

ISLANDS IN GRAPHS ON SURFACES

LOUIS ESPERET AND PASCAL OCHEM

ABSTRACT. An island in a graph is a set X of vertices, such that each element of X has few neighbors outside X . In this paper, we prove several bounds on the size of islands in large graphs embeddable on fixed surfaces. As direct consequences of our results, we obtain that:

- (1) Every graph of genus g can be colored from lists of size 5, in such a way that each monochromatic component has size $O(g)$. Moreover all but $O(g)$ vertices lie in monochromatic components of size at most 3.
- (2) Every triangle-free graph of genus g can be colored from lists of size 3, in such a way that each monochromatic component has size $O(g)$. Moreover all but $O(g)$ vertices lie in monochromatic components of size at most 10.
- (3) Every graph of girth at least 6 and genus g can be colored from lists of size 2, in such a way that each monochromatic component has size $O(g)$. Moreover all but $O(g)$ vertices lie in monochromatic components of size at most 16.

While (2) is optimal up to the size of the components, we conjecture that the size of the lists can be decreased to 4 in (1), and the girth can be decreased to 5 in (3). We also study the complexity of minimizing the size of monochromatic components in 2-colorings of planar graphs.

1. INTRODUCTION

In this paper we consider a relaxed version of the classical notion of proper coloring of a graph. We are interested in vertex colorings of graphs with the property that each color class consists of the disjoint union of (connected) components of bounded size. These components are said to be *monochromatic*, and the *size* of a monochromatic component is its number of vertices. A proper coloring is the same as a coloring in which every monochromatic component has size 1, so by allowing monochromatic components of larger size, one expects that the minimum number of colors needed might decrease significantly. For instance it was proved by Haxell, Szabó and Tardos [8] that every graph with maximum degree at most 5 can be 2-colored in such a way that all monochromatic components have size at most 20000 (such graphs have chromatic number as large as 6).

It was conjectured by Hadwiger that every graph with no K_t -minor has a proper coloring with $t - 1$ colors. The case $t = 5$ was shown to be equivalent to the famous 4 Color Theorem, which states that every planar graph has a proper 4-coloring. On the other hand, it was proved by Kleinberg, Motwani, Raghavan, and Venkatasubramanian [13], and independently by Alon, Ding,

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Oporowski and Vertigan [1] that there is no constant c such that every planar graph has a 3-coloring in which every monochromatic component has size at most c . More generally, for every t , there are graphs with no K_t -minor that cannot be colored with $t - 2$ colors such that all monochromatic components have size bounded by a function of t (see [14]). It follows that the bound predicted by Hadwiger's conjecture (and proved for $t = 5, 6$) on the chromatic number of a graph with no K_t -minor is best possible, even in our relaxed setting.

In this paper we prove that the bound can be significantly decreased in the specific case of graphs embeddable on surfaces of bounded genus. Given a graph G , a k -list assignment L (for the vertices of G) is a collection of lists $L(v)$, $v \in V(G)$, such that each list contains at least k elements. Given a list assignment L , an L -coloring c of G is the choice of an element $c(v) \in L(v)$ for each vertex $v \in V(G)$. Unless stated otherwise, such a coloring is not necessarily proper. It was proved by Thomassen [17] that for every planar graph G and every 5-list assignment L , the graph G has a proper L -coloring. We will prove that the same holds from any graph embeddable on a surface of genus g , provided that monochromatic components are only required to have size bounded by $O(g)$ (Theorem 2). The fact that cliques of order $\Omega(\sqrt{g})$ can be embedded on such surfaces shows that the size of monochromatic components has to depend on g . Moreover we will show that we can find a list-coloring in which all vertices except $O(g)$ of them lie in monochromatic components of size at most 3.

A theorem of Grötzsch [7] states that every triangle-free planar graph has a proper 3-coloring. Esperet and Joret [5] proved that there exist no constant c such that every triangle-free planar graph has a 2-coloring in which every monochromatic component has size at most c . Hence, it follows again that Grötzsch's theorem cannot be improved even in our relaxed setting. We will show however that every triangle-free graph embeddable on a surface of genus g can be colored from any 3-list assignment, in such a way that all monochromatic components have size $O(g)$, and all vertices except $O(g)$ lie in a monochromatic component of size at most 10 (Theorem 6). The case of triangle-free graph is particularly interesting because Voigt [18] proved that there exists a triangle-free planar graph G and a 3-list assignment L such that G is not L -colorable. So our result is non-trivial (and previously unknown, as far as we are aware of) even in the case of planar graphs.

The *girth* of a graph G is the smallest size of a cycle in G . We will also show that every graph of girth at least 6 embeddable on a surface of genus g can be colored from any 2-list assignment, in such a way that all monochromatic components have size $O(g)$, and all vertices except $O(g)$ lie in a monochromatic component of size at most 16 (Theorem 9).

All these results are direct consequences of purely structural results on (large) graphs embeddable on surfaces of bounded genus. Given a graph G , a k -island of G is a non-empty set X of vertices of G such that each vertex of X has at most k neighbors outside X in G . The *size* of a k -island is the number of vertices it contains. In Section 3 we show that

- (1) large graphs of bounded genus have a 4-island of size at most 3 (Theorem 1);
- (2) large triangle-free graphs of bounded genus have a 2-island of size at most 10 (Theorem 5);
- (3) large graphs of girth at least 6 and bounded genus have a 1-island of size at most 16 (Theorem 8).

The proofs of these three results use the discharging method and are very similar, but unfortunately each one has some particularities and therefore we have not been able to factorize them.

In Section 4, we study the computational aspects of minimizing the size of monochromatic components in 2-colorings of graphs. We show that approximating this minimum within a constant multiplicative factor is NP-hard, even when the input graph is a 2-degenerate graph of girth at least 8, or a 2-degenerate triangle-free planar graph.

2. GRAPHS ON SURFACES

All the graphs in this paper are simple (i.e., without loops and multiple edges).

In this paper, a *surface* is a non-null compact connected 2-manifold without boundary. We refer the reader to the monograph of Mohar and Thomassen [16] for background on graphs on surfaces.

A surface can be orientable or non-orientable. The *orientable surface* \mathbb{S}_h of genus h is obtained by adding $h \geq 0$ *handles* to the sphere; while the *non-orientable surface* \mathbb{N}_k of genus k is formed by adding $k \geq 1$ *cross-caps* to the sphere. The *Euler characteristic* $\chi(\Sigma)$ of a surface Σ is $2 - 2h$ if $\Sigma = \mathbb{S}_h$, and $2 - k$ if $\Sigma = \mathbb{N}_k$.

We say that an embedding is *cellular* if every face is homeomorphic to an open disc of \mathbb{R}^2 . Euler's Formula states that if G is a graph with a cellular embedding in Σ , with vertex-set V , edge-set E and face-set F , then $|V| - |E| + |F| = \chi(\Sigma)$.

Finally, if f is a face of a graph G cellularly embedded in a surface Σ , then a *boundary walk of f* is a walk consisting of vertices and edges as they are encountered when walking along the whole boundary of f , starting at some vertex and following some orientation of the face. The *degree of a face f* , denoted $d(f)$, is the number of edges on a boundary walk of f (note that some edges may be counted more than once).

3. ISLANDS IN GRAPHS ON SURFACES

Recall that a *k -island* in a graph G is a non-empty set X of vertices of G such that each vertex of X has at most k neighbors outside X in G , and that the *size* of X is its cardinality $|X|$.

Theorem 1. *Let χ be an integer, and let G be a connected graph that can be embedded on a surface of Euler characteristic χ . If G has more than -72χ vertices, then it contains a 4-island of size at most 3.*

Proof. For the sake of contradiction, we assume that there exists a χ and a connected graph G that can be embedded on a surface of Euler characteristic χ , and with more than -72χ vertices, but without any 4-island of size

at most 3. We choose such a graph G in such way that the integer χ is maximal. By maximality of χ , G has no embedding on a surface with higher Euler characteristic, and then using [16, Propositions 3.4.1 and 3.4.2] we can assume that G has a cellular embedding in Σ (in the non-orientable case we use the fact that G is not a tree, which easily follows from the fact that G has no 4-island of size at most 3). In the remainder, by a slight abuse of notation we identify G with its embedding in Σ .

We can assume that the embedding of G in Σ is edge-maximal (with respect to G being a simple graph), since if a graph obtained from G by adding an edge contains a 4-island of size at most 3, then so does G . In particular, we can assume that for every vertex v of G , there is a circular order on the neighbors of v such that any two consecutive vertices in the order are adjacent in G (note that it does not necessarily mean that G triangulates Σ , since the edge between two consecutive neighbors of v might not lie in a face containing v).

Since G does not contain any 4-island of size at most 3, **(1)** G has minimum degree at least 5; **(2)** G does not contain any path of at most 3 vertices of degree at most 6 in which the two end-vertices have degree 5, and **(3)** G does not contain any triangle whose vertices all have degree at most 6.

We now use the classical discharging method. First, every vertex v of G is assigned a charge $\rho(v) = d(v) - 6$ (since G is simple, all faces have degree at least 3 and then by Euler's Formula, the sum of the charge on all vertices is at most -6χ). Then, we locally move the charge as follows: **(R1)** Every vertex of degree at least 7 gives a charge of $\frac{1}{4}$ to every neighbor of degree 5, **(R2)** and also a charge of $\frac{1}{12}$ to every neighbor of degree 6. **(R3)** Every vertex of degree 6 gives a charge of $\frac{1}{6}$ to every neighbor of degree 5.

We now prove that after the discharging phase, the charge of each vertex is at least $\frac{1}{12}$.

Let v be any vertex of degree 5 (recall that by **(1)** G has minimum degree at least 5). By **(2)**, each neighbor of v has degree at least 6, and by **(3)** no two consecutive neighbors of v have both degree 6. It follows from rules **(R1)** and **(R3)** that v receives a charge of at least $3 \cdot \frac{1}{4} + 2 \cdot \frac{1}{6} = \frac{13}{12}$. The initial charge of v was $\rho(v) = -1$, so the new charge is $\rho'(v) \geq -1 + \frac{13}{12} = \frac{1}{12}$.

Let v be a vertex of degree 6. By **(3)**, no two consecutive neighbors of v have degree at most 6, so v has at least 3 neighbors of degree at least 7 (from which it receives a charge of at least $3 \cdot \frac{1}{12} = \frac{1}{4}$ by rule **(R2)**). On the other hand, by **(2)**, v has at most one neighbor of degree 5, to which it gives at charge of at most $\frac{1}{6}$ by rule **(R3)**. The initial charge of v was $\rho(v) = 6 - 6 = 0$, so the new charge is $\rho'(v) \geq \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$.

Let v be a vertex of degree 7. Assume first that v has at most two neighbors of degree 5. Then in this case v gives a charge of at most $2 \cdot \frac{1}{4} + 5 \cdot \frac{1}{12} = \frac{11}{12}$ by **(R1)** and **(R2)**. Assume now that v has at least three neighbors of degree 5, and let x_1, x_2, \dots, x_7 be the neighbors of v , in their circular order. Then, by **(2)**, v has precisely 3 neighbors of degree 5, say x_1, x_3, x_5 without loss of generality. Moreover, it follows from **(2)** again that x_2, x_4 have degree at least 7. Therefore, in this case v gives a charge of at

most $3 \cdot \frac{1}{4} + 2 \cdot \frac{1}{12} = \frac{11}{12}$ by **(R1)** and **(R2)**. In both cases, the new charge of v is $\rho'(v) \geq 7 - 6 - \frac{11}{12} = \frac{1}{12}$.

Assume now that v has degree $d \geq 8$. By **(2)**, no two consecutive neighbors have degree 5, so v gives a charge of at most $\frac{d}{2} \cdot \frac{1}{4} + \frac{d}{2} \cdot \frac{1}{12} = \frac{d}{6}$ by **(R1)** and **(R2)**. Therefore, the new charge of v is $\rho'(v) \geq d - 6 - \frac{d}{6} = d \cdot \frac{5}{6} - 6 \geq \frac{2}{3}$ whenever $d \geq 8$.

It follows that the charge of each vertex is at least $\frac{1}{12}$. Therefore, $-6\chi \geq \frac{n}{12}$ and so $n \leq -72\chi$, contradicting our initial assumption that $n > -72\chi$. \square

It was pointed out to us by a referee that Theorem 1 is close from a result of Jendrol' and Voss [10], who proved that if G has a cellular embedding in a surface of Euler characteristic χ , and has more than -83χ vertices, then it contains a vertex of degree at most 4, or a triangular face f such that the sum of the degrees of the vertices on f is at most 18. Because of triangular faces with vertices of degree 5, 6 and 7 respectively, it seems that Theorem 1 and this result are incomparable. More results on light subgraphs in graphs on surfaces can be found in two surveys of Jendrol' and Voss [9, 11]

Note that there exist planar graphs with minimum degree 5 in which the degree 5 vertices are arbitrarily far apart. This shows that our bound on the size of 4-islands is best possible, even in the case of planar graphs. Theorem 1 has the following direct consequence.

Theorem 2. *For any integer χ , for any graph G that can be embedded on a surface of Euler characteristic χ , and any 5-list assignment L , G has an L -coloring in which every monochromatic component has size at most $\max(3, -72\chi)$. Moreover, all vertices except at most -72χ of them lie in monochromatic components of size at most 3.*

Proof. Let G be a graph that can be embedded on a surface Σ of Euler characteristic χ , and let L be any 5-list assignment. The proof proceeds by induction on the number of vertices of G . If G contains at most -72χ vertices, then the theorem is certainly true. Assume now that G has more than -72χ vertices. We can assume that the embedding of G in Σ is edge-maximal, since proving the theorem for a supergraph of G also proves it for G . In particular, we can assume that G is connected, and therefore apply Theorem 1. It follows that G contains a 4-island X of size at most 3. Then by the induction hypothesis, the graph $G \setminus X$ has an L -coloring such that each monochromatic component has size at most $\max(3, -72\chi)$, and all vertices except at most -72χ of them lie in monochromatic components of size at most 3. We extend this coloring to G by choosing, for each vertex v of X , a color from $L(v)$ that is distinct from that of its neighbors outside X (if any). The coloring obtained is an L -coloring in which every monochromatic component has size at most $\max(3, -72\chi)$. Moreover, all vertices except at most -72χ of them lie in monochromatic components of size at most 3. This concludes the proof. \square

Cushing and Kierstead [4] proved that for every planar graph G and every 4-list assignment L to the vertices of G , there is an L -coloring of G in which each monochromatic component has size at most 2. Hence, Theorem 2

restricted to planar graphs is significantly weaker than their result. We conjecture the following:

Conjecture 3. *There is a function f such that for any integer χ , for any graph G that can be embedded on a surface of Euler characteristic χ , and any 4-list assignment L , G has an L -coloring in which every monochromatic component has size at most $f(\chi)$.*

We believe that any large graph of bounded genus contains a 3-island of bounded size, which would directly imply Conjecture 3, but we have not been able to prove it, even in the case of planar graphs.

Kawarabayashi and Thomassen [12] proved that every graph that has an embedding on a surface of Euler characteristic χ can be colored with colors 1, 2, 3, 4, 5, in such a way that each color $i \leq 4$ is an independent set, while color 5 induces a graph in which each connected component contains $O(\chi^2)$ vertices. A small variation in the proof of Theorem 2 shows the following corollary.

Corollary 4. *Every graph that has an embedding on a surface of Euler characteristic χ can be colored with colors 1, 2, 3, 4, 5, in such a way that each color $i \leq 4$ induces a graph in which each connected components has size at most 3, while color 5 induces a graph in which each connected component contains $O(|\chi|)$ vertices.*

We now prove a triangle-free version of Theorem 1.

Theorem 5. *Let χ be an integer, and let G be a connected triangle-free graph that can be embedded on a surface of Euler characteristic χ . If G has more than -72χ vertices, then it contains a 2-island of size at most 10.*

Proof. The proof is similar to that of Theorem 1. We consider a counterexample G (we can assume that it has a cellular embedding on some surface of Euler characteristic χ). Since G does not contain any 2-island of size at most 10, **(1)** G has minimum degree at least 3, and **(2)** G does not contain any path of at most 10 vertices of degree at most 4 in which the two end-vertices have degree 3 (the two end-vertices are allowed to coincide).

We now use the discharging method. First, every vertex v of G is assigned a charge $\rho(v) = d(v) - 4$, and every face f of G is assigned a charge $\rho(f) = d(f) - 4$ (by Euler's Formula, the sum of the charge on all vertices and faces is equal to -4χ). Then, we locally move the charge as described below.

We first choose, for every face f of G , an orientation of f and set it as the positive orientation of f (we do not need to have a consistent choice of positive orientations, therefore Σ is not required to be orientable). For any face f of G , for any orientation of f (positive or negative), and for any occurrence of a vertex v of degree 3 in a boundary walk of f according to the chosen orientation, take a maximal facial walk of f (a walk consisting only of vertices and edges incident to f) starting at v and going around f in the prescribed orientation of f , such that the inner vertices of the walk have degree precisely 4. Let u be the other end-vertex of the walk. If the walk contains at least 3 inner vertices, then the face f gives a charge of $\frac{1}{6}$ to v . Otherwise **(1)**, **(2)** and the maximality of the walk imply that u has degree at least 5. In this case u gives a charge of $\frac{1}{6}$ to v .

We now prove that after the discharging phase, all vertices and faces have nonnegative charge.

Let v be any vertex of degree 3 (recall that by **(1)** G has minimum degree at least 3). Then v appears 6 times in the union of all boundary walks of faces of G (for each face, we consider a boundary walk in the positive orientation and a boundary walk in the negative orientation of the face), and therefore receives 6 times a charge of $\frac{1}{6}$. The initial charge of v was $\rho(v) = -1$, so the new charge is $\rho'(v) = -1 + 6 \cdot \frac{1}{6} = 0$.

Vertices of degree 4 start with an initial charge of $4 - 4 = 0$ and do not give or receive any charge. Now let v be a vertex of degree $d \geq 5$. Consider the facial walks through which it gives a charge of $\frac{1}{6}$ to some vertices of degree 3, and observe that if a neighbor u of v is right after v in more than one such facial walk, then u has degree 3 (and receives exactly $2 \cdot \frac{1}{6} = \frac{1}{3}$ from v). For if u had degree at least 4 and was just after v in two facial walks starting at v as defined above, u would have degree exactly 4 and there would be two paths starting at u , each containing at most 1 inner vertex (of degree 4) and finishing at a vertex of degree 3, contradicting **(2)**. It follows that $\rho'(v) \geq d - 4 - 2d \cdot \frac{1}{6} = \frac{2}{3}d - 4 \geq \frac{2d}{21}$ whenever $d \geq 7$. If $d = 6$, then observe that v is adjacent to at most three vertices of degree 3 (otherwise G would contain a 2-island of size 5). Therefore, in this case we have $\rho'(v) \geq 2 - 3 \cdot \frac{1}{3} - 3 \cdot \frac{1}{6} = \frac{1}{2} = \frac{d}{12}$. If $d = 5$, then the walks through which v gives some charge contain at most two neighbors of v , since otherwise G would contain a 2-island of size at most 10. It follows that in this case we have $\rho'(v) \geq 1 - 2 \cdot \frac{1}{3} = \frac{1}{3} = \frac{d}{15}$.

Let f be a face of degree d in G . If $d = 4$, no vertex receives any charge from f , since otherwise f contains a vertex of degree 3 and three vertices of degree 4, and then the vertices of f form a 2-island of size 4. It follows that if $d = 4$, $\rho'(f) = 0$. Assume now that $d \geq 5$. For each occurrence of a vertex v of degree 3 that receives $\frac{1}{6}$ from f in the positive orientation, let $A^+(v)$ be the set consisting of the three vertices of degree exactly 4 following v in the positive orientation of f (the existence of these vertices follows from the definition of our discharging procedure). Similarly, define $A^-(v)$ for each occurrence of a vertex v of f receiving some charge from f in the negative orientation of f . Observe that all the sets $A^+(v)$ and $A^-(u)$ are pairwise disjoint: for a pair of sets $A^+(u)$ and $A^+(v)$, or $A^-(u)$ and $A^-(v)$, this follows from the definition of these sets and the fact that they exist only if u and v have degree three. For each pair $A^+(u), A^-(v)$, if these two sets have non-empty intersection then G contain a 2-island of size at most 7, which is a contradiction. It follows that f gives at most $\frac{1}{6} \cdot \lfloor \frac{d}{3} \rfloor$. Since $d \geq 5$ and the face f starts with an initial charge of $\rho(f) = d - 4$, in this case the new charge is $\rho'(f) \geq d - 4 - \frac{1}{6} \lfloor \frac{d}{3} \rfloor \geq \frac{d}{6}$.

We proved that all vertices and faces have nonnegative charge (if G is projective planar this is already a contradiction since in this case the total charge is negative). Moreover, vertices v with degree $d \geq 5$ have a charge $\rho'(v) \geq \frac{d}{15}$, while faces f with degree $d \geq 5$ have a charge $\rho'(f) \geq \frac{d}{6}$. We now redistribute the charge as follows: every vertex of degree at least 5 gives $\frac{1}{18}$ to every incident face, and then every face f gives $\frac{1}{54}$ to every occurrence of

a vertex of degree 3 or 4 in a boundary walk of f . Each face of degree $d \geq 5$ is left with at least $\frac{d}{6} - \frac{d}{54} \geq 0$. Note that a face of degree 4 is incident to at most 3 vertices of degree 3 or 4 (otherwise G would contain a 2-island of size 4), therefore such a face starts with a charge of 0, receives $\frac{1}{18}$ from a vertex of degree at least 5, and gives at most $3 \times \frac{1}{54} = \frac{1}{18}$ to the remaining vertices of its boundary. Therefore, each face has nonnegative charge.

Each vertex of degree $d \geq 5$ starts with a charge of at least $\frac{d}{15}$ and gives at most $\frac{d}{18}$, thus the remaining charge is at least $\frac{d}{90} \geq \frac{1}{18}$. Each vertex v of degree 3 or 4 starts with a charge of 0 and receives $\frac{1}{54}$ from each incident face, for a total of at least $\frac{1}{18}$ (note that since faces give charge to every occurrence of a vertex on their boundary, this holds even if the number of faces incident to v is less than $d(v)$ because then v appears several times in a boundary walk of some face). It follows that the charge of each vertex is at least $\frac{1}{18}$. Since all faces have nonnegative charge, we have $-4\chi \geq \frac{n}{18}$ and so $n \leq -72\chi$, contradicting our initial assumption that $n > -72\chi$. \square

Our bound on the size of 2-islands is not optimal in the case of planar graphs: it is possible to show, using a more detailed (and significantly longer) analysis, that every triangle-free planar graph contains a 2-island of size at most 5, which is best possible. It is likely that the result extends to higher surfaces as well, but we preferred to present a short and simple proof of a slightly weaker result instead (the most important part of the theorem being that the island is a 2-island).

Euler's formula shows that every triangle-free planar graph G contains a vertex of degree at most 3. It follows that for any 4-list assignment L , G has a proper L -coloring. On the other hand, Voigt [18] proved that there is a triangle-free planar graph G and a 3-list assignment L such that G is not L -colorable. Using the same proof as that of Theorem 2, Theorem 5 has the following direct consequence (which seems to have been previously unknown even for planar graphs).

Theorem 6. *For any integer χ , for any triangle-free graph G that can be embedded on a surface of Euler characteristic χ , and any 3-list assignment L , G has an L -coloring in which every monochromatic component has size at most $\max(10, -72\chi)$. Moreover, all vertices except at most -72χ of them lie in monochromatic components of size at most 10.*

Note that the size of the lists in Theorem 6 is best possible: Esperet and Joret [5] proved that triangle-free planar graphs G cannot be 2-colored such that each monochromatic component has bounded size. We conjecture the following:

Conjecture 7. *There is a function f such that for any integer χ , for any graph G of girth at least 5 that can be embedded on a surface of Euler characteristic χ , and any 2-list assignment L , G has an L -coloring in which every monochromatic component has size at most $f(\chi)$.*

We now prove a weaker version of this conjecture, for graphs of girth at least 6 (instead of 5).

Theorem 8. *Let χ be an integer, and let G be a connected graph of girth at least six that can be embedded on a surface of Euler characteristic χ . If*

G has more than -357χ vertices, then it contains a 1-island of size at most 16.

Proof. The proof is similar to that of Theorem 1. We consider a counterexample *G* (we can assume that it has a cellular embedding on some surface of Euler characteristic χ). Since *G* does not contain any 1-island of size at most 16, **(1)** *G* has minimum degree at least 2, and **(2)** *G* does not contain any path of at most 16 vertices of degree at most 3 in which the two end-vertices have degree two (the two end-vertices are allowed to coincide).

We now use the classical discharging method. First, every vertex v of *G* is assigned a charge $\rho(v) = 2d(v) - 6$, and every face f of *G* is assigned a charge $\rho(f) = d(f) - 6$ (by Euler's Formula, the sum of the charge on all vertices and faces is equal to -6χ). Then, we locally move the charge as described below.

We first choose, for every face f of *G*, an orientation of f and set it as the positive orientation of f . For any face f of *G*, for any orientation of f (positive or negative), and for any occurrence of a vertex v of degree two in a boundary walk of f according to the chosen orientation, take a maximal facial walk of f (a walk consisting only of vertices and edges incident to f) starting at v and going around f in the prescribed orientation of f , such that the inner vertices of the walk have degree precisely 3. Let u be the other end-vertex of the walk (note that possibly $u = v$ if for instance all vertices of f distinct from v have degree three; another extreme case is that there are no inner vertices at all and u and v are neighbors). If the walk contains at least 5 inner vertices, the face f gives a charge of $\frac{1}{2}$ to v . Otherwise **(1)**, **(2)** and the maximality of the walk imply that u has degree at least 4. In this case u gives a charge of $\frac{1}{2}$ to v .

We now prove that after the discharging phase, all vertices and faces have nonnegative charge.

Let v be any vertex of degree two (recall that by **(1)** *G* has minimum degree at least 2). Then v appears four times in the union of all boundary walks of faces of *G* (for each face, we consider a boundary walk in the positive orientation and a boundary walk in the negative orientation of the face), and therefore receives four times a charge of $\frac{1}{2}$. The initial charge of v was $\rho(v) = -2$, so the new charge is $\rho'(v) = -2 + 4 \cdot \frac{1}{2} = 0$.

Vertices of degree 3 start with an initial charge of 0, and neither give nor receive any charge, so after the discharging their charge is still 0. Now let v be a vertex of degree $d \geq 4$. Consider the facial walks through which it gives a charge of $\frac{1}{2}$ to some vertices of degree two, and observe that if a neighbor u of v is right after u in more than one such facial walk, then u has degree two. For if u had degree at least three and was just after v in two facial walks starting at v as defined above, u would have degree exactly three and there would be two paths starting at u , each containing at most 3 inner vertices (each of degree 3) and finishing at a vertex of degree two. Thus *G* would contain a path on at most 9 vertices, such that all vertices have degree at most 3 and the two endpoints have degree two, contradicting **(2)**.

It follows that v gives a charge of at most $d \cdot 2 \cdot \frac{1}{2}$. Since it starts with an initial charge of $\rho(v) = 2d - 6$, its new charge $\rho'(v)$ is at least $2d - 6 - d = d - 6 \geq \frac{d}{7}$ as soon as $d \geq 7$.

If $d = 6$ observe that v cannot be adjacent to at least 5 vertices of degree 2, since otherwise v together with these vertices would form a 1-island of size at most 6. Hence if $d = 6$, v gives at most $4 + 2 \cdot \frac{1}{2} = 5$. Since it starts with an initial charge $\rho(v) = 6$, we have $\rho'(v) \geq 6 - 5 = 1 \geq \frac{d}{6}$.

If $d = 5$, then by the same argument as above it cannot be adjacent to at least 4 vertices of degree 2. If it is adjacent to at most two vertices of degree 2, it gives a charge of at most $2 + 3 \cdot \frac{1}{2} = \frac{7}{2}$ and it follows that $\rho'(v) \geq 4 - \frac{7}{2} = \frac{1}{2} \geq \frac{d}{10}$. Otherwise, v is adjacent to exactly three vertices of degree 2. But then observe that v cannot give a charge of $\frac{1}{2}$ through any of its two neighbors of degree more than two, since otherwise G would contain a 1-island of size at most 9. Therefore in this case v gives a charge of 3, and then $\rho'(v) = 1 \geq \frac{d}{5}$.

If $d = 4$, then using again the same argument, v cannot be adjacent to more than two vertices of degree two. Moreover, if v is adjacent to two vertices of degree two, then it does not give any charge through its neighbors of degree more than two (in this case it follows that $\rho'(v) = 0$). If v has one neighbor of degree two then it cannot give charge through more than one neighbor of degree more than two (otherwise G contains a 1-island of size at most 12), so in this case we obtain $\rho'(v) \geq 2 - 1 - \frac{1}{2} = \frac{1}{2} \geq \frac{d}{8}$. If v has no neighbor of degree 2, then¹ v does not give charges through more than two of its neighbors, since otherwise G would contain a 1-island of size at most 16. Thus, in this case $\rho'(v) \geq 2 - 2 \cdot \frac{1}{2} = 1 \geq \frac{d}{4}$.

We proved that for any vertex v , $\rho'(v) \geq 0$, and if v has degree at least four and is not a vertex of degree four with exactly two neighbors of degree two, then $\rho'(v) \geq \frac{1}{10} d(v)$.

Let f be a face of degree d in G (since G has girth at least 6, $d \geq 6$). If $d = 6$, no vertex receives any charge from f , since otherwise f contains a vertex of degree two and 5 vertices of degree three, and then the vertices of f form a 1-island of size 6. It follows that if $d = 6$, $\rho'(f) = 0$. Assume now that $d \geq 7$. For each vertex v of degree two that receives $\frac{1}{2}$ from f in the positive orientation, let $A^+(v)$ be the set consisting of the five vertices of degree exactly 3 following v in the positive orientation of f (the existence of these vertices follows from the definition of our discharging procedure). Similarly, define $A^-(v)$ for each vertex v of f receiving some charge from f in the negative orientation of f . Observe that all the sets $A^+(v)$ and $A^-(u)$ are pairwise disjoint: for a pair of sets $A^+(u)$ and $A^+(v)$, or $A^-(u)$ and $A^-(v)$, this follows from the definition of these sets and the fact that they exist only if u and v have degree two. For each pair $A^+(u), A^-(v)$, if these two sets have non-empty intersection then G contain a 1-island of size at most 11, which is a contradiction. It follows that f gives at most $\frac{1}{2} \cdot \lfloor \frac{d}{5} \rfloor$.

¹In the planar case we can avoid this argument and simply say that in this case v gives at most $4 \cdot \frac{1}{2}$ and therefore its new charge is at least 0. This allows to find 1-islands of size at most 12 (instead of 16) in any 2-edge-connected planar graph of girth at least 6.

Since $d \geq 7$ and the face f starts with an initial charge of $\rho(f) = d - 6$, in this case the new charge is $\rho'(f) \geq d - 6 - \frac{1}{2} \lfloor \frac{d}{5} \rfloor \geq \frac{d}{14}$.

Recall that the total charge on the vertices and faces is -6χ , and we proved that the charge of every vertex and every face is nonnegative (if $\chi > 0$ this is already a contradiction, since in this case the total charge is negative). In the previous paragraphs we also proved that if a vertex or face of degree d has non-zero charge, then this charge is at least $\frac{d}{14}$.

A vertex that has non-zero charge, or is incident to a face of non-zero charge, or shares a face of degree 6 with a vertex with non-zero charge, is said to be *heavy*. Observe that every face f with non-zero charge defines at most $d(f)$ heavy vertices, and every vertex v with non-zero charge defines at most $4d(v) + 1 \leq \frac{17}{4}d(v)$ heavy vertices. So the number of heavy vertices is at most $\frac{17}{4}$ times the sum of the degrees of the vertices and faces with non-zero charge, which by the previous paragraphs is itself at most 14 times the total charge. Hence, there are at most $14 \cdot \frac{17}{4} \cdot (-6\chi) = -357\chi$ heavy vertices. Since G contains more than -357χ vertices, it contains a vertex v that is not heavy. By the definition of v , all the faces incident to v have degree 6, and all the vertices incident to these faces (including v) have degree 2, 3 or 4 (and if one of these vertices has degree 4, it has precisely two neighbors of degree 2).

Let f be any face incident to v . Since $d(f) = 6$, f contains at least one vertex of degree 4 (since otherwise the vertices of f would form a 1-island of size 6). By definition of v , any such vertex of degree 4 has exactly two neighbors of degree 2. In particular, two such vertices of degree four cannot be adjacent, otherwise they would form a 1-island of size at most 6 (together with their neighbors of degree 2). Thus, we can assume that f contains at most 3 vertices of degree 4. If each of these vertices has at least one neighbor of degree 2 outside f , then we obtain a 1-island of size at most 9. It follows that some vertex u_1 of degree four on the boundary of f has its two neighbors of degree two on f . Let P be the set of three vertices of f distinct from u_1 and its two neighbors of degree 2. Then P contains at least one vertex of degree 4, since otherwise we find a 1-island of size 5 in G . Using the same argument as above P contains a vertex u_2 of degree 4 such that its two neighbors of degree 2 belong to f . One of these neighbors is also a neighbor of degree two of u_1 , since otherwise we have a 1-island consisting of two adjacent vertices of degree two. It follows that f contains a third vertex u_3 of degree 4, having its two neighbors of degree two on f . Therefore, f contains only vertices of degree 2 and 4, that alternate on its boundary.

Note that the conclusion above holds for any face f incident to v . This implies that $d(v) \neq 4$, since otherwise v would have four neighbors of degree two, and $d(v) \neq 2$, since otherwise a neighbor of degree four of v has at least three neighbors of degree two.

This final contradiction concludes the proof of the theorem. \square

The bound on the size of 1-islands is certainly far from optimal. We were only able to construct large toroidal graphs of girth 6 with no 1-island of

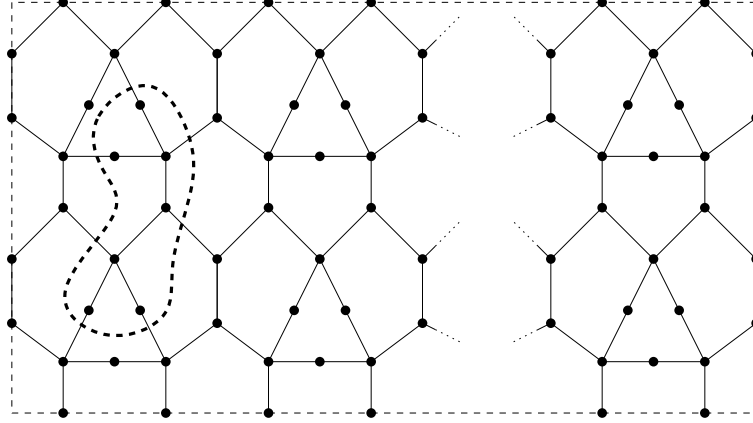


FIGURE 1. Arbitrarily large toroidal graphs of girth 6 with no 1-island of size at most 6. A 1-island on 7 vertices is highlighted.

size at most 6 (see Figure 1). Using the same proof as that of Theorem 2, the following is a direct consequence of Theorem 8.

Theorem 9. *For any integer χ , for any graph G of girth at least 6 that can be embedded on a surface of Euler characteristic χ , and any 2-list assignment L , G has an L -coloring in which every monochromatic component has size at most $\max(16, -357\chi)$. Moreover, all vertices except at most -357χ of them lie in monochromatic components of size at most 16.*

Using the argument mentioned in the footnote in the proof of Theorem 8, Theorem 9 can be slightly improved for planar graphs:

Theorem 10. *For any planar graph G of girth at least 6 and any 2-list assignment L , G has an L -coloring in which every monochromatic component has size at most 12.*

Note that it was proved by Borodin, Kostochka, and Yancey [3] that every planar graph of girth at least 7 has a 2-coloring in which every monochromatic component has size at most 2.

4. COMPLEXITY

In this section we show that it is NP-hard to approximate the minimum size of the largest monochromatic component in a 2-coloring of a graph within a constant multiplicative factor. Let us define an $MC(k)$ -coloring as a 2-coloring such that every monochromatic component has size at most k . Let $MC(k)$ be the class of graphs having an $MC(k)$ -coloring.

Theorem 11. *Let $k \geq 2$ be a fixed integer. The following problems are NP-complete.*

- (1) *Given a 2-degenerate graph with girth at least 8 that either is in $MC(2)$ or is not in $MC(k)$, determine whether it is in $MC(2)$.*
- (2) *Given a 2-degenerate triangle-free planar graph that either is in $MC(k)$ or is not in $MC(k(k-1))$, determine whether it is in $MC(k)$.*

Before proving Theorem 11, we first describe a gadget used in the proof and its properties. Let $t \geq 2$ be an integer. Let $T_{x,t}$ be the complete rooted tree of height 3 with root x such that every internal node has $5t$ children. We consider the planar embedding of T into 4 layers such that the root is on layer 0 and the leaves are on layer 3. We label the $(5t)^3$ leaves with the triples in $\{1, 2, \dots, 5t\}^3$ in lexicographical order from the leftmost leaf with label $(1, 1, 1)$ to the rightmost leaf with label $(5t, 5t, 5t)$. Let $J_{y,z,t}$ be the graph obtained from two copies $T_{y,t}$ and $T_{z,t}$ of $T_{x,t}$ by identifying the leaf labelled (l_1, l_2, l_3) in $T_{y,t}$ with the leaf labelled (l_3, l_2, l_1) in $T_{z,t}$, for every triple $(l_3, l_2, l_1) \in \{1, 2, \dots, 5t\}^3$.

Claim 12.

- (i) The graph $J_{y,z,t}$ is bipartite, 2-degenerate, and the distance between y and z is 6.
- (ii) The girth of $J_{y,z,t}$ is 8.
- (iii) Every $MC(t)$ -coloring of $J_{y,z,t}$ is such that y and z have the same color.

Proof.

- (i) Trivial.
- (ii) Since $J_{y,z,t}$ is bipartite, we suppose for contradiction that it contains a cycle C of length 4 or 6. Notice that C necessarily contains exactly 2 vertices u and v of degree 2. Let (u_1, u_2, u_3) and (v_1, v_2, v_3) be the labels of u and v in $T_{y,t}$. The cycle C consists in a path p_y contained in $T_{y,t}$ and a path p_z contained in $T_{z,t}$ that both link u to v . Since u and v are distinct, there exists an index i such that $u_i \neq v_i$. The length of p_y is at least $2(4 - i)$ and the length of p_z is at least $2i$. Thus, the length of C is at least $2(4 - i) + 2i = 8$, a contradiction.
- (iii) Suppose that $T_{x,t}$ has an $MC(t)$ -coloring using colors in $\{0, 1\}$ such that the root is colored 0. Notice that a vertex colored c in $T_{x,t}$ has at least $5t - (t - 1) = 4t + 1$ children with color $1 - c$. This implies that at least $(4t + 1)^i$ vertices in layer i are colored $i \pmod{2}$. Since $(4t + 1)^3 > \frac{1}{2}(5t)^3$, more than half of the leaves are colored 1. This forces y and z to have the same color in every $MC(t)$ -coloring of $J_{y,z,t}$. \square

Proof of Theorem 11. In each case, we make a reduction from 3-uniform hypergraph 2-colorability, which is a well-known NP-complete problem [15]. We consider a 3-uniform hypergraph H and construct a corresponding graph G as follows. For every vertex v of H , we consider a corresponding vertex v in G . These vertices are called the *primitive vertices* of G .

- (1) We describe the reduction for the first result. For every hyperedge $e = \{u_0, u_1, u_2\}$ of H , we add a path e_1, e_2, \dots, e_{k+1} in G . For every vertex e_j in this path, we take a new copy of $J_{y,z,k}$ and identify the vertex y with e_j and the vertex z with $u_j \pmod{3}$. By Claim 12, the girth of $J_{y,z,k}$ is 8 and thus the girth of G is also 8.

We now show that G is in $\mathcal{MC}(2)$ if H is 2-colorable and that G is not in $\mathcal{MC}(k)$ otherwise. If H is 2-colorable, then we consider a 2-coloring of H and color the primitive vertices of G accordingly. This colors the vertex z of every copy of $J_{y,z,k}$ and we extend this precoloring to all the vertices of

G by properly 2-coloring every copy of $J_{y,z,k}$. All the monochromatic edges belong to paths e_1, e_2, \dots, e_{k+1} corresponding to hyperedges e in H . In such a path, no three consecutive vertices can have the same color since it would correspond to a monochromatic hyperedge in H . This implies that G is in $\mathcal{MC}(2)$.

Now suppose for contradiction that H is not 2-colorable and that G is in $\mathcal{MC}(k)$. Since H is not 2-colorable, any 2-coloring of the primitive vertices of G is such that there exists three primitive vertices u, v , and w in G corresponding to a monochromatic hyperedge $e = \{u, v, w\}$ in H . By Claim 12, any $MC(k)$ -coloring of the gadgets $J_{y,z,k}$ containing u, v , or w and extending the precoloring of u, v , or w , is such that the path e_1, e_2, \dots, e_{k+1} corresponding to e is monochromatic. This gives a monochromatic component of size $k + 1$, which is a contradiction. So, if H is not 2-colorable then G is not in $\mathcal{MC}(k)$.

(2) The reduction for the second result is similar: for every hyperedge e in H , we add a path $e_1, e_2, \dots, e_{k(k-1)+1}$ in G . Such a path cannot be monochromatic in an $MC(k(k-1))$ -coloring of G . We now present the gadgets that are needed to transfer the color of the primitive vertices to the paths corresponding to the hyperedges of H .

Let $N_{y,z,k}$ be the bipartite graph obtained from two non-adjacent vertices y and z and a path $v_1, v_2, \dots, v_{3k^4}$ such that y is adjacent to all the vertices v_i with $i \equiv 0 \pmod{2}$ and z is adjacent to all the vertices v_i with $i \equiv 1 \pmod{2}$. Every $MC(k(k-1))$ -coloring of $N_{y,z,k}$ is such that y and z have distinct colors. For if y and z had the same color, say color 0, then at most $2k(k-1) - 2$ vertices on the path would be colored 0. Then the path would contain a monochromatic subpath colored 1 of length at least $\left\lceil \frac{3k^4 - (2k(k-1) - 2)}{2k(k-1) - 1} \right\rceil \geq k(k-1) + 1$.

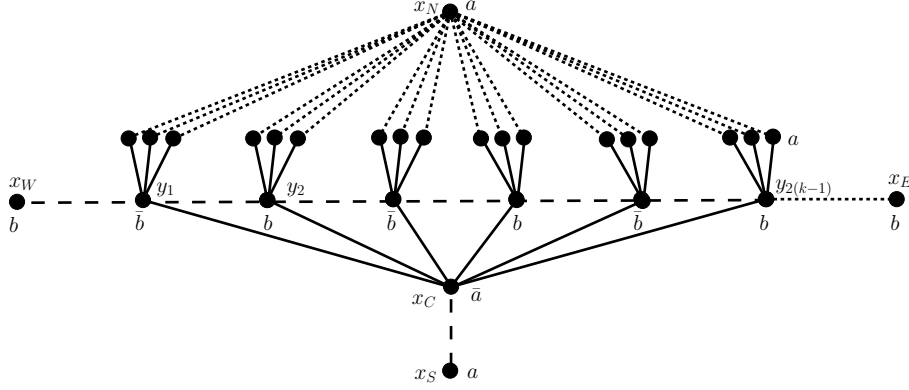
The gadget $N_{y,z,k}$ can thus be used to force two vertices to have distinct colors in an $MC(k(k-1))$ -coloring. To force two vertices to have the same color, we could simply chain two copies of $N_{y,z,k}$. We prefer to use a copy of $K_{2,2k(k-1)-1}$, since it is smaller.

In the last gadget U_k depicted in Figure 2, the dotted edges represent copies of $K_{2,2k(k-1)-1}$ and the dashed edges represent copies of $N_{y,z,k}$. Every vertex y_i , $1 \leq i \leq 2(k-1)$, has precisely $k-1$ neighbors connected to x_N by a dotted edge.

The gadget U_k has the following properties:

- (1) Every $MC(k(k-1))$ -coloring of U_k is such that x_N and x_S have the same color, x_W and x_E have the same color.
- (2) There exists an $MC(k)$ -coloring of U_k such that x_N and x_W have the same color, and there exists an $MC(k)$ -coloring of U_k such that x_N and x_W have distinct colors.

Consider an $MC(k(k-1))$ -coloring c of U_k such that $c(x_N) = a \in \{0, 1\}$ and $c(x_W) = b \in \{0, 1\}$ (note that possibly $a = b$). In what follows, we write \bar{a} instead of $1 - a$ and \bar{b} instead of $1 - b$. By the properties of the dotted and dashed edges, $c(y_{2i+1}) = \bar{b}$, $c(y_{2i}) = c(x_E) = b$. Every vertex y_i has $k-1$ neighbors that are linked to x_N with dotted edges. These

FIGURE 2. Uncrosser gadget U_k with $k = 4$.

$k - 1$ neighbors are thus colored a . The vertex x_C is adjacent to $k - 1$ vertices colored b and $k - 1$ vertices colored \bar{b} . In particular, x_C is adjacent to $k - 1$ vertices colored a , and each of them is adjacent to $k - 1$ vertices colored a . So x_C cannot be colored a , since it would create a monochromatic component of size $k(k - 1) + 1$. Thus we have $c(x_C) = \bar{a}$ and $c(x_S) = a$. This proves property (1). To prove property (2), observe that the 2-coloring we just considered contains only monochromatic components of size at most k , regardless whether $a = b$ or not.

To construct G , we use copies of $K_{2,2k(k-1)-1}$ to transfer the color of the primitive vertices to the vertices of the paths as we did in the in the previous proof. We can draw the graph in the plane in such a way that the edges of the paths do not cross any other edge (for instance by drawing each path on the line of equation $x = 0$). The obtained graph is not necessarily planar, so we replace each crossing of edges by a copy of U_k in order to obtain a planar graph G . \square

Remark. We can modify the gadget $J_{y,z,t}$ in the proof of Theorem 11(1), so that the same result holds for graphs with arbitrarily large (but fixed) girth. Note that in this case we lose the 2-degeneracy. The new gadget consists in the bipartite double cover of a good expander (for instance, a Ramanujan graph) having large girth and degree significantly larger than k . The vertices y and z are any pair of (far apart) vertices on the same side of the bipartition. Using the vertex expansion property, it can be proven that such a graph admits no $MC(k)$ -coloring other than the proper 2-coloring (and therefore y and z are always colored the same in such a coloring). We omit the details.

Related (recent) results. After this paper was submitted, two manuscripts dealing with similar topics appeared. In [6], Chappell and Gimbel conjectured that for any fixed surface Σ there is a constant k such that every graph embeddable on Σ can be 5-colored without monochromatic components of size more than k . Note that our Theorem 2 proves this conjecture in a strong sense. In [2], Axenovich, Ueckerdt, and Weiner proved the following strong variant of our Theorem 10: any planar graph of girth at least

6 has a 2-coloring such that each monochromatic component is a path on at most 14 vertices.

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REFERENCES

- [1] N. Alon, G. Ding, B. Oporowski, and D. Vertigan, *Partitioning into graphs with only small components*, J. Combin. Theory Ser. B **87** (2003), 231–243.
- [2] M. Axenovich, T. Ueckerdt, and P. Weiner, *Splitting Planar Graphs of Girth 6 into Two Linear Forests with Short Paths*, arXiv:1507.02815.
- [3] O.V. Borodin, A. V. Kostochka, and M. Yancey, *On 1-improper 2-coloring of sparse graphs*, Manuscript, 2013.
- [4] W. Cushing, H.A. Kierstead, *Planar graphs are 1-relaxed, 4-choosable*, European J. Combin. **31**(5) (2012), 1385–1397.
- [5] L. Esperet and G. Joret, *Coloring planar graphs with three colors and no large monochromatic components*, Combin. Prob. Comput. **23**(4) (2014), 551–570.
- [6] G. Chappell and J. Gimbel, *On Subgraphs Without Large Components*, to appear in Mathematica Bohemica.
- [7] H. Grötzsch, *Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel*, Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe **8** (1959) 109–120.
- [8] P. Haxell, T. Szabó, G. Tardos, *Bounded size components–partitions and transversals*, J. Combin. Theory Ser. B **88** (2003), 281–297.
- [9] S. Jendrol’ and H.-J. Voss, *Light subgraphs of graphs embedded in 2-dimensional manifolds of Euler characteristic ≤ 0 —a survey*, in: G. Halász, L. Lovász, M. Simonovits, V.T. Sós (Eds.), P. Erdős and his Mathematics II, Bolyai Society Mathematical Studies, vol. 11, Springer, Budapest, 2002, pp. 375–411.
- [10] S. Jendrol’ and H.-J. Voss, *Light subgraphs of order at most 3 in large maps of minimum degree 5 on compact 2-manifolds*, European J. Combin. **26** (2005), 457–471.
- [11] S. Jendrol’ and H.-J. Voss, *Light subgraphs of graphs embedded in the plane—A survey*, Discrete Math. **313**(4) (2013), 406–421.
- [12] K. Kawarabayashi and C. Thomassen, *From the plane to higher surfaces*, J. Combin. Theory Ser. B **102** (2012), 852–868.
- [13] J. Kleinberg, R. Motwani, P. Raghavan, and S. Venkatasubramanian, *Storage management for evolving databases*, Proceedings of the 38th Annual IEEE Symposium on Foundations of Computer Science (FOCS 1997), 353–362.
- [14] N. Linial, J. Matoušek, O. Sheffet, and G. Tardos, *Graph coloring with no large monochromatic components*, Combin. Prob. Comput. **17**(4) (2008), 577–589.
- [15] L. Lovász, *Coverings and colorings of hypergraphs*, Proceedings of the fourth south-eastern conference on combinatorics, graph theory, and computing, Boca Raton, Florida, 3–12, 1973.
- [16] B. Mohar and C. Thomassen, *Graphs on Surfaces*. Johns Hopkins University Press, Baltimore, 2001.
- [17] C. Thomassen, *Every planar graph is 5-choosable*, J. Combin. Theory Ser. B **62** (1994) 180–181.
- [18] M. Voigt, *A not 3-choosable planar graph without 3-cycles*, Discrete Math. **146** (1995) 325–328.

LABORATOIRE G-SCOP (CNRS, GRENOBLE-INP), GRENOBLE, FRANCE
E-mail address: louis.esperet@g-scop.fr

LIRMM (CNRS, UNIVERSITÉ DE MONTPELLIER), MONTPELLIER, FRANCE
E-mail address: ochem@lirmm.fr