# CONVEXIFYING POSITIVE POLYNOMIALS AND SUMS OF SQUARES APPROXIMATION 

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#### Abstract

We show that if a polynomial $f \in \mathbb{R}\left[x_{1}, \ldots, x_{n}\right]$ is nonnegative on a closed basic semialgebraic set $X=\left\{x \in \mathbb{R}^{n}: g_{1}(x) \geq 0, \ldots, g_{r}(x) \geq 0\right\}$, where $g_{1}, \ldots, g_{r} \in \mathbb{R}\left[x_{1}, \ldots, x_{n}\right]$, then $f$ can be approximated uniformly on compact sets by polynomials of the form $\sigma_{0}+\varphi\left(g_{1}\right) g_{1}+\cdots+\varphi\left(g_{r}\right) g_{r}$, where $\sigma_{0} \in \mathbb{R}\left[x_{1}, \ldots, x_{n}\right]$ and $\varphi \in \mathbb{R}[t]$ are sums of squares of polynomials. In particular, if $X$ is compact, and $h(x):=R^{2}-|x|^{2}$ is positive on $X$, then $f=\sigma_{0}+\sigma_{1} h+\varphi\left(g_{1}\right) g_{1}+\cdots+\varphi\left(g_{r}\right) g_{r}$ for some sums of squares $\sigma_{0}, \sigma_{1} \in$ $\mathbb{R}\left[x_{1}, \ldots, x_{n}\right]$ and $\varphi \in \mathbb{R}[t]$, where $|x|^{2}=x_{1}^{2}+\cdots+x_{n}^{2}$. We apply a quantitative version of those results to semidefinite optimization methods. Let $X$ be a convex closed semialgebraic subset of $\mathbb{R}^{n}$ and let $f$ be a polynomial which is positive on $X$. We give necessary and sufficient conditions for the existence of an exponent $N \in \mathbb{N}$ such that $\left(1+|x|^{2}\right)^{N} f(x)$ is a convex function on $X$. We apply this result to searching for lower critical points of polynomials on convex compact semialgebraic sets.


## Introduction

In the paper we study two types of problems for polynomials which are positive (or nonnegative) on subsets of $\mathbb{R}^{n}$. In the first part we prove stronger versions of known approximation and representation theorems with sums of squares of polynomials. Next we give quantitative versions of these results and explain some applications to semidefinite optimization methods. In the second part we prove that any polynomial $f$ which is positive on a convex closed set $X$ becomes strongly convex when multiplied by $\left(1+|x|^{2}\right)^{N}$ with $N$ large enough (the noncompact case requires some extra assumptions). In fact we give an explicit estimate for $N$, which depends on the size of the coefficients of $f$ and on the lower bound of $f$ on $X$. As an application of our convexification method we propose an algorithm which for a given polynomial $f$ on a compact semialgebraic set $X$ produces a sequence (starting from an arbitrary point in $X$ ) which converges to a critical point of $f$ on $X$. We also relate convexity and positivity issues.
0.1. Notation and state of the art. We denote by $\mathbb{R}[x]$ or $\mathbb{R}\left[x_{1}, \ldots, x_{n}\right]$ the ring of polynomials in $x=\left(x_{1}, \ldots, x_{n}\right)$ with coefficients in $\mathbb{R}$. Important problems of real algebraic geometry are representations of nonnegative polynomials on closed

[^0]semialgebraic sets. Recall Hilbert's 17th problem (solved by E. Artin [2]): if $f \in$ $\mathbb{R}[x]$ is nonnegative on $\mathbb{R}^{n}$, then
\[

$$
\begin{equation*}
f h^{2}=h_{1}^{2}+\cdots+h_{m}^{2} \quad \text { for some } h, h_{1}, \ldots, h_{m} \in \mathbb{R}[x], h \neq 0 \tag{AH}
\end{equation*}
$$

\]

that is, $f$ is a sum of squares of rational functions. With the additional assumptions that $f$ is homogeneous and $f(x)>0$ for $x \neq 0$, B. Reznick [24, Theorem 3.12] proved that there exists an integer $r_{0}$ such that for any $N \geq r_{0}$ the polynomial $\left(x_{1}^{2}+\cdots+x_{n}^{2}\right)^{N} f(x)$ is a sum of even powers of linear functions.

Let $X \subset \mathbb{R}^{n}$ be a closed basic semialgebraic set defined by $g_{1}, \ldots, g_{r} \in \mathbb{R}[x]$, i.e.,

$$
\begin{equation*}
X=\left\{x \in \mathbb{R}^{n}: g_{1}(x) \geq 0, \ldots, g_{r}(x) \geq 0\right\} \tag{0.1}
\end{equation*}
$$

The preordering generated by $g_{1}, \ldots, g_{r}$ is defined to be

$$
T\left(g_{1}, \ldots, g_{r}\right)=\left\{\sum_{e=\left(e_{1}, \ldots, e_{r}\right) \in\{0,1\}^{r}} \sigma_{e} g_{1}^{e_{1}} \cdots g_{r}^{e_{r}}: \sigma_{e} \in \sum \mathbb{R}[x]^{2} \text { for } e \in\{0,1\}^{r}\right\}
$$

where $\sum \mathbb{R}[x]^{2}$ denotes the set of sums of squares (s.o.s.) of polynomials from $\mathbb{R}[x]$. Natural generalizations of the above theorem of Artin are the Stellensätze of J.-L. Krivine [12], D. W. Dubois [9, and J.-J. Risler [26] (see also [7]). For references and a more detailed discussion of this subject see for instance [28], [20], [22]. When the set $X$ is compact, a very important result was obtained by K. Schmüdgen (see [29, [8]): every strictly positive polynomial $f$ on $X$ belongs to the preordering $T\left(g_{1}, \ldots, g_{r}\right)$. M. Schweighofer [30 studied degree bounds in the Schmüdgen Positivstellensatz representation

$$
f=\sum_{e \in\{0,1\}^{r}} \sigma_{e} g_{1}^{e_{1}} \cdots g_{r}^{e_{r}} \in T\left(g_{1}, \ldots, g_{r}\right)
$$

He obtained an upper bound for $\operatorname{deg} \sigma_{e} g_{1}^{e_{1}} \cdots g_{r}^{e_{r}}$ in terms of $\operatorname{deg} f, f^{*}:=\min \{f(x)$ : $x \in X\}$ and the coefficients of $f$, provided that $f^{*}>0$. As shown by C. Scheiderer [27], there is no such bound in terms of $\operatorname{deg} f$ unless $\operatorname{dim}(X) \leq 1$. Under some additional assumptions M. Putinar [23] proved that $f$ belongs to the quadratic module generated by $g_{1}, \ldots, g_{r}$,

$$
P\left(g_{1}, \ldots, g_{r}\right):=\left\{\sigma_{0}+\sigma_{1} g_{1}+\cdots+\sigma_{r} g_{r}: \sigma_{i} \in \sum \mathbb{R}[x]^{2}, i=0, \ldots, r\right\}
$$

The above results concern strictly positive polynomials. In the case of nonnegative polynomials C. Berg, J. P. R. Christensen and P. Ressel [4 and J. B. Lasserre and T. Netzer [19, Corollary 3.3] proved that any polynomial $f$ which is nonnegative on $[-1,1]^{n}$ can be approximated in the $l_{1}$-norm by sums of squares of polynomials. The $l_{1}$-norm of a polynomial is defined to be the sum of the absolute values of its coefficients (in the usual monomial basis). Hence we have

Fact 1. If a polynomial $f \in \mathbb{R}[x]$ is nonnegative on $[-R, R]^{n}, R>0$, then the polynomial $f(R x)$ can be approximated in the $l_{1}$-norm by sums of squares of polynomials. In particular $f(x)$ can be uniformly approximated on $[-R, R]^{n}$ by sums of squares of polynomials.
D. Hilbert [11] proved that for $n \geq 2$ there are nonnegative polynomials on $\mathbb{R}^{n}$ which are not sums of squares of polynomials. T. S. Motzkin 21 gave an explicit example of such a polynomial, $f\left(x_{1}, x_{2}\right)=1+x_{1}^{2} x_{2}^{2}\left(x_{1}^{2}+x_{2}^{2}-3\right)$, i.e., in the representation (AH) of $f$ the degree of $h$ must be positive. So in general the Schmüdgen Positivstellensatz does not hold on noncompact sets. For a polynomial
$f$ positive on a noncompact set $X$ the problem arises of approximation of $f$ by elements of the preordering $T\left(g_{1}, \ldots, g_{r}\right)$ or of the quadratic module $P\left(g_{1}, \ldots, g_{r}\right)$. In this connection J. B. Lasserre [17, Theorem 2.6] (see also [16]) proved that if $g_{1}, \ldots, g_{r}$ are concave polynomials such that $g_{1}(z)>0, \ldots g_{r}(z)>0$ for some $z \in X$, then any convex polynomial nonnegative on $X$ can be approximated in the $l_{1}$-norm by polynomials from the set
$L_{c}\left(g_{1}, \ldots, g_{r}\right):=\left\{\sigma_{0}+\lambda_{1}^{2} g_{1}+\cdots+\lambda_{r}^{2} g_{r}: \sigma_{0} \in \sum \mathbb{R}[x]^{2}\right.$ convex, $\left.\lambda_{1}, \ldots, \lambda_{r} \in \mathbb{R}\right\}$.
For $X=\mathbb{R}^{n}$ the approximation is uniform on compact sets. J. B. Lasserre [16] proved that if a polynomial $f \in \mathbb{R}[x]$ has a global minimum $f^{*} \geq 0$ then for every $\varepsilon>0$ there is $N \in \mathbb{N}$ such that the polynomial $f_{\varepsilon}:=f+\varepsilon \sum_{k=1}^{N} \sum_{j=1}^{n} \frac{x_{j}^{2 k}}{k!}$ is a sum of squares (see also 18 for polynomials on real algebraic sets).
0.2. Our contributions. In this article, we prove an analogue of the Schmüdgen and Putinar theorems for a smaller cone. Namely for $g \in \mathbb{R}[x]$ we put

$$
\begin{aligned}
\mathcal{K}\left(g, g_{1}, \ldots, g_{r}\right):=\left\{\sigma_{0}+\sigma_{1} g+\varphi\left(g_{1}\right) g_{1}+\cdots+\varphi\left(g_{r}\right) g_{r}: \sigma_{0}, \sigma_{1}\right. & \in \sum \mathbb{R}[x]^{2} \\
\varphi & \left.\in \sum \mathbb{R}[t]^{2}\right\},
\end{aligned}
$$

where $t$ is a single variable. Note that if we set

$$
\Phi\left(g_{1}, \ldots, g_{r}\right):=\left\{\varphi\left(g_{1}\right) g_{1}+\cdots+\varphi\left(g_{r}\right) g_{r}: \varphi \in \sum \mathbb{R}[t]^{2}\right\}
$$

then

$$
\mathcal{K}\left(g, g_{1}, \ldots, g_{r}\right)=T(g)+\Phi\left(g_{1}, \ldots, g_{r}\right),
$$

where $\mathcal{A}+\mathcal{B}=\{a+b: a \in \mathcal{A}, b \in \mathcal{B}\}$. In Section 1 we prove (Theorem 1.1) that for a closed basic semialgebraic set $X$ defined by $g_{1}, \ldots, g_{r} \in \mathbb{R}[x]$ and a polynomial $f \in \mathbb{R}[x]$ the following conditions are equivalent:
(i) $f$ is nonnegative on $X$,
(ii) $f$ can be uniformly approximated on compact sets by polynomials from the cone

$$
\mathcal{S}\left(g_{1}, \ldots, g_{r}\right):=\sum \mathbb{R}[x]^{2}+\Phi\left(g_{1}, \ldots, g_{r}\right)
$$

Moreover, $f$ can be approximated by polynomials from $\mathcal{S}\left(g_{1}, \ldots, g_{r}\right)$ in the $l_{1}$-norm.
In particular, if $X$ is a compact set and $g(x):=R^{2}-|x|^{2} \geq 0$ for $x \in X$, then (see Corollary 2.1)

$$
\begin{equation*}
f \text { is strictly positive on } X \quad \Longrightarrow \quad f \in \mathcal{K}\left(g, g_{1}, \ldots, g_{r}\right) \tag{0.2}
\end{equation*}
$$

0.3. Application to optimization. In 15] Lasserre gave a method of minimizing a polynomial $f$ on a compact basic semialgebraic set $X$ of the form (0.1). More precisely, let

$$
f^{*}:=\inf \{f(x): x \in X\}
$$

Then $f^{*}=\sup \{a \in \mathbb{R}: f(x)-a>0$ for $x \in X\}$, and by Putinar's result [23],

$$
f^{*}=\sup \left\{a \in \mathbb{R}: f-a \in P\left(g_{1}, \ldots, g_{r}\right)\right\}
$$

or equivalently

$$
f^{*}=\inf \left\{L(f): L: \mathbb{R}[x] \rightarrow \mathbb{R} \text { is linear, } L(1)=1, L\left(P\left(g_{1}, \ldots, g_{r}\right)\right) \subset[0, \infty)\right\}
$$

Denote

$$
P_{k}\left(g_{1}, \ldots, g_{r}\right):=\left\{\sigma_{0} g_{0}+\cdots+\sigma_{r} g_{r} \in P\left(g_{1}, \ldots, g_{r}\right): \operatorname{deg} \sigma_{i} g_{i} \leq k, i=0, \ldots, r\right\}
$$

where we set $g_{0}=1$. Lasserre considered the following optimization problems:
maximize $a \in \mathbb{R}: f-a \in P_{k}\left(g_{1}, \ldots, g_{r}\right)$,
minimize $L(f)$ for $L: \mathbb{R}[x]_{k} \rightarrow \mathbb{R}$, linear, $L(1)=1, L\left(P_{k}\left(g_{1}, \ldots, g_{r}\right)\right) \subset[0, \infty)$,
where $\mathbb{R}[x]_{k}$ is the linear space of polynomials $h \in \mathbb{R}[x]$ such that $\operatorname{deg} h \leq k$. Set

$$
\begin{aligned}
a_{k}^{*} & :=\sup \left\{a \in \mathbb{R}: f-a \in P_{k}\left(g_{1}, \ldots, g_{r}\right)\right\} \\
l_{k}^{*} & :=\inf \left\{L(f): L: \mathbb{R}[x]_{k} \rightarrow \mathbb{R} \text { is linear, } L(1)=1, L\left(P_{k}\left(g_{1}, \ldots, g_{r}\right)\right) \subset[0, \infty)\right\}
\end{aligned}
$$

for sufficiently large $k \in \mathbb{N}$. Lasserre proved that $\left(a_{k}^{*}\right),\left(l_{k}^{*}\right)$ are increasing sequences that converge to $f^{*}$ and $a_{k}^{*} \leq l_{k}^{*} \leq f^{*}$ for $k \in \mathbb{N}$.

We obtain a version of the Lasserre theorem for

$$
\begin{aligned}
\mathcal{K}_{k}\left(g, g_{1}, \ldots, g_{r}\right):=\left\{\sigma_{0}+\sigma_{1} g+\varphi( \right. & \left.g_{1}\right) g_{1}+\cdots+\varphi\left(g_{r}\right) g_{r} \in \mathcal{K}\left(g, g_{1}, \ldots, g_{r}\right): \\
& \left.\operatorname{deg} \sigma_{0}, \operatorname{deg} \sigma_{1} g, \operatorname{deg} g_{i} \varphi\left(g_{i}\right) \leq k\right\}, \quad k \in \mathbb{N} .
\end{aligned}
$$

The implication (0.2) allows us to apply the Lasserre algorithm of minimizing polynomials on basic compact semialgebraic sets by using $\mathcal{K}_{k}\left(g, g_{1}, \ldots, g_{r}\right)$ instead of $P_{k}\left(g, g_{1}, \ldots, g_{r}\right)$ (see Remark 2.2). Consideration of the cones $\mathcal{K}_{k}\left(g, g_{1}, \ldots, g_{r}\right)$ potentially simplifies the problem of minimizing polynomials on the set $X$, since these cones are properly contained in $P_{k}\left(g, g_{1}, \ldots, g_{r}\right)$.

In Proposition 2.3 we present another method of minimizing a polynomial $f$ on a compact basic semialgebraic set $X$, say $X \subset\left\{x \in \mathbb{R}^{n}:|x| \leq R\right\}$. Namely, for any $\epsilon>0$, we give an effective procedure for calculating a polynomial $h \in \Phi\left(g_{1}, \ldots, g_{r}\right)$ such that

$$
\forall_{|y| \leq R} \exists_{x \in X} f(y)-h(y) \geq f(x)-h(x)-\epsilon
$$

and $|h(x)|<\epsilon$ for $x \in X$. In particular,

$$
f^{*}-2 \epsilon \leq \inf \{f(y)-h(y):|y| \leq R\} \leq f^{*}+2 \epsilon
$$

Thus, the problem of approximate minimization of $f$ can be reduced to the simpler case when the set $X$ is described by one inequality $R^{2}-|x|^{2} \geq 0$ (see Remark 2.5). In this case M. Schweighofer [30] gave the rate of convergence of the sequence

$$
a_{k}^{* *}:=\sup \left\{a \in \mathbb{R}: f-h-a \in P_{k}\left(R^{2}-|y|^{2}\right)\right\} \rightarrow f^{* *}, \quad \text { as } k \rightarrow \infty
$$

where $f^{* *}:=\inf \{f(y)-h(y):|y| \leq R\}$.
0.4. Convexifying positive polynomials. We will prove Theorem 5.5 which, we believe, is of independent interest: for any polynomial $f$ positive on a convex closed set $X$, whose leading form is strictly positive in $\mathbb{R}^{n} \backslash\{0\}$, there exists $N_{0} \in \mathbb{N}$ such that for any integer $N \geq N_{0}$ the polynomial $\varphi_{N}(x)=\left(1+|x|^{2}\right)^{N} f(x)$ is a strictly convex function on $X$. In the case of homogeneous polynomials and $X=\mathbb{R}^{n}$ the same result was obtained by Reznick [25, Theorem 4.6], [24, Theorem 3.12].

First in Section 3 we consider the univariate case, and we give an explicit bound for $N_{0}$ in terms of the coefficients of $f$ and the infimum $f^{*}$. We also give an example to show that $N_{0}$ cannot be a function of the degree of $f$ alone.

In Section 5 we prove that the convexity at infinity of $\varphi_{N}(x)=\left(1+|x|^{2}\right)^{N} f(x)$ for sufficiently large $N$ is equivalent to the strict positivity of the leading form of $f$ (Proposition 5.3). Moreover, in Corollary 5.8 we obtain an interpretation of

Reznick's result [24, Theorem 3.12] in terms of convexity. As a consequence of Theorem 5.5 we prove in Corollary 5.7 that, if $X$ is a convex set containing at least two points, and $d>\operatorname{deg} f$ is an even integer, then the following conditions are equivalent:
(i) $f$ is nonnegative on $X$,
(iii) for any $a, b>0$ there exists $N_{0} \in \mathbb{N}$ such that for any integer $N \geq N_{0}$ the polynomial $\varphi_{N}(x)=\left(1+|x|^{2}\right)^{N}\left(f(x)+a|x|^{d}+b\right)$ is a strictly convex function on $X$.
Finally, we propose the following algorithm. Given a compact convex semialgebraic set $X$ and a polynomial $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$, assume that $f$ is positive on $X$. Then by our convexification result, there exists an integer $N$ such that $\varphi_{N, \xi}(x):=$ $\left(1+|x-\xi|^{2}\right)^{N} f(x)$ is a convex function for any $\xi \in X$. (Actually one can take $N=6$.) Choose any $a_{0} \in X$, and then by induction set $a_{\nu}:=\operatorname{argmin}_{X} \varphi_{N, a_{\nu-1}}$. In Theorem 6.5 we state that the limit $a^{*}=\lim _{\nu \rightarrow \infty} a_{\nu}$ exists; moreover, $a^{*}$ is a critical point of $f$ on $X$. The proof requires subtle estimates for the lengths of gradient trajectories of $f$ on $X$. Since the set of critical values is finite, this result gives a method for finding the minimum of $f$ on $X$.

## 1. Approximation of nonnegative polynomials

Let $X \subset \mathbb{R}^{n}$ be a closed basic semialgebraic set defined by $g_{1}, \ldots, g_{r} \in \mathbb{R}[x]$, i.e. of the form (0.1).

Theorem 1.1. Let $f \in \mathbb{R}[x]$ be nonnegative on the set $X$. Then there exists $a$ sequence $f_{\nu} \in P\left(g_{1}, \ldots, g_{r}\right), \nu \in \mathbb{N}$, that is uniformly convergent to $f$ on compact subsets. Moreover, $f_{\nu}$ can be chosen from the cone $\mathcal{S}\left(g_{1}, \ldots, g_{r}\right)$. In particular $f_{\nu}$ converges to $f$ in the $l_{1}$-norm.

Proof. Take any positive constants $\varepsilon, \delta, A, B$. By the Weierstrass Approximation Theorem there exists a polynomial $\varphi_{\varepsilon, \delta, A, B} \in \mathbb{R}[t]$ such that

$$
\begin{array}{ll}
\varphi_{\varepsilon, \delta, A, B}(t)>B & \text { for } t \in[-A,-\delta] \\
\varphi_{\varepsilon, \delta, A, B}(t)<\varepsilon & \text { for } t \in[0, A] . \tag{1.2}
\end{array}
$$

Taking $\varphi_{\varepsilon, \delta, A, B}^{2}$ if necessary, we may additionally assume that

$$
\begin{equation*}
\varphi_{\varepsilon, \delta, A, B}(t) \geq 0 \quad \text { for } t \in \mathbb{R} \tag{1.3}
\end{equation*}
$$

Set

$$
g_{i, \varepsilon, \delta, A, B}:=g_{i} \cdot \varphi_{\varepsilon, \delta, A, B} \circ g_{i} \quad \text { for } i=1, \ldots, r .
$$

Every nonnegative univariate polynomial is a sum of squares of polynomials, hence by (1.3) we have

$$
\begin{equation*}
\varphi_{\varepsilon, \delta, A, B} \circ g_{i} \in \sum \mathbb{R}[x]^{2} \quad \text { for } i=1, \ldots, r . \tag{1.4}
\end{equation*}
$$

Since the sequence $h_{\nu}=f+\frac{1}{\nu}, \nu \in \mathbb{N}$, uniformly converges to $f$, we may assume that $f$ is positive on $X$.

Fix an arbitrary $R>1$ and let $M>1$ be a constant such that

$$
\begin{equation*}
f(x) \geq-M \quad \text { for } x \in[-R, R]^{n} . \tag{1.5}
\end{equation*}
$$

[^1]Since $f$ is positive on $X$, we have

$$
X \cap[-R, R]^{n} \subset G_{1}
$$

where the set $G_{1}:=\left\{x \in[-R, R]^{n}: f(x)>0\right\}$ is open in $[-R, R]^{n}$. As $X \cap[-R, R]^{n}$ is a compact set, there exists $\eta>0$ such that

$$
G_{2}:=\left\{x \in[-R, R]^{n}: \operatorname{dist}(x, X) \leq \eta\right\} \subset G_{1}
$$

Since $\overline{[-R, R]^{n} \backslash G_{2}}=\left\{x \in[-R, R]^{n}: \operatorname{dist}(x, X) \geq \eta\right\}$ is also compact, by the definition of $X$ there exists $\delta \in(0,1]$ such that

$$
\begin{equation*}
G_{3}:=\left\{x \in[-R, R]^{n}: g_{i}(x) \geq-\delta \text { for } i=1, \ldots, r\right\} \subset G_{2} . \tag{1.6}
\end{equation*}
$$

Let

$$
f_{R}^{*}:=\min \left\{f(x): x \in G_{2}\right\}
$$

Obviously $f_{R}^{*}>0$.
Let $A \geq 1$ be a constant such that

$$
\left|g_{i}(x)\right| \leq A \quad \text { for } x \in[-R, R]^{n}, i=1, \ldots, r
$$

Take

$$
\begin{equation*}
\varepsilon:=\frac{f_{R}^{*}}{(r+1) A}, \quad B:=A \frac{M+r \varepsilon}{\delta} \tag{1.7}
\end{equation*}
$$

Lemma 1.2. For any $x \in[-R, R]^{n}$ we have $f(x)-\sum_{i=1}^{r} g_{i, \varepsilon, \delta, A, B}(x)>0$.
Proof. Take $x \in[-R, R]^{n}$.
If $x \in X$, then $g_{i}(x) \geq 0$ for $i=1, \ldots, r$, and by (1.2),

$$
g_{i, \varepsilon, \delta, A, B}(x)=g_{i}(x) \cdot \varphi_{\varepsilon, \delta, A, B} \circ\left(g_{i}(x)\right) \leq A \varepsilon<\frac{f_{R}^{*}}{r} \quad \text { for } i=1, \ldots, r .
$$

So

$$
f(x)-\sum_{i=1}^{r} g_{i, \varepsilon, \delta, A, B}(x)>f_{R}^{*}-r \frac{f_{R}^{*}}{r} \geq 0
$$

and the assertion holds.
Let $x \in G_{3} \backslash X$. Without loss of generality we may assume that

$$
g_{1}(x), \ldots, g_{k}(x) \geq 0 \quad \text { and } \quad g_{k+1}(x), \ldots, g_{r}(x)<0
$$

for some $0 \leq k<r$. Then by (1.2),

$$
g_{i, \varepsilon, \delta, A, B}(x) \leq A \varepsilon<\frac{f_{R}^{*}}{r} \quad \text { for } i=1, \ldots, k
$$

and by (1.3),

$$
g_{i, \varepsilon, \delta, A, B}(x)<0 \quad \text { for } i=k+1, \ldots, r
$$

Consequently, $f(x)-\sum_{i=1}^{r} g_{i, \varepsilon, \delta, A, B}(x)>f_{R}^{*}-k \frac{f_{R}^{*}}{r}>0$, and the assertion holds.
Let now $x \in[-R, R]^{n} \backslash G_{3}$. Without loss of generality we may assume that

$$
g_{1}(x), \ldots, g_{k}(x) \geq 0, \quad 0>g_{k+1}(x), \ldots, g_{l}(x) \geq-\delta, \quad g_{l+1}(x), \ldots, g_{r}(x)<-\delta
$$

where $0 \leq k \leq l<r$. Then

$$
g_{i, \varepsilon, \delta, A, B}(x)<\frac{f_{R}^{*}}{r+1} \quad \text { for } i=1, \ldots, k
$$

and

$$
g_{i, \varepsilon, \delta, A, B}(x)<0 \quad \text { for } i=k+1, \ldots, l
$$

By (1.1) we see that

$$
g_{i, \varepsilon, \delta, A, B}(x)<A(-M-r \varepsilon) \leq-M-\frac{r f_{R}^{*}}{r+1} \quad \text { for } i=l+1, \ldots, r
$$

Summing up,

$$
f(x)-\sum_{i=1}^{r} g_{i, \varepsilon, \delta, A, B}(x)>-M-k \frac{f_{R}^{*}}{r+1}+(r-l)\left(M+\frac{r f_{R}^{*}}{r+1}\right)>0
$$

as desired.
Remark 1.3. The polynomial $\varphi_{\varepsilon, \delta, A, B}(t)$ in the above proof can be chosen of the form

$$
\varphi(t)=\left(\frac{1}{A} t-1+\frac{\delta}{2 A}\right)^{2 N}
$$

with $N \log \left(1-\frac{\delta}{2 A}\right)^{2}<\log \varepsilon, N \log \left(\frac{\delta}{2 A}\right)^{2}<\log \varepsilon$ and $N \log \left(1+\frac{\delta}{2 A}\right)^{2}>\log B$. M. Schweighofer [31, Lemma 2.3] in a similar problem proposes a polynomial $\varphi$ of the form $\varphi(t)=a s(a t-1)^{2 N}$ for some $s \in \mathbb{N}$ and $a>0$.

By Lemma 1.2, for any $R>0$ there exists $\varphi_{R} \in \sum \mathbb{R}[t]^{2}$ such that

$$
f(x)-\sum_{i=1}^{r} \varphi_{R}\left(g_{i}(x)\right) g_{i}(x)>0 \quad \text { for } x \in[-R, R]^{n}
$$

By Fact 1 in the Introduction, it is easy to see that $f(x)-\sum_{i=1}^{r} \varphi_{R}\left(g_{i}(x)\right) g_{i}(x)$ can be approximated in the $l_{1}$-norm by sums of squares of polynomials and it can be approximated uniformly on $[-R, R]^{n}$ by sums of squares of polynomials. Consequently, $f$ can be approximated uniformly on $[-R, R]^{n}$ (in particular in the $l_{1}$-norm) by polynomials from the cone $\mathcal{S}\left(g_{1}, \ldots, g_{r}\right)$. Hence we deduce the assertion of Theorem 1.1

## 2. Quantitative aspects of Theorem 1.1

In order to estimate the rate of convergence in Lasserre's relaxation method 15 we show how to bound the degree of the polynomial $\varphi$ in Theorem 1.1. The key point is to find a lower bound for $\delta$ which satisfies the inclusion (1.6).

Assume now that $X$ is a compact set of the form

$$
\begin{equation*}
X=\left\{x \in \mathbb{R}^{n}: g_{1}(x) \geq 0, \ldots, g_{r}(x) \geq 0\right\} \tag{2.1}
\end{equation*}
$$

where $g_{1}, \ldots, g_{r} \in \mathbb{R}[x]$. Choose $R>0$ large enough so that $g_{0}(x)=R^{2}-|x|^{2}$ is nonnegative polynomial on $X$. We now define a cone

$$
\mathcal{K}\left(g_{0}, \ldots, g_{r}\right):=\left\{\sigma_{0}+\sigma_{1} g_{0}+\sum_{i=1}^{r} \varphi\left(g_{i}\right) g_{i}: \sigma_{0}, \sigma_{1} \in \sum \mathbb{R}[x]^{2}, \varphi \in \sum \mathbb{R}[t]^{2}\right\}
$$

By the argument in the proof of Theorem 1.1 we obtain
Corollary 2.1. If $f \in \mathbb{R}[x]$ is strictly positive on the set $X$, then $f \in \mathcal{K}\left(g_{0}, \ldots, g_{r}\right)$.
Proof. By Lemma 1.2 , there exists $\varphi \in \sum \mathbb{R}[t]^{2}$ such that

$$
h(x)=f(x)-\sum_{i=1}^{r} \varphi\left(g_{i}(x)\right) g_{i}(x)>0
$$

for $|x| \leq R$. Since $\left\{x \in \mathbb{R}^{n}:|x| \leq R\right\}=\left\{x \in \mathbb{R}^{n}: g_{0}(x) \geq 0\right\}$, Putinar's Positivstellensatz (or Schmüdgen's Positivstellensatz, because $\left.P\left(g_{0}\right)=T\left(g_{0}\right)\right)$ yields

$$
h \in\left\{\sigma_{0}+\sigma_{1} g_{0}: \sigma_{0}, \sigma_{1} \in \sum \mathbb{R}[x]^{2}\right\}=\mathcal{K}\left(g_{0}\right)
$$

which completes the proof.
Corollary 2.1 also follows from Schweighofer's result [31, Lemma 2.3] and the Putinar theorem.

Remark 2.2. We may use the Lasserre algorithm for minimization of a polynomial $f$ on a compact basic semialgebraic set $X$ by using $\mathcal{K}\left(g, g_{1}, \ldots, g_{r}\right)$ instead of $P\left(g, g_{1}, \ldots, g_{r}\right)$. In fact, we can use the set $\mathcal{K}_{k}\left(g, g_{1}, \ldots, g_{r}\right)$ consisting of all $\sigma_{0}+\sigma_{1} g+\sum_{i=1}^{r} \varphi\left(g_{i}\right) g_{i} \in \mathcal{K}\left(g, g_{1}, \ldots, g_{r}\right)$ such that $\operatorname{deg} \sigma_{0} \leq k, \operatorname{deg} \sigma_{1} g \leq k$ and $\operatorname{deg} \varphi\left(g_{i}\right) g_{i} \leq k$ for $i=1, \ldots, r$. Consider the following optimization problems:

- maximize $a \in \mathbb{R}$ such that $f-a \in \mathcal{K}_{k}\left(g, g_{1}, \ldots, g_{r}\right)$,
- minimize $L(f)$ for $L: \mathbb{R}[x]_{k} \rightarrow \mathbb{R}$, linear, $L(1)=1, L\left(\mathcal{K}_{k}\left(g, g_{1}, \ldots, g_{r}\right)\right) \subset$ $[0, \infty)$.
Denote
$u_{k}^{*}:=\sup \left\{a \in \mathbb{R}: f-a \in \mathcal{K}_{k}\left(g, g_{1}, \ldots, g_{r}\right)\right\}$,
$v_{k}^{*}:=\inf \left\{L(f): L: \mathbb{R}[x]_{k} \rightarrow \mathbb{R}\right.$ is linear, $\left.L(1)=1, L\left(\mathcal{K}_{k}\left(g, g_{1}, \ldots, g_{r}\right)\right) \subset[0, \infty)\right\}$,
for sufficiently large $k \in \mathbb{N}$. We see that $\left(u_{k}^{*}\right),\left(v_{k}^{*}\right)$ are increasing sequences that converge to $f^{*}$ (by Corollary 2.1) and $u_{k}^{*} \leq v_{k}^{*} \leq f^{*}$ for $k \in \mathbb{N}$.
2.1. Quantitative Lojasiewicz inequality. Let $g_{1}, \ldots, g_{r} \in \mathbb{R}[x]$, and let $G$ : $\mathbb{R}^{n} \rightarrow \mathbb{R}$ be defined by

$$
\begin{equation*}
G(x)=\max \left\{0,-g_{1}(x), \ldots,-g_{r}(x)\right\}, \quad x \in \mathbb{R}^{n} \tag{2.2}
\end{equation*}
$$

Then $X=\left\{x \in \mathbb{R}^{n}: g_{1}(x) \geq 0, \ldots, g_{r}(x) \geq 0\right\}=G^{-1}(0)$. Moreover,

$$
\operatorname{graph} G=Y_{0} \cup Y_{1} \cup \cdots \cup Y_{r}
$$

where $Y_{0}=X \times\{0\}$,

$$
Y_{i}=\left\{(x, y) \in \mathbb{R}^{n} \times \mathbb{R}: y=-g_{i}(x), g_{i}(x) \leq 0, g_{i}(x) \leq g_{j}(x) \text { for } j \neq i\right\}
$$

for $i=1, \ldots, r$. Note that each set $Y_{i}, i=0, \ldots, r$, is defined by $r$ inequalities and one equation. Let $d=\max \left\{\operatorname{deg} g_{1}, \ldots, \operatorname{deg} g_{r}\right\}$. We now state the well-known Eojasiewicz inequality in a quantitative version proved in [14, Corollary 2.3] (see also [13, Corollary 10]): there exist $C, \mathcal{L}>0$ such that

$$
\begin{equation*}
G(x) \geq C\left(\frac{\operatorname{dist}(x, X)}{1+|x|^{d}}\right)^{\mathcal{L}}, \quad x \in \mathbb{R}^{n} \tag{2.3}
\end{equation*}
$$

with

$$
\begin{equation*}
\mathcal{L} \leq d(6 d-3)^{n+r-1} \tag{2.4}
\end{equation*}
$$

It follows from (2.3) that for every $\rho>0$ there exists $C_{\rho}>0$ such that

$$
\begin{equation*}
G(x) \geq C_{\rho} \operatorname{dist}(x, X)^{\mathcal{L}} \quad \text { for any } x \in B(\rho) \tag{2.5}
\end{equation*}
$$

where $B(\rho)=\left\{x \in \mathbb{R}^{n}:|x| \leq \rho\right\}$. Fix $R>0$ such that $X \subset B(R)$. Assume that (2.5) holds with fixed $C^{\prime}=C_{R}$ and $\mathcal{L}$.

Fact 2. Let $\eta>0$. Set $\delta_{0}=C^{\prime} \eta^{\mathcal{L}}$. Then for any $0<\delta \leq \delta_{0}$,

$$
\left\{x \in B(R): g_{i}(x) \geq-\delta \text { for } i=1, \ldots, r\right\} \subset\{x \in B(R): \operatorname{dist}(x, X) \leq \eta\}
$$

Indeed, take $x \in B(R) \backslash X$ such that $g_{i}(x) \geq-\delta$ for $i=0, \ldots, r$. Let $G$ be the function defined by (2.2). Hence by (2.5),

$$
\delta \geq \max \left\{-g_{1}(x), \ldots,-g_{r}(x)\right\}=G(x) \geq C^{\prime} \operatorname{dist}(x, X)^{\mathcal{L}} .
$$

Thus for $0<\delta \leq \delta_{0}$ we deduce the assertion of Fact 2
2.2. Approximation. For $\nu=\left(\nu_{1}, \ldots, \nu_{n}\right) \in \mathbb{N}^{n}$ we set $|\nu|=\nu_{1}+\cdots+\nu_{n}$ and $a^{\nu}=a_{1}^{\nu_{1}} \cdots a_{n}^{\nu_{n}}$, where $a=\left(a_{1}, \ldots, a_{n}\right) \in \mathbb{R}^{n}$. For $h \in \mathbb{R}[x]$ of the form

$$
h(x)=\sum_{j=0}^{d} \sum_{|\nu|=j} a_{\nu} x^{\nu}
$$

we define

$$
\mathbb{A}(h, R)=\sum_{j=0}^{d} \sum_{|\nu|=j}\left|a_{\nu}\right| R^{j}, \quad \mathbb{B}(h, R)=\sum_{j=1}^{d} \sum_{|\nu|=j} j\left|a_{\nu}\right| R^{j-1} \quad \text { for } R>0 .
$$

Then for $x \in B(R)$ we have $|h(x)| \leq \mathbb{A}(h, R)$ and by the Euler formula for homogeneous functions, $|\nabla h(x)| \leq \mathbb{B}(h, R)$.

Using a similar argument to the one for Theorem 1.1] we obtain the following
Proposition 2.3. Let $f \in \mathbb{R}[x]$, let $X$ be a semialgebraic set of the form (0.1) such that $X \subset B(R), R>0$, and let $g_{1}, \ldots, g_{r} \in \mathbb{R}[x]$ be polynomials satisfying (2.5) with fixed $C, \mathcal{L}>0$. Take $M, A \in \mathbb{R}$ such that

$$
M \geq \max \{1, \mathbb{A}(f, R), \mathbb{B}(f, R)\}, \quad A \geq \max \left\{1, \mathbb{A}\left(g_{i}, R\right)\right\} \quad \text { for } i=1, \ldots, r
$$

Take $\epsilon>0$, and set

$$
\varphi(t)=\left(\frac{1}{A} t-1+\frac{\delta}{2 A}\right)^{2 N}
$$

where
$0<\delta \leq \min \left\{A, C\left(\frac{\epsilon}{2 M}\right)^{\mathcal{L}}\right\}, N \geq \max \left\{\frac{(r-1) A-1}{2}, \frac{A(2 M+1-\delta)}{\delta^{2}}, \frac{2 r A-\epsilon}{2 \epsilon}\right\}$.
Then the function

$$
h(x)=\sum_{i=1}^{r} \varphi\left(g_{i}(x)\right) g_{i}(x) \in \Phi\left(g_{1}, \ldots, g_{r}\right)
$$

satisfies the following conditions:

$$
\begin{gather*}
0 \leq h(x)<\epsilon \quad \text { for } x \in X  \tag{2.6}\\
\forall_{|y| \leq R} \exists_{x \in X} f(y)-h(y) \geq f(x)-h(x)-\epsilon \tag{2.7}
\end{gather*}
$$

Proof. It is easy to see that for the function

$$
\phi(t)=t\left(\frac{1}{A} t-1+\frac{\delta}{2 A}\right)^{2 N}
$$

where $0<\delta<A, N>\frac{(r-1) A-1}{2}, N>\frac{A(2 M+1-\delta)}{\delta^{2}}$, we have

$$
\begin{array}{ll}
\phi(t)<\frac{A}{2 N+1} & \text { for } t \in[0, A], \\
\phi(t) \leq-2 M-\frac{(r-1) A}{2 N+1} & \text { for } t \leq-\delta \tag{2.9}
\end{array}
$$

From the assumptions of $M$ and $A$ we have $|f(x)| \leq M,|\nabla f(x)| \leq M$ and $\left|g_{i}(x)\right| \leq A$ for $i=1, \ldots, r$ and $x \in \mathbb{R}^{n}$ such that $|x| \leq R$.

Take any $\epsilon>0$. Let

$$
\begin{aligned}
Y & :=\left\{y \in \mathbb{R}^{n}:|y| \leq R \wedge \exists_{x \in X} f(y) \geq f(x)-\epsilon / 2\right\} \\
\eta & :=\frac{\epsilon}{2 M} \\
Y_{1} & :=\left\{y \in \mathbb{R}^{n}:|y| \leq R \wedge \operatorname{dist}(y, X) \leq \eta\right\}
\end{aligned}
$$

By the Mean Value Theorem, $Y_{1} \subset Y$. From Fact 2 for $0<\delta \leq C \eta^{\mathcal{L}}$ we have

$$
Y_{2}:=\left\{x \in \mathbb{R}^{n}:|x| \leq R \wedge g_{i}(x) \geq-\delta \text { for } i=0, \ldots, r\right\} \subset Y_{1} \subset Y
$$

Obviously $h(x) \geq 0$ for $x \in X$. Since $h(x)=\sum_{i=1}^{r} \phi\left(g_{i}(x)\right)$ and $g_{i}(x) \in[0, A]$ for $x \in X$, by (2.8) and the assumption $N \geq \frac{2 r A-\epsilon}{2 \epsilon} \geq \frac{r A-\epsilon}{2 \epsilon}$ we obtain (2.6).

Now we prove (2.7). Obviously it holds for $y \in X$.
Take $y \in Y_{2} \backslash X$. Without loss of generality we may assume that

$$
g_{1}(y), \ldots, g_{k}(y) \geq 0 \quad \text { and } \quad g_{k+1}(y), \ldots, g_{r}(y)<0
$$

for some $0 \leq k<r$. Then there exists $x \in X$ such that $f(y) \geq f(x)-\frac{\epsilon}{2}$. So, (2.8) and the assumption $N \geq \frac{2 r A-\epsilon}{2 \epsilon}$ give
$f(y)-h(y) \geq f(x)-\frac{\epsilon}{2}-h(y) \geq f(x)-\frac{\epsilon}{2}-\sum_{i=1}^{k} \phi\left(g_{i}(y)\right) \geq f(x)-\epsilon \geq f(x)-h(x)-\epsilon$.
This proves (2.7) for $y \in Y_{2} \backslash X$.
Let now $y \in\left\{x \in \mathbb{R}^{n}:|x| \leq R, x \notin Y_{2}\right\}$. Without loss of generality we may assume that

$$
g_{1}(y), \ldots, g_{k}(y) \geq 0, \quad 0>g_{k+1}(y), \ldots, g_{l}(y) \geq-\delta, \quad g_{l+1}(y), \ldots, g_{r}(y)<-\delta
$$

where $0 \leq k \leq l<r$. Then, by the choice of $M$, the assumption $N \geq \frac{A(2 M+1-\delta)}{\delta^{2}}$ and (2.9) we see that $h(y) \leq-2 M$, and so for any $x \in X$ we have

$$
f(y)-h(y) \geq-M+2 M \geq f(x) \geq f(x)-h(x) \geq f(x)-h(x)-\epsilon
$$

This gives (2.7) in the case under consideration and ends the proof.
Remark 2.4. If we assume that $g_{1}, \ldots, g_{r}$ are $\mu$-strongly concave polynomials, i.e.,

$$
g_{i}(y) \leq g_{i}(x)+\left\langle y-x, \nabla g_{i}(x)\right\rangle-\frac{\mu}{2}|y-x|^{2} \quad \text { for } x, y \in \mathbb{R}^{n}
$$

where $\mu>0$ and $\langle\cdot, \cdot\rangle$ is the standard scalar product, then the assertion of Fact 2 holds with $\delta_{0}=\eta^{2} \mu / 2$. Hence, Proposition 2.3 holds with $0<\delta \leq \min \left\{A, \frac{\epsilon^{2} \mu}{8 M^{2}}\right\}$.

Remark 2.5. We can use Proposition 2.3 to minimize a polynomial $f$ on a compact basic semialgebraic set $X$. Let $X \subset\left\{x \in \mathbb{R}^{n}:|x| \leq R\right\}$. Then for any $\epsilon>0$, we can effectively compute a polynomial $h(x)=\sum_{i=i}^{r} \varphi\left(g_{i}(x)\right) g_{i}(x)$, where $\varphi \in \sum \mathbb{R}[t]^{2}$, such that

$$
f^{*}-2 \epsilon \leq \inf \{f(y)-h(y):|y| \leq R\} \leq f^{*}+2 \epsilon
$$

To approximate $f^{*}$, we can minimize $f-h$ on $B(R)$. To this end we may compute

$$
a_{k}^{* *}:=\sup \left\{a \in \mathbb{R}: f-h-a \in P_{k}\left(R^{2}-|y|^{2}\right)\right\} \quad \text { for } k \in \mathbb{N}
$$

By the Putinar Theorem (or the Schmüdgen Theorem) we see that

$$
a_{k}^{* *} \rightarrow f^{* *} \quad \text { as } k \rightarrow \infty
$$

where $f^{* *}:=\inf \{f(y)-h(y):|y| \leq R\}$.
Minimization of $f-h$ on $B(R)$ is much simpler than minimizing $f$ on $X$, because the set $B(R)$ is described by one inequality $R^{2}-|x|^{2} \geq 0$. In this case M. Schweighofer [30] gave the rate of convergence of the sequence $a_{k}^{* *}$ :

$$
f^{* *}-a_{k}^{* *} \leq \frac{c}{\sqrt[d]{k}}
$$

for some constant $c \in \mathbb{N}$ depending on $f$ and $R^{2}-|y|^{2}$ and some constant $d \in \mathbb{N}$ depending on $R^{2}-|y|^{2}$.

## 3. Convex polynomials in one variable

We denote by $\mathbb{N}^{*}$ the set of strictly positive integers. In this section $x$ denotes a single variable. Let $f \in \mathbb{R}[x]$ be a nonzero polynomial. For any $N \in \mathbb{N}^{*}$ we define the following polynomial:

$$
\begin{equation*}
\varphi_{N}(x):=\left(1+x^{2}\right)^{N} f(x) \tag{3.1}
\end{equation*}
$$

We will find $N_{0} \in \mathbb{N}^{*}$ such that for $N \geq N_{0}$ the polynomial $\varphi_{N}$ is strongly convex on a closed interval $I \subset \mathbb{R}$, provided $f$ is positive on $I$.

For positive numbers $m, R, D$ we set

$$
\begin{equation*}
\mathcal{N}(m, R, D):=\max \left\{\frac{D}{m}+\frac{m}{16 D}, \frac{\left(1+R^{2}\right) D}{R m}+1, \frac{4 D^{2}}{m^{2}}+2, \frac{\left(1+R^{2}\right) D}{2 m}\right\} \tag{3.2}
\end{equation*}
$$

We first prove that if $f$ is a $C^{2}$ function positive on a bounded interval $I$, then $\varphi_{N}(x)=\left(1+x^{2}\right)^{N} f(x)$ is convex for every $N$ sufficiently large. We formulate this lemma for $C^{2}$ functions because restricting to polynomials does not simplify the proof considerably.

Lemma 3.1. Let $f$ be a $C^{2}$ function positive on an interval $I=[a, b] \subset \mathbb{R}$, and let $R \geq \max \{|a|,|b|\}$. If $m, D>0$ satisfy the conditions

$$
\begin{gather*}
m \leq \min \{f(x): x \in I\}  \tag{3.3}\\
\left|f^{\prime}(x)\right| \leq D, \quad\left|f^{\prime \prime}(x)\right| \leq D \quad \text { for }|x| \leq R \tag{3.4}
\end{gather*}
$$

then for any $N \in \mathbb{N}$ satisfying

$$
\begin{equation*}
N>\mathcal{N}(m, R, D) \tag{3.5}
\end{equation*}
$$

we have $\varphi_{N}^{\prime \prime}(x)>0$ for $x \in I$, thus $\varphi_{N}(x)$ is strongly convex on $I$.
Proof. Denote $P_{N}=A_{N}+B_{N}+Q_{N}+T_{N}$, where

$$
\begin{array}{ll}
A_{N}(x)=4 N(N-1) x^{2} f(x), & B_{N}(x)=2 N\left(1+x^{2}\right) f(x) \\
Q_{N}(x)=4 N\left(1+x^{2}\right) x f^{\prime}(x), & T_{N}(x)=\left(1+x^{2}\right)^{2} f^{\prime \prime}(x)
\end{array}
$$

Then

$$
\begin{equation*}
\varphi_{N}^{\prime \prime}(x)=\left(1+x^{2}\right)^{N-2} P_{N}(x) \tag{3.6}
\end{equation*}
$$

Let $N \in \mathbb{N}$ satisfy (3.5). To prove that $\varphi_{N}$ is convex on $I$ we will proceed in several steps. From (3.3) and (3.4) we obtain

$$
\begin{array}{ll}
A_{N}(x) \geq 4 N(N-1) x^{2} m & \text { for } x \in I \\
B_{N}(x) \geq 2 N\left(1+x^{2}\right) m & \text { for } x \in I \\
Q_{N}(x) \geq-4 N\left(1+x^{2}\right)|x| D & \text { for }|x| \leq R \\
T_{N}(x) \geq-\left(1+x^{2}\right)^{2} D & \text { for }|x| \leq R \tag{3.10}
\end{array}
$$

Since $N$ satisfy (3.5), we have

$$
\begin{equation*}
N \geq \frac{D}{m}+\frac{m}{16 D} \tag{3.11}
\end{equation*}
$$

Note that then

$$
\begin{equation*}
\frac{m}{4 D} \leq \sqrt{\frac{N m-D}{D}} \tag{3.12}
\end{equation*}
$$

Assume now that $x \in I,|x|<\frac{m}{4 D}$. Then obviously $A_{N}(x) \geq 0$. By (3.8) and (3.9) we have

$$
\frac{1}{2} B_{N}(x)+Q_{N}(x)>0 .
$$

Also by (3.8), (3.10) and (3.12),

$$
\frac{1}{2} B_{N}(x)+T_{N}(x)>0 .
$$

So for $N$ satisfying (3.11) we have $P_{N}(x)>0$, and consequently by (3.6),

$$
\begin{equation*}
\varphi_{N}^{\prime \prime}(x)>0 \quad \text { for } x \in I,|x|<\frac{m}{4 D} \tag{3.13}
\end{equation*}
$$

We have to show now that $P_{N}(x)>0$ for $x \in I, \frac{m}{4 D} \leq|x| \leq R$. By (3.5) we have

$$
\begin{equation*}
N>\max \left\{\frac{\left(1+R^{2}\right) D}{R m}+1, \frac{4 D^{2}}{m^{2}}+2\right\} \tag{3.14}
\end{equation*}
$$

By (3.7) and (3.9) we see that
(3.15) $A_{N}(x)+Q_{N}(x) \geq\left(-D|x|^{2}+(N-1) m|x|-D\right) 4 N|x| \quad$ for $x \in I,|x| \leq R$,
and by (3.14),

$$
-D\left(\frac{m}{4 D}\right)^{2}+(N-1) m \frac{m}{4 D}-D>0
$$

and

$$
-D R^{2}+(N-1) m R-D>0
$$

Hence $-D|x|^{2}+(N-1) m|x|-D>0$ for $\frac{m}{4 D} \leq|x| \leq R$, and (3.15) gives

$$
\begin{equation*}
A_{N}(x)+Q_{N}(x)>0 \quad \text { for } x \in I, \frac{m}{4 D} \leq|x| \leq R \tag{3.16}
\end{equation*}
$$

By (3.5) we have

$$
N>\frac{\left(1+R^{2}\right) D}{2 m}
$$

then, by (3.8) and (3.10), we obtain

$$
\begin{equation*}
B_{N}(x)+T_{N}(x)>0 \quad \text { for } x \in I, \frac{m}{4 D} \leq|x| \leq R \tag{3.17}
\end{equation*}
$$

Consequently, by (3.16), (3.17) and (3.6), we have

$$
\begin{equation*}
\varphi_{N}^{\prime \prime}(x)>0 \quad \text { for } x \in I, \frac{m}{4 D} \leq|x| \leq R \tag{3.18}
\end{equation*}
$$

Summing up, for $N$ satisfying (3.5), by (3.13) and (3.18), we have $\varphi_{N}^{\prime \prime}(x)>0$, $x \in I$, which means that $\varphi_{N}$ is strongly convex on $I$ and Lemma 3.1 is proved.

Remark 3.2. Lemma 3.1 was proved under the assumption that the function $f$ is $C^{2}$. If we assume that $f$ is a polynomial which is positive except possibly at $0 \in \mathbb{R}$, then an analogous argument leads to a strictly convex function $\varphi_{N}$. More precisely, let $f \in \mathbb{R}[x]$ be a polynomial positive on $I=[a, b]$ except possibly at $0 \in \mathbb{R}$, where $0 \in(a, b)$. Then there exists $N_{0} \in \mathbb{N}$ such that for any $N \in \mathbb{N}$ with $N \geq N_{0}$ the polynomial $\varphi_{N}(x)$ is strictly convex on $I$.

For a polynomial of degree $d$ of the form

$$
\begin{equation*}
f=\sum_{i=0}^{d} a_{i} x^{d-i}, \quad a_{0}, \ldots, a_{d} \in \mathbb{R}, \quad a_{0} \neq 0 \tag{3.19}
\end{equation*}
$$

and $R>0$, we set

$$
D(f, R):=\max \left\{1, \sum_{i=0}^{d-1}(d-i)\left|a_{i}\right| R^{d-i-1}, \sum_{i=0}^{d-2}(d-i)(d-i-1)\left|a_{i}\right| R^{d-i-2}\right\}
$$

We easily see that for any $D \geq D(f, R)$ the assumption (3.4) of Lemma 3.1 holds. If $d>0$, we define

$$
K(f)=1+2 \max _{1 \leq i \leq d}\left|\frac{a_{i}}{a_{0}}\right|^{1 / i}
$$

Obviously $K(f)>0$. It is known that if $f(z)=0, z \in \mathbb{C}$ then $|z|<K(f)$. Since for $d \geq 2$ the complex zeroes of $f^{\prime}$ and $f^{\prime \prime}$ lie in the convex hull of the set of complex zeroes of $f$,

$$
\begin{equation*}
f, f^{\prime} \text { and } f^{\prime \prime} \text { have no zeroes } x \in \mathbb{R} \text { such that }|x| \geq K(f) \tag{3.20}
\end{equation*}
$$

We prove a version of Lemma 3.1 for a polynomial on an arbitrary interval. (A version of this lemma, without explicit bound for $N$, has been proven in the M.Sc. thesis of I. Fau [10].)

Lemma 3.3. Let $f \in \mathbb{R}[x]$ be positive on a closed interval $I \subset \mathbb{R}$. Let $m>0$ satisfy (3.3), and let $R \geq K(f)$ and $D \geq D(f, R)$ (or let $D$ satisfy (3.4)). Then for any integer $N>\mathcal{N}(m, R, D)$ the polynomial $\varphi_{N}(x)$ is strongly convex on $I$.

Proof. By the same argument as for (3.20), we deduce that $\varphi_{N}^{\prime \prime}(x)$ for $x \leq-R$ has the same sign as $\varphi_{N}^{\prime \prime}(-R)$. Analogously, $\varphi_{N}^{\prime \prime}(R)$ and $\varphi_{N}^{\prime \prime}(x)$ for $x \geq R$ have the same sign. Moreover, $\varphi_{N}^{\prime \prime}(-R) \neq 0$ and $\varphi_{N}^{\prime \prime}(R) \neq 0$. So considering the sign of $\varphi_{N}^{\prime \prime}$ on the intervals $J_{1}=I \cap[-R, R], J_{2}=I \cap[R,+\infty)$ and $J_{3}=I \cap(-\infty,-R]$ we deduce the assertion by Lemma 3.1. Note that the strong convexity of $\varphi_{N}$ is due to the fact that $f$ is a polynomial.

Remark 3.4. Under the assumptions of Lemma 3.1] and with the same argument, we obtain the assertion of this lemma for the function $\varphi_{N, \xi}(x)=\left(1+(x-\xi)^{2}\right)^{N} f(x)$ instead of $\varphi_{N}$, where $\xi \in[-R, R]$, with the bound $N>\mathcal{N}(m, 2 R, D)$. Hence, the assertion of Lemma 3.3 holds for the function $\varphi_{N, \xi}$ with the bound $N>\mathcal{N}(m, 2 R, D)$.

The exponent $N$ in Lemma 3.3 actually depends on the coefficients of $f$ even when the degree of $f$ is fixed.

Example 3.5. Let $f_{k}(x)=(x-k)^{2}+1$. Obviously $f_{k}$ is a convex function. We have $\varphi_{N}(x)=\left((x-k)^{2}+1\right)\left(1+x^{2}\right)^{N}$ and $\varphi_{N}(k)=\left(1+k^{2}\right)^{N}, \varphi(0)=k^{2}+1$, $\varphi_{N}\left(\frac{k}{2}\right)=\left(\frac{k^{2}}{4}+1\right)^{N+1}$. Assume that $\varphi_{N}$ is convex. Then

$$
\left(\frac{k^{2}}{4}+1\right)^{N+1} \leq \frac{1}{2}\left(k^{2}+1\right)+\frac{1}{2}\left(k^{2}+1\right)^{N}
$$

So the number $N$ (in Lemma 3.3) such that the function $\varphi_{N}$ is convex tends to infinity as $k \rightarrow \infty$.

Remark 3.6. By a similar argument to that for Lemmas 3.1 and 3.3 one can prove (see [10]): for any $f \in \mathbb{R}[x]$ positive on $\mathbb{R}$ and any $g \in \mathbb{R}[x]$ such that $g(x)>0$ and $g^{\prime \prime}(x)>0$ for $x \in \mathbb{R}$ there exists $N_{0} \in \mathbb{N}$ such that for any $N \geq N_{0}$ the polynomial $f g^{N}$ is strictly convex on $\mathbb{R}$.

## 4. Convexifying polynomials on compact sets

Let $x=\left(x_{1}, \ldots, x_{n}\right)$ be a system of variables and let $f \in \mathbb{R}[x]$ be a polynomial of the form

$$
\begin{equation*}
f=\sum_{j=0}^{d} \sum_{|\nu|=j} a_{\nu} x^{\nu} \tag{4.1}
\end{equation*}
$$

For $R \geq 0$ define $\mathbb{D}(f, R):=$

$$
\max \left\{1, \sqrt{1+R^{2}} \sum_{j=1}^{d} \sum_{|\nu|=j} j\left|a_{\nu}\right| R^{j-1},\left(1+R^{2}\right) \sum_{j=2}^{d} \sum_{|\nu|=j} j(j-1)\left|a_{\nu}\right| R^{j-2}\right\} .
$$

This will be a bound for the first and the second derivatives in (4.5) below.
Theorem 4.1. Let $f \in \mathbb{R}[x]$ be positive on a compact convex set $X \subset \mathbb{R}^{n}$ containing at least two points. Set $R=\max \{|x|: x \in X\}$, and let

$$
\begin{equation*}
0<m \leq \min \{f(x): x \in X\} \tag{4.2}
\end{equation*}
$$

Then for any $D \geq \mathbb{D}(f, R)$ and any integer $N \geq \mathcal{N}(m, R, D)$ the polynomial $\varphi_{N}(x)=\left(1+x_{1}^{2}+\cdots+x_{n}^{2}\right)^{N} f(x)$ is strongly convex in $X$.

Proof. Let

$$
\mathcal{A}=\left\{(\alpha, \beta) \in \mathbb{R}^{n} \times \mathbb{R}^{n}:\langle\alpha, \beta\rangle=0,|\beta|=1\right\}
$$

and let

$$
\begin{equation*}
\gamma_{\alpha, \beta}(t):=\sqrt{1+|\alpha|^{2}} \beta t+\alpha \tag{4.3}
\end{equation*}
$$

Clearly the family of all $\gamma_{\alpha, \beta}$ with $(\alpha, \beta) \in \mathcal{A}$ parametrizes all affine lines in $\mathbb{R}^{n}$. Denote by $\mathcal{B} \subset \mathcal{A}$ the set of all $(\alpha, \beta) \in \mathcal{A}$ for which the line parametrized by $\gamma_{\alpha, \beta}$ intersects $X$. It is easy to see that $\mathcal{B}$ is a compact set and

$$
\begin{equation*}
\mathcal{B} \subset\{(\alpha, \beta) \in \mathcal{A}:|\alpha| \leq R\} \tag{4.4}
\end{equation*}
$$

It suffices to prove that for any $(\alpha, \beta) \in \mathcal{B}$ and $N \geq \mathcal{N}(m, R, D)$ the function $f \circ \gamma_{\alpha, \beta}$ is strictly convex on $I_{\alpha, \beta}=\left\{t \in \mathbb{R}: \gamma_{\alpha, \beta}(t) \in X\right\}$. Since $X$ is a compact convex set, $I_{\alpha, \beta}$ is a compact interval or a point.

It is obvious that for $(\alpha, \beta) \in \mathcal{B}$ the set $\left\{t \in \mathbb{R}:\left|\gamma_{\alpha, \beta}(t)\right| \leq R\right\}$ is an interval centered at 0 (or a point), say $\left[-R_{\alpha, \beta}, R_{\alpha, \beta}\right]$. Moreover, we have $I_{\alpha, \beta} \subset$ $\left[-R_{\alpha, \beta}, R_{\alpha, \beta}\right] \subset[-R, R]$.

If $f$ is of the form (4.1), then we easily see that for $t \in \mathbb{R}$ such that $\left|\gamma_{\alpha, \beta}(t)\right| \leq R$ we have

$$
\left|\left(f \circ \gamma_{\alpha, \beta}\right)^{\prime}(t)\right| \leq \sqrt{1+R^{2}} \sum_{j=1}^{d} \sum_{|\nu|=j} j\left|a_{\nu}\right| R^{j-1}
$$

and
so

$$
\begin{equation*}
\left|\left(f \circ \gamma_{\alpha, \beta}\right)^{\prime}(t)\right| \leq D, \quad\left|\left(f \circ \gamma_{\alpha, \beta}\right)^{\prime \prime}(t)\right| \leq D \quad \text { for } t \in\left[-R_{\alpha, \beta}, R_{\alpha, \beta}\right] . \tag{4.5}
\end{equation*}
$$

A simple computation gives

$$
\begin{equation*}
1+\left|\gamma_{\alpha, \beta}(t)\right|^{2}=\left(1+|\alpha|^{2}\right)\left(1+t^{2}\right) \tag{4.6}
\end{equation*}
$$

hence

$$
\varphi_{N} \circ \gamma_{\alpha, \beta}(t)=\left(1+|\alpha|^{2}\right)^{N}\left(1+t^{2}\right)^{N} f \circ \gamma_{\alpha, \beta}(t)
$$

Obviously $\varphi_{N} \circ \gamma_{\alpha, \beta}$ is a strongly convex function on $I_{\alpha, \beta}$ if and only if the function $I_{\alpha, \beta} \ni t \mapsto\left(1+t^{2}\right)^{N} f \circ \gamma_{\alpha, \beta}(t)$ is strongly convex. Now applying Lemma 3.1 we deduce the assertion.

Remark 4.2. Under the assumptions of Theorem4.1, and with the same argument, we obtain the assertion of this theorem for the function $\varphi_{N, \xi}(x)=\left(1+|x-\xi|^{2}\right)^{N} f(x)$ instead of $\varphi_{N}$, where $\xi \in \mathbb{R}^{n}$, with the bound $N>\mathcal{N}(m, 2 R, D)$.

## 5. Convexity at infinity

We briefly recall basic definitions. For a $C^{2}$ function $f$ in an open subset of $\mathbb{R}^{n}$, $H_{x} f$ stands for the Hessian matrix of $f$ at $x$. The associated quadratic form $h_{x}$ : $\mathbb{R}^{n} \rightarrow \mathbb{R}$ reads

$$
\begin{equation*}
h_{x} f(y)=\left\langle H_{x} f(y), y\right\rangle \tag{5.1}
\end{equation*}
$$

Recall that the matrix $H_{x} f$ is said to be positive semidefinite (respectively positive definite) if $h_{x}(y) \geq 0$ for any $y \in \mathbb{R}^{n}$ (respectively $h_{x} f(y)>0$ for $\left.y \neq 0\right)$. Set, for $E \subset\{1, \ldots, n\}, E \neq \emptyset$,

$$
\Delta_{E}^{f}:=\operatorname{det}\left[\frac{\partial^{2} f}{\partial x_{i} \partial x_{j}}\right]_{i, j \in E}
$$

Recall a classical fact (Sylvester criterion):
Lemma 5.1. $H_{x} f$ is positive semidefinite (respectively positive definite) if and only if $\Delta_{E}^{f} \geq 0$ (respectively $\Delta_{E}^{f}>0$ ) for all nonempty $E \subset\{1, \ldots, n\}$.

Let $f \in \mathbb{R}[x]$ and $n \geq 2$. We call $f$ locally convex (respectively locally strictly convex or locally strongly convex) in an open set $G \subset \mathbb{R}^{n}$ if any point $x \in G$ has a convex neighbourhood $U \subset \mathbb{R}^{n}$ such that the restriction $\left.f\right|_{U}$ is convex (respectively strictly convex or strongly convex). In particular $f$ is locally convex in $G$ if and only if $H_{x} f$ is positive for any $x \in G$. We say that $f$ is convex at infinity (respectively strictly convex at infinity or strongly convex at infinity) if there exists $R \geq 0$ such that $f$ is locally convex (respectively locally strictly convex or locally strongly convex) in $G=\left\{x \in \mathbb{R}^{n}:|x|>R\right\}$. The analogous terminology will be used for concave functions.

Let $d=\operatorname{deg} f \geq 0$ and let $f_{0}, \ldots, f_{d} \in \mathbb{R}[x]$ be homogeneous polynomials such that $f_{i}=0$ or $\operatorname{deg} f_{i}=i$, and $f=f_{0}+\cdots+f_{d}$. Since $d=\operatorname{deg} f$, we have $f_{d} \neq 0$.

Lemma 5.2. If $f$ is convex at infinity, then $f_{d}$ is a convex function.
Proof. Assume that $f_{d}$ is not convex. Then for some nonempty $E \subset\{1, \ldots, n\}$ and $x_{0} \neq 0, \Delta_{E}^{f_{d}}\left(x_{0}\right)<0$. Since $\Delta_{E}^{f_{d}}$ is nonzero it must be a homogeneous polynomial of degree $k(d-2)$, where $k$ is the number of elements of $E$. Then

$$
\Delta_{E}^{f}\left(t x_{0}\right)=t^{k(d-2)} \Delta_{E}^{f_{d}}\left(x_{0}\right)+F(t)
$$

with some polynomial $F(t)$ of degree less than $k(d-2)$. So $\Delta_{E}^{f}\left(t x_{0}\right)<0$ as $t \rightarrow \infty$, hence $f$ is not convex at infinity, which contradicts the assumption.

To obtain the convexity of $\varphi_{N}$ we will assume that $f_{d}(x)>0$ for $x \in \mathbb{R}^{n} \backslash\{0\}$. This assumption is natural, as the following proposition shows.

Proposition 5.3. The following conditions are equivalent:
(a) $f_{d}(x)>0$ for $x \in \mathbb{R}^{n} \backslash\{0\}$,
(b) there exist $R>0$ and $N_{0} \in \mathbb{N}$ such that for any integer $N \geq N_{0}$ the polynomial

$$
\varphi_{N}(x)=\left(1+x_{1}^{2}+\cdots+x_{n}^{2}\right)^{N} f(x)
$$

is locally strongly convex on $G=\left\{x \in \mathbb{R}^{n}:|x|>R\right\}$,
(c) there exists $N_{0} \in \mathbb{N}$ such that for any integer $N \geq N_{0}$ the polynomial $\varphi_{N}$ is convex at infinity.

Proof. (a) $\Rightarrow(\mathrm{b})$. We use the notations (4.3) of the proof of Theorem 4.1 namely $\mathcal{A}=\left\{(\alpha, \beta) \in \mathbb{R}^{n} \times \mathbb{R}^{n}:|\beta|=1,\langle\alpha, \beta\rangle=0\right\}$ and $\gamma_{\alpha, \beta}(t):=\sqrt{1+|\alpha|^{2}} \beta t+\alpha$. We shall use a convenient renormalization of $f \circ \gamma_{\alpha, \beta}$. For $(\alpha, \beta) \in \mathcal{A}$ we set

$$
\begin{equation*}
g_{\alpha, \beta}(t):=\left(\sqrt{1+|\alpha|^{2}}\right)^{-d} f \circ \gamma_{\alpha, \beta}(t) \tag{5.2}
\end{equation*}
$$

The next crucial lemma gives an estimate on the size of the coefficients of

$$
f \circ \gamma_{\alpha, \beta}(t)=\sum_{i=0}^{d} c_{i}(\alpha, \beta) t^{d-i}
$$

Lemma 5.4. There exists a constant $C>0$ such that for any $(\alpha, \beta) \in \mathcal{A}$,

$$
\begin{equation*}
\left|c_{i}(\alpha, \beta)\right| \leq C\left(\sqrt{1+|\alpha|^{2}}\right)^{d} \quad \text { for } i=0, \ldots, d \tag{5.3}
\end{equation*}
$$

Proof. It is enough to check the assertion for a monomial $a x_{1}^{k_{1}} \cdots x_{n}^{k_{n}}$ with $k_{1}+$ $\cdots+k_{n} \leq d$.

Write $g_{\alpha, \beta}(t)=\left(\sqrt{1+|\alpha|^{2}}\right)^{-d} f \circ \gamma_{\alpha, \beta}(t)=\sum_{i=0}^{d} a_{i}(\alpha, \beta) t^{d-i}$. Lemma 5.4 yields a uniform estimate for the coefficients:

$$
\left|a_{i}(\alpha, \beta)\right| \leq C, \quad i=0, \ldots, d
$$

By the assumption that $f_{d}(x)>0$ for $x \neq 0$ it follows that

$$
a_{0}(\alpha, \beta)=f_{d}(\beta) \geq \inf _{|x|=1} f_{d}(x)=e>0
$$

so for $K=1+2 C / e$ we have $K \geq 1+2 \sup _{(\alpha, \beta) \in A} \max _{i=1, \ldots, d}\left|\frac{a_{i}(\alpha, \beta)}{a_{0}(\alpha, \beta)}\right|^{1 / i}$. Take $R \geq K$ and let

$$
D \geq \max \left\{1, C \sum_{i=0}^{d-1}(d-i) R^{d-i-1}, C \sum_{i=0}^{d-2}(d-i)(d-i-1) R^{d-i-2}\right\}
$$

Then $g_{\alpha, \beta}^{\prime}(t) \leq D$ and $g_{\alpha, \beta}^{\prime \prime}(t) \leq D$ for $t \in[-R, R]$. Again by the assumption that $f_{d}(x)>0$ for $x \neq 0$, one can assume that there exists $m>0$ such that for $|x| \geq R$ we have $f(x) \geq m\left(\sqrt{1+|x|^{2}}\right)^{d}$. So

$$
g_{\alpha, \beta}(t) \geq\left(\sqrt{1+|\alpha|^{2}}\right)^{-d} m\left(\sqrt{1+\left|\gamma_{\alpha, \beta}(t)\right|^{2}}\right)^{d} \geq m \quad \text { for }\left|\gamma_{\alpha, \beta}(t)\right| \geq R
$$

To end the proof of the implication $(\mathrm{a}) \Rightarrow(\mathrm{b})$ it is enough to apply Lemma 3.3
The implication $(\mathrm{b}) \Rightarrow(\mathrm{c})$ is trivial.
$(\mathrm{c}) \Rightarrow(\mathrm{a})$. Observe that

$$
\begin{equation*}
f_{d}(x) \geq 0 \quad \text { for } x \in \mathbb{R}^{n} \tag{5.4}
\end{equation*}
$$

Indeed, suppose there exists $x_{0} \in \mathbb{R}^{n} \backslash\{0\}$ such that $f_{d}\left(x_{0}\right)<0$. Let $t \in \mathbb{R}, t>0$, be such that $t x_{0} \in G$. Since $f_{d}$ is the leading form of $f$, we may assume that $f\left(t x_{0}\right)<0$. Let $H \subset G$ be a compact convex neighbourhood of $t x_{0}$ such that $f(x)<0$ for $x \in H$. By Theorem 4.1 there exists $N_{0} \in \mathbb{N}$ such that for $N \geq N_{0}$ the polynomial $\varphi_{N}$ is strictly concave on $H$. This contradicts (c) and gives (5.4).

Assume to the contrary that (a) fails. Then by (5.4), $f_{d}^{-1}(0) \neq\{0\}$. The leading form of $\varphi_{N}$ is equal to $\psi_{N}(x)=\left(x_{1}^{2}+\cdots+x_{n}^{2}\right)^{N} f_{d}(x)$ and by Lemma5.2 this form is convex. So $\psi_{N}^{-1}((-\infty, 0])$ is a convex set, and by (5.4), so is $f_{d}^{-1}(0)=f_{d}^{-1}((-\infty, 0])$. Consequently, the level set $f_{d}^{-1}(0)$ is a linear subspace, say of dimension $k>0$ (since $f_{d}$ is a homogeneous polynomial). By choosing a suitable coordinate system, we may assume that $\psi_{N}^{-1}(0)=f_{d}^{-1}(0)=\mathbb{R}^{k} \times\{0\}$. Since $f_{d} \neq 0$, we have $k<n$. As $\left.\psi_{N}\right|_{\mathbb{R}^{k+1} \times\{0\}}$ for $k+1<n$ is also a convex function, we may assume that $n=k+1$, and moreover that $n=2$ and $k=1$. Then

$$
f_{d}\left(x_{1}, x_{2}\right)=x_{2}^{s} \tilde{f}\left(x_{1}, x_{2}\right)
$$

for some $s \in \mathbb{N}^{*}$ and a homogeneous polynomial $\tilde{f}$ such that $\tilde{f}\left(x_{1}, x_{2}\right)>0$ for $\left(x_{1}, x_{2}\right) \in \mathbb{R}^{2} \backslash\{0\}$, and $\psi_{N}\left(x_{1}, x_{2}\right)=x_{2}^{s}\left(x_{1}^{2}+x_{2}^{2}\right)^{N} \tilde{f}\left(x_{1}, x_{2}\right)$. Observe that for $\psi_{N}\left(x_{1}, x_{2}\right)=1$ we have $x_{2} \rightarrow 0$ as $x_{1} \rightarrow \infty$ or $x_{1} \rightarrow-\infty$. Consequently, the set $\psi_{N}^{-1}((-\infty, 1])$ is not convex, which contradicts the convexity of $\psi_{N}$. This gives (a) and ends the proof of $(\mathrm{c}) \Rightarrow(\mathrm{a})$. The proof of Proposition 5.3 is complete.

Theorem 5.5. Let $X \subset \mathbb{R}^{n}$ be a convex closed set. Assume that $f$ is positive on $X$,

$$
\begin{equation*}
f_{d}^{-1}(0)=\{0\} \tag{5.5}
\end{equation*}
$$

and there exists $m \in \mathbb{R}$ such that

$$
\begin{equation*}
0<m \leq \inf \{f(x): x \in X\} \tag{5.6}
\end{equation*}
$$

Then there exists $N_{0} \in \mathbb{N}$ such that for any integer $N \geq N_{0}$ the polynomial $\varphi_{N}(x)=$ $\left(1+x_{1}^{2}+\cdots+x_{n}^{2}\right)^{N} f(x)$ is strongly convex on $X$.

Proof. If $f_{d}(x)<0$ for some $x \neq 0$, then $X$ is a compact set and the assertion follows from Theorem 4.1

Assume that $f_{d}(x)>0$ for any $x \neq 0$. If $X$ is a bounded set, then the assertion immediately follows from Theorem 4.1. So assume that $X$ is unbounded. Since
$f_{d}(x)>0$ for $x \neq 0$, by Proposition 5.3 there are $R \geq 0$ and $N_{1} \in \mathbb{N}$ such that for $N \geq N_{1}$ the polynomial $\varphi_{N}$ is strongly (locally) convex in $\left\{x \in \mathbb{R}^{n}:|x| \geq R\right\}$. By Theorem 4.1 one can assume that for $N \geq N_{1}$, the polynomial $\varphi_{N}$ is strongly convex on $\{x \in X:|x| \leq R+1\}$. Summing up, for $N \geq N_{1}$ the polynomial $\varphi_{N}$ is strongly convex on $X$.

Remark 5.6. If $X=\mathbb{R}^{n}$ then, for any $N$ large enough, $\varphi_{N}(x)$ is not only strictly convex, but it is a sum of squares of polynomials. More precisely, if $f$ satisfies the assumptions of Theorem 5.5 with $X=\mathbb{R}^{n}$, then its homogenization, denoted by $p$, satisfies the assumption of Reznick's theorem [24, Theorem 3.12]. So after dehomogenization of $\left(x_{0}^{2}+x_{1}^{2}+\cdots+x_{n}^{2}\right)^{N} p(x)$ we see that our function $\varphi_{N}$ is a sum of even powers of affine functions. Hence $\varphi_{N}$ is convex and it is a sum of squares of polynomials. However, this method cannot be applied if $X$ is a proper subset of $\mathbb{R}^{n}$.

Corollary 5.7. Let $X \subset \mathbb{R}^{n}$ be a closed convex semialgebraic set containing at least two points, let $f \in \mathbb{R}[x]$, and let $d>\operatorname{deg} f$ be an even integer. Then the following conditions are equivalent:
(1) $f$ is nonnegative on $X$,
(2) for any $a, b>0$ there exists $N_{0} \in \mathbb{N}$ such that for any integer $N \geq N_{0}$ the polynomial $\varphi_{N}(x)=\left(1+|x|^{2}\right)^{N}\left(f(x)+a|x|^{d}+b\right)$ is a strongly convex function on $X$.

Proof. The polynomial $f(x)+a|x|^{d}+b$ satisfies the assumptions of Theorem 5.5 if $a, b>0$. Hence the implication $(1) \Rightarrow(2)$ follows from Theorem 5.5 To prove the converse assume that $f\left(x_{0}\right)<0$ for some $x_{0} \in \operatorname{Int} X$. Note that $X$, being convex and containing at least two points, has nonempty (relative) interior. Then for sufficiently small $a, b$ and $N$ large enough, the function $-\varphi_{N}$ is strictly convex in a neighbourhood of $x_{0}$. So $\varphi_{N}$ is strongly concave in a neighbourhood of $x_{0}$, which is absurd.

For homogeneous polynomials on $\mathbb{R}^{n}$ we obtain the following extension of Reznick's result mentioned in the Introduction. For a fixed $f \in \mathbb{R}[x]$ and a positive integer $N$, we set $\psi_{N}(x):=\left(x_{1}^{2}+\cdots+x_{n}^{2}\right)^{N} f(x)$.

Corollary 5.8. Let $f \in \mathbb{R}[x]$ be a nonzero homogeneous polynomial. The following conditions are equivalent:
(a) $f(x)>0$ for $x \in \mathbb{R}^{n} \backslash\{0\}$,
(b) there exists $N_{1} \in \mathbb{N}$ such that for any $N \geq N_{1}$ the polynomial $\psi_{N}$ is a sum of even powers of linear functions,
(c) there exists $N_{2} \in \mathbb{N}$ such that for any $N \geq N_{2}$ the polynomial $\psi_{N}$ is a convex function,
(d) there exists $N_{3} \in \mathbb{N}$ such that for any $N \geq N_{3}$ the polynomial $\psi_{N}$ is a strictly convex function.

Proof. The implication $(\mathrm{a}) \Rightarrow(\mathrm{b})$ is Reznick's result (see [24, Theorem 3.12]). The implications $(\mathrm{b}) \Rightarrow(\mathrm{c})$ and $(\mathrm{d}) \Rightarrow(\mathrm{c})$ are trivial. The implication $(\mathrm{c}) \Rightarrow(\mathrm{a})$ follows by the same argument as $(\mathrm{c}) \Rightarrow(\mathrm{a})$ in Proposition 5.3.

To complete the proof it suffices to prove $(\mathrm{a}) \Rightarrow(\mathrm{d})$. We will investigate the convexity of $\psi_{N}$ on each line $l$ in $\mathbb{R}^{n}$. If $0 \in l$ then clearly $\left.\psi_{N}\right|_{l}$ is convex, so we
will check the convexity of $\psi_{N}$ on lines $l \subset \mathbb{R}^{n} \backslash\{0\}$. Since $f$ is homogeneous, it suffices to consider the convexity of $\psi_{N}$ on lines of the form

$$
l=\{a+b t: t \in \mathbb{R}\}, \quad(a, b) \in A
$$

where $A:=\left\{(a, b) \in \mathbb{R}^{n} \times \mathbb{R}^{n}:|a|=|b|=1,\langle a, b\rangle=0\right\}$. Clearly $A$ is compact. Denote $g(t, a, b)=f(a+b t)$ for $t \in \mathbb{R},(a, b) \in A$. Then

$$
g(t, a, b)=g_{0}(a, b) t^{d}+g_{1}(a, b) t^{d-1}+\cdots+g_{d}(a, b)
$$

and $g_{0}(a, b)=f(b)$. So by (a) there exists $m>0$ such that $g_{0}(a, b)>m$ for $(a, b) \in A$. Moreover, $f(x) \geq m$ for $x \in \mathbb{R}^{n},|x| \geq 1$, hence

$$
g(t, a, b) \geq m \quad \text { for } t \in \mathbb{R} \text { and }(a, b) \in A
$$

Take $R, D \in \mathbb{R}$ such that

$$
R \geq 1+2 \max _{(a, b) \in A} \max _{1 \leq i \leq d}\left|\frac{g_{i}(a, b)}{g_{0}(a, b)}\right|^{1 / i}
$$

and

$$
\begin{aligned}
& D \geq \max _{(a, b) \in A} \max \left\{1, \sum_{i=0}^{d-1}(d-i)\left|g_{i}(a, b)\right| R^{d-i-1},\right. \\
&\left.\sum_{i=0}^{d-2}(d-i)(d-i-1)\left|g_{i}(a, b)\right| R^{d-i-2}\right\} .
\end{aligned}
$$

Since for $(a, b) \in A$,

$$
\psi_{N}(a+b t)=\left(1+t^{2}\right)^{N} g(t, a, b)
$$

Lemma 3.3 implies that $\psi_{N}$ is a strictly convex function provided $N \geq \mathcal{N}(m, R, D)$. This gives the implication $(\mathrm{a}) \Rightarrow(\mathrm{d})$ and completes the proof.
Remark 5.9. Let $N_{1}, N_{2}, N_{3}$ be the minimal values in Corollary 5.8. Obviously $N_{2} \leq N_{1}$ and $N_{2} \leq N_{3}$. It is not clear to the authors whether the equalities $N_{1}=N_{2}=N_{3}$ hold.

By a result of Blekherman [5], 6] there exist strictly convex positive forms that are not sums of squares. However, this does not answer our question, because we are interested in the smallest numbers $N_{i}$ such that for every $N \geq N_{i}$ the polynomials $\psi_{N}$ are respectively: sum of even powers of linear functions, convex and strictly convex. Note that multiplying a convex form by $\left(x_{1}^{2}+\cdots+x_{n}^{2}\right)^{N}$ may produce a nonconvex form.

For instance the polynomial $f(x, y)=(x-k y)^{2}+y^{2}$ is a strictly convex sum of squares of linear forms. However for sufficiently large $k$ we can find $N \geq 1$ such that the polynomial $\psi_{N}$ is not convex (cf. Example 3.5) and consequently not a sum of even powers of linear functions.

## 6. A Proximity algorithm for a polynomial on a convex set

Let $X \subset \mathbb{R}^{n}$ be a compact convex semialgebraic set. We consider a polynomial $f$ restricted to $X$. We propose an algorithm, based on our convexification method, which produces a sequence converging to a critical point of $f$ on $X$.

Using a translation and a dilatation we may assume that $X$ is contained in a ball of radius $1 / 2$. Replacing $f$ by $f+c$, where $c$ is a constant large enough we may assume that $m=\inf \{f(x): x \in X\}=D>0$, where $D$ is a bound for the
absolute value of the first and the second directional derivatives of $f$ (along vectors of norm 1). Indeed, we may increase $D$ in such a way that $|f(x)| \leq D$ for $x \in X$. Then we put $c=2 D$, hence $f(x)+c \geq D$ for $x \in X$. Since now $m=D$ and $2 R=1$, by (3.2) we have $\mathcal{N}(m, 2 R, D)=6$.

By Remark 4.2, with $N=6$ and some $\mu>0$ the function

$$
\varphi_{N, \xi}(x):=\left(1+|x-\xi|^{2}\right)^{N} f(x)
$$

is $\mu$-strongly convex on $X$ for any $\xi \in X$. This means that

$$
\begin{equation*}
\varphi_{N, \xi}(y) \geq \varphi_{N, \xi}(x)+\left\langle y-x, \nabla \varphi_{N, \xi}(x)\right\rangle+\frac{\mu}{2}|y-x|^{2} \quad \text { for } x, y \in X \tag{6.1}
\end{equation*}
$$

Recall that any strictly convex, hence in particular any strongly convex, function $\varphi$ on a convex closed set $X$ admits a unique point, denoted by $\operatorname{argmin}_{X} \varphi$, at which $\varphi$ attains its minimum on $X$.

Choose an arbitrary point $a_{0} \in X$, and by induction set

$$
\begin{equation*}
a_{\nu}:=\operatorname{argmin}_{X} \varphi_{N, a_{\nu-1}} . \tag{6.2}
\end{equation*}
$$

Lemma 6.1. For any $\nu \in \mathbb{N}$ we have

$$
\left|a_{\nu+1}-a_{\nu}\right|=\operatorname{dist}\left(a_{\nu}, f^{-1}\left(f\left(a_{\nu+1}\right)\right) \cap X\right)
$$

Proof. If $\left|a^{\prime}-a_{\nu}\right|<\left|a_{\nu+1}-a_{\nu}\right|$ for some $a^{\prime} \in f^{-1}\left(f\left(a_{\nu+1}\right)\right) \cap X$, then by the definition of $\varphi_{N, a_{\nu}}$ we have $\varphi_{N, a_{\nu}}\left(a^{\prime}\right)<\varphi_{N, a_{\nu}}\left(a_{\nu+1}\right)$, which contradicts the definition of $a_{\nu+1}$. So, $\left|a_{\nu+1}-a_{\nu}\right| \leq \operatorname{dist}\left(a_{\nu}, f^{-1}\left(f\left(a_{\nu+1}\right)\right) \cap X\right)$. The opposite inequality is obvious.

Lemma 6.2. For any $\nu \in \mathbb{N}$ we have

$$
f\left(a_{\nu+1}\right) \leq \frac{f\left(a_{\nu}\right)-\frac{\mu}{2}\left|a_{\nu+1}-a_{\nu}\right|^{2}}{\left(1+\left|a_{\nu+1}-a_{\nu}\right|^{2}\right)^{N}} .
$$

In particular the sequence $f\left(a_{\nu}\right)$ is decreasing.
Proof. Since $\varphi_{N, a_{\nu}}$ is strongly convex, the definition of $a_{\nu+1}$ implies that the function

$$
[0,1] \ni t \mapsto \varphi_{N, a_{\nu}}\left(a_{\nu}+t\left(a_{\nu+1}-a_{\nu}\right)\right)
$$

decreases, so $\left\langle a_{\nu+1}-a_{\nu}, \nabla \varphi_{N, a_{\nu}}\left(a_{\nu+1}\right)\right\rangle \leq 0$. Thus, by (6.1) we see that

$$
\varphi_{N, a_{\nu}}\left(a_{\nu}\right) \geq \varphi_{N, a_{\nu}}\left(a_{\nu+1}\right)+\frac{\mu}{2}\left|a_{\nu}-a_{\nu+1}\right|^{2}
$$

Again, by the definition of $\varphi_{N, a_{\nu}}$ we have

$$
f\left(a_{\nu}\right) \geq\left(1+\left|a_{\nu+1}-a_{\nu}\right|^{2}\right)^{N} f\left(a_{\nu+1}\right)+\frac{\mu}{2}\left|a_{\nu+1}-a_{\nu}\right|^{2} .
$$

This ends the proof of the lemma.
Now we estimate from below the length of $\left|a_{\nu}-a_{\nu+1}\right|$, i.e., of the step in our sequence. It is enough to consider only the one-dimensional case with $a_{\nu}=0$. By a direct computation we obtain:

Lemma 6.3. Let $f:[0, \eta] \rightarrow \mathbb{R}$ be a $C^{1}$ function such that $0<f \leq C$ and $f^{\prime} \leq-\eta$ on $[0, \eta]$ for some $C \geq \frac{1}{2}$ and $\eta>0$. Assume that $\varphi_{N}(x)=\left(1+x^{2}\right)^{N} f(x)$ is strictly convex on $[0, \eta]$. Then $b_{1}:=\operatorname{argmin}_{[0, \eta]} \varphi_{N} \geq \frac{\eta}{2 N C}$. Hence $f(0)-f\left(b_{1}\right) \geq \frac{\eta^{2}}{2 N C}$.

Let $f$ be a $C^{1}$ function in a neighborhood $U$ of a closed set $X \subset \mathbb{R}^{n}$. Recall that $a \in X$ is a lower critical point of $f$ on $X$ if

$$
\begin{equation*}
\langle\nabla f(a), x-a\rangle \geq 0 \quad \text { for } x \in X \text { in a neighbourhood of } a \tag{6.3}
\end{equation*}
$$

We denote by $\Sigma_{X} f$ the set of lower critical points of $f$ on $X$, and by $\Sigma f:=\{x \in$ $U: \nabla f(x)=0\}$ the set of ordinary critical points of $f$. The following proposition recalls all the necessary properties of these sets.

Proposition 6.4. Assume that $X \subset \mathbb{R}^{n}$ is closed and $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ is a $C^{1}$ function. Then:
(1) $X \cap \Sigma f \subset \Sigma_{X} f$;
(2) if $f$ restricted to $X$ has a local minimum at a, then $a \in \Sigma_{X} f$;
(3) if $M \subset X$ is a smooth manifold and $a \in M \cap \Sigma_{X} f$, then for any $z \in T_{a} M$,

$$
\langle\nabla f(a), z\rangle=0
$$

(4) if $f$ is a polynomial and $X$ is semialgebraic, then $\Sigma_{X} f$ is a semialgebraic set and $f\left(\Sigma_{X} f\right)$ is a finite set.

Proof. The first three statements follow immediately from the definition. If $f$ is a polynomial and $X$ is semialgebraic then the set $\Sigma_{X} f$ is described by a first order formula (in the language of ordered fields) so it is semialgebraic as well (see e.g. [7, Chapter 2]). Hence $\Sigma_{X} f$ has finitely many connected components (in fact connected by piecewise $C^{1}$ semialgebraic arcs). Each such component is a finite union of smooth manifolds, hence by condition (3) the function $f$ is constant on it. So $f\left(\Sigma_{X} f\right)$ is a finite set.

Theorem 6.5. Let $X \subset \mathbb{R}^{n}$ be a compact convex semialgebraic set and $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ a positive polynomial on $X$. Let $a_{\nu}$ be the sequence defined by (6.2) with $a_{0} \in X$. Then the limit

$$
a^{*}=\lim _{\nu \rightarrow \infty} a_{\nu}
$$

exists, and $a^{*} \in \Sigma_{X} f$.
Remark 6.6. Note that Lemmas 6.1 and 6.2 hold true for any function of class $C^{2}$ in a neighborhood of $X$. However, they are not sufficient to prove the convergence of the sequence $a_{\nu}$ (at least we have not been able to do this). For $f$ polynomial the convergence of the sequence $a_{\nu}$ will follow from some fine properties of the gradient trajectories of polynomials.

Proof of Theorem 6.5. First, assuming that $a^{*}=\lim _{\nu \rightarrow \infty} a_{\nu}$ exists, we shall prove that $a^{*} \in \Sigma_{X} f$. Suppose that $a^{*} \notin \Sigma_{X} f$, so there exists $x \in X$ with $\left\langle\nabla f\left(a^{*}\right), x-\right.$ $\left.a^{*}\right\rangle<0$. Then there exists $\eta>0$ such that $\left\langle\nabla f\left(a^{*}+t\left(x-a^{*}\right)\right), x-a^{*}\right\rangle<-\eta$ for $t \in[0, \eta]$. By continuity the same holds with $a^{*}$ replaced by $a_{\nu}$ for $a_{\nu}$ sufficiently close to $a^{*}$. Moreover, we may assume that $\left|f\left(a_{\nu}\right)-f\left(a^{*}\right)\right|<\frac{\eta^{2}}{2 N C}$, where $C \geq f(x)$ for $x \in X$. Hence by continuity and Lemma 6.3 we obtain $f\left(a_{\nu+1}\right)<f\left(a^{*}\right)$, which is a contradiction.

Recall now the Comparison Principle [1 Lemma 4.2]. Let $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be a polynomial and let $M \subset \mathbb{R}^{n}$ be a smooth bounded semialgebraic set. Let $\nabla f(x)$ denote the gradient of $f$ with respect to the standard Euclidean scalar product, and $\nabla_{M} f(x)$ its projection on $T_{x} M$, the tangent space to $M$ at $x$.

Let $\Gamma_{M} \subset \bar{M}$ be a semialgebraic curve meeting each level set of $f$ and such that for every point $y \in \Gamma_{M}$ we have $\left|\nabla_{M} f(y)\right| \leq\left|\nabla_{M} f(x)\right|$ for all $x \in f^{-1}(f(y)) \cap \bar{M}$.

By standard arguments (semialgebraic choice) such a curve always exists; it is called a talweg or a ridge-valley line of $f$ in $X$. Then the following lemma holds.
Lemma 6.7 (Comparison Principle). For every pair of values $a<b$ taken by $f$, the length of any trajectory of $\nabla_{M} f$ lying in $f^{-1}((a, b)) \cap M$ is bounded by the length of $\Gamma_{M} \cap f^{-1}((a, b))$.

To prove that $\lim _{\nu \rightarrow \infty} a_{\nu}$ exists recall first that by Lemma 6.2 we have

$$
f\left(a_{\nu}\right) \geq f\left(a_{\nu+1}\right) \geq \cdots \geq f_{*}:=\lim _{\nu \rightarrow \infty} f\left(a_{\nu}\right)
$$

By Proposition 6.4(4) the set $f\left(\Sigma_{X} f\right)$ of critical values of $f$ on $X$ is finite, so we may assume that either the sequence $f\left(a_{\nu}\right)$ is eventually constant, or $\left(f\left(a_{\nu}\right), f_{*}\right) \cap$ $f\left(\Sigma_{X} f\right)=\emptyset$ for $\nu$ large enough. Clearly in the first case, by Lemma 6.2, also the sequence $a_{\nu}$ is eventually constant. So we assume from now on that the sequence $f\left(a_{\nu}\right)$ is strictly decreasing and $\left(f\left(a_{0}\right), f_{*}\right) \cap f\left(\Sigma_{X} f\right)=\emptyset$.

The set $X$ is semialgebraic, so there exists a stratification $X=\bigcup_{i \in I} M_{i}$, i.e., a finite disjoint union of connected smooth semialgebraic sets, called strata. Moreover $\bar{M}_{i} \backslash M_{i}$ is a union of some of the $M_{j}$ 's of dimension smaller than $\operatorname{dim} M_{i}$ (cf. [3, Chapter 2]). We can refine this stratification in such a way that $f$ is of constant rank on each $M_{i}, i \in I$; then our polynomial $f$ restricted to $M_{i}$ is either a constant or a submersion. Let $I^{*}=\left\{i \in I:\left.\operatorname{rank} f\right|_{M_{i}}=1\right\} ;$ note that $C_{X} f=\bigcup_{i \in I \backslash I^{*}} f\left(M_{i}\right)$ is a finite set. Since the sequence $f\left(a_{\nu}\right)$ is strictly decreasing we may assume that $\left(f\left(a_{0}\right), f_{*}\right) \cap C_{X} f=\emptyset$.

To each $M_{i}, i \in I^{*}$, we can associate a semialgebraic curve $\Gamma_{i}:=\Gamma_{M_{i}}$ which is a talweg of $f$ in $M_{i}$. Set $\Gamma:=\bigcup_{i \in I^{*}} \Gamma_{i}$.

Recall that, by Lemma 6.1], $a_{\nu+1}$ is the point closest to $a_{\nu}$ on the fiber $f^{-1}\left(f\left(a_{\nu+1}\right)\right)$ $\cap X$. To estimate $\left|a_{\nu+1}-a_{\nu}\right|$ we will construct a continuous curve $\gamma_{\nu}:\left[t_{\nu}, t_{\nu+1}\right] \rightarrow X$ such that $\gamma_{\nu}\left(t_{\nu}\right)=a_{\nu}$ and $f\left(\gamma_{\nu}\left(t_{\nu+1}\right)\right)=f\left(a_{\nu+1}\right)$. By Lemma 6.1 we will then have $\left|a_{\nu+1}-a_{\nu}\right| \leq$ length $\left(\gamma_{\nu}\right)$. The curve $\gamma_{\nu}$ will be a piecewise trajectory of $-\nabla_{M_{i}} f$ (more precisely, of $-\nabla_{M_{i}} f /\left|\nabla_{M_{i}} f\right|$ ). Hence, by the Comparison Principle,

$$
\left|a_{\nu+1}-a_{\nu}\right| \leq \operatorname{length}\left(\gamma_{\nu}\right) \leq \operatorname{length}\left(\Gamma \cap f^{-1}\left(\left(f\left(a_{\nu+1}\right), f\left(a_{\nu}\right)\right)\right)\right)
$$

Recall that $\Gamma$, being a bounded semialgebraic curve, has finite length (see e.g. [32, Corollary 5.2]); therefore

$$
\sum_{\nu=0}^{\infty}\left|a_{\nu+1}-a_{\nu}\right| \leq \operatorname{length}\left(\Gamma \cap f^{-1}\left(\left(f_{*}, f\left(a_{0}\right)\right)\right)\right)<\infty
$$

So the series $\sum_{\nu=0}^{\infty}\left|a_{\nu+1}-a_{\nu}\right|$ is convergent, which implies that $a^{*}=\lim _{\nu \rightarrow \infty} a_{\nu}$ exists.

Construction of the curve $\gamma_{\nu}$. Assume that $a_{\nu}$ belongs to a stratum $M_{i}$ for some $i \in I^{*}$. Let $\gamma_{\nu}:\left[t_{\nu}, t_{\nu}^{1}\right) \rightarrow M_{i}$ be a trajectory of $V_{i}:=-\nabla_{M_{i}} f /\left|\nabla_{M_{i}} f\right|$. By a trajectory we mean a maximal solution (to the right) of $\gamma^{\prime}=V_{i}$ in $M_{i}$. Note that

$$
b_{1}^{*}=\lim _{s \nearrow t_{\nu}^{1}} \gamma_{\nu}(s) \in \bar{M}_{i} \backslash M_{i}
$$

exists. Indeed, by Lemma 6.7 any maximal trajectory of $V_{i}$ has finite length so it has a limit in $\bar{M}_{i}$. But the vector field $V_{i}$ does not vanish on $M_{i}$, hence this limit belongs to $\bar{M}_{i} \backslash M_{i}$, which is a union of strata of smaller dimension.

If $f\left(b_{1}^{*}\right) \leq f\left(a_{\nu+1}\right)$ then there exists $t_{\nu+1} \in\left[t_{\nu}, t_{\nu}^{1}\right]$ such that $f\left(\gamma_{\nu}\left(t_{\nu+1}\right)\right)=$ $f\left(a_{\nu+1}\right)$, so $\gamma_{\nu}$ restricted to $\left[t_{\nu}, t_{\nu+1}\right]$ is the curve we are looking for. Now if $f\left(b_{1}^{*}\right)>$
$f\left(a_{\nu+1}\right)$, then $b_{1}^{*} \in M_{i_{1}}$ for some $i_{1} \in I^{*}$ such that $\operatorname{dim} M_{i_{1}}<\operatorname{dim} M_{i}$. We repeat the above construction on $M_{i_{1}}$ starting from the point $b_{1}^{*}$, then we glue it with the previous one. In this way the dimension of the stratum in which our curve $\gamma_{\nu}$ stays is strictly decreasing, but this dimension is always at least 1 . Finally we will reach the level $f^{-1}\left(a_{\nu+1}\right)$. Indeed, when our curve arrives at a point in a stratum of dimension 1 we follow this stratum until we arrive at the level $f^{-1}\left(a_{\nu+1}\right)$ since $\left(f\left(a_{0}\right), f_{*}\right) \cap C_{X} f=\emptyset$. The estimate

$$
\operatorname{length}\left(\gamma_{\nu}\right) \leq \operatorname{length}\left(\Gamma \cap f^{-1}\left(f\left(a_{\nu+1}\right), f\left(a_{\nu}\right)\right)\right)
$$

follows from Comparison Principle.
Remark 6.8. In the case when $X$ is a closed ball (or more generally when $X$ has a smooth boundary) the length of the curve $\Gamma$ can be effectively estimated (see [1).

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## References

[1] D. D'Acunto, K. Kurdyka, Bounds for gradient trajectories and geodesic diameter of real algebraic sets. Bull. London Math. Soc. 38 (2006), no. 6, 951-965.
[2] E. Artin, Über die Zerlegung definiter Functionen in Quadrate. Abh. Math. Sem. Univ. Hamburg 5 (1927), 100-115; Collected Papers, 273-288, Addison-Wesley, Reading, MA, 1965.
[3] R. Benedetti, J-J. Risler, Real algebraic and semialgebraic sets, Actualités Mathématiques, Hermann, Paris (1990).
[4] C. Berg, J. P. R. Christensen, P. Ressel, Positive definite functions on abelian semigroups. Math. Ann. 223 (1976), no. 3, 253-274.
[5] G. Blekherman, Convex forms that are not sums of squares. arXiv:0910.0656 (2009).
[6] G. Blekherman, Nonnegative polynomials and sums of squares. Chapter in: Semidefinite optimization and convex algebraic geometry. Edited by Grigoriy Blekherman, Pablo A. Parrilo and Rekha R. Thomas. MOS-SIAM Series on Optimization, 13. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA; Mathematical Optimization Society, Philadelphia, PA, 2013. xx +476 pp. 159-202.
[7] J. Bochnak, M. Coste, M-F. Roy, Real algebraic geometry. Springer-Verlag, Berlin, 1998.
[8] J. Cimprič, M. Marshall, T. Netzer, On the real multidimensional rational K-moment problem. Trans. Amer. Math. Soc. 363 (2011), no. 11, 5773-5788.
[9] D. W. Dubois, A Nullstellensatz for ordered fields. Ark. Mat. 8 (1969), 11-114.
[10] I. Fau, Some properties of convex functions. The convexity exponent of positive polynomials. University of Łódź M.Sc. thesis, 2013 (in Polish).
[11] D. Hilbert, Über die Darstellung definiter Formen als Summe von Formenquadraten. Math. Ann. 32 (1888), 342-350.
[12] J.-L. Krivine, Anneaux préordonnés. J. Analyse Math. 12 (1964), 307-326.
[13] K. Kurdyka, S. Spodzieja, Separation of real algebraic sets and the Łojasiewicz exponent. Proc. Amer. Math. Soc. 142 (2014), no. 9, 3089-3102.
[14] K. Kurdyka, S. Spodzieja, A. Szlachcińska, Metric properties of semialgebraic sets, arXiv:1412.5088 [math.AG], (2014).
[15] J. B. Lasserre, Global optimization with polynomials and the problem of moments. SIAM J. Optim. 11 (2001), no. 3, 796-817.
[16] J. B. Lasserre, A sum of squares approximation of nonnegative polynomials. SIAM J. Optim. 16 (2006), no. 3, 751-765.
[17] J. B. Lasserre, Representation of nonnegative convex polynomials. Arch. Math. 91 (2008), no. 2,126-130.
[18] J. B. Lasserre, Sum of squares approximation of polynomials, nonnegative on a real algebraic set. SIAM J. Optim. 16 (2005), no. 2, 610-628.
[19] J. B. Lasserre, T. Netzer, SOS approximations of nonnegative polynomials via simple high degree perturbations. Math. Z. 256 (2007), no. 1, 99 -112.
[20] M. Marshall, Positive polynomials and sums of squares. Mathematical Surveys and Monographs, 146. American Mathematical Society, Providence, RI, 2008.
[21] T. S. Motzkin, The arithmetic-geometric inequality. In: Inequalities (Ed. O. Shisha), Academic Press, 1967, 205-224.
[22] A. Prestel, Ch. N. Delzell, Positive polynomials. From Hilbert's $1^{\text {Th }}$ th problem to real algebra. Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2001.
[23] M. Putinar, Positive polynomials on compact semi-algebraic sets. Indiana Univ. Math. J. 42 (1993), no. 3, 969-984.
[24] B. Reznick, Uniform denominators in Hilbert's seventeenth problem. Math. Z. 220 (1995), no. 1, 75-97.
[25] B. Reznick, Blenders. Notions of positivity and the geometry of polynomials, 345-373, Trends Math., Birkhäuser/Springer Basel AG, Basel, 20111.
[26] J.-J. Risler, Une caractérisation des idéaux des variétés algébriques réelles. C. R. Acad. Sci. Paris Sér. A 271 (1970), 1171-1173.
[27] C. Scheiderer, Non-existence of degree bounds for weighted sums of squares representations. J. Complexity 21 (2005), no. 6, 823-844.
[28] C. Scheiderer, Positivity and sums of squares: A guide to recent results. In: Emerging applications of algebraic geometry, 271-324, IMA Vol. Math. Appl., 149, Springer, New York, 2009.
[29] K. Schmüdgen, The K-moment problem for compact semialgebraic sets. Math. Ann. 289 (1991), no. 2, 203-206.
[30] M. Schweighofer, On the complexity of Schmüdgen's Positivstellensatz. J. Complexity 20 (2004), 529-543.
[31] M. Schweighofer, Optimization of polynomials on compact semialgebraic sets. SIAM J. Optim. 15 (2005), no. 3, 805-825.
[32] Y. Yomdin, G. Comte, Tame geometry with applications in smooth analysis. Springer LNM 1834 (2004).

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[^1]:    ${ }^{1}$ Recall that $\mathcal{S}\left(g_{1}, \ldots, g_{r}\right)=\left\{\sigma_{0}+\varphi\left(g_{1}\right) g_{1}+\cdots+\varphi\left(g_{r}\right) g_{r}: \sigma_{0} \in \sum \mathbb{R}[x]^{2}, \varphi \in \sum \mathbb{R}[t]^{2}\right\}$.

