The existence of optimal bang-bang controls for GMxB contracts^{*}

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Abstract

A large collection of financial contracts offering guaranteed minimum benefits are often posed as control problems, in which at any point in the solution domain, a control is able to take any one of an uncountable number of values from the *admissible set*. Often, such contracts specify that the holder exert control at a finite number of deterministic times. The existence of an optimal bang-bang control, an optimal control taking on only a finite subset of values from the admissible set, is a common assumption in the literature. In this case, the numerical complexity of searching for an optimal control is considerably reduced. However, no rigorous treatment as to when an optimal bang-bang control exists is present in the literature. We provide the reader with a bang-bang principle from which the existence of such a control can be established for contracts satisfying some simple conditions. The bang-bang principle relies on the convexity and monotonicity of the solution and is developed using basic results in convex analysis and parabolic partial differential equations. We show that a *quaranteed lifelong withdrawal benefit* (GLWB) contract admits an optimal bang-bang control. In particular, we find that the holder of a GLWB can maximize a writer's losses by only ever performing nonwithdrawal, withdrawal at exactly the contract rate, or full surrender. We demonstrate that the related guaranteed minimum withdrawal benefit contract is not convexity preserving, and hence does not satisfy the bang-bang principle other than in certain degenerate cases.

Keywords: bang-bang controls, GMxB guarantees, convex optimization, optimal stochastic control

1 Introduction

1.1 Main results

A large collection of financial contracts offering guaranteed minimum benefits (GMxBs) are often posed as control problems [3], in which the control is able to take any one of an uncountable number of values from the *admissible set* at each point in its domain. For example, a contract featuring regular withdrawals may allow holders to withdraw any portion of their account. In the following, we consider a control which maximizes losses for the writer of the contract, hereafter referred to as an *optimal control*.

A typical example is a guaranteed minimum withdrawal benefit (GMWB). If withdrawals are allowed at any time (i.e. "continuously"), then the pricing problem can be formulated as a singular control [18, 9, 14, 15] or an impulse control [7] problem.

In practice, the contract usually specifies that the control can only be exercised at a finite number of deterministic *exercise times* $t_0 < t_1 < \cdots < t_{N-1}$ [3, 8]. The procedure for pricing such a contract using dynamic programming proceeds backwards from the expiry time t_N as follows:

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- 1. Given the solution as $t \to t_{n+1}^-$, the solution as $t \to t_n^+$ is acquired by solving an initial value problem.
- 2. The solution as $t \to t_n^-$ is then determined by applying an optimal control, which is found by considering a collection of optimization problems.

If, for example, a finite difference method is used to solve the initial value problem from t_{n+1}^- to t_n^+ , an optimal control is determined by solving an optimization problem at each grid node, in order to advance the solution to t_n^- . Continuing in this way, we determine the solution at the initial time.

If there exists an *optimal bang-bang* control, an optimal control taking on only a finite subset of values from the admissible set, the numerical algorithm simplifies considerably. The existence of such a control is a common assumption in insurance applications [2, 19, 13], although no rigorous treatment is present in the literature. In this paper, we will also consider a weaker condition, a *bang-bang principle*. In this case, although an optimal control is not necessarily a finite subset of values from the admissible set, we will see that a control having this property can result in a large reduction in computational complexity.

Our main result in this paper is the specification of sufficient conditions which can be used to guarantee the existence of an optimal bang-bang control. This result relies on the convexity and monotonicity of the solution and follows from a combination of basic results in convex analysis and parabolic partial differential equations (PDEs). We demonstrate our results on two common contracts in the GMxB family:

- The guaranteed lifelong withdrawal benefit (GLWB) (a.k.a. guaranteed minimum lifelong withdrawal benefits (GMLWB)) admits an optimal bang-bang control. In particular, we prove that a holder can maximize the writer's losses by only ever performing
 - nonwithdrawal,
 - withdrawal at the contract rate (i.e. never subject to a penalty), or
 - a full surrender (i.e. maximal withdrawal; may be subject to a penalty).
- On the other hand, the *guaranteed minimum withdrawal benefit* (GMWB) is not necessarily convexity preserving, and does not satisfy a bang-bang principle other than in certain degenerate cases.

In the event that it is not possible to determine an optimal control analytically, numerical methods are required. Standard techniques in optimization are not always applicable, since these methods cannot guarantee convergence to a global extremum. In particular, without a priori knowledge about the objective functions appearing in the family of optimization problems corresponding to optimal holder behavior at the exercise times, a numerical method needs to resort to a linear search over a discretization of the admissible set. Convergence to a desired tolerance is achieved by refining this partition [23]. Only with this approach can we be assured of a convergent algorithm. However, if an optimal bang-bang control exists, discretizing the control set becomes unnecessary. Theoretically, this simplifies convergence analysis. More importantly, in practice, this reduces the amount of work per local optimization problem, often the bottleneck of any numerical method.

1.2 Insurance applications

The GLWB is a response to a general reduction in the availability of defined benefit pension plans [6], allowing the buyer to replicate the security of such a plan via a substitute. The GLWB is bootstrapped via a lump sum payment w_0 to an insurer, which is invested in risky assets. We term this the *investment account*. Associated with the GLWB contract is the guaranteed withdrawal benefit account, referred to as the withdrawal benefit for brevity. This account is also initially set to w_0 . At a finite set of deterministic withdrawal times, the holder is entitled to withdraw a predetermined fraction of the withdrawal benefit (or any lesser amount), even if the investment account diminishes to zero. This predetermined fraction is referred to as the *contract* withdrawal rate. If holders wish to withdraw in excess of the contract withdrawal rate, they can do so upon the payment of a penalty. Typical GLWB contracts include penalty rates that are decreasing functions of time. These contracts are often bundled with *ratchets* (a.k.a. step-ups), a contract feature that periodically increases the withdrawal benefit to the investment account, provided that the latter has grown larger than the former. Moreover, *bonus* (a.k.a. roll-up) provisions are also often present, in which the withdrawal benefit is increased if the holder does not withdraw at a given withdrawal time. Upon death, the holder's estate receives the entirety of the investment account. We show that a holder can maximize the writer's costs by only ever performing *nonwithdrawal*, *withdrawal at exactly the contract rate*, or *surrendering the entirety of their account*. Such a holder will never withdraw a nonzero amount strictly below the contract rate or perform a partial surrender. However, this result requires a special form for the penalty and lapsation functions, which is not universal in all contracts. Pricing GLWB contracts has previously been considered in [21, 13, 10, 1].

Much like the GLWB contract, a GMWB is composed of an investment account and withdrawal benefit initially set to w_0 , in which w_0 is a lump sum payment to an insurer. At a finite set of withdrawal times, the holder is entitled to withdraw up to a predetermined amount. Note that this amount is not a fraction of the withdrawal benefit, as in the GLWB, but rather a constant amount irrespective of the withdrawal benefit's size. Furthermore, unlike the GLWB, the action of withdrawing decreases both the investment account and withdrawal benefit on a dollar-for-dollar basis.

The GMWB promises to return at least the entire original investment, regardless of the performance of the underlying risky investment. The holder may withdraw more than the predetermined amount subject to a penalty. Upon death, the contract is simply transferred to the holder's estate, and hence mortality risk need not be considered. Pricing GMWB contracts has been previously considered in [18, 9, 8, 14, 15].

1.3 Overview

In §2, we introduce the GLWB and GMWB contracts. In §3, we generalize this to model a contract that can be controlled at finitely many times, a typical case in insurance practice (e.g. yearly or quarterly exercise). In §4, we develop sufficient conditions for the existence of an optimal bang-bang control and show that the GLWB satisfies these conditions. §5 discusses a numerical method for finding the cost of funding GLWB and GMWB contracts, demonstrating the bang-bang principle for the former and providing an example of where it fails for the latter.

2 Guaranteed minimum benefits (GMxBs)

We introduce mathematical models for the GLWB and GMWB contracts in this section. Since most GMxB contracts offer withdrawals on anniversary dates, to simplify notation, we restrict our attention to annual withdrawals occurring at

$$\mathscr{T} \equiv \{0, 1, \dots, N-1\}$$

0 and N are referred to as the *initial* and *expiry* times, respectively (no withdrawal occurs at N).

In order to ensure that the writer can, at least in theory, hedge a short position in a GMxB with no risk, we assume that the holder will employ a loss-maximizing strategy. That is, the holder will act so as to maximize the cost of funding the GMxB. This represents the worst-case hedging cost for the writer. This worst-case cost is a function of the holder's investment account and withdrawal benefit. As such, we write $\mathbf{x} \equiv (x_1, x_2)$, where x_1 is the value of the investment account and x_2 is the value of the withdrawal benefit. Both of these quantities are nonnegative.

Let α denote the *hedging fee*, the rate continuously deducted from the investment account X_1 (while x_1 is used to denote a particular value of the investment account, the capital symbol X_1 is reserved for the corresponding stochastic process) to provide the premium for the contract. We assume that between exercise times, the investment account of the GMxBs follows geometric Brownian motion (GBM) as per

$$\frac{dX_1}{X_1} = (\mu - \alpha) \, dt + \sigma dZ$$

tracking the index \hat{X}_1 satisfying

$$\frac{d\hat{X}_1}{\hat{X}_1} = \mu dt + \sigma dZ$$

where Z is a Wiener process under the *real-world measure*. We assume that it is not possible to short the investment account X_1 for fiduciary reasons [8], so that the obvious arbitrage opportunity is prohibited.

The worst-case cost of a GMxB is posed as the solution to an initial value problem (IVP) specified by three conditions:

- 1. the worst-case cost of funding the contract at the expiry time (posed as a Cauchy boundary condition; see, for example, (2.1) and (2.11));
- 2. the evolution of the worst-case cost *across* withdrawals (posed as a supremum over the holder's actions, corresponding to the holder acting so as to maximize the writer's losses; see, for example, (2.2) and (2.12));
- 3. the evolution of the worst-case cost *between* withdrawals (posed as a conditional expectation; see, for example, (2.3) and (2.13)).

We begin by introducing the IVP for the GMWB before moving to the GLWB for ease of exposition. To distinguish the two contracts, we use the superscripts L and M to denote quantities that pertain to the GLWB and GMWB, respectively. In the following, we denote by $\tilde{\mathbb{E}}$ the expectation and by \tilde{Z} a Wiener process under the *risk-neutral measure*, that which renders the discounted index \hat{X}_1 into a martingale. For a function g whose domain is a subset of \mathbb{R} , we use the notations $g(t^-) \equiv \lim_{s\uparrow t} g(s)$ and $g(t^+) \equiv \lim_{s\downarrow t} g(s)$ to denote the one-sided limits at t.

2.1 Guaranteed minimum withdrawal benefit (GMWB)

Since the GMWB is transferred to the holder's estate upon death, mortality risk is not considered. The worst-case cost of funding a GMWB at time N (the expiry) is [9]

$$\varphi^{\mathrm{M}}(\mathbf{x}) \equiv \max\left(x_{1},\left(1-\kappa_{N}\right)x_{2}\right),$$

corresponding to the greater of the entirety of the investment account or a full surrender at the *penalty rate* at the Nth anniversary, $\kappa_N \in [0, 1]$. The worst-case cost of funding a GMWB at previous times is derived by a hedging argument in which the writer takes a position in the index \hat{X}_1 [8]. Equivalently, it is given by finding V (within the relevant space of functions; see Appendix A) such that (s.t.)

$$V(\mathbf{x}, N) = \varphi^{M}(\mathbf{x}) \qquad \text{on } [0, \infty)^{2} \qquad (2.1)$$

$$V\left(\mathbf{x}, n^{-}\right) = \sup_{\lambda \in [0,1]} \left[V\left(\mathbf{f}_{\mathbf{x},n}^{\mathrm{M}}\left(\lambda\right), n^{+}\right) + f_{\mathbf{x},n}^{\mathrm{M}}\left(\lambda\right) \right] \qquad \text{on } [0,\infty)^{2} \times \mathscr{T} \qquad (2.2)$$

$$V(\mathbf{x},t) = \tilde{\mathbb{E}}\left[e^{-\int_{t}^{n+1} r(\tau)d\tau} V\left(X_{1}\left((n+1)^{-}\right), x_{2}, (n+1)^{-}\right) | X_{1}\left(n^{+}\right) = x_{1}\right] \quad \text{on } [0,\infty)^{2} \times (n,n+1) \ \forall n \qquad (2.3)$$

where between exercise times

$$\frac{dX_1}{X_1} = (r - \alpha) dt + \sigma d\tilde{Z}.$$
(2.4)

r is the risk-free rate, $f^{\mathrm{M}}:[0,1] \to \mathbb{R}$ represents the cash flow from the writer to the holder, and $\mathbf{f}^{\mathrm{M}}:[0,1] \to [0,\infty)^2$ represents the state of the contract postwithdrawal. The construction of f^{M} and \mathbf{f}^{M} is outlined below. The holder is able to withdraw a fraction $\lambda \in [0,1]$ of the withdrawal benefit at each exercise time.

Intuitively, $V(\mathbf{x}, n^{-})$ and $V(\mathbf{x}, n^{+})$ can be thought of as the value of the contract "immediately before" and "immediately after" the exercise time n.

Let $G \ge 0$ denote the predetermined contract withdrawal amount associated with the GMWB so that $G \land x_2$ $(a \land b \equiv \min(a, b), a \lor b \equiv \max(a, b))$ is the maximum the holder can withdraw without incurring a penalty (both \land and \lor are understood to have lower operator precedence than the arithmetic operations). Consider the point (x_1, x_2, n) with $n \in \mathscr{T}$.

• The maximum a holder can withdraw without incurring a penalty is $G \wedge x_2$. If the holder withdraws the amount λx_2 with $\lambda x_2 \in [0, G \wedge x_2]$,

$$V\left(\mathbf{x}, n^{-}\right) = V\left(\underbrace{x_{1} - \lambda x_{2} \lor 0, x_{2} - \lambda x_{2}}_{\mathbf{f}^{\mathrm{M}}}, n^{+}\right) + \underbrace{\lambda x_{2}}_{f^{\mathrm{M}}}.$$
(2.5)

• Let $\kappa_n \in [0, 1]$ denote the *penalty rate* at the *n*th anniversary. If the holder withdraws the amount λx_2 with $\lambda x_2 \in (G \wedge x_2, x_2]$,

$$V\left(\mathbf{x}, n^{-}\right) = V\left(\underbrace{x_{1} - \lambda x_{2} \lor 0, x_{2} - \lambda x_{2}}_{\mathbf{f}^{\mathrm{M}}}, n^{+}\right) + \underbrace{\lambda x_{2} - \kappa_{n} \left(\lambda x_{2} - G\right)}_{f^{\mathrm{M}}}.$$
(2.6)

Here, $\lambda x_2 \in (G \wedge x_2, x_2)$ corresponds to a partial surrender and $\lambda x_2 = x_2$ (i.e. $\lambda = 1$) corresponds to a full surrender.

We can summarize (2.5) and (2.6) by taking

$$f_{\mathbf{x},n}^{\mathrm{M}}(\lambda) \equiv \begin{cases} \lambda x_{2} & \text{if } \lambda x_{2} \in [0, G \wedge x_{2}] \\ G + (1 - \kappa_{n}) (\lambda x_{2} - G) & \text{if } \lambda x_{2} \in (G \wedge x_{2}, x_{2}] \end{cases}$$
(2.7)

and

$$\mathbf{f}_{\mathbf{x},n}^{\mathrm{M}}\left(\lambda\right) \equiv \left(x_{1} - \lambda x_{2} \lor 0, \left(1 - \lambda\right) x_{2}\right).$$

It can be shown from (2.3) that the cost to fund the GMWB (between exercise times) satisfies¹ [8]

$$\partial_t V + \mathcal{L}V = 0 \text{ on } (0,\infty)^2 \times (n,n+1) \quad \forall n$$
(2.8)

where

$$\mathcal{L} \equiv \frac{1}{2}\sigma^2 x_1^2 \partial_{x_1 x_1} + (r - \alpha) x_1 \partial_{x_1} - r.$$
(2.9)

2.2 Guaranteed lifelong withdrawal benefit (GLWB)

Let $\mathcal{M}(t)$ be the mortality rate at time t (i.e. $\int_{t_1}^{t_2} \mathcal{M}(t) dt$ is the fraction of the original holders who pass away in the interval $[t_1, t_2]$), so that the survival probability at time t is

$$\mathcal{R}(t) = 1 - \int_{0}^{t} \mathcal{M}(s) \, ds.$$

We assume \mathcal{M} is continuous and nonnegative, along with $\mathcal{R}(t) \ge 0$ for all times t. We assume that mortality risk is diversifiable. Furthermore, we assume the existence of a time $t^* > 0$ s.t. $\mathcal{R}(t^*) = 0$. That is, survival beyond t^* is impossible (i.e. no holder lives forever). N is chosen s.t. $N \ge t^*$ to ensure that all holders have passed away at the expiry of the contract. As is often the case in practice, we assume ratchets are prescribed to occur on a subset of the anniversary dates (e.g. triennially).

As usual, we assume that the holder of a GLWB will employ a loss-maximizing strategy. Since N was picked sufficiently large, the insurer has no obligations at the Nth anniversary and the worst-case cost of funding a GLWB at time N is

$$\varphi^{\mathrm{L}}\left(\mathbf{x}\right) \equiv 0. \tag{2.10}$$

¹We discuss what it means for a function to satisfy this PDE in Appendix A.

As with the GMWB, the worst-case cost of funding a GLWB is derived by a hedging argument in which the writer takes a position in the index \hat{X}_1 [10]. Equivalently, it is given by finding V (within the relevant space of functions; see Appendix A) s.t.

$$V(\mathbf{x}, N) = \varphi^{\mathrm{L}}(\mathbf{x}) \qquad \text{on } [0, \infty)^{2} \qquad (2.11)$$

$$V(\mathbf{x}, n^{-}) = \sup_{\lambda \in [0,2]} \left[V\left(\mathbf{f}_{\mathbf{x},n}^{\mathrm{L}}(\lambda), n^{+} \right) + f_{\mathbf{x},n}^{\mathrm{L}}(\lambda) \right] \qquad \text{on } [0, \infty)^{2} \times \mathscr{T} \qquad (2.12)$$

$$V(\mathbf{x}, t) = \tilde{\mathbb{E}} \left[e^{-\int_{t}^{n+1} r(\tau) d\tau} V\left(X_{1}\left((n+1)^{-} \right), x_{2}, (n+1)^{-} \right) + \int_{t}^{n+1} e^{-\int_{t}^{s} r(\tau) d\tau} \mathcal{M}(s) X_{1}(s) ds \mid X_{1}\left(n^{+} \right) = x_{1} \right] \qquad \text{on } [0, \infty)^{2} \times (n, n+1) \ \forall n \qquad (2.13)$$

where between exercise times, X_1 is specified by (2.4).

 $f^{\mathrm{L}}:[0,2] \to \mathbb{R}$ represents the (mortality-adjusted [10]) cash flow from the writer to the holder and $\mathbf{f}^{\mathrm{L}}:[0,2] \to [0,\infty)^2$ represents the state of the contract postwithdrawal. In particular, $\lambda = 0$ corresponds to nonwithdrawal, $\lambda \in (0,1]$ corresponds to withdrawal at or below the contract rate, and $\lambda \in (1,2]$ corresponds to a partial or full surrender.

Remark 2.1. We remark that the admissible set of actions [0,2] is undesirably large (i.e. a continuum). We will apply the results established in §4 to show that an optimal strategy taking on values only from $\{0,1,2\}$ exists. In other words, an equivalent problem can be constructed by substituting the set $\{0,1,2\}$ for the original [0,2] in the optimization problem (2.12). The resulting problem has smaller computational complexity than the original one (i.e. successive refinements of [0,2] need not be considered to attain convergence).

The construction of f^{L} and \mathbf{f}^{L} is guided by the specification of the contract:

- Let β denote the *bonus rate*: if the holder does not withdraw, the withdrawal account is amplified by $1 + \beta$.
- Let δ denote the *contract withdrawal rate*; that is, δx_2 is the maximum a holder can withdraw without incurring a penalty.
- Let $\kappa_n \in [0, 1]$ denote the *penalty rate* at the *n*th anniversary, incurred if the holder withdraws above the contract withdrawal rate.
- Let

 $\mathbb{I}_n = \begin{cases} 1 & \text{if a ratchet is prescribed to occur on the } n \text{th anniversary} \\ 0 & \text{otherwise} \end{cases}$

Then,

$$f_{\mathbf{x},n}^{\mathrm{L}}(\lambda) \equiv \mathcal{R}(n) \cdot \begin{cases} 0 & \text{if } \lambda = 0\\ \lambda \delta x_{2} & \text{if } \lambda \in (0,1]\\ \delta x_{2} + (\lambda - 1)(1 - \kappa_{n})(x_{1} - \delta x_{2} \vee 0) & \text{if } \lambda \in (1,2] \end{cases}$$
(2.14)

and

$$\mathbf{f}_{\mathbf{x},n}^{\mathrm{L}}(\lambda) \equiv \begin{cases} (x_1, x_2 (1+\beta) \vee \mathbb{I}_n x_1) & \text{if } \lambda = 0\\ (x_1 - \lambda \delta x_2 \vee 0, x_2 \vee \mathbb{I}_n [x_1 - \lambda \delta x_2]) & \text{if } \lambda \in (0,1] \\ (2-\lambda) \mathbf{f}_{\mathbf{x},n}(1) & \text{if } \lambda \in (1,2] \end{cases}$$
(2.15)

It can be shown that the cost to fund the GLWB (between exercise times) satisfies [10]

$$\partial_t V + \mathcal{L}V + \mathcal{M}x_1 = 0 \text{ on } (0,\infty)^2 \times (n,n+1) \quad \forall n$$
(2.16)

where \mathcal{L} is defined in (2.9).

3 General formulation

We generalize now the above IVPs. Let $\mathscr{T} \equiv \{t_0, \ldots, t_{N-1}\}$ along with the order $0 \equiv t_0 < \cdots < t_N \equiv T$, in which T is referred to as the expiry time. Let Ω be a convex subset of \mathbb{R}^m . The set of all actions a holder can perform at an exercise time t_n is denoted by $\Lambda_n \subset \mathbb{R}^{m'}$, assumed to be nonempty, and referred to as an *admissible set*. For brevity, let

$$v_{\mathbf{x},n}\left(\lambda\right) \equiv V\left(\mathbf{f}_{\mathbf{x},n}\left(\lambda\right), t_{n}^{+}\right) + f_{\mathbf{x},n}\left(\lambda\right)$$

$$(3.1)$$

where $f_{\mathbf{x},n} \colon \Lambda_n \to \mathbb{R}$ and $\mathbf{f}_{\mathbf{x},n} \colon \Lambda_n \to \Omega$. We write $v_{\mathbf{x},n}(\lambda)$ to stress that for each fixed (\mathbf{x}, n) , we consider an optimization problem in the variable λ . The general problem is to find V satisfying the conditions

$$V(\mathbf{x}, T) = \varphi(\mathbf{x}) \qquad \text{on } \Omega \qquad (3.2)$$

$$V\left(\mathbf{x}, t_{n}^{-}\right) = \sup v_{\mathbf{x}, n}\left(\Lambda_{n}\right) \qquad \qquad \text{on } \Omega \times \mathscr{T}$$

$$(3.3)$$

along with a condition specifying the evolution of V from t_n^+ to t_{n+1}^- (see, for example, (2.3) and (2.13)).

Remark 3.1. Convexity preservation, a property that helps establish the bang-bang principle, depends on each admissible set Λ_n being independent of the state of the contract, **x**. This is discussed in Remark 4.18.

4 Control reduction

Definition 4.1 (Optimal bang-bang control). V, a solution to the general IVP introduced in §3, is said to admit an optimal bang-bang control at time $t_n \in \mathcal{T}$ whenever

$$V\left(\mathbf{x}, t_{n}^{-}\right) = \max v_{\mathbf{x}, n}\left(\hat{\Lambda}_{n}\right) on \ \Omega,$$

where $\hat{\Lambda}_n$ denotes a finite set independent of **x**.

The above condition is inherently simpler than (3.3), in which there are no guarantees on the cardinality of Λ_n .

§4.2 develops Corollary 4.13, establishing sufficient conditions for the existence of an optimal bang-bang control. This result requires that the relevant solution V be convex and monotone (CM). Given a CM initial condition (3.2), we seek to ensure that V preserves the CM property at all previous times. §4.3 develops conditions on the functions f and \mathbf{f} to ensure that the supremum (3.3) preserves the CM property. Similarly, §4.4 develops conditions on the dynamics of V (and hence the underlying stochastic process(es)) to ensure that the CM property is preserved between exercise times.

For the remainder of this work, we use the shorthand $V_n^+(\mathbf{x}) \equiv V(\mathbf{x}, t_n^+)$ and $V_n^-(\mathbf{x}) \equiv V(\mathbf{x}, t_n^-)$.

4.1 Preliminaries

In an effort to remain self-contained, we provide the reader with several elementary (but useful) definitions. In practice, we consider only vector spaces over \mathbb{R} and hence restrict our definitions to this case.

Definition 4.2 (convex set). Let W be a vector space over \mathbb{R} . $X \subset W$ is a convex set if for all $x, x' \in X$ and $\theta \in (0, 1)$, $\theta x + (1 - \theta) x' \in X$.

Definition 4.3 (convex function). Let X be a convex set and Y be a vector space over \mathbb{R} equipped with a partial order \leq_Y . h: $X \to Y$ is a convex function if for all $x, x' \in X$ and $\theta \in (0, 1)$,

$$h\left(\theta x + (1-\theta) x'\right) \leqslant_{Y} \theta h\left(x\right) + (1-\theta) h\left(x'\right).$$

Definition 4.4 (extreme point). An extreme point of a convex set X is a point $x \in X$ which cannot be written $x = \theta x' + (1 - \theta) x''$ for any $\theta \in (0, 1)$ and $x', x'' \in X$ with $x' \neq x''$.

Definition 4.5 (convex polytope). Let Y be a topological vector space over \mathbb{R} . $P \subset Y$ is a convex polytope if it is a compact convex set with finitely many extreme points. The extreme points of a convex polytope are referred to as its vertices.

Definition 4.6 (monotone function). Let X and Y be sets equipped with partial orders \leq_X and \leq_Y , respectively. $h: X \to Y$ is monotone if for all $x, x' \in X$, $x \leq_X x'$ implies $h(x) \leq_Y h(x')$.

Lemma 4.7. Let A be a convex set, and let B and C be vector spaces over \mathbb{R} equipped with partial orders \leq_B and \leq_C , respectively. If $h_1: A \to B$ and $h_2: B \to C$ are convex functions with h_2 monotone, then $h_2 \circ h_1$ is a convex function.

Remark 4.8. For the remainder of this work, we equip \mathbb{R}^m with the order \leq defined as follows: if $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, $\mathbf{x} \leq \mathbf{y}$ whenever $x_i \leq y_i$ for all *i*.

4.2 Bang-bang principle

Consider a particular exercise time t_n . Suppose the following:

- (A1) $\mathbf{x} \mapsto V_n^+(\mathbf{x})$ is CM.
- (A2) For each fixed $\mathbf{x} \in \Omega$, $v_{\mathbf{x},n}(\Lambda_n)$ is bounded above.

Throughout this section, we consider a particular point $\mathbf{y} \in \Omega$ in order to establish our result pointwise. For the results below, we require the following propositions:

- (B1) There exists a collection $\mathcal{P}_{n}(\mathbf{y}) \subset 2^{\Lambda_{n}}$ s.t. $\bigcup_{P \in \mathcal{P}_{n}(\mathbf{y})} P = \Lambda_{n}$ and each $P \in \mathcal{P}_{n}(\mathbf{y})$ is compact convex.
- (B2) For each $P \in \mathcal{P}_n(\mathbf{y})$, the restrictions $\lambda \mapsto f_{\mathbf{y},n}|_P(\lambda)$ and $\lambda \mapsto \mathbf{f}_{\mathbf{y},n}|_P(\lambda)$ are convex.

(B3) $\mathcal{P}_{n}(\mathbf{y})$ is a finite collection of convex polytopes.

Remark 4.9. (B1) simply states that we can "cut up" the admissible set Λ_n into (possibly overlapping) compact convex sets. (B2) states that the restrictions of $f_{\mathbf{y},n}$ and $\mathbf{f}_{\mathbf{y},n}$ on each of these sets are convex functions of λ .

Lemma 4.10. Suppose (A1), (B1), and (B2). For each $P \in \mathcal{P}_n(\mathbf{y})$, the restriction $\lambda \mapsto v_{\mathbf{y},n}|_P(\lambda)$ is convex.

Proof. The proof is by (3.1), (A1), (B2), and Lemma 4.7.

Lemma 4.11. Suppose (A1), (A2), (B1), and (B2). Let $P \in \mathcal{P}_n(\mathbf{y})$. Then,

$$\sup v_{\mathbf{y},n}\left(P\right) = \sup v_{\mathbf{y},n}\left(E\left(P\right)\right)$$

where E(P) denotes the set of extreme points of P.

Proof. Let $w \equiv v_{\mathbf{y},n}|_P$. Note that $w(P) = v_{\mathbf{y},n}(P)$, and hence no generality is lost in considering w. Lemma 4.10 establishes the convexity of w. Naturally, $\sup w(P)$ exists (and hence $\sup w(E(P))$ exists too) due to (A2). Finally, it is well known from elementary convex analysis that the supremum of a convex function on a compact convex set P lies on the extreme points of P, E(P). See [22, Chap. 32].

Theorem 4.12 (bang-bang principle). Suppose (A1), (A2), (B1), and (B2). Then,

$$\sup v_{\mathbf{y},n}\left(\Lambda_{n}\right) = \sup v_{\mathbf{y},n}\left(\bigcup_{P\in\mathcal{P}_{n}(\mathbf{y})}E\left(P\right)\right)$$

where E(P) denotes the set of extreme points of P.

Proof. By (B1), we have that $\Lambda_n = \bigcup_{P \in \mathcal{P}_n(\mathbf{y})} P$. We can, w.l.o.g., assume that all members of $\mathcal{P}_n(\mathbf{y})$ are nonempty (otherwise, remove all empty sets). $\sup v_{\mathbf{y},n}(\Lambda_n)$ exists due to (A2). Since for each $P \in \mathcal{P}_n(\mathbf{y})$, $\sup v_{\mathbf{y},n}(P) = \sup v_{\mathbf{y},n}(E(P))$ (Lemma 4.11), two applications of Lemma B.1 allow us to "commute" the supremum with the union to get

$$\sup v_{\mathbf{y},n} (\Lambda_n) = \sup v_{\mathbf{y},n} \left(\bigcup_{P \in \mathcal{P}_n(\mathbf{y})} P \right)$$
$$= \sup v_{\mathbf{y},n} \left(\bigcup_{P \in \mathcal{P}_n(\mathbf{y})} E(P) \right).$$

(4.1)

Theorem 4.12 reduces the region over which to search for an optimal control. When $\mathcal{P}_n(\mathbf{y})$ is a finite collection of convex polytopes, the situation is even nicer, as $\bigcup_{P \in \mathcal{P}_n(\mathbf{y})} E(P)$ is a finite set (a finite union of finite sets). If, in addition, \mathcal{P}_n is chosen independent of \mathbf{y} , we arrive at an optimal bang-bang control:

Corollary 4.13 (optimal bang-bang control). Suppose (A1) and (A2). Furthermore, suppose (B1), (B2), and (B3) for all $\mathbf{y} \in \Omega$. Finally, suppose that there exists \mathcal{P}_n s.t. $\mathcal{P}_n = \mathcal{P}_n(\mathbf{y})$ for all $\mathbf{y} \in \Omega$. Then, the general IVP introduced in §3 admits an optimal bang-bang control at time t_n (Definition 4.1) with

$$V\left(\mathbf{x}, t_{n}^{-}\right) = \sup v_{\mathbf{x}, n}\left(\Lambda_{n}\right) = \max v_{\mathbf{x}, n}\left(\hat{\Lambda}_{n}\right) \text{ on } \Omega$$

and

$$\hat{\Lambda}_n \equiv \bigcup_{P \in \mathcal{P}_n} E(P) \,.$$

Example 4.14. Let $\mathbf{y} \in [0, \infty)^2$. We now find $\mathcal{P}_n^L(\mathbf{y})$ s.t. (B1), (B2), and (B3) are satisfied for the GLWB. Take $P_1 \equiv [0, 1], P_2 \equiv [1, 2], and \mathcal{P}_n^L(\mathbf{y}) \equiv \{P_1, P_2\}$, satisfying (B3). Note that $\bigcup_{P \in \mathcal{P}_n^L(\mathbf{y})} P = [0, 2]$, satisfying (B1). It is trivial to show that the functions $f_{\mathbf{y},n}^L|_{P_j}$ and $\mathbf{f}_{\mathbf{y},n}^L|_{P_j}$ defined in (2.14) and (2.15) are convex as functions of λ (the maximum of convex functions is a convex function), thereby satisfying (B2). Since \mathbf{y} was arbitrary and \mathcal{P}_n^L was chosen independent of \mathbf{y} , we conclude (whenever (A1) and (A2) hold), by Corollary 4.13, that the supremum of $v_{\mathbf{y},n}^L$ occurs at

$$\hat{\Lambda}_{n}^{L} = E\left(P_{1}\right) \cup E\left(P_{2}\right) = E\left([0,1]\right) \cup E\left([1,2]\right) = \{0,1\} \cup \{1,2\} = \{0,1,2\}$$

(corresponding to nonwithdrawal, withdrawal at exactly the contract rate, and a full surrender).

Remark 4.15. When all the conditions required for Corollary 4.13 hold, with the exception that $\mathcal{P}_n(\mathbf{y})$ depends on \mathbf{y} , then an optimal control is not necessarily bang-bang, but does satisfy the bang-bang principle, Theorem 4.12. In many cases, this still results in considerable computational simplification (see Remark 5.3).

4.3 Preservation of convexity and monotonicity across exercise times

Since the convexity and monotonicity of V are desirable properties upon which the bang-bang principle depends (i.e. (A1)), we would like to ensure that they are preserved "across" exercise times (i.e. from t_n^+ to t_n^-).

Consider the *n*th exercise time, t_n . Suppose the following:

(C1) For each fixed $\lambda \in \Lambda_n$, $\mathbf{x} \mapsto \mathbf{f}_{\mathbf{x},n}(\lambda)$ and $\mathbf{x} \mapsto f_{\mathbf{x},n}(\lambda)$ are convex.².

²Note that this is not the same as (B2) Here, we mean that for each fixed
$$\lambda \in \Lambda_n$$
 and for all $\mathbf{x}, \mathbf{x}' \in \Omega$ and $\theta \in (0, 1)$,
 $f_{\theta \mathbf{x} + (1-\theta)\mathbf{x}', n}(\lambda) \leq \theta f_{\mathbf{x}, n}(\lambda) + (1-\theta) f_{\mathbf{x}', n}(\lambda)$

and

$$\mathbf{f}_{\theta\mathbf{x}+(1-\theta)\mathbf{x}',n}\left(\lambda\right) \leqslant \theta \mathbf{f}_{\mathbf{x},n}\left(\lambda\right) + (1-\theta) \,\mathbf{f}_{\mathbf{x}',n}\left(\lambda\right).$$

The order \leq used in (4.1) is that on $\Omega \subset \mathbb{R}^m$, inherited from the order on \mathbb{R}^m established in Remark 4.8.

(C2) For each $\mathbf{x}, \mathbf{x}' \in \Omega$ s.t. $\mathbf{x} \leq \mathbf{x}'$, there exist sequences $\{\lambda_k\}, \{\lambda'_k\} \in \Lambda_n^{\mathbb{N}}$ s.t. $v_{\mathbf{x},n}(\lambda_k) \to V_n^-(\mathbf{x})$, and for all $k, f_{\mathbf{x},n}(\lambda_k) \leq f_{\mathbf{x}',n}(\lambda'_k)$ and $\mathbf{f}_{\mathbf{x},n}(\lambda_k) \leq \mathbf{f}_{\mathbf{x}',n}(\lambda'_k)$.

Remark 4.16. (C2) simplifies greatly if for all \mathbf{x} , $v_{\mathbf{x},n}$ (Λ_n) contains its supremum.³ Denote this supremum $v_{\mathbf{x},n}$ ($\lambda_{\mathbf{x}}$), where $\lambda_{\mathbf{x}} \in \Lambda_n$ is an optimal action at \mathbf{x} . In this case, the following simpler assumption yields (C2): for each $\mathbf{x}, \mathbf{x}' \in \Omega$ s.t. $\mathbf{x} \leq \mathbf{x}'$, there exists $\lambda' \in \Lambda_n$ s.t. $f_{\mathbf{x},n}$ ($\lambda_{\mathbf{x}}$) $\leq f_{\mathbf{x}',n}$ (λ') and $\mathbf{f}_{\mathbf{x},n}$ ($\lambda_{\mathbf{x}}$) $\leq \mathbf{f}_{\mathbf{x}',n}$ (λ') (take $\lambda_k = \lambda_{\mathbf{x}}$ and $\lambda'_k = \lambda'$ for all k to arrive at (C2)).

This simpler condition states that for each pair of positions $\mathbf{x} \leq \mathbf{x}'$, there is an action λ' s.t. the position and cash flow after the event at \mathbf{x}' under action λ' are greater than (or equal to) the position and cash flow after the event at \mathbf{x} under an optimal action $\lambda_{\mathbf{x}}$. Intuitively, this guarantees us that the position \mathbf{x}' is more desirable than \mathbf{x} (from the holder's perspective). This is not a particularly restrictive assumption, and it should hold true for any model of a contract in which a larger position is more desirable than a smaller one.

Lemma 4.17. Suppose (A1), (A2), and (C1). Then, $\mathbf{x} \mapsto V_n^-(\mathbf{x})$ is convex.

Proof. Fix $\mathbf{x}, \mathbf{x}' \in \Omega$ and $\theta \in (0, 1)$, and let $\mathbf{z} \equiv \theta \mathbf{x} + (1 - \theta) \mathbf{x}'$. Then, by (A1) and (C1),

$$\begin{split} V_{n}^{-}\left(\mathbf{z}\right) &= \sup v_{\mathbf{z},n}\left(\Lambda_{n}\right) \\ &= \sup_{\lambda \in \Lambda_{n}}\left[V_{n}^{+}\left(\mathbf{f}_{\mathbf{z},n}\left(\lambda\right)\right) + f_{\mathbf{z},n}\left(\lambda\right)\right] \\ &\leqslant \sup_{\lambda \in \Lambda_{n}}\left[V_{n}^{+}\left(\theta\mathbf{f}_{\mathbf{x},n}\left(\lambda\right) + \left(1-\theta\right)\mathbf{f}_{\mathbf{x}',n}\left(\lambda\right)\right) + \theta f_{\mathbf{x},n}\left(\lambda\right) + \left(1-\theta\right)f_{\mathbf{x}',n}\left(\lambda\right)\right] \\ &\leqslant \theta \sup_{\lambda \in \Lambda_{n}}\left[V_{n}^{+}\left(\mathbf{f}_{\mathbf{x},n}\left(\lambda\right)\right) + f_{\mathbf{x},n}\left(\lambda\right)\right] + \left(1-\theta\right)\sup_{\lambda \in \Lambda_{n}}\left[V_{n}^{+}\left(\mathbf{f}_{\mathbf{x}',n}\left(\lambda\right)\right) + f_{\mathbf{x}',n}\left(\lambda\right)\right] \\ &= \theta \sup v_{\mathbf{x},n}\left(\Lambda_{n}\right) + \left(1-\theta\right)\sup v_{\mathbf{x}',n}\left(\Lambda_{n}\right) \\ &= \theta V_{n}^{-}\left(\mathbf{x}\right) + \left(1-\theta\right)V_{n}^{-}\left(\mathbf{x}'\right). \end{split}$$

Remark 4.18. Note that the proof of Lemma 4.17 involves using $V_n^-(\mathbf{y}) = \sup v_{\mathbf{y},n}(\Lambda_n)$ for $\mathbf{y} = \mathbf{x}, \mathbf{x}'$. If Λ_n is instead a function of the contract state (i.e. $\Lambda_n \equiv \Lambda_n(\mathbf{x})$), then the above proof methodology does not work since it is not necessarily true that $V_n^-(\mathbf{y}) = \sup v_{\mathbf{y},n}(\Lambda_n(\mathbf{z}))$ for $\mathbf{y} = \mathbf{x}, \mathbf{x}'$.

Lemma 4.19. Suppose (A1), (A2), and (C2). Then, $\mathbf{x} \mapsto V_n^-(\mathbf{x})$ is monotone.

Proof. Let $\mathbf{x}, \mathbf{x}' \in \Omega$ s.t. $\mathbf{x} \leq \mathbf{x}'$. By (A1) (specifically, since V_n^+ is monotone) and (C2), for each k,

$$\begin{aligned} v_{\mathbf{x},n}\left(\lambda_{k}\right) &= V_{n}^{+}\left(\mathbf{f}_{\mathbf{x}},n\left(\lambda_{k}\right)\right) + f_{\mathbf{x}},n\left(\lambda_{k}\right) \\ &\leqslant V_{n}^{+}\left(\mathbf{f}_{\mathbf{x}',n}\left(\lambda_{k}'\right)\right) + f_{\mathbf{x}',n}\left(\lambda_{k}'\right) \\ &= v_{\mathbf{x}',n}\left(\lambda_{k}'\right) \end{aligned}$$

Then,

$$V_{n}^{-}(\mathbf{x}) = \lim_{k \to \infty} v_{\mathbf{x},n}(\lambda_{k}) \leqslant \limsup_{k \to \infty} v_{\mathbf{x}',n}(\lambda_{k}') \leqslant \sup v_{\mathbf{x}',n}(\Lambda_{n}) = V_{n}^{-}(\mathbf{x}'),$$

as desired.

Example 4.20. We now show that the GLWB satisfies (C1) and (C2) given (A1) and (A2). It is trivial to show that the functions $f_{\mathbf{x},n}^{L}(\lambda)$ and $\mathbf{f}_{\mathbf{x},n}^{L}(\lambda)$ defined in (2.14) and (2.15) are convex in \mathbf{x} (the maximum of convex functions is a convex function), thereby satisfying (C1). (C2) is slightly more tedious to verify. Let $\mathbf{x}, \mathbf{x}' \in \Omega$ s.t. $\mathbf{x} \leq \mathbf{x}'$. By (A1), (A2) and the argument in Example 4.14, we can, w.l.o.g., assume $\lambda_{\mathbf{x}} \in \{0, 1, 2\}$, where $\lambda_{\mathbf{x}}$ denotes an optimal action at \mathbf{x} . Hence, we need only consider three cases:

³It is worthwhile to note that in practice, this is often the case; for fixed n, consider Λ_n compact and $\lambda \mapsto v_{\mathbf{x},n}(\lambda)$ continuous for all \mathbf{x} .

- 1. Suppose $\lambda_{\mathbf{x}} = 0$. Take $\lambda' = 0$ to get $f_{\mathbf{x},n}^{L}(0) = f_{\mathbf{x}',n}^{L}(\lambda')$ and $\mathbf{f}_{\mathbf{x},n}^{L}(0) \leq \mathbf{f}_{\mathbf{x}',n}^{L}(\lambda')$.
- 2. Suppose $\lambda_{\mathbf{x}} = 1$. W.l.o.g., we can assume $x'_2 \ge x_2 > 0$. Take $\lambda' = x_2/x'_2$ to get $f_{\mathbf{x},n}^L(1) = f_{\mathbf{x}',n}^L(\lambda')$ and $\mathbf{f}_{\mathbf{x},n}^L(1) \le \mathbf{f}_{\mathbf{x}',n}^L(\lambda')$.
- 3. Suppose $\lambda_{\mathbf{x}} = 2$. If $x_1 \leq \delta x_2$, then $f_{\mathbf{x},n}^L(2) = f_{\mathbf{x},n}^L(1)$ and $\mathbf{f}_{\mathbf{x},n}^L(2) = (0,0) \leq \mathbf{f}_{\mathbf{x},n}^L(1)$, and we can w.l.o.g. assume $x'_2 \geq x_2 > 0$ and once again take $\lambda' = x_2/x'_2$ to get $f_{\mathbf{x},n}^L(2) = f_{\mathbf{x}',n}^L(\lambda')$ and $\mathbf{f}_{\mathbf{x},n}^L(2) = (0,0) \leq \mathbf{f}_{\mathbf{x}',n}^L(\lambda')$. Therefore, we can safely assume that $x_1 > \delta x_2$ so that

$$f_{\mathbf{x},n}^{L}(2) = \mathcal{R}(n) \left[(1-\kappa) x_{1} + \kappa \delta x_{2} \right] \leqslant \mathcal{R}(n) x_{1}.$$

$$(4.2)$$

(a) Suppose $x'_1 \leq \delta x'_2$. Take $\lambda' = 1$ to get $\mathbf{f}^L_{\mathbf{x},n}(2) = (0,0) \leq \mathbf{f}^L_{\mathbf{x}',n}(1)$ and

$$f_{\mathbf{x},n}^{L}\left(2\right) \leqslant \mathcal{R}\left(n\right) x_{1} \leqslant \mathcal{R}\left(n\right) \delta x_{2}^{\prime} = f_{\mathbf{x}^{\prime},n}^{L}\left(1\right)$$

by (4.2).

(b) Suppose $x'_1 > \delta x'_2$. Take $\lambda' = 2$ to get $\mathbf{f}^L_{\mathbf{x},n}(2) = (0,0) = \mathbf{f}^L_{\mathbf{x}',n}(2)$ and

$$f_{\mathbf{x},n}^{L}(2) \leqslant \mathcal{R}(n) \left[(1 - \kappa_n) x_1' + \kappa \delta x_2' \right] = f_{\mathbf{x}',n}^{L}(2).$$

4.4 Preservation of convexity and monotonicity between exercise times

As previously mentioned, to apply Theorem 4.12, we need to check the validity of (A1) (i.e. that the solution is CM at t_n^+). In light of this, we would like to identify scenarios in which V_n^+ is CM provided that V_{n+1}^- is CM (i.e. convexity and monotonicity are preserved between exercise times).

Example 4.21. If we assume that both GLWB and GMWB are written on an asset that follows GBM, then Appendix A establishes the convexity and monotonicity (under sufficient regularity) of V_n^+ given the convexity and monotonicity of V_{n+1}^- . The general argument is applicable to contracts written on assets whose returns follow multidimensional drift-diffusions with parameters independent of the level of the asset (a local volatility model, for example, is not included in this class). Convexity and monotonicity preservation are retrieved directly from a property of the corresponding Green's function.

Although the methodology in Appendix A relates convexity and monotonicity to a general property of the Green's function (including the class of contracts driven by GBM), in the interest of intuition, we provide the reader with an alternate proof below using the linearity of the expectation operator along with the linearity of the stochastic process w.r.t. its initial value. Consider, in particular, the GLWB. Equation (2.13) stipulates

$$V_{n}^{+}(\mathbf{x}) = \tilde{\mathbb{E}} \Big[e^{-\int_{n}^{n+1} r(\tau) d\tau} V_{n+1}^{-} \left(X_{1} \left((n+1)^{-} \right), x_{2} \right) \\ + \int_{n}^{n+1} e^{-\int_{n}^{s} r(\tau) d\tau} \mathcal{M}(s) X_{1}(s) ds \mid X_{1}(n^{+}) = x_{1} \Big] \qquad on \ [0, \infty)^{2} \times \mathscr{T}.$$

Linearity allows us to consider the two terms appearing in the sum inside the conditional expectation separately. If each is convex in \mathbf{x} , so too is the entire expression. If $X_1(n^+) = x_1$,

$$X_1\left(s\right) = x_1Y\left(s\right)$$

between n and n+1, where

$$Y(s) \equiv \exp\left(\int_{n}^{s} \left[r(\tau) - \alpha(\tau) - \frac{1}{2}\sigma^{2}(\tau)\right] d\tau + \int_{n}^{s} \sigma(\tau) d\tilde{Z}(\tau)\right),$$

from which it is evident that $X_1(s)$ is convex in x_1 since Y depends only on time (note that the parameters appearing in Y are independent of the level of the asset, precluding a local volatility model). It remains to

show that the first term is also convex. Fix $\mathbf{y}, \mathbf{y}' \in [0, \infty)^2$, $\theta \in (0, 1)$, and let $\mathbf{x} \equiv \theta \mathbf{y} + (1 - \theta) \mathbf{y}'$. Then, by assuming that $V_{n+1}^-(\mathbf{x})$ is convex in \mathbf{x} ,

$$\begin{split} V_{n+1}^{-} \left(x_1 Y \left(\left(n+1 \right)^{-} \right), x_2 \right) &= V_{n+1}^{-} \left(\left(\theta y_1 + \left(1-\theta \right) y_1' \right) Y \left(\left(n+1 \right)^{-} \right), \theta y_2 + \left(1-\theta \right) y_2' \right) \\ &\leq \theta V_{n+1}^{-} \left(y_1 Y \left(\left(n+1 \right)^{-} \right), y_2 \right) + \left(1-\theta \right) V_{n+1}^{-} \left(y_1' Y \left(\left(n+1 \right)^{-} \right), y_2' \right). \end{split}$$

One can use the same technique to show that monotonicity is preserved. An identical argument can be carried out for the GMWB.

Convexity and monotonicity preservation are established for a stochastic volatility model in [4]. For the case of general parabolic equations, convexity preservation is established in [16]. This result is further generalized to parabolic integro-differential equations, arising from problems involving assets whose returns follow jump-diffusion processes [5].

4.5 Existence of an optimal bang-bang control

Once we have established that convexity and monotonicity are preserved across and between exercise times (i.e. §4.3 and §4.4, respectively), we need only apply our argument inductively to establish the existence of an optimal bang-bang control. Instead of providing a proof for the general case, we simply focus on the GLWB contract here. For the case of a general contract, assuming the dynamics followed by the assets preserve the convexity and monotonicity of the cost of funding the contract between exercise times (e.g. GBM, as in Appendix A), the reader can apply the same techniques to establish the existence of a bang-bang control.

Example 4.22. Consider the GLWB. Suppose that for some $n \text{ s.t. } 0 \leq n < N, V_{n+1}^-$ is CM. By Example 4.21, V_n^+ is also CM. Under sufficient regularity (see Appendix A), for fixed \mathbf{x} , $v_{\mathbf{x},n}$ is bounded above (satisfying (A2)). Since (A1) and (A2) are satisfied, we can use Example 4.14 to conclude that the supremum of $v_{\mathbf{x},n}$, for each $\mathbf{x} \in \Omega$, occurs on $\{0,1,2\}$. By Example 4.20, V_n^- is convex and monotone.

By (2.10) and (2.11), $V(\mathbf{x}, N) = 0$. Since $V(\mathbf{x}, N)$ is trivially CM as a function of \mathbf{x} , we can apply the above argument inductively to establish the existence of an optimal bang-bang control.

5 Numerical Examples

To demonstrate the bang-bang principle in practice, we implement a numerical method to solve the GLWB and GMWB problems and examine loss-maximizing withdrawal strategies.

5.1 Contract pricing algorithm

Algorithm 1 highlights the usual dynamic programming approach to pricing contracts with finitely many exercise times. Note that line 2 is purposely non-specific; the algorithm does not presume anything about the underlying dynamics of the stochastic process(es) that V is a function of, and as such does not make mention of a particular numerical method used to solve V_n^+ given V_{n+1}^- . Establishing that the control is bang-bang for a particular contract allows us to replace Λ_n appearing on line 4 with a finite subset of itself.

5.2 Numerical method

The numerical method discussed here applies to both GLWB and GMWB contracts. Each contract is originally posed on $\Omega = [0, \infty)^2$. We employ Algorithm 1 but instead approximate the solution using a finite difference method on the truncated domain $[0, x_1^{\max}] \times [0, x_2^{\max}]$. As such, since $\mathbf{f}_{\mathbf{x},n}(\lambda)$ will not necessarily land on a mesh node, linear interpolation is used to approximate $V_n^+(\mathbf{f}_{\mathbf{x},n}(\lambda))$ on line 4. A local optimization problem is solved for each point on the finite difference grid. Details of the numerical scheme can be found in [1, 10].

Data: payoff at the expiry, $V_N = \varphi$ Result: price of the contract at time zero, $V_0 \equiv V_0^-$ 1 for $n \leftarrow N - 1$ to 0 do 2 | use V_{n+1}^- to determine V_n^+ 3 | for $\mathbf{x} \in \Omega$ do 4 | $V_n^-(\mathbf{x}) \equiv \sup_{\lambda \in \Lambda_n} V_n^+(\mathbf{f}_{\mathbf{x},n}(\lambda)) + f_{\mathbf{x},n}(\lambda)$ 5 | end 6 end

Algorithm 1: Dynamic programming for pricing contracts with finitely many exercise times.

Between exercise times, the cost of funding each contract satisfies one of (2.8) or (2.16). Corresponding to line 2 of Algorithm 1, we determine V_n^+ from V_{n+1}^- using an implicit finite difference discretization. No additional boundary condition is needed at $x_1 = 0$ or $x_2 = 0$ ((2.8) and (2.16) hold along $\partial \Omega \times [t_n, t_{n+1})$). The same is true of $x_2 = x_2^{\max} \gg 0$. At $x_1 = x_1^{\max} \gg 0$, we impose

$$V(x_1^{\max}, x_2, t) = g(t) x_1^{\max}$$
(5.1)

for some function g differentiable everywhere but possibly at the exercise times n. Substituting the above into (2.8) or (2.16) yields an ordinary differential equation which is solved numerically alongside the rest of the domain. Errors introduced by the above approximations are small in the region of interest, as verified by numerical experiments.

Remark 5.1. Since we advance the numerical solution from n^- to $(n-1)^+$ using a convergent method, the numerical solution converges pointwise to a solution V that is convexity and monotonicity preserving. Although it is possible to show—for special cases—that convexity and monotonicity are preserved for finite mesh sizes, this is not necessarily true unconditionally.

Remark 5.2. Although we have shown that an optimal bang-bang control exists for the GLWB problem, we do not replace Λ_n with $\{0, 1, 2\}$ on line 4 of Algorithm 1 when computing the cost to fund a GLWB in §5.3.1 so as to demonstrate that our numerical method, having preserved convexity and monotonicity, selects an optimal bang-bang control. For both GLWB and GMWB, We assume that nothing is known about $v_{\mathbf{x},n}$ and hence form a partition

$$\lambda_1 < \lambda_2 < \cdots < \lambda_p$$

of the admissible set and perform a linear search

5.3 Results

5.3.1 Guaranteed Lifelong Withdrawal Benefits. Figure 5.1 shows withdrawal strategies for the holder under the parameters in Table 5.1 on the first four contract anniversaries. We can clearly see that the optimal control is bang-bang from the figures. At any point (\mathbf{x}, n) , we see that the holder performs one of nonwithdrawal, withdrawal at exactly the contract rate, or a full surrender (despite being afforded the opportunity to withdraw any amount between nonwithdrawal and a full surrender).

When the withdrawal benefit is much larger than the investment account, the optimal strategy is withdrawal at the contract rate (the guarantee is in the money). Conversely, when the investment account is much larger than the withdrawal benefit, the optimal strategy is surrender (the guarantee is out of the money), save for when the holder is anticipating the triennial ratchet (times n = 2 and n = 3). Otherwise, the optimal strategy is constant along any straight line through the origin since the solution is homogeneous of order one in \mathbf{x} , as discussed in [10].

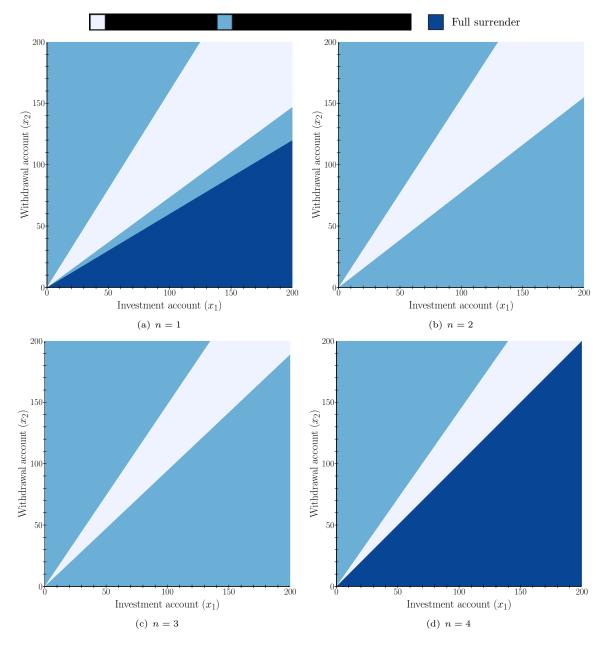


FIGURE 5.1: Optimal control for the GLWB for data in Table 5.1. As predicted, there exists an optimal control consisting only of nonwithdrawal, withdrawal at the contract rate, and a full surrender.

Parameter		Value
Volatility	σ	0.20
Risk-free rate	r	0.04
Hedging fee	α	0.015
Contract rate	δ	0.05
Bonus rate	β	0.06
Expiry	N	57
Initial investment	w_0	100
Initial age at time zero		65
Mortality data		[20]
Ratchets		Triennial
Withdrawals		Annual

TABLE 5.1: GLWB parameters.

Penalty κ_n
0.03
0.02
0.01
0.00

5.3.2 Guaranteed minimum withdrawal benefit. For the GMWB, (C1) is violated. In particular, for $\kappa_n > 0$, the function $f_{\mathbf{x},n}^{\mathrm{M}}(\lambda)$ is concave as a function of \mathbf{x} . However, when $\kappa_n = 0$ or G = 0 (G = 0 is considered in [15]), the function $f_{\mathbf{x},n}^{\mathrm{M}}(\lambda)$ (see (2.7)) is linear in \mathbf{x} , and hence the convexity of V_n^- can be guaranteed given V_n^+ CM. In this case, it is possible to use the same machinery as was used in the GLWB case to arrive at a bang-bang principle (see Theorem 4.12). The case of $\kappa_n = 0$ corresponds to zero surrender charges at the *n*th anniversary, while G = 0 corresponds to enforcing that all withdrawals (regardless of size) be charged at the penalty rate.

Now, consider the data in Table 5.2. Since $\kappa_n = 0$ for all $n \ge 7$, the convexity of V in **x** is preserved for all $t \in (6, N]$. However, since $\kappa_6 > 0$, the convexity is violated as $t \to 6^-$. Figure 5.2 demonstrates this preservation and violation of convexity. As a consequence, V will not necessarily be convex in **x** as $t \to 5^+$, and the contract fails to satisfy the bang-bang principle at each anniversary date $n \le 5$.

Note that for $x_2 > 0$, the conditions $\lambda x_2 \in [0, G \wedge x_2]$ and $\lambda x_2 \in (G \wedge x_2, x_2]$ appearing in (2.7) are equivalent to $\lambda \in [0, G/x_2 \wedge 1]$ and $\lambda \in (G/x_2 \wedge 1, 1]$, respectively. Assuming that V_n^+ is CM and taking $\mathcal{P}_n^{\mathrm{M}}(\mathbf{x}) \equiv \{P_1, P_2\}$ with $P_1 \equiv [0, G/x_2 \wedge 1]$ and $P_2 \equiv [G/x_2 \wedge 1, 1]$ yields that there exists an optimal control taking on one of the values in $\{0, G/x_2, 1\}$ at any point (\mathbf{x}, n) with $x_2 > 0$. These three actions correspond to nonwithdrawal, withdrawing the predetermined amount G, or performing a full surrender. This is verified by Figure 5.3, which shows withdrawal strategies under the parameters in Table 5.2 at times n = 6 and n = 7. As predicted, along any line $x_2 = \text{const.}$, the optimal control takes on one of a finite number of values. Since at n = 6, $\kappa_n > 0$, we see that the holder is more hesitant to surrender the contract whenever $x_1 \gg x_2$ (compare with the same region at n = 7). Control figures for GMWBs not satisfying the bang-bang principle can be seen in the numerical results in [9, 7].

Remark 5.3. Consider a GMWB with $\kappa_n = 0$ for all withdrawal times n. As suggested by the above, this contract satisfies the bang-bang principle (in particular, Theorem 4.12 is satisfied) everywhere. However, the GMWB does not necessarily yield an optimal bang-bang control since $\mathcal{P}_n^M(\mathbf{x})$ depends on x_2 (Corollary 4.13 is not satisfied). For example, consider an optimal control for the GMWB taking on the value G/x_2 at each \mathbf{x} with $x_2 > 0$. Such a control's range is a superset of $(0, \infty)^2$ (not a finite set). However, in this case, the bang-bang principle guarantees that for fixed x_2 , a finite subset of the admissible set need only be considered in the corresponding optimization problem. Computationally, this is just as desirable as the case of an optimal bang-bang control.

FIGURE 5.2: $V(\mathbf{x},t)$ for fixed $x_1 = 100$ under the data in Table 5.2. Points where $V(\mathbf{x},n^-) = V(\mathbf{x},n^+)$ correspond to nonwithdrawal. To the left of these points, the holder performs withdrawal (see Figure 5.3).

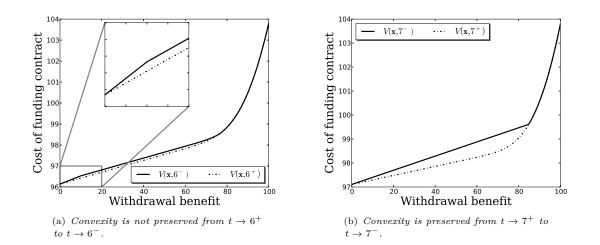
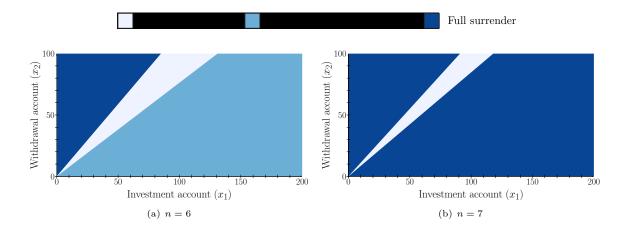


TABLE 5.2: GMWB parameters [8].

Parameter		Value	Anniversary n	Penalty κ_n
Volatility	σ	0.15	1	0.08
Risk-free rate	r	0.05	2	0.07
Hedging fee	α	0.01	3	0.06
Contract rate	G	10	4	0.05
Expiry	N	10	5	0.04
Initial investment	w_0	100	6	0.03
Withdrawals		Annual	$\geqslant 7$	0.00

FIGURE 5.3: Optimal control $\lambda_{\mathbf{x}}$ scaled by x_2 for the data in Table 5.2.



6 Conclusion

Although it is commonplace in the insurance literature to assume the existence of optimal bang-bang controls, there does not appear to be a rigorous statement of this result. We have rigorously derived sufficient conditions which guarantee the existence of optimal bang-bang controls for GMxB guarantees.

These conditions require that the contract features be such that the solution to the optimal control can be formulated as maximizing a convex objective function, and that the underlying stochastic process assumed for the risky assets preserves convexity and monotonicity.

These conditions are non-trivial, in that the conditions are satisfied for the GLWB contract but not for the GMWB contract with typical contract parameters. From a practical point of view, the existence of optimal bang-bang controls allows for the use of very efficient numerical methods.

Although we have focused specifically on the application of our results to GMxB guarantees, the reader will have no difficulty in applying the sufficient conditions to other optimal control problems in finance. We believe that we can also use an approach similar to that used here to establish the existence of optimal bang-bang controls for general impulse control problems. In the impulse control case, these conditions will require that the intervention operator have a particular form and that the stochastic process (without intervention) preserve convexity and monotonicity. We leave this generalization for future work.

A Preservation of convexity and monotonicity

In this appendix, we establish the convexity and monotonicity of a contract whose payoff is CM and written on assets whose returns follow (multidimensional, possibly correlated) GBM. We do so by considering the PDE satisfied by V and the fundamental solution corresponding to the operator appearing in the log-transformed version of this PDE. Considering the log-transformed PDE allows us to eliminate the parabolic degeneracy at the boundaries and to argue that the resulting fundamental solution for the log-transformed operator should be of the form $\Gamma(\mathbf{y}, \mathbf{y}', \tau, \tau') \equiv \Gamma(\mathbf{y} - \mathbf{y}', \tau, \tau')$.

We begin by describing some of the notation used in this appendix:

- Let $\Omega \equiv \Omega_1 \times \Omega_2$ where $\Omega_1 \equiv (0, \infty)^m$ and Ω_2 is a convex subset of a partially ordered vector space A over \mathbb{R} with order \leq_A . Ω can thus be considered as a convex subset of the vector space $\mathcal{A} \equiv \mathbb{R}^m \times A$ over \mathbb{R} .
- We write an element of Ω in the form $(\mathbf{x}, x_{m+1}) \equiv (x_1, \ldots, x_m, x_{m+1})$ with $\mathbf{x} \in \Omega_1$ and $x_{m+1} \in \Omega_2$ in order to distinguish between elements of Ω_1 and Ω_2 .
- The partial order we consider on \mathcal{A} is simply inherited from the orders \leq on \mathbb{R}^m (Remark 4.8) and \leq_A . Specifically, $(\mathbf{x}, x_{m+1}) \leq_{\mathcal{A}} (\mathbf{x}', x'_{m+1})$ if and only if $\mathbf{x} \leq_{\mathbb{R}^m} \mathbf{x}'$ and $x_{m+1} \leq_A x'_{m+1}$.

Suppose V satisfies

$$\partial_t V + \mathcal{L}V + \omega = 0 \text{ on } \Omega \times (t_n, t_{n+1})$$
(A.1)

and

$$V\left(\mathbf{x}, x_{m+1}, t_{n+1}^{-}\right) = \varphi\left(\mathbf{x}, x_{m+1}\right) \text{ on } \Omega$$
(A.2)

where

$$\mathcal{L} \equiv \frac{1}{2} \sum_{i,j=1}^{m} a_{i,j} x_i x_j \partial_{x_i} \partial_{x_j} + \sum_{i=1}^{m} b_i x_i \partial_{x_i} + c.$$
(A.3)

In the above, $\omega \equiv \omega(\mathbf{x}, t)$. We will, for the remainder of this appendix, assume the following:

(D1) $a_{i,j} \equiv a_{i,j}(t), b_i \equiv b_i(t)$, and $c \equiv c(t)$ (i.e. the functions $a_{i,j}, b_i$ and c are independent of (\mathbf{x}, x_{m+1})). (D2) $\sum_{i,j=1}^{m} a_{i,j} \partial_{x_i} \partial_{x_j}$ is uniformly elliptic. **Example A.1.** For the GLWB guarantee, \mathcal{L} is given in (2.9) and $\omega = \mathcal{M}(t) x_1$.

Remark A.2. We say V satisfies (A.1) if V is twice differentiable in (the components of) \mathbf{x} and once differentiable in t on $\Omega \times (t_n, t_{n+1})$,⁴ continuous on $\Omega \times (t_n, t_{n+1}]$,⁵ and satisfies (A.1) pointwise.

We now describe the log-transformed problem. For ease of notation, let

$$e^{\mathbf{y}} \equiv (e^{y_1}, \dots, e^{y_m}) \qquad \qquad a'_{i,j}(\tau) \equiv a_{i,j}(t_{n+1} - \tau)$$

$$\varphi'(\mathbf{y}, y_{m+1}) \equiv \varphi(e^{\mathbf{y}}, y_{m+1}) \qquad \qquad b'_i(\tau) \equiv b_i(t_{n+1} - \tau)$$

$$\omega'(\mathbf{y}, y_{m+1}, \tau) \equiv \omega(e^{\mathbf{y}}, y_{m+1}, t_{n+1} - \tau) \qquad \qquad c'(\tau) \equiv c(t_{n+1} - \tau)$$

and $\Omega' \equiv \mathbb{R}^m \times \Omega_2$. Let V be a solution of the Cauchy problem (A.1) and (A.2). Let

$$u\left(\mathbf{y}, y_{m+1}, \tau\right) \equiv V\left(e^{\mathbf{y}}, y_{m+1}, t_{n+1} - \tau\right)$$

and $\Delta \equiv t_{n+1} - t_n$. Then, *u* satisfies

$$\mathcal{L}'u - \partial_{\tau}u + \omega' = 0 \text{ on } \Omega' \times (0, \Delta)$$
(A.4)

and

$$u\left(\mathbf{y}, y_{m+1}, 0\right) = \varphi'\left(\mathbf{y}, y_{m+1}\right) \tag{A.5}$$

where

$$\mathcal{L}' \equiv \frac{1}{2} \sum_{i,j=1}^{m} a'_{i,j} \partial_{y_i} \partial_{y_j} + \sum_{i=1}^{m} b'_i \partial_{y_i} + c'.$$

Note that (D2) implies that \mathcal{L}' is uniformly elliptic.

In order to guarantee that a solution u to the log-transformed Cauchy problem (A.4) and (A.5) exists, and is unique, sufficient regularity must be imposed on φ' , \mathcal{L}' , and ω' . We summarize below.

- (E1) For each y_{m+1} , $\mathbf{y} \mapsto \varphi'(\mathbf{y}, y_{m+1})$ is continuous on \mathbb{R}^m .
- (E2) The coefficients of \mathcal{L}' are sufficiently regular.
- (E3) For each $y_{m+1} \in \Omega_2$, $(\mathbf{y}, \tau) \mapsto \omega'(\mathbf{y}, y_{m+1}, \tau)$ is sufficiently regular.
- (E4) u satisfies a growth condition as $|x| \to \infty$.

For an accurate detailing of the required regularity, see [11, Chap. 1: Thms. 12 and 16].

When (D2) and (E1)—(E4) are satisfied, the solution u can be written as

$$u\left(\mathbf{y}, y_{m+1}, \tau\right) = \int_{\mathbb{R}^m} \Gamma\left(\mathbf{y}, \mathbf{y}', \tau, 0\right) \varphi'\left(\mathbf{y}', y_{m+1}\right) d\mathbf{y}' + \int_0^\Delta \int_{\mathbb{R}^m} \Gamma\left(\mathbf{y}, \mathbf{y}', \tau, \tau'\right) \omega'\left(\mathbf{y}', y_{m+1}, \tau'\right) d\mathbf{y}' d\tau' \text{ on } \mathbb{R}^m \times (0, \Delta) \quad (A.6)$$

where Γ is the fundamental solution for \mathcal{L}' (whose construction was first detailed by [17]). We first note that (E1) follows immediately if φ is convex, as shown below.

Lemma A.3. If φ is convex w.r.t. the order $\leq_{\mathcal{A}}$ (see Definition 4.3), then for all x_{m+1} , $\mathbf{y} \mapsto \phi'(\mathbf{y}; x_{m+1})$ is continuous on \mathbb{R}^m .

⁴i.e. $V|_{\Omega \times (t_n, t_{n+1})} \in C^{2,1}(\Omega \times (t_n, t_{n+1})).$ ⁵i.e. $V|_{\Omega \times (t_n, t_{n+1}]} \in C(\Omega \times (t_n, t_{n+1}]).$

Proof. We have assumed that $\varphi \equiv \varphi(\mathbf{x}, x_{m+1})$ is convex w.r.t. $\leq_{\mathcal{A}}$ on Ω . From this it follows that for all $x_{m+1} \in \Omega_2, \varphi$ is convex in \mathbf{x} on $\Omega_1 \equiv [0, \infty)^m$ w.r.t. to the order \leq on \mathbb{R}^m . This in turn yields that for all $x_{m+1} \in \Omega_2, \varphi$ is continuous in \mathbf{x} on Ω_1 . Therefore, $\varphi' \equiv \varphi'(\mathbf{y}; x_{m+1})$ is continuous in \mathbf{y} on \mathbb{R}^m . \Box

Theorem A.4. Suppose (D1), (D2) and (E2)—(E4). Suppose that φ is CM w.r.t. the order $\leq_{\mathcal{A}}$ (see Definition 4.3 and Lemma 4.7). Suppose further that for all $t \in (t_n, t_{n+1}]$, ω is CM in (\mathbf{x}, x_{m+1}) on Ω w.r.t. the order $\leq_{\mathcal{A}}$. Then, for all $t \in (t_n, t_{n+1}]$, V is CM in (\mathbf{x}, x_{m+1}) on Ω w.r.t. the order $\leq_{\mathcal{A}}$. In particular, V_n^+ is CM.

Proof. Γ appearing in (A.6) depends on \mathbf{y}' and \mathbf{y} through $\mathbf{y}' - \mathbf{y}$ alone since by (D1), $a'_{i,j}$, b'_i and c' are independent of the spatial variables [11, Chap. 9: Thm. 1]. Therefore

$$\begin{split} u\left(\mathbf{y}, y_{m+1}, \tau\right) &= \int_{\mathbb{R}^m} \Gamma\left(\mathbf{y}' - \mathbf{y}, \tau, 0\right) \varphi'\left(\mathbf{y}', y_{m+1}\right) d\mathbf{y}' \\ &+ \int_0^\Delta \int_{\mathbb{R}^m} \Gamma\left(\mathbf{y}' - \mathbf{y}, \tau, \tau'\right) \omega'\left(\mathbf{y}', y_{m+1}, \tau'\right) d\mathbf{y}' d\tau' \text{ on } \mathbb{R}^m \times (0, \Delta) \,. \end{split}$$

Let $\log \mathbf{x} \equiv (\log x_1, \ldots, \log x_m)$. From the above, whenever $x_i > 0$ for all $i \leq m$,

$$V(\mathbf{x}, x_{m+1}, t) = \int_{\mathbb{R}^m} \Gamma(\mathbf{y}' - \log \mathbf{x}, t_{n+1} - t, 0) \varphi\left(e^{\mathbf{y}'}, x_{m+1}\right) d\mathbf{y}' + \int_0^\Delta \int_{\mathbb{R}^m} \Gamma(\mathbf{y}' - \log \mathbf{x}, t_{n+1} - t, \tau') \omega\left(e^{\mathbf{y}'}, x_{m+1}, t_{n+1} - \tau'\right) d\mathbf{y}' d\tau' \text{ on } \Omega \times (t_n, t_{n+1}).$$

Denote by $\mathbf{x} \circ \mathbf{x}' \equiv (x_1 x'_1, \dots, x_m x'_m)$ the elementwise product of \mathbf{x} and \mathbf{x}' . The substitution $\mathbf{y}' = \log(\mathbf{x} \circ \mathbf{x}')$ into the above yields

$$V(\mathbf{x}, x_{m+1}, t) = \int_0^\infty \dots \int_0^\infty \Gamma\left(\log \mathbf{x}', t_{n+1} - t, 0\right) \varphi\left(\mathbf{x} \circ \mathbf{x}', x_{m+1}\right) \frac{1}{\prod_i x_i'} d\mathbf{x}' + \int_0^\Delta \int_0^\infty \dots \int_0^\infty \Gamma\left(\log \mathbf{x}', t_{n+1} - t, \tau'\right) \omega\left(\mathbf{x} \circ \mathbf{x}', x_{m+1}, t_{n+1} - \tau'\right) \frac{1}{\prod_i x_i'} d\mathbf{x}' d\tau' \text{ on } \Omega \times (t_n, t_{n+1}).$$

Since Γ is > 0 [11, Chap. 2: Thm. 11] (a related, arguably more general result is given in [12, Chap. IV: Prop. 1.11]), from the convexity and monotonicity of V_{n+1}^- and ω , it follows immediately that $V_n(\mathbf{x}, x_{m+1}, t)$ is CM on Ω for any $t \in (t_n, t_{n+1})$.

Remark A.5. We can extend our construction to $\overline{\Omega_1} \times \Omega_2$ by taking limits of V up to the boundary. Since the codomain of V is Hausdorff, this extension is unique.

B Commutativity of union and supremum

Let T be a poset with order \leq satisfying the least-upper-bound property. All supremums are taken w.r.t. T. Lemma B.1. Let $S \equiv \{S_{\alpha}\}_{\alpha \in \mathcal{A}}$ be an indexed family of nonempty subsets of T. Let $S \equiv \bigcup_{\alpha \in \mathcal{A}} S_{\alpha}$ and

$$U \equiv \{\sup S_{\alpha}\}_{\alpha \in \mathcal{A}}.$$

Then, $\sup S = \sup U$ whenever S is bounded above.

Proof. Suppose \mathcal{A} is empty. Then both S and U are empty, and hence the expressions agree.

Suppose \mathcal{A} is nonempty and that S is bounded above. Since S is bounded above, its supremum u must occur in T. For each α , u is an upper bound of S_{α} , and since S_{α} is a nonempty subset of T, sup $S_{\alpha} = u_{\alpha}$ for some $u_{\alpha} \in T$. Thus, $U = \{u_{\alpha}\}_{\alpha \in \mathcal{A}} \subset T$. Since $u_{\alpha} \leq u$ for each α , u is an upper bound of U. Since \mathcal{A} is nonempty, U is nonempty and hence U has a least upper bound $u' \in T$ with $u' \leq u$. Let $x \in S$. Then $x \in S_{\beta}$ for some β , and hence $x \leq u_{\beta} \leq u'$ so that u' is an upper bound of S. Since $\sup S = u$, $u \leq u'$ and hence u' = u.

References

- P. Azimzadeh, P. A. Forsyth, and K. R. Vetzal. Hedging costs for variable annuities under regimeswitching. In R. Mamon and R. Elliot, editors, *Hidden Markov Models in Finance Volume II: Further Developments and Applications*, pages 503–528. Springer, New York, 2014.
- [2] A. R. Bacinello, P. Millossovich, A. Olivieri, and E. Pitacco. Variable annuities: A unifying valuation approach. *Insurance Math. Econom.*, 49(3):285–297, 2011.
- [3] D. Bauer, A. Kling, and J. Russ. A universal pricing framework for guaranteed minimum benefits in variable annuities. ASTIN Bulletin-Actuarial Studies in Non Life Insurance, 38(2):621–651, 2008.
- [4] Y. Z. Bergman, B. D. Grundy, and Z. Wiener. General properties of option prices. J. Finance, 51(5):1573-1610, 1996.
- [5] B. Bian and P. Guan. Convexity preserving for fully nonlinear parabolic integro-differential equations. Methods Appl. Anal, 15:39–51, 2008.
- [6] B. A. Butrica, H. M. Iams, K. E. Smith, and E. J. Toder. The disappearing defined benefit pension and its potential impact on the retirement incomes of baby boomers. *Social Security Bulletin*, 69, 2009.
- [7] Z. Chen and P. A. Forsyth. A numerical scheme for the impulse control formulation for pricing variable annuities with a guaranteed minimum withdrawal benefit (GMWB). *Numer. Math.*, 109:535–569, 2008.
- [8] Z. Chen, K. R. Vetzal, and P. A. Forsyth. The effect of modelling parameters on the value of GMWB guarantees. *Insurance Math. Econom.*, 43(1):165–173, 2008.
- M. Dai, Y. K. Kwok, and J. Zong. Guaranteed minimum withdrawal benefit in variable annuities. *Math. Finance*, 18(4):595–611, 2008.
- [10] P. A. Forsyth and K. R. Vetzal. An optimal stochastic control framework for determining the cost of hedging of variable annuities. J. Econom. Dynam. Control, 44:29–53, 2014.
- [11] A. Friedman. Partial differential equations of parabolic type. 1964.
- [12] M. G. Garroni and J. L. Menaldi. Green functions for second order parabolic integro-differential problems. Pitman Res. Notes Math. Ser. 275, Longman Scientific & Technical, Harlow, UK, 1992.
- [13] D. Holz, A. Kling, and J. Ruß. GMWB for life an analysis of lifelong withdrawal guarantees. Zeitschrift für die gesamte Versicherungswissenschaft, 101(3):305–325, 2012.
- [14] Y. Huang and P. A. Forsyth. Analysis of a penalty method for pricing a guaranteed minimum withdrawal benefit (GMWB). IMA J. Numer. Anal., 32:320–351, 2012.
- [15] Y. T. Huang and Y. K. Kwok. Analysis of optimal dynamic withdrawal policies in withdrawal guarantee products. J. Econom. Dynam., 45:19–43, 2014.
- [16] S. Janson and J. Tysk. Preservation of convexity of solutions to parabolic equations. J. Differential Equations, 206(1):182–226, 2004.

- [17] E. E. Levi. Sulle equazioni lineari totalmente ellittiche alle derivate parziali. Rendiconti del circolo Matematico di Palermo, 24(1):275–317, 1907.
- [18] M. A. Milevsky and T. S. Salisbury. Financial valuation of guaranteed minimum withdrawal benefits. *Insurance Math. Econom.*, 38(1):21–38, 2006.
- [19] A. Ngai and M. Sherris. Longevity risk management for life and variable annuities: The effectiveness of static hedging using longevity bonds and derivatives. *Insurance Math. Econom.*, 49(1):100–114, 2011.
- [20] U. Pasdika and J. Wolff. Coping with longevity: The new German annuity valuation table DAV 2004 R. In The Living to 100 and Beyond Symposium, Orlando Florida, 2005.
- [21] G. Piscopo and S. Haberman. The valuation of guaranteed lifelong withdrawal benefit options in variable annuity contracts and the impact of mortality risk. North American Actuarial Journal, 15(1):59–76, 2011.
- [22] R. T. Rockafellar. Convex analysis. Princeton University press, Princeton, N.J., 1997.
- [23] J. Wang and P. A. Forsyth. Maximal use of central differencing for Hamilton-Jacobi-Bellman PDEs in finance. SIAM J. Numer. Anal., 46(3):1580–1601, 2008.