathbb{P}^\lambda_1-$a.s. on $\mathscr{A}_\lambda$ by (7.58), we can deduce from (7.57), (7.62) and (7.59) that

$$\overline{Z}_t(\omega)\le\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathscr{Z}^{\mathbb{P}^\lambda_1}_{\zeta^n_k\wedge\tau_\lambda}\Big]=\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{\mathscr{A}_\lambda}\mathscr{Z}^{\mathbb{P}^\lambda_1}_{\zeta^n_k}+\mathbf{1}_{\mathscr{A}^c_\lambda}\mathcal{Y}_{\tau_\lambda}\Big]\le\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{\mathscr{A}_\lambda}\mathcal{Z}_{\zeta^n_k}+\mathbf{1}_{\mathscr{A}^c_\lambda}\mathcal{Y}_{\tau_\lambda}\Big]+\varepsilon.$$

2d) *In the next step, we replace* $\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{\mathscr{A}_\lambda}\mathcal{Z}_{\zeta^n_k}+\mathbf{1}_{\mathscr{A}^c_\lambda}\mathcal{Y}_{\tau_\lambda}\Big]$ *on the right-hand-side of* (7.61) *by an expectation under* $\mathbb{P}$.

For $i=1,\cdots,k-1$, as $\mathscr{A}_\lambda\in\mathcal{F}^t_{\zeta^n_k\wedge\zeta^*}\subset\mathcal{F}^t_{\zeta^n_k}$, one has $\mathscr{A}^i_\lambda:=\mathscr{A}_\lambda\cap\{\zeta^n_k=t_i\}=\{\zeta^n_k\le\zeta^*\}\cap(\mathcal{A}^i_0)^c\in\mathcal{F}^t_{t_i}$. By (7.56), (7.55), Remark 4.2 and Proposition 4.2,

$$\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{\mathscr{A}^i_\lambda}\mathcal{Z}_{t_i}\Big]=\cdots=\mathbb{E}_{\mathbb{P}^\lambda_i}\Big[\mathbf{1}_{\mathscr{A}^i_\lambda}\mathcal{Z}_{t_i}\Big]=\mathbb{E}_{\mathbb{P}^\lambda_{i+1}}\Big[\mathbf{1}_{\mathscr{A}^i_\lambda}\mathcal{Z}_{t_i}\Big]=\cdots=\mathbb{E}_{\mathbb{P}^\lambda_k}\Big[\mathbf{1}_{\mathscr{A}^i_\lambda}\mathcal{Z}_{t_i}\Big]=\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\mathscr{A}^i_\lambda}\mathcal{Z}_{t_i}\Big].$$

Their sum over $i\in\{1,\cdots,k-1\}$ is

$$\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{\mathscr{A}_\lambda}\mathcal{Z}_{\zeta^n_k}\Big]=\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\mathscr{A}_\lambda}\mathcal{Z}_{\zeta^n_k}\Big]. \tag{7.63}$$

Using (7.58) and the fact that $\mathcal{Z}_T=\mathcal{Y}_T$ $\big($see (7.45)$\big)$, we obtain

$$\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{\{T=\zeta^n_k\le\zeta^*\}}\mathcal{Y}_{\tau_\lambda}\Big]=\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{\{T=\zeta^n_k\le\zeta^*\}}\mathcal{Y}_T\Big]=\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{\{T=\zeta^n_k\le\zeta^*\}}\mathcal{Z}_T\Big]=\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{\{T=\zeta^n_k\le\zeta^*\}}\mathcal{Z}_{\zeta^n_k}\Big]. \tag{7.64}$$

Since $\{T=\zeta^n_k\le\zeta^*\}\subset\{\zeta^n_k=T\}\subset\overset{k-1}{\underset{i=1}{\cap}}\mathcal{A}^i_0$, one can deduce from (7.56) and Proposition 4.2 again that

$$\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{\{T=\zeta^n_k\le\zeta^*\}}\mathcal{Z}_{\zeta^n_k}\Big]=\mathbb{E}_{\mathbb{P}^\lambda_2}\Big[\mathbf{1}_{\{T=\zeta^n_k\le\zeta^*\}}\mathcal{Z}_{\zeta^n_k}\Big]=\cdots=\mathbb{E}_{\mathbb{P}^\lambda_k}\Big[\mathbf{1}_{\{T=\zeta^n_k\le\zeta^*\}}\mathcal{Z}_{\zeta^n_k}\Big]=\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{T=\zeta^n_k\le\zeta^*\}}\mathcal{Z}_{\zeta^n_k}\Big], \tag{7.65}$$

and similarly that

$$\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{\big(\overset{k-1}{\underset{i=1}{\cap}}\mathcal{A}^i_0\big)\backslash\{T=\zeta^n_k\le\zeta^*\}}\mathcal{Y}_{\tau_\lambda}\Big]=\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\big(\overset{k-1}{\underset{i=1}{\cap}}\mathcal{A}^i_0\big)\backslash\{T=\zeta^n_k\le\zeta^*\}}\mathcal{Y}_{\tau_\lambda}\Big]\le\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\big(\overset{k-1}{\underset{i=1}{\cap}}\mathcal{A}^i_0\big)\backslash\{T=\zeta^n_k\le\zeta^*\}}\mathcal{Y}_*\Big]. \tag{7.66}$$

Similar to (7.27), one can find a sequence $\{\tau^\ell_\lambda\}_{\ell\in\mathbb{N}}$ of $\mathcal{T}^t$ such that $\lim\limits_{\ell\to\infty}\mathbb{E}_{\mathbb{P}^\lambda_1}\big[\big|\mathcal{Y}_{\tau^\ell_\lambda}-\mathcal{Y}_{\tau_\lambda}\big|\big]=0$. Let $\ell\in\mathbb{N}$ and $(i,j)\in\{1,\cdots,k-1\}\times\{1,\cdots,\lambda\}$. Since $\{\zeta^*<\zeta^n_k\}\in\mathcal{F}^t_{\zeta^*\wedge\zeta^n_k}\subset\mathcal{F}^t_{\zeta^n_k}$, we have $\{\zeta^*<\zeta^n_k\}\cap\mathcal{A}^i_j=\{\zeta^*<\zeta^n_k\}\cap\{\zeta^n_k=t_i\}\cap\mathcal{A}^i_j\in\mathcal{F}^t_{t_i}$. As $\mathcal{A}^i_j\subset\mathcal{A}^{i'}_0$ for $i'\in\{1,\cdots N-1\}\backslash\{i\}$, we can deduce from (3.2) and (7.54)$-$(7.56) that

$$\begin{aligned}\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{\{\zeta^*<\zeta^n_k\}\cap\mathcal{A}^i_j}\mathcal{Y}_{\tau^\ell_\lambda}\Big]&=\cdots=\mathbb{E}_{\mathbb{P}^\lambda_i}\Big[\mathbf{1}_{\{\zeta^*<\zeta^n_k\}\cap\mathcal{A}^i_j}\mathcal{Y}_{\tau^\ell_\lambda}\Big]=\mathbb{E}_{\mathbb{P}^\lambda_i}\Big[\mathbf{1}_{\{\zeta^*<\zeta^n_k\}\cap\mathcal{A}^i_j\cap\{\tau^\ell_\lambda\le t_i\}}\mathcal{Y}_{\tau^\ell_\lambda\wedge t_i}+\mathbf{1}_{\{\zeta^*<\zeta^n_k\}\cap\mathcal{A}^i_j\cap\{\tau^\ell_\lambda>t_i\}}\mathcal{Y}_{\tau^\ell_\lambda\vee t_i}\Big]\\&\le\mathbb{E}_{\mathbb{P}^\lambda_{i+1}}\bigg[\mathbf{1}_{\{\zeta^*<\zeta^n_k\}\cap\mathcal{A}^i_j\cap\{\tau^\ell_\lambda\le t_i\}}\mathcal{Y}_{\tau^\ell_\lambda\wedge t_i}+\mathbf{1}_{\{\zeta^*(\widetilde{\omega})<\zeta^n_k(\widetilde{\omega})\}}\mathbf{1}_{\{\widetilde{\omega}\in\mathcal{A}^i_j\}}\mathbf{1}_{\{\tau^\ell_\lambda(\widetilde{\omega})>t_i\}}\bigg(\sup_{\zeta\in\mathcal{T}^{t_i}}\mathbb{E}_{\mathbb{P}^i_j}\Big[Y^{t_i,\omega\otimes_t\widetilde{\omega}}_\zeta\Big]+\widehat{\rho}_0(\delta)\bigg)\bigg].\end{aligned} \tag{7.67}$$

If $M:=\sup\limits_{(t,\omega')\in[0,T]\times\Omega}Y_t(\omega')<\infty$, it follows that

$$\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{\{\zeta^*<\zeta^n_k\}\cap\mathcal{A}^i_j}\mathcal{Y}_{\tau^\ell_\lambda}\Big]\le\mathbb{E}_{\mathbb{P}^\lambda_{i+1}}\Big[\mathbf{1}_{\{\zeta^*<\zeta^n_k\}\cap\mathcal{A}^i_j}(1+M^+)\Big]. \tag{7.68}$$

Suppose otherwise that $M=\infty$. The right continuity of process Y and Proposition 2.1 (2) imply that $\xi_i:=\sup\limits_{r\in[t,t_i]}|\mathcal{Y}_r|=\bigg(\sup\limits_{r\in\mathbb{Q}\cap[t,t_i)}|\mathcal{Y}_r|\bigg)\vee|\mathcal{Y}_{t_i}|$ is $\mathcal{F}^t_{t_i}-$measurable. For any $\zeta\in\mathcal{T}^{t_i}$, $\widetilde{\omega}\in\Omega^t$ and $\widehat{\omega}\in\Omega^{t_i}$, since $\widehat{t}:=\zeta(\widehat{\omega})\ge t_i$ and since $Y_r\big(\omega\otimes_t(\widetilde{\omega}\otimes_{t_i}\widehat{\omega})\big)=Y_r(\omega)$ for any $r\in[0,t]$ by (2.2) again, (5.2) implies that

$$\begin{aligned}Y^{t_i,\omega\otimes_t\widetilde{\omega}}_\zeta(\widehat{\omega})&=Y\big(\widehat{t},\omega\otimes_t(\widetilde{\omega}\otimes_{t_i}\widehat{\omega})\big)\le Y\big(t_i,\omega\otimes_t(\widetilde{\omega}\otimes_{t_i}\widehat{\omega})\big)+L+\sup_{r\in[0,t_i]}\big|Y\big(r,\omega\otimes_t(\widetilde{\omega}\otimes_{t_i}\widehat{\omega})\big)\big|+\rho_1\Big(\sup_{r\in[t_i,\widehat{t}]}\big|\widehat{\omega}(r)\big|\Big)\\&=\mathcal{Y}\big(t_i,\widetilde{\omega}\otimes_{t_i}\widehat{\omega}\big)+L+\sup_{r\in[0,t]}\big|Y\big(r,\omega\big)\big|\vee\sup_{r\in[t,t_i]}\big|\mathcal{Y}\big(r,\widetilde{\omega}\otimes_{t_i}\widehat{\omega}\big)\big|+\rho_1\Big(\sup_{r\in[t_i,\widehat{t}]}\big|B^{t_i}_r(\widehat{\omega})\big|\Big)\\&\le L+2\xi_i(\widetilde{\omega}\otimes_{t_i}\widehat{\omega})+\sup_{r\in[0,t]}\big|Y_r(\omega)\big|+\rho_1\Big(\sup_{r\in[t_i,T]}\big|B^{t_i}_r(\widehat{\omega})\big|\Big)=L+2\xi_i(\widetilde{\omega})+\sup_{r\in[0,t]}\big|Y_r(\omega)\big|+\rho_1\Big(\sup_{r\in[t_i,T]}\big|B^{t_i}_r(\widehat{\omega})\big|\Big).\end{aligned}$$

Since $\|\omega\otimes_t \omega_j^\alpha\|_{0,t_i} \le \|\omega\|_{0,t}+\|\omega_j^\alpha\|_{t,t_i} \le \|\omega\|_{0,t}+\|\omega_j^\alpha\|_{t,T} < \|\omega\|_{0,t}+\alpha := \alpha'$, (4.4) shows that $\mathbb{E}_{\mathbb{P}_j^i}\Big[Y_\zeta^{t_i,\omega\otimes_t\widetilde{\omega}}\Big] \le \widetilde{L}+2\mathcal{Y}_*+\rho_{\alpha'}(T-t_i)$, where $\widetilde{L} := L+\sup\limits_{r\in[0,t]}\big|Y_r(\omega)\big| < \infty$ by Lemma A.9. Plugging this into (7.67) yields that $\mathbb{E}_{\mathbb{P}_1^\lambda}\Big[\mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap\mathcal{A}_j^i}\mathcal{Y}_{\tau_\lambda^\ell}\Big] \le \mathbb{E}_{\mathbb{P}_{i+1}^\lambda}\Big[\mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap\mathcal{A}_j^i}\big(1+\widetilde{L}+2\mathcal{Y}_*+\rho_{\alpha'}(T-t_i)\big)\Big]$, which together with (7.68), (7.56) and (3.2) shows that

$$\mathbb{E}_{\mathbb{P}_1^\lambda}\Big[\mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap\mathcal{A}_j^i}\mathcal{Y}_{\tau_\lambda^\ell}\Big] \le \mathbb{E}_{\mathbb{P}_{i+1}^\lambda}\Big[\mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap\mathcal{A}_j^i}(1+\eta_{\alpha'})\Big] = \cdots = \mathbb{E}_{\mathbb{P}_k^\lambda}\Big[\mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap\mathcal{A}_j^i}(1+\eta_{\alpha'})\Big] = \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap\mathcal{A}_j^i}(1+\eta_{\alpha'})\Big]$$

for $\eta_{\alpha'} := \mathbf{1}_{\{M<\infty\}}M^+ + \mathbf{1}_{\{M=\infty\}}\big(\widetilde{L}+2\mathcal{Y}_*+\rho_{\alpha'}(T)\big)$. Summing them up over $j\in\{1,\cdots,\lambda\}$ and then over $i\in\{1,\cdots,k-1\}$ gives that

$$\begin{aligned}\mathbb{E}_{\mathbb{P}_1^\lambda}\Big[\mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap\big(\overset{k-1}{\underset{i=1}{\cup}}(\mathcal{A}_0^i)^c\big)}\mathcal{Y}_{\tau_\lambda}\Big] &\le \mathbb{E}_{\mathbb{P}_1^\lambda}\Big[\mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap\big(\overset{k-1}{\underset{i=1}{\cup}}(\mathcal{A}_0^i)^c\big)}\mathcal{Y}_{\tau_\lambda^\ell}\Big] + \mathbb{E}_{\mathbb{P}_1^\lambda}\big[\big|\mathcal{Y}_{\tau_\lambda}-\mathcal{Y}_{\tau_\lambda^\ell}\big|\big]\\ &= \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap\big(\overset{k-1}{\underset{i=1}{\cup}}(\mathcal{A}_0^i)^c\big)}(1+\eta_{\alpha'})\Big] + \mathbb{E}_{\mathbb{P}_1^\lambda}\big[\big|\mathcal{Y}_{\tau_\lambda}-\mathcal{Y}_{\tau_\lambda^\ell}\big|\big].\end{aligned}$$

As $\ell\to\infty$, we obtain $\mathbb{E}_{\mathbb{P}_1^\lambda}\Big[\mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap\big(\overset{k-1}{\underset{i=1}{\cup}}(\mathcal{A}_0^i)^c\big)}\mathcal{Y}_{\tau_\lambda}\Big] \le \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap\big(\overset{k-1}{\underset{i=1}{\cup}}(\mathcal{A}_0^i)^c\big)}(1+\eta_{\alpha'})\Big]$.

Putting this and (7.63)-(7.66) back into (7.61) yields that

$$\overline{Z}_t(\omega) \le \mathbb{E}_{\mathbb{P}}\bigg[\Big(\mathbf{1}_{\{\zeta_k^n\le\zeta^*\}\cap\big(\overset{k-1}{\underset{i=1}{\cup}}(\mathcal{A}_0^i)^c\big)} + \mathbf{1}_{\{T=\zeta_k^n\le\zeta^*\}}\Big)\mathcal{Z}_{\zeta_k^n} + \mathbf{1}_{\big(\overset{k-1}{\underset{i=1}{\cap}}\mathcal{A}_0^i\big)\backslash\{T=\zeta_k^n\le\zeta^*\}}\mathcal{Y}_* + \mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap\big(\overset{k-1}{\underset{i=1}{\cup}}(\mathcal{A}_0^i)^c\big)}(1+\eta_{\alpha'})\bigg] + \varepsilon. \quad (7.69)$$

2e) *In the last step, we will gradually send the parameters λ, k, n, α to ∞ to obtain* (7.50).

Let $A_{n,k}^\alpha := \underset{\lambda\in\mathbb{N}}{\cup}\overset{k-1}{\underset{i=1}{\cup}}\big(\mathcal{A}_0^i\big)^c$ and $\mathfrak{O}_\delta^\alpha := \underset{j\in\mathbb{N}}{\cup} O_\delta(\omega_j^\alpha)$. As $O_\delta(\omega_j^\alpha)\subset O_\delta^{t_i}(\omega_j^\alpha)$ for $(i,j)\in\{1,\cdots,k-1\}\times\mathbb{N}$, one can deduce that

$$\begin{aligned}&A_{n,k}^\alpha = \overset{k-1}{\underset{i=1}{\cup}}\underset{\lambda\in\mathbb{N}}{\cup}\big(\mathcal{A}_0^i\big)^c = \overset{k-1}{\underset{i=1}{\cup}}\underset{j\in\mathbb{N}}{\cup}\mathcal{A}_j^i = \overset{k-1}{\underset{i=1}{\cup}}\Big(\{\zeta_k^n=t_i\}\cap\Big(\underset{j\in\mathbb{N}}{\cup}O_\delta^{t_i}(\omega_j^\alpha)\Big)\Big) \subset \overset{k-1}{\underset{i=1}{\cup}}\{\zeta_k^n=t_i\} = \{\zeta_k^n<T\} \quad \text{and}\\ &A_{n,k}^\alpha = \overset{k-1}{\underset{i=1}{\cup}}\Big(\{\zeta_k^n=t_i\}\cap\Big(\underset{j\in\mathbb{N}}{\cup}O_\delta^{t_i}(\omega_j^\alpha)\Big)\Big) \supset \overset{k-1}{\underset{i=1}{\cup}}\big(\{\zeta_k^n=t_i\}\cap\mathfrak{O}_\delta^\alpha\big) = \Big(\overset{k-1}{\underset{i=1}{\cup}}\{\zeta_k^n=t_i\}\Big)\cap\mathfrak{O}_\delta^\alpha = \{\zeta_k^n<T\}\cap\mathfrak{O}_\delta^\alpha.\end{aligned} \quad (7.70)$$

Since $\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_*+\eta_{\alpha'}\big]<\infty$ by (3.2), and since $\big\{\mathcal{Z}_{\zeta_k^n}\big\}_{n,k\in\mathbb{N}}$ is $\mathbb{P}-$uniformly integrable by Proposition 4.2, letting $\lambda\to\infty$ in (7.69) and applying the dominated convergence theorem yield that

$$\begin{aligned}\overline{Z}_t(\omega) &\le \mathbb{E}_{\mathbb{P}}\Big[\big(\mathbf{1}_{\{\zeta_k^n\le\zeta^*\}\cap A_{n,k}^\alpha}+\mathbf{1}_{\{T=\zeta_k^n\le\zeta^*\}}\big)\mathcal{Z}_{\zeta_k^n}+\mathbf{1}_{(A_{n,k}^\alpha)^c\backslash\{T=\zeta_k^n\le\zeta^*\}}\mathcal{Y}_*+\mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap A_{n,k}^\alpha}(1+\eta_{\alpha'})\Big]+\varepsilon\\ &\le \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\zeta_k^n\le\zeta^*\}}\mathcal{Z}_{\zeta_k^n}+\mathbf{1}_{(\mathfrak{O}_\delta^\alpha)^c}\mathcal{Y}_*+\mathbf{1}_{(\mathfrak{O}_\delta^\alpha)^c\cup\{T=\zeta_k^n>\zeta^*\}}\mathcal{Y}_*+\mathbf{1}_{\{\zeta^*<\zeta_k^n\}}(1+\eta_{\alpha'})\Big]+\varepsilon,\end{aligned} \quad (7.71)$$

where the second inequality is due to the fact that

$$\begin{aligned}\mathbf{1}_{\{\zeta_k^n\le\zeta^*\}\cap A_{n,k}^\alpha}\mathcal{Z}_{\zeta_k^n} &= \mathbf{1}_{\{\zeta_k^n\le\zeta^*\}\cap\{\zeta_k^n<T\}}\mathcal{Z}_{\zeta_k^n}-\mathbf{1}_{\{\zeta_k^n\le\zeta^*\}\cap(\{\zeta_k^n<T\}\backslash A_{n,k}^\alpha)}\mathcal{Z}_{\zeta_k^n} \le \mathbf{1}_{\{\zeta_k^n\le\zeta^*\}\cap\{\zeta_k^n<T\}}\mathcal{Z}_{\zeta_k^n}-\mathbf{1}_{\{\zeta_k^n\le\zeta^*\}\cap(\{\zeta_k^n<T\}\backslash A_{n,k}^\alpha)}\mathcal{Y}_{\zeta_k^n}\\ &\le \mathbf{1}_{\{\zeta_k^n\le\zeta^*\}\cap\{\zeta_k^n<T\}}\mathcal{Z}_{\zeta_k^n}+\mathbf{1}_{\{\zeta_k^n\le\zeta^*\}\cap\{\zeta_k^n<T\}\cap(\mathfrak{O}_\delta^\alpha)^c}\mathcal{Y}_* \le \mathbf{1}_{\{\zeta_k^n\le\zeta^*\}\cap\{\zeta_k^n<T\}}\mathcal{Z}_{\zeta_k^n}+\mathbf{1}_{(\mathfrak{O}_\delta^\alpha)^c}\mathcal{Y}_*.\end{aligned}$$

As $\zeta^*=(\tau^*)^{t,\omega}>t$ by (7.49), we see that $\lim\limits_{k\to\infty}\zeta_k^n=\zeta^n\le(\tau^n\vee t)^{t,\omega}=(\tau^n)^{t,\omega}\vee t<\zeta^*\le T$. Then letting $k\to\infty$ in (7.71), using the continuity of $\overline{Z}$ (Proposition 4.2), and applying the dominated convergence theorem again yield that

$$\overline{Z}_t(\omega)\le\mathbb{E}_{\mathbb{P}}\big[\mathcal{Z}_{\zeta^n}+\mathbf{1}_{(\mathfrak{O}_\delta^\alpha)^c}2\mathcal{Y}_*\big]+\varepsilon = \mathbb{E}_{\mathbb{P}}\big[\mathcal{Z}_{(\gamma\wedge(\tau^n\vee t))^{t,\omega}}+\mathbf{1}_{(\mathfrak{O}_\delta^\alpha)^c}2\mathcal{Y}_*\big]+\varepsilon. \quad (7.72)$$

Clearly, $\tau' := \lim\limits_{n\to\infty}\uparrow\tau^n\le\inf\{t\in[0,T]:\overline{Z}_t=Y_t\}=\tau^*$. For any $n\in\mathbb{N}$, $\overline{Z}_{\tau^n}\le Y_{\tau^n}+1/n$. As $n\to\infty$, the continuity of $\overline{Z}$ and Remark 3.1 (1) show that $\overline{Z}_{\tau'}\le Y_{\tau'-}\le Y_{\tau'}\le\overline{Z}_{\tau'}$, which implies that $\tau^*=\tau'=\lim\limits_{n\to\infty}\uparrow\tau^n$. Since $\underset{\alpha\in\mathbb{N}}{\cup}\mathfrak{O}_\delta^\alpha=\Omega^t$, letting $n\to\infty$, $\alpha\to\infty$ and then letting $\varepsilon\to0$ in (7.72), we can deduce from the continuity of $\overline{Z}$ and (7.49) that

$$\big(\overline{Z}_{\tau^*\wedge\gamma\wedge t}\big)(\omega)=\overline{Z}_t(\omega)\le\mathbb{E}_{\mathbb{P}}\big[\mathcal{Z}_{(\gamma\wedge(\tau^*\vee t))^{t,\omega}}\big]=\mathbb{E}_{\mathbb{P}}\big[\mathcal{Z}_{(\gamma\wedge\tau^*)^{t,\omega}}\big]=\mathbb{E}_{\mathbb{P}}\Big[\big(\overline{Z}_{\tau^*\wedge\gamma}\big)^{t,\omega}\Big],$$

where we used the fact that for any $\widetilde{\omega} \in \Omega^t$

$$\mathcal{Z}_{(\gamma\wedge\tau^*)^{t,\omega}}(\widetilde{\omega}) = \overline{Z}^{t,\omega}\big((\gamma\wedge\tau^*)^{t,\omega}(\widetilde{\omega}),\widetilde{\omega}\big) = \overline{Z}\big((\gamma\wedge\tau^*)(\omega\otimes_t\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big) = \big(\overline{Z}_{\tau^*\wedge\gamma}\big)(\omega\otimes_t\widetilde{\omega}) = \big(\overline{Z}_{\tau^*\wedge\gamma}\big)^{t,\omega}(\widetilde{\omega}).$$

Eventually, taking infimum over $\mathbb{P}\in\mathcal{P}(t,\omega)$ yields (7.50), which together with (7.48) leads to (7.50). Therefore, $\big\{\overline{Z}_{\tau^*\wedge t}\big\}_{t\in[0,T]}$ is an $\underline{\mathscr{E}}$−submartingale and it follows that

$$\inf_{\mathbb{P}\in\mathcal{P}}\sup_{\tau\in\mathcal{T}}\mathbb{E}_\mathbb{P}\big[Y_\tau\big] = \overline{Z}_0 \le \underline{\mathscr{E}}_0\big[\overline{Z}_{\tau^*}\big] = \inf_{\mathbb{P}\in\mathcal{P}}\mathbb{E}_\mathbb{P}\big[\overline{Z}_{\tau^*}\big] = \inf_{\mathbb{P}\in\mathcal{P}}\mathbb{E}_\mathbb{P}\big[Y_{\tau^*}\big] \le \sup_{\tau\in\mathcal{T}}\inf_{\mathbb{P}\in\mathcal{P}}\mathbb{E}_\mathbb{P}[Y_\tau] \le \inf_{\mathbb{P}\in\mathcal{P}}\sup_{\tau\in\mathcal{T}}\mathbb{E}_\mathbb{P}\big[Y_\tau\big]. \qquad \square$$

7.5 Proofs of the results in Section 6

Proof of Lemma 6.1: Define a mapping $\Psi:[t,T]\times\Omega^t\times\mathbb{R}^{d\times d}\to[t,T]\times\Omega\times\mathbb{R}^{d\times d}$ by $\Psi(r,\widetilde{\omega},u)=(r,\omega\otimes_t\widetilde{\omega},u)$, $\forall\,(r,\widetilde{\omega},u)\in[t,T]\times\Omega^t\times\mathbb{R}^{d\times d}$. Given $\mathcal{D}\in\mathscr{P}$ and $U\in\mathscr{B}(\mathbb{R}^{d\times d})$, one can deduce from Proposition 2.1 (3) that

$$\Psi^{-1}\big(\mathcal{D}\times U\big)=\big\{\big(r,\widetilde{\omega},u\big)\in[t,T]\times\Omega^t\times\mathbb{R}^{d\times d}:\big(r,\omega\otimes_t\widetilde{\omega},u\big)\in\mathcal{D}\times U\big\}=\mathcal{D}^{t,\omega}\times U\in\mathscr{P}^t\otimes\mathscr{B}(\mathbb{R}^{d\times d}).$$

So $\mathcal{D}\times U\in\Lambda:=\big\{\mathcal{J}\subset[0,T]\times\Omega\times\mathbb{R}^{d\times d}:\Psi^{-1}(\mathcal{J})\in\mathscr{P}^t\otimes\mathscr{B}(\mathbb{R}^{d\times d})\big\}$, which is clearly a σ−field of $[0,T]\times\Omega\times\mathbb{R}^{d\times d}$. It follows that $\mathscr{P}\otimes\mathscr{B}(\mathbb{R}^{d\times d})\subset\Lambda$, i.e., $\Psi^{-1}(\mathcal{J})\in\mathscr{P}^t\otimes\mathscr{B}(\mathbb{R}^{d\times d})$ for any $\mathcal{J}\in\mathscr{P}\otimes\mathscr{B}(\mathbb{R}^{d\times d})$.

For any $\mathcal{E}\in\mathscr{B}\big(\mathbb{R}^d\big)$, the measurability of b assures that $\widetilde{\mathcal{J}}:=\big\{(r,\omega',u)\in[0,T]\times\Omega\times\mathbb{R}^{d\times d}: b(r,\omega',u)\in\mathcal{E}\big\}\in\mathscr{P}\otimes\mathscr{B}(\mathbb{R}^{d\times d})$. Thus, $\big\{\big(r,\widetilde{\omega},u\big)\in[t,T]\times\Omega^t\times\mathbb{R}^{d\times d}: b^{t,\omega}\big(r,\widetilde{\omega},u\big)=b(r,\omega\otimes_t\widetilde{\omega},u)\in\mathcal{E}\big\}=\Psi^{-1}(\widetilde{\mathcal{J}})\in\mathscr{P}^t\otimes\mathscr{B}(\mathbb{R}^{d\times d})$, which gives the measurability of $b^{t,\omega}$. $\square$

Proof of the wellposedness of SDE (6.2)**:**

1) Fix $t\in[0,T]$. Let $\mathbb{S}^2_{\overline{\mathbf{F}}^t}([t,T];\mathbb{R}^d)$ denote the space of all $\mathbb{R}^d$−valued, $\overline{\mathbf{F}}^t$−adapted continuous processes X with $E_t[X_*^2]=E_t\big[\|X\|^2_{t,T}\big]<\infty$, and let us consider the following norm on $\mathbb{S}^2_{\overline{\mathbf{F}}^t}([t,T];\mathbb{R}^d)$:

$$\big\|X\big\|_\kappa:=\bigg(E_t\bigg[\sup_{s\in[t,T]}e^{-2\kappa^2Ts}|X_s|^2\bigg]\bigg)^{1/2},\qquad\forall\,X\in\mathbb{S}^2_{\overline{\mathbf{F}}^t}([t,T];\mathbb{R}^d).$$

Also, fix $\omega\in\Omega$ and $\mu\in\mathscr{U}_t$. Given $X\in\mathbb{S}^2_{\overline{\mathbf{F}}^t}([t,T];\mathbb{R}^d)$,

$$\mathcal{X}_s:=\int_t^s b^{t,\omega}(r,X,\mu_r)dr+\int_t^s\mu_r\,dB^t_r,\quad s\in[t,T]$$

defines an $\mathbb{R}^d$−valued, $\overline{\mathbf{F}}^t$−adapted continuous process. Since

$$\|\omega\otimes_tX\|_{0,r}\le\|\omega\|_{0,t}+\|X\|_{t,r},\qquad\forall\,r\in[t,T],\tag{7.73}$$

(6.1) implies that

$$\begin{aligned}\|\mathcal{X}\|_{t,T}&=\sup_{s\in[t,T]}\big|\mathcal{X}_s\big|\le\int_t^T\Big(\big|b(s,\omega\otimes_tX,\mu_s)-b(s,\mathbf{0},\mu_s)\big|+\big|b(s,\mathbf{0},\mu_s)\big|\Big)ds+\sup_{s\in[t,T]}\bigg|\int_t^s\mu_rdB^t_r\bigg|\\&\le\kappa\big(\|\omega\|_{0,t}+\|X\|_{t,T}+1+\kappa\big)(T-t)+\sup_{s\in[t,T]}\bigg|\int_t^s\mu_rdB^t_r\bigg|,\quad\mathbb{P}^t_0-a.s.\end{aligned}$$

The Doob's martingale inequality then shows that

$$E_t\big[\|\mathcal{X}\|^2_{t,T}\big]\le2\kappa^2T^2E_t\Big[\big(\|\omega\|_{0,t}+\|X\|_{t,T}+1+\kappa\big)^2\Big]+8E_t\int_t^T|\mu_s|^2ds\le4\kappa^2T^2\Big(\big(\|\omega\|_{0,t}+1+\kappa\big)^2+E_t\big[\|X\|^2_{t,T}\big]\Big)+8\kappa^2T<\infty.$$

So $\mathcal{X}\in\mathbb{S}^2_{\overline{\mathbf{F}}^t}([t,T];\mathbb{R}^d)$.

We set $\Psi^{t,\omega,\mu}(X):=\mathcal{X}$. To see that $\Psi^{t,\omega,\mu}$ defines a contraction map on $\mathbb{S}^2_{\overline{\mathbf{F}}^t}([t,T];\mathbb{R}^d)$ under the norm $\|\cdot\|_\kappa$, let $\widetilde{X}$ be another process in $\mathbb{S}^2_{\overline{\mathbf{F}}^t}([t,T];\mathbb{R}^d)$ and let $\widetilde{\mathcal{X}}:=\Psi^{t,\omega,\mu}\big(\widetilde{X}\big)$. Setting $\Delta X:=X-\widetilde{X}$, $\Delta\mathcal{X}:=\mathcal{X}-\widetilde{\mathcal{X}}$ and applying

Itô's formula to process $e^{-2\kappa^2 Ts}|\Delta\mathcal{X}_s|^2$ over the interval $[t,T]$, we can deduce from (6.1) that $\mathbb{P}^t_0-$a.s.

$$\begin{aligned} e^{-2\kappa^2 Ts}|\Delta\mathcal{X}_s|^2 &= \int_t^s e^{-2\kappa^2 Tr}\Big[2\big\langle\Delta\mathcal{X}_r, b^{t,\omega}(r,X,\mu_r)-b^{t,\omega}(r,\widetilde{X},\mu_r)\big\rangle-2\kappa^2 T|\Delta\mathcal{X}_r|^2\Big]dr \\ &\le \int_t^s e^{-2\kappa^2 Tr}\Big[2\kappa|\Delta\mathcal{X}_r|\big\|\omega\otimes_t X-\omega\otimes_t\widetilde{X}\big\|_{0,r}-2\kappa^2 T|\Delta\mathcal{X}_r|^2\Big]dr \\ &\le \frac{1}{2T}\int_t^s e^{-2\kappa^2 Tr}\big\|X-\widetilde{X}\big\|^2_{t,r}dr \le \frac12\sup_{r\in[t,T]} e^{-2\kappa^2 Tr}|\Delta X_r|^2, \quad s\in[t,T]. \end{aligned}$$

It follows that $\|\Delta\mathcal{X}\|^2_k = E_t\Big[\sup\limits_{s\in[t,T]} e^{-2\kappa^2 Ts}|\Delta\mathcal{X}_s|^2\Big] \le \frac12 E_t\Big[\sup\limits_{s\in[t,T]} e^{-2\kappa^2 Ts}|\Delta X_s|^2\Big] = \frac12\|\Delta X\|^2_k$.

Hence, $\Psi^{t,\omega,\mu}$ is a contraction mapping on $\mathbb{S}^2_{\overline{\mathbf{F}}^t}([t,T];\mathbb{R}^d)$ under the norm $\|\cdot\|_\kappa$. Then the unique fixed point $X^{t,\omega,\mu}$ of $\Psi^{t,\omega,\mu}$ forms a unique solution of (6.2) in $\mathbb{S}^2_{\overline{\mathbf{F}}^t}([t,T];\mathbb{R}^d)$.

2) Now, let $p\ge 1$ and $s\in[t,T]$. Since (6.2), (6.1) and (7.73) show that

$$\begin{aligned} \|X^{t,\omega,\mu}\|_{t,s} &= \sup_{r\in[t,s]}\big|X^{t,\omega,\mu}_r\big| \le \int_t^s\Big(\big|b(r,\omega\otimes_t X^{t,\omega,\mu},\mu_r)-b(r,\mathbf{0},\mu_r)\big|+\big|b(r,\mathbf{0},\mu_r)\big|\Big)dr + \sup_{r\in[t,s]}\Big|\int_t^r \mu_{r'}dB^t_{r'}\Big| \\ &\le \kappa\int_t^s\big(\|\omega\|_{0,t}+\|X^{t,\omega,\mu}\|_{t,r}+1+\kappa\big)dr + \sup_{r\in[t,s]}\Big|\int_t^r \mu_{r'}dB^t_{r'}\Big|, \quad \mathbb{P}^t_0-a.s., \end{aligned}$$

Using the inequality

$$\Big(\sum_{i=1}^n a_i\Big)^p \le n^{p-1}\sum_{i=1}^n a_i^p, \quad \forall\, a_1,\cdots,a_n\in(0,\infty), \tag{7.74}$$

we can deduce from Hölders inequality, the Burkholder-Davis-Gundy inequality and Fubini's Theorem that for some constant $c_p>0$

$$\begin{aligned} E_t\big[\|X^{t,\omega,\mu}\|^p_{t,s}\big] &\le 3^{p-1}\kappa^p\big(\|\omega\|_{0,t}+1+\kappa\big)^p(s-t)^p+3^{p-1}\kappa^p E_t\Big[\Big(\int_t^s\|X^{t,\omega,\mu}\|_{t,r}dr\Big)^p\Big]+c_pE_t\Big[\Big(\int_t^s|\mu_r|^2dr\Big)^{p/2}\Big] \\ &\le \kappa^p\big[3^{p-1}\big(\|\omega\|_{0,t}+1+\kappa\big)^p(s-t)^p+c_p(s-t)^{p/2}\big]+3^{p-1}\kappa^p(s-t)^{p-1}\int_t^s E_t\|X^{t,\omega,\mu}\|^p_{t,r}dr. \end{aligned}$$

Then an application of Gronwall's inequality shows that

$$E_t\big[\|X^{t,\omega,\mu}\|^p_{t,s}\big] \le \big[3^{p-1}\big(\|\omega\|_{0,t}+1+\kappa\big)^p(s-t)^p+c_p(s-t)^{p/2}\big]\exp\{3^{p-1}\kappa^p(s-t)^p\}<\infty, \quad \forall\, s\in[t,T]. \tag{7.75}$$

Proof of (6.3)**:** Let $t\in[0,T]$, $\omega,\omega'\in\Omega$ and $\mu\in\mathscr{U}_t$. For any $r\in[t,T]$, we set $\Delta X_r := X^{t,\omega,\mu}_r - X^{t,\omega',\mu}_r$. Given $s\in[t,T]$, since (6.2) and (6.1) show that

$$\|\Delta X\|_{t,s} = \sup_{r\in[t,s]}\big|\Delta X_r\big| \le \kappa\int_t^s\big\|\omega\otimes_t X^{t,\omega,\mu}-\omega'\otimes_t X^{t,\omega',\mu}\big\|_{0,r}dr \le \kappa\int_t^s\big(\|\omega-\omega'\|_{0,t}+\|\Delta X\|_{t,r}\big)dr, \quad \mathbb{P}^t_0-a.s.$$

we can deduce from (7.74), Hölder's inequality and Fubini's Theorem that

$$E_t\big[\|\Delta X\|^p_{t,s}\big] \le 2^{p-1}\kappa^p\Big\{\|\omega-\omega'\|^p_{0,t}(s-t)^p+(s-t)^{p-1}\int_t^s E_t\|\Delta X\|^p_{t,r}dr\Big\}.$$

Similar to (7.75), Gronwall's inequality implies that (6.3) holds for $C_p := 2^{p-1}\kappa^p\exp\{2^{p-1}\kappa^p T^p\}$. □

Proof of (6.4)**:** Fix $(t,\omega)\in[0,T]\times\Omega$ and $\mu\in\mathscr{U}_t$. Let ζ be an $\overline{\mathbf{F}}^t-$stopping time and $\delta>0$.

Given $s\in[t,T]$, set $\nu_s := (\zeta\vee s)\wedge(\zeta+\delta)$. Since an analogy to (7.73), (6.2) and (6.1) show that

$$\begin{aligned} \big|X^{t,\omega,\mu}_{\nu_s}-X^{t,\omega,\mu}_\zeta\big| &\le \int_\zeta^{\nu_s}\big(|b(r,\omega\otimes_t X^{t,\omega,\mu},\mu_r)-b(r,\mathbf{0},\mu_r)|+|b(r,\mathbf{0},\mu_r)|\big)dr+\Big|\int_\zeta^{\nu_s}\mu_r\,dB^t_r\Big| \\ &\le \kappa\big(\|\omega\|_{0,t}+\big\|X^{t,\omega,\mu}\big\|_{t,T}+1+\kappa\big)(\nu_s-\zeta)+\Big|\int_t^s \mathbf{1}_{\{\zeta\le r\le(\zeta+\delta)\wedge T\}}\mu_r\,dB^t_r\Big|, \quad \mathbb{P}^t_0-a.s., \end{aligned}$$

we see from $0 \le \nu_s - \zeta \le \delta$ that $\mathbb{P}^t_0$−a.s.

$$\sup_{r\in[\zeta,(\zeta+\delta)\wedge T]} \big|X^{t,\omega,\mu}_r - X^{t,\omega,\mu}_\zeta\big| = \sup_{s\in[t,T]} \big|X^{t,\omega,\mu}_{\nu_s} - X^{t,\omega,\mu}_\zeta\big| \le \kappa\big(\|\omega\|_{0,t} + \big\|X^{t,\omega,\mu}\big\|_{t,T} + 1 + \kappa\big)\delta + \sup_{s\in[t,T]} \bigg| \int_t^s \mathbf{1}_{\{\zeta \le r \le (\zeta+\delta)\wedge T\}} \mu_r \, dB^t_r \bigg|.$$

Using (7.74) again, we can deduce from Hölders inequality, the Burkholder-Davis-Gundy inequality, Fubini's Theorem and (7.75) that

$$\begin{aligned} E_t\bigg[\sup_{r\in[\zeta,(\zeta+\delta)\wedge T]} \big|X^{t,\omega,\mu}_r - X^{t,\omega,\mu}_\zeta\big|^p\bigg] &\le 3^{p-1}\kappa^p\delta^p \Big\{ (\|\omega\|_{0,t} + 1 + \kappa)^p + E_t\big[\|X^{t,\omega,\mu}\|^p_{t,T}\big] \Big\} + c_p E_t\bigg[\Big(\int_t^T \mathbf{1}_{\{\zeta\le r\le(\zeta+\delta)\wedge T\}} |\mu_r|^2 dr\Big)^{p/2}\bigg] \\ &\le \varphi_p\big(\|\omega\|_{0,t}\big)\delta^{p/2} \end{aligned}$$

for the continuous function $\varphi_p(x) := 3^{p-1}\kappa^p T^{p/2}\Big\{ (x+1+\kappa)^p + \big[3^{p-1}\big(x+1+\kappa\big)^p T^p + c_p T^{p/2}\big] \exp\{3^{p-1}\kappa^p T^p\} \Big\} + c_p\kappa^p$, $\forall\, x > 0$. □

Proof of Proposition 6.1: The conclusion clearly holds when $t = s$. So let us just consider the case $t < s$.

1) *In the first step, we will apply* (6.2) *to path* $\widetilde{\omega}\otimes_s\widehat{\omega}$ *so as to get a rough version* (7.80) *of the shifted SDE.*

By (6.2), it holds except on an $\mathcal{N}_1 \in \overline{\mathscr{N}}^t$ that

$$\mathcal{X}_r - \mathcal{X}_s = \int_s^r b^{t,\omega}\big(r', \mathcal{X}, \mu_{r'}\big) dr' + \int_s^r \mu_{r'} dB^t_{r'}, \quad r \in [s,T]. \tag{7.76}$$

Applying Lemma A.4 (3) with $(\mathbb{P}, X) = (\mathbb{P}^t_0, B^t)$ shows that $\mathcal{X}$ has a $(\mathbf{F}^t, \mathbb{P}^t_0)$−version $\widetilde{\mathcal{X}}$. Set $\mathcal{N}_2 := \big\{\widetilde{\omega} \in \Omega^t : \mathcal{X}_r(\widetilde{\omega}) \neq \widetilde{\mathcal{X}}_r(\widetilde{\omega})$ for some $r \in [t,T]\big\} \in \overline{\mathscr{N}}^t$ and let $\mathcal{N} := \mathcal{N}_1 \cup \mathcal{N}_2 \in \overline{\mathscr{N}}^t$. Since $\mathcal{D} := \{(r,\widetilde{\omega}) \in [t,T] \times \Omega^t : |\mu_r(\widetilde{\omega})| > \kappa\}$ satisfies $(dr \times d\mathbb{P}^t_0)(\mathcal{D}) = 0$, Lemma 2.5 shows that for all $\widetilde{\omega} \in \Omega^t$ except on some $\mathcal{N}_3 \in \overline{\mathscr{N}}^t$,

$$\mathcal{N}^{s,\widetilde{\omega}} \in \overline{\mathscr{N}}^s \quad \text{and} \quad (dr \times d\mathbb{P}^s_0)\big(\mathcal{D}^{s,\widetilde{\omega}}\big) = 0. \tag{7.77}$$

Fix $\widetilde{\omega} \in \big(\mathcal{N}_2 \cup \mathcal{N}_3\big)^c$ and set $\mathfrak{X}^{\widetilde{\omega}}_r(\widehat{\omega}) := \mathcal{X}^{s,\widetilde{\omega}}_r(\widehat{\omega}) - \mathcal{X}_s(\widetilde{\omega})$, $(r,\widehat{\omega}) \in [s,T] \times \Omega^s$. Since the shifted process $\widetilde{\mathcal{X}}^{s,\widetilde{\omega}}$ is $\mathbf{F}^s$−adapted by Proposition 2.1 (2), we can deduce from (7.77) that for any $(r,\mathcal{E}) \in [s,T] \times \mathscr{B}(\mathbb{R}^d)$

$$\big\{\widehat{\omega} \in \Omega^s : \mathfrak{X}^{\widetilde{\omega}}_r(\widehat{\omega}) \in \mathcal{E}\big\} = \big\{\widehat{\omega} \in \mathcal{N}^{s,\widetilde{\omega}} : \mathfrak{X}^{\widetilde{\omega}}_r(\widehat{\omega}) \in \mathcal{E}\big\} \cup \big\{\widehat{\omega} \in \big(\mathcal{N}^{s,\widetilde{\omega}}\big)^c = (\mathcal{N}^c)^{s,\widetilde{\omega}} : \widetilde{\mathcal{X}}^{s,\widetilde{\omega}}_r(\widehat{\omega}) \in \mathcal{E} + \mathcal{X}_s(\widetilde{\omega})\big\} \in \overline{\mathcal{F}}^s_r.$$

So $\mathfrak{X}^{\widetilde{\omega}}$ is $\overline{\mathbf{F}}^s$−adapted.

For any $r \in [t,s]$, since $\widetilde{\mathcal{X}}_r \in \mathcal{F}^t_r \subset \mathcal{F}^t_s$, we see from (2.2) that

$$\mathcal{X}_r(\widetilde{\omega} \otimes_s \widehat{\omega}) = \widetilde{\mathcal{X}}_r(\widetilde{\omega} \otimes_s \widehat{\omega}) = \widetilde{\mathcal{X}}_r(\widetilde{\omega}) = \mathcal{X}_r(\widetilde{\omega}), \quad \forall\, \widehat{\omega} \in \big(\mathcal{N}^{s,\widetilde{\omega}}\big)^c. \tag{7.78}$$

Let $\widehat{\omega} \in \big(\mathcal{N}^{s,\widetilde{\omega}}\big)^c$. The equality (7.78) implies that $\mathfrak{X}^{\widetilde{\omega}}_s(\widehat{\omega}) = 0$ and thus $\mathfrak{X}^{\widetilde{\omega}}(\widehat{\omega}) \in \Omega^s$. By (7.78) again

$$\begin{aligned} \big(\omega \otimes_t \mathcal{X}(\widetilde{\omega} \otimes_s \widehat{\omega})\big)(r) &= \mathbf{1}_{\{r\in[0,t)\}}\omega(r) + \mathbf{1}_{\{r\in[t,T]\}}\big(\mathcal{X}_r(\widetilde{\omega} \otimes_s \widehat{\omega}) + \omega(t)\big) \\ &= \mathbf{1}_{\{r\in[0,t)\}}\omega(r) + \mathbf{1}_{\{r\in[t,s)\}}\big(\mathcal{X}_r(\widetilde{\omega}) + \omega(t)\big) + \mathbf{1}_{\{r\in[s,T]\}}\big(\mathfrak{X}^{\widetilde{\omega}}_r(\widehat{\omega}) + \mathcal{X}_s(\widetilde{\omega}) + \omega(t)\big) \\ &= \mathbf{1}_{\{r\in[0,s)\}}\big(\omega \otimes_t \mathcal{X}(\widetilde{\omega})\big)(r) + \mathbf{1}_{\{r\in[s,T]\}}\big(\mathfrak{X}^{\widetilde{\omega}}_r(\widehat{\omega}) + \big(\omega \otimes_t \mathcal{X}(\widetilde{\omega})\big)(s)\big) = \Big(\big(\omega \otimes_t \mathcal{X}(\widetilde{\omega})\big) \otimes_s \mathfrak{X}^{\widetilde{\omega}}(\widehat{\omega})\Big)(r), \quad \forall\, r \in [0,T]. \end{aligned}$$

It follows that

$$b^{t,\omega}\big(r, \mathcal{X}(\widetilde{\omega} \otimes_s \widehat{\omega}), \mu_r(\widetilde{\omega} \otimes_s \widehat{\omega})\big) = b\big(r, \omega \otimes_t \mathcal{X}(\widetilde{\omega} \otimes_s \widehat{\omega}), \mu^{s,\widetilde{\omega}}_r(\widehat{\omega})\big) = b^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega})}\big(r, \mathfrak{X}^{\widetilde{\omega}}(\widehat{\omega}), \mu^{s,\widetilde{\omega}}_r(\widehat{\omega})\big), \quad \forall\, r \in [s,T]. \tag{7.79}$$

Applying (7.76) to path $\widetilde{\omega}\otimes_s\widehat{\omega}$ and using (7.78), (7.79) yield that

$$\mathfrak{X}^{\widetilde{\omega}}_r(\widehat{\omega}) = \mathcal{X}^{s,\widetilde{\omega}}_r(\widehat{\omega}) - \mathcal{X}_s(\widetilde{\omega}) = \int_s^r b^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega})}\big(r', \mathfrak{X}^{\widetilde{\omega}}(\widehat{\omega}), \mu^{s,\widetilde{\omega}}_{r'}(\widehat{\omega})\big) dr' + \Big(\int_s^r \mu_{r'} dB^t_{r'}\Big)(\widetilde{\omega} \otimes_s \widehat{\omega}), \quad \forall\, r \in [s,T]. \tag{7.80}$$

2) *Next, we show that for* $\mathbb{P}^s_0$*−a.s.* $\widehat{\omega} \in \Omega^s$, $\big(\int_s^r \mu_{r'} dB^t_{r'}\big)(\widetilde{\omega}\otimes_s\widehat{\omega}) = \big(\int_s^r \mu_{r'} dB^t_{r'}\big)^{s,\widetilde{\omega}} = \big(\int_s^r \mu^{s,\widetilde{\omega}}_{r'} dB^s_{r'}\big)(\widehat{\omega})$, $\forall\, r \in [s,T]$. *This is quite technically involved since the stochastic integral* $\int_s^r \mu_{r'} dB^t_{r'}$ *is not constructed pathwisely.*

Clearly, $\mathfrak{M}_r := \int_t^r \mu_{r'} dB^t_{r'},\ r \in [t,T]$ is a martingale with respect to $\big(\overline{\mathbf{F}}^t, \mathbb{P}^t_0\big)$. Applying Lemma A.4 (3) with $(\mathbb{P}, X) = (\mathbb{P}^t_0, B^t)$ shows that $\mathfrak{M}$ has a $(\mathbf{F}^t, \mathbb{P}^t_0)-$version $\widetilde{\mathfrak{M}}$. Let $\mathcal{N}_4 := \big\{\widetilde{\omega} \in \Omega^t:$ the path $\mathfrak{M}_\cdot(\widetilde{\omega})$ is not continuous$\big\} \cup \big\{\widetilde{\omega} \in \Omega^t : \mathfrak{M}_r(\widetilde{\omega}) \neq \widetilde{\mathfrak{M}}_r(\widetilde{\omega})$ for some $r \in [t,T]\big\} \in \overline{\mathscr{N}}^t$. Similar to (7.77), it holds for all $\widetilde{\omega} \in \Omega^t$ except on an $\mathcal{N}_5 \in \overline{\mathscr{N}}^t$

$$\mathcal{N}_4^{s,\widetilde{\omega}} \in \overline{\mathscr{N}}^s. \tag{7.81}$$

We know that (see e.g. Problem 3.2.27 of [19]) there is a sequence of $\mathcal{S}^{>0}_d-$valued, $\overline{\mathbf{F}}^t-$simple processes $\Big\{\overline{\Phi}^n_r = \sum_{i=1}^{\ell_n} \overline{\xi}^n_i \, \mathbf{1}_{\{r \in (t^n_i, t^n_{i+1}]\}},\ r \in [t,T]\Big\}_{n \in \mathbb{N}}$ $\big($where $t = t^n_1 < \cdots < t^n_{\ell_n+1} = T$ and $\overline{\xi}^n_i \in \overline{\mathcal{F}}^t_{t^n_i}$ for $i = 1, \cdots, \ell_n\big)$ such that

$$\mathbb{P}^t_0 - \lim_{n\to\infty} \int_t^T trace\Big\{\big(\overline{\Phi}^n_r - \mu_r\big)\big(\overline{\Phi}^n_r - \mu_r\big)^T\Big\}\, dr = 0 \quad \text{and} \quad \mathbb{P}^t_0 - \lim_{n\to\infty} \sup_{r\in[t,T]} \big|\overline{\mathfrak{M}}^n_r - \widetilde{\mathfrak{M}}_r\big| = \mathbb{P}^t_0 - \lim_{n\to\infty} \sup_{r\in[t,T]} \big|\overline{\mathfrak{M}}^n_r - \mathfrak{M}_r\big| = 0,$$

where $\overline{\mathfrak{M}}^n_r := \int_t^r \overline{\Phi}^n_{r'} dB^t_{r'} = \sum_{i=1}^{\ell_n} \overline{\xi}^n_i \big(B^t_{r\wedge t^n_{i+1}} - B^t_{r \wedge t^n_i}\big)$. Given $n \in \mathbb{N}$, applying Lemma A.4 (2) with $(\mathbb{P}, X) = (\mathbb{P}^t_0, B^t)$ shows that there exists an $\mathbb{R}^{d\times d}-$valued, $\mathcal{F}^t_{t^n_i}-$measurable random variable ξ^n_i such that $\xi^n_i = \overline{\xi}^n_i$, $\mathbb{P}^t_0-$a.s. for any $i = 1, \cdots, \ell_n$. Then the $\mathbf{F}^t-$simple processes $\Big\{\Phi^n_r = \sum_{i=1}^{\ell_n} \xi^n_i \, \mathbf{1}_{\{r\in(t^n_i, t^n_{i+1}]\}},\ r \in [t,T]\Big\}_{n\in\mathbb{N}}$ satisfy

$$\mathbb{P}^t_0 - \lim_{n\to\infty} \int_t^T trace\Big\{\big(\Phi^n_r - \mu_r\big)\big(\Phi^n_r - \mu_r\big)^T\Big\}\, dr = 0 \quad \text{and} \quad \mathbb{P}^t_0 - \lim_{n\to\infty} \sup_{r\in[t,T]} \big|\mathfrak{M}^n_r - \widetilde{\mathfrak{M}}_r\big| = 0,$$

where $\mathfrak{M}^n_r := \int_t^r \Phi^n_{r'} dB^t_{r'} = \sum_{i=1}^{\ell_n} \xi^n_i \big(B^t_{r\wedge t^n_{i+1}} - B^t_{r\wedge t^n_i}\big)$. Since $\int_t^T trace\Big\{\big(\Phi^n_r - \mu_r\big)\big(\Phi^n_r - \mu_r\big)^T\Big\}\, dr$ and $\sup\limits_{r\in[t,T]\cap\mathbb{Q}} \big|\mathfrak{M}^n_r - \widetilde{\mathfrak{M}}_r\big|$ are both $\mathcal{F}^t_T-$measurable, Lemma A.10 shows that $\{\Phi^n\}_{n\in\mathbb{N}}$ has a subsequence $\Big\{\widehat{\Phi}^n_r = \sum_{i=1}^{\widehat{\ell}_n} \widehat{\xi}^{\,n}_i \mathbf{1}_{\big\{r\in\big(\widehat{t}^{\,n}_i, \widehat{t}^{\,n}_{i+1}\big]\big\}},\ r \in [t,T]\Big\}_{n\in\mathbb{N}}$ such that for any $\widetilde{\omega} \in \Omega^t$ except on some $\mathcal{N}_6 \in \overline{\mathscr{N}}^t$

$$0 = \mathbb{P}^s_0 - \lim_{n\to\infty} \int_s^T trace\Big\{\Big(\big(\widehat{\Phi}^n\big)^{s,\widetilde{\omega}}_r - \mu^{s,\widetilde{\omega}}_r\Big)\Big(\big(\widehat{\Phi}^n\big)^{s,\widetilde{\omega}}_r - \mu^{s,\widetilde{\omega}}_r\Big)^T\Big\}\, dr \tag{7.82}$$

$$\text{and} \quad 0 = \mathbb{P}^s_0 - \lim_{n\to\infty} \sup_{r\in[s,T]\cap\mathbb{Q}} \Big|\big(\widehat{\mathfrak{M}}^n\big)^{s,\widetilde{\omega}}_r - \big(\widehat{\mathfrak{M}}^n\big)^{s,\widetilde{\omega}}_s - \widetilde{\mathfrak{M}}^{s,\widetilde{\omega}}_r + \widetilde{\mathfrak{M}}^{s,\widetilde{\omega}}_s\Big|, \tag{7.83}$$

where $\widehat{\mathfrak{M}}^{\,n}_r := \int_t^r \widehat{\Phi}^n_{r'} dB^t_{r'} = \sum_{i=1}^{\widehat{\ell}_n} \widehat{\xi}^{\,n}_i \Big(B^t_{r\wedge \widehat{t}^{\,n}_{i+1}} - B^t_{r\wedge \widehat{t}^{\,n}_i}\Big)$.

Fix $\widetilde{\omega} \in \big(\mathcal{N}_5 \cup \mathcal{N}_6\big)^c$. For any $\widehat{\omega} \in (\mathcal{N}_4^{s,\widetilde{\omega}})^c = (\mathcal{N}^c_4)^{s,\widetilde{\omega}}$, the path $\widetilde{\mathfrak{M}}_\cdot(\widetilde{\omega} \otimes_s \widehat{\omega}) = \mathfrak{M}_\cdot(\widetilde{\omega} \otimes_s \widehat{\omega})$ is continuous, so

$$\sup_{r\in[s,T]\cap\mathbb{Q}} \Big|\big(\widehat{\mathfrak{M}}^n\big)^{s,\widetilde{\omega}}_r - \big(\widehat{\mathfrak{M}}^n\big)^{s,\widetilde{\omega}}_s - \widetilde{\mathfrak{M}}^{s,\widetilde{\omega}}_r + \widetilde{\mathfrak{M}}^{s,\widetilde{\omega}}_s\Big|(\widehat{\omega}) = \sup_{r\in[s,T]} \Big|\big(\widehat{\mathfrak{M}}^n\big)^{s,\widetilde{\omega}}_r - \big(\widehat{\mathfrak{M}}^n\big)^{s,\widetilde{\omega}}_s - \widetilde{\mathfrak{M}}^{s,\widetilde{\omega}}_r + \widetilde{\mathfrak{M}}^{s,\widetilde{\omega}}_s\Big|(\widehat{\omega}), \quad \forall\, n \in \mathbb{N}.$$

As $\mathcal{N}_4^{s,\widetilde{\omega}} \in \overline{\mathscr{N}}^s$ by (7.81), it follows from (7.83) that

$$0 = \mathbb{P}^s_0 - \lim_{n\to\infty} \sup_{r\in[s,T]} \Big|\big(\widehat{\mathfrak{M}}^n\big)^{s,\widetilde{\omega}}_r - \big(\widehat{\mathfrak{M}}^n\big)^{s,\widetilde{\omega}}_s - \widetilde{\mathfrak{M}}^{s,\widetilde{\omega}}_r + \widetilde{\mathfrak{M}}^{s,\widetilde{\omega}}_s\Big|. \tag{7.84}$$

Given $n \in \mathbb{N}$, there exists some $j_n \in \{1, \cdots, \widehat{\ell}_n\}$ such that $s \in \big(\widehat{t}^{\,n}_{j_n}, \widehat{t}^{\,n}_{j_n+1}\big]$. Since $\widehat{\xi}^{\,n}_{j_n} \in \mathcal{F}^t_{\widehat{t}^{\,n}_{j_n}} \subset \mathcal{F}^t_s$, (2.2) shows that $\big(\widehat{\xi}^{\,n}_{j_n}\big)^{s,\widetilde{\omega}} = \widehat{\xi}^{\,n}_{j_n}(\widetilde{\omega})$ and Proposition 2.1 (1) shows that $\big(\widehat{\xi}^{\,n}_i\big)^{s,\widetilde{\omega}} \in \mathcal{F}^s_{\widehat{t}^{\,n}_i}$ for $i = j_n + 1, \cdots, \widehat{\ell}_n$. It then holds for any $(r, \widehat{\omega}) \in [s,T] \times \Omega^s$ that

$$\begin{aligned}\big(\widehat{\Phi}^n\big)^{s,\widetilde{\omega}}_r(\widehat{\omega}) &= \widehat{\Phi}^n_r(\widetilde{\omega} \otimes_s \widehat{\omega}) = \widehat{\xi}^{\,n}_{j_n}(\widetilde{\omega} \otimes_s \widehat{\omega})\, \mathbf{1}_{\big\{r\in\big[s, \widehat{t}^{\,n}_{j_n+1}\big]\big\}} + \sum_{i=j_n+1}^{\widehat{\ell}_n} \widehat{\xi}^{\,n}_i(\widetilde{\omega} \otimes_s \widehat{\omega})\, \mathbf{1}_{\big\{r\in\big(\widehat{t}^{\,n}_i, \widehat{t}^{\,n}_{i+1}\big]\big\}} \\ &= \widehat{\xi}^{\,n}_{j_n}(\widetilde{\omega})\, \mathbf{1}_{\big\{r\in\big[s, \widehat{t}^{\,n}_{j_n+1}\big]\big\}} + \sum_{i=j_n+1}^{\widehat{\ell}_n} \big(\widehat{\xi}^{\,n}_i\big)^{s,\widetilde{\omega}}(\widehat{\omega})\, \mathbf{1}_{\big\{r\in\big(\widehat{t}^{\,n}_i, \widehat{t}^{\,n}_{i+1}\big]\big\}},\end{aligned}$$

so $\big\{\big(\widehat{\Phi}^n\big)^{s,\widetilde{\omega}}_r\big\}_{r\in[s,T]}$ is an $\mathbf{F}^s$−simple process. Applying Proposition 3.2.26 of [19], we see from (7.82) that

$$0=\mathbb{P}^s_0-\lim_{n\to\infty}\sup_{r\in[s,T]}\bigg|\int_s^r\big(\widehat{\Phi}^n\big)^{s,\widetilde{\omega}}_{r'}dB^s_{r'}-\int_s^r\mu^{s,\widetilde{\omega}}_{r'}dB^s_{r'}\bigg|. \tag{7.85}$$

For any $n\in\mathbb{N}$ and $\widehat{\omega}\in\Omega^s$, one can deduce that for any $r\in[s,T]$

$$\begin{aligned}\Big(\big(\widehat{\mathfrak{M}}^n\big)^{s,\widetilde{\omega}}_r-\big(\widehat{\mathfrak{M}}^n\big)^{s,\widetilde{\omega}}_s\Big)(\widehat{\omega})&=\bigg[\widehat{\xi}^n_{j_n}\big(B^t_{r\wedge\widehat{t}^n_{j_n+1}}-B^t_s\big)+\sum_{i=j_n+1}^{\widehat{\ell}_n}\widehat{\xi}^n_i\Big(B^t_{r\wedge\widehat{t}^n_{i+1}}-B^t_{r\wedge\widehat{t}^n_i}\Big)\bigg](\widetilde{\omega}\otimes_s\widehat{\omega})\\&=\widehat{\xi}^n_{j_n}(\widetilde{\omega})\cdot\widehat{\omega}\big(r\wedge\widehat{t}^n_{j_n+1}\big)+\sum_{i=j_n+1}^{\widehat{\ell}_n}\big(\widehat{\xi}^n_i\big)^{s,\widetilde{\omega}}(\widehat{\omega})\Big(\widehat{\omega}\big(r\wedge\widehat{t}^n_{i+1}\big)-\widehat{\omega}\big(r\wedge\widehat{t}^n_i\big)\Big)\\&=\bigg[\widehat{\xi}^n_{j_n}(\widetilde{\omega})\cdot B^s_{r\wedge\widehat{t}^n_{j_n+1}}+\sum_{i=j_n+1}^{\widehat{\ell}_n}\big(\widehat{\xi}^n_i\big)^{s,\widetilde{\omega}}\Big(B^s_{r\wedge\widehat{t}^n_{i+1}}-B^s_{r\wedge\widehat{t}^n_i}\Big)\bigg](\widehat{\omega})=\bigg(\int_s^r\big(\widehat{\Phi}^n\big)^{s,\widetilde{\omega}}_{r'}dB^s_{r'}\bigg)(\widehat{\omega}),\end{aligned}$$

which together with (7.84), (7.85) and (7.81) shows that $\mathbb{P}^s_0$−a.s.

$$\int_s^r\mu^{s,\widetilde{\omega}}_{r'}dB^s_{r'}=\widetilde{\mathfrak{M}}^{s,\widetilde{\omega}}_r-\widetilde{\mathfrak{M}}^{s,\widetilde{\omega}}_s=\mathfrak{M}^{s,\widetilde{\omega}}_r-\mathfrak{M}^{s,\widetilde{\omega}}_s=\Big(\int_s^r\mu_{r'}dB^t_{r'}\Big)^{s,\widetilde{\omega}},\quad r\in[s,T]. \tag{7.86}$$

3) Let $\widetilde{\omega}\in\big(\mathcal{N}_2\cup\mathcal{N}_3\cup\mathcal{N}_5\cup\mathcal{N}_6\big)^c$. Proposition 2.1 (2) shows the shift process $\mu^{s,\widetilde{\omega}}$ is $\mathbf{F}^s$−progressively measurable. And (7.77) implies that

$$(dr\times d\mathbb{P}^s_0)\{(r,\widehat{\omega})\in[s,T]\times\Omega^s:|\mu^{s,\widetilde{\omega}}_r(\widehat{\omega})|>\kappa\}=(dr\times d\mathbb{P}^s_0)\{(r,\widehat{\omega})\in[s,T]\times\Omega^s:(r,\widetilde{\omega}\otimes_s\widehat{\omega})\in\mathcal{D}\}=(dr\times d\mathbb{P}^s_0)\big(\mathcal{D}^{s,\widetilde{\omega}}\big)=0.$$

So $\mu^{s,\widetilde{\omega}}\in\mathcal{U}_s$. In light of (7.86) and (7.80), it holds $\mathbb{P}^s_0$−a.s. that

$$\mathfrak{X}^{\widetilde{\omega}}_r=\int_s^r b^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega})}\big(r',\mathfrak{X}^{\widetilde{\omega}},\mu^{s,\widetilde{\omega}}_{r'}\big)dr'+\int_s^r\mu^{s,\widetilde{\omega}}_{r'}\,dB^s_{r'},\quad r\in[s,T].$$

Then the uniqueness of solutions to the SDE (6.2) over period $[s,T]$ with drift $b^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega})}$ and control $\mu^{s,\widetilde{\omega}}$ leads to that $\mathcal{X}^{s,\widetilde{\omega}}-\mathcal{X}_s(\widetilde{\omega})=\mathfrak{X}^{\widetilde{\omega}}=X^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega}),\mu^{s,\widetilde{\omega}}}$. □

Proof of Proposition 6.2: Fix $(t,\omega)\in[0,T]\times\Omega$ and $\mu\in\mathcal{U}_t$. Let us set $\mathcal{X}=X^{t,\omega,\mu}$ and consider the induced filtration $\mathcal{X}^{-1}(\mathbf{F}^t)=\big\{\mathcal{X}^{-1}(\mathcal{F}^t_s):=\{\mathcal{X}^{-1}(A):A\in\mathcal{F}^t_s\}\big\}_{s\in[t,T]}$. Also, we define a mapping $\Psi^{\mathcal{X}}:[t,T]\times\Omega^t\to[t,T]\times\Omega^t$ by $\Psi^{\mathcal{X}}(r,\widetilde{\omega}):=\big(r,\mathcal{X}(\widetilde{\omega})\big)$, $\forall\,(r,\widetilde{\omega})\in[t,T]\times\Omega^t$. Clearly, $\sigma^{\mathcal{X}}:=(\Psi^{\mathcal{X}})^{-1}(\mathscr{P}^t)=\{(\Psi^{\mathcal{X}})^{-1}(\mathcal{D}):\mathcal{D}\in\mathscr{P}^t\}$ is a σ−field of $[t,T]\times\Omega^t$. A process $K=\{K_s\}_{s\in[t,T]}$ on Ω^t is called $\mathbb{P}^t_0$*−a.s.* $\mathcal{X}^{-1}(\mathbf{F}^t)$*−progressively measurable* (*resp.* $\mathbb{P}^t_0$*−a.s.* $\sigma^{\mathcal{X}}$*−measurable*) if K has a $\mathbb{P}^t_0$−indistinguishable version that is $\mathcal{X}^{-1}(\mathbf{F}^t)$−progressively measurable (resp. $\sigma^{\mathcal{X}}$−measurable).

1) *We first show that* B^t *is* $\mathbb{P}^t_0$*−a.s.* $\sigma^{\mathcal{X}}$*−measurable.*

1a) *In the first step, we show that the inverse of the* $\mathcal{S}^{>0}_d$*−valued control process* $\{\mu_s\}_{s\in[t,T]}$ *is* $ds\times d\mathbb{P}^t_0$*−a.s. equal to an* $\mathcal{X}^{-1}(\mathbf{F}^t)$*−progressively measurable process.*

Given $i,j\in\{1,\cdots,d\}$, let $\mathcal{X}^i$ be the i^{th} component of $\mathcal{X}$. It is known that (see e.g. Proposition IV.2.13 of [31])

$$\mathbb{P}^t_0-\lim_{n\to\infty}\sup_{s\in[t,T]}\bigg|M^n_s-\int_t^s\mathcal{X}^i_r d\mathcal{X}^j_r\bigg|=0, \tag{7.87}$$

where $M^n_s=M^{i,j,n}_s:=\sum_{\ell=0}^{n-1}\mathcal{X}^i_{s\wedge t^n_\ell}\big(\mathcal{X}^j_{s\wedge t^n_{\ell+1}}-\mathcal{X}^j_{s\wedge t^n_\ell}\big)$ and $t^n_\ell:=t+\frac{\ell}{n}(T-t)$. Clearly, $\mathcal{X}$ is $\mathcal{X}^{-1}(\mathbf{F}^t)$−adapted, so is $\mathcal{X}^i$. For any $t'\in[t,T]$, the continuity of $\mathcal{X}$ implies that

$$\text{the process }\{\mathcal{X}^i_{s\wedge t'}\}_{s\in[t,T]}\text{ is }\mathcal{X}^{-1}(\mathbf{F}^t)\text{−progressively measurable.} \tag{7.88}$$

So each process M^n is $\mathcal{X}^{-1}(\mathbf{F}^t)-$progressively measurable. Then we can deduce from (7.87) that the $\mathbb{P}^t_0-$stochastic integral $\int_t^\cdot \mathcal{X}^i_r d\mathcal{X}^j_r$ is $\mathbb{P}^t_0-$a.s. $\mathcal{X}^{-1}(\mathbf{F}^t)-$progressively measurable, so is the process $\Upsilon^{i,j}_s := \mathcal{X}^i_s\mathcal{X}^j_s - \int_t^s \mathcal{X}^i_r d\mathcal{X}^j_r - \int_t^s \mathcal{X}^j_r d\mathcal{X}^i_r$, $s \in [t,T]$. It follows that for any $n \in \mathbb{N}$, the process $\Upsilon^{n,i,j}_s := n\big(\Upsilon^{i,j}_s - \Upsilon^{i,j}_{(s-1/n)\vee t}\big)$, $s \in [t,T]$ is $\mathbb{P}^t_0-$a.s. $\mathcal{X}^{-1}(\mathbf{F}^t)-$progressively measurable. Hence, $\widetilde{\Upsilon}^{i,j}_s := \Big(\overline{\lim_{n\to\infty}}\,\Upsilon^{n,i,j}_s\Big)\mathbf{1}_{\big\{\overline{\lim\limits_{n\to\infty}}\,\Upsilon^{n,i,j}_s<\infty\big\}}$, $s \in [t,T]$ is still a $\mathbb{P}^t_0-$a.s. $\mathcal{X}^{-1}(\mathbf{F}^t)-$progressively measurable process.

Let μ^i denote the i^{th} row of μ. Since it holds except on an $\mathcal{N}_{i,j} \in \overline{\mathscr{N}}^t$ that $\int_t^s \mu^i_r \cdot \mu^j_r dr = \langle \mathcal{X}^i, \mathcal{X}^j\rangle^{\mathbb{P}^t_0}_s = \Upsilon^{i,j}_s$ for any $s \in [t,T]$, the Lebesgue differentiation theorem implies that for any $\widetilde{\omega} \in \mathcal{N}^c_{i,j}$,

$$\big(\mu^i_s \cdot \mu^j_s\big)(\widetilde{\omega}) = \lim_{n\to\infty} n\big(\Upsilon^{i,j}_s - \Upsilon^{i,j}_{(s-1/n)\vee t}\big)(\widetilde{\omega}) = \lim_{n\to\infty} \Upsilon^{n,i,j}_s(\widetilde{\omega}), \quad \text{for a.e. } s \in [t,T],$$

which implies that

$$\mu^2 = \widetilde{\Upsilon}, \quad ds \times d\mathbb{P}^t_0 - a.s. \tag{7.89}$$

For any $\ell \in \mathbb{N}$, let $c_\ell := -\dfrac{1\times 3\times\cdots\times(2\ell-3)}{2^\ell\,\ell!}$, which is the $\ell-$th coefficient of the power series of $\sqrt{1-x}$, $x \in [-1,1]$. Given $\Gamma \in \mathcal{S}^{>0}_d$ with $|\Gamma| \le 1$, we know (see e.g. Theorem VI.9 of [30]) that $\widehat{\Gamma} := I_{d\times d} + \sum_{\ell\in\mathbb{N}} c_\ell (I_{d\times d} - \Gamma)^\ell$ is the unique element in $\mathcal{S}^{>0}_d$ such that $\widehat{\Gamma}^2 = \widehat{\Gamma}\cdot\widehat{\Gamma} = \Gamma$. Given $(s,\widetilde{\omega}) \in [t,T]\times\Omega^t$, since $\mathfrak{n}_s(\widetilde{\omega}) := \dfrac{\mu^2(\widetilde{\omega})}{|\mu(\widetilde{\omega})|^2} \in \mathcal{S}^{>0}_d$, $\widehat{\mathfrak{n}}_s(\widetilde{\omega}) := I_{d\times d} + \sum_{\ell\in\mathbb{N}} c_\ell(I_{d\times d} - \mathfrak{n}_s(\widetilde{\omega}))^\ell$ is the unique element in $\mathcal{S}^{>0}_d$ such that $\widehat{\mathfrak{n}}^2_s(\widetilde{\omega}) = \mathfrak{n}_s(\widetilde{\omega}) = \dfrac{\mu^2_s(\widetilde{\omega})}{|\mu_s(\widetilde{\omega})|^2}$, thus

$$\widehat{\mathfrak{n}}_s(\widetilde{\omega}) = \frac{\mu_s(\widetilde{\omega})}{|\mu_s(\widetilde{\omega})|}. \tag{7.90}$$

On the other hand, since $\widetilde{\Upsilon}$ is an $\mathbb{R}^{d\times d}-$valued, $\mathbb{P}^t_0-$a.s. $\mathcal{X}^{-1}(\mathbf{F}^t)-$progressively measurable, so is the process $\widehat{\Upsilon}_s := \mathbf{1}_{\{|\widetilde{\Upsilon}_s|>0\}}\dfrac{\widetilde{\Upsilon}_s}{|\widetilde{\Upsilon}_s|}$, $s \in [t,T]$. It follows that $\mathfrak{u}_s(\widetilde{\omega}) := I_{d\times d} + \sum_{\ell\in\mathbb{N}} c_\ell(I_{d\times d} - \widehat{\Upsilon}_s(\widetilde{\omega}))^\ell$, $s \in [t,T]$ is also an $\mathbb{R}^{d\times d}-$valued, $\mathbb{P}^t_0-$a.s. $\mathcal{X}^{-1}(\mathbf{F}^t)-$progressively measurable process. By (7.89), we see that $\widehat{\Upsilon}_s = \mathfrak{n}_s$, $ds\times d\mathbb{P}^t_0-$a.s. and thus $\mathfrak{u}_s = \widehat{\mathfrak{n}}_s$, $ds \times d\mathbb{P}^t_0-$a.s. Then (7.90) and (7.89) imply that $\mu_s = \widehat{\mathfrak{n}}_s|\mu_s| = \mathfrak{u}_s\sqrt{|\widetilde{\Upsilon}_s|}$, $ds\times d\mathbb{P}^t_0-$a.s. Clearly, $\mathfrak{u}\sqrt{|\widetilde{\Upsilon}|}$ is still an $\mathbb{R}^{d\times d}-$valued, $\mathbb{P}^t_0-$a.s. $\mathcal{X}^{-1}(\mathbf{F}^t)-$progressively measurable process. Let $\widetilde{\mu}$ be its $\mathbb{P}^t_0-$indistinguishable version that is $\mathcal{X}^{-1}(\mathbf{F}^t)-$progressively measurable, so

$$\mu_s = \widetilde{\mu}_s, \quad ds \times d\mathbb{P}^t_0 - a.s. \tag{7.91}$$

Let a^{ij} (resp. $\widetilde{a}^{ij}$) denote the determinant of the $(d–1)\times(d–1)$ matrix that results from deleting row i and column j of μ (resp. $\widetilde{\mu}$). As $det(\widetilde{\mu})$ and $\widetilde{a}^{ij}$'s are all $\mathcal{X}^{-1}(\mathbf{F}^t)-$progressively measurable, the $\mathbb{R}^{d\times d}-$valued process

$$\mathfrak{q}_s := \mathbf{1}_{\{det(\widetilde{\mu}_s)\neq 0\}}\frac{1}{det(\widetilde{\mu}_s)}\big[(-1)^{i+j}\,\widetilde{a}^{ji}_s\big]_{d\times d}, \quad \forall\, s\in[t,T]$$

is also $\mathcal{X}^{-1}(\mathbf{F}^t)-$progressively measurable. Then we see from (7.91) that

$$\mu^{-1}_s = \mathbf{1}_{\{det(\mu_s)\neq 0\}}\frac{1}{det(\mu_s)}\big[(-1)^{i+j}\,a^{ji}_s\big]_{d\times d} = \mathfrak{q}_s, \quad ds\times d\mathbb{P}^t_0 - a.s. \tag{7.92}$$

1b) *In the second step, we show that the $\mathbb{P}^t_0-$stochastic integral $\int_t^\cdot \mathfrak{q}_r d\mathcal{X}_r$ is $\mathbb{P}^t_0-$a.s. $\sigma^{\mathcal{X}}-$measurable.*

Let ϕ be an $\mathbb{R}^{d\times d}-$valued, $\mathcal{X}^{-1}(\mathbf{F}^t)-$progressively measurable bounded processes such that $\sup\limits_{s\in[t,T]}|\phi_s| \le C_\phi$, $\mathbb{P}^t_0-$a.s. for some $C_\phi > 0$. Given $i,j\in\{1,\cdots,d\}$, since $\Phi^{i,j}_s := \int_t^s \phi^{i,j}_r dr$, $s\in[t,T]$ defines a real$-$valued, $\mathcal{X}^{-1}(\mathbf{F}^t)-$adapted continuous process, for any $n\in\mathbb{N}$ the process $\Phi^{n,i,j}_s := n\big(\Phi^{i,j}_s - \Phi^{i,j}_{(s-1/n)\vee t}\big)$ is again a real$-$valued, $\mathcal{X}^{-1}(\mathbf{F}^t)-$adapted continuous process with $\sup\limits_{s\in[t,T]}\big|\Phi^{n,i,j}_s\big| \le C_\phi$, $\mathbb{P}^t_0-$a.s. In light of the Lebesgue differentiation theorem, it holds for $\mathbb{P}^t_0-$a.s. $\widetilde{\omega}\in\Omega^t$ that

$$\phi^{i,j}_s(\widetilde{\omega}) = \lim_{n\to\infty} n\big(\Phi^{i,j}_s - \Phi^{i,j}_{(s-1/n)\vee t}\big)(\widetilde{\omega}) = \lim_{n\to\infty}\Phi^{n,i,j}_s(\widetilde{\omega}), \quad \text{for a.e. } s\in[t,T].$$

The bounded convergence theorem then implies that

$$\lim_{n\to\infty}\sum_{i=1}^d E_t\bigg[\Big\langle\int_t^\cdot(\Phi^{n,i}_r-\phi^i_r)d\mathcal{X}_r\Big\rangle^{\mathbb{P}^t_0}_T\bigg]=\lim_{n\to\infty}\sum_{i,j,k=1}^d E_t\bigg[\int_t^T(\Phi^{n,i,j}_r-\phi^{i,j}_r)(\Phi^{n,i,k}_r-\phi^{i,k}_r)d\big\langle\mathcal{X}^j,\mathcal{X}^k\big\rangle^{\mathbb{P}^t_0}_r\bigg]$$
$$=\lim_{n\to\infty}\sum_{i,j,k,l=1}^d E_t\int_t^T(\Phi^{n,i,j}_r-\phi^{i,j}_r)(\Phi^{n,i,k}_r-\phi^{i,k}_r)\mu^{j,l}_r\mu^{k,l}_r dr=\lim_{n\to\infty}E_t\int_t^T\big|(\Phi^n_r-\phi_r)\mu_r\big|^2dr \tag{7.93}$$
$$\le\kappa^2\lim_{n\to\infty}E_t\int_t^T\big|\Phi^n_r-\phi_r\big|^2dr=0.$$

It follows that (see e.g. Problem 1.5.25 of [19])

$$\mathbb{P}^t_0-\lim_{n\to\infty}\sup_{s\in[t,T]}\Big|\int_t^s(\Phi^n_r-\phi_r)d\mathcal{X}_r\Big|=0. \tag{7.94}$$

Given $n\in\mathbb{N}$, since the process Φ^n is continuous, using Proposition IV.2.13 of [31] again yields that

$$\mathbb{P}^t_0-\lim_{m\to\infty}\sup_{s\in[t,T]}\Big|\widetilde{M}^{n,m}_s-\int_t^s\Phi^n_r d\mathcal{X}_r\Big|=0, \tag{7.95}$$

where $\widetilde{M}^{n,m}_s:=\sum_{\ell=0}^{m-1}\Phi^n_{s\land t^m_\ell}\big(\mathcal{X}_{s\land t^m_{\ell+1}}-\mathcal{X}_{s\land t^m_\ell}\big)=\sum_{\ell=0}^{m-1}\mathbf{1}_{\{s>t^m_\ell\}}\Phi^n_{t^m_\ell}\big(\mathcal{X}_{s\land t^m_{\ell+1}}-\mathcal{X}_{s\land t^m_\ell}\big)$ and $t^m_\ell:=t+\frac{\ell}{m}(T-t)$. For any $m\in\mathbb{N}$ and $\ell=0,\cdots,m-1$, since $\big\{\mathbf{1}_{\{s>t^m_\ell\}}\Phi^n_{t^m_\ell}\big\}_{s\in[t,T]}$ is a $\mathcal{X}^{-1}(\mathbf{F}^t)-$adapted process with all left-continuous paths. Lemma A.11 and (7.88) show that $\big\{\mathbf{1}_{\{s>t^m_\ell\}}\Phi^n_{t^m_\ell}\big\}_{s\in[t,T]}$ is $\sigma^{\mathcal{X}}-$measurable, and so is $\widetilde{M}^{n,m}$. It follows from (7.95) that each $\mathbb{P}^t_0-$stochastic integral $\int_t^\cdot\Phi^n_r d\mathcal{X}_r$ is $\mathbb{P}^t_0-$a.s. $\sigma^{\mathcal{X}}-$measurable, and so is $\int_t^\cdot\phi_r d\mathcal{X}_r$ thanks to (7.94).

Now for $\alpha\in\mathbb{N}$, taking $\phi=\Big\{\mathfrak{q}^\alpha_s:=\dfrac{\alpha}{|\mathfrak{q}_s|\lor\alpha}\mathfrak{q}_s\Big\}_{s\in[t,T]}$ shows that $\int_t^\cdot\mathfrak{q}^\alpha_r d\mathcal{X}_r$ is $\mathbb{P}^t_0-$a.s. $\sigma^{\mathcal{X}}-$measurable. Similar to (7.93), we can deduce that $\lim_{\alpha\to\infty}\sum_{i=1}^d E_t\Big[\Big\langle\int_t^\cdot(\mathfrak{q}^\alpha_r-\mathfrak{q}_r)d\mathcal{X}_r\Big\rangle^{\mathbb{P}^t_0}_T\Big]=\lim_{\alpha\to\infty}E_t\int_t^T\big|(\mathfrak{q}^\alpha_r-\mathfrak{q}_r)\mu_r\big|^2dr$. Since $\big|(\mathfrak{q}^\alpha_s-\mathfrak{q}_s)\mu_s\big|=\Big(1-\dfrac{\alpha}{|\mathfrak{q}_s|\lor\alpha}\Big)\big|\mathfrak{q}_s\mu_s\big|\le\big|\mathfrak{q}_s\mu_s\big|=\big|\mu^{-1}_s\mu_s\big|=\big|I_{d\times d}\big|=\sqrt{d}$, $ds\times d\mathbb{P}^t_0-$a.s. by (7.92), the bounded convergence theorem implies that $\lim_{\alpha\to\infty}\sum_{i=1}^d E_t\Big[\Big\langle\int_t^\cdot(\mathfrak{q}^\alpha_r-\mathfrak{q}_r)d\mathcal{X}_r\Big\rangle^{\mathbb{P}^t_0}_T\Big]=0$. Then applying Problem 1.5.26 of [19] again shows that $\mathbb{P}^t_0-\lim_{\alpha\to\infty}\sup_{s\in[t,T]}\Big|\int_t^s(\mathfrak{q}^\alpha_r-\mathfrak{q}_r)d\mathcal{X}_r\Big|=0$. It follows that the $\mathbb{P}^t_0-$stochastic integral $\int_t^\cdot\mathfrak{q}_r d\mathcal{X}_r$ is also $\mathbb{P}^t_0-$a.s. $\sigma^{\mathcal{X}}-$measurable. Let K^1 be its $\mathbb{P}^t_0-$indistinguishable version that is $\sigma^{\mathcal{X}}-$measurable. (As we have seen from (6.6) that any $\mathcal{X}^{-1}(\mathbf{F}^t)-$progressively measurable process is also $\overline{\mathbf{F}}^t-$progressively measurable, the $\mathbb{P}^t_0-$stochastic integrals mentioned in this part are all well-defined.)

1c) Fix $U\in\mathscr{B}(\mathbb{R}^{d\times d})$. For any $s\in[t,T]$, we define a mapping $\widehat{\Psi}_s:[t,s]\times\Omega^t\to[t,s]\times\Omega^t\times\mathbb{R}^{d\times d}$ by $\widehat{\Psi}_s(r,\widetilde{\omega}):=\big(r,\mathcal{X}(\widetilde{\omega}),\widetilde{\mu}_r(\widetilde{\omega})\big)$, $\forall\,(r,\widetilde{\omega})\in[t,s]\times\Omega^t$. Given $\mathcal{E}\in\mathscr{B}\big([t,s]\big)$ and $A\in\mathcal{F}^t_s$, one can deduce from the $\mathcal{X}^{-1}(\mathbf{F}^t)-$progressive measurability of $\widetilde{\mu}$ that

$$\begin{aligned}\widehat{\Psi}^{-1}_s(\mathcal{E}\times A\times U)&=\big\{(r,\widetilde{\omega})\in[t,s]\times\Omega^t:\big(r,\mathcal{X}(\widetilde{\omega}),\widetilde{\mu}_r(\widetilde{\omega})\big)\in\mathcal{E}\times A\times U\big\}\\&=\big(\mathcal{E}\times\mathcal{X}^{-1}(A)\big)\cap\big\{(r,\widetilde{\omega})\in[t,s]\times\Omega^t:\widetilde{\mu}_r(\widetilde{\omega})\in U\big\}\in\mathscr{B}\big([t,s]\big)\otimes\mathcal{X}^{-1}(\mathcal{F}^t_s).\end{aligned}$$

So $\mathcal{E}\times A\in\Lambda_U:=\big\{\mathcal{D}\subset[t,s]\times\Omega^t:\widehat{\Psi}^{-1}_s(\mathcal{D}\times U)\in\mathscr{B}\big([t,s]\big)\otimes\mathcal{X}^{-1}(\mathcal{F}^t_s)\big\}$, which is clearly a $\sigma-$field of $[t,s]\times\Omega^t$. It follows that $\mathscr{B}\big([t,s]\big)\otimes\mathcal{F}^t_s\in\Lambda_U$, i.e., $\widehat{\Psi}^{-1}_s(\mathcal{D}\times U)\in\mathscr{B}\big([t,s]\big)\otimes\mathcal{X}^{-1}(\mathcal{F}^t_s)$ for any $\mathcal{D}\in\mathscr{B}\big([t,s]\big)\otimes\mathcal{F}^t_s$.

Now, let $\widetilde{\mathcal{D}}\in\mathscr{P}^t$. For any $s\in[t,T]$, as $\widetilde{\mathcal{D}}\cap\big([t,s]\times\Omega^t\big)\in\mathscr{B}\big([t,s]\big)\otimes\mathcal{F}^t_s$, one can deduce that

$$\widehat{\Psi}^{-1}_T(\widetilde{\mathcal{D}}\times U)\cap\big([t,s]\times\Omega^t\big)=\big\{(r,\widetilde{\omega})\in[t,s]\times\Omega^t:\big(r,\mathcal{X}(\widetilde{\omega}),\widetilde{\mu}_r(\widetilde{\omega})\big)\in\widetilde{\mathcal{D}}\times U\big\}$$
$$=\big\{(r,\widetilde{\omega})\in[t,s]\times\Omega^t:\big(r,\mathcal{X}(\widetilde{\omega}),\widetilde{\mu}_r(\widetilde{\omega})\big)\in\big(\widetilde{\mathcal{D}}\cap([t,s]\times\Omega^t)\big)\times U\big\}=\widehat{\Psi}^{-1}_s\Big(\big(\widetilde{\mathcal{D}}\cap([t,s]\times\Omega^t)\big)\times U\Big)\in\mathscr{B}\big([t,s]\big)\otimes\mathcal{X}^{-1}(\mathcal{F}^t_s).$$

So $\widehat{\Psi}_T^{-1}(\widetilde{\mathcal{D}}\times U)\in\mathscr{P}_{\mathcal{X}^{-1}}$, the $\mathcal{X}^{-1}(\mathbf{F}^t)-$progressively measurable $\sigma-$field of $[t,T]\times\Omega^t$. Then $\widetilde{\mathcal{D}}\times U\in\widehat{\Lambda}:=\big\{\mathcal{J}\in[t,T]\times\Omega^t\times\mathbb{R}^{d\times d}:\widehat{\Psi}_T^{-1}(\mathcal{J})\in\mathscr{P}_{\mathcal{X}^{-1}}\big\}$, which is clearly a $\sigma-$field of $[t,T]\times\Omega^t\times\mathbb{R}^{d\times d}$. It follows that $\mathscr{P}^t\otimes\mathscr{B}(\mathbb{R}^{d\times d})\in\widehat{\Lambda}$, i.e., $\widehat{\Psi}_T^{-1}(\mathcal{J})\in\mathscr{P}_{\mathcal{X}^{-1}}$ for any $\mathcal{J}\in\mathscr{P}^t\otimes\mathscr{B}(\mathbb{R}^{d\times d})$. Hence, the mapping $\widehat{\Psi}_T$ is $\mathscr{P}_{\mathcal{X}^{-1}}\big/\mathscr{P}^t\otimes\mathscr{B}(\mathbb{R}^{d\times d})-$measurable. Then we see from Lemma 6.1 that the mapping

$$(r,\widetilde{\omega})\to b^{t,\omega}\big(r,\mathcal{X}(\widetilde{\omega}),\widetilde{\mu}_r(\widetilde{\omega})\big)=b^{t,\omega}\big(\widehat{\Psi}_T(r,\widetilde{\omega})\big)\text{ is }\mathscr{P}_{\mathcal{X}^{-1}}\big/\mathscr{B}(\mathbb{R}^d)-\text{measurable},$$

which together with the $\mathcal{X}^{-1}(\mathbf{F}^t)-$progressive measurability of $\mathfrak{q}$ shows that the integral $K_s^2:=\int_t^s\mathfrak{q}_r b^{t,\omega}(r,\mathcal{X},\widetilde{\mu}_r)dr$, $s\in[t,T]$ is $\mathcal{X}^{-1}(\mathbf{F}^t)-$adapted. By Lemma A.11 again, K^2 is also $\sigma^{\mathcal{X}}-$measurable. Then we can deduce from (7.91) and (7.92) that $\mathbb{P}_0^t-$a.s.

$$B_s^t=\int_t^s\mathfrak{q}_r d\mathcal{X}_r-\int_t^s\mathfrak{q}_r b^{t,\omega}(r,\mathcal{X},\widetilde{\mu}_r)dr=K_s^1+K_s^2,\quad s\in[t,T].\tag{7.96}$$

Since the process K^1+K^2 is $\sigma^{\mathcal{X}}-$measurable, an application of Doob-Dynkin Lemma shows that there exists a $\mathscr{P}^t-$measurable (or $\mathbf{F}^t-$progressively measurable) process $\mathcal{W}=W^{t,\omega,\mu}$ satisfying $(K^1+K^2)(s,\widetilde{\omega})=\mathcal{W}\big(\Psi^{\mathcal{X}}(s,\widetilde{\omega})\big)=\mathcal{W}\big(s,\mathcal{X}(\widetilde{\omega})\big)$, $\forall\,(s,\widetilde{\omega})\in[t,T]\times\Omega^t$, which together with (7.96) shows that for all $\widetilde{\omega}\in\Omega^t$ except on a $\mathbb{P}_0^t-$null set $\mathcal{N}_{\mathcal{X}}$

$$B_s^t(\widetilde{\omega})=\mathcal{W}_s\big(\mathcal{X}(\widetilde{\omega})\big),\quad\forall\,s\in[t,T].\tag{7.97}$$

2) *Setting* $(\mathbb{P},\mathfrak{p})=\big(\mathbb{P}^{t,\omega,\mu},\mathfrak{p}^{t,\omega,\mu}\big)$, *we next show that the filtration* $\mathbf{F}^{\mathbb{P}}$ *is right-continuous and thus* $\mathbb{P}\in\mathfrak{P}_t$.

2a) *We first claim that* $\mathcal{W}$ *is actually a Brownian motion on* Ω^t *under* $\mathfrak{p}$*:*

By (7.97), it holds for any $\widetilde{\omega}\in\mathcal{N}_{\mathcal{X}}^c$ that $\mathcal{X}_s(\widetilde{\omega})=\mathcal{X}_s\big(\mathcal{W}\big(\mathcal{X}(\widetilde{\omega})\big)\big)$, $\forall\,s\in[t,T]$. It follows that for any $\widetilde{\omega}'\in A_{\mathcal{X}}:=\{\widetilde{\omega}'\in\Omega^t:\exists\,\widetilde{\omega}\in\mathcal{N}_{\mathcal{X}}^c\text{ such that }\widetilde{\omega}'=\mathcal{X}(\widetilde{\omega})\}=\{\widetilde{\omega}'\in\Omega^t:\mathcal{N}_{\mathcal{X}}^c\cap\mathcal{X}^{-1}(\widetilde{\omega}')\neq\emptyset\}$, one has

$$B_s^t(\widetilde{\omega}')=\mathcal{X}_s\big(\mathcal{W}(\widetilde{\omega}')\big),\quad\forall\,s\in[t,T].\tag{7.98}$$

As $A_{\mathcal{X}}^c=\{\widetilde{\omega}'\in\Omega^t:\mathcal{X}^{-1}(\widetilde{\omega}')\subset\mathcal{N}_{\mathcal{X}}\}$, we see that $\mathcal{X}^{-1}(A_{\mathcal{X}}^c)\subset\mathcal{N}_{\mathcal{X}}$, i.e. $\mathcal{X}^{-1}(A_{\mathcal{X}}^c)\in\overline{\mathscr{N}}^t\subset\overline{\mathcal{F}}_T^t$. So $A_{\mathcal{X}}^c\in\mathcal{G}_T^{\mathcal{X}}=\big\{A\subset\Omega^t:\mathcal{X}^{-1}(A)\in\overline{\mathcal{F}}_T^t\big\}$ with $\mathfrak{p}(A_{\mathcal{X}}^c)=\mathbb{P}_0^t\big(\mathcal{X}^{-1}(A_{\mathcal{X}}^c)\big)=0$, namely, $A_{\mathcal{X}}^c$ is a $\mathfrak{p}-$null set. (It is worth pointing out that $A_{\mathcal{X}}^c$ may not belong to $\mathcal{F}_T^{\mathbb{P}}$ though $\mathcal{X}^{-1}(A_{\mathcal{X}}^c)\in\overline{\mathcal{F}}_T^t$. In general, the inverse conclusion of (6.6) may not be true.) Since

$$A_{\mathcal{X}}=\{\widetilde{\omega}'\in\Omega^t:\exists\,\widetilde{\omega}\in\mathcal{N}_{\mathcal{X}}^c\text{ such that }\widetilde{\omega}'=\mathcal{X}(\widetilde{\omega})\}\subset\{\widetilde{\omega}'\in\Omega^t:\mathcal{W}_\cdot(\widetilde{\omega}')\in\Omega^t\}\tag{7.99}$$

by (7.97), the process $\mathcal{W}$ has $\mathfrak{p}-$a.s. continuous paths starting from 0.

(i) Given $t\le s\le r\le T$, (7.97) implies that for any $\mathcal{E}\in\mathscr{B}(\mathbb{R}^d)$

$$\begin{aligned}\mathfrak{p}\big\{\widetilde{\omega}\in\Omega^t:\mathcal{W}_r(\widetilde{\omega})-\mathcal{W}_s(\widetilde{\omega})\in\mathcal{E}\big\}&=\mathbb{P}_0^t\big\{\widetilde{\omega}\in\Omega^t:\mathcal{W}_r\big(\mathcal{X}(\widetilde{\omega})\big)-\mathcal{W}_s\big(\mathcal{X}(\widetilde{\omega})\big)\in\mathcal{E}\big\}\\&=\mathbb{P}_0^t\big\{\widetilde{\omega}\in\Omega^t:B_r^t(\widetilde{\omega})-B_s^t(\widetilde{\omega})\in\mathcal{E}\big\},\end{aligned}\tag{7.100}$$

which shows that the distribution of $\mathcal{W}_r-\mathcal{W}_s$ under $\mathfrak{p}$ is the same as that of $B_r^t-B_s^t$ under $\mathbb{P}_0^t$ (a $d-$dimensional normal distribution with mean 0 and variance matrix $(r-s)I_{d\times d}$).

(ii) Given $t\le s_1\le r_1\le s_2\le r_2\le T$, similar to (7.100), it holds for any $\mathcal{E}_1,\mathcal{E}_2\in\mathscr{B}(\mathbb{R}^d)$ that

$$\begin{aligned}&\mathfrak{p}\big\{\widetilde{\omega}\in\Omega^t:\mathcal{W}_{r_i}(\widetilde{\omega})-\mathcal{W}_{s_i}(\widetilde{\omega})\in\mathcal{E}_i,i=1,2\big\}=\mathbb{P}_0^t\big\{\widetilde{\omega}\in\Omega^t:B_{r_i}^t(\widetilde{\omega})-B_{s_i}^t(\widetilde{\omega})\in\mathcal{E}_i,i=1,2\big\}\\&=\prod_{i=1}^2\mathbb{P}_0^t\big\{\widetilde{\omega}\in\Omega^t:B_{r_i}^t(\widetilde{\omega})-B_{s_i}^t(\widetilde{\omega})\in\mathcal{E}_i\big\}=\prod_{i=1}^2\mathfrak{p}\big\{\widetilde{\omega}\in\Omega^t:\mathcal{W}_{r_i}(\widetilde{\omega})-\mathcal{W}_{s_i}(\widetilde{\omega})\in\mathcal{E}_i\big\},\end{aligned}$$

which shows that $\mathcal{W}_{r_1}-\mathcal{W}_{s_1}$ is independent of $\mathcal{W}_{r_2}-\mathcal{W}_{s_2}$ under $\mathfrak{p}$. Hence, $\mathcal{W}$ is a $d-$dimensional standard Brownian motion on Ω^t under $\mathfrak{p}$ and the corresponding augmented Brownian filtration

$$\widetilde{\mathcal{F}}_s^{\mathcal{W},\mathfrak{p}}:=\sigma\Big(\mathcal{F}_s^{\mathcal{W}}\cup\mathscr{N}^{\mathcal{W},\mathfrak{p}}\Big),\quad s\in[t,T]\tag{7.101}$$

is right-continuous, where $\mathscr{N}^{\mathcal{W},\mathfrak{p}}:=\big\{\mathcal{N}'\subset\Omega^t:\mathcal{N}'\subset A\text{ for some }A\in\mathcal{F}_T^{\mathcal{W}}\text{ with }\mathfrak{p}(A)=0\big\}$ (see e.g. Proposition 2.7.7 of [19]).

2b) *In the second step, we show that the right-continuity of the augmented Brownian filtration* $\big\{\widetilde{\mathcal{F}}^{\mathcal{W},\mathfrak{p}}_s\big\}_{s\in[t,T]}$ *implies that of the filtration* $\mathbf{F}^{\mathcal{W},\mathbb{P}}$.

Since $\mathcal{F}^{\mathcal{W}}_T\subset\mathcal{F}^t_T$ by the $\mathbf{F}^t-$adaptedness of $\mathcal{W}$, we see from Lemma A.12 (1) that $\mathscr{N}^{\mathcal{W},\mathfrak{p}}=\big\{\mathcal{N}'\subset\Omega^t:\mathcal{N}'\subset A$ for some $A\in\mathcal{F}^{\mathcal{W}}_T$ with $\mathbb{P}(A)=0\big\}\subset\big\{\mathcal{N}'\subset\Omega^t:\mathcal{N}'\subset A$ for some $A\in\mathcal{F}^t_T$ with $\mathbb{P}(A)=0\big\}=\mathscr{N}^{\mathbb{P}}$. It follows that

$$\sigma\big(\widetilde{\mathcal{F}}^{\mathcal{W},\mathfrak{p}}_s\cup\mathscr{N}^{\mathbb{P}}\big)=\sigma\big(\mathcal{F}^{\mathcal{W}}_s\cup\mathscr{N}^{\mathbb{P}}\big)=\mathcal{F}^{\mathcal{W},\mathbb{P}}_s,\quad\forall\,s\in[t,T].$$

Similar to Problem 2.7.3 of [19], one can show that

$$\mathcal{F}^{\mathcal{W},\mathbb{P}}_s=\Big\{A\subset\Omega^t:A\Delta\widetilde{A}\in\mathscr{N}^{\mathbb{P}}\text{ for some }\widetilde{A}\in\widetilde{\mathcal{F}}^{\mathcal{W},\mathfrak{p}}_s\Big\},\quad\forall\,s\in[t,T].\tag{7.102}$$

Let $s\in[t,T)$ and $A\in\mathcal{F}^{\mathcal{W},\mathbb{P}}_{s+}:=\underset{s'\in(s,T]}{\cap}\mathcal{F}^{\mathcal{W},\mathbb{P}}_{s'}$. For any $n\ge n_s:=\big\lceil\frac{1}{T-s}\big\rceil$, as $A\in\mathcal{F}^{\mathcal{W},\mathbb{P}}_{s+1/n}$, there exists $A_n\in\widetilde{\mathcal{F}}^{\mathcal{W},\mathfrak{p}}_{s+1/n}$ such that $A\Delta A_n\in\mathscr{N}^{\mathbb{P}}$. By (7.101), $\widetilde{A}:=\underset{n\ge n_s}{\cap}\,\underset{i\ge n}{\cup}A_i\in\widetilde{\mathcal{F}}^{\mathcal{W},\mathfrak{p}}_{s+}=\widetilde{\mathcal{F}}^{\mathcal{W},\mathfrak{p}}_s$. Since $\widetilde{A}\backslash A\subset\underset{n\ge n_s}{\cap}\,\underset{i\ge n}{\cup}(A_i\backslash A)\subset\underset{n\ge n_s}{\cap}\,\underset{i\ge n}{\cup}(A\Delta A_i)$ and since $A\backslash\widetilde{A}=\underset{n\ge n_s}{\cup}\,\underset{i\ge n}{\cap}\big(A\backslash A_i\big)\subset\underset{n\ge n_s}{\cup}\,\underset{i\ge n}{\cap}\big(A\Delta A_i\big)$, we see that $A\Delta\widetilde{A}\subset\underset{n\ge n_s}{\cup}\big(A\Delta A_n\big)\in\mathscr{N}^{\mathbb{P}}$, namely $A\in\mathcal{F}^{\mathcal{W},\mathbb{P}}_s$ by (7.102). So $\mathcal{F}^{\mathcal{W},\mathbb{P}}_{s+}=\mathcal{F}^{\mathcal{W},\mathbb{P}}_s$, which shows that

$$\mathbf{F}^{\mathcal{W},\mathbb{P}}=\big\{\mathcal{F}^{\mathcal{W},\mathbb{P}}_s\big\}_{s\in[t,T]}\text{ is also a right-continuous filtration.}\tag{7.103}$$

2c) *In the last step, we show that the filtration* $\mathbf{F}^{\mathcal{W},\mathbb{P}}$ *is exactly* $\mathbf{F}^{\mathbb{P}}$.

Let $s\in[t,T]$. Since $\mathcal{W}$ is $\mathbf{F}^t-$adapted, it is clear that $\mathcal{F}^{\mathcal{W},\mathbb{P}}_s=\sigma\Big(\mathcal{F}^{\mathcal{W}}_s\cup\mathscr{N}^{\mathbb{P}}\Big)\subset\sigma\Big(\mathcal{F}^t_s\cup\mathscr{N}^{\mathbb{P}}\Big)=\mathcal{F}^{\mathbb{P}}_s$. So we only need to show the reverse inclusion. For any $r\in[t,s]$ and $\mathcal{E}\in\mathscr{B}(\mathbb{R}^d)$, (7.97) implies that $\{\widetilde{\omega}\in\Omega^t:B^t_r(\widetilde{\omega})\in\mathcal{E}\}\,\Delta\,\{\widetilde{\omega}\in\Omega^t:\mathcal{W}_r\big(\mathcal{X}(\widetilde{\omega})\big)\in\mathcal{E}\}\subset\mathcal{N}_{\mathcal{X}}\in\overline{\mathscr{N}}^t$, which shows that $\big(B^t_r\big)^{-1}(\mathcal{E})\in\widehat{\Lambda}_s:=\Big\{A\subset\Omega^t:A\Delta\widetilde{A}\in\overline{\mathscr{N}}^t\text{ for some }\widetilde{A}\in\mathcal{X}^{-1}(\mathcal{F}^{\mathcal{W}}_s)\Big\}$. As $\mathcal{X}^{-1}(\mathcal{F}^{\mathcal{W}}_s)$ is a $\sigma-$field of Ω^t, an analogy to Problem 2.7.3 of [19] yields that $\widehat{\Lambda}_s$ forms a $\sigma-$field of Ω^t. It follows that $\mathcal{F}^t_s\subset\widehat{\Lambda}_s$. Clearly, $\overline{\mathscr{N}}^t\subset\widehat{\Lambda}_s$, so we further have $\overline{\mathcal{F}}^t_s\subset\widehat{\Lambda}_s$.

For any $A\in\mathcal{F}^{\mathbb{P}}_s$, Lemma A.12 (1) shows that $\mathcal{X}^{-1}(A)\in\overline{\mathcal{F}}^t_s\subset\widehat{\Lambda}_s$, i.e., for some $\widetilde{A}\in\mathcal{F}^{\mathcal{W}}_s\subset\mathcal{F}^t_s$, one has $\mathcal{X}^{-1}\big(A\,\Delta\,\widetilde{A}\big)=\big(\mathcal{X}^{-1}(A)\big)\Delta\big(\mathcal{X}^{-1}\big(\widetilde{A}\big)\big)\in\overline{\mathscr{N}}^t$. As $A\,\Delta\,\widetilde{A}\in\mathcal{F}^{\mathbb{P}}_s\subset\mathcal{F}^{\mathbb{P}}_T$, applying Lemma A.12 (1) again yields that $\mathbb{P}\big(A\,\Delta\,\widetilde{A}\big)=\mathfrak{p}\big(A\,\Delta\,\widetilde{A}\big)=\mathbb{P}^t_0\big(\mathcal{X}^{-1}(A\,\Delta\,\widetilde{A})\big)=0$, i.e., $A\,\Delta\,\widetilde{A}\in\mathscr{N}^{\mathbb{P}}$. It follows that $A=\widetilde{A}\,\Delta\,\big(A\,\Delta\,\widetilde{A}\big)\in\mathcal{F}^{\mathcal{W},\mathbb{P}}_s$. Therefore, $\mathcal{F}^{\mathbb{P}}_s=\mathcal{F}^{\mathcal{W},\mathbb{P}}_s$, which together with (7.103) shows that $\mathbb{P}\in\mathfrak{P}_t$. □

Proof of Lemma 6.2: Fix $(t,\omega)\in[0,T]\times\Omega$ and $\mu\in\mathcal{U}_t$. We set $(\mathcal{X},\mathbb{P})=\big(X^{t,\omega,\mu},\mathbb{P}^{t,\omega,\mu}\big)$. Given $\widetilde{\omega}\in\Omega^t$, (3.1) shows

$$\big|Y^{t,\mathbf{0}}_r(\mathcal{X}(\widetilde{\omega}))-Y_r(\mathbf{0})\big|=\big|Y_r(\mathbf{0}\otimes_t\mathcal{X}(\widetilde{\omega}))-Y_r(\mathbf{0})\big|\le\rho_0\big(\|\mathbf{0}\otimes_t\mathcal{X}(\widetilde{\omega})\|_{0,r}\big)\le\kappa\big(1+\|\mathcal{X}(\widetilde{\omega})\|^{\varpi}_{t,r}\big),\ \ \forall\,r\in[t,T].$$

It follows that $Y^{t,\mathbf{0}}_*(\mathcal{X}(\widetilde{\omega}))=\underset{r\in[t,T]}{\sup}\big|Y^{t,\mathbf{0}}_r(\mathcal{X}(\widetilde{\omega}))\big|\le\kappa\big(1+\|\mathcal{X}(\widetilde{\omega})\|^{\varpi}_{t,T}\big)+\mathfrak{m}_Y$, where $\mathfrak{m}_Y:=\underset{r\in[t,T]}{\sup}\big|Y_r(\mathbf{0})\big|<\infty$ by Lemma A.9. Then we can deduce from (6.4) that

$$\mathbb{E}_{\mathbb{P}}\big[Y^{t,\mathbf{0}}_*\big]=\mathbb{E}_t\big[Y^{t,\mathbf{0}}_*(\mathcal{X})\big]\le\kappa\Big(1+\mathbb{E}_t\big[\|\mathcal{X}\|^{\varpi}_{t,T}\big]\Big)+\mathfrak{m}_Y\le\kappa\big(1+\varphi_{\varpi}\big(\|\omega\|_{0,t}\big)\,T^{\varpi/2}\big)+\mathfrak{m}_Y<\infty.$$

Namely, $Y^{t,\mathbf{0}}\in\mathbb{D}(\mathbf{F}^t,\mathbb{P})$, which together with Proposition 6.2 shows that $\mathbb{P}=\mathbb{P}^{t,\omega,\mu}\in\mathfrak{P}^Y_t$. □

Proof of Proposition 6.3: Fix $0\le t<s\le T$, $\omega\in\Omega$ and $\mu\in\mathcal{U}_t$. We will denote $(\mathbb{P}^{t,\omega,\mu},\mathfrak{p}^{t,\omega,\mu},X^{t,\omega,\mu},W^{t,\omega,\mu})$ by $(\mathbb{P},\mathfrak{p},\mathcal{X},\mathcal{W})$. For any $r\in[t,T]$, (6.6) and Lemma A.12 (2) show that $\mathfrak{F}_r:=\sigma\big(\mathcal{F}^t_r\cup\mathscr{N}^{\mathfrak{p}}\big)\subset\mathcal{G}^{\mathcal{X}}_r$.

Let $A_{\mathcal{X}}$ as defined in (7.98). As $A^c_{\mathcal{X}}\in\mathscr{N}^{\mathfrak{p}}$, we see from the $\mathbf{F}^t-$adaptedness of $\mathcal{W}$ and (7.99) that the process $\widetilde{\mathcal{W}}_r(\widetilde{\omega}):=\mathbf{1}_{\{\widetilde{\omega}\in A_{\mathcal{X}}\}}\mathcal{W}_r(\widetilde{\omega})$, $\forall\,(r,\widetilde{\omega})\in[t,T]\times\Omega^t$ is adapted to the filtration $\{\mathfrak{F}_r\}_{r\in[t,T]}$ and all its paths belong to Ω^t. Given $r\in[t,T]$, for any $r'\in[t,r]$ and $\mathcal{E}\in\mathscr{B}(\mathbb{R}^d)$, an analogy to (6.5) shows that $\widetilde{\mathcal{W}}^{-1}\big((B^t_{r'})^{-1}(\mathcal{E})\big)=\big\{\widetilde{\omega}\in\Omega^t:\widetilde{\mathcal{W}}(\widetilde{\omega})\in(B^t_r)^{-1}(\mathcal{E})\big\}=\big\{\widetilde{\omega}\in\Omega^t:\widetilde{\mathcal{W}}_{r'}(\widetilde{\omega})\in\mathcal{E}\big\}\in\mathcal{F}^{\widetilde{\mathcal{W}}}_r$. Thus, $\big(B^t_{r'}\big)^{-1}(\mathcal{E})\in\Lambda_r:=\big\{A\subset\Omega^t:\widetilde{\mathcal{W}}^{-1}(A)\in\mathcal{F}^{\widetilde{\mathcal{W}}}_r\big\}$, which is clearly a $\sigma-$field of Ω^t. It follows that $\mathcal{F}^t_r\subset\Lambda_r$, i.e.,

$$\widetilde{\mathcal{W}}^{-1}(A)\in\mathcal{F}^{\widetilde{\mathcal{W}}}_r\subset\mathfrak{F}_r,\quad\forall\,A\in\mathcal{F}^t_r,\ \ \forall\,r\in[t,T].\tag{7.104}$$

1) *We first show that for $\mathfrak{p}-$a.s. $\widetilde{\omega} \in \Omega^t$, $\mathbb{P}^{s,\widetilde{\omega}} = \mathbb{P}^{s,\omega\otimes_t\widetilde{\omega},\mu^{s,\mathcal{W}(\widetilde{\omega})}} \in \mathcal{P}(s,\omega\otimes_t\widetilde{\omega})$, and thus the probability class $\{\mathcal{P}(t,\omega)\}_{(t,\omega)\in[0,T]\times\Omega}$ satisfies (P1).*

1a) *In the first step, we show that for a given set $A \in \mathcal{F}^s_T$, its shifted probability $\mathbb{P}^{s,\widetilde{\omega}}(A)$ is equal to $\xi_A\big(\widetilde{\mathcal{W}}(\widetilde{\omega})\big)$ for $\mathfrak{p}-$a.s. $\widetilde{\omega}\in\Omega^t$, where $\xi_A := \mathbb{E}_t\big[\mathbf{1}_{\mathcal{X}^{-1}(\overline{A})}\big|\mathcal{F}^t_s\big]$ and $\overline{A} := (\Pi^t_s)^{-1}(A)$.*

Since $\overline{A} = (\Pi^t_s)^{-1}(A) \in \mathcal{F}^t_T$ by Lemma A.1, applying (2.6) yield that for $\mathbb{P}-$a.s. $\widetilde{\omega}\in\Omega^t$

$$\mathbb{P}^{s,\widetilde{\omega}}(A) = \mathbb{P}^{s,\widetilde{\omega}}\Big(\overline{A}^{s,\widetilde{\omega}}\Big) = \mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\big[\mathbf{1}_{\overline{A}^{s,\widetilde{\omega}}}\big] = \mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\Big[(\mathbf{1}_{\overline{A}})^{s,\widetilde{\omega}}\Big] = \mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\overline{A}}\big|\mathcal{F}^t_s\big](\widetilde{\omega}). \tag{7.105}$$

For any $\widetilde{\omega}\in\mathcal{N}^c_{\mathcal{X}}$, set $\widetilde{\omega}' := \mathcal{X}(\widetilde{\omega})$. As $\widetilde{\omega}\in\mathcal{N}^c_{\mathcal{X}}\cap\mathcal{X}^{-1}(\widetilde{\omega}')$, we see that $\mathcal{X}(\widetilde{\omega}) = \widetilde{\omega}' \in A_{\mathcal{X}}$. Then (7.97) shows that

$$\widetilde{\omega} = B^t(\widetilde{\omega}) = \mathcal{W}\big(\mathcal{X}(\widetilde{\omega})\big) = \widetilde{\mathcal{W}}\big(\mathcal{X}(\widetilde{\omega})\big), \quad \forall\,\widetilde{\omega}\in\mathcal{N}^c_{\mathcal{X}}. \tag{7.106}$$

Given $\mathcal{N}' \in \overline{\mathscr{N}}^t$, there exists an $A\in\mathcal{F}^t_T$ with $\mathbb{P}^t_0(A) = 0$ such that $\mathcal{N}'\subset A$. Since $\widetilde{\mathcal{W}}^{-1}(A)\in\mathfrak{F}_T\subset\mathcal{G}^{\mathcal{X}}_T$ by (7.104), one can deduce from (7.106) that

$$\mathfrak{p}\big(\widetilde{\mathcal{W}}^{-1}(A)\big) = \mathbb{P}^t_0\Big(\mathcal{X}^{-1}\big(\widetilde{\mathcal{W}}^{-1}(A)\big)\Big) = \mathbb{P}^t_0\big\{\widetilde{\mathcal{W}}(\mathcal{X})\in A\big\} = \mathbb{P}^t_0(A) = 0,$$

which implies that $\widetilde{\mathcal{W}}^{-1}(A)\in\mathscr{N}^{\mathfrak{p}}$ and thus

$$\widetilde{\mathcal{W}}^{-1}(\mathcal{N}')\in\mathscr{N}^{\mathfrak{p}}. \tag{7.107}$$

Hence, it holds for any $r\in[t,T]$ that $\overline{\mathscr{N}}^t \in \widetilde{\Lambda}_r := \{A'\subset\Omega^t : \widetilde{\mathcal{W}}^{-1}(A')\in\mathfrak{F}_r\}$. Clearly $\widetilde{\Lambda}_r$ is a $\sigma-$field of Ω^t, then we see from (7.104) that $\overline{\mathcal{F}}^t_r\subset\widetilde{\Lambda}_r$, i.e.

$$\widetilde{\mathcal{W}}^{-1}(A')\in\mathfrak{F}_r, \quad \forall\,A'\in\overline{\mathcal{F}}^t_r,\ \ \forall\,r\in[t,T]. \tag{7.108}$$

Let $\mathcal{A}\in\mathfrak{F}_s$. Similar to Problem 2.7.3 of [19], there exists an $\mathcal{A}'\in\mathcal{F}^t_s$ such that $\mathcal{A}\,\Delta\,\mathcal{A}'\in\mathscr{N}^{\mathfrak{p}}$. Then

$$\int_{\mathcal{A}}\mathbf{1}_{\overline{A}}\,d\mathfrak{p} = \int_{\mathcal{A}'}\mathbf{1}_{\overline{A}}\,d\mathfrak{p} = \int_{\mathcal{A}'}\mathbf{1}_{\overline{A}}\,d\mathbb{P} = \int_{\mathcal{A}'}\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\overline{A}}\big|\mathcal{F}^t_s\big]d\mathbb{P} = \int_{\mathcal{A}'}\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\overline{A}}\big|\mathcal{F}^t_s\big]d\mathfrak{p} = \int_{\mathcal{A}}\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\overline{A}}\big|\mathcal{F}^t_s\big]d\mathfrak{p}. \tag{7.109}$$

As $\mathcal{X}^{-1}(\overline{A})\in\overline{\mathcal{F}}^t_T$ by (6.6), applying Lemma A.4 (1) again with $(\mathbb{P},X) = (\mathbb{P}^t_0, B^t)$ shows that $\xi_A = \mathbb{E}_t\big[\mathbf{1}_{\mathcal{X}^{-1}(\overline{A})}\big|\mathcal{F}^t_s\big] = \mathbb{E}_t\big[\mathbf{1}_{\mathcal{X}^{-1}(\overline{A})}\big|\overline{\mathcal{F}}^t_s\big]$, $\mathbb{P}^t_0-$a.s. Since $\mathcal{A}\in\mathfrak{F}_s\subset\mathcal{G}^{\mathcal{X}}_s$, i.e. $\mathcal{X}^{-1}(\mathcal{A})\in\overline{\mathcal{F}}^t_s$, we can deduce from (7.106) that

$$\begin{aligned}\mathbb{E}_{\mathfrak{p}}\big[\mathbf{1}_{\mathcal{A}\cap\overline{A}}\big] &= \mathbb{E}_t\big[\mathbf{1}_{\mathcal{X}^{-1}(\mathcal{A}\cap\overline{A})}\big] = \mathbb{E}_t\big[\mathbf{1}_{\mathcal{X}^{-1}(\mathcal{A})\cap\mathcal{X}^{-1}(\overline{A})}\big] = \mathbb{E}_t\Big[\mathbf{1}_{\mathcal{X}^{-1}(\mathcal{A})}\mathbb{E}_t\big[\mathbf{1}_{\mathcal{X}^{-1}(\overline{A})}\big|\overline{\mathcal{F}}^t_s\big]\Big] = \mathbb{E}_t\big[\mathbf{1}_{\mathcal{X}^{-1}(\mathcal{A})}\xi_A\big]\\ &= \mathbb{E}_t\big[\mathbf{1}_{\mathcal{X}^{-1}(\mathcal{A})}\xi_A(\widetilde{\mathcal{W}}(\mathcal{X}))\big] = \mathbb{E}_{\mathfrak{p}}\big[\mathbf{1}_{\mathcal{A}}\xi_A(\widetilde{\mathcal{W}})\big].\end{aligned} \tag{7.110}$$

Given $\mathcal{E}\in\mathscr{B}(\mathbb{R})$, as $\xi^{-1}_A(\mathcal{E})\in\mathcal{F}^t_s$, (7.108) shows that $\big\{\widetilde{\omega}\in\Omega^t : \xi_A\big(\widetilde{\mathcal{W}}(\widetilde{\omega})\big)\in\mathcal{E}\big\} = \widetilde{\mathcal{W}}^{-1}\big(\xi^{-1}_A(\mathcal{E})\big)\in\mathfrak{F}_s$, namely the random variable $\xi_A(\widetilde{\mathcal{W}})$ is $\mathfrak{F}_s-$measurable. So letting $\mathcal{A}$ vary over $\mathfrak{F}_s$ in (7.109) and (7.110), we see from (7.105) that

$$\xi_A\big(\widetilde{\mathcal{W}}(\widetilde{\omega})\big) = \mathbb{E}_{\mathfrak{p}}\big[\mathbf{1}_{\overline{A}}\big|\mathfrak{F}_s\big](\widetilde{\omega}) = \mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\overline{A}}\big|\mathcal{F}^t_s\big](\widetilde{\omega}) = \mathbb{P}^{s,\widetilde{\omega}}(A) \tag{7.111}$$

holds for all $\widetilde{\omega}\in\Omega^t$ except on some $\mathfrak{N}(A)\in\mathscr{N}^{\mathfrak{p}}$.

1b) *In the second step, we show that for $\mathbb{P}^t_0-$a.s. $\widetilde{\omega}\in\Omega^t$, $\xi_A(\widetilde{\omega})$ is equal to $\mathbb{P}^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega}),\mu^{s,\widetilde{\omega}}}(A)$.*

Since $\mathcal{X}^{-1}(\overline{A})\in\overline{\mathcal{F}}^t_T$, Proposition 2.3 and Lemma 2.4 yield that for all $\widetilde{\omega}\in\Omega^t$ except on an $\mathcal{N}_1(A)\in\overline{\mathscr{N}}^t$

$$\xi_A(\widetilde{\omega}) = \mathbb{E}_t\big[\mathbf{1}_{\mathcal{X}^{-1}(\overline{A})}\big|\mathcal{F}^t_s\big](\widetilde{\omega}) = \mathbb{E}_s\Big[\big(\mathbf{1}_{\mathcal{X}^{-1}(\overline{A})}\big)^{s,\widetilde{\omega}}\Big]. \tag{7.112}$$

By (7.78), there exists $\mathcal{N}_2\in\overline{\mathscr{N}}^t$ such that for any $\widetilde{\omega}\in\mathcal{N}^c_2$, it holds for $\mathbb{P}^s_0-$a.s. $\widehat{\omega}\in\Omega^s$ that $\mathcal{X}_s(\widetilde{\omega}\otimes_s\widehat{\omega}) = \mathcal{X}_s(\widetilde{\omega})$, so

$$\Pi^t_s(\mathcal{X}(\widetilde{\omega}\otimes_s\widehat{\omega}))(r) = \mathcal{X}_r(\widetilde{\omega}\otimes_s\widehat{\omega}) - \mathcal{X}_s(\widetilde{\omega}\otimes_s\widehat{\omega}) = \mathcal{X}^{s,\widetilde{\omega}}_r(\widehat{\omega}) - \mathcal{X}_s(\widetilde{\omega}), \quad \forall\,r\in[s,T]. \tag{7.113}$$

Moreover, Proposition 6.1 shows that for all $\widetilde{\omega}\in\Omega^t$ except on an $\mathcal{N}_3\in\overline{\mathscr{N}}^t$

$$\mu^{s,\widetilde{\omega}}\in\mathcal{U}_s \quad\text{and}\quad \mathfrak{X}^{\widetilde{\omega}} := X^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega}),\mu^{s,\widetilde{\omega}}} = \mathcal{X}^{s,\widetilde{\omega}} - \mathcal{X}_s(\widetilde{\omega}). \tag{7.114}$$

For any $\widetilde{\omega} \in \mathcal{N}_3^c$, we set $\mathbb{P}^{\widetilde{\omega}} := \mathbb{P}_0^s \circ \big(\mathfrak{X}^{\widetilde{\omega}}\big)^{-1} = \mathbb{P}^{s,\omega\otimes_t \mathcal{X}(\widetilde{\omega}),\mu^{s,\widetilde{\omega}}}$.

Let $\mathcal{N}(A) := \mathcal{N}_1(A) \cup \mathcal{N}_2 \cup \mathcal{N}_3 \in \overline{\mathscr{N}}^t$. For any $\widetilde{\omega} \in (\mathcal{N}(A))^c$, we can deduce from (7.113) and (7.114) that for $\mathbb{P}_0^s-$a.s. $\widehat{\omega} \in \Omega^s$, $\big(\mathbf{1}_{\mathcal{X}^{-1}(\overline{A})}\big)^{s,\widetilde{\omega}}(\widehat{\omega}) = \mathbf{1}_{\big\{\widetilde{\omega}\otimes_s\widehat{\omega}\in\mathcal{X}^{-1}(\overline{A})\big\}} = \mathbf{1}_{\{\mathcal{X}(\widetilde{\omega}\otimes_s\widehat{\omega})\in\overline{A}\}} = \mathbf{1}_{\{\Pi_s^t(\mathcal{X}(\widetilde{\omega}\otimes_s\widehat{\omega}))\in A\}} = \mathbf{1}_{\{\mathcal{X}^{s,\widetilde{\omega}}(\widehat{\omega})-\mathcal{X}_s(\widetilde{\omega})\in A\}} = \mathbf{1}_{\{\mathfrak{X}^{\widetilde{\omega}}(\widehat{\omega})\in A\}}$. Plugging this into (7.112) yields that

$$\xi_A(\widetilde{\omega}) = \mathbb{E}_s\big[\mathbf{1}_{\{\mathfrak{X}^{\widetilde{\omega}}\in A\}}\big] = \mathbb{E}_{\mathbb{P}^{\widetilde{\omega}}}[\mathbf{1}_A] = \mathbb{P}^{\widetilde{\omega}}(A). \tag{7.115}$$

1c) *Now, we will combine the above two steps to obtain the conclusion:*

By (7.107), $\widehat{\mathfrak{N}}(A) := A_{\mathcal{X}}^c \cup \mathfrak{N}(A) \cup \widetilde{\mathcal{W}}^{-1}\big(\mathcal{N}(A)\big) \in \mathscr{N}^{\mathfrak{p}}$. Given $\widetilde{\omega} \in \big(\widehat{\mathfrak{N}}(A)\big)^c = A_{\mathcal{X}} \cap \big(\mathfrak{N}(A)\big)^c \cap \widetilde{\mathcal{W}}^{-1}\big((\mathcal{N}(A))^c\big)$, (7.111) and (7.115) imply that $\mathbb{P}^{s,\widetilde{\omega}}(A) = \xi_A\big(\widetilde{\mathcal{W}}(\widetilde{\omega})\big) = \mathbb{P}^{\widetilde{\mathcal{W}}(\widetilde{\omega})}(A)$.

Since $\mathscr{C}_T^s$ is a countable set, $\mathfrak{N}_* := \underset{A\in\mathscr{C}_T^s}{\cup} \widehat{\mathfrak{N}}(A)$ belongs to $\mathscr{N}^{\mathfrak{p}}$. Then $\mathscr{C}_T^s \subset \Lambda := \big\{A \in \mathcal{F}_T^s : \mathbb{P}^{s,\widetilde{\omega}}(A) = \mathbb{P}^{\widetilde{\mathcal{W}}(\widetilde{\omega})}(A),\ \forall\, \widetilde{\omega} \in \mathfrak{N}_*^c\big\}$, which is clearly a Dynkin system. As $\mathscr{C}_T^s$ is closed under intersection, Lemma A.2 and Dynkin System Theorem show that $\mathcal{F}_T^s = \sigma\big(\mathscr{C}_T^s\big) \subset \Lambda \subset \mathcal{F}_T^s$. To wit, it holds for any $\widetilde{\omega} \in \mathfrak{N}_*^c$ that $\mathbb{P}^{s,\widetilde{\omega}} = \mathbb{P}^{\widetilde{\mathcal{W}}(\widetilde{\omega})}$ on $\mathcal{F}_T^s$, which together with (7.98) and (7.114) leads to that

$$\mathbb{P}^{s,\widetilde{\omega}} = \mathbb{P}^{\widetilde{\mathcal{W}}(\widetilde{\omega})} = \mathbb{P}^{s,\omega\otimes_t\mathcal{X}(\widetilde{\mathcal{W}}(\widetilde{\omega})),\mu^{s,\widetilde{\mathcal{W}}(\widetilde{\omega})}} = \mathbb{P}^{s,\omega\otimes_t\mathcal{X}(\mathcal{W}(\widetilde{\omega})),\mu^{s,\widetilde{\mathcal{W}}(\widetilde{\omega})}} = \mathbb{P}^{s,\omega\otimes_t\widetilde{\omega},\mu^{s,\widetilde{\mathcal{W}}(\widetilde{\omega})}} \in \mathcal{P}(s,\omega\otimes_t\widetilde{\omega}), \quad \forall\, \widetilde{\omega} \in \mathfrak{N}_*^c.$$

Hence the probability class $\{\mathcal{P}(t,\omega)\}_{(t,\omega)\in[0,T]\times\Omega}$ satisfies (P1) with $(\mathcal{F}',\mathbb{P}',\Omega') = \big(\mathcal{G}_T^{\mathcal{X}},\mathfrak{p},\mathfrak{N}_*^c\big)$.

2) *We next show that the probability class $\{\mathcal{P}(t,\omega)\}_{(t,\omega)\in[0,T]\times\Omega}$ satisfies (P2). Given $\delta\in\mathbb{Q}_+$ and $\lambda\in\mathbb{N}$, let $\{\mathcal{A}_j\}_{j=0}^\lambda$ be a $\mathcal{F}_s^t-$partition of Ω^t such that for $j=1,\cdots,\lambda$, $\mathcal{A}_j \subset O_{\delta_j}^s(\widetilde{\omega}_j)$ for some $\delta_j \in \big((0,\delta]\cap\mathbb{Q}\big)\cup\{\delta\}$ and $\widetilde{\omega}_j \in \Omega^t$, and let $\{\mu^j\}_{j=1}^\lambda \subset \mathcal{U}_s$. We will paste these $\mathcal{U}_s-$controls $\{\mu^j\}_{j=1}^\lambda$ with the given $\mathcal{U}_t-$control μ to form a new $\mathcal{U}_t-$control $\widehat{\mu}$, see* (7.118) *below. Then we will use the uniqueness of controlled SDE* (6.2), *the continuity* (3.1) *of Y and the estimates* (6.3) *of $X^{t,\omega,\mu}$ to show that $\{\mathcal{P}(t,\omega)\}_{(t,\omega)\in[0,T]\times\Omega}$ satisfies the conditions (P2) (i) and (ii).*

Given $j=1,\cdots,\lambda$, (6.6) shows that $\mathcal{A}_j^{\mathcal{X}} := \mathcal{X}^{-1}(\mathcal{A}_j) \in \overline{\mathcal{F}}_s^t$. So there exists an $A_j \in \mathcal{F}_s^t$ such that $\mathcal{A}_j^{\mathcal{X}} \,\Delta\, A_j \in \overline{\mathscr{N}}^t$ (see e.g. Problem 2.7.3 of [19]). Set $\widetilde{A}_j := A_j \backslash \underset{j'<j}{\cup} A_{j'} \in \mathcal{F}_s^t$. As $\big\{\mathcal{A}_j^{\mathcal{X}}\big\}_{j=0}^\lambda$ is a partition of Ω^t with $\mathcal{A}_0^{\mathcal{X}} := \mathcal{X}^{-1}(\mathcal{A}_0) \in \overline{\mathcal{F}}_s^t$, an analogy to (7.6) shows that $\mathcal{A}_j^{\mathcal{X}} \backslash \widetilde{A}_j \subset \underset{j'\le j}{\cup} \big(\mathcal{A}_{j'}^{\mathcal{X}} \Delta A_{j'}\big) \in \overline{\mathscr{N}}^t$. On the other hand, it is clear that $\widetilde{A}_j \backslash \mathcal{A}_j^{\mathcal{X}} \subset A_j \backslash \mathcal{A}_j^{\mathcal{X}} \subset \mathcal{A}_j^{\mathcal{X}} \,\Delta\, A_j \in \overline{\mathscr{N}}^t$. Thus

$$\mathcal{A}_j^{\mathcal{X}} \Delta \widetilde{A}_j \in \overline{\mathscr{N}}^t. \tag{7.116}$$

Let $\widetilde{A}_0 := \Big(\overset{\lambda}{\underset{j=1}{\cup}} \widetilde{A}_j\Big)^c \in \mathcal{F}_s^t$. As $\mathcal{A}_0^{\mathcal{X}} = \Big(\overset{\lambda}{\underset{j=1}{\cup}} \mathcal{A}_j^{\mathcal{X}}\Big)^c$, one can deduce that

$$\widetilde{A}_0 \backslash \mathcal{A}_0^{\mathcal{X}} = \widetilde{A}_0 \cap \Big(\overset{\lambda}{\underset{j=1}{\cup}} \mathcal{A}_j^{\mathcal{X}}\Big) = \overset{\lambda}{\underset{j=1}{\cup}} \big(\widetilde{A}_0 \cap \mathcal{A}_j^{\mathcal{X}}\big) \subset \overset{\lambda}{\underset{j=1}{\cup}} \big(\widetilde{A}_j^c \cap \mathcal{A}_j^{\mathcal{X}}\big) \subset \overset{\lambda}{\underset{j=1}{\cup}} \big(\mathcal{A}_j^{\mathcal{X}} \Delta \widetilde{A}_j\big) \in \overline{\mathscr{N}}^t$$

$$\text{and} \qquad \mathcal{A}_0^{\mathcal{X}} \backslash \widetilde{A}_0 = \mathcal{A}_0^{\mathcal{X}} \cap \Big(\overset{\lambda}{\underset{j=1}{\cup}} \widetilde{A}_j\Big) = \overset{\lambda}{\underset{j=1}{\cup}} \big(\mathcal{A}_0^{\mathcal{X}} \cap \widetilde{A}_j\big) \subset \overset{\lambda}{\underset{j=1}{\cup}} \big((\mathcal{A}_j^{\mathcal{X}})^c \cap \widetilde{A}_j\big) \subset \overset{\lambda}{\underset{j=1}{\cup}} \big(\mathcal{A}_j^{\mathcal{X}} \Delta \widetilde{A}_j\big) \in \overline{\mathscr{N}}^t.$$

Hence,

$$\mathcal{A}_0^{\mathcal{X}} \Delta \widetilde{A}_0 \in \overline{\mathscr{N}}^t. \tag{7.117}$$

(2a) *In the first step, we show that the pasted control*

$$\widehat{\mu}_r(\widetilde{\omega}) := \mathbf{1}_{\{r\in[t,s)\}}\mu_r(\widetilde{\omega}) + \mathbf{1}_{\{r\in[s,T]\}}\Big(\mathbf{1}_{\{\widetilde{\omega}\in\widetilde{A}_0\}}\mu_r(\widetilde{\omega}) + \sum_{j=1}^{\lambda} \mathbf{1}_{\{\widetilde{\omega}\in\widetilde{A}_j\}}\mu_r^j(\Pi_s^t(\widetilde{\omega}))\Big), \quad \forall\,(r,\widetilde{\omega}) \in [t,T]\times\Omega^t \tag{7.118}$$

belongs to $\mathcal{U}_t$.

We start with demonstrating the $\mathbf{F}^t-$progressive measurability of $\widehat{\mu}$: Let $r \in [t,T]$ and $U \in \mathscr{B}(\mathcal{S}_d^{>0})$. The $\mathbf{F}^t-$progressive measurability of μ implies that for any $\mathcal{D} \in \mathscr{B}\big([t,r]\big) \otimes \mathcal{F}_r^t$

$$\big\{(r',\widetilde{\omega}) \in \mathcal{D} :\, \mu_{r'}(\widetilde{\omega}) \in U\big\} = \big\{(r',\widetilde{\omega}) \in [t,r]\times\Omega^t :\, \mu_{r'}(\widetilde{\omega}) \in U\big\} \cap \mathcal{D} \in \mathscr{B}\big([t,r]\big) \otimes \mathcal{F}_r^t. \tag{7.119}$$

If $r<s$, this shows that

$$\big\{(r',\widetilde{\omega})\in[t,r]\times\Omega^t:\widehat{\mu}_{r'}(\widetilde{\omega})\in U\big\}=\big\{(r',\widetilde{\omega})\in[t,r]\times\Omega^t:\mu_{r'}(\widetilde{\omega})\in U\big\}\in\mathscr{B}\big([t,r]\big)\otimes\mathcal{F}^t_r.$$

On the other hand, suppose $r\ge s$. Since $\widetilde{A}_0\in\mathcal{F}^t_s\subset\mathcal{F}^t_r$, applying (7.119) with $\mathcal{D}=[t,r]\times\widetilde{A}_0$, we obtain

$$\big\{(r',\widetilde{\omega})\in[t,r]\times\widetilde{A}_0:\,\widehat{\mu}_{r'}(\widetilde{\omega})\in U\big\}=\big\{(r',\widetilde{\omega})\in[t,r]\times\widetilde{A}_0:\,\mu_{r'}(\widetilde{\omega})\in U\big\}\in\mathscr{B}\big([t,r]\big)\otimes\mathcal{F}^t_r. \tag{7.120}$$

Given $j=1,\cdots,\lambda$, as $\widetilde{A}_j\in\mathcal{F}^t_s\subset\mathcal{F}^t_r$, applying (7.119) with $\mathcal{D}=[t,s)\times\widetilde{A}_j$ gives that

$$\big\{(r',\widetilde{\omega})\in[t,s)\times\widetilde{A}_j:\widehat{\mu}_{r'}(\widetilde{\omega})\in U\big\}=\big\{(r',\widetilde{\omega})\in[t,s)\times\widetilde{A}_j:\mu_{r'}(\widetilde{\omega})\in U\big\}\in\mathscr{B}\big([t,r]\big)\otimes\mathcal{F}^t_r. \tag{7.121}$$

Since $\mathcal{D}_j:=\big\{(r',\widetilde{\omega})\in[s,r]\times\Omega^s:\,\mu^j_{r'}(\widetilde{\omega})\in U\big\}\in\mathscr{B}\big([s,r]\big)\otimes\mathcal{F}^s_r$ by the $\mathbf{F}^s-$progressive measurability of μ^j, one can deduce from Lemma A.13 that

$$\begin{aligned}\big\{(r',\widetilde{\omega})\in[s,r]\times\widetilde{A}_j:\widehat{\mu}_{r'}(\widetilde{\omega})\in U\big\}&=\big\{(r',\widetilde{\omega})\in[s,r]\times\widetilde{A}_j:\mu^j_{r'}\big(\Pi^t_s(\widetilde{\omega})\big)\in U\big\}=\big\{(r',\widetilde{\omega})\in[s,r]\times\widetilde{A}_j:\big(r',\Pi^t_s(\widetilde{\omega})\big)\in\mathcal{D}_j\big\}\\&=\big\{(r',\widetilde{\omega})\in[s,T]\times\Omega^t:\widehat{\Pi}^t_s(r',\widetilde{\omega})\in\mathcal{D}_j\big\}\cap\big([s,r]\times\widetilde{A}_j\big)=(\widehat{\Pi}^t_s)^{-1}\big(\mathcal{D}_j\big)\cap\big([s,r]\times\widetilde{A}_j\big)\in\mathscr{B}\big([s,r]\big)\otimes\mathcal{F}^t_r\subset\mathscr{B}\big([t,r]\big)\otimes\mathcal{F}^t_r,\end{aligned}$$

which together with (7.121) shows that $\big\{(r',\widetilde{\omega})\in[t,r]\times\widetilde{A}_j:\widehat{\mu}_{r'}(\widetilde{\omega})\in U\big\}\in\mathscr{B}\big([t,r]\big)\otimes\mathcal{F}^t_r$. Then taking union over $j\in\{1,\cdots,\lambda\}$ and combining with (7.120) lead to that $\big\{(r',\widetilde{\omega})\in[t,r]\times\Omega^t:\widehat{\mu}_{r'}(\widetilde{\omega})\in U\big\}\in\mathscr{B}\big([t,r]\big)\otimes\mathcal{F}^t_r$. Hence, $\widehat{\mu}$ is $\mathbf{F}^t-$progressively measurable.

For any $j=1,\cdots,\lambda$, since $\widetilde{\mathcal{D}}_j:=\big\{(r,\widehat{\omega})\in[s,T]\times\Omega^s:\,|\mu^j_r(\widehat{\omega})|>\kappa\big\}$ is a $dr\times d\mathbb{P}^s_0-$null set, we can deduce that

$$\big\{(r,\widetilde{\omega})\in[s,T]\times\widetilde{A}_j:|\widehat{\mu}_r(\widetilde{\omega})|>\kappa\big\}=\big([s,T]\times\widetilde{A}_j\big)\cap\big\{(r,\widetilde{\omega})\in[s,T]\times\Omega^t:(r,\Pi^t_s(\widetilde{\omega}))\in\widetilde{\mathcal{D}}_j\big\}=\big([s,T]\times\widetilde{A}_j\big)\cap(\widehat{\Pi}^t_s)^{-1}(\widetilde{\mathcal{D}}_j).$$

Lemma A.13 again implies that

$$(dr\times d\mathbb{P}^t_0)\big\{(r,\widetilde{\omega})\in[s,T]\times\widetilde{A}_j:\,|\widehat{\mu}_r(\widetilde{\omega})|>\kappa\big\}\le(dr\times d\mathbb{P}^t_0)\big((\widehat{\Pi}^t_s)^{-1}(\widetilde{\mathcal{D}}_j)\big)=(dr\times d\mathbb{P}^s_0)(\widetilde{\mathcal{D}}_j)=0. \tag{7.122}$$

Clearly, $(dr\times d\mathbb{P}^t_0)\big\{(r,\widetilde{\omega})\in([t,s)\times\Omega^t)\cup([s,T]\times\widetilde{A}_0):|\widehat{\mu}_r(\widetilde{\omega})|>\kappa\big\}\le(dr\times d\mathbb{P}^t_0)\big\{(r,\widetilde{\omega})\in[t,T]\times\Omega^t:|\mu_r(\widetilde{\omega})|>\kappa\big\}=0$, which together with (7.122) shows that $|\widehat{\mu}_r|\le\kappa$, $dr\times d\mathbb{P}^t_0-$a.s. Therefore, $\widehat{\mu}\in\mathcal{U}_t$.

Let $(r,\widetilde{\omega})\in[s,T]\times\widetilde{A}_j$ for some $j=0,\cdots,\lambda$. For any $\widehat{\omega}\in\Omega^s$, since $\widetilde{\omega}\otimes_s\widehat{\omega}\in\widetilde{A}_j$ by Lemma 2.1, (7.118) shows that

$$\widehat{\mu}^{s,\widetilde{\omega}}_r(\widehat{\omega})=\widehat{\mu}_r\big(\widetilde{\omega}\otimes_s\widehat{\omega}\big)=\left\{\begin{array}{ll}\mu_r\big(\widetilde{\omega}\otimes_s\widehat{\omega}\big)=\mu^{s,\widetilde{\omega}}_r(\widehat{\omega}), & \text{if } j=0\,;\\ \mu^j_r\big(\Pi^t_s(\widetilde{\omega}\otimes_s\widehat{\omega})\big)=\mu^j_r(\widehat{\omega}), & \text{if } j=1,\cdots,\lambda.\end{array}\right. \tag{7.123}$$

(2b) *In the second step, we use the uniqueness of controlled SDE (6.2) to show that the equality $\widehat{\mu}=\mu$ over $\big([t,s]\times\Omega^t\big)\cup\big([s,T]\times\widetilde{A}_0\big)$ implies the equality $\widehat{\mathcal{X}}:=X^{t,\omega,\widehat{\mu}}=\mathcal{X}$ over $\big([t,s]\times\Omega^t\big)\cup\big([s,T]\times\widetilde{A}_0\big)$. It follows that $\widehat{\mathbb{P}}:=\mathbb{P}^{t,\omega,\widehat{\mu}}$ satisfies (P2) (i) and the first part of (P2) (ii).*

Since both $\big\{X^{t,\omega,\mu}_r\big\}_{r\in[t,s]}$ and $\big\{X^{t,\omega,\widehat{\mu}}_r\big\}_{r\in[t,s]}$ satisfy the same SDE:

$$X_r=\int_t^r b^{t,\omega}(r',X,\mu_{r'})\,dr'+\int_t^r\mu_{r'}\,dB^t_{r'},\quad r\in[t,s],$$

the uniqueness of solution to such a SDE shows that except on an $\widehat{\mathcal{N}}\in\overline{\mathscr{N}}^t$

$$\mathcal{X}_r=X^{t,\omega,\mu}_r=X^{t,\omega,\widehat{\mu}}_r=\widehat{\mathcal{X}}_r,\quad\forall\,r\in[t,s]. \tag{7.124}$$

Given $A\in\mathcal{F}^t_s$, we claim that $\mathcal{X}^{-1}(A)\cap\widehat{\mathcal{N}}^c\cap(\widehat{\mathcal{X}}^{-1}(A))^c=\emptyset$: Without loss of generality, assume that $\mathcal{X}^{-1}(A)\cap\widehat{\mathcal{N}}^c$ is not empty and contains some $\widetilde{\omega}$. By (7.124) and Lemma 2.1, $\widehat{\mathcal{X}}(\widetilde{\omega})\in\mathcal{X}(\widetilde{\omega})\otimes_s\Omega^s\subset A$, i.e., $\widetilde{\omega}\in\widehat{\mathcal{X}}^{-1}(A)$. So $\mathcal{X}^{-1}(A)\cap\widehat{\mathcal{N}}^c\subset\widehat{\mathcal{X}}^{-1}(A)$, which shows that $\mathcal{X}^{-1}(A)\cap\widehat{\mathcal{N}}^c\cap(\widehat{\mathcal{X}}^{-1}(A))^c=\emptyset$, proving the claim. It then follows that $\mathcal{X}^{-1}(A)\cap(\widehat{\mathcal{X}}^{-1}(A))^c\subset\widehat{\mathcal{N}}$. Exchanging the role of $\mathcal{X}^{-1}(A)$ and $\widehat{\mathcal{X}}^{-1}(A)$ gives that $\widehat{\mathcal{X}}^{-1}(A)\cap(\mathcal{X}^{-1}(A))^c\subset\widehat{\mathcal{N}}$. Hence,

$$\mathcal{X}^{-1}(A)\Delta\widehat{\mathcal{X}}^{-1}(A)\in\overline{\mathscr{N}}^t,\quad\forall\,A\in\mathcal{F}^t_s. \tag{7.125}$$

Multiplying $\mathbf{1}_{\widetilde{A}_0}$ to the SDE (6.2) for $\mathcal{X} = X^{t,\omega,\mu}$ and $\widehat{\mathcal{X}} = X^{t,\omega,\widehat{\mu}}$ over period $[s,T]$ yields that

$$\begin{aligned}
\mathbf{1}_{\widetilde{A}_0}(\mathcal{X}_r-\mathcal{X}_s) &= \int_s^r \mathbf{1}_{\widetilde{A}_0} b^{t,\omega}(r',\mathbf{1}_{\widetilde{A}_0}\mathcal{X},\mu_{r'})\,dr' + \int_s^r \mathbf{1}_{\widetilde{A}_0}\mu_{r'}\,dB^t_{r'}, \quad r\in[s,T],\\
\text{and}\quad \mathbf{1}_{\widetilde{A}_0}(\widehat{\mathcal{X}}_r-\widehat{\mathcal{X}}_s) &= \int_s^r \mathbf{1}_{\widetilde{A}_0} b^{t,\omega}(r',\mathbf{1}_{\widetilde{A}_0}\widehat{\mathcal{X}},\widehat{\mu}_{r'})\,dr' + \int_s^r \mathbf{1}_{\widetilde{A}_0}\widehat{\mu}_{r'}\,dB^t_{r'}\\
&= \int_s^r \mathbf{1}_{\widetilde{A}_0} b^{t,\omega}(r',\mathbf{1}_{\widetilde{A}_0}\widehat{\mathcal{X}},\mu_{r'})\,dr' + \int_s^r \mathbf{1}_{\widetilde{A}_0}\mu_{r'}\,dB^t_{r'}, \quad r\in[s,T].
\end{aligned}$$

By (7.124), $\big\{\mathbf{1}_{\widetilde{A}_0}\mathcal{X}_r\big\}_{r\in[s,T]}$ and $\big\{\mathbf{1}_{\widetilde{A}_0}\widehat{\mathcal{X}}_r\big\}_{r\in[s,T]}$ satisfy the same SDE:

$$X'_r = \mathbf{1}_{\widetilde{A}_0}\mathcal{X}_s + \int_t^r \mathbf{1}_{\widetilde{A}_0} b^{t,\omega}(r',X',\mathbf{1}_{\widetilde{A}_0}\mu_{r'})\,dr' + \int_t^r \mathbf{1}_{\widetilde{A}_0}\mu_{r'}\,dB^t_{r'}, \quad r\in[s,T].$$

Similar to (6.2), this SDE admits a unique solution. So it holds $\mathbb{P}^t_0-$a.s. on $\widetilde{A}_0$ that

$$\mathcal{X}_r = \widehat{\mathcal{X}}_r, \quad \forall\, r\in[s,T]. \tag{7.126}$$

Let $j=1,\cdots,\lambda$. Proposition 6.1, (7.124) and (7.123) show that for all $\widetilde{\omega}\in\widetilde{A}_j$ except on an $\mathcal{N}_j\in\overline{\mathscr{N}}^t$

$$\widehat{\mathcal{X}}^{s,\widetilde{\omega}} = X^{s,\omega\otimes_t\widehat{\mathcal{X}}(\widetilde{\omega}),\widehat{\mu}^{s,\widetilde{\omega}}} + \widehat{\mathcal{X}}_s(\widetilde{\omega}) = X^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega}),\mu^j} + \mathcal{X}_s(\widetilde{\omega}), \tag{7.127}$$

where we used the fact that $X^{s,\omega\otimes_t\widehat{\mathcal{X}}(\widetilde{\omega}),\widehat{\mu}^{s,\widetilde{\omega}}}$ depends only on $\omega\otimes_t\widehat{\mathcal{X}}(\widetilde{\omega})\big|_{[0,s]}$. Lemma 2.5 (1), an analogy to (7.78) and the continuity of $\mathcal{X}$ imply that for all $\widetilde{\omega}\in\Omega^t$ except on an $\widehat{\mathcal{N}}'\in\overline{\mathscr{N}}^t$

$$\widehat{\mathcal{N}}^{s,\widetilde{\omega}}\in\overline{\mathscr{N}}^s \quad\text{and}\quad \mathbb{P}^s_0\big\{\widehat{\omega}\in\Omega^s:\mathcal{X}_r(\widetilde{\omega}\otimes_s\widehat{\omega}) = \mathcal{X}_r(\widetilde{\omega}),\ \forall\, r\in[t,s]\big\}=1. \tag{7.128}$$

Set $\widetilde{\mathcal{N}}_j := \mathcal{N}_j\cup\widehat{\mathcal{N}}'\in\overline{\mathscr{N}}^t$. Given $\widetilde{\omega}\in\widetilde{A}_j\cap\widetilde{\mathcal{N}}^c_j$, since

$$\big\{\widehat{\omega}\in\Omega^s:\mathcal{X}_r(\widetilde{\omega}\otimes_s\widehat{\omega})\neq\widehat{\mathcal{X}}_r(\widetilde{\omega}\otimes_s\widehat{\omega}) \text{ for some } r\in[t,s]\big\} = \{\widehat{\omega}\in\Omega^s:\widetilde{\omega}\otimes_s\widehat{\omega}\in\widehat{\mathcal{N}}\} = \widehat{\mathcal{N}}^{s,\widetilde{\omega}}\in\overline{\mathscr{N}}^s,$$

we can deduce from (7.127) and (7.128) that for all $\widehat{\omega}\in\Omega^s$ except on some $\mathcal{N}_{\widetilde{\omega}}\in\overline{\mathscr{N}}^s$

$$\begin{aligned}
\widehat{\mathcal{X}}_r(\widetilde{\omega}\otimes_s\widehat{\omega}) &= \mathbf{1}_{\{r\in[t,s)\}}\mathcal{X}_r(\widetilde{\omega}\otimes_s\widehat{\omega}) + \mathbf{1}_{\{r\in[s,T]\}}\big(X^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega}),\mu^j}_r(\widehat{\omega})+\mathcal{X}_s(\widetilde{\omega})\big)\\
&= \mathbf{1}_{\{r\in[t,s)\}}\mathcal{X}_r(\widetilde{\omega}) + \mathbf{1}_{\{r\in[s,T]\}}\big(X^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega}),\mu^j}_r(\widehat{\omega})+\mathcal{X}_s(\widetilde{\omega})\big) = \big(\mathcal{X}(\widetilde{\omega})\otimes_s X^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega}),\mu^j}(\widehat{\omega})\big)(r), \quad \forall\, r\in[t,T].
\end{aligned} \tag{7.129}$$

For any $A\in\mathcal{F}^t_T$, applying (7.125) with $A=\mathcal{A}_0$, we can deduce from (7.117), (7.124) and (7.126) that

$$\begin{aligned}
\widehat{\mathbb{P}}(A\cap\mathcal{A}_0) &= \mathbb{P}^t_0\big(\widehat{\mathcal{X}}^{-1}(A\cap\mathcal{A}_0)\big) = \mathbb{P}^t_0\big(\widehat{\mathcal{X}}^{-1}(A)\cap\widehat{\mathcal{X}}^{-1}(\mathcal{A}_0)\big) = \mathbb{P}^t_0\big(\widehat{\mathcal{X}}^{-1}(A)\cap\mathcal{X}^{-1}(\mathcal{A}_0)\big) = \mathbb{P}^t_0\big(\widehat{\mathcal{X}}^{-1}(A)\cap\widetilde{A}_0\big)\\
&= \mathbb{P}^t_0\big\{\widetilde{\omega}\in\widetilde{A}_0:\widehat{\mathcal{X}}(\widetilde{\omega})\in A\big\} = \mathbb{P}^t_0\big\{\widetilde{\omega}\in\widetilde{A}_0:\mathcal{X}(\widetilde{\omega})\in A\big\} = \mathbb{P}^t_0\big(\mathcal{X}^{-1}(A)\cap\widetilde{A}_0\big) = \mathbb{P}^t_0\big(\mathcal{X}^{-1}(A)\cap\mathcal{X}^{-1}(\mathcal{A}_0)\big)\\
&= \mathbb{P}^t_0\big(\mathcal{X}^{-1}(A\cap\mathcal{A}_0)\big) = \mathbb{P}(A\cap\mathcal{A}_0).
\end{aligned}$$

On the other hand, for any $A\in\mathcal{F}^t_s$ and $j=1,\cdots,\lambda$, applying (7.125) with $A=A\cap\mathcal{A}_j$ yields that

$$\widehat{\mathbb{P}}(A\cap\mathcal{A}_j) = \mathbb{P}^t_0\big(\widehat{\mathcal{X}}^{-1}(A\cap\mathcal{A}_j)\big) = \mathbb{P}^t_0\big(\mathcal{X}^{-1}(A\cap\mathcal{A}_j)\big) = \mathbb{P}(A\cap\mathcal{A}_j).$$

(2c) *In the last step, we use the continuity* (3.1) *of* Y *and the estimates* (6.3) *of* $X^{t,\omega,\mu}$ *to verify* (3.4) *for* $\widehat{\mathbb{P}}$.

Fix $j=1,\cdots,\lambda$. We set $(\mathbb{P}_j,\mathfrak{p}_j,\mathcal{X}^j,\mathcal{W}^j) := \big(\mathbb{P}^{s,\omega\otimes_t\widetilde{\omega}_j,\mu^j},\mathfrak{p}^{s,\omega\otimes_t\widetilde{\omega}_j,\mu^j},X^{s,\omega\otimes_t\widetilde{\omega}_j,\mu^j},W^{s,\omega\otimes_t\widetilde{\omega}_j,\mu^j}\big)$. Similar to (7.97), it holds for all $\widehat{\omega}\in\Omega^s$ except on a $\mathbb{P}^s_0-$null set $\mathcal{N}_{\mathcal{X}^j}$ that

$$B^s_r(\widehat{\omega}) = \mathcal{W}^j_r\big(\mathcal{X}^j(\widehat{\omega})\big), \quad \forall\, r\in[s,T]. \tag{7.130}$$

Set $A_{\mathcal{X}^j} := \{\widehat{\omega}'\in\Omega^s:\mathcal{N}^c_{\mathcal{X}^j}\cap(\mathcal{X}^j)^{-1}(\widehat{\omega}')\neq\emptyset\}$ and $\mathfrak{F}^j_r := \sigma\big(\mathcal{F}^s_r\cup\mathscr{N}^{\mathfrak{p}_j}\big)\subset\mathcal{G}^{\mathcal{X}^j}_r$, $\forall\, r\in[s,T]$. The process $\widetilde{\mathcal{W}}^j_r(\widehat{\omega}) := \mathbf{1}_{\{\widehat{\omega}\in A_{\mathcal{X}^j}\}}\mathcal{W}^j_r(\widehat{\omega})$, $\forall\,(r,\widehat{\omega})\in[s,T]\times\Omega^s$ is adapted to the filtration $\{\mathfrak{F}^j_r\}_{r\in[s,T]}$ and all its paths belong to Ω^s.

By Proposition 2.1 (2) and Remark 3.1 (1), the shifted process $\mathcal{Y}_r := Y^{t,\omega}_r$, $r \in [t,T]$ as defined in (7.17) is $\mathbf{F}^t-$adapted and its paths are all RCLL. Then (6.6) implies that $\mathcal{Y}(\widehat{\mathcal{X}})$ is an $\overline{\mathbf{F}}^t-$adapted process whose paths are all RCLL. Applying Lemma A.4 (3) with $(\mathbb{P},X)=(\mathbb{P}^t_0,B^t)$ shows that $\mathcal{Y}(\widehat{\mathcal{X}})$ has an $(\mathbf{F}^t,\mathbb{P}^t_0)-$version $\mathscr{Y}$, which is $\mathbf{F}^t-$progressively measurable process with $\mathcal{N}_Y := \{\widetilde{\omega}\in\Omega^t : \mathscr{Y}_r(\widetilde{\omega}) \neq \mathcal{Y}_r(\widehat{\mathcal{X}}(\widetilde{\omega}))$ for some $r\in[t,T]\}\in\overline{\mathscr{N}}^t$. By Lemma 2.5 (1), it holds for all $\widetilde{\omega}\in\Omega^t$ except on an $\widetilde{\mathcal{N}}_Y \in \overline{\mathscr{N}}^t$ that $\mathcal{N}^{s,\widetilde{\omega}}_Y \in \overline{\mathscr{N}}^s$.

Fix $A\in\mathcal{F}^t_s$, $\tau\in\mathcal{T}^t_s$ and set $\widehat{\tau}=\tau(\widehat{\mathcal{X}})$. For any $r\in[s,T]$, since $A_r := \{\tau\le r\}\in\mathcal{F}^t_r$, (6.6) shows that

$$\{\widehat{\tau}\le r\}=\big\{\widetilde{\omega}\in\Omega^t:\tau\big(\widehat{\mathcal{X}}(\widetilde{\omega})\big)\le r\big\}=\{\widetilde{\omega}\in\Omega^t:\widehat{\mathcal{X}}(\widetilde{\omega})\in A_r\}=\widehat{\mathcal{X}}^{-1}(A_r)\in\overline{\mathcal{F}}^t_r,\ \text{namely}\ \widehat{\tau}\in\overline{\mathcal{T}}^t_s.$$

By Lemma 2.5 (3), it holds for all $\widetilde{\omega}\in\Omega^t$ except on a $\mathcal{N}_\tau\in\overline{\mathscr{N}}^t$ that $\widehat{\tau}^{s,\widetilde{\omega}}\in\overline{\mathcal{T}}^s$.

For any $\widetilde{\omega}\in\mathcal{N}^c_Y$, we have

$$\mathscr{Y}(r,\widetilde{\omega})=\mathcal{Y}\big(r,\widehat{\mathcal{X}}(\widetilde{\omega})\big),\quad\forall\,r\in[t,T].\tag{7.131}$$

In particular, taking $r=\widehat{\tau}(\widetilde{\omega})$ gives that $\mathscr{Y}_{\widehat{\tau}}(\widetilde{\omega})=\mathscr{Y}\big(\widehat{\tau}(\widetilde{\omega}),\widetilde{\omega}\big)=\mathcal{Y}\big(\widehat{\tau}(\widetilde{\omega}),\widehat{\mathcal{X}}(\widetilde{\omega})\big)=\mathcal{Y}\big(\tau\big(\widehat{\mathcal{X}}(\widetilde{\omega})\big),\widehat{\mathcal{X}}(\widetilde{\omega})\big)=\mathcal{Y}_\tau\big(\widehat{\mathcal{X}}(\widetilde{\omega})\big)$. So

$$\mathbb{E}_{\widehat{\mathbb{P}}}\big[\mathbf{1}_{A\cap\mathcal{A}_j}Y^{t,\omega}_\tau\big]=\mathbb{E}_{\widehat{\mathbb{P}}}\big[\mathbf{1}_{A\cap\mathcal{A}_j}\mathcal{Y}_\tau\big]=\mathbb{E}_t\Big[\mathbf{1}_{\widehat{\mathcal{X}}^{-1}(A\cap\mathcal{A}_j)}\mathcal{Y}_\tau\big(\widehat{\mathcal{X}}\big)\Big]=\mathbb{E}_t\Big[\mathbf{1}_{\widehat{\mathcal{X}}^{-1}(A\cap\mathcal{A}_j)}\mathscr{Y}_{\widehat{\tau}}\Big].\tag{7.132}$$

Also, one can deduce from (7.131), Lemma 6.2 and (3.2) that

$$\mathbb{E}_t[\mathscr{Y}_*]=\mathbb{E}_t\big[\mathcal{Y}_*\big(\widehat{\mathcal{X}}\big)\big]=\mathbb{E}_{\widehat{\mathbb{P}}}\big[\mathcal{Y}_*\big]=\mathbb{E}_{\widehat{\mathbb{P}}}\big[Y^{t,\omega}_*\big]<\infty.\tag{7.133}$$

Since $\widehat{\mathcal{X}}^{-1}(A\cap\mathcal{A}_j)\in\overline{\mathcal{F}}^t_s$ by (6.6) and since $\mathscr{Y}_{\widehat{\tau}}\in L^1(\overline{\mathcal{F}}^t_T,\mathbb{P}^t_0)$ by (7.133), applying Lemma A.4 (1) and Proposition 2.3 with $(\mathbb{P},X,\xi)=\big(\mathbb{P}^t_0,B^t,\mathscr{Y}_{\widehat{\tau}}\big)$ as well as using (7.125) with $A=A\cap\mathcal{A}_j$, we can deduce from (7.132), Lemma 2.4 and (7.116) that

$$\begin{aligned}\mathbb{E}_{\widehat{\mathbb{P}}}\big[\mathbf{1}_{A\cap\mathcal{A}_j}Y^{t,\omega}_\tau\big]\ &=\ \mathbb{E}_t\Big[\mathbf{1}_{\widehat{\mathcal{X}}^{-1}(A\cap\mathcal{A}_j)}\mathscr{Y}_{\widehat{\tau}}\Big]=\mathbb{E}_t\Big[\mathbf{1}_{\widehat{\mathcal{X}}^{-1}(A\cap\mathcal{A}_j)}\mathbb{E}_t\big[\mathscr{Y}_{\widehat{\tau}}\big|\overline{\mathcal{F}}^t_s\big]\Big]=\mathbb{E}_t\Big[\mathbf{1}_{\mathcal{X}^{-1}(A\cap\mathcal{A}_j)}\mathbb{E}_t\big[\mathscr{Y}_{\widehat{\tau}}\big|\mathcal{F}^t_s\big]\Big]\\&=\mathbb{E}_t\Big[\mathbf{1}_{\{\widetilde{\omega}\in\mathcal{X}^{-1}(A)\cap\mathcal{A}^{\mathcal{X}}_j\}}\mathbb{E}_s\big[(\mathscr{Y}_{\widehat{\tau}})^{s,\widetilde{\omega}}\big]\Big]=\mathbb{E}_t\Big[\mathbf{1}_{\{\widetilde{\omega}\in\mathcal{X}^{-1}(A)\cap\mathcal{A}^{\mathcal{X}}_j\cap\widetilde{A}_j\}}\mathbb{E}_s\big[(\mathscr{Y}_{\widehat{\tau}})^{s,\widetilde{\omega}}\big]\Big].\end{aligned}\tag{7.134}$$

Let $\widetilde{\omega}\in\mathcal{A}^{\mathcal{X}}_j\cap\widetilde{A}_j\cap\widetilde{\mathcal{N}}^c_j\cap\widetilde{\mathcal{N}}^c_Y\cap\mathcal{N}^c_\tau$. Then one has

$$\big\{\widehat{\omega}\in\Omega^s:\mathscr{Y}_r(\widetilde{\omega}\otimes_s\widehat{\omega})\neq\mathcal{Y}_r\big(\widehat{\mathcal{X}}(\widetilde{\omega}\otimes_s\widehat{\omega})\big)\ \text{for some}\ r\in[t,T]\big\}=\{\widehat{\omega}\in\Omega^s:\widetilde{\omega}\otimes_s\widehat{\omega}\in\mathcal{N}_Y\}=\mathcal{N}^{s,\widetilde{\omega}}_Y\in\overline{\mathscr{N}}^s.\tag{7.135}$$

For any $\widehat{\omega}\in\Omega^s$ except on $\mathcal{N}^{s,\widetilde{\omega}}_Y\cup\mathcal{N}_{\mathcal{X}^j}\cup\mathcal{N}_{\widetilde{\omega}}\in\overline{\mathscr{N}}^s$, similar to (7.106), we see that $\mathcal{X}^j(\widehat{\omega})\in A_{\mathcal{X}^j}$, and can deduce from (7.130) that $\widehat{\omega}=B^s(\widehat{\omega})=\mathcal{W}^j\big(\mathcal{X}^j(\widehat{\omega})\big)=\widetilde{\mathcal{W}}^j\big(\mathcal{X}^j(\widehat{\omega})\big)$. Then (7.135), (7.129) and (3.1) imply that

$$\begin{aligned}(\mathscr{Y}_{\widehat{\tau}})^{s,\widetilde{\omega}}(\widehat{\omega})&=\mathscr{Y}\big(\widehat{\tau}(\widetilde{\omega}\otimes_s\widehat{\omega}),\widetilde{\omega}\otimes_s\widehat{\omega}\big)=\mathcal{Y}\big(\widehat{\tau}^{s,\widetilde{\omega}}(\widehat{\omega}),\widehat{\mathcal{X}}(\widetilde{\omega}\otimes_s\widehat{\omega})\big)=Y\Big(\zeta_{\widetilde{\omega}}\big(\mathcal{X}^j(\widehat{\omega})\big),\omega\otimes_t\big(\mathcal{X}(\widetilde{\omega})\otimes_s X^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega}),\mu^j}(\widehat{\omega})\big)\Big)\\&\le Y\big(\zeta_{\widetilde{\omega}}\big(\mathcal{X}^j(\widehat{\omega})\big),\omega\otimes_t\big(\mathcal{X}(\widetilde{\omega})\otimes_s\mathcal{X}^j(\widehat{\omega})\big)\big)+\rho_0\big(\Delta X^j_{\widetilde{\omega}}(\widehat{\omega})\big)=Y^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega})}_{\zeta_{\widetilde{\omega}}}\big(\mathcal{X}^j(\widehat{\omega})\big)+\rho_0\big(\Delta X^j_{\widetilde{\omega}}(\widehat{\omega})\big)\\&\le Y^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega})}_{\zeta_{\widetilde{\omega}}}\big(\mathcal{X}^j(\widehat{\omega})\big)+\mathbf{1}_{\big\{\Delta X^j_{\widetilde{\omega}}(\widehat{\omega})\le\delta^{1/2}\big\}}\rho_0\big(\delta^{1/2}\big)+\mathbf{1}_{\big\{\Delta X^j_{\widetilde{\omega}}(\widehat{\omega})>\delta^{1/2}\big\}}\kappa\delta^{-1/2}\Big(\Delta X^j_{\widetilde{\omega}}(\widehat{\omega})+\big(\Delta X^j_{\widetilde{\omega}}(\widehat{\omega})\big)^{\varpi+1}\Big),\end{aligned}\tag{7.136}$$

where $\zeta_{\widetilde{\omega}}(\widehat{\omega}') := \widehat{\tau}^{s,\widetilde{\omega}}\big(\widetilde{\mathcal{W}}^j(\widehat{\omega}')\big)$, $\forall\,\widehat{\omega}'\in\Omega^s$ and $\Delta X^j_{\widetilde{\omega}}(\widehat{\omega}) := \big\|X^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega}),\mu^j}(\widehat{\omega})-\mathcal{X}^j(\widehat{\omega})\big\|_{s,T}$.

For any $r\in[s,T]$, as $\widetilde{A}_r := \{\widehat{\tau}^{s,\widetilde{\omega}}\le r\}\in\overline{\mathcal{F}}^s_r$, an analogy to (7.108) shows that $\big\{\zeta_{\widetilde{\omega}}\le r\big\}=\big\{\widehat{\omega}\in\Omega^s:\widetilde{\mathcal{W}}^j(\widehat{\omega})\in\widetilde{A}_r\big\}=(\widetilde{\mathcal{W}}^j)^{-1}(\widetilde{A}_r)\in\mathfrak{F}^j_r$. So $\zeta_{\widetilde{\omega}}$ is a $\mathfrak{F}^j-$stopping time.

Given $\varepsilon>0$, similar to (7.26), there exists some $\zeta'_{\widetilde{\omega}}\in\mathcal{T}^s$ such that

$$\mathbb{E}_{\mathfrak{p}_j}\Big[\Big|Y^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega})}_{\zeta'_{\widetilde{\omega}}}-Y^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega})}_{\zeta_{\widetilde{\omega}}}\Big|\Big]<\varepsilon.\tag{7.137}$$

As $\widetilde{\omega}\in\mathcal{A}^{\mathcal{X}}_j=\mathcal{X}^{-1}(\mathcal{A}_j)$, i.e. $\mathcal{X}(\widetilde{\omega})\in\mathcal{A}_j\subset O^s_{\delta_j}(\widetilde{\omega}_j)$, we see that $\|\omega\otimes_t\mathcal{X}(\widetilde{\omega})-\omega\otimes_t\widetilde{\omega}_j\|_{0,s}=\|\mathcal{X}(\widetilde{\omega})-\widetilde{\omega}_j\|_{t,s}<\delta_j\le\delta$. It then follows from (7.136) and (6.3) that

$$\begin{aligned}\mathbb{E}_s\big[(\mathscr{Y}_{\widehat{\tau}})^{s,\widetilde{\omega}}\big]&\le\mathbb{E}_s\Big[Y^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega})}_{\zeta_{\widetilde{\omega}}}\big(\mathcal{X}^j\big)\Big]+\rho_0\big(\delta^{1/2}\big)+\kappa\delta^{-1/2}\big(C_1T\|\omega\otimes_t\mathcal{X}(\widetilde{\omega})-\omega\otimes_t\widetilde{\omega}_j\|_{0,s}+C_{\varpi+1}T^{\varpi+1}\|\omega\otimes_t\mathcal{X}(\widetilde{\omega})-\omega\otimes_t\widetilde{\omega}_j\|^{\varpi+1}_{0,s}\big)\\&\le\mathbb{E}_{\mathfrak{p}_j}\Big[Y^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega})}_{\zeta_{\widetilde{\omega}}}\Big]+\rho_0\big(\delta^{1/2}\big)+\kappa\big(C_1T\delta^{1/2}+C_{\varpi+1}T^{\varpi+1}\delta^{\varpi+1/2}\big)\le\mathbb{E}_{\mathfrak{p}_j}\Big[Y^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega})}_{\zeta'_{\widetilde{\omega}}}\Big]+\widehat{\rho}_0(\delta)+\varepsilon,\end{aligned}\tag{7.138}$$

where $\widehat{\rho}(\delta) := \rho_0\big(\delta^{1/2}\big)+\kappa\big(C_1T\delta^{1/2}+C_{\varpi+1}T^{\varpi+1}\delta^{\varpi+1/2}\big)$. Since $\zeta'_{\widetilde{\omega}} \in \mathcal{T}^s$, the $\mathbf{F}-$adaptedness of Y and Proposition 2.1 (2) show that $Y^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega})}_{\zeta'_{\widetilde{\omega}}} \in \mathcal{F}^s_T$, and thus

$$\mathbb{E}_{\mathfrak{p}_j}\Big[Y^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega})}_{\zeta'_{\widetilde{\omega}}}\Big] = \mathbb{E}_{\mathbb{P}_j}\Big[Y^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega})}_{\zeta'_{\widetilde{\omega}}}\Big] \le \sup_{\zeta\in\mathcal{T}^s}\mathbb{E}_{\mathbb{P}_j}\Big[Y^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega})}_{\zeta}\Big]. \tag{7.139}$$

Then plugging (7.138) into (7.134), we can deduce from (7.116) and Lemma A.12 (1) that

$$\begin{aligned}\mathbb{E}_{\widehat{\mathbb{P}}}\big[\mathbf{1}_{A\cap\mathcal{A}_j}Y^{t,\omega}_\tau\big] &\le \mathbb{E}_t\bigg[\mathbf{1}_{\{\widetilde{\omega}\in\mathcal{X}^{-1}(A)\cap\mathcal{X}^{-1}(\mathcal{A}_j)\cap\widetilde{A}_j\}}\Big(\sup_{\zeta\in\mathcal{T}^s}\mathbb{E}_{\mathbb{P}_j}\Big[Y^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega})}_\zeta\Big]+\widehat{\rho}_0(\delta)+\varepsilon\Big)\bigg]\\ &= \mathbb{E}_t\bigg[\mathbf{1}_{\{\widetilde{\omega}\in\mathcal{X}^{-1}(A\cap\mathcal{A}_j)\}}\Big(\sup_{\zeta\in\mathcal{T}^s}\mathbb{E}_{\mathbb{P}_j}\Big[Y^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega})}_\zeta\Big]+\widehat{\rho}_0(\delta)+\varepsilon\Big)\bigg] = \mathbb{E}_{\mathbb{P}}\bigg[\mathbf{1}_{\{\widetilde{\omega}\in A\cap\mathcal{A}_j\}}\Big(\sup_{\zeta\in\mathcal{T}^s}\mathbb{E}_{\mathbb{P}_j}\Big[Y^{s,\omega\otimes_t\widetilde{\omega}}_\zeta\Big]+\widehat{\rho}_0(\delta)+\varepsilon\Big)\bigg],\end{aligned}$$

where we used the fact that the mapping $\widetilde{\omega} \to \sup_{\zeta\in\mathcal{T}^s}\mathbb{E}_{\mathbb{P}_j}\big[Y^{s,\omega\otimes_t\widetilde{\omega}}_\zeta\big]$ is continuous by Remark 3.3 (2). Letting $\varepsilon \to 0$ and taking supremum over $\tau\in\mathcal{T}^t_s$, we see that (3.4) holds.

3) *In this part, we still use the continuity* (3.1) *of* Y *and the estimates* (6.3) *of* $X^{t,\omega,\mu}$ *to show that* $\{\mathcal{P}(t,\omega)\}_{(t,\omega)\in[0,T]\times\Omega}$ *satisfies Assumption 4.1.*

Let $\omega' \in \Omega$. We set $(\mathcal{X}',\mathbb{P}') = \big(X^{t,\omega',\mu},\mathbb{P}^{t,\omega',\mu}\big)$ and $\delta := \|\omega'-\omega\|_{0,t}$. For any $\widetilde{\omega} \in \Omega^t$, define $\Delta X(\widetilde{\omega}) := \|\mathcal{X}'(\widetilde{\omega})-\mathcal{X}(\widetilde{\omega})\|_{t,T}$. Similar to (7.136), we can deduce from (3.1) that for any $r \in [t,T]$

$$\begin{aligned}&Y\big(r,\omega'\otimes_t\mathcal{X}'(\widetilde{\omega})\big)-Y\big(r,\omega\otimes_t\mathcal{X}(\widetilde{\omega})\big) \le \rho_0\big(\|\omega'\otimes_t\mathcal{X}'(\widetilde{\omega})-\omega\otimes_t\mathcal{X}(\widetilde{\omega})\|_{0,r}\big) \le \rho_0\big(\|\omega'-\omega\|_{0,t}+\|\mathcal{X}'(\widetilde{\omega})-\mathcal{X}(\widetilde{\omega})\|_{t,r}\big)\\ &\quad\le \rho_0\big(\delta+\Delta X(\widetilde{\omega})\big) \le \mathbf{1}_{\{\Delta X(\widetilde{\omega})\le\delta^{1/2}\}}\rho_0\big(\delta+\delta^{1/2}\big)+\mathbf{1}_{\{\Delta X(\widetilde{\omega})>\delta^{1/2}\}}\kappa\delta^{-1/2}\big((1+2^{\varpi-1}\delta^{\varpi})\Delta X(\widetilde{\omega})+2^{\varpi-1}(\Delta X(\widetilde{\omega}))^{\varpi+1}\big).\end{aligned}$$

Given $\tau \in \mathcal{T}^t$, it follows from (6.3) that

$$\mathbb{E}_t\Big[Y\big(\tau(\mathcal{X}'),\omega'\otimes_t\mathcal{X}'\big)-Y\big(\tau(\mathcal{X}'),\omega\otimes_t\mathcal{X}\big)\Big] \le \rho_0(\delta+\delta^{1/2})+\kappa\big(1+2^{\varpi-1}\delta^{\varpi}\big)C_1T\delta^{1/2}+\kappa2^{\varpi-1}C_{\varpi+1}T^{\varpi+1}\delta^{\varpi+1/2} := \rho_1(\delta).$$

Clearly, ρ_1 is a modulus of continuity function greater than ρ_0. Then (7.106) implies that

$$\begin{aligned}\mathbb{E}_{\mathbb{P}'}\big[Y^{t,\omega'}_\tau\big] &= \mathbb{E}_t\big[Y^{t,\omega'}_\tau(\mathcal{X}')\big] = \mathbb{E}_t\big[Y^{t,\omega'}\big(\tau(\mathcal{X}'),\mathcal{X}'\big)\big] = \mathbb{E}_t\Big[Y\big(\tau(\mathcal{X}'),\omega'\otimes_t\mathcal{X}'\big)\Big]\\ &\le \mathbb{E}_t\Big[Y\big(\tau\big(\mathcal{X}'\big),\omega\otimes_t\mathcal{X}\big)\big)\Big]+\rho_1(\delta) = \mathbb{E}_t\Big[Y\Big(\tau\big(\mathcal{X}'(\widetilde{\mathcal{W}}(\mathcal{X}))\big),\omega\otimes_t\mathcal{X}\Big)\Big]+\rho_1(\delta)\\ &= \mathbb{E}_t\Big[Y\big(\zeta(\mathcal{X}),\omega\otimes_t\mathcal{X}\big)\Big]+\rho_1(\delta) = \mathbb{E}_t\big[Y^{t,\omega}_\zeta(\mathcal{X})\big]+\rho_1(\delta) = \mathbb{E}_{\mathfrak{p}}\big[Y^{t,\omega}_\zeta\big]+\rho_1(\delta),\end{aligned} \tag{7.140}$$

where $\zeta := \tau\big(\mathcal{X}'(\widetilde{\mathcal{W}})\big)$. For any $r\in[t,T]$, as $\widehat{A}_r := \{\tau \le r\} \in \mathcal{F}^t_r$, (6.6) shows that $(\mathcal{X}')^{-1}(\widehat{A}_r) \in \overline{\mathcal{F}}^t_r$. Then (7.108) implies

$$\{\zeta \le r\} = \big\{\widetilde{\omega}\in\Omega^t : \mathcal{X}'\big(\widetilde{\mathcal{W}}(\widetilde{\omega})\big) \in \widehat{A}_r\big\} = \widetilde{\mathcal{W}}^{-1}\big((\mathcal{X}')^{-1}(\widehat{A}_r)\big) \in \mathfrak{F}_r.$$

So ζ is a $\mathfrak{F}-$stopping time. Given $\varepsilon>0$, similar to (7.137) and (7.139), there exists a $\zeta'\in\mathcal{T}^t$ such that $\mathbb{E}_{\mathfrak{p}}\Big[\big|Y^{t,\omega}_{\zeta'}-Y^{t,\omega}_\zeta\big|\Big]<\varepsilon$ and $\mathbb{E}_{\mathfrak{p}}\Big[Y^{t,\omega}_{\zeta'}\Big]=\mathbb{E}_{\mathbb{P}}\Big[Y^{t,\omega}_{\zeta'}\Big] \le \sup_{\tau'\in\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}\big[Y^{t,\omega}_{\tau'}\big]$, which together with (7.140) shows that

$$\mathbb{E}_{\mathbb{P}'}\big[Y^{t,\omega'}_\tau\big] \le \mathbb{E}_{\mathfrak{p}}\big[Y^{t,\omega}_\zeta\big]+\rho_1(\delta) \le \sup_{\tau'\in\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}\big[Y^{t,\omega}_{\tau'}\big]+\rho_1(\delta)+\varepsilon.$$

Letting $\varepsilon \to 0$, taking supremum over $\tau \in \mathcal{T}^t$ on the left-hand-side and then taking infimum over $\mu\in\mathcal{U}_t$ yield that

$$\overline{Z}_t(\omega') = \inf_{\mu\in\mathcal{U}_t}\sup_{\tau\in\mathcal{T}^t}\mathbb{E}_{\mathbb{P}^{t,\omega',\mu}}\big[Y^{t,\omega'}_\tau\big] \le \inf_{\mu\in\mathcal{U}_t}\sup_{\tau'\in\mathcal{T}^t}\mathbb{E}_{\mathbb{P}^{t,\omega,\mu}}\big[Y^{t,\omega}_{\tau'}\big]+\rho_1\big(\|\omega'-\omega\|_{0,t}\big) = \overline{Z}_t(\omega)+\rho_1\big(\|\omega'-\omega\|_{0,t}\big).$$

Exchanging the roles of ω' and ω shows that $\{\mathcal{P}(t,\omega)\}_{(t,\omega)\in[0,T]\times\Omega}$ satisfies Assumption 4.1.

4) *In last part of the proof, we use the estimates* (6.3) *once again to show that* $\{\mathcal{P}(t,\omega)\}_{(t,\omega)\in[0,T]\times\Omega}$ *satisfies Assumption 4.2.*

There exists a constant $\widetilde{C}_\varpi$ depending on ϖ and T such that $\rho_1(\delta) \le \kappa \widetilde{C}_\varpi (1+\delta^{\varpi+1/2}),\ \forall\, \delta>0$. Let $\alpha > \|\omega\|_{0,t}$ and $\delta \in (0,T]$. We can deduce from (6.4) that

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}}\Big[\rho_1\Big(\delta+2 \sup_{r\in[t,(t+\delta)\wedge T]}\big|B^t_r\big|\Big)\Big] &= \mathbb{E}_t\Big[\rho_1\Big(\delta+2 \sup_{r\in[t,(t+\delta)\wedge T]}\big|\mathcal{X}_r\big|\Big)\Big] \\
&\le \rho_1(\delta+2\delta^{1/4}) + \kappa \widetilde{C}_\varpi \mathbb{E}_t\Big[\mathbf{1}_{\big\{\sup\limits_{r\in[t,(t+\delta)\wedge T]}|\mathcal{X}_r|>\delta^{1/4}\big\}}\Big(1+2^{\varpi-1/2}\delta^{\varpi+1/2}+2^{2\varpi}\sup_{r\in[t,(t+\delta)\wedge T]}\big|\mathcal{X}_r\big|^{\varpi+1/2}\Big)\Big] \\
&\le \rho_1(\delta+2\delta^{1/4}) + \kappa \widetilde{C}_\varpi \delta^{-1/4}\mathbb{E}_t\Big[\big(1+2^{\varpi-1/2}\delta^{\varpi+1/2}\big)\sup_{r\in[t,(t+\delta)\wedge T]}\big|\mathcal{X}_r\big| + 2^{2\varpi}\sup_{r\in[t,(t+\delta)\wedge T]}\big|\mathcal{X}_r\big|^{\varpi+3/2}\Big)\Big] \\
&\le \rho_1(\delta+2\delta^{1/4}) + \kappa \widetilde{C}_\varpi \big(1+2^{\varpi-1/2}\delta^{\varpi+1/2}\big)\varphi_1(\alpha)\,\delta^{1/4} + \kappa\widetilde{C}_\varpi 2^{2\varpi}\varphi_{\varpi+\frac{3}{2}}(\alpha)\,\delta^{\varpi/2+1/2} := \rho_\alpha(\delta).
\end{aligned}
$$

Clearly, ρ_α is a modulus of continuity function. Hence, $\{\mathcal{P}(t,\omega)\}_{(t,\omega)\in[0,T]\times\Omega}$ satisfies Assumption 4.2. $\square$

A Appendix: Technical Lemmata

Lemma A.1. *Let $0\le t\le s\le S\le T<\infty$. The mapping $\Pi^{t,T}_{s,S}$ is continuous (under the uniform norms) and is $\mathcal{F}^{t,T}_r/\mathcal{F}^{s,S}_r-$measurable for any $r\in[s,S]$. The law of $\Pi^{t,T}_{s,S}$ under $\mathbb{P}^{t,T}_0$ is exactly $\mathbb{P}^{s,S}_0$, i.e.,*

$$
\mathbb{P}^{t,T}_0\Big(\big(\Pi^{t,T}_{s,S}\big)^{-1}(A)\Big) = \mathbb{P}^{s,S}_0(A), \qquad \forall\, A\in\mathcal{F}^{s,S}_S. \tag{A.1}
$$

It also holds for any $r\in[s,S]$ and $\tau\in\mathcal{T}^{s,S}_r$ that $\tau\big(\Pi^{t,T}_{s,S}\big)\in\mathcal{T}^{t,T}_r$.

Proof: For simplicity, let us denote $\Pi^{t,T}_{s,S}$ by Π.

1) We first show the continuity of Π. Let A be an open subset of $\Omega^{s,S}$. Given $\omega\in\Pi^{-1}(A)$, since $\Pi(\omega)\in A$, there exist a $\delta>0$ such that $O_\delta\big(\Pi(\omega)\big)=\big\{\widetilde{\omega}\in\Omega^{s,S}:\|\widetilde{\omega}-\Pi(\omega)\|_{s,S}<\delta\big\}\subset A$. For any $\omega'\in O_{\delta/2}(\omega)$, one can deduce that

$$
\big\|\Pi(\omega')-\Pi(\omega)\big\|_{s,S} \le \big|\omega'(s)-\omega(s)\big| + \big\|\omega'-\omega\big\|_{s,S} \le 2\|\omega'-\omega\|_{t,T} < \delta,
$$

which shows that $\Pi(\omega')\in O_\delta\big(\Pi(\omega)\big)\subset A$ or $\omega'\in\Pi^{-1}(A)$. Hence, $\Pi^{-1}(A)$ is an open subset of $\Omega^{t,T}$.

Let $r\in[s,S]$. For any $s'\in[s,r]$ and $\mathcal{E}\in\mathscr{B}(\mathbb{R}^d)$, one can deduce that

$$
\Pi^{-1}\Big(\big(B^{s,S}_{s'}\big)^{-1}(\mathcal{E})\Big) = \Big\{\omega\in\Omega^{t,T}: B^{s,S}_{s'}\big(\Pi(\omega)\big)\in\mathcal{E}\Big\} = \big\{\omega\in\Omega^{t,T}: \omega(s')-\omega(s)\in\mathcal{E}\big\} = (B^{t,T}_{s'}-B^{t,T}_s)^{-1}(\mathcal{E})\in\mathcal{F}^{t,T}_r. \tag{A.2}
$$

Thus all the generating sets of $\mathcal{F}^{s,S}_r$ belong to $\Lambda:=\big\{A\subset\Omega^{s,S}:\Pi^{-1}(A)\in\mathcal{F}^{t,T}_r\big\}$, which is clearly a $\sigma-$field of $\Omega^{s,S}$. It follows that $\mathcal{F}^{s,S}_r\subset\Lambda$, i.e., $\Pi^{-1}(A)\in\mathcal{F}^{t,T}_r$ for any $A\in\mathcal{F}^{s,S}_r$.

2) Next, let us show that the induced probability $\widetilde{\mathbb{P}}:=\mathbb{P}^{t,T}_0\circ\Pi^{-1}$ equals to $\mathbb{P}^{s,S}_0$ on $\mathcal{F}^{s,S}_S$: Since the Wiener measure on $\big(\Omega^{s,S},\mathscr{B}(\Omega^{s,S})\big)$ is unique (see e.g. Proposition I.3.3 of [31]), it suffices to show that the canonical process $B^{s,S}$ is a Brownian motion on $\Omega^{s,S}$ under $\widetilde{\mathbb{P}}$: Let $s\le r\le r'\le S$. For any $\mathcal{E}\in\mathscr{B}(\mathbb{R}^d)$, similar to (A.2), one can deduce that

$$
\Pi^{-1}\big(\big(B^{s,S}_{r'}-B^{s,S}_r\big)^{-1}(\mathcal{E})\big) = (B^{t,T}_{r'}-B^{t,T}_r)^{-1}(\mathcal{E}). \tag{A.3}
$$

Thus, $\widetilde{\mathbb{P}}\Big(\big(B^{s,S}_{r'}-B^{s,S}_r\big)^{-1}(\mathcal{E})\Big) = \mathbb{P}^{t,T}_0\Big(\Pi^{-1}\big(\big(B^{s,S}_{r'}-B^{s,S}_r\big)^{-1}(\mathcal{E})\big)\Big) = \mathbb{P}^{t,T}_0\Big((B^{t,T}_{r'}-B^{t,T}_r)^{-1}(\mathcal{E})\Big)$, which shows that the distribution of $B^{s,S}_{r'}-B^{s,S}_r$ under $\widetilde{\mathbb{P}}$ is the same as that of $B^{t,T}_{r'}-B^{t,T}_r$ under $\mathbb{P}^{t,T}_0$ (a $d-$dimensional normal distribution with mean 0 and variance matrix $(r'-r)I_{d\times d}$).

On the other hand, for any $A\in\mathcal{F}^{s,S}_r$, since $\Pi^{-1}(A)$ belongs to $\mathcal{F}^{t,T}_r$, its independence from $B^{t,T}_{r'}-B^{t,T}_r$ under $\mathbb{P}^{t,T}_0$ and (A.3) yield that for any $\mathcal{E}\in\mathscr{B}(\mathbb{R}^d)$

$$
\begin{aligned}
\widetilde{\mathbb{P}}\Big(A\cap\big(B^{s,S}_{r'}-B^{s,S}_r\big)^{-1}(\mathcal{E})\Big) &= \mathbb{P}^{t,T}_0\Big(\Pi^{-1}(A)\cap\Pi^{-1}\big(\big(B^{s,S}_{r'}-B^{s,S}_r\big)^{-1}(\mathcal{E})\big)\Big) \\
&= \mathbb{P}^{t,T}_0\Big(\Pi^{-1}(A)\Big)\cdot\mathbb{P}^{t,T}_0\Big(\Pi^{-1}\big(\big(B^{s,S}_{r'}-B^{s,S}_r\big)^{-1}(\mathcal{E})\big)\Big) = \widetilde{\mathbb{P}}(A)\cdot\widetilde{\mathbb{P}}\Big(\big(B^{s,S}_{r'}-B^{s,S}_r\big)^{-1}(\mathcal{E})\Big).
\end{aligned}
$$

Hence, $B^{s,S}_{r'} - B^{s,S}_{r}$ is independent of $\mathcal{F}^{s,S}_r$ under $\widetilde{\mathbb{P}}$.

3) Now, let $r \in [s,S]$ and $\tau \in \mathcal{T}^{s,S}_r$. For any $r' \in [r,S]$, as $\widetilde{A} := \{\widetilde{\omega} \in \Omega^{s,S} : \tau(\widetilde{\omega}) \le r'\} \in \mathcal{F}^{s,S}_{r'}$, one can deduce that $\big\{\omega \in \Omega^{t,T} : \tau\big(\Pi^{t,T}_{s,S}(\omega)\big) \le r'\big\} = \big\{\omega \in \Omega^{t,T} : \Pi^{t,T}_{s,S}(\omega) \in \widetilde{A}\big\} = \big(\Pi^{t,T}_{s,S}\big)^{-1}(\widetilde{A}) \in \mathcal{F}^{t,T}_{r'}$. So $\tau\big(\Pi^{t,T}_{s,S}\big) \in \mathcal{T}^{t,T}_r$. □

Lemma A.2. *Let $t \in [0,T]$. For any $s \in [t,T]$, the $\sigma-$field $\mathcal{F}^t_s$ is countably generated by*

$$\mathscr{C}^t_s := \Big\{ \bigcap_{i=1}^{m} \big(B^t_{t_i}\big)^{-1}\big(O_{\lambda_i}(x_i)\big) : m \in \mathbb{N},\, t_i \in \mathbb{Q} \text{ with } t \le t_1 \le \cdots \le t_m \le s,\, x_i \in \mathbb{Q}^d,\, \lambda_i \in \mathbb{Q}_+ \Big\}.$$

Proof: For any $s \in [t,T]$, it is clear that $\sigma\big(\mathscr{C}^t_s\big) \subset \sigma\Big\{ \big(B^t_r\big)^{-1}(\mathcal{E}) : r \in [t,s], \mathcal{E} \in \mathscr{B}(\mathbb{R}^d) \Big\} = \mathcal{F}^t_s$. To see the reverse, we fix $r \in [t,s]$. For any $x \in \mathbb{Q}^d$ and $\lambda \in \mathbb{Q}_+$, let $\{s_j\}_{j\in\mathbb{N}} \subset (r,s) \cap \mathbb{Q}$ with $\lim\limits_{j\to\infty} \downarrow s_j = r$. The continuity of paths in Ω^t implies that

$$\big(B^t_r\big)^{-1}\big(O_\lambda(x)\big) = \bigcup_{n=\lceil \frac{2}{\lambda} \rceil}^{\infty} \bigcup_{m\in\mathbb{N}} \bigcap_{j>m} \Big(\big(B^t_{s_j}\big)^{-1}\big(O_{\lambda - \frac{1}{n}}(x)\big) \Big) \in \sigma\big(\mathscr{C}^t_s\big),$$

which shows that $\mathcal{O} := \big\{O_\lambda(x) : x \in \mathbb{Q}^d, \lambda \in \mathbb{Q}_+\big\} \subset \Lambda_r := \Big\{ \mathcal{E} \subset \mathbb{R}^d : \big(B^t_r\big)^{-1}(\mathcal{E}) \in \sigma\big(\mathscr{C}^t_s\big) \Big\}$. Clearly, $\mathcal{O}$ generates $\mathscr{B}(\mathbb{R}^d)$ and Λ_r is a $\sigma-$field of $\mathbb{R}^d$. So one has $\mathscr{B}(\mathbb{R}^d) \subset \Lambda_r$. Then $\mathcal{F}^t_s = \sigma\Big\{ \big(B^t_r\big)^{-1}(\mathcal{E}) : r \in [t,s], \mathcal{E} \in \mathscr{B}(\mathbb{R}^d) \Big\} \subset \sigma\big(\mathscr{C}^t_s\big)$. □

Lemma A.3. *Let $0 \le t \le s \le T$. For any $r \in [s,T]$, The mapping Π^t_s is further $\overline{\mathcal{F}}^t_r / \overline{\mathcal{F}}^s_r-$measurable: i.e. $(\Pi^t_s)^{-1}(A) \in \overline{\mathcal{F}}^t_r$, $\forall\, A \in \overline{\mathcal{F}}^s_r$.*

Proof: Let $r \in [s,T]$ and $A \in \overline{\mathcal{F}}^s_r$. By e.g. Problem 2.7.3 of [19], there exists a $A' \in \mathcal{F}^s_r$ such that $A \,\Delta\, A' \in \overline{\mathscr{N}}^s$, i.e. $A \,\Delta\, A' \subset \mathcal{N}$ for some $\mathcal{N} \in \mathcal{F}^s_T$ with $\mathbb{P}^s_0(\mathcal{N}) = 0$. Since $(\Pi^t_s)^{-1}(\mathcal{N}) \in \mathcal{F}^t_T$ by Lemma A.1 and since

$$\big(\mathbf{1}_{(\Pi^t_s)^{-1}(\mathcal{N})}\big)^{s,\omega}(\widetilde{\omega}) = \mathbf{1}_{\{\omega \otimes_s \widetilde{\omega} \in (\Pi^t_s)^{-1}(\mathcal{N})\}} = \mathbf{1}_{\{\Pi^t_s(\omega \otimes_s \widetilde{\omega}) \in \mathcal{N}\}} = \mathbf{1}_{\{\widetilde{\omega} \in \mathcal{N}\}} = \mathbf{1}_{\mathcal{N}}(\widetilde{\omega}), \quad \forall\, \omega \in \Omega^t, \ \forall\, \widetilde{\omega} \in \Omega^s,$$

Lemma 2.4 and Proposition 2.2 (1) imply that

$$\mathbb{P}^t_0\big((\Pi^t_s)^{-1}(\mathcal{N})\big) = \mathbb{E}_t\big[\mathbf{1}_{(\Pi^t_s)^{-1}(\mathcal{N})}\big] = \mathbb{E}_t\big[\mathbb{E}_t\big[\mathbf{1}_{(\Pi^t_s)^{-1}(\mathcal{N})} \big| \mathcal{F}^t_s\big]\big] = \mathbb{E}_t\big[\mathbb{E}_s\big[\big(\mathbf{1}_{(\Pi^t_s)^{-1}(\mathcal{N})}\big)^{s,\omega}\big]\big] = \mathbb{E}_t\big[\mathbb{P}^s_0(\mathcal{N})\big] = \mathbb{P}^s_0(\mathcal{N}) = 0.$$

It follows that $(\Pi^t_s)^{-1}(A) \,\Delta\, (\Pi^t_s)^{-1}(A') = (\Pi^t_s)^{-1}(A \,\Delta\, A') \in \overline{\mathscr{N}}^t$. As Lemma A.1 also shows that $(\Pi^t_s)^{-1}(A') \in \mathcal{F}^t_r$, one can deduce that $(\Pi^t_s)^{-1}(A) \in \overline{\mathcal{F}}^t_r$. □

Lemma A.4. *Given $t \in [0,T]$ and $\widetilde{d}, \widetilde{d}' \in \mathbb{N}$, let $\mathbb{P}$ be a probability on $\big(\Omega^t, \mathscr{B}(\Omega^t)\big)$ and let $\{X_s\}_{s\in[t,T]}$ be an $\mathbb{R}^{\widetilde{d}}-$valued, $\mathbf{F}^{\mathbb{P}}-$adapted process.*

1) For any $s \in [t,T]$ and any $\mathbb{R}^{\widetilde{d}'}-$valued, $\mathcal{F}^{X,\mathbb{P}}_T-$measurable random variable ξ with $\mathbb{E}_{\mathbb{P}}\big[|\xi|\big] < \infty$, $\mathbb{E}_{\mathbb{P}}\big[\xi \big| \mathcal{F}^{X,\mathbb{P}}_s\big] = \mathbb{E}_{\mathbb{P}}\big[\xi \big| \mathcal{F}^X_s\big]$, $\mathbb{P}-$a.s.

2) For any $s \in [t,T]$ and any $\mathbb{R}^{\widetilde{d}'}-$valued, $\mathcal{F}^{X,\mathbb{P}}_s-$measurable random variable ξ, there exists an $\mathbb{R}^{\widetilde{d}'}-$valued, $\mathcal{F}^X_s-$measurable random variable $\widetilde{\xi}$ such that $\widetilde{\xi} = \xi$, $\mathbb{P}-$a.s.

3) For any $\mathbb{R}^{\widetilde{d}'}-$valued, $\mathbf{F}^{X,\mathbb{P}}-$adapted process $\{K_s\}_{s\in[t,T]}$ with $\mathbb{P}-$a.s. right-continuous paths, there exists an $\mathbb{R}^{\widetilde{d}'}-$valued, $\mathbf{F}^X-$progressively measurable process $\{\widetilde{K}_s\}_{s\in[t,T]}$ such that $\big\{\omega \in \Omega^t : \widetilde{K}_s(\omega) \ne K_s(\omega) \text{ for some } s \in [t,T]\big\} \in \mathscr{N}^{\mathbb{P}}$. We call $\widetilde{K}$ the $(\mathbf{F}^X, \mathbb{P})-$version of K.

Proof: 1) Let $s \in [t,T]$ and let ξ be an $\mathbb{R}^{\widetilde{d}'}-$valued, $\mathcal{F}^{X,\mathbb{P}}_T-$measurable random variable with $\mathbb{E}_{\mathbb{P}}\big[|\xi|\big] < \infty$. For any $A \in \mathcal{F}^{X,\mathbb{P}}_s$, similar to Problem 2.7.3 of [19], there exists an $\widetilde{A} \in \mathcal{F}^X_s$ such that $A \,\Delta\, \widetilde{A} \in \mathscr{N}^{\mathbb{P}}$. Thus we can deduce that $\int_A \xi d\mathbb{P} = \int_{\widetilde{A}} \xi d\mathbb{P} = \int_{\widetilde{A}} \mathbb{E}_{\mathbb{P}}\big[\xi \big| \mathcal{F}^X_s\big] d\mathbb{P} = \int_A \mathbb{E}_{\mathbb{P}}\big[\xi \big| \mathcal{F}^X_s\big] d\mathbb{P}$, which implies that $\mathbb{E}_{\mathbb{P}}\big[\xi \big| \mathcal{F}^{X,\mathbb{P}}_s\big] = \mathbb{E}_{\mathbb{P}}\big[\xi \big| \mathcal{F}^X_s\big]$, $\mathbb{P}-$a.s.

2) Let $s \in [t,T]$ and let ξ be an $\mathbb{R}^{\widetilde{d}'}-$valued, $\mathcal{F}^{X,\mathbb{P}}_s-$measurable random variable. We first assume $\widetilde{d}' = 1$. For any $n \in \mathbb{N}$, we set $\xi_n := (\xi \wedge n) \vee (-n) \in \mathcal{F}^{X,\mathbb{P}}_s$ and see from part (1) that $\widetilde{\xi}_n := \mathbb{E}_{\mathbb{P}}\big[\xi_n \big| \mathcal{F}^X_s\big] = \mathbb{E}_{\mathbb{P}}\big[\xi_n \big| \mathcal{F}^{X,\mathbb{P}}_s\big] = \xi_n$, $\mathbb{P}-$a.s. Clearly, the random variable $\widetilde{\xi} := \Big(\varlimsup\limits_{n\to\infty} \widetilde{\xi}_n\Big) \mathbf{1}_{\big\{\varlimsup\limits_{n\to\infty} \widetilde{\xi}_n < \infty\big\}}$ is $\mathcal{F}^X_s-$measurable and satisfies $\widetilde{\xi} = \lim\limits_{n\to\infty} \xi_n = \xi$, $\mathbb{P}-$a.s. When $\widetilde{d}' > 1$, let ξ^i be the i-th component of ξ, $i = 1, \cdots, \widetilde{d}'$. We denote by $\widetilde{\xi}^i$ the real-valued, $\mathcal{F}^X_s-$measurable random variable such that $\widetilde{\xi}^i = \xi^i$, $\mathbb{P}-$a.s. Then $\widetilde{\xi} = (\widetilde{\xi}^1, \cdots, \widetilde{\xi}^{\widetilde{d}'})$ is an $\mathbb{R}^{\widetilde{d}'}-$valued, $\mathcal{F}^X_s-$measurable random variable such that $\widetilde{\xi} = \xi$, $\mathbb{P}-$a.s.

3) Let $\{K_s\}_{s\in[t,T]}$ be an $\mathbb{R}^{\widetilde{d}'}-$valued, $\mathbf{F}^{X,\mathbb{P}}-$adapted process with $\mathbb{P}-$a.s. right-continuous paths. Like part (2), it suffices to discuss the case of $\widetilde{d}' = 1$. For any $s \in \mathbb{Q}_{t,T} := \{s \in [t,T] : s - t \in \mathbb{Q}\} \cup \{T\}$, part (2) shows that there exists a real-valued, $\mathcal{F}^X_s-$measurable random variable $\mathcal{K}_s$ such that $\mathcal{K}_s = K_s$, $\mathbb{P}-$a.s. Set $\mathcal{N} := \big\{\omega \in \Omega^t : \text{the path } K_\cdot(\omega) \text{ is not right-continuous}\big\} \cup \Big(\underset{s\in\mathbb{Q}_{t,T}}{\cup} \{K_s \neq \mathcal{K}_s\}\Big) \in \mathscr{N}^{\mathbb{P}}$. Since

$$\widetilde{K}^n_s := \mathcal{K}_t \mathbf{1}_{\{s=t\}} + \sum_{i=1}^{\lceil n(T-t)\rceil} \mathcal{K}_{(t+\frac{i}{n})\wedge T} \mathbf{1}_{\{s\in(t+\frac{i-1}{n},(t+\frac{i}{n})\wedge T]\}}, \quad s \in [t,T]$$

is a real$-$valued, $\mathbf{F}^X-$progressively measurable process for any $n \in \mathbb{N}$, we see that $\widetilde{K}_s := \Big(\overline{\lim_{n\to\infty}} \widetilde{K}^n_s\Big)\mathbf{1}_{\big\{\overline{\lim}_{n\to\infty} K^n_s<\infty\big\}}$, $s \in [t,T]$ also defines a real$-$valued, $\mathbf{F}^X-$progressively measurable process.

Let $\omega \in \mathcal{N}^c$ and $s \in (t,T]$. For any $n \in \mathbb{N}$, since $s \in (s_n - \frac{1}{n}, s_n \wedge T]$ with $s_n := t + \dfrac{\lceil n(s-t)\rceil}{n}$, one has $\widetilde{K}^n_s(\omega) = \mathcal{K}_{s_n\wedge T}(\omega) = K_{s_n\wedge T}(\omega)$. Clearly, $\lim_{n\to\infty} s_n \wedge T = s$. As $n \to \infty$, the right-continuity of K shows that $\lim_{n\to\infty} \widetilde{K}^n_s(\omega) = \lim_{n\to\infty} K_{s_n\wedge T}(\omega) = K_s(\omega)$, which implies that $\mathcal{N}^c \subset \big\{\omega \in \Omega^t : \widetilde{K}_s(\omega) = K_s(\omega),\ \forall\, s \in [t,T]\big\}$. $\square$

Lemma A.5. *Let $0 \le t \le r \le s \le T < \infty$. For any $A \in \mathcal{F}^t_r$, $\widetilde{A} := \Pi^{t,T}_{t,s}(A) = \big\{\Pi^{t,T}_{t,s}(\omega) : \omega \in A\big\}$ belongs to $\mathcal{F}^{t,s}_r$ and satisfies $\big(\Pi^{t,T}_{t,s}\big)^{-1}(\widetilde{A}) = A$. Then $\Pi^{t,T}_{t,s}$ induces an one-to-one correspondence between $\mathcal{F}^t_r$ and $\mathcal{F}^{t,s}_r$.*

Proof: Let $\Lambda := \big\{A \in \mathcal{F}^t_r : \Pi^{t,T}_{t,s}(A) \in \mathcal{F}^{t,s}_r\big\}$. Clearly, $\Pi^{t,T}_{t,s}(\emptyset) = \emptyset$ and $\Pi^{t,T}_{t,s}(\Omega^t) = \Omega^{t,s}$, so $\emptyset, \Omega^t \in \Lambda$. Given $A \in \Lambda$, if $\Pi^{t,T}_{t,s}(A)$ intersected $\Pi^{t,T}_{t,s}(A^c)$ at some $\widetilde{\omega} \in \Omega^{t,s}$, there would exist $\omega \in A$ and $\omega' \in A^c$ such that $\widetilde{\omega} = \omega\big|_{[t,s]} = \omega'\big|_{[t,s]}$. It would then follow from Lemma 2.1 that $\omega' \in \omega \otimes_r \Omega^r \subset A$, a contradiction appears. So $\Pi^{t,T}_{t,s}(A) \cap \Pi^{t,T}_{t,s}(A^c) = \emptyset$. On the other hand, for any $\widetilde{\omega} \in \Omega^{t,s}$, the continuous path

$$\omega(s') := \widetilde{\omega}(s' \wedge s), \quad s' \in [t,T] \tag{A.4}$$

is either in A or in A^c, which shows that $\widetilde{\omega} = \Pi^{t,T}_{t,s}(\omega) \in \Pi^{t,T}_{t,s}(A) \cup \Pi^{t,T}_{t,s}(A^c)$. So $\Pi^{t,T}_{t,s}(A^c) = \Omega^{t,s} \backslash \Pi^{t,T}_{t,s}(A) \in \mathcal{F}^{t,s}_r$, i.e., $A^c \in \Lambda$. For any $\{A_n\}_{n\in\mathbb{N}} \subset \Lambda$, as $\Pi^{t,T}_{t,s}\Big(\underset{n\in\mathbb{N}}{\cup} A_n\Big) = \underset{n\in\mathbb{N}}{\cup} \Pi^{t,T}_{t,s}(A_n) \in \mathcal{F}^{t,s}_r$, we see that $\underset{n\in\mathbb{N}}{\cup} A_n \in \Lambda$. Hence, Λ is a $\sigma-$field of Ω^t.

Let $r' \in [t,r]$ and $\varepsilon \in \mathscr{B}(\mathbb{R}^d)$. For any $\widetilde{\omega} \in (B^{t,s}_{r'})^{-1}(\mathcal{E})$, we set the path $\omega \in \Omega^t$ as in (A.4). Since $B^t_{r'}(\omega) = \omega(r') = \widetilde{\omega}(r') = B^{t,s}_{r'}(\widetilde{\omega}) \in \mathcal{E}$, one can deduce that $\widetilde{\omega} = \Pi^{t,T}_{t,s}(\omega) \in \Pi^{t,T}_{t,s}\big((B^t_{r'})^{-1}(\mathcal{E})\big)$. On the other hand, for any $\widetilde{\omega}' \in \Pi^{t,T}_{t,s}\big((B^t_{r'})^{-1}(\mathcal{E})\big)$, there exists $\omega' \in \big(B^t_{r'}\big)^{-1}(\mathcal{E})$ such that $\widetilde{\omega}' = \Pi^{t,T}_{t,s}(\omega')$. So $B^{t,s}_{r'}(\widetilde{\omega}') = \widetilde{\omega}'(r') = \omega'(r') = B^t_{r'}(\omega') \in \mathcal{E}$, i.e., $\widetilde{\omega}' \in \big(B^{t,s}_{r'}\big)^{-1}(\mathcal{E})$. Then $\Pi^{t,T}_{t,s}\big((B^t_{r'})^{-1}(\mathcal{E})\big) = (B^{t,s}_{r'})^{-1}(\mathcal{E}) \in \mathcal{F}^{t,s}_r$, which shows that all the generating sets of $\mathcal{F}^t_r$ belong to Λ. It follows that $\Lambda = \mathcal{F}^t_r$. Moreover, for any $\widetilde{A}' \in \mathcal{F}^{t,s}_r$, since $\Pi^{t,T}_{t,s}$ is $\mathcal{F}^t_r/\mathcal{F}^{t,s}_r-$measurable by Lemma A.1, one has $A' = \big(\Pi^{t,T}_{t,s}\big)^{-1}(\widetilde{A}') \in \mathcal{F}^t_r$ and $\Pi^{t,T}_{t,s}(A') = \widetilde{A}'$. Hence we can then regard $\Pi^{t,T}_{t,s}$ as a surjective mapping from $\mathcal{F}^t_r$ to $\mathcal{F}^{t,s}_r$.

Next, let $A \in \mathcal{F}^t_r$ and set $\widetilde{A} := \Pi^{t,T}_{t,s}(A)$. Clearly, $A \subset \big(\Pi^{t,T}_{t,s}\big)^{-1}(\widetilde{A})$. For any $\omega \in \big(\Pi^{t,T}_{t,s}\big)^{-1}(\widetilde{A})$, $\Pi^{t,T}_{t,s}(\omega) \in \widetilde{A} = \Pi^{t,T}_{t,s}(A)$. So there exists a $\omega' \in A$ such that $\Pi^{t,T}_{t,s}(\omega) = \Pi^{t,T}_{t,s}(\omega')$. Applying Lemma 2.1 again yields that $\omega \in \omega' \otimes_r \Omega^r \subset A$. Thus $A = \big(\Pi^{t,T}_{t,s}\big)^{-1}(\widetilde{A})$, which implies that the mapping $\Pi^{t,T}_{t,s}$ from $\mathcal{F}^t_r$ to $\mathcal{F}^{t,s}_r$ is also injective. $\square$

Lemma A.6. *For any $0 \le t \le T < \infty$, $\mathscr{B}(\Omega^t) = \sigma(\Theta^t_T) = \sigma\big\{O_\delta(\widehat{\omega}^t_j) : \delta \in \mathbb{Q}_+,\ j \in \mathbb{N}\big\}$.*

Proof: We only need to show that any open subset $\mathcal{O}$ of Ω^t under $\|\cdot\|_{t,T}$ is a union of some open balls in Θ^t_T: For any $j \in \mathbb{N}$, if $\widehat{\omega}^t_j \notin \mathcal{O}$, we set $O_j := \emptyset$; otherwise, we choose a $q_j \in \mathbb{Q}_+ \cap (\widetilde{\delta}_j/2, \widetilde{\delta}_j)$ $\Big($with $\widetilde{\delta}_j := \text{dist}\big(\widehat{\omega}^t_j, \mathcal{O}^c\big) = \inf_{\omega\in\mathcal{O}^c} \|\omega - \widehat{\omega}^t_j\|_{t,T}\Big)$ and set $O_j := O_{q_j}(\widehat{\omega}^t_j) \subset O_{\widetilde{\delta}_j}(\widehat{\omega}^t_j) \subset \mathcal{O}$. Given $\omega \in \mathcal{O}$, let $\delta := \text{dist}\big(\omega, \mathcal{O}^c\big)$. There exists an $J \in \mathbb{N}$ such that $\widehat{\omega}^t_J \in O_{\delta/3}(\omega) \subset \mathcal{O}$. As $\text{dist}\big(\widehat{\omega}^t_J, \mathcal{O}^c\big) \ge \text{dist}\big(\omega, \mathcal{O}^c\big) - \big\|\widehat{\omega}^t_J - \omega\big\|_{t,T} > \frac{2}{3}\delta$, we see that $q_J > \delta_J/2 > \delta/3$ and thus $\omega \in O_{\delta/3}\big(\widehat{\omega}^t_J\big) \subset O_{q_J}\big(\widehat{\omega}^t_J\big) = O_J$. It follows that $\mathcal{O} = \underset{j\in\mathbb{N}}{\cup} O_j$. $\square$

Lemma A.7. *Given $0 \le t \le T < \infty$, let $\mathbb{P}$ be a probability on $\big(\Omega^t, \mathscr{B}(\Omega^t)\big)$. For any $A \in \mathscr{B}(\Omega^t)$ and $\varepsilon > 0$, there exist a closed subset F and an open subset O of Ω^t such that $F \subset A \subset O$ and that $\mathbb{P}(A \backslash F) \vee \mathbb{P}(O \backslash A) < \varepsilon$.*

Proof: Let $\Lambda := \{A \in \mathscr{B}(\Omega^t)$: for any $\varepsilon > 0$, there exist a closed F and an open O of Ω^t such that $F \subset A \subset O$ and that $\mathbb{P}(A\backslash F) \vee \mathbb{P}(O\backslash A) < \varepsilon\}$. Clearly, $\emptyset, \Omega^t \in \Lambda$ as they are both open and closed. It is also easy to see that $A^c \in \Lambda$ if $A \in \Lambda$. Given $\{A_n\}_{n\in\mathbb{N}} \subset \Lambda$, let $\varepsilon > 0$. For any $n \in \mathbb{N}$, there exist a closed F_n and an open O_n such that $F_n \subset A_n \subset O_n$ and that $\mathbb{P}(A_n\backslash F_n) \vee \mathbb{P}(O_n\backslash A_n) < \varepsilon 2^{-(1+n)}$. The open set $O := \underset{n\in\mathbb{N}}{\cup} O_n$ contains $\widetilde{A} := \underset{n\in\mathbb{N}}{\cup} A_n$ and satisfies $\mathbb{P}(O\backslash \widetilde{A}) \le \sum\limits_{n\in\mathbb{N}} \mathbb{P}(O_n\backslash \widetilde{A}) \le \sum\limits_{n\in\mathbb{N}} \mathbb{P}(O_n\backslash A_n) < \varepsilon/2$. Similarly, it holds for $F_o = \underset{n\in\mathbb{N}}{\cup} F_n$ that $\mathbb{P}(\widetilde{A}\backslash F_o) \le \sum\limits_{n\in\mathbb{N}} \mathbb{P}(A_n\backslash F_n) < \varepsilon/2$. We can find an $N \in \mathbb{N}$ such that $\mathbb{P}\Big(\overset{N}{\underset{n=1}{\cup}} F_n\Big) > \mathbb{P}(F_o) - \varepsilon/2$. Then $F := \overset{N}{\underset{n=1}{\cup}} F_n$ is a closed set included in $\widetilde{A}$ such that $\mathbb{P}(\widetilde{A}\backslash F) \le \mathbb{P}(\widetilde{A}\backslash F_o) + \mathbb{P}(F_o\backslash F) < \varepsilon$, which shows $\widetilde{A} = \underset{n\in\mathbb{N}}{\cup} A_n \in \Lambda$. Thus Λ is a $\sigma-$field of Ω^t.

For any $\delta \in \mathbb{Q}_+$, $j \in \mathbb{N}$ and $\varepsilon > 0$, since $O_\delta(\widehat{\omega}^t_j) = \underset{k\in\mathbb{N}}{\cup} \overline{O}_{\delta-\delta/k}(\widehat{\omega}^t_j)$, there exists a $k \in \mathbb{N}$ such that $\mathbb{P}\big(\overline{O}_{\delta-\delta/k}(\widehat{\omega}^t_j)\big) > \mathbb{P}\big(O_\delta(\widehat{\omega}^t_j)\big) - \varepsilon$. So $\Theta^t_T = \big\{O_\delta(\widehat{\omega}^t_j) : \delta \in \mathbb{Q}_+, j \in \mathbb{N}\big\} \subset \Lambda$. Lemma A.6 then implies that $\mathscr{B}(\Omega^t) = \sigma(\Theta^t_T) \subset \Lambda \subset \mathscr{B}(\Omega^t)$, proving the lemma. □

Lemma A.8. *Given $0 \le t \le s \le T < \infty$, let $\mathbb{P}$ be a probability on $\big(\Omega^t, \mathscr{B}(\Omega^t)\big)$. For any $A \in \mathcal{F}^t_s$ and $\varepsilon > 0$, the countable subset $\Theta^t_s = \big\{O^s_\delta(\widehat{\omega}^t_j) : \delta \in \mathbb{Q}_+, j \in \mathbb{N}\big\}$ of $\mathcal{F}^t_s$ has a sequence $\big\{O_i\big\}_{i\in\mathbb{N}}$ such that $A \subset \underset{i\in\mathbb{N}}{\cup} O_i$ and that $\mathbb{P}(A) > \mathbb{P}\Big(\underset{i\in\mathbb{N}}{\cup} O_i\Big) - \varepsilon$.*

Proof: Let $A \in \mathcal{F}^t_s$ and $\varepsilon > 0$. We consider the induced probability $\widehat{\mathbb{P}} := \mathbb{P} \circ \big(\Pi^{t,T}_{t,s}\big)^{-1}$ on $\big(\Omega^{t,s}, \mathscr{B}(\Omega^{t,s})\big)$. Since $\widetilde{A} = \Pi^{t,T}_{t,s}(A) \in \mathcal{F}^{t,s}_s$ by Lemma A.5, applying Lemma A.7 with $T = s$ shows that there exists an open subset O of $\Omega^{t,s}$ such that $\widetilde{A} \subset O$ and $\widehat{\mathbb{P}}(O) - \widehat{\mathbb{P}}(\widetilde{A}) < \varepsilon$.

For any $j \in \mathbb{N}$, set $\widetilde{\omega}_j := \widehat{\omega}^t_j\big|_{[t,s]} \in \Omega^{t,s}$. Given $\widetilde{\omega} \in \Omega^{t,s}$ and $\widetilde{\varepsilon} > 0$, still setting the path $\omega \in \Omega^t$ as in (A.4), we can find an $J \in \mathbb{N}$ such that $\big\|\omega - \widehat{\omega}^t_J\big\|_{t,T} < \widetilde{\varepsilon}$. It follows that $\|\widetilde{\omega} - \widetilde{\omega}_J\|_{t,s} = \big\|\omega - \widehat{\omega}^t_J\big\|_{t,s} \le \big\|\omega - \widehat{\omega}^t_J\big\|_{t,T} < \widetilde{\varepsilon}$, which shows that $\{\widetilde{\omega}_j\}_{j\in\mathbb{N}}$ is a dense subset of $\Omega^{t,s}$. Similar to the proof of Lemma A.6, one can show that O is the union of some open balls in $\widetilde{\Theta} := \big\{O_\delta(\widetilde{\omega}_j) : \delta \in \mathbb{Q}_+, j \in \mathbb{N}\big\}$.

For any $\delta \in \mathbb{Q}_+$ and $j \in \mathbb{N}$, one can deduce that

$$\Pi^{t,T}_{t,s}\big(O^s_\delta(\widehat{\omega}^t_j)\big) = \Big\{\Pi^{t,T}_{t,s}(\omega) : \omega \in \Omega^t,\ \|\omega - \widehat{\omega}^t_j\|_{t,s} < \delta\Big\} = \Big\{\widetilde{\omega} \in \Omega^{t,s} \colon \|\widetilde{\omega} - \widetilde{\omega}_j\|_{t,s} < \delta\Big\} = O_\delta(\widetilde{\omega}_j).$$

Since $\Pi^{t,T}_{t,s}$ induces an one-to-one correspondence between $\mathcal{F}^t_s$ and $\mathcal{F}^{t,s}_s$ by Lemma A.5, we see that $\big(\Pi^{t,T}_{t,s}\big)^{-1}(\widetilde{A}) = A$ and Lemma A.1 implies that

$$\big(\Pi^{t,T}_{t,s}\big)^{-1}\big(O_\delta(\widetilde{\omega}_j)\big) = O^s_\delta(\widehat{\omega}^t_j) \text{ is an open set of } \Omega^t. \tag{A.5}$$

Thus, $\big(\Pi^{t,T}_{t,s}\big)^{-1}(O)$ is the union of some sequence $\big\{O_i\big\}_{i\in\mathbb{N}}$ in $\big(\Pi^{t,T}_{t,s}\big)^{-1}(\widetilde{\Theta}) = \Big\{\big(\Pi^{t,T}_{t,s}\big)^{-1}\big(O_\delta(\widetilde{\omega}_j)\big) : \delta \in \mathbb{Q}_+, j \in \mathbb{N}\Big\} = \Theta^t_s$. It follows that $A = \big(\Pi^{t,T}_{t,s}\big)^{-1}(\widetilde{A}) \subset \big(\Pi^{t,T}_{t,s}\big)^{-1}(O) = \underset{i\in\mathbb{N}}{\cup} O_i$ and that

$$\mathbb{P}(A) = \widehat{\mathbb{P}}(\widetilde{A}) > \widehat{\mathbb{P}}(O) - \varepsilon = \mathbb{P}\Big(\big(\Pi^{t,T}_{t,s}\big)^{-1}(O)\Big) - \varepsilon = \mathbb{P}\Big(\underset{i\in\mathbb{N}}{\cup} O_i\Big) - \varepsilon. \qquad □$$

Lemma A.9. *It holds for any $\omega \in \Omega$ that $Y_*(\omega) = \sup\limits_{r\in[0,T]} \big|Y_r(\omega)\big| < \infty$.*

Proof: Let us fist show $Y_*(\mathbf{0}) < \infty$: Assume not, then $\lim\limits_{n\to\infty} \uparrow \big|Y_{r_n}(\mathbf{0})\big| = \infty$ for some sequence $\{r_n\}_{n\in\mathbb{N}}$ of $[0,T]$, from which one can pick up a convergent subsequence (we still denote it by $\{r_n\}_{n\in\mathbb{N}}$) with limit $r_* \in [0,T]$. If $\{r_n\}_{n\in\mathbb{N}}$ had a subsequence $\{r'_n\}_{n\in\mathbb{N}} \subset [r_*, T]$, then the RCLL property of path $Y_\cdot(\mathbf{0})$ by Remark 3.1 (1) would imply that $|Y_{r_*}(\mathbf{0})| = \lim\limits_{n\to\infty} \uparrow \big|Y_{r'_n}(\mathbf{0})\big| = \infty$. A contradiction appear. On the other hand, if $\{r_n\}_{n\in\mathbb{N}}$ had a subsequence $\{\widetilde{r}_n\}_{n\in\mathbb{N}} \subset [0, r_*]$, then one would have $\lim\limits_{n\to\infty} \uparrow \big|Y_{\widetilde{r}_n}(\mathbf{0})\big| = \infty$. For any $n \in \mathbb{N}$, (3.1) implies that $Y_{\widetilde{r}_1}(\mathbf{0}) - Y_{\widetilde{r}_n}(\mathbf{0}) \le \rho_0(\widetilde{r}_n - \widetilde{r}_1) \le \rho_0(r_* - \widetilde{r}_1)$.

This together with Remark 3.1 (1) shows that $Y_{\widetilde{r}_1}(\mathbf{0}) - \rho_0(r_* - \widetilde{r}_1) \le \lim_{n\to\infty} Y_{\widetilde{r}_n}(\mathbf{0}) \le Y_{r_*}(\mathbf{0})$, which contradicts with $\lim_{n\to\infty}\uparrow \big|Y_{\widetilde{r}_n}(\mathbf{0})\big| = \infty$. Hence, $Y_*(\mathbf{0}) < \infty$.

Given $\omega\in\Omega$, since $|Y_r(\omega) - Y_r(\mathbf{0})| \le \rho_0\big(\|\omega\|_{0,r}\big)$, $\forall\, r\in[0,T]$ by (3.1), we can deduce that $Y_*(\omega) = \sup_{r\in[0,T]} \big|Y_r(\omega)\big| \le \sup_{r\in[0,T]} \big|Y_r(\mathbf{0})\big| + \rho_0\big(\|\omega\|_{0,T}\big) = Y_*(\mathbf{0}) + \rho_0\big(\|\omega\|_{0,T}\big) < \infty$. □

Lemma A.10. *Given $0 \le t \le s \le T$ and $\widetilde{d} \in \mathbb{N}$, for any sequence $\{\xi_i\}_{i\in\mathbb{N}}$ of $\mathbb{R}^{\widetilde{d}}-$valued, $\mathcal{F}^t_T-$measurable random variables that converges to 0 in probability $\mathbb{P}^t_0$, we can find a subsequence $\big\{\widehat{\xi}_i\big\}_{i\in\mathbb{N}}$ of it such that for $\mathbb{P}^t_0-$a.s. $\omega\in\Omega^t$, $\big\{\widehat{\xi}^{s,\omega}_i\big\}_{i\in\mathbb{N}}$ converges to 0 in probability $\mathbb{P}^s_0$.*

Proof: Let $\{\xi_i\}_{i\in\mathbb{N}}$ be a sequence of $\mathbb{R}^{\widetilde{d}}-$valued, $\mathcal{F}^t_T-$measurable random variables that converges to 0 in probability $\mathbb{P}^t_0$, i.e.

$$\lim_{i\to\infty}\downarrow \mathbb{E}_t\big[\mathbf{1}_{\{|\xi_i|>1/n\}}\big] = \lim_{i\to\infty}\downarrow \mathbb{P}^t_0\big(|\xi_i| > 1/n\big) = 0, \quad \forall\, n\in\mathbb{N}. \tag{A.6}$$

In particular, $\lim_{i\to\infty}\downarrow \mathbb{E}_t\big[\mathbf{1}_{\{|\xi_i|>1\}}\big] = 0$ allows us to extract a subsequence $S_1 = \big\{\xi^1_i\big\}_{i\in\mathbb{N}}$ from $\{\xi_i\}_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty} \mathbf{1}_{\{|\xi^1_i|>1\}} = 0$, $\mathbb{P}^t_0-$a.s. Clearly, S_1 also satisfies (A.6). Then as $\lim_{i\to\infty}\downarrow \mathbb{E}_t\big[\mathbf{1}_{\{|\xi^1_i|>1/2\}}\big] = 0$, we can find a subsequence $S_2 = \big\{\xi^2_i\big\}_{i\in\mathbb{N}}$ of S_1 such that $\lim_{i\to\infty} \mathbf{1}_{\{|\xi^2_i|>1/2\}} = 0$, $\mathbb{P}^t_0-$a.s. Inductively, for each $n\in\mathbb{N}$ we can select a subsequence $S_{n+1} = \{\xi^{n+1}_i\}_{i\in\mathbb{N}}$ of $S_n = \{\xi^n_i\}_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty} \mathbf{1}_{\big\{|\xi^{n+1}_i| > \frac{1}{n+1}\big\}} = 0$, $\mathbb{P}^t_0-$a.s.

For any $i\in\mathbb{N}$, we set $\widehat{\xi}_i := \xi^i_i$, which belongs to S_n for $n = 1,\cdots,i$. Given $n\in\mathbb{N}$, since $\big\{\widehat{\xi}_i\big\}_{i=n}^\infty \subset S_n$, it holds $\mathbb{P}^t_0-$a.s. that $\lim_{i\to\infty} \mathbf{1}_{\big\{|\widehat{\xi}_i| > \frac{1}{n}\big\}} = 0$. Then Bound Convergence Theorem, (2.6) and Lemma 2.4 imply that

$$0 = \lim_{i\to\infty} \mathbb{E}_t\Big[\mathbf{1}_{\{|\widehat{\xi}_i|>1/n\}}\big|\mathcal{F}^t_s\Big](\omega) = \lim_{i\to\infty} \mathbb{E}_s\Big[\big(\mathbf{1}_{\{|\widehat{\xi}_i|>1/n\}}\big)^{s,\omega}\Big] \tag{A.7}$$

holds for all $\omega\in\Omega^t$ except on some $\mathcal{N}_n \in \overline{\mathscr{N}}^t$. Let $\omega \in \Big(\underset{n\in\mathbb{N}}{\cup}\mathcal{N}_n\Big)^c$. For any $n\in\mathbb{N}$, one can deduce that

$$\big(\mathbf{1}_{\{|\widehat{\xi}_i|>1/n\}}\big)^{s,\omega}(\widetilde{\omega}) = \mathbf{1}_{\big\{|\widehat{\xi}_i(\omega\otimes_s\widetilde{\omega}))|>1/n\big\}} = \mathbf{1}_{\big\{\big|\widehat{\xi}^{s,\omega}_i(\widetilde{\omega})\big|>1/n\big\}} = \Big(\mathbf{1}_{\big\{|\widehat{\xi}^{s,\omega}_i|>1/n\big\}}\Big)(\widetilde{\omega}), \quad \forall\, \widetilde{\omega}\in\Omega^s,$$

which together with (A.7) leads to that $\lim_{i\to\infty} \mathbb{P}^s_0\Big(|\widehat{\xi}^{s,\omega}_i| > 1/n\Big) = \lim_{i\to\infty} \mathbb{E}_s\Big[\big(\mathbf{1}_{\{|\widehat{\xi}_i|>1/n\}}\big)^{s,\omega}\Big] = 0$. □

Lemma A.11. *Given $t\in[0,T]$ and a metric space $\mathbb{M}$, let $\{X_s\}_{s\in[t,T]}$ be an $\mathbb{R}^d-$valued process on Ω^t such that all its paths are continuous and starting from 0. Define a mapping $\Psi^X\colon [t,T]\times\Omega^t \to [t,T]\times\Omega^t$ by $\Psi^X(r,\omega) := \big(r, X(\omega)\big)$, $\forall\,(r,\omega)\in[t,T]\times\Omega^t$. Clearly, $\sigma^X := (\Psi^X)^{-1}(\mathscr{P}^t) = \{(\Psi^X)^{-1}(\mathcal{D})\colon \mathcal{D}\in\mathscr{P}^t\}$ is a $\sigma-$field of $[t,T]\times\Omega^t$. If an $\mathbb{M}-$valued process K is adapted to the induced filtration $X^{-1}(\mathbf{F}^t) = \big\{X^{-1}(\mathcal{F}^t_s) := \{X^{-1}(A) : A\in\mathcal{F}^t_s\}\big\}_{s\in[t,T]}$ and all its paths are left-continuous, then K is σ^X-measurable. In particular, X is σ^X-measurable.*

Proof: Let $x_0\in\mathbb{R}^d$ and $\delta>0$. Since the path $K_\cdot(\omega)$ is left-continuous for each $\omega\in\Omega^t$, one can deduce that

$$\big\{(s,\omega)\in[t,T]\times\Omega^t : K(s,\omega)\in\overline{O}_\delta(x_0)\big\} = \underset{n\in\mathbb{N}}{\cap}\ \underset{m\in\mathbb{N}}{\cup}\ \underset{i\ge m}{\cap}\ \overset{i-1}{\underset{j=0}{\cup}} \big\{(s,\omega)\in[t^i_j, t^i_{j+1}]\times\Omega^t : K_{t^i_j}(\omega)\in\overline{O}_{\delta+1/n}(x_0)\big\},$$

where $t^i_j := t + \frac{j}{i}(T-t)$. For any $n,i\in\mathbb{N}$ and $j = 0,\cdots,i-1$, since $\big\{K_{t^i_j}\in\overline{O}_{\delta+1/n}(x_0)\big\} = X^{-1}\big(A^n_{i,j}\big)$ for some $A^n_{i,j}\in\mathcal{F}^t_{t^i_j}$, and since $[t^i_j, t^i_{j+1}]\times A^n_{i,j}\in\mathscr{P}^t$, we see that

$$\big\{(s,\omega)\in[t^i_j,t^i_{j+1}]\times\Omega^t\colon K_{t^i_j}(\omega)\in\overline{O}_{\delta+1/n}(x_0)\big\} = \big\{(s,\omega)\in[t^i_j,t^i_{j+1}]\times\Omega^t\colon X(\omega)\in A^n_{i,j}\big\} = (\Psi^X)^{-1}\big([t^i_j,t^i_{j+1}]\times A^n_{i,j}\big)\in\sigma^X.$$

So $\big\{(s,\omega)\in[t,T]\times\Omega^t : K(s,\omega)\in\overline{O}_\delta(x_0)\big\}\in\sigma^X$, which shows that $\overline{O}_\delta(x_0)\in\Lambda := \Big\{\mathcal{E}\subset\mathbb{R}^d : \big\{(s,\omega)\in[t,T]\times\Omega^t : K(s,\omega)\in\mathcal{E}\big\}\in\sigma^X\Big\}$. Clearly, Λ is a $\sigma-$field on $\mathbb{R}^d$, it follows that $\mathscr{B}(\mathbb{R}^d)\subset\Lambda$. To wit, K is σ^X-measurable.

For any $s\in[t,T]$ and $\mathcal{E}\in\mathscr{B}(\mathbb{R}^d)$, since $A_s := (B^t_s)^{-1}(\mathcal{E})\in\mathcal{F}^t_s$,

$$X_s^{-1}(\mathcal{E}) = \{\omega\in\Omega^t : X_s(\omega)\in\mathcal{E}\} = \{\omega\in\Omega^t : B^t_s(X(\omega))\in\mathcal{E}\} = \{\omega\in\Omega^t : X(\omega)\in A_s\} = X^{-1}(A_s)\in X^{-1}\big(\mathcal{F}^t_s\big),$$

which shows that X is in particular adapted to the filtration $X^{-1}(\mathbf{F}^t)$. By its continuity, X is σ^X-measurable. □

Lemma A.12. *Let $(t,\omega)\in[0,T]\times\Omega$ and let μ be a $\mathcal{U}_t-$control considered in Section 6.*
(1) It holds for any $s\in[t,T]$ that $\mathcal{F}_s^{\mathbb{P}^{t,\omega,\mu}}\subset\mathcal{G}_s^{X^{t,\omega,\mu}}$, and $\mathfrak{p}^{t,\omega,\mu}$ coincides with $\mathbb{P}^{t,\omega,\mu}$ on $\mathcal{F}_T^{\mathbb{P}^{t,\omega,\mu}}$.
(2) The $\sigma-$field $\mathcal{G}_T^{X^{t,\omega,\mu}}$ is complete under $\mathfrak{p}^{t,\omega,\mu}$, and $\mathscr{N}^{\mathbb{P}^{t,\omega,\mu}}\subset\mathscr{N}^{\mathfrak{p}^{t,\omega,\mu}}:=\big\{A\in\mathcal{G}_T^{X^{t,\omega,\mu}}:\mathfrak{p}^{t,\omega,\mu}(A)=0\big\}\subset\mathcal{G}_t^{X^{t,\omega,\mu}}$.

Proof: 1) Set $\vartheta=(t,\omega,\mu)$ and let $s\in[t,T]$. For any $\mathcal{N}\in\mathscr{N}^{\mathbb{P}^\vartheta}$, there exists an $A\in\mathcal{F}_T^t$ with $\mathbb{P}^\vartheta(A)=0$ such that $\mathcal{N}\subset A$. By (6.6), $\big(X^\vartheta\big)^{-1}(A)\in\overline{\mathcal{F}}_T^t$ and thus $\mathbb{P}_0^t\big((X^\vartheta)^{-1}(A)\big)=\mathbb{P}^\vartheta(A)=0$. Then, as a subset of $\big(X^\vartheta\big)^{-1}(A)$,

$$\big(X^\vartheta\big)^{-1}\big(\mathcal{N}\big)\in\overline{\mathscr{N}}^t\subset\overline{\mathcal{F}}_s^t. \tag{A.8}$$

So $\mathscr{N}^{\mathbb{P}^\vartheta}\subset\mathcal{G}_s^{X^\vartheta}$, which already contains $\mathcal{F}_s^t$ by (6.6). It follows that $\mathcal{F}_s^{\mathbb{P}^\vartheta}\subset\mathcal{G}_s^{X^\vartheta}$.

Given $A\in\mathcal{F}_T^{\mathbb{P}^\vartheta}\subset\mathcal{G}_T^{X^\vartheta}$, we know (see e.g. Proposition 11.4 of [34]) that $A=\widetilde{A}\cup\mathcal{N}$ for some $\widetilde{A}\in\mathcal{F}_T^t$ and $\mathcal{N}\in\mathscr{N}^{\mathbb{P}^\vartheta}$. Since $(X^\vartheta)^{-1}\big(\widetilde{A}\big)\in\overline{\mathcal{F}}_T^t$ by (6.6) and since $(X^\vartheta)^{-1}\big(\mathcal{N}\big)\in\overline{\mathscr{N}}^t$ by (A.8), one can deduce that

$$\mathfrak{p}^\vartheta(A)=\mathbb{P}_0^t\big((X^\vartheta)^{-1}(A)\big)=\mathbb{P}_0^t\Big((X^\vartheta)^{-1}\big(\widetilde{A}\big)\cup(X^\vartheta)^{-1}\big(\mathcal{N}\big)\Big)=\mathbb{P}_0^t\Big((X^\vartheta)^{-1}\big(\widetilde{A}\big)\Big)=\mathbb{P}^\vartheta\big(\widetilde{A}\big)=\mathbb{P}^\vartheta(A).$$

2) Let $\mathfrak{N}\subset A$ for some $A\in\mathcal{G}_T^{X^\vartheta}$ with $\mathfrak{p}^\vartheta(A)=0$. As $(X^\vartheta)^{-1}(\mathfrak{N})\subset(X^\vartheta)^{-1}(A)\in\overline{\mathcal{F}}_T^t$ and $0=\mathfrak{p}^\vartheta(A)=\mathbb{P}_0^t\Big((X^\vartheta)^{-1}(A)\Big)$, we see that

$$(X^\vartheta)^{-1}(\mathfrak{N})\in\overline{\mathscr{N}}^t. \tag{A.9}$$

In particular, $\mathfrak{N}\in\mathcal{G}_T^{X^\vartheta}$, so the $\sigma-$field $\mathcal{G}_T^{X^\vartheta}$ is complete under $\mathfrak{p}^\vartheta$. Then it easily follows from part (1) that $\mathscr{N}^{\mathbb{P}^\vartheta}=\big\{A\in\mathcal{F}_T^{\mathbb{P}^\vartheta}:\mathbb{P}^\vartheta(A)=0\big\}=\big\{A\in\mathcal{F}_T^{\mathbb{P}^\vartheta}:\mathfrak{p}^\vartheta(A)=0\big\}\subset\big\{A\in\mathcal{G}_T^{X^\vartheta}:\mathfrak{p}^\vartheta(A)=0\big\}=\mathscr{N}^{\mathfrak{p}^\vartheta}$. Moreover, taking $\mathfrak{N}=A$ for any $A\in\mathcal{G}_T^{X^\vartheta}$ with $\mathfrak{p}^\vartheta(A)=0$ in (A.9) shows that $\mathscr{N}^{\mathfrak{p}^\vartheta}\subset\mathcal{G}_t^{X^\vartheta}$. □

Lemma A.13. *Let $0\le t\le s\le T$ and define $\widehat{\Pi}_s^t(r,\omega):=\big(r,\Pi_s^t(\omega)\big)$, $\forall\,(r,\omega)\in[s,T]\times\Omega^t$. For any $r\in[s,T]$ and $\mathcal{D}\in\mathscr{B}([s,r])\otimes\mathcal{F}_r^s$, we have $(\widehat{\Pi}_s^t)^{-1}(\mathcal{D})\in\mathscr{B}([s,r])\otimes\mathcal{F}_r^t$ and $\big(dr\times d\mathbb{P}_0^t\big)\big((\widehat{\Pi}_s^t)^{-1}(\mathcal{D})\big)=\big(dr\times d\mathbb{P}_0^s\big)(\mathcal{D})$.*

Proof: Given $r\in[s,T]$, for any $\mathcal{E}\in\mathscr{B}([s,r])$ and $A\in\mathcal{F}_r^s$, applying Lemma A.1 with $S=T$ yields that

$$(\widehat{\Pi}_s^t)^{-1}\big(\mathcal{E}\times A\big)=\big\{(r,\omega)\in[s,T]\times\Omega^t:\big(r,\Pi_s^t(\omega)\big)\in\mathcal{E}\times A\big\}=\mathcal{E}\times(\Pi_s^t)^{-1}(A)\in\mathscr{B}([s,r])\otimes\mathcal{F}_r^t. \tag{A.10}$$

So all rectangular measurable sets of $\mathscr{B}([s,r])\otimes\mathcal{F}_r^s$ belongs to $\Lambda:=\big\{\mathcal{D}\subset[s,r]\times\Omega^s:(\widehat{\Pi}_s^t)^{-1}(\mathcal{D})\in\mathscr{B}([s,r])\otimes\mathcal{F}_r^t\big\}$, which is a $\sigma-$field of $[s,r]\times\Omega^s$. It follows that $\mathscr{B}([s,r])\otimes\mathcal{F}_r^s\subset\Lambda$, i.e.,

$$(\widehat{\Pi}_s^t)^{-1}(\mathcal{D})\in\mathscr{B}([s,r])\otimes\mathcal{F}_r^t,\quad\forall\,\mathcal{D}\in\mathscr{B}([s,r])\otimes\mathcal{F}_r^s.$$

Next, we show that $\big(dr\times d\mathbb{P}_0^t\big)\circ(\widehat{\Pi}_s^t)^{-1}=\big(dr\times d\mathbb{P}_0^s\big)$ on $\mathscr{B}([s,T])\otimes\mathcal{F}_T^s$: For any $\widetilde{\mathcal{E}}\in\mathscr{B}\big([s,T]\big)$ and $\widetilde{A}\in\mathcal{F}_T^s$, using (A.10) with $r=T$ and (A.1) with $S=T$ gives that

$$\big(dr\times d\mathbb{P}_0^t\big)\big((\widehat{\Pi}_s^t)^{-1}(\widetilde{\mathcal{E}}\times\widetilde{A})\big)=\big(dr\times d\mathbb{P}_0^t\big)\big(\widetilde{\mathcal{E}}\times(\Pi_s^t)^{-1}(\widetilde{A})\big)=|\widetilde{\mathcal{E}}|\times\mathbb{P}_0^t\big((\Pi_s^t)^{-1}(\widetilde{A})\big)=|\widetilde{\mathcal{E}}|\times\mathbb{P}_0^s(\widetilde{A})=\big(dr\times d\mathbb{P}_0^s\big)(\widetilde{\mathcal{E}}\times\widetilde{A}),$$

where $|\widetilde{\mathcal{E}}|$ denotes the Lebesgue measure of $\widetilde{\mathcal{E}}$. Thus the collection $\mathfrak{C}_s$ of all rectangular measurable sets of $\mathscr{B}\big([s,T]\big)\otimes\mathcal{F}_T^s$ is contained in $\widetilde{\Lambda}:=\big\{\mathcal{D}\subset[s,T]\times\Omega^s:\big(dr\times d\mathbb{P}_0^s\big)(\mathcal{D})=\big(dr\times d\mathbb{P}_0^t\big)\big((\widehat{\Pi}_s^t)^{-1}(\mathcal{D})\big)\big\}$. In particular, $\emptyset\times\emptyset\in\widetilde{\Lambda}$ and $[s,T]\times\Omega^s\in\widetilde{\Lambda}$. For any $\mathcal{D}\in\widetilde{\Lambda}$, one can deduce that

$$\begin{aligned}\big(dr\times d\mathbb{P}_0^s\big)\big(([s,T]\times\Omega^s)\backslash\mathcal{D}\big)&=\big(dr\times d\mathbb{P}_0^s\big)\big([s,T]\times\Omega^s\big)-\big(dr\times d\mathbb{P}_0^s\big)(\mathcal{D})=\big(dr\times d\mathbb{P}_0^t\big)\big((\widehat{\Pi}_s^t)^{-1}\big([s,T]\times\Omega^s\big)\big)-\big(dr\times d\mathbb{P}_0^t\big)\big((\widehat{\Pi}_s^t)^{-1}(\mathcal{D})\big)\\&=\big(dr\times d\mathbb{P}_0^t\big)\big((\widehat{\Pi}_s^t)^{-1}\big([s,T]\times\Omega^s\big)-(\widehat{\Pi}_s^t)^{-1}(\mathcal{D})\big)=\big(dr\times d\mathbb{P}_0^t\big)\big((\widehat{\Pi}_s^t)^{-1}\big(([s,T]\times\Omega^s)\backslash\mathcal{D}\big)\big).\end{aligned}$$

On the other hand, for any pairwisely-disjoint sequence $\{\mathcal{D}_n\}_{n\in\mathbb{N}}$ of $\widetilde{\Lambda}$ (i.e. $\mathcal{D}_m\cap\mathcal{D}_n=\emptyset$ if $m\neq n$), it is clear that $\big\{(\widehat{\Pi}_s^t)^{-1}(\mathcal{D}_n)\big\}_{n\in\mathbb{N}}$ is also a pairwisely-disjoint sequence. It follows that

$$\begin{aligned}\big(dr\times d\mathbb{P}_0^s\big)\Big(\underset{n\in\mathbb{N}}{\cup}\mathcal{D}_n\Big)&=\sum_{n\in\mathbb{N}}\big(dr\times d\mathbb{P}_0^s\big)\big(\mathcal{D}_n\big)=\sum_{n\in\mathbb{N}}\big(dr\times d\mathbb{P}_0^t\big)\big((\widehat{\Pi}_s^t)^{-1}(\mathcal{D}_n)\big)\\&=\big(dr\times d\mathbb{P}_0^t\big)\Big(\underset{n\in\mathbb{N}}{\cup}(\widehat{\Pi}_s^t)^{-1}(\mathcal{D}_n)\Big)=\big(dr\times d\mathbb{P}_0^t\big)\Big((\widehat{\Pi}_s^t)^{-1}\big(\underset{n\in\mathbb{N}}{\cup}\mathcal{D}_n\big)\Big),\end{aligned}$$

thus $\widetilde{\Lambda}$ is a Dynkin system. Since $\mathfrak{C}_s$ is closed under intersection, the Dynkin System Theorem shows that $\mathscr{B}\big([s,T]\big)\otimes\mathcal{F}_T^s=\sigma(\mathfrak{C}_s)\subset\widetilde{\Lambda}$, i.e. $\big(dr\times d\mathbb{P}_0^t\big)\circ(\widehat{\Pi}_s^t)^{-1}=\big(dr\times d\mathbb{P}_0^s\big)$ on $\mathscr{B}([s,T])\otimes\mathcal{F}_T^s$. □

Lemma A.14. *Let $t \in [0,T]$, $\delta \in \mathbb{R}$ and let X be an $\mathbf{F}^t-$adapted process.*

(1) If all paths of X are left-lower-semicontinuous and right-continuous, then $\tau_\delta := \inf\big\{s \in [t,T] : X_s \le \delta\big\} \wedge T$ is an $\mathbf{F}^t-$stopping time.

(2) If all paths of X satisfy

$$X_t(\omega) \ge \varlimsup_{s \nearrow t} X_s(\omega) \wedge \varlimsup_{s \searrow t} X_s(\omega), \quad \forall\, (t,\omega) \in [0,T] \times \Omega, \tag{A.11}$$

then $\nu_\delta := \inf\big\{s \in [t,T] : X_s < \delta\big\} \wedge T$ is an $\mathbf{F}^t-$optional time.

Proof: 1) Suppose that all paths of X are left-lower-semicontinuous and right-continuous. Let $s \in [t,T]$. We first claim that for any $\omega \in \Omega$

$$\text{if } X_r(\omega) > 0, \ \forall\, r \in [t,s], \text{ then } \inf_{r \in [t,s]} X_r(\omega) > 0. \tag{A.12}$$

Assume not, i.e. there exists a $\omega' \in \Omega^t$ such that $X_r(\omega') > 0$, $\forall\, r \in [t,s]$ and $\inf\limits_{r \in [t,s]} X_r(\omega') \le 0$. Then one can find a sequence $\{r_n = r_n(t,\omega')\}_{n \in \mathbb{N}}$ of $[t,s]$ such that $\lim\limits_{n \to \infty} \downarrow X_{r_n}(\omega') = \inf\limits_{r \in [t,s]} X_r(\omega')$. Clearly, $\{r_n\}_{n \in \mathbb{N}}$ has a convergent subsequence $\{r_{n_i}\}_{i \in \mathbb{N}}$ with limit $r_* \in [t,s]$. We can deduce from the lower-semicontinuity of X that $0 < X_{r_*}(\omega') \le \varliminf\limits_{r \to r_*} X_r(\omega') \le \lim\limits_{i \to \infty} \downarrow X_{r_{n_i}}(\omega') = \inf\limits_{r \in [t,s]} X_r(\omega') \le 0$. An contradiction appears. So (A.12) holds and it follows that

$$\{\tau_\delta > s\} \ = \ \{\omega \in \Omega^t : X_r(\omega) > \delta, \ \forall\, r \in [t,s]\} = \underset{n \in \mathbb{N}}{\cup} \{\omega \in \Omega^t : X_r(\omega) \ge \delta + 1/n, \ \forall\, r \in [t,s]\}. \tag{A.13}$$

For any $n \in \mathbb{N}$, the right-continuity of X implies that $\{\omega \in \Omega^t : X_r(\omega) \ge \delta + 1/n, \ \forall\, r \in [t,s]\} = \{\omega \in \Omega^t : X_r(\omega) \ge \delta + 1/n, \ \forall\, r \in \mathbb{Q}_{t,s}\}$, where $\mathbb{Q}_{t,s} := \big([t,s] \cap \mathbb{Q}\big) \cup \{t,s\}$. Putting these equalities back into (A.13) yields that

$$\{\tau_\delta > s\} \ = \ \underset{n \in \mathbb{N}}{\cup} \{\omega \in \Omega^t : X_r(\omega) \ge 1/n, \ \forall\, r \in \mathbb{Q}_{t,s}\} = \underset{n \in \mathbb{N}}{\cup} \ \underset{s \in \mathbb{Q}_{t,s}}{\cap} \{\omega \in \Omega^t : X_r(\omega) \ge 1/n\} \in \mathcal{F}^t_s,$$

which shows that τ_δ is an $\mathbf{F}^t-$stopping time.

2) Under (A.11), it holds for any $s \in [t,T]$ that

$$\{\nu_\delta \ge s\} = \{\widetilde{\omega} \in \Omega^t : X_r(\omega) \ge \delta, \ \forall\, r \in [t,s)\} = \{\omega \in \Omega^t : X_r(\omega) \ge \delta, \ \forall\, r \in \mathbb{Q}_{t,s}\} = \underset{r \in \mathbb{Q}_{t,s}}{\cap} \{\omega \in \Omega^t : X_r(\omega) \ge \delta\} \in \mathcal{F}^t_s.$$

Thus ν_δ is an $\mathbf{F}^t-$optional time. □

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