Front Propagation in Stochastic Neural Fields: A Rigorous Mathematical Framework

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Abstract. We develop a complete and rigorous mathematical framework for the analysis of stochastic neural field equations under the influence of spatially extended additive noise. By comparing a solution to a fixed deterministic front profile it is possible to realise the difference as strong solution to an $L^2(\mathbb{R})$ -valued SDE. A multiscale analysis of this process then allows us to obtain rigorous stability results. Here a new representation formula for stochastic convolutions in the semigroup approach to linear function-valued SDE with adapted random drift is applied. Additionally, we introduce a dynamic phaseadaption process of gradient type.

Keywords. travelling fronts, stochastic convolution, additive noise, strong solution, dynamic phase-adaption, stability

AMS subject classifications. 60H20, 60H25, 92C20

1 Introduction

Neural field equations are used to model the spatiotemporal evolution of neural activity in thin layers of cortical tissue from a macroscopic point of view. Under the influence of spatially extended additive noise ξ they can be formulated as

$$\partial_t u(x,t) = -u(x,t) + \int_{-\infty}^{\infty} w(x-y) \ F(u(y,t)) \ dy + \sqrt{\varepsilon} \ \dot{\xi}(x,t) \tag{1}$$

where u is meant to describe the activity of a neuron at position x and time t. The probability kernel w models the strength of nonlocal excitatory synaptic connections, whereas $F : \mathbb{R} \to \mathbb{R}$ constitutes a nonlinear firing rate function. A precise interpretation of the stochastic forcing term ξ , characterising extrinsic fluctuations of the system, will be given below. Several approaches (e.g. in [2], [13] and [20]) have been developed to derive neural field equations as continuum limit of spatially extended, synaptically coupled neural systems under different assumptions on the underlying microscopic network structure. Here, the particular importance of a macroscopic perspective on neural activity primarily arises from the capability of field equations like (1) to model a wide range of neurophysiological phenomena. In the deterministic theory ($\varepsilon = 0$) Amari first provided a complete taxonomy of pattern dynamics exhibited by such fields (cf. [1]), which includes the propagation of activity in form of travelling waves. For the simplified case that each neuron only switches between a spiking and a non-spiking state, i.e. that

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the firing rate function F is given by a Heaviside nonlinearity, even an explicit travelling wave solution of (1) can be constructed (cf. [6]), whereas for more general input data wand F sufficient criteria for the existence and uniqueness of such solutions have been formulated in [10] resp. [7]. Extending the deterministic field equation to scalar stochastic neural fields, the objective of Bressloff and Webber in [6] was to investigate the effect of spatially extended, extrinsic stochastic perturbations on such travelling wave dynamics. As it turned out, random forcing terms result in two distinct structural phenomena: 'fast' perturbations of the front shape as well as a 'slow' horizontal displacement of the wave profile from its uniformly translating position. Under these observations a separation of time-scales method was applied to solve a neural field equation (subject to multiplicative noise) by decomposing the solution into a fixed deterministic front profile, a diffusion-like horizontal translation process as well as time- and space-dependent fluctuations. However, as the authors of [6] themselves state, they "base [their] analysis on formal perturbation methods developed previously for PDEs, since a rigorous mathematical framework is currently lacking".

The purpose of our paper therefore is to introduce such a complete and mathematically rigorous framework, which qualitatively captures the above phenomena and - by comparing a solution of (1) to a fixed reference profile - allows us to realise a stochastic neural field as stochastic evolution equation on a suitable function space (for the underlying theory refer to [9], [16]). Here, ξ will be chosen as Q-Wiener process on the Hilbert space $L^2(\mathbb{R})$. In contrast to the more direct rigorous approach recently suggested in [11], where a stochastic neural field is interpreted as SDE on a suitable weighted L^2 -space, our approach via decomposition has the advantage of providing information on the precise structure of solutions and even more, allows us to prove new stability results.

The paper is structured as follows: Section 2 provides a brief introduction to the representation of stochastic travelling waves as developed in [6]. These ideas are then given rigorous meaning by the decompositions presented in Section 3. As part of this framework, in Subsection 3.2 we introduce a gradient-descent-type ODE for a dynamic phase-adaption and prove existence and uniqueness of a classical solution to this ODE in Theorem 3.5. Moreover, the fluctuations are further specified by a suitable decomposition into processes of different order, namely into an Ornstein-Uhlenbeck-like process satisfying a linear SDE with non-autonomous adopted random drift as well as a lowerorder differentiable remainder process on $L^2(\mathbb{R})$. As a consequence of the randomness of the linear drift term it is not possible to directly define a stochastic convolution of the (possibly non-adapted) semigroup with the driving Wiener process as required for a mild-solution concept. We therefore represent the Ornstein-Uhlenbeck process via a stochastic convolution of new type (recently introduced in [17]) that is applicable to general adapted random drifts and allows us to derive a locally uniform (in time) pathwise control on the Ornstein-Uhlenbeck component. A similar approach based on a pathwise control of the stochastic convolution has also been taken in [19]. Section 4 then contains our main results, Theorem 4.2 and Theorem 4.3, on the stability of stochastic travelling waves. The crucial assumption to obtain this type of stochastic stability is the existence of a spectral gap of the time-dependent random drift.

An interesting issue for future research would be to investigate the underlying statistics of the dynamic wave speed as well as the Ornstein-Uhlenbeck component of the fluctuations, which, unlike in the case of deterministic associated drift operators, cannot be expected to be a Gaussian process. Moreover, our hope is that the presented approach can be carried over to spatially discrete models in order to examine the stochastic stability of fluctuating travelling waves in discrete neural networks.

2 Phenomenological Motivation

Interpreting (1) as stochastic evolution equation on a suitable function space we now analyse the following nonlinear, scalar neural field equation

$$\begin{cases} du_t &= \left[-u_t + w * F(u_t)\right] dt + \sqrt{\varepsilon} \ dW_t^Q, \\ u(0) &= u_0 \end{cases}$$
(2)

where the additive stochastic forcing term is given by a Q-Wiener process W^Q on $L^2(\mathbb{R})$ (for a detailed introduction of Q-Wiener processes on general Hilbert spaces refer to [9]). Let the weight kernel w be a bounded probability density function, i.e. in particular

$$\int_{\mathbb{R}} w(x) \, dx = 1$$

and let the nonlinear gain function F be in $C_b^2(\mathbb{R})$, i.e. twice continuously differentiable with bounded derivatives.

Even though [6] examined the effect of extrinsic multiplicative noise on the above field equation, the structural separation of the dynamics on different time-scales is also adoptable to the setting of additive extrinsic forcing terms as assumed in (2). Basically, the analysis in [6] splits up the dynamics into its behaviour on short and long timescales, i.e. the process is represented via a "slow" diffusive-like displacement of the front from its uniformly translating position and "fast" fluctuations in the front profile. More precisely, a fixed deterministic wave profile U_0 with uniformly translating position $\xi = x - ct$ is horizontally displaced by $\Delta(t)$, a diffusive-like stochastic process on the real line. Already taking account of the correct order w.r.t. ε the ansatz in [6] is to solve (2) by decomposing

$$u(x,t) = U_0(\xi - \Delta(t)) + \sqrt{\varepsilon} \Phi(\xi - \Delta(t), t),$$
(3)

where U_0 is a travelling wave solution of (2) in the deterministic case $\varepsilon = 0$ and Φ can - at least formally - be derived as a process on $L^2(\mathbb{R})$. For more detailed insights into the nature of these processes as well as the applied methodology we refer to the original paper. In the following sections this phenomenological motivation is translated into a rigorous mathematical analysis, which will partly proceed along lines of a similar approach having been developed in the context of stochastic reaction diffusion equations (cf. [18], [21]).

3 Mathematical Modelling on $L^2(\mathbb{R})$

In the sequel assume the input data w and F to allow for a unique travelling wave solution \hat{u} with intrinsic wave speed c solving the deterministic neural field equation (2) in the case $\varepsilon = 0$. In addition, the wave profile should connect two stable fixed points 0 and 1 of the dynamics, i.e. we impose the boundary conditions $\lim_{\xi \to \infty} \hat{u}(\xi) = 1$, $\lim_{\xi \to -\infty} \hat{u}(\xi) = 0$. Sufficient criteria on w and F ensuring the existence and uniqueness of such solutions are stated in [10] resp. [7].

3.1 Decomposition of the solution

In analogy with [6] we decompose the solution of the stochastic neural field (2) into

$$u_t = \hat{u}_t + v_t, \tag{4}$$

with $\hat{u}_t = \hat{u}(\cdot - ct)$ being a travelling wave solution of the deterministic ODE

$$-c\hat{u}_x = -\hat{u} + w * F(\hat{u}) \tag{5}$$

Since by classical calculus \hat{u}_t is a solution of

$$d\hat{u}_{t}(x) = \partial_{t}\hat{u}(x - ct) \ dt = -c\hat{u}_{x}(x - ct) \ dt$$

= $[-\hat{u}_{t}(x) + (w * F(\hat{u}_{t}))(x)] \ dt$ (6)

 v_t satisfies the following evolution equation

$$dv_t = du_t - d\hat{u}_t$$

= $\left[-v_t + w * \left(F(v_t + \hat{u}_t) - F(\hat{u}_t)\right)\right] dt + \sqrt{\varepsilon} dW_t^Q$ (7)

Expanding to first order yields

$$dv_t = \left[-v_t + w * \left(F'(\hat{u}_t) \ v_t\right)\right] dt + R_t(\hat{u}_t, v_t) \ dt + \sqrt{\varepsilon} dW_t^Q,\tag{8}$$

where the remainder is given by

$$R_t(\hat{u}_t, v_t) = w * \left(\frac{1}{2} F''(\xi(\hat{u}_t, v_t)) \; v_t^2\right)$$
(9)

with $\xi(\hat{u}_t, v_t)$ denoting an intermediate point between \hat{u}_t and v_t . For a given travelling wave solution \hat{u} the remainder $R_t(\hat{u}_t, \cdot)$ is indeed a well-defined map on $L^2(\mathbb{R})$ satisfying the estimate

$$\|R_t(\hat{u}_t, v_t)\|_{L^2}^2 \le c \int_{\mathbb{R}} \left(\int_{\mathbb{R}} w(x-y) \ v_t^2(y) \ dy \right)^2 dx = \tilde{c} \ \|v_t\|_{L^2}^4.$$
(10)

Theorem 3.1. For $v_0 \in L^2(\mathbb{R})$ equation (7) has a unique mild solution v with $v \in L^{\infty}([0,T]; L^2(\mathbb{R}))$ almost surely. This solution is also a strong solution and admits a continuous modification.

Proof. For proving existence of a unique mild solution we define

$$B(t, v) := -v + w * (F(v + \hat{u}_t) - F(\hat{u}_t)).$$

Applying Jensen's inequality we obtain that B is well-defined as a map $[0,T] \times L^2(\mathbb{R}) \to L^2(\mathbb{R})$ satisfying the following estimate

$$||B(t,v)||_{L^{2}(\mathbb{R})}^{2} \leq 2||v||_{L^{2}(\mathbb{R})}^{2} + 2||F'||_{\infty}^{2} ||v||_{L^{2}(\mathbb{R})}^{2}$$

For a given topological space X let $\mathcal{B}(X)$ denote the Borelian σ -algebra on X. Then, B is obviously measurable from the measurable space $([0,T] \times L^2(\mathbb{R}), \mathcal{B}([0,T]) \otimes \mathcal{B}(L^2(\mathbb{R})))$ to $(L^2(\mathbb{R}), \mathcal{B}(L^2(\mathbb{R})))$.

As one can easily verify for $v_1, v_2 \in L^2(\mathbb{R}), t \in [0, T]$, we obtain the Lipschitz property

$$||B(t,v_1) - B(t,v_2)||_{L^2(\mathbb{R})}^2 \le 2 (1 + ||F'||_{\infty}^2) ||v_1 - v_2||_{L^2(\mathbb{R})}^2$$

which, together with the property

$$B(t,0) = w * (F(0 + \hat{u}_t) - F(\hat{u}_t)) = 0,$$

directly implies a linear growth condition for B. Considering (7) as semilinear evolution equation with linear part Av_t , A = 0, [9, Theorem 7.4] provides the existence of a unique mild solution $v \in L^{\infty}([0,T], L^2(\mathbb{R}))$ a.s., which allows for a continuous modification and can be represented as

$$v_t = v_0 + \int_0^t B(s, v_s) \, ds + \sqrt{\varepsilon} W_t^Q. \tag{11}$$

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This immediately shows that v is also a strong solution of (7).

It is clear from the above that if the initial condition u_0 of problem (2) is chosen such that $v_0 := u_0 - \hat{u}_0 \in L^2(\mathbb{R})$, then $u_t := \hat{u}_t + v_t$ is a solution of (2). Uniqueness of the components only holds after we have fixed an initial phase for the travelling wave solution \hat{u}_t .

3.2 Gradient-descent type ODE for random phase-shift

The intention of this subsection is to find a suitable derivation of a slow horizontal translation process $(C(t))_{t \in [0,T]}$, as mentioned in Section 2, such that the initial representation $u_t = \hat{u}_t + v_t$ can be replaced by

$$u_t = \hat{u}(\cdot - ct - C(t)) + \tilde{v}_t, \tag{12}$$

where \tilde{v}_t denotes a new L^2 -component. The phase-shift process (C(t)) should have the effect of dynamically matching the deterministic profile \hat{u} with the stochastic travelling wave in the sense of minimising the L^2 -distance between the solution u and all possible translations of \hat{u} , for which reason (as already proposed in [14], [18]) we consider the gradient-descent-type (pathwise) ODE

$$\begin{cases} \dot{C}(t) = -m \ \langle \hat{u}_x(\cdot - ct - C(t)), u(t, \cdot) - \hat{u}(\cdot - ct - C(t)) \rangle_{L^2} \\ C(0) = 0 \end{cases}$$
(13)

with relaxation rate m > 0. This phase-adaption can be seen as an alternative approach to the phase conditions specified by certain algebraic constraints in the classical stability analysis (refer to [12]). Existence and uniqueness of a classical global solution to the ODE (13) are proven under the following assumptions on the weight kernel:

Assumption 3.2. (a) w is piecewise continuously differentiable

(b) w satisfies

$$\int_{\mathbb{R}} \frac{w_x^2}{w}(x) \, dx < \infty$$

- **Remark.** (i) Note that Assumption 3.2(b) ensures that in the case c = 0 the travelling wave satisfies $\hat{u}_{xx} \in L^2(\mathbb{R})$, which is used in the proof of Lemma 3.4(ii). In the case $c \neq 0$ part (b) of the above assumption is not needed.
- (ii) The above conditions are satisfied for the exponential weight function $w(x) = \frac{1}{2\sigma}e^{-\frac{|x|}{\sigma}}$ as well as for a Gaussian kernel $w(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}$, which are both common choices in modelling synaptic excitatory connections.

Lemma 3.3. For bounded weight kernel w and $F \in C_b^1(\mathbb{R})$ we obtain

$$\hat{u}_x \in L^2(\mathbb{R}).$$

Proof. In the case $c \neq 0$ multiplying (5) with the gradient \hat{u}_x yields

$$\hat{u}_x^2 = \frac{1}{c}\hat{u} \,\,\hat{u}_x - \frac{1}{c} \left(w * F(\hat{u})\right)\hat{u}_x$$

such that by integration by parts for $y, z \in \mathbb{R}$

$$\begin{split} \int_{y}^{z} \hat{u}_{x}^{2}(r) dr &\leq \frac{1}{2|c|} \left(\hat{u}^{2}(z) + \hat{u}^{2}(y) \right) + \frac{1}{|c|} \|w * F(\hat{u})\|_{\infty} (\hat{u}(z) + \hat{u}(y)) \\ &\longrightarrow \frac{1}{2|c|} + \frac{1}{|c|} \|w * F(\hat{u})\|_{\infty} \quad \text{as } y \to -\infty, \ z \to \infty. \end{split}$$

Here, the boundary conditions of \hat{u} were applied.

In the case c = 0 (5) yields $\hat{u} = w * F(\hat{u})$ and under Assumption 3.2 (b) one obtains

$$\hat{u}_x = w * F'(\hat{u})\hat{u}_x.$$

Note that trivially for all $c \in \mathbb{R}$ we have $\hat{u}_x \in L^1(\mathbb{R})$. This suffices to obtain

$$\begin{aligned} \|\hat{u}_{x}\|_{L^{2}(\mathbb{R})}^{2} &= \int \left| \int w(x-y)F'(\hat{u}(y))\hat{u}_{x}(y) \ dy \right|^{2} dx \\ &= \int \int \int w(x-y)w(x-\tilde{y})F'(\hat{u}(y))F'(\hat{u}(\tilde{y}))\hat{u}_{x}(y)\hat{u}_{x}(\tilde{y}) \ dyd\tilde{y}dx \\ &\leq \left(\sup_{y,\tilde{y}} \int w(x-y)w(x-\tilde{y}) \ dx \right) \|F'\|_{\infty}^{2} \|\hat{u}_{x}\|_{L^{1}(\mathbb{R})}^{2} \\ &\leq \|w\|_{\infty} \|F'\|_{\infty}^{2} \|\hat{u}_{x}\|_{L^{1}(\mathbb{R})}^{2} \end{aligned}$$

With the above considerations the following Lipschitz properties for \hat{u} as well as \hat{u}_x are obtained:

Lemma 3.4. Given Assumption 3.2 let $C_1, C_2 \in \mathbb{R}$. Then

(i)

$$\|\hat{u}(\cdot - C_1) - \hat{u}(\cdot - C_2)\|_{L^2} \le \|\hat{u}_x\|_{L^2} |C_1 - C_2|$$

(ii) There exists a constant $\tilde{c} > 0$ such that

$$\|\hat{u}_x(\cdot - C_1) - \hat{u}_x(\cdot - C_2)\|_{L^2} \le \tilde{c} |C_1 - C_2|$$

Proof. The proof of property (i) only requires Hölder's inequality, whereas for (ii) note that in the case $c \neq 0$ equation (5) determines the gradient of the travelling wave by

$$\hat{u}_x = \frac{1}{c}(\hat{u} - w * F(\hat{u})).$$

Then, (i) and Jensen's inequality suffice to verify the above Lipschitz property. In the case c = 0 equation (5) yields the identity $\hat{u} = w * F(\hat{u})$ such that by differentiating we see that the gradient satisfies $\hat{u}_x = w * F'(\hat{u})\hat{u}_x$. The rest again follows from straightforward calculations and Assumption 3.2.

After these preparatory considerations we state the main existence and uniqueness result of this subsection:

Proposition 3.5. Let m > 0 and let Assumption 3.2 be satisfied. Then P-almost surely there exists a unique global solution of the (pathwise) ODE (13).

Proof. Let

$$B(t,C) := \langle \hat{u}_x(\cdot - ct - C), u(t, \cdot) - \hat{u}(\cdot - ct - C) \rangle_{L^2}.$$

Since (13) constitutes an initial value problem of first order it suffices to show that the map $(t, C) \in [0, T] \times \mathbb{R} \mapsto B(t, C)$ is continuous and Lipschitz continuous w.r.t. C uniformly in t. Decomposing

$$B(t,C) = \langle \hat{u}_x(\cdot - ct - C), u(t, \cdot) - \hat{u}(\cdot - ct) \rangle_{L^2} + \langle \hat{u}_x(\cdot - ct - C), \hat{u}(\cdot - ct) - \hat{u}(\cdot - ct - C) \rangle_{L^2}$$

observe that the maps

- (i) $(t,C) \mapsto \hat{u}_x(\cdot ct C)$
- (ii) $(t, C) \mapsto u(t, \cdot) \hat{u}(\cdot ct)$
- (iii) $(t, C) \mapsto \hat{u}(\cdot ct) \hat{u}(\cdot ct C)$

are continuous in $L^2(\mathbb{R})$. By Lemma 3.4 this holds true due to the continuity of \hat{u} , \hat{u}_x and $v_t = u(t, \cdot) - \hat{u}_t$, where we take the continuous modification provided by Theorem 3.1. Next, for any $C_1, C_2 \in \mathbb{R}$, $t \in [0, T]$,

$$B(t, C_1) - B(t, C_2) = \langle \hat{u}_x(\cdot - ct - C_1) - \hat{u}_x(\cdot - ct - C_2), u(t, \cdot) - \hat{u}(\cdot - ct) \rangle_{L^2} + \langle \hat{u}_x(\cdot - ct - C_1), \hat{u}(\cdot - ct) - \hat{u}(\cdot - ct - C_1) \rangle_{L^2} - \langle \hat{u}_x(\cdot - ct - C_2), \hat{u}(\cdot - ct) - \hat{u}(\cdot - ct - C_2) \rangle_{L^2} = I + II + III, \text{ say.}$$

Applying Cauchy-Schwarz as well as Lemma 3.4 the first summand is P-a.s. controlled by

$$|I| \leq \|\hat{u}_x(\cdot - ct - C_1) - \hat{u}_x(\cdot - ct - C_2)\|_{L^2} \|v_t\|_{L^2}$$

$$\leq c \|v\|_{C([0,T];L^2(\mathbb{R}))} \|C_1 - C_2\|.$$

After the substitution $\cdot - C_1 \rightarrow \cdot$ (analogously for C_2) the second and third part can similarly be estimated by

$$|II + III| = |\langle \hat{u}_x(\cdot - ct), \hat{u}(\cdot - ct + C_1) - \hat{u}(\cdot - ct) \rangle_{L^2} - \langle \hat{u}_x(\cdot - ct), \hat{u}(\cdot - ct + C_2) - \hat{u}(\cdot - ct) \rangle_{L^2}| = |\langle \hat{u}_x(\cdot - ct), \hat{u}(\cdot - ct + C_1) - \hat{u}(\cdot - ct + C_2) \rangle_{L^2}| \leq ||\hat{u}_x(\cdot - ct)||_{L^2} ||\hat{u}(\cdot - ct + C_1) - \hat{u}(\cdot - ct + C_2)||_{L^2} \leq ||\hat{u}_x||_{L^2}^2 |C_1 - C_2|.$$

Overall, this provides us with the *P*-a.s. existence and uniqueness of a classical global solution $C \in C^1([0,T];\mathbb{R})$.

The representation

$$u_t = \hat{u}(\cdot - ct - C(t)) + \tilde{v}_t,$$

i.e.

$$\tilde{v}_t = u_t - \hat{u}(\cdot - ct - C(t)) = v_t + (\hat{u}(\cdot - ct) - \hat{u}(\cdot - ct - C(t)))$$
(14)

should now phenomenologically correspond to the representation established by Bressloff and Webber in [6].

Introducing the notation

$$\tilde{\hat{u}}(t) := \hat{u}(\cdot - ct - C(t)), \quad \tilde{\hat{u}}_x(t) := \hat{u}_x(\cdot - ct - C(t))$$

it is obvious that $(\tilde{v}_t)_{t\in[0,T]}$ is again a process on $L^2(\mathbb{R})$ and strong solution of the function-valued SDE

$$\begin{split} d\tilde{v}_{t} &= dv_{t} + \partial_{t}\hat{u}(\cdot - ct) \ dt - \partial_{t}\hat{u}(\cdot - ct - C(t)) \ dt \\ &= \left[-v_{t} + w * (F(v_{t} + \hat{u}_{t}) - F(\hat{u}_{t})) \right] \ dt - c\hat{u}_{x}(\cdot - ct) \ dt \\ &+ \left(c + \dot{C}(t) \right) \hat{u}_{x}(\cdot - ct - C(t)) \ dt + \sqrt{\varepsilon} dW_{t}^{Q} \\ &= \left[-v_{t} + w * (F(v_{t} + \hat{u}_{t}) - F(\hat{u}_{t})) - \hat{u}_{t} + w * F(\hat{u}_{t}) + \hat{u}(\cdot - ct - C(t)) \right] \\ &- w * F(\hat{u}(\cdot - ct - C(t))) + \dot{C}(t)\hat{u}_{x}(\cdot - ct - C(t)) \right] \ dt + \sqrt{\varepsilon} dW_{t}^{Q} \\ &= \left[-\tilde{v}_{t} + w * (F(v_{t} + \hat{u}_{t}) - F(\hat{u}_{t})) + w * (F(\hat{u}_{t}) - F(\hat{u}(\cdot - ct - C(t)))) \right] \ dt \\ &+ \dot{C}(t) \ \hat{u}_{x}(\cdot - ct - C(t)) \ dt + \sqrt{\varepsilon} dW_{t}^{Q} \\ &= \left[-\tilde{v}_{t} + w * \left(F(\tilde{v}_{t} + \tilde{\hat{u}}_{t}) - F(\tilde{\hat{u}}_{t}) \right) \right] \ dt + \dot{C}(t) \ \hat{u}_{x}(\cdot - ct - C(t)) \ dt + \sqrt{\varepsilon} dW_{t}^{Q} \\ &= \left[-\tilde{v}_{t} + w * \left(F(\tilde{v}_{t} + \tilde{\hat{u}}_{t}) - F(\tilde{\hat{u}}_{t}) \right) \right] \ dt + \dot{C}(t) \ \hat{u}_{x}(\cdot - ct - C(t)) \ dt + \sqrt{\varepsilon} dW_{t}^{Q} \\ &= \left[-\tilde{v}_{t} + w * \left(F(\tilde{v}_{t} + \tilde{\hat{u}}_{t}) - F(\tilde{\hat{u}}_{t}) \right) \right] \ dt + \dot{C}(t) \ \hat{u}_{x}(\cdot - ct - C(t)) \ dt + \sqrt{\varepsilon} dW_{t}^{Q} \end{aligned}$$
(15)

for the time- and ω -dependent linear operator

$$A(t)v = -v + w * \left(F'(\tilde{\hat{u}}_t) v\right) - m \left\langle \hat{u}_x(\cdot - ct - C(t)), v \right\rangle_{L^2} \cdot \hat{u}_x(\cdot - ct - C(t))$$
(16)

and the remainder R_t as introduced in (9). Here, the ω -dependence of A results from the ω -dependence of the process C. Note that $\langle \hat{u}_x, v \rangle \cdot \hat{u}_x$ in the above formula denotes an orthogonal projection of v onto the linear span generated by $\hat{u}_x(\cdot - ct - C(t))$ that, in the case $c \neq 0$, describes the infinitesimal tangential direction of the travelling wave \hat{u} w.r.t. time, because

$$\partial_t \hat{u} = -\hat{u} + w * F(\hat{u}) = -c\hat{u}_x$$

To obtain an even deeper understanding of the behaviour of the stochastic travelling wave u, its stability and the exact influence of the noise strength $\sqrt{\varepsilon}$ the following analysis works out the inner structure of the fluctuations \tilde{v} . To this end it is necessary to more thoroughly examine the family of linear operators $(A(t))_{t\in[0,T]}$.

3.3 Properties of the associated family of linear operators

For $t \in [0,T]$ let $A^0(t)$ denote the linear operator on $L^2(\mathbb{R})$ defined by

$$A^{0}(t)z = -z + w * \left(F'(\tilde{\hat{u}}_{t}) z\right), \quad z \in L^{2}(\mathbb{R}).$$

We work under the following

Assumption 3.6. There exist $\kappa_* > 0$, $C_* > 0$ such that

$$\langle A^{0}(t)z, z \rangle \leq -\kappa_{*} \|z\|^{2} + C_{*} \left(\langle \tilde{\hat{u}}_{x}(t), z \rangle \right)^{2} \quad \forall z \in L^{2}(\mathbb{R}).$$
(A1)

Furthermore, in the sequel the relaxation rate m is chosen such that

$$m > C_*. \tag{A2}$$

In the following Lemma relevant properties of the family of linear operators (A(t)) are proven. We denote by $\mathscr{L}(L^2(\mathbb{R}))$ the space of all bounded linear operators on $L^2(\mathbb{R})$.

Lemma 3.7. $A: [0,T] \times \Omega \to \mathscr{L}(L^2(\mathbb{R}))$ is well-defined, strongly measurable and strongly adapted. Furthermore,

- (i) For all $t \in [0,T]$, almost all $\omega \in \Omega$, $A(t)(\omega)$ is a bounded linear operator on $L^2(\mathbb{R})$.
- (ii) For almost all $\omega \in \Omega$ the map $t \mapsto A(t)(\omega)$ is continuous in the uniform operator topology.

In particular, A P-a.s. generates an evolution family $(P(t,s))_{0 \le s \le t \le T}$ on $L^2(\mathbb{R})$ and for $m > C_*$:

- (*iii*) $\langle v, A(t)v \rangle_{L^2} \le -\kappa_* \|v\|_{L^2}^2$.
- (iv) $||P(t,s)||_{\mathscr{L}(L^2(\mathbb{R}))} \le e^{-\kappa_*(t-s)} \quad \forall \ 0 \le s \le t \le T.$

Proof. Let $z \in L^2(\mathbb{R})$. As a combination of measurable, adapted functions $A(\cdot, \cdot)z$ itself is measurable and adapted.

(i) Linearity of the operator is obvious. Furthermore, straightforward calculations yield the bound

$$\sup_{r \in [0,T]} \|A(r)\| \le 1 + \|F'\|_{\infty} + m \|\hat{u}_x\|_{L^2}^2.$$

(ii) Let $s, t \in [0, T]$. We estimate

$$\begin{split} \|A(t)z - A(s)z\| \\ &\leq \|w * \left(F'(\tilde{\hat{u}}_t) z\right) - w * \left(F'(\tilde{\hat{u}}_s) z\right)\| + \| - m \langle \tilde{\hat{u}}_x(t), z \rangle \ \tilde{\hat{u}}_x(t) + m \langle \tilde{\hat{u}}_x(s), z \rangle \ \tilde{\hat{u}}_x(s)\| \\ &= \|w * \left(F'(\tilde{\hat{u}}_t) - F'(\tilde{\hat{u}}_s)\right) z\| \\ &+ m \|\langle \tilde{\hat{u}}_x(t), z \rangle \ \tilde{\hat{u}}_x(t) - \langle \tilde{\hat{u}}_x(t), z \rangle \ \tilde{\hat{u}}_x(s) + \langle \tilde{\hat{u}}_x(t), z \rangle \ \tilde{\hat{u}}_x(s) - \langle \tilde{\hat{u}}_x(s), z \rangle \ \tilde{\hat{u}}_x(s)\| \\ &=: I + II \end{split}$$

Applying Jensen's inequality one obtains

$$\begin{split} I^{2} &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} w(x-y) \left| F'(\tilde{\hat{u}}_{t}(y)) - F'(\tilde{\hat{u}}_{s}(y)) \right|^{2} z^{2}(y) \, dy \, dx \\ &\leq \|F''\|_{\infty}^{2} \int_{\mathbb{R}} \int_{\mathbb{R}} w(x-y) \left| \tilde{\hat{u}}_{t}(y) - \tilde{\hat{u}}_{s}(y) \right|^{2} z^{2}(y) \, dy \, dx \\ &\leq \|F''\|_{\infty}^{2} \int_{\mathbb{R}} \int_{\mathbb{R}} w(x-y) \|\hat{u}\|_{C^{1}}^{2} \left| ct - cs + C(t) - C(s) \right|^{2} z^{2}(y) \, dy \, dx \\ &\leq \|F''\|_{\infty}^{2} \|\hat{u}\|_{C^{1}}^{2} \left(c \left| t - s \right| + \|C\|_{C^{1}} |t - s| \right)^{2} \int_{\mathbb{R}} z^{2}(y) \int_{\mathbb{R}} w(x-y) \, dx \, dy \\ &\leq \tilde{c} \left| t - s \right|^{2} \|z\|_{L^{2}}^{2}. \end{split}$$

The second summand is estimated as follows:

$$II \le m \left(\| \langle \tilde{\hat{u}}_x(t), z \rangle \left(\tilde{\hat{u}}_x(t) - \tilde{\hat{u}}_x(s) \right) \|_{L^2} + \| \langle \tilde{\hat{u}}_x(t) - \tilde{\hat{u}}_x(s), z \rangle \left\| \tilde{\hat{u}}_x(s) \|_{L^2} \right) \\ \le m \left(\| \tilde{\hat{u}}_x(t) \|_{L^2} \| z \|_{L^2} \| \tilde{\hat{u}}_x(t) - \tilde{\hat{u}}_x(s) \|_{L^2} + \| \tilde{\hat{u}}_x(t) - \tilde{\hat{u}}_x(s) \|_{L^2} \| z \|_{L^2} \| \tilde{\hat{u}}_x(s) \|_{L^2} \right)$$

According to Lemma 3.4 there exists a constant $\tilde{c} > 0$ such that

$$\|\tilde{\hat{u}}_x(t) - \tilde{\hat{u}}_x(s)\|_{L^2} \le \tilde{c} |ct + C(t) - cs - C(s)| \le \tilde{c} (c |t - s| + \|C\|_{C^1} |t - s|)$$

Thus, II is also of order |t - s|, which then yields the continuity (even Lipschitz continuity) of $t \mapsto A(t)$ in the uniform operator norm.

Given (i) and (ii) the existence of an evolution family $(P(t,s))_{0 \le s \le t \le T}$ on $L^2(\mathbb{R})$ generated by $(A(t))_{t \in [0,T]}$ is now provided by [15, Chapter 5, Theorem 5.1].

(iii) The negative definiteness of the operator is a direct consequence of Assumption 3.6. (iv) Let $Q(t,s) = e^{\kappa_*(t-s)}P(t,s), s \leq t$. Then,

$$\partial_t \|Q(t,s)z\|^2 = 2 \langle Q(t,s)z, \kappa_* e^{\kappa_*(t-s)} P(t,s)z + e^{\kappa_*(t-s)} A(t) P(t,s)z \rangle$$

= 2 \langle Q(t,s)z, \langle A(t) + \kappa_* \rangle Q(t,s)z \rangle
\leq -2 \kappa_* \|Q(t,s)z\|^2 + 2 \kappa_* \|Q(t,s)z\|^2
= 0

which implies

$$||Q(t,s)z||^2 \le ||Q(s,s)z||^2 = ||z||^2$$

3.4 Ornstein-Uhlenbeck decomposition of the L²-component

To even better characterise the behaviour of the $L^2(\mathbb{R})$ -valued fluctuations \tilde{v} we will derive a decomposition into an Ornstein-Uhlenbeck process $(Z_t)_{t\in[0,T]}$ and a corresponding remainder process $(y_t^{\varepsilon})_{t\in[0,T]}$, which turn out to display different orders of the intrinsic dynamics of \tilde{v} . To this end let us first introduce

$$\tilde{v}_t^{\varepsilon} := \frac{1}{\sqrt{\varepsilon}} \, \tilde{v}_t, \tag{17}$$

which by equation (15) satisfies the SDE

$$\begin{cases} d\tilde{v}_t^{\varepsilon} &= \left[A(t) \ \tilde{v}_t^{\varepsilon} + \sqrt{\varepsilon} \ R_t^{\varepsilon} (\tilde{\hat{u}}_t, \tilde{v}_t^{\varepsilon}) \right] \ dt + dW_t^Q \\ \tilde{v}^{\varepsilon}(0) &= \frac{1}{\sqrt{\varepsilon}} \ v_0 \end{cases}$$

with

$$R_t^{\varepsilon}(\tilde{\hat{u}}_t, \tilde{v}_t^{\varepsilon}) = w * \left(\frac{1}{2}F''(\xi(\tilde{\hat{u}}_t, \sqrt{\varepsilon}\tilde{v}_t^{\varepsilon}))(\tilde{v}_t^{\varepsilon})^2\right).$$

As before, $\xi(u, v)$ denotes an intermediate point between u and v.

To work out the inner structure of the solution w.r.t. different order terms we introduce the decomposition

 $\tilde{v}_t^{\varepsilon} = Z_t + y_t^{\varepsilon},$

where (Z_t) satisfies the linear SDE

$$\begin{cases} dZ_t = A(t)Z_t \ dt + dW_t^Q \\ Z_0 = 0 \end{cases}$$
(18)

with ω -dependent operator A(t) and (y_t^{ε}) consequently solves the pathwise functionvalued ODE

$$\begin{cases} y_{\varepsilon}'(t) = A(t)y_{\varepsilon}(t) + \sqrt{\varepsilon} \ R_{t}^{\varepsilon}(\tilde{\hat{u}}_{t}, Z_{t} + y_{\varepsilon}(t)) \\ y_{\varepsilon}(0) = \frac{1}{\sqrt{\varepsilon}}v_{0} \end{cases}$$
(19)

The well-posedness of this decomposition is provided by the following existence and uniqueness theorems:

Theorem 3.8. There exists a unique mild solution Z of equation (18) satisfying

$$Z_t = \int_0^t A(s) Z_s \, ds + W_t^Q.$$
(20)

Thus, Z is also a strong solution.

Proof. To prove existence and uniqueness of a mild solution to equation (18) set

$$B(t, \omega, Z) := A(t)(\omega)Z$$

and verify that B satisfies a Lipschitz- as well as linear growth condition as required in the standard existence and uniqueness result for mild solutions ([9, Theorem 7.4]). Since one can formally enhance equation (18) by the linear operator A = 0, generating the semigroup $T_t = \text{Id}, t \in [0, T]$, the unique mild solution $Z \in L^{\infty}([0, T]; L^2(\mathbb{R}))$ even satisfies the identity

$$Z_t = \int_0^t A(s) Z_s \ ds + W_t^Q.$$

Given the unique strong solutions $(v_t^{\varepsilon})_{t \in [0,T]}$ and $(Z_t)_{t \in [0,T]}$ the following theorem is a direct consequence:

Theorem 3.9. For all T > 0 there exists a differentiable $L^2(\mathbb{R})$ -valued process (y_t^{ε}) , which is the unique strong solution of the pathwise ODE (19).

Even though Theorem (3.8) ensures existence and uniqueness of a strong solution (Z_t) , equation (20) only yields an implicit representation of the process. As already shown in Lemma 3.7, the family (A(t)) *P*-almost surely generates an evolution family (P(t,s)), which will even allow us to find an explicit mild-solution-like representation of (Z_t) via the following representation formula for weak solutions. This formula has been introduced as so-called "pathwise mild solution" in a much more general setting (cf. [17]) and manifests a way to pass around the difficulty of defining a mild-solution-like stochastic convolution in the case where the integrand cannot be assumed to be adapted. This indeed occurs if the operator A(t) depends on the underlying probability space.

Theorem 3.10. The process $Z : [0,T] \times \Omega \to L^2(\mathbb{R})$ defined by

$$Z_t = P(t,0)Z_0 + \int_0^t P(t,r)A(r)W_r^Q dr + W_t^Q$$

= $P(t,0)Z_0 + \int_0^t P(t,r)A(r)(W_r^Q - W_t^Q) dr + P(t,0)W_t^Q$

is an adapted weak solution of the linear SDE

$$dZ_t = A(t)Z_t \ dt + dW_t^Q$$

with $Z_0 \in L^2(\mathbb{R})$.

Using this representation, the Ornstein-Uhlenbeck process can now be controlled by the following pathwise order estimate, which yields a significant ε -independent bound on (Z_t) . This is not trivial since the operator A(t) indirectly depends on ε :

Lemma 3.11. Let $\eta \in (0, \frac{1}{2})$. The unique mild (and also strong) solution of equation (18) satisfies

$$\sup_{t\in[0,T]} \|Z_t\|_{L^2(\mathbb{R})} \le C \xi,$$

with a constant $C = C(\kappa_*, \eta, \|F'\|_{\infty}, \|\hat{u}_x\|_{L^2})$ and $\xi = \|W^Q\|_{C^{\eta}([0,T];L^2(\mathbb{R}))}$. In particular, we have $\xi < \infty$ almost surely.

Proof. By Theorem 3.10 the unique strong solution can be represented as

$$Z_t = \int_0^t P(t, r) A(r) (W_r^Q - W_t^Q) \, dr + P(t, 0) W_t^Q$$

and for each $\eta \in (0, \frac{1}{2})$ we have $W^Q \in C^{\eta}([0, T]; L^2(\mathbb{R}))$ a.s. (refer to [9]). Consequently, for $t \in [0, T]$ and $\xi = \|W^Q\|_{C^{\eta}([0, T]; L^2(\mathbb{R}))}$:

$$\begin{split} \|Z_t\|_{L^2} &\leq \int_0^t \|P(t,r)\| \, \|A(r)(W_r^Q - W_t^Q)\| \, dr + \|P(t,0)W_t^Q\| \\ &\leq \sup_{r \in [0,T]} \|A(r)\| \int_0^t e^{-\kappa_*(t-r)} \, \|W_r^Q - W_t^Q\| \, dr + e^{-\kappa_*t} \, \|W_t^Q\| \\ &\leq \sup_{r \in [0,T]} \|A(r)\| \int_0^t e^{-\kappa_*(t-r)} \, \|W^Q\|_{C^\eta} \, |t-r|^\eta \, dr + e^{-\kappa_*t} \, \|W^Q\|_{C^\eta} \, t^\eta \\ &\leq \sup_{r \in [0,T]} \|A(r)\| \left[\int_0^t e^{-\kappa_*(t-r)} \, |t-r|^\eta \, dr + e^{-\kappa_*t} \, t^\eta \right] \, \xi \\ &=: \sup_{r \in [0,T]} \|A(r)\| \left[I + II \right] \, \xi. \end{split}$$

Lemma 3.7(i) yields the bound

$$\sup_{r \in [0,T]} \|A(r)\| \le 1 + \|F'\|_{\infty} + m \|\hat{u}_x\|_{L^2}^2,$$

which, in particular, does not depend on ω . Furthermore, after suitable substitutions

$$I = \int_0^t e^{-\kappa_* y} y^\eta \, dy \le \kappa_*^{-\eta - 1} \, \Gamma(\eta + 1),$$

where Γ denotes the Gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad \text{for } x \in \mathbb{R}_+.$$

Likewise, term II is a bounded function on [0, T], thus

$$II \le \sup_{[0,T]} e^{-\kappa_* t} t^\eta < \infty.$$

Combining all results, there exists a positive constant $C = C(\kappa_*, \eta, \|F'\|_{\infty}, \|\hat{u}_x\|_{L^2})$ such that

$$\sup_{t \in [0,T]} \|Z_t\|_{L^2(\mathbb{R})} \le C \xi.$$

4 Stability results

Summing up, we derived a decomposition

$$\tilde{v}_t = \sqrt{\varepsilon} \ Z_t + \sqrt{\varepsilon} \ y_t^{\varepsilon}$$

such that (Z_t) is *P*-a.s. of order $\mathcal{O}(1)$. Characterising the behaviour of the stochastic travelling wave *u* for small noise strength $\sqrt{\varepsilon}$ our main theorem of this section will show that the remainder process (y_t^{ε}) is of lower order than $\mathcal{O}(1)$, i.e. $\lim_{\varepsilon \to 0} y_t^{\varepsilon} = 0$ in $L^2(\mathbb{R})$

uniformly w.r.t. $t \in [0, T]$ *P*-a.s. The relevance of this result is that with high probability we have the decomposition

$$u_t = \tilde{\hat{u}}_t + \sqrt{\varepsilon} Z_t + \text{ lower order terms.}$$

which, in comparison to the classical stability analysis via Evans functions (as conducted in [8] for a Heaviside nonlinearity), yields an alternative approach to the stability of fluctuating travelling waves.

From now on let us assume $v_0 = 0$, hence also $y_{\varepsilon}(0) = 0$.

Lemma 4.1. There exists a constant c > 0 such that

$$\forall v \in L^2(\mathbb{R}), \ t \in [0,T]: \quad \|R_t^{\varepsilon}(\tilde{\hat{u}}_t,v)\|_{L^2} \le c \ \|v\|_{L^2}^2.$$

Explicitly, the constant is given by

$$c = \frac{1}{2} \|F''\|_{\infty} \left(\sup_{y,\tilde{y}\in\mathbb{R}} \int_{\mathbb{R}} w(x-y)w(x-\tilde{y}) \ dx \right)^{1/2} \le \frac{1}{2} \|F''\|_{\infty} \|w\|_{\infty}^{1/2}.$$

Proof. Straightforward calculations.

For the proof of our main stability result (Theorem 4.3) the following bound on (y_t^{ε}) is already a crucial achievement:

Theorem 4.2. Let $\sqrt{\varepsilon} < \frac{\kappa_*}{4c}$, where c is the constant from Lemma 4.1 and define

$$Z := \sup_{t \in [0,T]} \|Z_t\|_{L^2}^2.$$

On the set $\Omega_{\varepsilon} = \left\{ \omega \in \Omega \mid Z < \frac{\kappa_*}{8c\sqrt{\varepsilon}} \right\}$ we obtain the following uniform bound on y_t^{ε} :

$$\sup_{t \in [0,T]} \|y_t^{\varepsilon}\|_{L^2}^2 \le \frac{3}{2} \ Z.$$

In the limit $\varepsilon \downarrow 0$ this bound even holds for P-almost all paths $\omega \in \Omega$, more precisely:

$$\lim_{\varepsilon \to 0} P[\Omega_{\varepsilon}] = 1$$

Proof. Applying Lemma 3.7 and Lemma 4.1, the process $||y_t^{\varepsilon}||^2$ satisfies the following differential inequality:

$$\frac{1}{2} \partial_t \|y_t^{\varepsilon}\|_{L^2}^2 = \langle y_t^{\varepsilon}, A(t)y_t^{\varepsilon} \rangle_{L^2} + \sqrt{\varepsilon} \langle y_t^{\varepsilon}, R_t^{\varepsilon}(\tilde{\hat{u}}_t, Z_t + y_t^{\varepsilon}) \rangle_{L^2} \\
\leq -\kappa_* \|y_t^{\varepsilon}\|_{L^2}^2 + \sqrt{\varepsilon} \|y_t^{\varepsilon}\|_{L^2} c \|Z_t + y_t^{\varepsilon}\|_{L^2}^2 \\
\leq -\kappa_* \|y_t^{\varepsilon}\|_{L^2}^2 + 2c\sqrt{\varepsilon} \|y_t^{\varepsilon}\|_{L^2} \left(\|Z_t\|_{L^2}^2 + \|y_t^{\varepsilon}\|_{L^2}^2\right)$$
(21)

W.l.o.g. assume c = 1. For $\sqrt{\varepsilon} < \frac{\kappa_*}{4}$ the right-hand side of (21) can be further estimated from above by

$$\frac{1}{2} \partial_t \|y_t^{\varepsilon}\|_{L^2}^2 \leq -\kappa_* \|y_t^{\varepsilon}\|_{L^2}^2 + \sqrt{\varepsilon} \left(\|y_t^{\varepsilon}\|_{L^2}^2 + \|Z_t\|_{L^2}^4\right) + \sqrt{\varepsilon} \left(\|y_t^{\varepsilon}\|_{L^2}^2 + \|y_t^{\varepsilon}\|_{L^2}^4\right) \\
\leq -\frac{\kappa_*}{2} \|y_t^{\varepsilon}\|_{L^2}^2 + \sqrt{\varepsilon} \|Z_t\|_{L^2}^4 + \sqrt{\varepsilon} \|y_t^{\varepsilon}\|_{L^2}^4$$

Writing $g_{\varepsilon}(t) := \|y_t^{\varepsilon}\|_{L^2}^2$ one obtains the following differential inequality

$$\dot{g}_{\varepsilon}(t) \le -\kappa_* g_{\varepsilon}(t) + 2\sqrt{\varepsilon} Z^2 + 2\sqrt{\varepsilon} g_{\varepsilon}^2(t)$$
(22)

By the comparison principle for ODE this problem is now solved in the case of true equality, i.e.

$$\frac{\dot{g}_{\varepsilon}(t)}{-\kappa_* g_{\varepsilon}(t) + 2\sqrt{\varepsilon} g_{\varepsilon}^2(t) + 2\sqrt{\varepsilon} Z^2} = 1$$

Integrating over [0, t] and carrying out a suitable substitution yields

$$t = \int_0^{g_{\varepsilon}(t)} \frac{1}{-\kappa_* \ g + 2\sqrt{\varepsilon} \ g^2 + 2\sqrt{\varepsilon}Z^2} \ dg \tag{23}$$

Note that $\Delta := 16\varepsilon Z^2 - \kappa_*^2 < 0$ on the set Ω_{ε} . Thus, on this particular set of paths the integral (23) is given by

$$t = \frac{1}{\sqrt{-\Delta}} \log \left(\frac{4\sqrt{\varepsilon}g_{\varepsilon}(t) - \kappa_* - \sqrt{-\Delta}}{4\sqrt{\varepsilon}g_{\varepsilon}(t) - \kappa_* + \sqrt{-\Delta}} \right) - \frac{1}{\sqrt{-\Delta}} \log \left(\frac{-\kappa_* - \sqrt{-\Delta}}{-\kappa_* + \sqrt{-\Delta}} \right)$$
(24)

since $g_{\varepsilon}(0) = 0$. Let

$$M_{\varepsilon} := \frac{\kappa_* + \sqrt{-\Delta}}{\kappa_* - \sqrt{-\Delta}}.$$

It is important to remark that $M_{\varepsilon} \in (1, \infty)$ P-a.s. since Z > 0 a.s. and hence $-\Delta = \kappa_*^2 - 16\varepsilon Z^2 < \kappa_*^2$ a.s. Now, equation (24) is equivalent to

$$g_{\varepsilon}(t) = \frac{-(\kappa_* + \sqrt{-\Delta}) + (\kappa_* - \sqrt{-\Delta})e^{\sqrt{-\Delta} t}M_{\varepsilon}}{4\sqrt{\varepsilon} \left(e^{\sqrt{-\Delta} t}M_{\varepsilon} - 1\right)}$$

Note that an explosion of $g_{\varepsilon}(t)$ is excluded by the fact that $M_{\varepsilon} > 1$ a.s. Given these considerations and dropping the first negative summand in the numerator of $g_{\varepsilon}(t)$ we further estimate

$$g_{\varepsilon}(t) \leq \frac{(\kappa_* - \sqrt{-\Delta}) e^{\sqrt{-\Delta} t} M_{\varepsilon}}{4\sqrt{\varepsilon} \left(e^{\sqrt{-\Delta} t} M_{\varepsilon} - 1\right)} =: f_{\varepsilon}(t)$$

It is easy to see that $f_{\varepsilon}(t)$ is a non-increasing function, hence attains its supremum in t = 0. This implies

$$\sup_{t \in [0,T]} g_{\varepsilon}(t) \le f_{\varepsilon}(0) = \frac{(\kappa_* - \sqrt{-\Delta})}{4\sqrt{\varepsilon}} \cdot \frac{M_{\varepsilon}}{M_{\varepsilon} - 1} = I \cdot II, \text{ say.}$$

On Ω_{ε} one is able to estimate

$$\sqrt{\kappa_*^2 - 16\varepsilon Z^2} \ge \sqrt{\kappa_*^2} - \sqrt{16\varepsilon Z^2} = \kappa_* - 4\sqrt{\varepsilon}Z$$

which allows us to bound I as follows:

$$I = \frac{\kappa_* - \sqrt{\kappa_*^2 - 16\varepsilon Z^2}}{4\sqrt{\varepsilon}} \le \frac{\kappa_* - (\kappa_* - 4\sqrt{\varepsilon}Z)}{4\sqrt{\varepsilon}} = Z.$$

Turning our attention to II and again restricting ourselves to the paths in Ω_{ε} we obtain

$$II = \frac{\kappa_* + \sqrt{-\Delta}}{2\sqrt{-\Delta}} = \frac{\kappa_*}{2\sqrt{\kappa_*^2 - 16\varepsilon Z^2}} + \frac{1}{2} \le \frac{\kappa_*}{2\kappa_* - 8\sqrt{\varepsilon}Z} + \frac{1}{2} \le \frac{3}{2}$$

To prove that in the limit $\varepsilon \to 0$ the obtained pathwise bound even holds for almost every $\omega \in \Omega$ note that for decreasing ε the sets Ω_{ε} constitute an ascending sequence of sets converging to the event $\{Z < \infty\}$. Therefore,

$$P[\Omega_{\varepsilon}] \underset{\varepsilon \to 0}{\longrightarrow} P[Z < \infty] = 1,$$

since Z is an integrable random variable.

This auxiliary result now allows us to state the main theorem of this section. Similar results on the stability of certain macroscopic dynamics modelled by stochastic partial differential equations on bounded domain have also been obtained by Blömker in [3], [4], [5].

Theorem 4.3. Let $q \in (0, \frac{1}{2})$ and let c denote the constant from Lemma 4.1. Define the stopping time τ by

$$\tau = \inf \left\{ t \ge 0 \Big| \int_0^t \| \tilde{v}_s^{\varepsilon} \|_{L^2}^2 \, ds > \varepsilon^{-q} \right\} \wedge T$$

Then we obtain the order estimate

$$\sup_{t \in [0,\tau]} \|y_t^{\varepsilon}\|_{L^2} \le c \ \varepsilon^{1/2-q}.$$

Moreover,

$$\lim_{\varepsilon \to 0} P[\tau = T] = 1$$

Proof. Let $t < \tau$. The process y^{ε} satisfies

$$y_t^{\varepsilon} = \sqrt{\varepsilon} \int_0^t P(t,s) R_s^{\varepsilon}(\tilde{\hat{u}}_s, \tilde{v}_s^{\varepsilon}) \ ds.$$

With Lemma 3.7 and 4.1 we are able to estimate

$$\|y_t^{\varepsilon}\|_{L^2} \leq \sqrt{\varepsilon} \int_0^t e^{-\kappa_*(t-s)} c \|\tilde{v}_s^{\varepsilon}\|_{L^2}^2 ds \leq c \varepsilon^{1/2-q}$$

Hence,

$$\sup_{t \in [0,\tau]} \|y_t^{\varepsilon}\|_{L^2} \le c \ \varepsilon^{1/2-q}.$$

To prove the convergence $P[\tau=T] \rightarrow 1$ for $\varepsilon \rightarrow 0$ define the set

$$\Omega^* := \left\{ \omega \in \Omega \ \left| \ \sup_{t \in [0,T]} \|Z_t\|_{L^2}^2 \le \frac{\varepsilon^{-q}}{4T}, \ \sup_{t \in [0,T]} \|y_t^\varepsilon\|_{L^2}^2 \le \frac{\varepsilon^{-q}}{4T} \right\} \right.$$

which can be shown to be a subset of $\{\tau = T\}$. Indeed,

$$\{\tau = T\} = \left\{ \omega \in \Omega \right| \int_0^T \|\tilde{v}_s^\varepsilon\|_{L^2}^2 \, ds \le \varepsilon^{-q} \right\}$$

and on Ω^* one is able to estimate

$$\int_0^T \|\tilde{v}_s^{\varepsilon}\|_{L^2}^2 \, ds \le 2T \left(\sup_{t \in [0,T]} \|Z_t\|_{L^2}^2 + \sup_{t \in [0,T]} \|y_t^{\varepsilon}\|_{L^2}^2 \right) \le \varepsilon^{-q}.$$

In the following step we prove that in the limit $\varepsilon \downarrow 0$ even the smaller set Ω^* has full measure: Let $P_{\Omega_{\varepsilon}} = P[\cdot |\Omega_{\varepsilon}]$ denote the conditional probability distribution on the set Ω_{ε} with corresponding conditional expectation $E_{\Omega_{\varepsilon}} = E[\cdot |\Omega_{\varepsilon}]$.

By Lemma 3.11 the random variable $Z = \sup_{t \in [0,T]} ||Z_t||_{L^2}^2$ is bounded by the integrable majorant $C^2 \xi^2$, which, in particular, is independent of ε . With Markov's inequality and Proposition 4.2 one obtains

$$\begin{split} P_{\Omega_{\varepsilon}}[\Omega^*] &\geq 1 - P_{\Omega_{\varepsilon}}\left[\sup_{t\in[0,T]} \|Z_t\|_{L^2}^2 > \frac{1}{4 T\varepsilon^q}\right] - P_{\Omega_{\varepsilon}}\left[\sup_{t\in[0,T]} \|y_t^{\varepsilon}\|_{L^2}^2 > \frac{1}{4 T\varepsilon^q}\right] \\ &\geq 1 - 4 T\varepsilon^q \ E_{\Omega_{\varepsilon}}\left[\sup_{t\in[0,T]} \|Z_t\|_{L^2}^2\right] - 4 T\varepsilon^q \ E_{\Omega_{\varepsilon}}\left[\sup_{t\in[0,T]} \|y_t^{\varepsilon}\|_{L^2}^2\right] \\ &\geq 1 - 4 T\varepsilon^q \ \frac{E[C^2\xi^2;\Omega_{\varepsilon}]}{P[\Omega_{\varepsilon}]} - 4 T\varepsilon^q \ \frac{E[\sup_{t\in[0,T]} \|y_t^{\varepsilon}\|_{L^2}^2;\Omega_{\varepsilon}]}{P[\Omega_{\varepsilon}]} \\ &\geq 1 - \frac{4 T\varepsilon^q}{P[\Omega_{\varepsilon}]} \ C^2 \ E[\xi^2] - \frac{4 T\varepsilon^q}{P[\Omega_{\varepsilon}]} \ \frac{3}{2} \ E[Z] \\ &\longrightarrow 1 \quad \text{as } \varepsilon \to 0. \end{split}$$

In conclusion, the limit behaviour of the original probability measure P is immediately determined by

$$P[\Omega^*] \ge P_{\Omega_{\varepsilon}}[\Omega^*] \ P[\Omega_{\varepsilon}] \underset{\varepsilon \to 0}{\longrightarrow} 1,$$

which suffices to prove the assertion.

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