Isothermal Navier-Stokes Equations and Radon Transform

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Abstract

In the paper we prove the existence results for initial-value boundary value problems for compressible isothermal Navier-Stokes equations. We restrict ourselves to 2D case of a problem with no-slip condition for nonstationary motion of viscous compressible isothermal fluid. However, the technique of modeling and analysis presented here is general and can be used for 3D problems.

Key words: Navier–Stokes equations, compressible fluids, Radon transform

AMS:35Q30, 49J20, 76N10

1 Introduction

1.1 Problem formulation

Suppose a viscous compressible fluid occupies a bounded domain $\Omega \subset \mathbb{R}^2$. The state of the fluid is characterized by the macroscopic quantities: the density $\rho(x, t)$ and the velocity $\mathbf{u}(x, t)$. The problem is to find $\mathbf{u}(x, t)$ and $\rho(x, t)$ satisfying the following equations and boundary conditions in the cylinder

 $Q_T = \Omega \times (0,T)$.

$$
\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla \varrho = \operatorname{div} \mathbb{S}(\mathbf{u}) + \varrho \mathbf{f} \quad \text{in} \quad Q_T, \tag{1a}
$$

$$
\partial_t \varrho + \text{div}(\varrho \mathbf{u}) = 0 \quad \text{in} \ \ Q_T,
$$
 (1b)

$$
\mathbf{u} = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{1c}
$$

$$
\mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \varrho(x,0) = \varrho_0(x) \quad \text{in } \Omega. \tag{1d}
$$

Here, the vector field f denotes the density of external mass forces, the viscous stress tensor $\mathbb{S}(\mathbf{u})$ has the form

$$
\mathbb{S}(\mathbf{u}) = \nu_1 (\nabla \mathbf{u} + \nabla \mathbf{u}^\top) + \nu_2 \text{div } \mathbf{u} \mathbb{I}, \tag{1e}
$$

in which the viscosity coefficients satisfy the inequalities $\nu_1 > 0$, $\nu_1 + \nu_2 \geq 0$. It is necessary to notice that problem [\(1\)](#page-1-0) is the simplest multidimensional boundary value problem for the compressible Navier-Stokes equations. In 1986 Padula, see [\[8\]](#page-38-0), formulated the result on existence of a weak solution to problem [\(1\)](#page-1-0), but the proof presented was incomplete, see [\[9\]](#page-38-1). The first nonlocal results concerning the mathematical theory of compressible Navier-Stokes equations are due to P.-L. Lions. In monograph [\[6\]](#page-38-2) he established the existence of a renormalized solution to nonstationary boundary value problem for the Navier-Stokes equations with the pressure function $p \sim \varrho^{\gamma}$ for all $\gamma > 5/3$ in 3D case and for all $\gamma > 3/2$ in 2D case. More recently, Feireisl, Novotný, and Petzeltova', see [\[4\]](#page-38-3), proved the existence result for all $\gamma > 3/2$ in 3D case and for all $\gamma > 1$ in 2D case, see also monographs [\[5\]](#page-38-4), [\[7\]](#page-38-5), and [\[10\]](#page-38-6) for references and details. The question on solvability of problem [\(1\)](#page-1-0) remained open. The main difficulty is the so called concentration problem, see [\[6\]](#page-38-2) ch.6.6. This means that the finite kinetic energy can be concentrated in very small domains. Our goal is to relax the restriction $\gamma > 1$ and to prove the existence of solutions to problem [\(1\)](#page-1-0). In order to make the presentation clearer and avoid unnecessary technical difficulties, we assume that the flow domain and the given data satisfy the hypotheses:

Condition 1.1. • The flow domain $\Omega \subset \mathbb{R}^2$ is a bounded domain with C^{∞} boundary.

• The data satisfy $\varrho_0, \mathbf{u}_0 \in L^{\infty}(\Omega)$, $\mathbf{f} \in L^{\infty}(Q_T)$, and

$$
\|\mathbf{u}_0\|_{W_0^{1,2}(\Omega)} + \|\varrho_0\|_{L^\infty(\Omega)} + \|\mathbf{f}\|_{L^\infty(Q_T)} \le c_e, \quad \varrho_0 > c > 0,\tag{2}
$$

where c_e , c are positive constants.

Remark 1.1. Further, we denote by E generic constants depending only on $\Omega, T, ||\varrho_0||_{L^{\infty}(\Omega)}, ||\mathbf{u}_0||_{L^2(\Omega)}, ||\mathbf{f}||_{L^{\infty}(Q_T)},$ and ν_i .

We claim that problem [\(1\)](#page-1-0) admits a weak solution which is defined as follows:

Definition 1.1. A couple

$$
\varrho \in L^{\infty}(0,T; L^{1}(\Omega)), \quad \mathbf{u} \in L^{2}(0,T; W_{0}^{1,2}(\Omega))
$$

is said to be a weak solution to problem [\(1\)](#page-1-0) if (ϱ, \mathbf{u}) satisfies

- The kinetic energy is bounded, i.e., $\varrho|\mathbf{u}|^2 \in L^\infty(0,T; L^1(\Omega))$. The density function is non-negative $\rho \geq 0$.
- The integral identity

$$
\int_{Q_T} \left(\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\xi} + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla \boldsymbol{\xi} + \varrho \text{div } \boldsymbol{\xi} - \mathbb{S}(\mathbf{u}) : \nabla \boldsymbol{\xi} \right) dx dt
$$
\n
$$
+ \int_{Q_T} \varrho \mathbf{f} \cdot \boldsymbol{\xi} \, dx dt + \int_{\Omega} (\varrho_0 \mathbf{u}_0)(x) \cdot \boldsymbol{\xi})(x,0) \, dx = 0 \quad (3)
$$

holds for all vector fields $\xi \in C^{\infty}(Q_T)$ vanishing in a neighborhood of $\partial\Omega \times [0,T]$ and of $\Omega \times \{t = T\}.$

• The integral identity

$$
\int_{Q_T} \left(\varrho \partial_t \psi + \varrho \mathbf{u} \cdot \nabla \psi \right) dx dt + \int_{\Omega} \varrho_0(x) \psi(x,0) dx = 0 \tag{4}
$$

holds for all $\psi \in C^{\infty}(Q_T)$ vanishing in a neighborhood of the top $\Omega \times$ $\{t = T\}.$

The following existence theorem is the main result of the paper.

Theorem 1.1. Assume that Condition 1.1 is fulfilled. Then problem [\(1\)](#page-1-0) has a weak solution which meets all requirements of Definition [1.1](#page-2-0) and satisfies the estimate

$$
\|\mathbf{u}\|_{L^{2}(0,T;W_{0}^{1,2}(\Omega))} + \|\varrho|\mathbf{u}|^{2}\|_{L^{\infty}(0,T;L^{1}(\Omega))} + \|\varrho\log(1+\varrho)\|_{L^{\infty}(0,T;L^{1}(\Omega))} \leq E,
$$
\n(5)

where the constant E is as in Remark [1.1.](#page-2-1)

The next theorem, which is the second main result of the paper, shows that a weak solution to problem [\(1\)](#page-1-0) has extra regularity properties.

Theorem 1.2. Let Condition 1.1 be satisfied. Assume that (ϱ, \mathbf{u}) meets all requirements of Theorem [1.1.](#page-2-2) Furthermore assume that \bf{u} and ρ are extended by 0 to $\mathbb{R}^2 \times (0,T)$. Then for every nonnegative function $\zeta \in C_0^{\infty}(\mathbb{R}^2)$ with spt $\zeta \in \Omega$,

$$
\text{ess}\sup_{\boldsymbol{\omega}\in\mathbb{S}^1}\int\limits_{0}^{T}\int\limits_{-\infty}^{\infty}\Phi(\boldsymbol{\omega},\tau,t)^2\,d\tau dt\leq c(\zeta)E,\tag{6}
$$

where Φ is the Radon transform of $\zeta(x)\rho(x,t)$,

$$
\Phi(\omega, \tau, t) = \int_{\omega \cdot x = \tau} \zeta(x) \varrho(x, t) \, dl. \tag{7}
$$

Moreover, the function $\zeta_{\mathcal{Q}}$ admits the estimates

$$
\|\zeta \varrho\|_{L^2(0,T;H^{-1/2}(\mathbb{R}^2))} \le c(\zeta)E,
$$

$$
\|\zeta \varrho\|_{L^{1+\lambda}(Q_T)} \le c(\zeta,\lambda)E \quad \text{for all} \quad \lambda \in [0,1/6).
$$
 (8)

Here $c(\zeta)$ depends only on ζ and $c(\zeta, \lambda)$ depends only on ζ , λ .

The remaining part of the paper is devoted to the proof of these theorems. In sections [2](#page-3-0) and [3](#page-5-0) we collect basic facts on Sobolev spaces, the Radon transform, and the isentropic Navier-Stokes equations. Section [4](#page-7-0) is the heart of the work. Here we derive the L^2 -estimates for the Radon transform of the density function ρ . In sections [5](#page-16-0) and [6](#page-21-0) we prove that the density is locally integrable with exponent $1 + \lambda < 7/6$ $1 + \lambda < 7/6$ $1 + \lambda < 7/6$. In section 7 we complete the proof of Theorems [1.1](#page-2-2) and [1.2.](#page-3-1)

2 Preliminaries

2.1 Sobolev spaces. Radon transform. Multiplicators

For every $s \in \mathbb{R}$, denote by $H^s(\mathbb{R}^2)$ the Sobolev space of all tempered distributions u in \mathbb{R}^2 with the finite norm

$$
||u||_{H^{s}(\mathbb{R}^{2})} = ||(1+|\xi|^{2})^{s/2} \mathfrak{F}u||_{L^{2}(\mathbb{R}^{2})},
$$
\n(9)

where $\mathfrak{F}u(\xi)$ is the Fourier transform of u. For all nonnegative integers k, the space $H^k(\mathbb{R}^2)$ coincides with $W^{k,2}(\mathbb{R}^2)$. For every $u \in L^2(\mathbb{R}^2)$ and $s \geq 0$ we have

$$
||u||_{H^{-s}(\mathbb{R}^2)} = \sup_{g \in H^s(\mathbb{R}^2)} \frac{\int_{\mathbb{R}^2} ug \, dx}{||g||_{H^s(\mathbb{R}^2)}}.
$$
(10)

Introduce the Bessel kernel $G_1 = \mathfrak{F}^{-1}(1+|\xi|^2)^{-1/2}$. It is well-known that it is strictly positive and analytic in $\mathbb{R}^2 \setminus \{0\}$. Moreover, the Bessel kernel admits the estimates

$$
c^{-1}|z|^{-1} \le G_1(z) \le c|z|^{-1} \quad \text{for} \quad |z| \le 1, \quad G_1(z) \le c|z|^{-1}e^{-|z|} \quad \text{for} \quad |z| \ge 1. \tag{11}
$$

In particular, for every $N > 0$ there exists a constant $e(N) > 0$ with the property

$$
e(N)|z|^{-1} \le G_1(z) \le c|z|^{-1} \quad \text{for} \quad |z| \le N. \tag{12}
$$

The equality

$$
||G_1 * u||_{H^{s+1}(\mathbb{R}^2)} = ||u||_{H^s(\mathbb{R}^2)}.
$$
\n(13)

holds true for all $u \in H^s(\mathbb{R}^2)$, $s \in \mathbb{R}$.

The next lemma constitutes Sobolev estimates for the functions with integrable Radon transform.

Lemma 2.1. Let $g \in L^2(\mathbb{R}^2)$ be a compactly supported. Then

$$
||g||_{H^{-1/2}(\mathbb{R}^2)}^2 \le \frac{1}{4\pi} \int_{\mathbb{S}^1 \times \mathbb{R}} \Phi(\boldsymbol{\omega}, \tau)^2 d\boldsymbol{\omega} d\tau, \text{ where } \Phi(\boldsymbol{\omega}, \tau) = \int_{\boldsymbol{\omega} \cdot x = \tau} g(x) dl.
$$
\n(14)

Proof. The proof is in Appendix [A](#page-36-0)

The last lemma concerns multiplicative properties of Sobolev spaces.

Lemma 2.2. Let $s > 1/2$, $g \in L^2(\mathbb{R}^2)$ and $u \in H^1(\mathbb{R}^2)$. Then there is $c(s) > 0$ such that

$$
||gu||_{H^{-s}(\mathbb{R}^2)} \le c(s)||g||_{H^{-1/2}(\mathbb{R}^2)} ||u||_{H^1(\mathbb{R}^2)}.
$$
\n(15)

Proof. The proof is in Appendix [A](#page-36-0)

 \Box

2.2 Poisson equation

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $r \in (1, \infty)$. Let $f \in L^r(\mathbb{R}^2)$ be an arbitrary function such that spt $f \subset \Omega$. Then, see [\[3\]](#page-38-7), the Poisson equation

$$
\Delta u = f \quad \text{in } \mathbb{R}^2,\tag{16}
$$

has a solution with the properties: This solution is analytic outside of Ω , and satisfies

$$
\limsup_{|x|\to\infty} (\log |x|)^{-1} |u(x)| < \infty, \quad ||u||_{W^{1,2}(B_R)} \le c ||f||_{L^r(\mathbb{R}^d)}.
$$

Here B_R is the ball $\{x \in \mathbb{R}^d : |x| < R\}$ of an arbitrary radius $R < \infty$, and the constant c depends only on R and Ω . The relation $f \to u$ determines a linear operator Δ^{-1} . In this framework we can define the linear operators

$$
A_j = \partial_{x_j} \Delta^{-1}, \quad R_j = \partial_{x_j} (-\Delta)^{-1/2}, \quad j = 1, 2.
$$

The Riesz operator R_j is a singular integral operator and by the Zygmund-Calderón theorem it is bounded in any space $L^p(\mathbb{R}^d)$ with $1 < p < \infty$. In particular we have

$$
||A_j f||_{W^{1,r}(B_R)} \le c(R, \Omega) ||f||_{L^r(\mathbb{R}^2)} \text{ when } \operatorname{spt} f \subset \Omega, ||R_j f||_{L^p(\mathbb{R}^2)} \le c(r) ||f||_{L^r(\mathbb{R}^2)}.
$$

Notice that these operators have integral representations. In particular, we have

$$
A_i f(x) = c \int_{\mathbb{R}^2} |x - y|^{-2} (x_i - y_i) f(y) dy.
$$
 (17)

3 Regularized problem

In order to regularize problem [\(1\)](#page-1-0) we use the artificial pressure method and replace equations [\(1\)](#page-1-0) by regularized equations

$$
\partial_t(\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho) = \mathrm{div}\,\mathbb{S}(\mathbf{u}) + \varrho \mathbf{f} \quad \text{in} \quad Q_T,\tag{18a}
$$

$$
\partial_t \varrho + \text{div}(\varrho \mathbf{u}) = 0 \quad \text{in} \ \ Q_T,\tag{18b}
$$

$$
\mathbf{u} = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{18c}
$$

$$
\mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \varrho(x,0) = \varrho_0(x) \quad \text{in } \Omega. \tag{18d}
$$

Here, the artificial pressure function is given by

$$
p(\varrho) = \varrho + \varepsilon \varrho^{\gamma}, \quad \varepsilon \in (0, 1], \quad \gamma \ge 6. \tag{18e}
$$

The existence of weak renormalized solutions to problem [\(18\)](#page-5-1) was established in monographs [\[5\]](#page-38-4) and [\[6\]](#page-38-2). The following proposition is a consequence of these results.

Proposition 3.1. Let domain Ω , and functions \mathbf{u}_0 , ϱ_0 , f satisfy Condition 1.1. Then problem [\(18\)](#page-5-1) has a weak solution (ϱ, \mathbf{u}) with the following properties:

(i) The functions $\rho \geq 0$ and **u** satisfy the energy inequality

$$
\underset{t\in(0,T)}{\mathrm{ess\,sup}} \int_{\Omega} \left\{ \varrho |\mathbf{u}|^{2} + \varrho \ln(1+\varrho) + \varepsilon \varrho^{\gamma} \right\} (x,t) \, dx + \int_{Q_{T}} |\nabla \mathbf{u}|^{2} \, dx dt \le cE. \tag{19}
$$

The constant E is as in Remark [1.1.](#page-2-1)

(ii) The integral identity

$$
\int_{Q_T} (\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\xi}) \, dx dt + \int_{Q_T} (\varrho \mathbf{u} \otimes \mathbf{u} + p(\varrho) \mathbb{I} - \mathbb{S}(\mathbf{u})) : \nabla \boldsymbol{\xi} \, dx dt +
$$
\n
$$
\int_{Q_T} \varrho \mathbf{f} \cdot \boldsymbol{\xi} \, dx dt + \int_{\Omega} \varrho_0(x) \mathbf{u}_0(x) \cdot \boldsymbol{\xi}(x, 0) \, dx = 0 \quad (20)
$$

holds for all vector fields $\xi \in C^{\infty}(Q)$ satisfying

$$
\boldsymbol{\xi}(x,T) = 0 \quad in \ \Omega, \quad \boldsymbol{\xi}(x,t) = 0 \quad on \ \partial\Omega \times (0,T). \tag{21}
$$

(iii) The integral identity

$$
\int_{Q_T} \left(\varphi(\varrho) \partial_t \psi + \left(\varphi(\varrho) \mathbf{u} \right) \cdot \nabla \psi - \psi \left(\varphi'(\varrho) \varrho - \varphi(\varrho) \right) \operatorname{div} \mathbf{u} \right) dx dt
$$

$$
+ \int_{\Omega} (\psi \varphi(\varrho_0)) (x, 0) dx = 0 \quad (22)
$$

holds for all smooth functions ψ , vanishing in a neighborhood of the top $\Omega \times \{t = T\}$, and for all functions $\varphi \in C^2[0,\infty)$ satisfying the growth condition

$$
|\varphi(\varrho)| + |\varphi'(\varrho)\varrho| + |\varphi''(\varrho)\varrho^2| \le C(1 + \varrho^2). \tag{23}
$$

Remark 3.1. Further we will assume that (ϱ, \mathbf{u}) and $(\varrho_0, \mathbf{u}_0)$ are extended by 0 to the layer

$$
\Pi = \mathbb{R}^2 \times (0, T). \tag{24}
$$

The following consequences of Proposition [3.1](#page-6-0) will be used throughout the paper.

Corollary [3.1.](#page-6-0) Assume that (ϱ, \mathbf{u}) meets all requirements of Proposition 3.1. Then there is a constant $c(E, \varepsilon)$, depending only on E, β , α , γ , and ε , such that

$$
\| \varrho \mathbf{u} \|_{L^{\infty}(0,T;L^{2\gamma/(\gamma+1)}(\Omega))} \le c(E,\varepsilon),
$$

\n
$$
\| \varrho \mathbf{u} \|_{L^{2}(0,T;L^{\beta}(\Omega))} \le c(E,\varepsilon) \text{ for all } \beta \in [1,\infty] \text{ with } \beta < \gamma,
$$

\n
$$
\| \varrho \mathbf{u} \|_{L^{\alpha}(0,T;L^{2}(\Omega))} \le c(E,\varepsilon) \text{ for all } \alpha \in [1,2\gamma-2).
$$

Proof. It follows from [\(19\)](#page-6-1) that

$$
\|\varrho\|_{L^{\infty}(0,T;L^{\gamma}(\Omega))} + \|\varrho|\mathbf{u}|^{2}\|_{L^{\infty}(0,T;L^{1}(\Omega))} \leq c(E,\varepsilon),
$$

$$
\|\mathbf{u}\|_{L^{2}(0,T;W_{0}^{1,2}(\Omega))} \leq c(E,\varepsilon).
$$
 (25)

Hence (ϱ, \mathbf{u}) are bounded energy functions and the corollary is a particular case of Corollary 4.2.2 in [\[10\]](#page-38-6). \Box

Corollary 3.2. Assume that (ϱ, \mathbf{u}) meets all requirements of Proposition [3.1.](#page-6-0) Then there is a constant $c(E, \varepsilon)$, depending only on E, γ , and ε , such that

$$
\|\varrho|\mathbf{u}|^2\|_{L^2(0,T;L^\tau(\Omega))} \le c(E,\epsilon) \quad \text{for all } \tau \in [1,2\gamma/(\gamma+1)),\tag{26}
$$

$$
\|\varrho|\mathbf{u}|^2\|_{L^1(0,T;L^\tau(\Omega))} \le c(E,\varepsilon) \quad \text{for all } \tau \in [1,\gamma). \tag{27}
$$

Proof. In view of [\(25\)](#page-7-1) the corollary is a particular case of Corollary 4.2.3 in [\[10\]](#page-38-6). \Box

4 Radon transform

In this section we estimate the Radon transform of solutions to regularized equations [\(18\)](#page-5-1). The corresponding result is given by the following theorem. Fix an arbitrary function ζ with the properties

$$
\zeta \in C_0^{\infty}(\mathbb{R}^2), \quad \text{spt } \zeta \Subset \Omega, \quad \zeta \ge 0. \tag{28}
$$

Theorem 4.1. Assume that a renormalized solution to problem [\(18\)](#page-5-1) meets all requirements of Proposition [3.1.](#page-6-0) Furthermore, assume that \bf{u} and ρ are extended by 0 to the layer Π . Then for every unit vector $\boldsymbol{\omega} \in \mathbb{R}^2$,

$$
\int_0^T \int_{-\infty}^\infty \left\{ \int_{\omega \cdot x = \tau} \zeta(x) \varrho(x, t) \, dt \right\}^2 d\tau dt \le c(\zeta) E,\tag{29}
$$

where $c(\zeta)$ depends only on ζ , and E is specified by Remark [1.1.](#page-2-1) Notice that $c(\zeta)$ and E are independent of ω and ε .

Since the Navier- Stokes equations are invariant with respect to rotations, it suffices to prove [\(29\)](#page-8-0) for $\boldsymbol{\omega} = (1, 0)$, i.e., to prove the inequality

$$
\int_0^T \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} \zeta(x) \varrho(x, t) \, dx_2 \right\}^2 dx_1 dt \le c(\zeta) E. \tag{30}
$$

We split the proof of [\(30\)](#page-8-1) into a sequence of lemmas.

Lemma [4.1](#page-8-2). Let all hypotheses of Theorem 4.1 be satisfied. Then for every function $\varphi \in C^{\infty}(Q_T)$ vanishing in a neighborhood of $\partial \Omega \times (0,T)$,

$$
\int_{Q_T} \left(\varrho u_1 \cdot \partial_t \varphi + \left(\varrho u_1 u_i - \mathbb{S}(\mathbf{u})_{i1} \right) \frac{\partial \varphi}{\partial x_i} + p \frac{\partial \varphi}{\partial x_1} \right) dx dt + \int_{Q_T} \varrho f_1 \varphi dx dt \le c(\zeta) E \|\varphi\|_{L^\infty(Q_T)}.
$$
 (31)

Proof. Set

$$
\eta_h(t) = 1
$$
 for $t \le T - h$, $\eta = \frac{1}{h}(T - t)$ for $t \in [T - h, T]$. (32)

Substituting $\xi = (\eta_h \varphi, 0)$ into [\(20\)](#page-6-2) we arrive at the identity

$$
\int_{Q_T} \eta_h \varrho u_1 \cdot \partial_t \varphi \, dx dt + \int_{Q_T} \eta_h \Big(\Big(\varrho u_1 u_i - \mathbb{S}(\mathbf{u})_{i1} \Big) \frac{\partial \varphi}{\partial x_i} + p \frac{\partial \varphi}{\partial x_1} \Big) \, dx dt +
$$
\n
$$
\int_{Q_T} \eta_h \varrho f_1 \varphi \, dx dt = \frac{1}{h} \int_{T-h}^T \int_{\Omega} \varrho \mathbf{u}_1 \varphi \, dx dt - \int_{\Omega} \varrho_0 \mathbf{u}_{1,0} \varphi(x,0) \, dx. \tag{33}
$$

Next notice that

$$
\left|\frac{1}{h}\int_{T-h}^{T}\int_{\Omega}\varrho\mathbf{u}_{1}\varphi\,dxdt\right| + \left|\int_{\Omega}\varrho_{0}\mathbf{u}_{1,0}\varphi(x,0)\,dx\right| \le
$$

\n
$$
\|\varrho\mathbf{u}\|_{L^{\infty}(0,T;L^{1}(\Omega))}\|\varphi\|_{L^{\infty}(Q_{T})} + \|\varrho_{0}\mathbf{u}_{0}\|_{L^{1}(\Omega)}\|\varphi\|_{L^{\infty}(Q_{T})} \le E\|\varphi\|_{L^{\infty}(Q_{T})}.
$$

\netting $h \to 0$ in (33) we arrive at (31)

Letting $h \to 0$ in [\(33\)](#page-8-3) we arrive at [\(31\)](#page-8-4)

Now we specify the test function φ . Choose $\omega : \mathbb{R} \to \mathbb{R}^+$ satisfying

$$
\omega \in C_0^{\infty}(\mathbb{R}), \text{ spt } \omega \subset [-1, 1], \omega \text{ is even }, \int_{\mathbb{R}} \omega(s) ds = 1,
$$

For every $f \in L^1_{loc}(\mathbb{R}^2)$ define the mollifiers

$$
\[f\]_h = \frac{1}{h^2} \int_{\mathbb{R}^2} \omega\left(\frac{x_1 - y_1}{h}\right) \omega\left(\frac{x_2 - y_2}{h}\right) f(y) \, dy. \tag{34}
$$

Introduce the auxiliary functions

$$
H(x_1, t) = \int_{-\infty}^{x_1} \Psi(s, t) \, ds, \quad \Psi(x_1, t) = \int_{\mathbb{R}} \left[\zeta \varrho \right]_h(x_1, x_2, t) \, dx_2,\tag{35a}
$$

and take the test function φ in the form

$$
\varphi(x,t) = \zeta(x) \left[H \right]_h(x_1,t). \tag{35b}
$$

The following lemma constitutes properties of Ψ and H .

Lemma 4.2. $\Psi, H \in L^{\infty}(0,T; C^{k}(\mathbb{R}))$ for every integer $k \geq 0$, and

$$
||H||_{L^{\infty}(\mathbb{R}\times(0,T))} \leq c(\zeta)E. \tag{36}
$$

Proof. Notice that $\zeta_{\ell} \in L^{\infty}(0,T; L^{1}(\mathbb{R}^{2}))$ and its norm in this space does not exceed E. Hence for a.e. $t \in (0, T)$,

$$
\|[\zeta \varrho]_h(t)\|_{C^k(\mathbb{R}^2)} \le c(k) \|\zeta \varrho(t)\|_{L^1(\mathbb{R}^2)} \le c(k)E.
$$

Hence $\left[\zeta \varrho\right]_h$ belongs to $L^{\infty}(0,T; C^k(\mathbb{R}))$. Next, there is N such that the square $[-N+1, N-1]^2$ contains domain Ω . Hence the function $\left[\zeta \varrho\right]_h(t)$ is compactly supported in the square $[-N, N]^2$. It follows that $\Psi \in L^{\infty}(0, T; C^k(\mathbb{R}))$ and $\Psi(\cdot, t)$ is supported in the interval $[-N, N]$. From this and [\(35\)](#page-9-0) we conclude that $H \in L^{\infty}(0,T; C^{k}(\mathbb{R}))$. It remains to note that

$$
|H(x_1,t)| \leq \int_{\mathbb{R}} \Psi(x_1,t) dx_1 \leq \int_{\mathbb{R}^2} \left[\zeta \varrho\right]_h dx dt = \int_{\mathbb{R}^2} \zeta \varrho dx dt = \int_{\Omega} \zeta(x) \varrho(x,t) dx \leq cE.
$$

Now we investigate in details the time dependence of H.

Lemma 4.3. The function $\partial_t H$ belongs to the class $L^{\infty}(0,T; C^k(\Omega))$ for every integer $k \geq 0$. Moreover, it has the representation

$$
\partial_t H = -v_h + J_0
$$
, where $v_h(x_1, t) = \int_{\mathbb{R}} \left[\zeta \varrho u_1 \right]_h(x_1, x_2, t) dx_2$, (37)

and the reminder admits the estimate

$$
|J_0| \le c(\zeta)E. \tag{38}
$$

Proof. Integral identity [\(22\)](#page-6-3) with $\varphi(\varrho) = \varrho$ and ψ replaced by $\zeta \psi$ reads

$$
\int_{\Pi} \left(\zeta \varrho \, \partial_t \psi + \zeta \varrho \mathbf{u} \cdot \nabla \psi + \psi \varrho \nabla \zeta \mathbf{u} \right) dx dt + \int_{\mathbb{R}^2} \psi(x, 0) \zeta(x) \varrho_0(x) dx = 0. \tag{39}
$$

This identity holds true for all functions $\psi \in C^{\infty}(\mathbb{R}^2 \times (0,T))$, vanishing in a neighborhood of the top $\mathbb{R}^2 \times \{t = T\}$. Now choose an arbitrary $\xi \in C_0^{\infty}(0,T)$ and $y \in \mathbb{R}^2$. Inserting

$$
\psi = \xi(t) h^{-2} \omega \left(\frac{x_1 - y_1}{h}\right) \omega \left(\frac{x_2 - y_2}{h}\right)
$$

into [\(39\)](#page-10-0) we arrive at

$$
\int_0^T \Big(\big[\zeta \varrho\big]_h(y,t)\xi'(t) - \xi \operatorname{div} \big[\zeta \varrho \mathbf{u}\big]_h(y,t) + \xi \big[\varrho \nabla \zeta\big]_h(y,t) \Big) dt = 0,
$$

which yields

$$
\partial_t [\zeta \varrho]_h = - \operatorname{div} [\zeta \varrho \mathbf{u}]_h + [\varrho \nabla \zeta \cdot \mathbf{u}]_h \quad \text{in} \quad \mathbb{R}^2 \times [0, T]. \tag{40}
$$

Next, Corollary [3.1](#page-7-2) implies that $\zeta \varrho \mathbf{u}$ and $\varrho \nabla \zeta \cdot \mathbf{u}$ belong to $L^{\infty}(0,T; L^{1}(\mathbb{R}^{2})).$ Hence the functions div $\left[\zeta \varrho \mathbf{u}\right]_h$ and $\left[\varrho \nabla \zeta \cdot \mathbf{u}\right]_h$ belong to $L^{\infty}(0,T; C^k(\mathbb{R}^2))$ for all integer $k \geq 0$. Moreover, they are supported in Q_T . It follows that $\partial_t [\zeta \varrho]_h$ belongs to $L^{\infty}(0,T; C^k(\mathbb{R}^2))$ and is supported in Q_T . Therefore, the function

$$
\partial_t H = \int_{-\infty}^{x_1} \int_{-\infty}^{\infty} \partial_t [\zeta \varrho]_h(s, x_2, t) ds dx_2
$$

belongs to the class $L^{\infty}(0,T; C^{k}(\mathbb{R}^{2}))$. Integrating both sides of [\(40\)](#page-10-1) over $(-\infty, x_1] \times \mathbb{R}$ we obtain representation [\(37\)](#page-10-2) with the reminder

$$
J_0(x_1,t) = \int_{-\infty}^{x_1} \left\{ \int_{\mathbb{R}} \left[\varrho \nabla \zeta \cdot \mathbf{u} \right]_h(s,x_2,t) dx_2 \right\} ds.
$$

It remains to note that for a. e. $t \in (0, T)$, we have

$$
|J_0(x_1, t)| \leq \int_{\mathbb{R}^2} \left[\varrho |\nabla \zeta| |\mathbf{u}| \right]_h(x, t) dx = \int_{\mathbb{R}^2} \varrho |\nabla \zeta| |\mathbf{u}| (x, t) dx \leq
$$

$$
c(\zeta) \int_{\Omega} \varrho |\mathbf{u}| (x, t) dx \leq c(\zeta) ||\varrho \mathbf{u}||_{L^{\infty}(0, T; L^1(\Omega))} \leq c(\zeta) E.
$$

In view of Lemma [4.3](#page-10-3) the function φ given by formula [\(35\)](#page-9-0) belongs to $L^{\infty}(0,T; C^{k}(\mathbb{R}^{2}))$ and is supported in Q_T . Moreover, we have

$$
\|\varphi\|_{L^{\infty}(Q_T)} \le c(\zeta) \|H\|_{L^{\infty}(\mathbb{R}\times(0,T))} \le c(\zeta)E. \tag{41}
$$

Substituting φ in [\(31\)](#page-8-4) and using [\(41\)](#page-11-0) we obtain

$$
I_1 + I_2 + I_3 + I_4 + I_5 \le c(\zeta)E \tag{42}
$$

where

$$
I_1 = \int_{\Pi} \varrho u_1 \partial_t \varphi dx dt, \quad I_2 = \int_{\Pi} \varrho u_1 u_i \partial_{x_i} \varphi dx dt, \quad I_3 = \int_{\Pi} p \partial_{x_1} \varphi dx dt,
$$
\n
$$
I_4 = -\int_{\Pi} \mathbb{S}_{i1} \partial_{x_i} \varphi dx dt, \quad I_5 = \int_{\Pi} \varrho f_1 \varphi dx dt.
$$
\n(43)

Let us consider each term in [\(42\)](#page-11-1) separately.

Lemma 4.4.

$$
I_1 = -\int_0^T \int_{\mathbb{R}} v_h^2 dx_1 dt + J_1, \text{ where } |J_1| \le c(\zeta) E. \tag{44}
$$

Proof. Since $\varrho u_1 \in L^2(\Pi)$ and the mollifying operator is symmetric, it follows from [\(35\)](#page-9-0) that

$$
I_1 = \int_{\Pi} \left[\zeta \varrho u_1 \right]_h (x_1, x_2, t) \partial_t H(x_1, t) \, dx dt.
$$

Next, the function $\left[\zeta \varrho u_1\right]$ is supported in $\Omega \times [0, T]$. Therefore, the function v_h is supported in every rectangular $[-N, N] \times [0, T]$ such that $[-N, N]^2 \supset \Omega$. From this we conclude that

$$
I_1 = \int_0^T \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} \left[\zeta \varrho u_1 \right]_h dx_2 \right\} \partial_t H(x_1, t) dx_1 dt = \int_0^T \int_{\mathbb{R}} v_h(x_1, t) \partial_t H(x_1, t) dx_1 dt.
$$

Inserting expression [\(37\)](#page-10-2) for $\partial_t H$ we obtain representation [\(44\)](#page-11-2) with the reminder

$$
J_1 = \int_0^T \int_{\mathbb{R}} J_0 v_h \, dx_1 dt.
$$

It remains to note that in view of [\(38\)](#page-10-4),

$$
|J_1| \le c(\zeta)E \int_0^T \int_{\mathbb{R}} |v_h| dx_1 dt \le c(\zeta)E \int_{\Pi} [\zeta \varrho | \mathbf{u}]_h dx dt
$$

= $cE \int_{\Pi} \zeta \varrho |\mathbf{u}| dx dt = cE \int_{Q_T} \zeta \varrho |\mathbf{u}| dx dt \le c(\zeta)E \int_{Q_T} \varrho |\mathbf{u}| dx dt \le c(\zeta)E.$

Lemma 4.5.

$$
I_2 = \int_0^T \int_{\mathbb{R}} \Upsilon_h(x_1, t) \Psi(x_1, t) dx_1 dt + J_2, \text{ where } \Upsilon_h = \int_{\mathbb{R}} \left[\zeta \varrho u_1^2 \right]_h(x_1, x_2, t) dx_2,
$$
\n(45)

and the reminder J_2 admits the estimate

$$
|J_2| \le c(\zeta)E. \tag{46}
$$

Moreover, the function Υ_h belongs to the class $L^{\infty}(0,T; C^k(\mathbb{R}))$ for every integer $k \geq 0$. It is supported in any rectangular $[-N, N] \times [0, T]$ such that $[-N, N]^2 \supset \Omega.$

Proof. Notice that $\zeta \varrho u_i u_1 \in L^\infty(0,T; L^1(\Omega))$ is supported in Q_T . It follows from [\(35\)](#page-9-0) that

$$
I_2 = \int\limits_{\Pi} \zeta \varrho u_1^2 \left[\frac{\partial H}{\partial x_1}\right]_h dx dt + J_2, \text{ where } J_2 = \int\limits_{\Pi} \varrho u_1 (\nabla \zeta \cdot \mathbf{u}) \left[H\right]_h dx dt.
$$

Since $\partial_{x_1}H = \Psi$ is independent of x_2 , we have

$$
\int_{\Pi} \zeta \varrho u_1^2 \left[\frac{\partial H}{\partial x_1}\right]_h dx dt = \int_{\Pi} \zeta \varrho u_1^2 \left[\Psi\right]_h dx dt = \int_{\Pi} \left[\zeta \varrho u_1^2\right]_h \Psi dx dt =
$$

$$
\int_0^T \int_{\mathbb{R}} \Psi(x_1, t) \left\{ \int_{\mathbb{R}} \left[\zeta \varrho u_1^2\right]_h dx_2 \right\} dx_1 dt = \int_0^T \int_{\mathbb{R}} \Upsilon_h(x_1, t) \Psi(x_1, t) dx_1 dt.
$$

This leads to the desired representation [\(45\)](#page-12-0). In order to estimate J_2 notice that in view of [\(36\)](#page-9-1), we have $\sup |[H]_h| \leq \sup H \leq c(\zeta)E$. This gives

$$
|J_2| \le c(\zeta)E \int_{Q_T} \varrho |\mathbf{u}|^2 dx dt \le c(\zeta)E.
$$

Lemma 4.6.

$$
I_3 \ge \int_0^T \int_{\mathbb{R}} \Psi^2 dx_1 dt + J_3, \quad \text{where} \quad |J_3| \le c(\zeta, E). \tag{47}
$$

Proof. We have

$$
\partial_{x_1}\varphi = \partial_{x_1}\zeta \left[H\right]_h + \zeta \left[\Psi\right]_h.
$$

Hence

$$
I_3 = \int_{\Pi} \zeta p \big[\Psi \big]_h \, dx dt + J_3, \quad \text{where} \quad J_3 = \int_{\Pi} \partial_{x_1} \zeta p \big[H \big]_h \, dx dt.
$$

Since Ψ is nonnegative we have

$$
\int_{\Pi} \zeta p[\Psi]_h \, dxdt = \int_{\Pi} \zeta \varrho[\Psi]_h \, dxdt + \varepsilon \int_{\Pi} \zeta \varrho^{\gamma}[\Psi]_h \, dxdt \ge
$$

$$
\int_{\Pi} \zeta \varrho[\Psi]_h \, dxdt = \int_{\Pi} [\zeta \varrho]_h \Psi \, dxdt = \int_0^T \int_{\mathbb{R}} \Psi^2 dx_1 dt,
$$

which leads to (47) . It remains to estimate J_3 . To this end notice that in view of [\(36\)](#page-9-1) and [\(19\)](#page-6-1),

$$
|J_3| \le c(\zeta) ||H||_{L^{\infty}(\mathbb{R} \times (0,T))} \int_0^T \int_{\Omega} p \, dx dt \le c(\zeta) E.
$$

Lemma 4.7.

$$
|I_4| \le c(\zeta)E + c(\zeta) E\left(\int_0^T \int_{\mathbb{R}} \Psi^2 dx_1 dt\right)^{1/2}, \quad |I_5| \le c(\zeta)E. \tag{48}
$$

Proof. It follows from formulae [\(35\)](#page-9-0) that

$$
I_4 = \int_{\Pi} \zeta \mathbb{S}_{11}(\mathbf{u}) \big[\Psi \big]_h dx dt + \int_{\Pi} (\partial_{x_i} \zeta) \mathbb{S}_{i1}(\mathbf{u}) \big[H \big]_h dx dt. \tag{49}
$$

Notice that $[\zeta \mathbb{S}]_h$ is compactly supported in $\Omega \times [0, T]$. Hence it is supported in the slab $[-N, N]^2 \times [0, T]$ such that $[-N, N]^2 \supset \Omega$. Thus we get

$$
\Big| \int_{\Pi} \zeta \mathbb{S}_{11} [\Psi]_{h} dxdt \Big| = \Big| \int_{\Pi} [\zeta \mathbb{S}_{11}]_{h} \Psi dxdt \Big| = \Big| \int_{0}^{T} \int_{[-N,N]^{2}} [\zeta \mathbb{S}_{11}]_{h} \Psi dxdt \Big| \le
$$

$$
\Big(\int_{0}^{T} \int_{[-N,N]^{2}} [\zeta \mathbb{S}_{11}]_{h}^{2} dxdt \Big)^{1/2} \Big(\int_{0}^{T} \int_{[-N,N]^{2}} \Psi^{2} dxdt \Big)^{1/2} \le
$$

$$
\Big(\int_{0}^{T} \int_{[-N,N]^{2}} (\zeta \mathbb{S}_{11})^{2} dxdt \Big)^{1/2} \Big(N \int_{0}^{T} \int_{[-N,N]} \Psi^{2} dx_{1} dt \Big)^{1/2} \le
$$

$$
c(\zeta) N^{1/2} \Big(\int_{Q_{T}} |\nabla \mathbf{u}|^{2} dxdt \Big)^{1/2} \Big(\int_{0}^{T} \int_{\mathbb{R}} \Psi^{2} dx_{1} dt \Big)^{1/2}
$$

Since N depends only on Ω , these inequalities along with estimate [\(19\)](#page-6-1) imply

$$
\left| \int_{\Pi} \zeta \mathbb{S}_{11} \left[\Psi \right]_h dx dt \right| \le c(\zeta) E \left(\int_0^T \int_{\mathbb{R}} \Psi^2 dx_1 dt \right)^{1/2} \tag{50}
$$

We also have

$$
\left| \int_{\Pi} (\partial_{x_i} \zeta) \, \mathbb{S}_{i1}(\mathbf{u}) \big[H \big]_h dx dt \right| \le c(\zeta) \| H \|_{L^{\infty}(\mathbb{R} \times (0,T))} \int_{Q_T} |\nabla \mathbf{u}| dx dt \le c(\zeta) E. \tag{51}
$$

Inserting [\(50\)](#page-14-0) and [\(51\)](#page-14-1) into [\(49\)](#page-14-2) we arrive at the first inequality in [\(48\)](#page-13-1). It remains to notice that

$$
|I_5| \le c \|f\|_{L^{\infty}(Q_T)} \|H\|_{L^{\infty}(\mathbb{R}\times(0,T))} \int_{\Pi} \zeta \varrho \, dx dt \le c(\zeta) E \int_{Q_T} \varrho \, dx dt \le c(\zeta) E.
$$

Lemma 4.8.

$$
\frac{1}{2} \int_0^T \int_{\mathbb{R}} \Psi^2 dx_1 dt + \int_0^T \int_{\mathbb{R}} (\Upsilon_h \Psi - v_h^2) dx_1 dt \le c(\zeta) E. \tag{52}
$$

Proof. Estimate [\(48\)](#page-13-1) for I_5 and inequality [\(42\)](#page-11-1) imply

$$
I_1 + I_2 + I_3 + I_4 \leq c(\zeta)E.
$$

From this and Lemmas [4.4,](#page-11-3) [4.5](#page-12-1) we obtain

$$
I_3 + \int_0^T \int_{\mathbb{R}} (\Upsilon_h \Psi - v_h^2) dx_1 dt \le c(\zeta) E + |I_4|
$$

Applying Lemmas [4.6](#page-13-2) and [4.7](#page-13-3) we arrive at

$$
\int_0^T \int_{\mathbb{R}} \Psi^2 dx_1 dt + \int_0^T \int_{\mathbb{R}} (\Upsilon_h \Psi - v_h^2) dx_1 dt \le c(\zeta) E + c(\zeta) E \left(\int_0^T \int_{\mathbb{R}} \Psi^2 dx_1 dt \right)^{1/2},
$$

which obviously leads to (52) .

Lemma 4.9. $\Upsilon_h \Psi - v_h^2 \geq 0$ in Π .

Proof. We begin with the observation that the inequality

$$
\left[fg\right]_h^2 \le \left[f^2\right]_h \left[g^2\right]_h
$$

holds for all functions $f(x)$, $g(x)$ locally integrable with square in \mathbb{R}^2 . Setting $f = \sqrt{\zeta \varrho(\cdot, t)}$ and $g = \sqrt{\zeta \varrho} |u_1|(\cdot, t)$ we obtain for a.e. x, t and all $\delta > 0$,

$$
[\zeta \varrho |u_1|]_h(x_1, x_2, t) \le \sqrt{[\zeta \varrho u_1^2]_h}(x_1, x_2, t) \sqrt{[\zeta \varrho]_h}(x_1, x_2, t) \le
$$

$$
\frac{1}{2} \delta [\zeta \varrho u_1^2]_h(x_1, x_2, t) + \frac{1}{2} \delta^{-1} [\zeta \varrho]_h(x_1, x_2, t)
$$

Integrating both sides with respect to x_2 over R and recalling formulae [\(37\)](#page-10-2) and [\(45\)](#page-12-0) for v_h and Υ_h we arrive at

$$
v_h(x_1, t) \leq \frac{1}{2} \delta \Upsilon_h(x_1, t) + \frac{1}{2} \delta^{-1} \Psi(x_1, t).
$$

Recall that Υ_h and Ψ are nonnegative. Setting $\delta = (\Psi/\Upsilon_h)^{1/2}$ we obtain the desired inequality. \Box

We are now in a position to complete the proof of Theorem [4.1.](#page-8-2) Introduce the function

$$
\Phi_1(x_1,t) = \int_{\mathbb{R}} \zeta \varrho(x_1,x_2,t) dx_2 \equiv \Phi(\mathbf{e}_1,x_1,t).
$$

Recall that $\zeta \varrho$ is supported in Q_T . It suffices to prove that

$$
\int_0^T \int_{\mathbb{R}} \Phi_1^2 dx_1 dt \le c(\zeta) E. \tag{53}
$$

By virtue of the energy estimate [\(19\)](#page-6-1), the function Φ_1 belongs to the class $L^2(0,T;\mathbb{R})$. It is supported in the rectangular $[-N, N] \times [0,T]$ for every N such that $[-N, N]^2 \supset \Omega$. It obviously follows from this and definition [\(34\)](#page-9-2) of the mollifier that

$$
\Psi = [\Phi_1]_h^{(1)}, \text{ where } [\Phi_1]_h^{(1)}(x_1, t) = \frac{1}{h} \int_{\mathbb{R}} \omega \left(\frac{x_1 - y_1}{h} \right) \Phi_1(y_1, t) dy_1.
$$

In other words, $[\Phi_1]_h^{(1)}$ is the mollifying of Φ_1 with respect to x_1 . Lemmas [4.8](#page-14-4) and [4.9](#page-15-0) imply the inequality

$$
\int_0^T \int_{\mathbb{R}} \left(\left[\Phi_1 \right]_h^{(1)} \right)^2 dx_1 dt \le c(\zeta) E. \tag{54}
$$

Notice that $\left[\Phi_1\right]_h^{(1)} \to \Phi_1$ a.e. in $\mathbb{R} \times (0,T)$. Letting $h \to 0$ in [\(54\)](#page-16-1) and applying the Fatou Theorem we arrive at (53) . \Box

5 Momentum estimates

In this section we prove auxiliary estimates for solutions (ρ, \mathbf{u}) to regularized equations [\(18\)](#page-5-1). We start with the estimating of norms of ρ and ρ **u** in negative Sobolev spaces.

Proposition 5.1. Let a solution (ρ, \mathbf{u}) to problem [\(18\)](#page-5-1) meets all require-ments of Proposition [3.1.](#page-6-0) Let $\zeta \in C_0^{\infty}(\mathbb{R}^2)$ be an arbitrary nonnegative compactly supported in Ω function and $s > 1/2$. Then

$$
\|\zeta \varrho\|_{L^2(0,T;H^{-1/2}(\mathbb{R}^2))} \le c(\zeta)E,\tag{55}
$$

$$
\|\zeta \varrho \mathbf{u}\|_{L^1(0,T;H^{-s}(\mathbb{R}^2))} \le c(\zeta)c(s)E,\tag{56}
$$

where $c(\zeta)$ depends only on ζ , $c(s)$ depends only on s, and E is specified by Remark [1.1.](#page-2-1)

Proof. Lemma [2.1](#page-4-0) and Theorem [4.1](#page-8-2) imply the estimates

$$
\int_0^T \|\zeta \varrho\|_{H^{-1/2}(\mathbb{R}^2)}^2 dt \le \int_0^T \int_{\mathbb{S}^1} \int_{-\infty}^\infty \left\{ \int_{\omega \cdot x = \tau} \zeta \varrho \, dl \right\}^2 d\tau d\omega dt =
$$

$$
\int_{\mathbb{S}^1} \left\{ \int_0^T \int_{-\infty}^\infty \left\{ \int_{\omega \cdot x = \tau} \zeta \varrho \, dl \right\}^2 d\tau dt \right\} d\omega \le c(\zeta) E \int_{\mathbb{S}^1} d\omega \le c(\zeta) E,
$$

which yield [\(55\)](#page-16-3). Next, Lemma [2.2](#page-4-1) implies the inequality

$$
\|\zeta \varrho(t) \mathbf{u}(t)\|_{H^{-s}(\mathbb{R}^2)} \leq c(\zeta,s) \|\zeta \varrho(t)\|_{H^{-1/2}(\mathbb{R}^2)} \|\mathbf{u}(t)\|_{H^1(\mathbb{R}^2)}.
$$

From this, [\(55\)](#page-16-3), and estimate [\(19\)](#page-6-1) we finally obtain

$$
\|\zeta \varrho \mathbf{u}\|_{L^1(0,T;H^{-s}(\mathbb{R}^2))} \leq c(\zeta,s) \|\zeta \varrho\|_{L^2(0,T;H^{-1/2}(\mathbb{R}^2))} \|\mathbf{u}\|_{L^2(0,T;H^1(\mathbb{R}^2))} \leq c(\zeta,s)E.
$$

 \Box

5.1 Cauchy-Riemann equations

Further notation ∇^{\perp} and rot stands for the differential operators

$$
\nabla^{\perp} f = (-\partial_{x_2} f, \partial_{x_1} f), \quad \text{rot } \mathbf{w} = \partial_{x_2} w_1 - \partial_{x_1} w_2.
$$

Denote by $\mathbf{F} = (F_1, F_2)$ a solution to the inhomogeneous Cauchy-Riemann equation

$$
\nabla F_1 + \nabla^{\perp} F_2 = \zeta \varrho \mathbf{u} \text{ in } \Pi. \tag{57}
$$

It is easily seen that

$$
F_1 = \text{div } \Delta^{-1}(\zeta \varrho \mathbf{u}), \quad F_2 = -\text{rot } \Delta^{-1}(\zeta \varrho \mathbf{u}). \tag{58}
$$

The following two auxiliary lemmas give L^p - estimates for a vector function F.

Lemma 5.1. Under the assumptions of Proposition [5.1,](#page-16-4) for every positive δ and R, there is a constant $c(\delta, \zeta, R)$ such that

$$
\|\mathbf{F}\|_{L^{4}(0,T;L^{\frac{8}{3+\delta}}(B_R))} \leq c(\delta,\zeta,R)E. \tag{59}
$$

Proof. Fix an arbitrary positive δ and R. Without loss of generality we may assume that $\delta < 1$ and $B_R \supset \Omega$. Integral representation [\(17\)](#page-5-2) for the operator $\partial_x\Delta^{-1}$ and formulae [\(58\)](#page-17-0) for solutions to inhomogeneous Cauchy-Riemann equations imply the inequalities

$$
|\mathbf{F}(x,t)| \leq c \int_{\Omega} |x-y|^{-1} \zeta \varrho |\mathbf{u}|(y,t) dy \leq \frac{c}{2} b(x,t) \mathcal{L}(x,t) + \frac{c}{2} b^{-1}(x,t) \mathcal{Q}(x,t),
$$

where

$$
\mathcal{L}(x,t) = \int_{\Omega} |x-y|^{-1} \zeta \varrho |\mathbf{u}|^2(y,t) \, dy, \quad \mathcal{Q}(x,t) = \int_{\Omega} |x-y|^{-1} \zeta \varrho \, dy,
$$

b is an arbitrary positive function. Notice that if $\mathcal{L}(x, t)$ or $\mathcal{Q}(x, t)$ vanishes at least at one point (x, t) , then $\mathbf{F}(\cdot, t)$ vanishes in \mathbb{R}^2 . In opposite case we can take $b = \sqrt{\mathcal{Q}/\mathcal{L}}$. Thus we get

$$
|\mathbf{F}(x,t)| \le c\sqrt{\mathcal{L}(x,t)} \sqrt{\mathcal{Q}(x,t)} \quad \text{a.e. in } \mathbb{R}^2 \times (0,T) \tag{60}
$$

Now our task is to estimate $\mathcal L$ and $\mathcal Q$. We have

$$
\mathcal{L}(x,t) = \int_{B_R} \left(\zeta \varrho |\mathbf{u}|^2 \right)^{\frac{1+\delta}{2}} |x-y|^{-1-\delta+\alpha} \left(\zeta \varrho |\mathbf{u}|^2 \right)^{\frac{1-\delta}{2}} |x-y|^\alpha \, dy,
$$

where $\alpha = \delta/2 > 0$. It follows that

$$
\mathcal{L}(x,t) \le c(R) \int_{B_R} \left(\zeta \varrho |\mathbf{u}|^2\right)^{\frac{1+\delta}{2}} |x-y|^{-1-\delta+\alpha} \left(\zeta \varrho |\mathbf{u}|^2\right)^{\frac{1-\delta}{2}} dy \text{ for } x \in B_R.
$$

Applying the Hölder inequality we obtain

$$
\mathcal{L}(x,t) \leq c \Big(\int_{B_R} \zeta \varrho |\mathbf{u}|^2 |x-y|^{-2+\frac{2\alpha}{1+\delta}} dy \Big)^{\frac{1+\delta}{2}} \Big(\int_{B_R} \zeta \varrho |\mathbf{u}|^2 dy \Big)^{\frac{1-\delta}{2}}.
$$

This leads to the inequality

$$
\int_{B_R} \mathcal{L}(x,t)^{\frac{2}{1+\delta}} dx \leq \Big(\int_{B_R} \zeta \varrho |\mathbf{u}|^2 dy\Big)^{\frac{1-\delta}{1+\delta}} \int_{B_R} \int_{B_R} \zeta \varrho |\mathbf{u}|^2(y,t) |x-y|^{-2+\frac{2\alpha}{1+\delta}} dxdy \leq
$$

$$
c\Big(\int_{B_R} \zeta \varrho |\mathbf{u}|^2(y,t) dy\Big)^{\frac{2}{1+\delta}} \leq c\Big(\|\varrho |\mathbf{u}|^2(t)\|_{L^1(\Omega)}\Big)^{2/(1+\delta)}.
$$

Recalling the energy estimate [\(19\)](#page-6-1) we finally obtain

$$
\|\mathcal{L}\|_{L^{\infty}(0,T;L^{\frac{2}{1+\delta}}(B_R))} \le c(R,\zeta)\|\varrho|\mathbf{u}|^2\|_{L^{\infty}(0,T;L^1(\Omega))} \le c(\zeta,R)E. \tag{61}
$$

Now our task is to estimate Q. Notice that $|x-y| \leq 2R$ for all $x, y \in B_R$. It follows from this and [\(12\)](#page-4-2) that $|x - y|^{-1} \le c(R)G_1(x - y)$ for all $x, y \in B_R$, where G_1 is the Bessel kernel. We thus get

$$
\mathcal{Q}(x,t) \le c(R) \int_{\mathbb{R}^2} G_1(x-y) \zeta \varrho(y,t) \, dy := G(x,t).
$$

Estimate [\(13\)](#page-4-3) for the Bessel kernel and inequality [\(55\)](#page-16-3) yield

$$
||G||_{L^2(0,T;H^{1/2}(\mathbb{R}^2)} \leq ||\zeta \varrho||_{L(0,T;H^{-1/2}(\mathbb{R}^2))} \leq c(\zeta)E.
$$

Since the embedding $H^{1/2}(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)$ is bounded, see [\[1\]](#page-38-8), thm. 7.57, we obtain

$$
\|\mathcal{Q}\|_{L^2(0,T;L^4(B_R))} \le c(R) \|G\|_{L^2(0,T;L^4(B_R))} \le c(R) \|G\|_{L^2(0,T;H^{1/2}(\mathbb{R}^2))} \le c(R,\zeta)E.
$$

Combining these inequalities with [\(61\)](#page-19-0) we arrive at the estimates

$$
\|\sqrt{\mathcal{L}}\|_{L^{\infty}(0,T;L^{\frac{4}{1+\delta}}(B_R))} \le c(R,\zeta)E, \quad \|\sqrt{\mathcal{Q}}\|_{L^4(0,T;L^8(B_R))} \le c(R,\zeta)E. \quad (62)
$$

Next, the Hölder inequality implies that

$$
\|\sqrt{\mathcal{L}}\sqrt{\mathcal{Q}}\|_{L^{\tau}(0,T;L^{r}(B_{R}))}\leq \|\sqrt{\mathcal{L}}\|_{L^{\tau_{1}}(0,T;L^{r_{1}}(B_{R}))}\|\sqrt{\mathcal{Q}}\|_{L^{\tau_{2}}(0,T;L^{r_{2}}(B_{R}))}
$$

for all $\tau, r, \tau_i, r_i \in [1, \infty]$ satisfying the condition

$$
\tau^{-1} = \tau_1^{-1} + \tau_2^{-1}, \quad r^{-1} = r_1^{-1} + r_2^{-1}.
$$

Setting $\tau = \tau_2 = 4$, $\tau_1 = \infty$, $r_1 = 4/(1+\delta)$, $r_2 = 8$, $r = 8/(3+2\delta)$, and recalling inequalities [\(62\)](#page-19-1) we obtain

$$
\|\sqrt{\mathcal{L}}\sqrt{\mathcal{Q}}\|_{L^4(0,T;L^{8/(3+2\delta)}(B_R))} \le c(\zeta,\delta,R)E.
$$

Combining this result with [\(60\)](#page-18-0) we arrive at [\(59\)](#page-17-1).

Lemma 5.2. . Under the assumptions of Proposition [5.1,](#page-16-4) for every positive $\nu < 3$ and R, there is a constant $c(\nu, \zeta, R)$ such that

$$
\|\mathbf{F}\|_{L^{2}(0,T;L^{3-\nu}(B_R))} \le c(\nu,\zeta,R)E. \tag{63}
$$

Proof. Assume that $\Omega \subset B_R$. It follows from [\(12\)](#page-4-2) that $|x-y|^{-1} \le c(R)G_1(x-\lambda)$ y) for all $x, y \in B_R$. Thus we get

$$
|\mathbf{F}(x,t)| \le c \int_{\mathbb{R}^2} G_1(x-y)\zeta \varrho |\mathbf{u}|(y,t) \, dy := M(x,t) \quad \text{for all} \quad x \in B_R. \tag{64}
$$

Now choose an arbitrary $\mu \in (0, 1/2)$. It follows from [\(13\)](#page-4-3) that

$$
||M(t)||_{H^{1/2-\mu}(\mathbb{R}^2)} \leq ||\zeta \varrho| \mathbf{u}|(t) ||_{H^{-1/2-\mu}(\mathbb{R}^2)}.
$$

Applying inequality [\(56\)](#page-16-5) we obtain

$$
||M||_{L^1(0,T;H^{1/2-\mu}(\mathbb{R}^2))} \leq c(\mu,\zeta)E.
$$

Since the embedding $H^{1/2-\mu}(\mathbb{R}^2) \hookrightarrow L^{\frac{4}{1+2\mu}}(\mathbb{R}^2)$ is bounded, [\[1\]](#page-38-8), thm. 7.57, we get

$$
||M||_{L^1(0,T;L^{\frac{4}{1+2\mu}}(\mathbb{R}^2))} \leq c(\mu,\zeta)E.
$$

Combining this result with [\(64\)](#page-20-0) we arrive at

$$
\|\mathbf{F}\|_{L^1(0,T;L^{\frac{4}{1+2\mu}}(B_R))} \le c(\mu,\zeta)E. \tag{65}
$$

Next notice that by the interpolation inequality,

$$
\|\mathbf{F}\|_{L^r(0,T;L^s(B_R))} \le \|\mathbf{F}\|_{L^4(0,T;L^{\frac{8}{3+\delta}}(B_R))}^{\alpha} \|\mathbf{F}\|_{L^1(0,T;L^{\frac{4}{1+2\mu}}(B_R))}^{1-\alpha}
$$

holds for all $\alpha \in (0,1)$ and

$$
r^{-1} = \frac{\alpha}{4} + \frac{1-\alpha}{1}, \quad s^{-1} = \frac{3+\delta}{8}\alpha + \frac{1+2\mu}{4}(1-\alpha).
$$

Setting $\alpha = 2/3$ and recalling inequalities [\(59\)](#page-17-1), [\(65\)](#page-20-1) we obtain

$$
\|\mathbf{F}\|_{L^2(0,T;L^{\frac{12}{4+\mu+\delta}}(B_R))} \leq c(\zeta,\delta,\mu,R).
$$

Choosing μ and δ so small that $3 - \nu < 12/(4 + \mu + \delta)$ we finally arrive at [\(63\)](#page-19-2).

 6 L^p estimates

In this section we investigate properties of solutions (ϱ, \mathbf{u}) to regularized problem [\(18\)](#page-5-1) and prove that the pressure function $p(\rho)$ is locally integrable with an exponent greater than 1. The corresponding result is given by the following

Theorem 6.1. Let a solution (ρ, \mathbf{u}) to problem [\(18\)](#page-5-1) meets all requirements of Proposition [3.1.](#page-6-0) Let $\zeta \in C_0^{\infty}(\mathbb{R}^2)$ be an arbitrary nonnegative compactly supported in Ω function and $\lambda \in (0, 1/6)$. Then

$$
\int_{Q_T} \zeta^2 p(\varrho) \varrho^{\lambda} dx dt \le c(\zeta, \lambda) E,\tag{66}
$$

where $c(\zeta, \lambda)$ depends only on ζ and λ .

The rest of the section is devoted to the proof of this theorem. Our strategy is the following. First we construct a special test function ξ such that p div $\xi \sim p(\varrho)\varrho^{\lambda}$. Next we insert ξ into [\(20\)](#page-6-2) to obtain special integral identity containing the vector field F. Finally we employ Lemmas [5.1](#page-17-2) and [5.2](#page-19-3) to obtain estimate [\(66\)](#page-21-1). Hence the proof of Theorem [6.1](#page-21-2) falls into four steps.

6.1 Step 1. Test functions

Fix an arbitrary $\lambda \in (0, 1/6)$ and choose a function $\psi \in C^{\infty}(\mathbb{R})$ with the properties

$$
\psi(0) = 0, \quad \psi(s) \ge 0, \quad c^{-1}|s|^\lambda - 1 \le \psi(s) \le c|s|^\lambda, \quad |s\psi'(s)| \le c|s|^\lambda, \quad (67)
$$

where c is some positive constant. Next choose an arbitrary function $\zeta \in$ $C_0^{\infty}(\mathbb{R}^2)$ such that ζ is nonnegative and is compactly supported in Ω . Recall the definition of the mollifier $\left[\cdot\right]_h$ and introduce the auxiliary function

$$
g(x,t) = \left[\zeta\psi(\varrho)\right]_h(x,t) \quad \text{in} \quad R^2 \times [0,T]. \tag{68}
$$

We will assume that h is less than the distance between the support of ζ and the boundary of Ω . Finally, introduce the test vector field

$$
\boldsymbol{\xi}(x,t) = \zeta(x) \mathbf{H}(x,t), \text{ where } \mathbf{H} = \nabla \Delta^{-1} g. \tag{69}
$$

The following lemmas constitute the basic properties of $\psi(\varrho)$, g, and **H**.

Lemma 6.1. Under the assumptions of Theorem [6.1,](#page-21-2) there is a constant $c(\lambda)$, depending only on λ and ψ , such that

$$
\|\psi(\varrho)\nabla \mathbf{u}\|_{L^{2}(0,T;L^{\frac{2}{1+2\lambda}}(\mathbb{R}^{2}))} + \|(\psi(\varrho) - \varrho\psi'(\varrho))\nabla \mathbf{u}\|_{L^{2}(0,T;L^{\frac{2}{1+2\lambda}}(\mathbb{R}^{2}))} \le c(\lambda)E,
$$
\n(70)\n
$$
\|\psi(\varrho)\mathbf{u}\|_{L^{2}(0,T;L^{3}(\mathbb{R}^{2}))} \le c(\lambda)E.
$$

Proof. Notice that

$$
|\psi(\varrho)\nabla \mathbf{u}| + |(\psi(\varrho) - \varrho\psi'(\varrho))\nabla \mathbf{u}| \le c\varrho^{\lambda}|\nabla \mathbf{u}|.
$$
 (72)

Recall that **u** and ρ vanish outside of $\Omega \times [0, T]$. From this and relations

$$
1/2 + 1/\infty = 1/2, \quad 1/2 + 1/(1/\lambda) = (1 + 2\lambda)/2
$$

we obtain

$$
\|\varrho^{\lambda}\nabla\mathbf{u}\|_{L^{2}(0,T;L^{\frac{2}{1+2\lambda}}(\Omega))} \leq \|\nabla\mathbf{u}\|_{L^{2}(0,T;L^{2}(\Omega))} \|\varrho^{\lambda}\|_{L^{\infty}(0,T;L^{1/\lambda}(\Omega))} \leq
$$

$$
c(\lambda)\|\nabla\mathbf{u}\|_{L^{2}(0,T;L^{2}(\Omega))} \|\varrho\|_{L^{\infty}(0,T;L^{1}(\Omega))}^{\lambda} \leq c(\lambda)E,
$$

which along with [\(72\)](#page-22-0) yields [\(70\)](#page-22-1). Next set $q = 3/(1 - 3\lambda)$. Since $1/q$ + $1/(1/\lambda) = 1/3$, we have

$$
\|\psi(\varrho)\mathbf{u}\|_{L^{2}(0,T;L^{3}(\mathbb{R}^{2}))} \leq c\|\varrho^{\lambda}\mathbf{u}\|_{L^{2}(0,T;L^{3}(\Omega))} \leq c\|\varrho^{\lambda}\|_{L^{\infty}(0,T;L^{1/\lambda}(\Omega))} \|\mathbf{u}\|_{L^{2}(0,T;L^{q}(\Omega))}
$$

$$
\leq c(\lambda)\|\varrho\|_{L^{\infty}(0,T;L^{1}(\Omega))}^{\lambda}\|\mathbf{u}\|_{L^{2}(0,T;L^{q}(\Omega))} \leq c(\lambda)\|\varrho\|_{L^{\infty}(0,T;L^{1}(\Omega))}^{\lambda}\|\mathbf{u}\|_{L^{2}(0,T;W_{0}^{1,2}(\Omega))} \leq cE,
$$

and the lemma follows.

Lemma 6.2. Under the assumptions of Theorem [6.1,](#page-21-2) the function g belongs to the class $L^{\infty}(0,T; C^{k}(\mathbb{R}^{2}))$ for every integer $k \geq 0$. It is compactly supported in $\Omega \times [0, T]$ and admits the estimate

$$
||g||_{L^{\infty}(0,T;L^{1/\lambda}(\mathbb{R}^2))} \leq c(\zeta)E. \tag{73}
$$

Moreover, $\partial_t g$ belongs to the class $L^2(0,T;C^k(\mathbb{R}^2))$ and has the representation

$$
\partial_t g = -\text{div}\left[\zeta \psi(\varrho) \mathbf{u}\right]_h + \left[\psi(\varrho)\nabla \zeta \mathbf{u}\right]_h + \left[\zeta(\psi(\varrho) - \psi'(\varrho)\varrho) \text{ div } \mathbf{u}\right]_h. \tag{74}
$$

Proof. Since $\zeta \psi(\varrho) \in L^{\infty}(0,T; L^{1}(\mathbb{R}^{2}))$, it follows from general properties of the mollifier that $g \in L^{\infty}(0,T; C^k(\mathbb{R}^2))$. Since h is less than the distance between spt ζ and $\mathbb{R}^2 \setminus \Omega$, the function g is supported in $\Omega \times [0, T]$. Next, inequality [\(67\)](#page-21-3) implies the estimate

$$
||g(t)||_{L^{1/\lambda}(\mathbb{R}^2)} \leq ||\zeta \psi(\varrho)(t)||_{L^{1/\lambda}(\mathbb{R}^2)} \leq c(\zeta) ||\varrho(t)||_{L^1(\mathbb{R}^2)}^{\lambda},
$$

which along with energy inequality [\(19\)](#page-6-1) yields [\(73\)](#page-22-2). Let us consider the time derivative of q . In view of (22) the integral identity

$$
\int_{Q_T} \left(\psi(\varrho) \partial_t \varsigma + \left(\psi(\varrho) \mathbf{u} \right) \cdot \nabla \varsigma - \varsigma \left(\psi'(\varrho) \varrho - \psi(\varrho) \right) \mathrm{div} \, \mathbf{u} \right) dx dt = 0 \quad (75)
$$

holds for every function $\varsigma \in C^{\infty}(\Pi)$ which is supported in Q_T . Choose an arbitrary $\xi \in C_0^{\infty}(0,T)$, $y \in \mathbb{R}^2$, and set

$$
\varsigma(x,t) = \xi(t)\zeta(x) h^{-2}\omega\left(\frac{x_1 - y_1}{h}\right)\omega\left(\frac{x_2 - y_2}{h}\right).
$$

Substituting ς into [\(75\)](#page-23-0) we obtain

$$
\int_0^T \xi'(t)g(y,t) dt - \int_0^T \xi(t) \operatorname{div} \left[\zeta \psi(\varrho) \mathbf{u} \right]_h(y,t) dt + \int_0^T \xi(t) \left[\psi(\varrho) \nabla \zeta \mathbf{u} \right]_h(y,t) dt +
$$

$$
\int_0^T \xi(t) \left[\zeta(\psi(\varrho) - \psi'(\varrho)\varrho) \operatorname{div} \mathbf{u} \right]_h(y,t) dt = 0,
$$

which yields [\(74\)](#page-22-3). In view of Lemma [6.1,](#page-22-4) the functions $\psi(\varrho)\zeta\mathbf{u}, \psi(\varrho)\nabla\zeta\cdot\mathbf{u}$ belong to the class $L^2(0,T;L^3(\mathbb{R}^2))$, and the function $\zeta(\psi(\rho)-\psi'(\rho)\rho)$ div u belong to $L^2(0,T; L^{2/(1+2\lambda)}(\mathbb{R}^2)$. Since the mollifier $[\cdot]_h: L^p(\mathbb{R}^2) \to C^k(\mathbb{R}^2)$ is bounded for all $p \ge 1$ and $k \ge 0$, the representation [\(35\)](#page-9-0) yields the inclusion $\partial_t q \in L^2(0, T; C^k(\mathbb{R}^2))$. $\partial_t g \in L^2(0,T;C^k(\mathbb{R}^2))$.

Lemma 6.3. Under the assumptions of Theorem [6.1,](#page-21-2) H belongs to the class class $L^{\infty}(0,T; C^{k}(\mathbb{R}^{2}))$, and $\partial_{t}H$ belongs to the class $L^{2}(0,T; C^{k}(\mathbb{R}^{2}))$ for every integer $k \geq 0$. Moreover, **H** admits the estimates.

$$
\|\mathbf{H}\|_{L^{\infty}(0,T;L^{\infty}(\Omega))} \le c(\zeta)E, \quad \|\nabla \mathbf{H}\|_{L^{\infty}(0,T;L^{1/\lambda}(\Omega))} \le c(\zeta)E. \tag{76}
$$

Proof. Since the function g is supported in Q_T , the inclusions $\mathbf{H} \in L^{\infty}(0,T; C^k(\mathbb{R}^2))$ and $\partial_t \mathbf{H} \in L^2(0,T; C^k(\mathbb{R}^2))$ obviously follow from [\(6.2\)](#page-22-5). Now choose an arbitrary R such that $\Omega \in B_{R/2}$. Since spt $g(t) \subset \Omega$, we have

$$
\|\mathbf{H}(t)\|_{W^{1/\lambda}(B_R)} \le c(R,\Omega) \|g(t)\|_{L^{1/\lambda}(\Omega)}
$$

The embedding $W^{1/\lambda}(B_R) \hookrightarrow C(B_R)$ is bounded for every $\lambda < 1/2$. It follows that

$$
\|\mathbf{H}\|_{L^{\infty}(0,T;C(B_R))} \le c(R,\lambda) \|\mathbf{H}\|_{L^{\infty}(0,T;W^{1/\lambda}(B_R))} \le
$$

$$
\|g\|_{L^{\infty}(0,T;L^{1/\lambda}(\Omega)} \le c(R,\zeta,\lambda)E \quad (77)
$$

Since $\Omega \subset B_{R/2}$, representation [\(17\)](#page-5-2) implies

$$
|\mathbf{H}(x,t)| \le c(R,\Omega) \|g(t)\|_{L^1(\Omega)} \text{ for } x \in \mathbb{R}^2 \setminus B_R
$$

Thus we get

$$
\|\mathbf{H}\|_{L^{\infty}(0,T;C(\mathbb{R}^2\setminus B_R))} \le c(R,\lambda) \|g\|_{L^{\infty}(0,T;L^1(\Omega))} \le c(R,\zeta,\lambda)E.
$$

Combining this result with [\(77\)](#page-24-0) gives the first inequality in [\(76\)](#page-23-1). Next notice that

$$
\partial_{x_i} H_j = \partial_{x_i} \partial_{x_j} \Delta^{-1} g = R_i R_j g,
$$

where R_i , R_j are the Riesz singular operators. Since the Riesz operators are bounded in $L^{1/\lambda}(\mathbb{R}^2)$, the second inequality in [\(76\)](#page-23-1) is a straightforward consequence of estimate [\(73\)](#page-22-2).

6.2 Step 2. Integral identities

The proof of Theorem [6.1](#page-21-2) is based on the special integral identity which is given by the following proposition.

Proposition 6.1. Under the assumptions of Theorem [6.1,](#page-21-2) we have

$$
\int_{\Pi} \zeta p(\varrho) \left[\zeta \psi(\varrho) \right]_h dx dt = \sum_{i=1}^7 \Gamma^{(i)} + I_h,\tag{78}
$$

where

$$
\Gamma^{(1)} = \int_{Q_T} F_1 \left[\zeta(\psi(\varrho) - \varrho \psi'(\varrho)) \operatorname{div} \mathbf{u} \right]_h dx dt,
$$

$$
\Gamma^{(3)} = \int_{Q_T} \left(g F_1 \operatorname{div} \mathbf{u} - F_1 \nabla u_j \cdot \frac{\partial \mathbf{H}}{\partial x_j} - F_2 \nabla^{\perp} u_j \cdot \frac{\partial \mathbf{H}}{\partial x_j} \right) dx dt,
$$

$$
\Gamma^{(2)} = \int_{Q_T} F_1 \left[\psi(\varrho) \nabla \zeta \cdot \mathbf{u} \right]_h dx dt, \quad \Gamma^{(4)} = -\int_{Q_T} \varrho(\mathbf{u} \cdot \nabla \zeta) (\mathbf{u} \cdot \mathbf{H}) dx dt,
$$

$$
\Gamma^{(5)} = -\int_{Q_T} \zeta p \nabla \zeta \cdot \mathbf{H} dx dt, \quad \Gamma^{(6)} = \int_{Q_T} \left(\mathbb{S}(\mathbf{u}) : \nabla \xi - \varrho \mathbf{f} \cdot \xi \right) dx dt,
$$

$$
\Gamma^{(7)} = \lim_{\tau \to 0} \frac{1}{\tau} \int_{T-\tau}^T \int_{\Omega} \zeta \varrho \mathbf{u} \cdot \mathbf{H} dx dt - \int_{\Omega} \varrho_0(x) \zeta \mathbf{u}_0(x) \mathbf{H}(x, 0) dx
$$
 (79)

$$
I_h = -\int_{\mathbb{R}^2 \times [0,T]} \zeta \varrho \mathbf{u} \cdot \nabla \operatorname{div} \Delta^{-1} \left(\left[\zeta \psi(\varrho) \right]_h \mathbf{u} - \left[\zeta \psi(\varrho) \mathbf{u} \right]_h \right) dx dt. \tag{80}
$$

Here \bf{F} is a solution to the Cauchy-Riemann equations [\(57\)](#page-17-3), and \bf{H} is given $by (69).$ $by (69).$ $by (69).$

Proof. Recall formulae [\(32\)](#page-8-5) and [\(69\)](#page-21-4) for the cut-off function η_{τ} and the vector field ξ . Notice that the function $\eta_{\tau}\xi$ and its time derivative belong to $L^{\infty}(0,T; C^{k}(\Omega))$ and $L^{2}(0,T; C^{k}(\Omega))$ respectively for all integer $k \geq 0$. Moreover, $\eta_{\tau} \xi$ vanishes at the lateral side and the top of the cylinder Q_T . Hence we can use this function as a test function in integral identity [\(20\)](#page-6-2) to obtain

$$
\int_{Q_T} \eta_\tau(t) \left(\varrho \mathbf{u} \cdot \partial_t \xi \right) dx dt + \int_{Q_T} \eta_\tau \left(\varrho \mathbf{u} \otimes \mathbf{u} + p(\varrho) \mathbb{I} - \mathbb{S}(\mathbf{u}) \right) : \nabla \xi \, dx dt +
$$
\n
$$
\int_{Q_T} \eta_\tau \varrho \mathbf{f} \cdot \xi \, dx dt = \Gamma_T(\tau), \quad (81)
$$

where

$$
\Gamma_T(\tau) = \frac{1}{\tau} \int_{T-\tau}^T \int_{\Omega} \zeta \varrho \mathbf{u} \cdot \mathbf{H} \, dx dt - \int_{\Omega} \varrho_0(x) \zeta \mathbf{u}_0(x) \mathbf{H}(x,0) dx \qquad (82)
$$

Letting $\tau \to 0$ in [\(81\)](#page-25-0) we arrive at

$$
\int_{Q_T} (\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\xi}) \, dx dt + \int_{Q_T} (\varrho \mathbf{u} \otimes \mathbf{u} + p(\varrho) \, \mathbb{I} - \mathbb{S}(\mathbf{u})) : \nabla \boldsymbol{\xi} \, dx dt + \int_{Q_T} \varrho \mathbf{f} \cdot \boldsymbol{\xi} \, dx dt = \Gamma^{(7)}.
$$

The limit $\Gamma^{(7)} = \lim_{\tau \to 0} \Gamma_T(\tau)$ exists since there exists the limit of the left hand side of [\(81\)](#page-25-0). We can rewrite the latter identity in the form

$$
\int_{Q_T} p(\varrho) \operatorname{div} \xi \, dx dt = \Gamma^{(6)} + \Gamma^{(7)} - \int_{Q_T} \varrho \mathbf{u} \cdot \partial_t \xi \, dx dt - \int_{Q_T} \varrho \mathbf{u} \otimes \mathbf{u} : \nabla \xi \, dx dt. \tag{83}
$$

It follows from the expression [\(69\)](#page-21-4) for ξ that $\partial_t \xi = \zeta \nabla \Delta^{-1} \partial_t g$, where g is given by [\(68\)](#page-21-5). From this and and the representation [\(74\)](#page-22-3) in Lemma [6.2](#page-22-5) we obtain the identity

$$
\int_{Q_T} \varrho \mathbf{u} \cdot \partial_t \boldsymbol{\xi} \, dx dt = \int_{Q_T} \zeta \varrho \mathbf{u} \cdot \nabla \Delta^{-1} \left[\zeta(\psi(\varrho) - \varrho \psi'(\varrho)) \, \mathrm{div} \, \mathbf{u} \right]_h dx dt +
$$
\n
$$
\int_{Q_T} \zeta \varrho \mathbf{u} \cdot \nabla \Delta^{-1} \left[\psi(\varrho) \nabla \zeta \cdot \mathbf{u} \right]_h dx dt - \int_{Q_T} \zeta \varrho \mathbf{u} \cdot \nabla \mathrm{div} \, \Delta^{-1} \left[\zeta \psi(\varrho) \mathbf{u} \right]_h dx dt \tag{84}
$$

Since ζ , $[\zeta(\psi(\varrho) - \varrho\psi'(\varrho))$ div $\mathbf{u}]_h$, and $[\psi(\varrho)\nabla\zeta \cdot \mathbf{u}]_h$ are supported in Q_T , we have

$$
\int_{Q_T} \zeta \varrho \mathbf{u} \cdot \nabla \Delta^{-1} \left[\zeta(\psi(\varrho) - \varrho \psi'(\varrho)) \, \mathrm{div} \, \mathbf{u} \right]_h dx dt = \int_{\Pi} \zeta \varrho \mathbf{u} \cdot \nabla \Delta^{-1} \left[\zeta(\psi(\varrho) - \varrho \psi'(\varrho)) \, \mathrm{div} \, \mathbf{u} \right]_h dx dt
$$
\n
$$
= - \int_{\Pi} F_1 \left[\zeta(\psi(\varrho) - \varrho \psi'(\varrho)) \, \mathrm{div} \, \mathbf{u} \right]_h dx dt = -\Gamma^{(1)},
$$
\n
$$
\int_{Q_T} \zeta \varrho \mathbf{u} \cdot \nabla \Delta^{-1} \left[\psi(\varrho) \nabla \zeta \cdot \mathbf{u} \right]_h dx dt = \int_{\Pi} \zeta \varrho \mathbf{u} \cdot \nabla \Delta^{-1} \left[\psi(\varrho) \nabla \zeta \cdot \mathbf{u} \right]_h dx dt =
$$
\n
$$
- \int_{\Pi} F_1 \left[\psi(\varrho) \nabla \zeta \cdot \mathbf{u} \right]_h dx dt = -\Gamma^{(2)}.
$$

Inserting these equalities into [\(84\)](#page-26-0) we arrive at

$$
\int_{\Pi} \varrho \mathbf{u} \cdot \partial_t \boldsymbol{\xi} \, dx dt = -\Gamma^{(1)} - \Gamma^{(2)} - \int_{Q_T} \zeta \varrho \mathbf{u} \cdot \nabla \text{div } \Delta^{-1} \big[\zeta \psi(\varrho) \mathbf{u} \big]_h \, dx dt. \tag{85}
$$

Next, expression (69) for **H** implies

$$
\int_{Q_T} \varrho u_i u_j \frac{\partial \xi_i}{\partial x_j} dx dt = \int_{\Pi} u_j \frac{\partial \mathbf{H}}{\partial x_j} \cdot (\zeta \varrho \mathbf{u}) dx dt +
$$
\n
$$
\int_{\Pi} \varrho (\mathbf{u} \cdot \nabla \zeta) (\mathbf{u} \cdot \mathbf{H}) dx dt = \int_{\Pi} u_j \frac{\partial \mathbf{H}}{\partial x_j} \cdot (\zeta \varrho \mathbf{u}) dx dt - \Gamma^{(4)} \quad (86)
$$

Using equation [\(57\)](#page-17-3) we obtain the identity

$$
\int_{\Pi} u_j \frac{\partial \mathbf{H}}{\partial x_j} \cdot (\zeta \varrho \mathbf{u}) \, dx dt = \int_{\Pi} u_j \left(\nabla F_1 + \nabla^{\perp} F_2 \right) \cdot \frac{\partial \mathbf{H}}{\partial x_j} \, dx dt.
$$

In view of Corollary [3.1,](#page-7-2) the function ϱ **u** belongs to the class $L^2(\Pi)$. Obviously it is supported in $\Omega \times (0,T)$. From this and formula [\(59\)](#page-17-1) we conclude that $\mathbf{F} \in L^2(0,T;W^{1,2}(B_R))$ for every ball $B_R \subset \mathbb{R}^2$. Recall that $\mathbf{u} \in L^2(0,T;W^{1,2}(\mathbb{R}^2))$ is compactly supported in $\Omega \times (0,T)$. Finally notice that $\mathbf{H} \in L^{\infty}(0,T; C^{k}(\mathbb{R}^{2}))$ for every $k \geq 0$. Hence we can integrate by parts to obtain

$$
\int_{\Pi} u_j \frac{\partial \mathbf{H}}{\partial x_j} \cdot (\zeta \varrho \mathbf{u}) \, dx dt = \int_{\Pi} \left(F_2 \text{rot } \frac{\partial \mathbf{H}}{\partial x_j} - F_1 \text{div } \frac{\partial \mathbf{H}}{\partial x_j} \right) u_j \, dx dt -
$$
\n
$$
\int_{\Pi} \left(F_1 \nabla u_j \cdot \frac{\partial \mathbf{H}}{\partial x_j} + F_2 \nabla^{\perp} u_j \cdot \frac{\partial \mathbf{H}}{\partial x_j} \right) dx dt. \tag{87}
$$

Recall that $\mathbf{H} = \nabla \Delta^{-1} g$, where $g = [(\zeta \psi(\varrho)]_h \in L^{\infty}(0,T; C^3(\mathbb{R}^2))$ is supported in Q_T . It follows that

$$
\text{rot } \frac{\partial \mathbf{H}}{\partial x_j} = 0, \quad \text{div } \frac{\partial \mathbf{H}}{\partial x_j} = \partial_{x_j} \big[\zeta \psi(\varrho) \big]_h.
$$

We thus get

$$
\int_{\Pi} \left(F_2 \text{rot } \frac{\partial \mathbf{H}}{\partial x_j} - F_1 \text{div } \frac{\partial \mathbf{H}}{\partial x_j} \right) u_j \, dx dt = - \int_{\Pi} F_1 u_j \partial_{x_j} \left[\zeta \psi(\varrho) \right]_h = - \int_{Q_T} F_1 \text{div } \left(\left[\zeta \psi(\varrho) \right]_h \mathbf{u} \right) dx dt + \int_{Q_T} g F_1 \text{div } \mathbf{u} \, dx dt.
$$

Noting that $F_1 = \text{div } \Delta^{-1}(\zeta \varrho \mathbf{u})$ we arrive at the identity

$$
\int_{\Pi} \left(F_2 \text{rot } \frac{\partial \mathbf{H}}{\partial x_j} - F_1 \text{div } \frac{\partial \mathbf{H}}{\partial x_j} \right) u_j \, dx dt =
$$
\n
$$
\int_{\Pi} \zeta \varrho \mathbf{u} \cdot \nabla \text{div } \Delta^{-1} \big(\big[\zeta \psi(\varrho) \big]_h \mathbf{u} \big) \, dx dt + \int_{\Pi} g F_1 \text{div } \mathbf{u} \, dx dt.
$$

Inserting this equality into [\(87\)](#page-27-0) and recalling the expression [\(81\)](#page-25-0) for $\Gamma^{(3)}$ we get

$$
\int_{\Pi} \zeta \varrho u_j \frac{\partial \mathbf{H}}{\partial x_j} \cdot \mathbf{u} \, dx dt = -\Gamma^{(3)} - \int_{\Pi} \zeta \varrho \mathbf{u} \cdot \nabla \text{div } \Delta^{-1} \big(\big[\zeta \psi(\varrho) \big]_h \mathbf{u} \big) \, dx dt.
$$

Inserting this result into [\(86\)](#page-26-1) we finally obtain

$$
\int_{\Pi} \varrho u_i u_j \frac{\partial \xi_i}{\partial x_j} dx dt = -\Gamma^{(3)} - \Gamma^{(4)} + \int_{\Pi} \zeta \varrho \mathbf{u} \cdot \nabla \text{div } \Delta^{-1} \big(\big[\zeta \psi(\varrho) \big]_h \mathbf{u} \big) dx dt.
$$
\n(88)

It remains to note that in view of [\(79\)](#page-25-1) and [\(69\)](#page-21-4) we have

$$
\int_{\Pi} p \operatorname{div} \xi dx dt = \int_{\Pi} \zeta p \big[\zeta \psi(\varrho]_h dx dt - \Gamma^{(5)}.
$$
\n(89)

Inserting [\(85\)](#page-26-2), [\(88\)](#page-28-0), and [\(89\)](#page-28-1) into [\(83\)](#page-26-3) we obtain the desired identity [\(78\)](#page-24-1). \Box

6.3 Step 3. Estimates of $\Gamma^{(i)}$

In this section we show that the quantities $\Gamma^{(i)}$ in the basic integral identity [\(78\)](#page-24-1) are bounded by a constant, depending only on the exponent λ , the cut-off function ζ , and the constant E specified by Remark [1.1.](#page-2-1)

Proposition 6.2. Under the assumptions of Theorem [6.1,](#page-21-2)

$$
\Gamma^{(i)} \le c(\zeta, \lambda)E,\tag{90}
$$

where c depends only on ζ and λ .

Proof. Let us estimate $\Gamma^{(1)}$ and $\Gamma^{(2)}$. Since $\lambda < 1/6$, we can choose $\nu > 0$, depending on λ , such that $(3 - \nu)^{-1} + (1 + 2\lambda)2^{-1} = 1$. Lemmas [5.2,](#page-19-3) [6.1](#page-22-4) and the Hölder inequality imply

$$
|\Gamma^{(1)}| \leq ||F_1||_{L^2(0,T;L^{3-\nu}(\mathbb{R}^2))} ||[\zeta(\psi(\varrho) - \varrho\psi'(\varrho)) \operatorname{div} \mathbf{u}]_h ||_{L^2(0,T;L^{2/(1+2\lambda)}(\mathbb{R}^2))} \leq
$$

$$
||F_1||_{L^2(0,T;L^{3-\nu}(\mathbb{R}^2))} ||\zeta(\psi(\varrho) - \varrho\psi'(\varrho)) \operatorname{div} \mathbf{u}||_{L^2(0,T;L^{2/(1+2\lambda)}(\mathbb{R}^2))} \leq c(\lambda,\zeta)E.
$$

(91)

Applying Lemmas [5.2](#page-19-3) and [6.1](#page-22-4) once more we obtain

$$
|\Gamma^{(2)}| \leq ||F_1||_{L^2(0,T;L^{3-\nu}(\mathbb{R}^2))} ||[\psi(\varrho)\nabla\zeta \cdot \mathbf{u}]_h||_{L^2(0,T;L^2(\mathbb{R}^2))} \leq
$$

$$
||F_1||_{L^2(0,T;L^{3-\nu}(\mathbb{R}^2))} ||\psi(\varrho)\nabla\zeta \cdot \mathbf{u}||_{L^2(0,T;L^2(\mathbb{R}^2))} \leq c(\lambda,\zeta)E.
$$
 (92)

Our next task is to estimate $\Gamma^{(3)}$ and $\Gamma^{(4)}$. Since **u** is supported in $\Omega \times [0, T]$, we have

$$
|\Gamma^{(3)}| \le \int_{\Omega \times (0,T)} |\mathbf{F}| |\nabla \mathbf{u}| (|g| + |\nabla \mathbf{H}|) dx dt \qquad (93)
$$

It follows from Lemmas [6.2](#page-22-5) and [6.3](#page-23-2) that

$$
\|g\|_{L^{\infty}(0,T;L^{1/\lambda}(\mathbb{R}^2))} + \|\nabla \mathbf{H}\|_{L^{\infty}(0,T;L^{1/\lambda}(\mathbb{R}^2))} \le c(\zeta,\lambda)E. \tag{94}
$$

On the other hand, energy estimate [\(19\)](#page-6-1) and Lemma [5.2](#page-19-3) imply

$$
\|\mathbf{F}\|_{L^2(0,T;L^{3-\kappa}(\Omega))} \le c(\zeta,\kappa)E, \quad \|\nabla \mathbf{u}\|_{L^2(0,T;L^2(\Omega))} \le c(\zeta,\lambda)E,\qquad(95)
$$

where κ is an arbitrary positive number. Since $\lambda < 1/6$, we can choose κ such that $(3 - \kappa)^{-1} + \lambda + 2^{-1} = 1$. Applying the Hölder inequality and using [\(94\)](#page-29-0), [\(95\)](#page-29-1) we obtain

$$
\int_{\Omega\times(0,T)}|\mathbf{F}||\nabla\mathbf{u}|(|g|+|\nabla\mathbf{H}|)\,dxdt\le
$$

$$
\|\mathbf{F}\|_{L^{2}(0,T;L^{3-\kappa}(\Omega))}\|\nabla\mathbf{u}\|_{L^{2}(0,T;L^{2}(\Omega))}(\|g\|_{L^{\infty}(0,T;L^{1/\lambda}(\mathbb{R}^2))}+\|\nabla\mathbf{H}\|_{L^{\infty}(0,T;L^{1/\lambda}(\mathbb{R}^2))})\leq c(\zeta,\lambda)E.
$$

which leads to estimate [\(90\)](#page-28-2) for $\Gamma^{(3)}$. In order to estimate $\Gamma^{(4)}$ notice that in view of Lemma [6.3](#page-23-2) and energy estimate [\(19\)](#page-6-1), we have

$$
\begin{aligned} |\Gamma^{(4)}| &\le c(\zeta) \int_{\Omega \times (0,T)} |\mathbf{H}| \varrho |\mathbf{u}|^2 \, dx dt \le \\ &\le c(\zeta) \|\mathbf{H}\|_{L^{\infty}(\Omega \times (0,T))} \int_{\Omega \times (0,T)} \varrho |\mathbf{u}|^2 \, dx dt \le c(\zeta, \lambda) E. \end{aligned}
$$

Next we employ Lemma [6.3](#page-23-2) and estimate [\(19\)](#page-6-1) to obtain

$$
|\Gamma^{(5)}| \leq \int_{\mathbb{R}^2 \times (0,T)} \zeta p |\nabla \zeta| |\mathbf{H}| dx dt \leq c(\zeta) \|\mathbf{H}\|_{L^{\infty}(\Omega \times (0,T))} \int_{\Omega \times (0,T)} p dx dt \leq c(\zeta, \lambda) E.
$$

It remains to estimate $\Gamma^{(6)}$ and $\Gamma^{(7)}$. Expression [\(69\)](#page-21-4) for the vector field ξ , and expression [\(79\)](#page-25-1) for $\Gamma^{(6)}$ yield the estimate

$$
|\Gamma^{(6)}| \leq \Big| \int_{\Pi} \mathbb{S}(\mathbf{u}) : \nabla \xi \, dx dt \Big| + \Big| \int_{\Pi} \varrho \mathbf{f} \cdot \xi \, dx dt \Big| \leq
$$

$$
\int_{\Pi} \zeta |\mathbb{S}(\mathbf{u})| \, |H| \, dx dt + \int_{\Pi} (\zeta + |\nabla \zeta|) |\mathbf{H}| (|\mathbb{S}(\mathbf{u})| + \varrho |\mathbf{f}|) \, dx dt \leq
$$

$$
c(\zeta) \int_{Q_T} |\nabla \mathbf{u}| (|\mathbf{H}| + |\nabla \mathbf{H}|) \, dx dt + c(\zeta) E \int_{Q_T} \varrho |\mathbf{H}| \, dx dt. \tag{96}
$$

On the other hand, energy estimate [\(19\)](#page-6-1) yields

$$
\int_{Q_T} |\nabla \mathbf{u}| (|\mathbf{H}| + |\nabla \mathbf{H}|) dx dt \le c(\zeta) ||\nabla \mathbf{u}||_{L^2(Q_T)} ||\mathbf{H}||_{L^2(0,T;W^{1,2}(\Omega))} \le
$$

 $c(\zeta) E ||\mathbf{H}||_{L^2(0,T;W^{1,2}(\Omega))}$

Next, Lemma [6.3](#page-23-2) and the inequality $1/\lambda > 2$ imply

 $\|\mathbf{H}\|_{L^2(0,T;W^{1,2}(\Omega))} \le \|\mathbf{H}\|_{L^{\infty}(0,T;L^{\infty}(\Omega))} + \|\nabla \mathbf{H}\|_{L^{\infty}(0,T;L^{1/\lambda}(\Omega))} \le c(\zeta,\lambda)E.$

We thus get

$$
\int_{Q_T} |\nabla \mathbf{u}| (|\mathbf{H}| + |\nabla \mathbf{H}|) dx dt \le c(\zeta, \lambda) E
$$
\n(97)

Finally notice that

$$
\int_{Q_T} \varrho |\mathbf{H}| \, dx dt \leq \|\mathbf{H}\|_{L^{\infty}(Q_T)} \, \|\varrho\|_{L^{\infty}(0,T;L^1(\Omega))} \leq c(\zeta,\lambda) E.
$$

Inserting this estimate along with [\(97\)](#page-30-0) into [\(96\)](#page-29-2) we arrive at the desired estimate [\(90\)](#page-28-2) for $\Gamma^{(6)}$. Finally, expression [\(82\)](#page-25-2) for Γ_T , Lemma [6.3,](#page-23-2) and the energy estimate [\(19\)](#page-6-1) imply

$$
|\Gamma_T(\tau)| \le c(\zeta) \|\mathbf{H}\|_{L^\infty(Q_T)} \|\varrho\|_{L^\infty(0,T;L^1(\Omega))} \le c(\zeta) E. \tag{98}
$$

It follows from this that $\Gamma^{(7)} = \lim_{\tau \to 0} \Gamma_T(\tau)$ satisfies inequality [\(90\)](#page-28-2).

6.4 Step 4. Proof of Theorem [6.1](#page-21-2)

The proof is based on Propositions [6.1](#page-24-2) and [6.2.](#page-28-3) First we show that the quantity I_h in identity [\(78\)](#page-24-1) tends to zero as $h \to 0$. We begin with the observation that

$$
I_h = \int_{\Pi} \zeta \varrho \mathbf{u} \cdot \nabla \operatorname{div} \Delta^{-1} \Big(\left[\zeta \psi(\varrho) \mathbf{u} \right]_h - \zeta \psi(\varrho) \mathbf{u} \Big) dx dt -
$$

$$
\int_{\Pi} \zeta \varrho \mathbf{u} \cdot \nabla \operatorname{div} \Delta^{-1} \Big(\left(\left[\zeta \psi(\varrho) \right]_h - \zeta \psi(\varrho) \right) \mathbf{u} \Big) dx dt.
$$

Since the Riesz operator ∇ div $\Delta^{-1}: L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is bounded and ζ is supported in Ω , we have

$$
|I_h|^2 \le c \| \varrho \mathbf{u} \|_{L^2(Q_T)}^2 \int_0^T (J_h(t) + L_h(t)) dt,
$$
\n(99)

where

$$
J_h(t) = \|\left[\zeta \psi(\varrho) \mathbf{u}\right]_h(t) - \zeta \psi(\varrho) \mathbf{u}(t)\|_{L^2(\Omega)}^2, \ L_h(t) = \|\left[\zeta \psi(\varrho)\right]_h \mathbf{u}(t) - \zeta \psi(\varrho) \mathbf{u}(t)\|_{L^2(\Omega)}^2.
$$

In view of Corollary [3.2](#page-7-3) the vector field ϱ **u** belongs to the class $L^2(Q_T)$. Hence it suffices to prove that the sequence $J_h + L_h$ has an integrable majorant and tends to 0 a.e. in $(0, T)$ as $h \to 0$. Inequality [\(71\)](#page-22-6) in Lemma [6.1](#page-22-4) implies that $\zeta\psi(\varrho)\mathbf{u}\in L^2(0,T;L^2(\mathbb{R}^2)).$ It follows from properties of the mollifier that $J_h(t) \to 0$ as $h \to 0$ a.e. in $(0, T)$. On the other hand, we have

$$
J_h(t) \leq \|\big[\psi(\varrho)\mathbf{u}\big]_h(t)\|_{L^2(\Omega)} + \|\psi(\varrho)\mathbf{u}(t)\|_{L^2(\Omega)} \leq 2\|\psi(\varrho)\mathbf{u}(t)\|_{L^2(\Omega)}
$$

By [\(71\)](#page-22-6), the right hand side is integrable over $[0, T]$ and is independent of h. Hence the sequence $J_h \to 0$ a.e. in $(0, T)$ and has an integrable majorant. Next, the Hölder inequality implies

$$
L_h(t) \leq ||\left[\zeta \psi(\varrho)\right]_h(t) - \zeta \psi(\varrho)(t)||^2_{L^{1/\lambda}(\Omega)} ||\mathbf{u}(t)||^2_{L^{2/(1-2\lambda)}(\Omega)}
$$

Since the embedding $W_0^{1,2}$ $L^{1,2}(\Omega) \hookrightarrow L^{2/(1-2\lambda)}(\Omega)$ is bounded, we have

$$
L_h(t) \le c(\lambda) \|\big[\zeta \psi(\varrho)\big]_h(t) - \zeta \psi(\varrho)(t) \|^2_{L^{1/\lambda}(\Omega)} \|\mathbf{u}(t)\|^2_{W_0^{1,2}(\Omega)}.
$$
 (100)

Since $\zeta \psi(\varrho) \in L^{\infty}(0,T; L^{1/\lambda}(\Omega))$, we have

$$
\|\left[\zeta\psi(\varrho)\right]_h(t) - \zeta\psi(\varrho)(t)\|_{L^{1/\lambda}(\Omega)}^2 \to 0 \text{ as } h \to 0 \text{ for a.e. } t \in (0, T).
$$

Hence $L_h(t) \to 0$ a.e. in $(0, T)$. Notice that

$$
\begin{aligned} \|\left[\zeta\psi(\varrho)\right]_h(t) - \zeta\psi(\varrho)(t)\|_{L^{1/\lambda}(\Omega)} &\leq\\ \|\left[\zeta\psi(\varrho)\right]_h(t)\|_{L^{1/\lambda}(\Omega)} + \|\zeta\psi(\varrho)(t)\|_{L^{1/\lambda}(\Omega)} &\leq 2\|\zeta\psi(\varrho)(t)\|_{L^{1/\lambda}(\Omega)}. \end{aligned}
$$

Combing this result with [\(100\)](#page-31-0) and recalling that $\zeta \psi(\varrho) \in L^{\infty}(0,T; L^{1/\lambda}(\Omega))$ we obtain $L_h(t) \leq c ||\mathbf{u}(t)||^2_{W_0^{1,2}(\Omega)}$. In view of the energy estimate [\(19\)](#page-6-1) the right side of this inequality is integrable over $(0, T)$. Hence the sequence L_h has an integrable majorant. Applying the Lebesgue dominant convergence Theorem we arrive at the relation

$$
\int_0^T (J_h(t) + L_h(t)) dt \to 0 \text{ as } h \to 0.
$$

From this and [\(99\)](#page-30-1) we conclude that $I_h \to 0$ as $h \to 0$. Next, Propositions [6.1](#page-24-2) and [6.2](#page-28-3) imply

$$
\int_{Q_T} \zeta p \big[\zeta \psi(\varrho)\big]_h \, dx dt \le I_h + c(\zeta, \lambda) E. \tag{101}
$$

The functions $[\zeta \psi(\varrho)]_h$ are nonnegative and converge a.e. in Q_T to $\zeta \psi(\varrho)$. Letting $h \to 0$ in [\(101\)](#page-32-1) and applying the Fatou Theorem we arrive at the inequality

$$
\int_{Q_T} \zeta^2 p\psi(\varrho) \, dx dt \le c(\zeta, \lambda) E.
$$

It remains to note that $\psi(\varrho) \geq c\varrho^{\lambda} - 1$ and the theorem follows.

7 Proof of Theorems [1.1](#page-2-2) and [1.2](#page-3-1)

7.1 Proof of Theorem [1.1](#page-2-2)

By Proposition [3.1,](#page-6-0) for every $\varepsilon > 0$ regularized problem [\(18\)](#page-5-1) has a solution $(\varrho_{\epsilon}, \mathbf{u}_{\epsilon})$ which admits estimates [\(19\)](#page-6-1) and satisfies integral identities [\(20\)](#page-6-2), [\(21\)](#page-6-4).

Lemma 7.1. Let $\lambda \in [0, 1/6)$ and $\Omega' \in \Omega$. Then there are exponent $r, p \in \Omega$. $(2, \infty)$ and $q, s, z \in (1, \infty)$ such that

$$
\|\varrho_{\varepsilon}\|_{L^{1+\lambda}(\Omega'\times(0,T))} + \varepsilon \int_{\Omega'\times(0,T)} \varrho_{\varepsilon}^{\gamma+\lambda} \le C,
$$
\n(102)

$$
\|\varrho_{\varepsilon}\|_{L^{r}(0,T;L^{s}(\Omega'))}+\|\varrho_{\varepsilon}\mathbf{u}_{\varepsilon}\|_{L^{p}(0,T;L^{z}(\Omega'))}+\|\varrho_{\varepsilon}|\mathbf{u}_{\varepsilon}|^{2}\|_{L^{q}(\Omega'\times(0,T))}\leq C,\qquad(103)
$$

where C is independent of ε . Moreover, the sequences $\varrho_{\varepsilon} \mathbf{u}_{\varepsilon}$ and ϱ_{ε} are equiintegrable.

Proof. Fix an arbitrary $\Omega' \in \Omega$. Choose a nonnegative function $\zeta \in C_0^{\infty}(\mathbb{R}^2)$ with the properties: ζ is compactly supported in Ω and $\zeta = 1$ in Ω' . Notice that λ , ζ , and ϱ_{ε} meet all requirements of Theorem [6.1.](#page-21-2) Hence $p(\varrho_{\varepsilon})$ satisfy inequality [\(66\)](#page-21-1). It is easy to see that estimates [\(102\)](#page-32-2) follows from [\(66\)](#page-21-1) and the formula $p(\varrho) = \varrho_{\varepsilon} + \varrho_{\varepsilon}^{\gamma}$. Next choose an arbitrary $r \in (2,\infty)$ and set $s = r/(r - \lambda) > 1$ and $\alpha = (1 + \lambda)/r \in (0, 1)$. Obviously

$$
(1 - \alpha)/\infty + \alpha/(1 + \lambda) = 1/r, \quad 1 - \alpha + \alpha/(1 + \lambda) = 1/s.
$$

From this, inequality [\(102\)](#page-32-2), and the interpolation inequality we obtain

$$
\|\varrho_{\varepsilon}\|_{L^r(0,T;L^s(\Omega'))} \le \|\varrho_{\varepsilon}\|_{L^{\infty}(0,T;L^1(\Omega'))}^{1-\alpha} \|\varrho_{\varepsilon}\|_{L^{1+\lambda}(0,T;L^{1+\lambda}(\Omega'))}^{\alpha} < C,
$$
 (104)

which gives the estimate [\(103\)](#page-32-3) for ϱ_{ε} . In order to estimate the quantity $\varrho_{\varepsilon}|\mathbf{u}_{\varepsilon}|$, represent it in the form

$$
\varrho_{\varepsilon}|\mathbf{u}_{\varepsilon}| = \varrho_{\varepsilon}^{\mu}(\varrho_{\varepsilon}|\mathbf{u}_{\varepsilon}|^{2})^{\beta}|\mathbf{u}_{\varepsilon}|^{\nu}.
$$
\n(105)

Let us show that there exist exponents $\mu \in (1/2, 1), \beta, \nu \in (0, 1)$ and $p, z, \sigma \in$ $(1, \infty)$ with the properties

$$
\beta = 1 - \mu, \quad \nu + 2\beta = 1, \quad \text{i.e.,} \quad \nu = 2\mu - 1, \n\mu/r + \nu/2 = 1/p, \quad \mu/s + \beta + \nu/\sigma = 1/z.
$$
\n(106)

To this end notice that these relations can be equivalently rewritten in the form

$$
1/p = (2\mu - 1)/2 + \mu/r, \ 1/z = 1 + \mu(1/s - 1) + (2\mu - 1)/\sigma, \ \beta = 1 - \mu, \ \nu = 2\mu - 1,
$$

which gives $\mu = (1/2 + 1/p)(1 + 1/r)^{-1}$. The inequalities $1/2 < \mu < 1$ are fulfilled if and only if $2r/(r+2) < p < 2r$. Since $r > 2$, there exists $p > 2$ satisfying these inequalities. On the other hand, it follows from $s > 1$ that $0 < 1 + \mu(1/s - 1) < 1$. Hence there is $\sigma \in (1, \infty)$ such that $z \in (1, \infty)$. This completes the proof of the existence of exponents satisfying [\(106\)](#page-33-0). The Hölder inequality, estimate (102) , and energy estimate (19) imply

$$
\|\varrho_{\varepsilon}\mathbf{u}_{\varepsilon}\|_{L^{p}(0,T;L^{s}(\Omega'))} \leq
$$

$$
\|\varrho_{\varepsilon}\|_{L^{r/\mu}(0,T;L^{s/\mu}(\Omega'))} \|\varrho_{\varepsilon}^{\beta}|\mathbf{u}_{\varepsilon}|^{2\beta} \|_{L^{\infty}(0,T;L^{1/\beta}(\Omega'))} \|\|\mathbf{u}_{\varepsilon}\|^{p} \|_{L^{2/\nu}(0,T;L^{\sigma/\nu}(\Omega'))}
$$

$$
= \|\varrho_{\varepsilon}\|_{L^{r}(0,T;L^{s}(\Omega'))}^{\mu} \|\varrho_{\varepsilon}\mathbf{u}_{\varepsilon}\|^{2} \|\varrho_{\infty}(0,T;L^{1}(\Omega'))} \|\mathbf{u}_{\varepsilon}\|_{L^{2}(0,T;L^{\sigma}(\Omega'))}^{\nu} \leq C \|\mathbf{u}_{\varepsilon}\|_{L^{2}(0,T;L^{\sigma}(\Omega'))}^{\nu}.
$$

Recall that the embedding $W_0^{1,2}$ $L^{\sigma}(\Omega) \hookrightarrow L^{\sigma}(\Omega)$ is bounded for every $\sigma \in$ $[1, \infty)$. It follows from this and energy estimate [\(19\)](#page-6-1) that

$$
\|\mathbf{u}_{\varepsilon}\|_{L^2(0,T;L^{\sigma}(\Omega'))}\leq c\|\mathbf{u}_{\varepsilon}\|_{L^2(0,T;W_0^{1,2}(\Omega))}\leq C,
$$

which leads to the estimate for $\rho_{\varepsilon} u_{\varepsilon}$ in [\(103\)](#page-32-3). Now our task is to estimate the kinetic energy density $\varrho_{\varepsilon}|\mathbf{u}_{\varepsilon}|^2$ Since $p>2$ and $z>1$ there are $\kappa_1, \kappa_2, \omega \in$

 $(1, \infty)$ such that $1/p + 1/2 = 1/\kappa_1$ and $1/z + 1/\omega = 1/\kappa_2$. Set $q = \min\{\kappa_i\}$. Applying the Hölder inequality and using estimate [\(103\)](#page-32-3) for $\rho_{\varepsilon} \mathbf{u}_{\varepsilon}$ we obtain

$$
\|\varrho_{\varepsilon}|\mathbf{u}_{\varepsilon}|^{2}\|_{L^{q}(\Omega'\times(0,T))}\leq C\|\varrho_{\varepsilon}|\mathbf{u}_{\varepsilon}|^{2}\|_{L^{\kappa_{1}}(0,T;L^{\kappa_{2}}(\Omega'))}
$$

$$
\leq c\|\varrho_{\varepsilon}\mathbf{u}_{\varepsilon}\|\|_{L^{p}(0,T;L^{z}(\Omega'))}\|\mathbf{u}_{\varepsilon}\|_{L^{2}(0,T;L^{\omega}(\Omega'))}\leq C\|\mathbf{u}_{\varepsilon}\|_{L^{2}(0,T;W_{0}^{1,2}(\Omega))}\leq C.
$$

This completes the proof of [\(103\)](#page-32-3). It remains to show that the sequences ϱ_{ε} and $\varrho_{\varepsilon}u_{\varepsilon}$ are equi-integrable in Q_T . By energy estimate [\(19\)](#page-6-1), the sequence ϱ_{ε} log(1 + ϱ_{ε}) is bounded in $L^1(Q_T)$. Hence this sequence is equi-integrable. This means that for every $\varkappa > 0$ there is $\delta(\varkappa)$, depending on ϵ only, such that the inequality

$$
\int_A \varrho_\varepsilon\,dxdt \le \varkappa
$$

hold for every $A \subset Q_T$ such that meas $A < \delta(\varkappa)$. By the Cauchy inequality and energy estimate [\(19\)](#page-6-1), we have

$$
\int_A \varrho_\varepsilon |\mathbf{u}_\varepsilon| \, dx dt \le \Big(\int_A \varrho_\varepsilon \, dx dt \Big)^{1/2} \Big(\int_{Q_T} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 \, dx dt \Big)^{1/2} \le E\sqrt{\varkappa}.
$$

which yields the equi-integrability of the sequence $\varrho_{\varepsilon} \mathbf{u}_{\varepsilon}$.

Let us turn to the proof of Theorem 1.1. It follows from energy estimates (19) and Lemma 7.1 that, after passing to a subsequence if necessary, we can assume that there are integrable functions
$$
\varrho
$$
, **u**, $\overline{\varrho} \overline{u}$, and $\overline{\varrho} \overline{u} \otimes \overline{u}$ with the properties

$$
\varrho_{\varepsilon} \rightharpoonup \varrho, \quad \varrho_{\varepsilon} \mathbf{u}_{\varepsilon} \rightharpoonup \overline{\varrho} \overline{\mathbf{u}} \quad \text{weakly in } L^{1}(\Omega \times (0, T)),
$$
\n
$$
\mathbf{u}_{\varepsilon} \rightharpoonup \mathbf{u} \quad \text{weakly in } L^{2}(0, T; W_{0}^{1,2}(\Omega)), \tag{107}
$$

For every compact set $\Omega' \subset \Omega$, we have

$$
\varrho_{\varepsilon} \rightharpoonup \varrho \text{ weakly in } L^{r}(0,T;L^{s}(\Omega')), \ \varrho_{\varepsilon} \mathbf{u}_{\varepsilon} \rightharpoonup \overline{\varrho \mathbf{u}} \text{ weakly in } L^{p}(0,T;L^{z}(\Omega')),
$$
\n
$$
\varrho_{\varepsilon} \mathbf{u}_{\varepsilon} \otimes \mathbf{u}_{\varepsilon} \rightharpoonup \overline{\varrho \mathbf{u} \otimes \mathbf{u}} \text{ weakly in } L^{q}(\Omega' \times (0,T)).
$$
\n(108)

Here $r, p \in (2, \infty)$ and $q, s, z \in (1, \infty)$ are given by Lemma [7.1.](#page-32-4) It follows from energy estimate [\(19\)](#page-6-1) and convexity of the function $\varrho \log(1+\varrho)$ that ϱ and **u** satisfies inequalities [\(5\)](#page-2-3). Moreover, $\varrho \in L^r(0,T; L^s(\Omega'))$, $\overline{\varrho}$ **u** \in

$$
\Box
$$

 $L^p(0,T; L^z(\Omega'))$ and $\overline{\varrho \mathbf{u} \otimes \mathbf{u}} \in L^q(\Omega' \times (0,T))$ for ever $\Omega' \Subset \Omega$. Finally notice that in view of estimates [\(102\)](#page-32-2) we have

$$
\varepsilon \int_{\Omega' \times (0,T)} \varrho \varepsilon^{\gamma} \to 0 \text{ as } \varepsilon \to 0.
$$

Substituting $(\varrho_{\varepsilon}, \mathbf{u}_{\varepsilon})$ and $\varphi(\varrho_{\varepsilon}) := \varrho_{\varepsilon}$ into [\(20\)](#page-6-2), [\(22\)](#page-6-3) and letting $\varepsilon \to 0$ we obtain that the integral identities

$$
\int_{Q_T} \left(\overline{\varrho} \overline{\mathbf{u}} \cdot \partial_t \xi + \overline{\varrho} \overline{\mathbf{u}} \otimes \overline{\mathbf{u}} : \nabla \xi + \varrho \operatorname{div} \xi - \mathbb{S}(\mathbf{u}) : \nabla \xi \right) dx dt
$$
\n
$$
+ \int_Q \varrho \mathbf{f} \cdot \xi \, dx dt + \int_\Omega (\varrho_0 \mathbf{u}_0 \cdot \xi)(x, 0) dx = 0 \quad (109)
$$
\n
$$
\int (\varrho \partial_t \psi + \overline{\varrho} \overline{\mathbf{u}} \cdot \nabla \psi) dx dt + \int \varrho_0(x) \psi(x, 0) dx = 0 \quad (110)
$$

$$
\int_{Q_T} \left(\varrho \partial_t \psi + \overline{\varrho \mathbf{u}} \cdot \nabla \psi \right) dx dt + \int_{\Omega} \varrho_0(x) \psi(x, 0) dx = 0 \tag{110}
$$

hold for all vector fields $\xi \in C^{\infty}(Q_T)$ equal 0 in a neighborhood of $\partial\Omega \times$ [0, T] and of the top $\Omega \times \{t = T\}$ and for all $\psi \in C^{\infty}(Q_T)$ vanishing in a neighborhood of the top $\Omega \times \{t = T\}$. It remains to prove that

$$
\overline{\varrho \mathbf{u}} = \varrho \mathbf{u}, \quad \overline{\varrho \mathbf{u} \otimes \mathbf{u}} = \varrho \mathbf{u} \otimes \mathbf{u} \quad \text{a.e. in } Q_T \tag{111}
$$

The proof is standard, see [\[5\]](#page-38-4). We begin with the observation that ϱ_{ε} and \mathbf{u}_{ε} satisfies the equations

$$
\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) = \text{ div } \left(\mathbf{S}(\mathbf{u}_\varepsilon) - \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \right) - \nabla p(\varrho_\varepsilon) + \varrho_\varepsilon \mathbf{f}, \ \partial_t \varrho_\varepsilon = - \text{ div } \big(\varrho_\varepsilon \mathbf{u}_\varepsilon \big), \tag{112}
$$

which are understood in the sense of the distribution theory.Notice that in view of the energy estimate [\(19\)](#page-6-1), the sequences ϱ_{ε} , $\mathbf{S}(\mathbf{u}_{\varepsilon})$, $\varrho_{\varepsilon}\mathbf{u}_{\varepsilon} \otimes \mathbf{u}_{\varepsilon}$, $p(\varrho_{\varepsilon})$ are bounded in the space $L^2(0,T;L^1(\Omega))$. Choose an arbitrary function $\xi \in C_0^{\infty}(Q_T)$. Since the embedding $W_0^{3,2}$ $O_0^{3,2}(\Omega) \rightarrow C_0^1(\Omega)$ is bounded, it fol-lows from [\(112\)](#page-35-0) that the sequences $\partial_t(\xi_{\ell_{\varepsilon}})$ and $\partial_t(\xi_{\ell_{\varepsilon}}\mathbf{u}_{\varepsilon})$ are bounded in $L^2(0,T;W^{-3,2}(\Omega))$. On the other hand, Lemma [7.1](#page-32-4) implies that the sequences $\xi \varrho_{\varepsilon}$ and $\xi \varrho_{\varepsilon} \mathbf{u}_{\varepsilon}$ are bounded in $L^r(0,T;L^s(\Omega))$ and $L^p(0,T;L^z(\Omega))$ respectively. Notice that $r, p > 2$ and the embedding $W^{-1,2}(\Omega) \hookrightarrow L^s(\Omega)$, $W^{-1}(\Omega) \hookrightarrow L^z(\Omega)$ is compact for $s, z > 1$. Applying the Dubinskii-Lions compactness Theorem we conclude that the sequences $\xi_{\ell_{\varepsilon}}$ and $\xi_{\ell_{\varepsilon}}\mathbf{u}_{\varepsilon}$ are relatively compact in $L^2(0,T;W^{-1,2}(\Omega))$. From this and [\(107\)](#page-34-0) we obtain

$$
\int_{Q_T} \xi \varrho_{\varepsilon} \mathbf{u}_{\varepsilon} dxdt \to \int_{Q_T} \xi \varrho \mathbf{u} dxdt, \quad \int_{Q_T} \xi \varrho_{\varepsilon} \mathbf{u}_{\varepsilon} \otimes \mathbf{u}_{\varepsilon} dxdt \to \int_{Q_T} \xi \varrho \mathbf{u} \otimes \mathbf{u} dxdt
$$

as $\varepsilon \to 0$, which yields [\(111\)](#page-35-1). This completes the proof of Theorem [1.1.](#page-2-2)

7.2 Proof of Theorem [1.2](#page-3-1)

It suffices to note that estimate [\(6\)](#page-3-2) follows directly from Theorem [4.1,](#page-8-2) and estimates [\(8\)](#page-3-3) follow from Proposition [5.1](#page-16-4) and Theorem [6.1.](#page-21-2)

A Proof of Lemmas [2.1](#page-4-0) and [2.2](#page-4-1)

Proof of Lemma [2.1](#page-4-0) Introduce the polar coordinates $\lambda = |\xi| \in \mathbb{R}^+$ and $\boldsymbol{\omega} = |\xi|^{-1} \xi \in \mathbb{S}^1$. Applying the Fubini Theorem we obtain

$$
\mathfrak{F}g(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-ix\cdot\xi} g(x) dx = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-i\lambda \omega \cdot x} g(x) dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda \tau} \left\{ \int_{\omega \cdot x = \tau} g(x) dx \right\} d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda \tau} \Phi(\omega, \tau) d\tau = \frac{1}{\sqrt{2\pi}} \mathfrak{F}_{\lambda} \Phi(\omega, \lambda),
$$

where \mathfrak{F}_{λ} is the Fourier transform with respect to $\tau.$ We thus get

$$
|\mathfrak{F}g(\lambda\omega)|^2 = \frac{1}{2\pi} |\mathfrak{F}_{\lambda}\Phi(\boldsymbol{\omega},\lambda)|^2
$$
 (113)

Since Φ is a real valued function, the Plancherel identity yields

$$
\int_0^\infty |\mathfrak{F}_{\lambda}\Phi(\boldsymbol{\omega},\lambda)|^2 d\lambda = \frac{1}{2} \int_{-\infty}^\infty |\mathfrak{F}_{\lambda}\Phi(\boldsymbol{\omega},\lambda)|^2 d\lambda = \frac{1}{2} \int_{-\infty}^\infty |\Phi(\boldsymbol{\omega},\tau)|^2 d\tau.
$$

Integrating both sides of [\(113\)](#page-36-1) by λ we conclude that

$$
\int_0^\infty |\mathfrak{F}g(\lambda\boldsymbol{\omega})|^2 d\lambda = \frac{1}{4\pi} \int_{-\infty}^\infty |\Phi(\boldsymbol{\omega}, \tau)|^2 d\tau
$$

It follows that

$$
\int_{\mathbb{R}^2} |\xi|^{-1} |\mathfrak{F}g|^2 d\xi = \int_{\mathbb{S}^1} \int_{0}^{\infty} \frac{1}{\lambda} |\mathfrak{F}g(\lambda\omega)|^2 \lambda d\lambda d\omega = \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \int_{0}^{\infty} |\mathfrak{F}g(\lambda\omega)|^2 d\lambda d\omega = \frac{1}{4\pi} \int_{\mathbb{S}^1} \int_{-\infty}^{\infty} |\Phi(\omega,\tau)|^2 d\tau d\omega.
$$

Recalling expression [\(9\)](#page-3-4) for H^s - norm we obtain the desired estimate [\(14\)](#page-4-4).

Proof of Lemma [2.2](#page-4-1) Let $s > 1/2$. It suffices to prove that

$$
||uv||_{H^{1/2}(\mathbb{R}^2)} \leq c||u||_{H^1(\mathbb{R}^2)} ||v||_{H^s(\mathbb{R}^2)}
$$

for all $u \in H^1(\mathbb{R}^2)$ and for all $v \in H^s(\mathbb{R}^2)$. Choose an arbitrary $u \in H^1(\mathbb{R}^2)$ and consider the linear operator $\mathbf{U}: v \mapsto u v$. Set $\delta = s-1/2 > 0$. Recall that $H^s(\mathbb{R}^2)$ coincides with $W^{s,2}(\mathbb{R}^2)$. Since the embedding $H^{1+\delta}(\mathbb{R}^2) \hookrightarrow L^{\infty}(\mathbb{R}^2)$ is bounded, we have

$$
||uv||_{L^{2}(\mathbb{R}^{2})} \leq ||v||_{L^{\infty}(\mathbb{R}^{2})} ||u||_{L^{2}(\mathbb{R}^{2})} \leq c||v||_{H^{1+\delta}(\mathbb{R}^{2})} ||u||_{H^{1}(\mathbb{R}^{2})}
$$
(114)

Since the embedding $H^{1+\delta}(\mathbb{R}^2) \hookrightarrow W^{1,2/(1-\delta)}(\mathbb{R}^2)$ and $H^1(\mathbb{R}^2) \hookrightarrow L^{2/\delta}(\mathbb{R}^2)$ is bounded, see [\[1\]](#page-38-8), thm. 7.57, we have

$$
\|\nabla(uv)\|_{L^{2}(\mathbb{R}^{2})} \leq \|v\|_{L^{\infty}(\mathbb{R}^{2})} \|\nabla u\|_{L^{2}(\mathbb{R}^{2})} + \|u\nabla v\|_{L^{2}(\Omega)} \leq
$$

\n
$$
c\|v\|_{H^{1+\delta}} \|u\|_{H^{1}(\mathbb{R}^{2})} + \|\nabla v\|_{L^{2/(1-\delta)}(\mathbb{R}^{2})} \|u\|_{L^{2/\delta}(\mathbb{R}^{2})} \leq (115)
$$

\n
$$
c\|v\|_{H^{1+\delta}(\mathbb{R}^{2})} + \|v\|_{W^{1,2/(1-\delta)}(\mathbb{R}^{2})}) \|u\|_{H^{1}(\mathbb{R}^{2})} \leq c\|v\|_{H^{1+\delta}(\mathbb{R}^{2})} \|u\|_{H^{1}(\mathbb{R}^{2})}.
$$

Combining [\(114\)](#page-37-0)and [\(115\)](#page-37-1) we obtain

$$
\|\mathbf{U}v\|_{H^1(\mathbb{R}^2)} \le c\|u\|_{H^1(\mathbb{R}^2)}\|v\|_{H^{1+\delta}(\mathbb{R}^2)}.\tag{116}
$$

On the other hand, the boundedness of the embedding $H^{\delta} \hookrightarrow L^{2/(1-\delta)}(\mathbb{R}^2)$ implies

$$
||uv||_{L^2(\mathbb{R}^2)} \leq ||v||_{L^{2/(1-\delta)}(\mathbb{R}^2)} ||u||_{L^{2/\delta}(\mathbb{R}^2)} \leq c||v||_{H^{\delta}(\mathbb{R}^2)} ||u||_{H^1(\mathbb{R}^2)},
$$

which yields the estimate

$$
||\mathbf{U}v||_{L^2(\mathbb{R}^2)} \leq c||u||_{H^1(\mathbb{R}^2)}||v||_{H^{\delta}(\mathbb{R}^2)}.
$$

From this and [\(116\)](#page-37-2) we conclude that **U** is a bounded operator from $H^{\delta}(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$ and from $H^{1+\delta}(\mathbb{R}^2)$ to $H^1(\mathbb{R}^2)$. Moreover, its norm does not exceed $c||u||_{H^1(\mathbb{R}^2)}$. Applying the interpolation theorem, [\[2\]](#page-38-9) Sec. 2.4, Sec. 6.4 Thm. 6.4.5, and noting that $1/2 + \delta = s$ we obtain that the desired inequality

$$
||uv||_{H^{1/2}(\mathbb{R}^2)} \equiv ||\mathbf{U}v||_{H^{1/2}(\mathbb{R}^2)} \le c||u||_{H^1(\mathbb{R}^2)} ||v||_{H^{(\delta+1+\delta)/2}(\mathbb{R}^2)} = c||u||_{H^1(\mathbb{R}^2)} ||v||_{H^s(\mathbb{R}^2)}
$$

holds for all $u \in H^1(\mathbb{R}^2)$ and all $v \in H^s(\mathbb{R}^2)$.

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