# On the Metric s-t Path Traveling Salesman Problem

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#### Abstract

We study the metric s-t path Traveling Salesman Problem (TSP). [An, Kleinberg, and Shmoys, STOC 2012] improved on the long standing  $\frac{5}{3}$ -approximation factor and presented an algorithm that achieves an approximation factor of  $\frac{1+\sqrt{5}}{2}\approx 1.61803$ . Later [Sebő, IPCO 2013] further improved the approximation factor to  $\frac{8}{5}$ . We present a simple, self-contained analysis that unifies both results; our main contribution is a unified correction vector. Additionally, we compare two different linear programming (LP) relaxations of the s-t path TSP, namely, the path version of the Held-Karp LP relaxation for TSP and a weaker LP relaxation, and we show that both LPs have the same (fractional) optimal value. Also, we show that the minimum cost of integral solutions of the two LPs are within a factor of  $\frac{3}{2}$  of each other. Furthermore, we prove that a half-integral solution of the stronger LP-relaxation of cost c can be rounded to an integral solution of cost at most  $\frac{3}{2}c$ . Finally, we give an instance that presents obstructions to two natural methods that aim for an approximation factor of  $\frac{3}{2}$ .

# 1 Introduction

The metric Traveling Salesman Problem (TSP) is a celebrated problem in Combinatorial Optimization, see [Sch03, Chapter 58], [BB08]. One important variant of TSP is the (metric) s-t path TSP. Let G be a complete graph G with nonnegative metric edge costs c, i.e., c satisfies the triangle inequality. Given two fixed vertices s, t in G, the s-t path TSP is to find a minimum-cost Hamiltonian path from s to t in G.

Hoogeveen [Hoo91] gave an s-t path TSP variant of Christofides' approximation algorithm for the TSP [Chr76], and obtained an approximation factor of  $\frac{5}{3}$ . There was no improvement in this approximation factor for

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over two decades until An, Kleinberg, and Shmoys [AKS12] improved the approximation factor to  $\frac{1+\sqrt{5}}{2} \approx 1.61803$ . One of the key new contributions of [AKS12] is to design and analyse a randomized version of Christofides' algorithm. The analysis introduced the notion of a correction vector for the s-t path TSP. Most recently, Sebő [Seb13] further improved the analysis and obtained a better approximation factor of  $\frac{8}{5}$ . [Seb13] introduced a correction vector different from that of [AKS12], and this is one reason why the analysis in [Seb13] gives a better approximation factor. Informally speaking, a better correction vector provides a better approximation factor. In this paper, we give a unified presentation of the results from both [AKS12] and [Seb13] by introducing a new correction vector that we call the unified correction vector. Our correction vector is simple and it leads to short derivations of the approximation factors of both [AKS12] and [Seb13]. The difference between our correction vector and the previous ones is that it assigns the value one to the minimum-cost edge in each so-called  $\tau$ -narrow cut, whereas the correction vectors used in [AKS12] and [Seb13] are fractional on each  $\tau$ -narrow cut. We mention that Vygen's [Vyg12] comprehensive recent survey discusses the common points of the analysis of [AKS12] and [Seb13], and the survey sketches short proofs of both approximation factors; however, [Vyg12] uses the same correction vectors as [AKS12] and [Seb13].

An et al. [AKS12] and Sebő [Seb13] use two different LP relaxations of the s-t path TSP in their algorithms. [AKS12] uses the path version of the Held-Karp LP relaxation for TSP, whereas [Seb13] uses a weaker LP relaxation. This motivates a comparison of these two LP relaxations. We mention that Sebő proves an approximation factor of  $\frac{8}{5}$  for a more general problem, namely, the connected T-join problem, and the LP in his paper is a relaxation of this problem. We show that both LPs for the s-t path TSP have the same (fractional) optimal value. Also, we show that the minimum cost of integral solutions of the two LPs are within a factor of  $\frac{3}{2}$  of each other; moreover, we present an example to show that the factor of  $\frac{3}{2}$  is tight. We prove this result by showing that a half-integral solution of the stronger LP-relaxation of cost c can be rounded to an integral solution of cost at most  $\frac{3}{2}c$ .

For the s-t path TSP, it is known that the integrality ratio of the path version of the Held-Karp LP relaxation has a lower bound of  $\frac{3}{2}$ . All of the algorithms mentioned above are LP-based. This leads to the best known upper bound  $\frac{8}{5}$  on the integrality ratio of the LP relaxation. A natural open question is to close this gap by designing an LP-based  $\frac{3}{2}$ -approximation algorithm for the s-t path TSP. Given a connected graph H with unit edge costs and

two fixed vertices s and t, the s-t path graph-TSP is to find a minimum-cost Hamiltonian path from s to t in the metric completion of H. For this critical special case of the s-t path TSP, the integrality ratio of the corresponding LP relaxation has been resolved already. The first  $\frac{3}{2}$ -approximation algorithm was given by Sebő and Vygen [SV14] using ear decompositions. Gao [Gao13] designed another, conceptually simpler, LP-based  $\frac{3}{2}$ -approximation algorithm. The analysis of the  $\frac{3}{2}$ -approximation factor of [Gao13] uses the graphic property only for one point: to guarantee that the cost of a special spanning tree constructed in the algorithm is at most the optimum of the LP relaxation. A natural question is whether we can extend this graphic LP-based approximation algorithm and analysis to the general metric case. Unfortunately, we present an instance that shows that that is not possible. Moreover, our instance also illustrates that probabilistic methods are relevant for the analysis of improved LP-based approximation algorithms. This instance may shed some light on how to design a better approximation algorithm for the s-t path TSP.

The paper is organized as follows. Section 2 has some notation and basic results. Section 3 presents our unified correction vector. Section 4 shows the relationship of two different LP relaxations of the s-t path TSP. Section 5 discusses an instance that points to some of the obstructions for obtaining better approximation factors.

### 2 Preliminaries

Let G = (V, E) be a complete graph. Let s, t be two fixed vertices in G. We call a nonempty, proper subset of vertices S a cut; thus,  $\emptyset \subsetneq S \subsetneq V$ . In particular, if  $|S \cap \{s, t\}| = 1$ , then we call S an s-t cut. For  $S \subseteq V$ , let  $\delta(S)$  denote the set of edges that have one end in S, thus,  $\delta(S) = \{(u, v) \in E : u \in S, v \notin S\}$ . If  $S = \{v\}$ , then we use  $\delta(v)$  instead of  $\delta(\{v\})$ . Let E(S) denote the set of edges induced by S, thus,  $E(S) = \{(u, v) \in E : u, v \in S\}$ . For any two sets A and B, we use  $A \setminus B$  to denote  $\{a \in A : a \notin B\}$ . For a vector  $x \in \mathbb{R}^A$ , we define  $x(D) = \sum_{e \in D} x(e)$  for any subset D of A. When there is no risk of confusion, we will use the same notation H for a subgraph H and its edge set E(H).

For any probabilistic event A, we use Pr(A) to denote the probability of occurrence of A. For a random variable R, the expectation of R is denoted by  $\mathbb{E}(R)$ .

#### 2.1 Linear programs

The path version of the Held-Karp relaxation for the s-t path TSP is defined as follows:

(L.P.1) minimize: 
$$\sum_{e \in E} c_e x_e$$
 subject to:  $x(\delta(s)) = x(\delta(t)) = 1$  
$$x(\delta(v)) = 2 \qquad \forall v \neq s, t$$
 
$$x(\delta(S)) \geq 1 \qquad \forall s\text{-}t \text{ cut } S$$
 
$$x(\delta(S)) \geq 2 \qquad \forall \emptyset \subsetneq S \subsetneq V, |S \cap \{s, t\}| \text{ even}$$
 
$$1 \geq x_e \geq 0 \qquad \forall e \in E$$

The spanning tree polytope is shown as follows:

(**L.P.2**) minimize: 
$$\sum_{e \in E} c_e x_e$$
 subject to: 
$$x(E) = |V| - 1$$
 
$$x(E(S)) \le |S| - 1 \quad \forall \emptyset \subsetneq S \subsetneq V$$
 
$$x_e \ge 0 \qquad \forall e \in E$$

**Lemma 2.1** Every solution x of (L.P.1) lies in the spanning tree polytope (L.P.2).

**Proof.** By the degree constraint for each vertex in (L.P.1), we have x(E) = |V| - 1. Now consider the second set of constraint in (L.P.2). If  $|S \cap \{s,t\}|$  is even, by the degree and cut constraints in (L.P.1),  $x(E(S)) = \frac{\sum_{v \in S} x(\delta(v)) - x(\delta(S))}{2} \le \frac{2|S| - 2}{2} = |S| - 1$ . Otherwise,  $|S \cap \{s,t\}| = 1$ . Similarly,  $x(E(S)) = \frac{\sum_{v \in S} x(\delta(v)) - x(\delta(S))}{2} \le \frac{(2|S| - 1) - 1}{2} = |S| - 1$ . This completes the proof.

#### **2.2** *T*-joins

Let T be a nonempty subset of V with |T| even. For  $F \subseteq E$ , if the set of odd degree vertices of the graph (V, F) is T, then we call F a T-join. For any  $\emptyset \subsetneq S \subseteq V$ , if  $|S \cap T|$  is odd (even), then we call S a T-odd cut (T-even cut). The following LP formulates the problem of finding a T-join of minimum cost:

(**L.P.3**) minimize : 
$$\sum_{e \in E} c_e x_e$$
 subject to :  $x(\delta(S)) \ge 1 \quad \forall T\text{-odd } S$   $x_e \ge 0 \quad \forall e \in E$ 

**Lemma 2.2** [EJ01] The optimal value of (L.P.3) is the same as the minimum cost of a T-join.

Let K be a spanning tree. The set of wrong degree vertices of K is defined as  $\{v \in \{s,t\} : |\delta(v) \cap K| \text{ even }\} \cup \{v \in V \setminus \{s,t\} : |\delta(v) \cap K| \text{ odd }\}.$ 

**Lemma 2.3** [AKS12] Let T be the set of wrong degree vertices of a spanning tree K. Let S be an s-t cut. If S is T-odd, then  $|\delta(S) \cap K|$  is even.

The proof can be also found in [CFG12, Lemma 2.1]. But for the sake of completeness, we present a proof here.

**Proof.** Since  $\sum_{v \in S} |\delta(v) \cap K| = 2|E(S) \cap K| + |\delta(S) \cap K|$ , we have  $|\delta(S) \cap K|$  has the same parity as  $\sum_{v \in S} |\delta(v) \cap K|$ . Without loss of the generality, we assume  $s \in S, t \notin S$ . By the definition of T, we know that  $(S \setminus \{s\}) \cap T$  is the set of vertices v in  $S \setminus \{s\}$  such that  $|\delta(v) \cap K|$  is odd. If  $|\delta(s) \cap K|$  is odd, then  $s \notin T$ . In this case, since S is T-odd,  $|(S \setminus \{s\}) \cap T|$  is odd. Hence, we have an even number of vertices v in S such that  $|\delta(v) \cap K|$  is odd, which implies that  $\sum_{v \in S} |\delta(v) \cap K|$  is even. Otherwise,  $|\delta(s) \cap K|$  is even. Then,  $s \in T$ . This implies that  $|(S \setminus \{s\}) \cap T|$  is even. Similarly,  $\sum_{v \in S} |\delta(v) \cap K|$  is even.

# 2.3 Polyhedra and convex decomposition

Let

$$\mathcal{P} := \{x : Ax \le b\}$$
 where  $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$ .

Let x' be a feasible solution of  $\mathcal{P}$ . For a constraint  $a_i^{\mathsf{T}} x \leq b_i$  in  $\mathcal{P}$ , we say x' is *tight* at this constraint if  $a_i^{\mathsf{T}} x' = b_i$ . Let  $x_1, x_2$  be two distinct feasible solutions of  $\mathcal{P}$ . If there exists a  $0 < \lambda < 1$  and  $y \in \mathcal{P}$  such that  $\lambda x_1 + (1 - \lambda)y = x_2$ , we say  $x_1$  is in some convex decomposition of  $x_2$  in  $\mathcal{P}$ .

From the geometry of polyhedra, we have the following characterization of the convex decompositions.

**Lemma 2.4** The solution  $x_1$  is in some convex decomposition of  $x_2$  in  $\mathcal{P}$  if and only if  $x_1$  is tight at the constraints of  $\mathcal{P}$  where  $x_2$  is tight.

#### 2.4 Christofides' algorithm for s-t path TSP

Hoogeveen [Hoo91] gave a variant of Christofides' algorithm to achieve the first approximation factor of  $\frac{5}{3}$  for the s-t path TSP.

#### Christofides' algorithm for s-t path TSP

Compute a minimum-cost spanning tree  $J^*$ . Let T be the set of wrong degree vertices of  $J^*$ . Find a minimum-cost T-join  $F^*$ . Then, the union  $J^*\dot{\cup}F^*$  of  $J^*$  and  $F^*$  (that keeps the duplicated edges) forms a connected graph that has even degree at all nodes except s and t. One can then take the Eulerian traversal that starts at s and ends at t, and shortcut it, to obtain an s-t path visiting all vertices of no greater cost.

**Theorem 2.5** [Hoo91] Christofides' algorithm for s-t path TSP achieves an approximation factor of  $\frac{5}{3}$ .

For the sake of completeness, we present a nice proof from Sebő and Vygen [SV14].

**Proof.** Let  $P^*$  be an optimal solution of s-t path TSP. Let  $T, J^*, F^*$  be as in the algorithm. Let R be the s-t path in  $J^*$  and  $F_{P^*}$  be the T-join in  $P^*$ . Since  $P^*$  is a spanning tree, we know  $c(J^*) \leq c(P^*)$ . So, we only need to prove  $c(F^*) \leq \frac{2}{3}c(P^*)$ . This follows from the fact that  $J^*\dot{\cup}P^*$  can be partitioned into three T-joins: one is  $J^*\backslash R$ , one is  $F_{P^*}$ , and one is the union of R and  $P^*\backslash F_{P^*}$ . One can check that each of these edge sets is a T-join by using the fact that T is the set of wrong degree vertices of  $J^*$ . Then,  $3c(F^*) \leq c(J^*) + c(P^*) \leq 2c(P^*)$ . This completes the proof.

#### 3 Unified correction vector

An et al. [AKS12] designed a randomized Christofides' algorithm for the s-t-path TSP, and they proved an approximation factor of  $\frac{1+\sqrt{5}}{2}$  by analysing this algorithm. Their algorithm and their analysis were based on the LP relaxation (L.P.1). Sebő [Seb13] presented a new analysis of this randomized algorithm and improved the approximation factor to  $\frac{8}{5}$ . The algorithm and analysis of [Seb13] were based on a different LP relaxation, see (L.P.4) in Section 4. In Section 4, we prove that (L.P.1) and (L.P.4) have the same optimal value. This result together with a few more observations implies that (L.P.4) can be replaced by (L.P.1) in the algorithm and analysis of [Seb13] to achieve the same approximation factor of  $\frac{8}{5}$ . In this section, we prove the approximation factor of [AKS12]; also, we prove the  $\frac{8}{5}$ -approximation factor of [Seb13] based on (L.P.1) rather than (L.P.4).

#### Randomized Christofides' algorithm:

Solve the LP relaxation (L.P.1) to get an optimal solution  $x^*$ . Since  $x^*$  is in

the spanning tree polytope, there exists a convex decomposition of spanning trees  $J_1, J_2, \ldots, J_l$  such that  $\sum_{1 \leq i \leq l} \lambda_i \mathcal{X}^{J_i} = x^*$  where  $\sum_{1 \leq i \leq l} \lambda_i = 1$ ,  $\lambda_i > 0$  and  $\mathcal{X}^{J_i}$  is the edge incidence vector of  $J_i$ . Such a decomposition can be found in polynomial time, see Theorem 51.5 of [Sch03]. We sample a spanning tree J from these spanning trees according to the probability defined by the coefficient  $\lambda_i$  of each spanning tree in the convex combination. Let T denote the set of the wrong degree vertices of J. Then, as in the Christofides' algorithm, a minimum-cost T-join F is added to fix the wrong degree vertices of J.

The expected cost of the random solution of the algorithm is the sum of the expected cost of J, which is the cost of  $x^*$ , and the expected cost of the T-join F. Any feasible solution of the T-join polyhedron provides a cost upper bound for the T-join F. An et al. [AKS12] introduced correction vectors to construct a special type of fractional T-join. A correction vector for a  $\tau$ -narrow cut S is an edge vector z that satisfies  $\sum_{e \in \delta(S)} z_e \geq 1$ , where the definition of  $\tau$ -narrow cut will be given next. The correction vectors were further analyzed in [Seb13] to obtain a better approximation factor. In this section, we present a unified correction vector to derive the results of both [AKS12] and [Seb13].

The following key definition is introduced in [AKS12]. Let  $0 < \tau \le 1$ . If an s-t cut Q satisfies  $x^*(\delta(Q)) < 1 + \tau$ , we call it a  $\tau$ -narrow cut. Let  $\mathcal{C}_{\tau}$  be the set of all  $\tau$ -narrow cuts that contain s. It turns out that  $\tau$ -narrow cuts have a nice structural property.

**Lemma 3.1** [AKS12] Let  $Q_1$ ,  $Q_2$  be two distinct cuts in  $C_{\tau}$ . Then either  $Q_1 \subseteq Q_2$  or  $Q_2 \subseteq Q_1$ .

For the sake of completeness, we present a proof.

**Proof.** Suppose that the statement is false. Then both  $Q_1 \backslash Q_2$  and  $Q_2 \backslash Q_1$  are nonempty. Note that both  $Q_1 \backslash Q_2$  and  $Q_2 \backslash Q_1$  are  $\{s,t\}$ -even. Hence,  $x^*(\delta(Q_1)) + x^*(\delta(Q_2)) \geq x^*(\delta(Q_1 \backslash Q_2)) + x^*(\delta(Q_2 \backslash Q_1)) \geq 4$  by the constraints in (L.P.1). However,  $x^*(\delta(Q_1)) + x^*(\delta(Q_2)) < 2 + 2\tau \leq 4$ . This is a contradiction.

Thus, we can use  $Q_1,Q_2,\ldots,Q_k$  to denote all of the  $\tau$ -narrow cuts containing s such that  $s\in Q_1\subsetneq Q_2\subsetneq Q_3\cdots\subsetneq Q_k\subsetneq V$ . Note that  $\mathcal{C}_{\tau}=\{Q_i\}_{1\leq i\leq k}$ . Define  $L_i=Q_i\backslash Q_{i-1}$  for  $i=1,2,\ldots,k,k+1$  where  $Q_0=\emptyset$  and  $Q_{k+1}=V$ . Each  $L_i$  is nonempty and  $\bigcup_{1\leq i\leq k+1}L_i=V$ . We call  $\{L_i\}$  the partition derived by the  $\tau$ -narrow cuts  $\mathcal{C}_{\tau}$ .

Let  $\mathcal{X}^J$  denote the edge incidence vector of the edge set of J. For any  $Q \in \mathcal{C}_{\tau}$ , we let  $e_Q$  be an edge in  $\delta(Q)$  of minimum cost. Let  $\mathcal{X}^{e_Q}$  denote the

edge incidence vector of  $\{e_Q\}$ , i.e.,  $\mathcal{X}_{e_Q}^{e_Q} = 1$ , and  $\mathcal{X}_e^{e_Q} = 0$  if  $e \neq e_Q$ . Our unified correction vector is defined as  $\mathcal{X}^{e_Q}$  for each  $Q \in \mathcal{C}_{\tau}$ , i.e., the unified correction vector simply assigns the value one to the minimum-cost edge in each  $\tau$ -narrow cut. In contrast, the correction vectors used in [AKS12] and [Seb13] are fractional but sum up to at least one for each  $\tau$ -narrow cut.

Let  $\alpha, \beta$  and  $\tau$  be real parameters between 0 and 1, whose specific values are given later. Recall that J is the random spanning tree in the randomized Christofides' algorithm. Our fractional feasible T-join solution with unified correction vectors, called *unified fractional* T-join, is as follows:

#### Unified fractional T-join:

$$f = \alpha \mathcal{X}^J + \beta x^* + \sum_{Q \in \mathcal{C}_{\tau}, \ Q \text{ is } T\text{-odd}} (1 - 2\alpha - \beta x^*(\delta(Q))) \mathcal{X}^{e_Q}.$$

where  $\alpha, \beta, \tau$  satisfy the following condition:

$$\alpha + 2\beta = 1, \tau = \frac{1 - 2\alpha}{\beta} - 1, \alpha \ge 0 \text{ and } \beta \ge 0.$$
 (1)

Let us derive the settings of  $\alpha$ ,  $\beta$  and  $\tau$  in (1). The purpose of the unified fractional T-join f is to provide an upper bound on the cost of the minimum-cost T-join F in the randomized Christofides' algorithm. By Lemma 2.2, it suffices to make f feasible for the T-join polyhedron (L.P.3). This requires special settings of  $\alpha$ ,  $\beta$  and  $\tau$ .

Consider the cut constraints in (L.P.3). Let S be a T-odd cut. First we need to make sure that for any  $Q \in \mathcal{C}_{\tau}$ , the coefficient  $1 - 2\alpha - \beta x^*(\delta(Q))$  is nonnegative. Since  $x^*(\delta(Q)) < 1 + \tau$  for any  $Q \in \mathcal{C}_{\tau}$ , it suffices to set  $1 - 2\alpha - \beta(1 + \tau) = 0$ , i.e.,  $\tau = \frac{1-2\alpha}{\beta} - 1$ .

Suppose that S is an s-t cut. Note that S is T-odd. Hence, by Lemma 2.3,  $|\delta(S)\cap J|$  is even. If S is not a  $\tau$ -narrow cut, then  $f(\delta(S)) \geq \alpha \mathcal{X}^J(\delta(S)) + \beta x^*(\delta(S)) \geq 2\alpha + \beta(1+\tau)$ . By the assumption that  $\tau = \frac{1-2\alpha}{\beta} - 1$ , we have  $f(\delta(S)) \geq 1$  in this case. If S is a cut in  $C_\tau$ , then  $f(\delta(S)) \geq 2\alpha + \beta x^*(\delta(S)) + (1-2\alpha-\beta x^*(\delta(S)))\mathcal{X}^{e_S}(\delta(S)) \geq 1$ .

Now the only remaining case is that S is  $\{s,t\}$ -even. Then  $x^*(\delta(S)) \geq 2$  by (L.P.1). Since J is a spanning tree, we have  $\mathcal{X}^J(\delta(S)) \geq 1$ . This implies  $f(\delta(S)) \geq \alpha \mathcal{X}^J(\delta(S)) + \beta x^*(\delta(S)) \geq \alpha + 2\beta$ . Hence, in this case, it suffices to set  $\alpha + 2\beta = 1$ .

Hence, we have the following result by the analysis above.

**Lemma 3.2** The unified fractional T-join f is a feasible solution of the T-join polyhedron (L.P.3).

Lemma 3.2 shows that the expected cost of the minimum-cost T-join F computed by the randomized Christofides' algorithm is at most the expected cost of the unified fractional T-join. Hence, the expected cost of the solution of the randomized Christofides' algorithm is upper bounded by the optimal value of (L.P.1) plus the expected cost of the unified fractional T-join. In Section 3.1 and Section 3.2, we will present two different analyses of the expected cost of the unified fractional T-join to derive two different approximation factors from [AKS12] and [Seb13] for the randomized Christofides' algorithm.

**Remark 3.3** From the analysis above, the cost analysis of the unified fractional T-join is critical for proving an approximation factor for the randomized Christofides' algorithm. If we can get a better upper bound on the cost of the unified fractional T-join, then the approximation factor can be further improved.

The following lemma is used in the analysis of the expected cost of the unified fractional T-join in Section 3.1 and Section 3.2.

**Lemma 3.4** [AKS12][Seb13] Let J be the random spanning tree and T be the set of wrong degree vertices of J in the randomized Christofides' algorithm. Let  $Q \in \mathcal{C}_{\tau}$ , i.e., Q is a  $\tau$ -narrow cut. Then

(i) 
$$\Pr(|\delta(Q) \cap J| = 1) \ge 2 - x^*(\delta(Q))$$
, and

(ii) 
$$Pr(Q \text{ is } T\text{-odd}) \leq x^*(\delta(Q)) - 1.$$

For the sake of completeness, we present a proof.

**Proof.** Since J is a spanning tree,  $|\delta(Q) \cap J| \geq 1$  always holds. So  $\sum_{i\geq 1} \Pr(|\delta(Q) \cap J| = i) = 1$ . Then

$$\begin{split} \Pr(|\delta(Q)\cap J| \geq 2) & \leq & \sum_{i\geq 1} i * \Pr(|\delta(Q)\cap J| = i) - \sum_{i\geq 1} \Pr(|\delta(Q)\cap J| = i) \\ & = & \mathbb{E}(|\delta(Q)\cap J|) - \sum_{i\geq 1} \Pr(|\delta(Q)\cap J| = i) \\ & = & x^*(\delta(Q)) - 1. \end{split}$$

Note that  $\mathbb{E}(|\delta(Q) \cap J|) = x^*(\delta(Q))$  follows from the fact that  $\mathbb{E}(\mathcal{X}^J) = x^*$  since J is a random tree in the convex decomposition of spanning trees for  $x^*$  where the coefficients of the spanning trees define the probability

distribution. Thus, we have  $\Pr(|\delta(Q) \cap J| = 1) = 1 - \Pr(|\delta(Q) \cap J| \ge 2) \ge 2 - x^*(\delta(Q))$ . This proves the first inequality.

Now consider the second inequality. By Lemma 2.3,  $|\delta(Q) \cap J|$  is even if Q is T-odd. This means  $\Pr(Q \text{ is } T\text{-odd}) \leq \Pr(|\delta(Q) \cap J| \text{ is even}) \leq \Pr(|\delta(Q) \cap J| \geq 2) \leq x^*(\delta(Q)) - 1$ .

# 3.1 AKS' $\frac{1+\sqrt{5}}{2}$ -approximation via unified correction vector

First, we present two lemmas needed for the cost analysis of the randomized Christofides' algorithm.

**Lemma 3.5** Let K be a spanning tree with n vertices. Let  $S = \{S_i : 1 \le i \le n-1\}$  be a family of subsets of the vertex set of K such that  $|S_i| = i$  and  $S_i \subsetneq S_{i+1}$ . There exists a bijection from S to E(K) such that each cut  $S_i$  is mapped to an edge of K in  $\delta(S_i)$ .

**Proof.** Without loss of generality, we can assume that the vertex set of K is  $\{v_1, v_2, \ldots, v_n\}$  and  $S_i = \{v_1, v_2, \ldots, v_i\}$  for  $1 \le i \le n-1$ . We prove the result by induction on n. The statement is clearly true for n = 2. Suppose  $n \ge 3$ . Consider the vertex  $v_n$ .

We first pick the edge e of K incident with  $v_n$  in the unique path of K between  $v_{n-1}$  and  $v_n$ . We map  $S_{n-1}$  to this edge e. Let K' be the graph obtained from  $K \setminus \{e\}$  by contracting  $v_{n-1}$  and  $v_n$  into a single vertex  $v'_{n-1}$ . Note that K' is a connected graph with n-2 edges. This implies that K' is a spanning tree with n-1 vertices  $\{w_1, w_2, \ldots, w_{n-1}\}$  where  $w_i = v_i$  for  $1 \le i \le n-2$  and  $w_{n-1} = v'_{n-1}$ . Note that  $\delta(\{w_1, w_2, \ldots, w_i\})$  is a subset of  $\delta(S_i)$  for  $1 \le i \le n-2$ . Hence, we can define the rest of the bijection by applying the induction hypothesis to the spanning tree K' on these n-1 vertices.

Lemma 3.6

$$\sum_{Q \in \mathcal{C}_{\tau}} c(e_Q) \le c(x^*) \tag{2}$$

**Proof.** Let  $K_{min}$  be a minimum-cost spanning tree on G. Consider the partition  $\{L_i\}$  derived by  $\mathcal{C}_{\tau}$ . We contract every  $L_i$  into a single vertex. Then the resulting graph obtained from  $K_{min}$  is connected. Let K be a spanning tree of the contracted graph. Applying Lemma 3.5 to K, we construct an injective mapping  $\phi$  from  $\mathcal{C}_{\tau}$  to the edge set of K such that  $\phi(Q) \in \delta(Q)$  for each  $Q \in \mathcal{C}_{\tau}$ . Note that  $K \subseteq K_{min}$ . Then

 $\sum_{Q \in \mathcal{C}_{\tau}} c(e_Q) \leq \sum_{Q \in \mathcal{C}_{\tau}} c(\phi(Q)) \leq c(K_{min}) \leq c(x^*)$  since  $x^*$  is in the spanning tree polytope. The first inequality follows from the fact that  $e_Q$  is the minimum-cost edge in  $\delta(Q)$ .

**Theorem 3.7** [AKS12] The randomized Christofides' algorithm achieves an approximation factor of  $\frac{1+\sqrt{5}}{2}$ .

**Proof.** Since J is a random spanning tree based on the convex decomposition of spanning trees for  $x^*$ , we have  $\mathbb{E}(\mathcal{X}^J) = x^*$ . Hence, the expected cost of the solution of the randomized Christofides' algorithm is upper bounded by the optimal value of (L.P.1) plus the expected cost of the minimum-cost T-join F. By Lemma 2.2 and Lemma 3.2, the expected cost of F is at most the expected cost of the unified fractional T-join.

$$\mathbb{E}[c(\alpha \mathcal{X}^{J} + \beta x^{*} + \sum_{Q \in \mathcal{C}_{\tau}, Q \text{ is } T\text{-odd}} (1 - 2\alpha - \beta x^{*}(\delta(Q)))\mathcal{X}^{e_{Q}})]$$

$$\stackrel{Lemma \ 3.4}{\leq} (\alpha + \beta)c(x^{*}) + \sum_{Q \in \mathcal{C}_{\tau}} (x^{*}(\delta(Q)) - 1)(1 - 2\alpha - \beta x^{*}(\delta(Q)))c(e_{Q})$$

$$\stackrel{\leq}{\leq} (\alpha + \beta)c(x^{*}) + \max_{0 \leq z < \tau} z(1 - 2\alpha - \beta z - \beta) \sum_{Q \in \mathcal{C}_{\tau}} c(e_{Q})$$

$$\stackrel{Lemma \ 3.6}{\leq} (\alpha + \beta + \max_{0 \leq z < \tau} z(1 - 2\alpha - \beta z - \beta))c(x^{*})$$

$$\stackrel{By \ (1)}{=} (\alpha + \beta + \beta \max_{0 \leq z < \tau} z(\tau - z))c(x^{*}).$$

The last equality follows from the fact that  $1-2\alpha=\beta(\tau+1)$  by (1). The value of z that maximizes the expression is  $\frac{\tau}{2}$ . Hence, the upper bound on the expected cost of the unified fractional T-join is at most  $(\alpha+\beta+\beta(\frac{\tau}{2})^2)c(x^*)$ . Substitute  $\tau=\frac{1-2\alpha}{\beta}-1$ ,  $\alpha=1-2\beta$  from (1) into the upper bound. Minimizing with respect to  $\beta$  gives  $\frac{\sqrt{5}-1}{2}c(x^*)$  with optimal settings:  $\beta=\frac{1}{\sqrt{5}}, \ \alpha=1-\frac{2}{\sqrt{5}}, \ \tau=3-\sqrt{5}$ . Therefore, the optimal value of (L.P.1) plus this upper bound  $\frac{\sqrt{5}-1}{2}c(x^*)$  on the expected cost of the unified fractional T-join leads to the approximation factor of  $\frac{1+\sqrt{5}}{2}$  that was first proved in [AKS12].

In [AKS12], the correction vector is constructed by using flow computations to map the optimal LP solution  $x^*$  to the  $\tau$ -narrow cuts. In contrast, our unified correction vector simply assigns the value one to the minimum-cost edge in each  $\tau$ -narrow cut. We avoid the flow computation argument of [AKS12] by using Lemma 3.5.

# 3.2 Sebő's $\frac{8}{5}$ -approximation via unified correction vector

Let P be the s-t path in J. Sebő [Seb13] points out the crucial fact that  $J \setminus P$  is a T-join for the set of wrong degree vertices T of J. Recall that F is the minimum-cost T-join in the randomized Christofides' algorithm. This implies that  $\mathbb{E}(c(F)) \leq \mathbb{E}(c(J \setminus P))$ . Note that  $c(x^*) = \mathbb{E}(c(J)) = \mathbb{E}(c(J \setminus P)) + \mathbb{E}(c(P))$ .

It turns out that  $\mathbb{E}(c(P))$  also serves as an upper bound in another cost inequality similar to (2); see the following lemma.

#### Lemma 3.8

$$\sum_{Q \in \mathcal{C}_{\tau}} (2 - x^*(\delta(Q))) c(e_Q) \le \mathbb{E}(c(P)). \tag{3}$$

**Proof.** Let  $Q \in \mathcal{C}_{\tau}$ ; thus, Q is a  $\tau$ -narrow cut. If  $|\delta(Q) \cap J| = 1$ , then let  $e'_Q$  denote the unique edge in  $\delta(Q) \cap J$ . Recall that a  $\tau$ -narrow cut is an s-t cut, and therefore  $e'_Q$  must be in P since P is the s-t path in J. Moreover, observe that Q is one of the two connected components of  $J \setminus \{e'_Q\}$ . Hence, for distinct  $Q_1, Q_2 \in \mathcal{C}_{\tau}$  such that  $|\delta(Q_1) \cap J| = 1$  and  $|\delta(Q_2) \cap J| = 1$ , the edges  $e'_{Q_1}$  and  $e'_{Q_2}$  must be distinct (otherwise,  $J \setminus \{e'_{Q_1}\}$  and  $J \setminus \{e'_{Q_2}\}$  would have the same connected components, contradicting the fact that  $Q_1, Q_2$  are distinct sets containing s). Then

$$c(P) \ge \sum_{|\delta(Q) \cap J| = 1, Q \in \mathcal{C}_{\tau}} c(e'_Q) \ge \sum_{|\delta(Q) \cap J| = 1, Q \in \mathcal{C}_{\tau}} c(e_Q).$$

By Lemma 3.4,

$$\mathbb{E}(c(P)) \ge \sum_{Q \in \mathcal{C}_{\tau}} \Pr(|\delta(Q) \cap J| = 1) c(e_Q) \ge \sum_{Q \in \mathcal{C}_{\tau}} (2 - x^*(\delta(Q))) c(e_Q).$$

**Theorem 3.9** [Seb13] The randomized Christofides' algorithm achieves an approximation factor of  $\frac{8}{5}$ .

**Proof.** By an argument similar to the one in the proof of Theorem 3.7, we are only concerned with the expected cost of the unified fractional T-join, which bounds the expected cost of the minimum-cost T-join F in the

randomized Christofides' algorithm.

$$\mathbb{E}[c(\alpha \mathcal{X}^{J} + \beta x^{*} + \sum_{Q \in \mathcal{C}_{\tau}, Q \text{ is } T\text{-odd}} (1 - 2\alpha - \beta x^{*}(\delta(Q)))\mathcal{X}^{e_{Q}})]$$

$$\stackrel{Lemma 3.4}{\leq} (\alpha + \beta)c(x^{*}) + \sum_{Q \in \mathcal{C}_{\tau}} (x^{*}(\delta(Q)) - 1)(1 - 2\alpha - \beta x^{*}(\delta(Q)))c(e_{Q})$$

$$\leq (\alpha + \beta)c(x^{*}) + \sum_{Q \in \mathcal{C}_{\tau}} \frac{(x^{*}(\delta(Q)) - 1)(1 - 2\alpha - \beta x^{*}(\delta(Q)))}{2 - x^{*}(\delta(Q))}(2 - x^{*}(\delta(Q)))c(e_{Q})$$

$$\leq (\alpha + \beta)c(x^{*}) + \max_{0 \leq z < \tau} \frac{z(1 - 2\alpha - \beta z - \beta)}{1 - z} \sum_{Q \in \mathcal{C}_{\tau}} (2 - x^{*}(\delta(Q)))c(e_{Q})$$

$$\stackrel{Lemma 3.8}{\leq} (\alpha + \beta)c(x^{*}) + \max_{0 \leq z < \tau} \frac{z(1 - 2\alpha - \beta z - \beta)}{1 - z} \mathbb{E}(c(P))$$

$$\stackrel{By}{=} (1) (\alpha + \beta)c(x^{*}) + \beta \max_{0 \leq z < \tau} \frac{z(\tau - z)}{1 - z} \mathbb{E}(c(P)).$$
(4)

The last equality follows from the fact that  $1-2\alpha=\beta(\tau+1)$  by (1). The value of z that maximizes the expression is  $1-\sqrt{1-\tau}$ . Hence, the upper bound on the expected cost of the unified fractional T-join is at most  $(\alpha+\beta)c(x^*)+\beta(1-\sqrt{1-\tau})^2\mathbb{E}(c(P))$ . Substitute  $\tau=\frac{1-2\alpha}{\beta}-1$ ,  $\alpha=1-2\beta$  from (1) into (4). Then the coefficients of the terms in (4) only depend on  $\beta$ . Denote the coefficient of the last term in (4) by  $h(\beta)$  where  $h(\beta)=(\sqrt{\beta}-\sqrt{1-2\beta})^2$ . Then the bound can be written as  $(1-\beta)c(x^*)+h(\beta)\mathbb{E}(c(P))$ . Note that  $c(x^*)=\mathbb{E}(c(J\backslash P))+\mathbb{E}(c(P))$ . Assume  $\mathbb{E}(c(P))=\lambda_0 c(x^*)$ . So  $0\leq \lambda_0\leq 1$  and  $\mathbb{E}(c(J\backslash P))=(1-\lambda_0)c(x^*)$ . Since  $\mathbb{E}(c(J\backslash P))\geq \mathbb{E}(c(F))$ , we have

$$\mathbb{E}(c(F)) \leq \min\{(1 - \lambda_0)c(x^*), (1 - \beta + h(\beta)\lambda_0)c(x^*)\}$$

$$\leq \max_{0 \leq \lambda \leq 1} \{\min\{(1 - \lambda)c(x^*), (1 - \beta + h(\beta)\lambda)c(x^*)\}\}.$$
 (5)

 $\lambda$  maximizes the expression when  $(1-\lambda)c(x^*)=(1-\beta+h(\beta)\lambda)c(x^*)$ . So  $\lambda=\frac{\beta}{h(\beta)+1}$ . Minimizing the upper bound in (5) with respect to  $\beta$  gives  $\frac{3}{5}c(x^*)$  with optimal settings:  $\beta=\frac{4}{9}, \alpha=\frac{1}{9}, \tau=\frac{3}{4}$ ; moreover,  $\lambda=\frac{2}{5}$ . Therefore, the optimal value of (L.P.1) plus this upper bound  $\frac{3}{5}c(x^*)$  leads to the approximation factor of  $\frac{8}{5}$  that was first proved in [Seb13].

# 4 Linear programming relaxations of the s-t path TSP

In this section, we investigate the relationship between two different LP relaxations of the s-t path TSP. Let H = (V, E(H)) be a connected graph with nonnegative edge costs c', and let s and t be two fixed vertices. For a partition  $\mathcal{W} = \{W_1, W_2, \ldots, W_\ell\}$  of the vertex set V, let  $\delta(\mathcal{W})$  denote  $\bigcup_{1 \leq i \leq \ell} \delta(W_i)$ . Let G = (V, E) be the metric completion of H with metric costs c. As mentioned in Section 3, (L.P.1) is a linear programming relaxation of the s-t path TSP on G. Let 2H be the graph obtained from H by doubling every edge of H. The s-t path TSP on G is equivalent to the problem of finding a minimum-cost trail in 2H from s to t visiting every vertex at least once (multiple visits are allowed for the vertices but not the edges). Thus, the problem is to find a minimum-cost connected spanning subgraph of 2H with  $\{s,t\}$  as the odd-degree vertex set. Hence, the following (L.P.4) is another LP relaxation of the s-t path TSP.

```
\begin{array}{lll} \textbf{(L.P.4)} & \text{minimize}: & \sum_{e \in E(H)} c'_e x_e \\ & \text{subject to}: & x(\delta(\mathcal{W})) & \geq & |\mathcal{W}|-1 & \forall \text{ partition } \mathcal{W} \text{ of } V \\ & & x(\delta(S)) & \geq & 2 & \forall \emptyset \subsetneq S \subsetneq V, |S \cap \{s,t\}| \text{ even} \\ & & x_e & \geq & 0 & \forall e \in E(H) \end{array}
```

Note that (L.P.4) is defined on the original graph H but (L.P.1) is defined on the metric completion G of H.

In this section, we show that both LPs, (L.P.1) and (L.P.4), have the same (fractional) optimal value, see Corollary 4.3. But these two LPs can differ with respect to integral solutions. Observe that the integral solutions of (L.P.1) are exactly the s-t Hamiltonian paths of G; this follows because an integral solution induces a graph that is connected, has degree one at s, t, and has degree two at all other vertices. The integral solutions of (L.P.4) need not correspond to the s-t Eulerian paths of H; see the example shown in Figure 1.

Let  $Opt(LP_k)$  denote the optimal value of (L.P.k), for k=1,4. Let  $Opt_{int}(LP_k)$  denote the minimum cost of an integral solution that satisfies all constraints of (L.P.k), for k=1,4. We call  $Opt_{int}(LP_k)$  the optimal integral value of (L.P.k). The following table summarizes the relationship between the two LPs; the new results of this section appear in the last two columns.

LPs	Graph	Costs	Optimum	Optimal Integral Value
(L.P.1)	G: metric completion of $H$	c: metric extension of $c'$	$Opt(LP_1)$	$Opt_{int}(LP_1)$
(L.P.4)	H	$c' \ge 0$	$Opt(LP_1)$	$Opt_{int}(LP_4) \le Opt_{int}(LP_1) \le \frac{3}{2}Opt_{int}(LP_4)$

To obtain these results, we need an edge-splitting lemma. Let K be a multigraph, i.e., two adjacent vertices in K may be connected by one or more edges. Let  $(u, v), (v, w) \in E(K)$ . The *splitting operation* on (u, v), (v, w) at the vertex v is defined as follows:

• Remove (u, v), (v, w) and then add (u, w) if  $u \neq w$ .

If u = w, then we remove the loop formed by adding (u, w); note that this removal of the loop has no effect on the edge-connectivity of the graph. We use the following result to prove Lemma 4.2; see [Fra92, Theorem A'].

**Lemma 4.1** [Lov74][Lov79, Ex. 6.51] Let K be a multigraph with even degree at each vertex. Let  $v \in V(K)$  and let  $U = V(K) \setminus \{v\}$ . Let d be a positive integer. If

$$|\delta(S)| \ge d \text{ for each } \emptyset \subsetneq S \subsetneq U$$
 (6)

then the edges incident with v can be partitioned into  $\frac{|\delta(v)|}{2}$  disjoint edge pairs (p, v), (v, q) such that the multigraph obtained by applying the splitting operation to any one of these edge pairs (at the vertex v) still satisfies (6).

**Lemma 4.2** Let x be a rational solution of (L.P.4) of cost c'(x). Then there exists a solution x' of (L.P.1) with cost at most c'(x). Moreover, if x is an integral solution, then x' is half-integral.

**Proof.** The first part of this statement follows from the parsimonious property shown in [BT97]. However, to show the second part of the statement, we present a proof for the first part as well.

Define an edge vector y on G as follows:

$$y_e = \begin{cases} x_e, & \text{if } e \in E(H), \\ 0, & \text{otherwise.} \end{cases}$$

Since G is the metric completion of H, we know  $c(y) \leq c'(x)$ . Then we construct y' from y as follows:

$$y'_e = \begin{cases} 1 + y_e, & \text{if } e = (s, t), \\ y_e, & \text{otherwise.} \end{cases}$$

By the constraints of (L.P.4) and the fact that  $y'_{(s,t)} = y_{(s,t)} + 1$ , we have  $y'(\delta(S)) \geq 2$  for each cut S. Let C be a positive integer such that

Cy' is integral. Consider the multigraph  $K_{2C}$  with  $2Cy'_{(u,v)}$  number of edges between u and v. Then  $|\delta_{K_{2C}}(S)| \geq 4C$ .

By using Lemma 4.1, we apply splitting operations at every vertex until the degree of every vertex is exactly 4C. We claim that this procedure can be applied such that the number of edges between s and t is  $\geq 2C$ . To see this, consider a splitting operation at s or t, say s; note that splitting at other vertices does not decrease the number of edges between s and t. There are at least 2C+1 feasible splitting pairs available at s (since otherwise there is no need to do a splitting operation at s, i.e.,  $|\delta_{K_{2C}}(s)| = 4C$ ). This implies that we can always choose a splitting pair such that at least 2C edges between s and t are preserved.

Let z be the edge vector associated with the resulting graph after splitting, i.e.,  $z_{(u,v)}$  equals the number of edges between u and v in the resulting graph. Furthermore, let z'=z/2C. Then  $z'(\delta(S))\geq 2$  for each cut S,  $z'(\delta(v))=2$  for each vertex v, and  $z'_{(s,t)}\geq 1$ . Consider two different vertices u,v. We know  $z'(\delta(u))=z'(\delta(v))=2$  and  $z'(\delta(\{u,v\}))\geq 2$ . This implies  $z'_{(u,v)}\leq 1$ . In particular,  $z'_{(s,t)}=1$ . Construct x' from z' as follows:

$$x'_e = \begin{cases} z'_e - 1 = 0, & \text{if } e = (s, t), \\ z'_e, & \text{otherwise.} \end{cases}$$

By the properties obtained for z', we have x' is a feasible solution of (L.P.1). Note that the splitting operations never increase the total cost since the edge costs are metric on G. Therefore, the cost of x' is at most c'(x). In particular, if x is integral, we can set C = 1 in the procedure. In this case, x' is half-integral.

Conversely, any feasible solution of (L.P.1) can be transformed to a feasible solution of (L.P.4): the idea is to replace each edge (u,v) in E(G) by a shortest u-v path in H. Note that every solution of (L.P.1) is a feasible solution of the spanning tree polytope. Hence, it can be seen that the transformed solution is feasible for (L.P.4), and, in particular, it satisfies the partition constraints in (L.P.4). Hence,

$$Opt(LP_4) \le Opt(LP_1), \quad Opt_{int}(LP_4) \le Opt_{int}(LP_1).$$
 (7)

By Lemma 4.2, we have the following result.

#### Corollary 4.3 $Opt(LP_4) = Opt(LP_1)$ .

However, (L.P.1) and (L.P.4) may differ in terms of the integral optimal value. Consider the graph with unit edge costs in Figure 1; this is meant to be the original graph H in the instance of the s-t path TSP.



Figure 1: Tight Example

Note that (L.P.4) is defined on the original graph but (L.P.1) is defined on the metric completion. Let  $\ell$  be the length of the middle path in Figure 1. It is not hard to see that  $Opt_{int}(LP_1) \approx 3\ell$  but  $Opt_{int}(LP_4) \approx 2\ell$  when  $\ell$  is sufficiently large. (For (L.P.4), consider the integral solution with value 1 for every edge of the original graph.) In this case,  $\frac{Opt_{int}(LP_1)}{Opt_{int}(LP_4)} \approx \frac{3}{2}$ . Interestingly,  $\frac{3}{2}$  can be proved to be an upper bound for this ratio. This example shows that the upper bound of  $\frac{3}{2}$  is tight. To prove this upper bound, we present an algorithm to round a half-integral solution of (L.P.1) to an integral one by increasing the cost by a factor of at most  $\frac{3}{2}$ .

Apply the randomized Christofides' algorithm to a half-integral solution x of (L.P.1). Let J be the random spanning tree obtained from x. Let F be a minimum-cost T-join for the set of wrong degree vertices T of J.

**Lemma 4.4**  $x(\delta(S)) \geq 2$  for any T-odd cut S.

**Proof.** For any vertex  $v \in V$ ,  $x(\delta(v))$  is integral by the constraints of (L.P.1). Since  $x_e$  is half-integral,  $x(\delta(S)) = \sum_{v \in S} x(\delta(v)) - 2x(E(S))$  implies that  $x(\delta(S))$  is integral. Suppose  $x(\delta(S)) < 2$  for some T-odd cut S. Then we have  $x(\delta(S)) = 1$ . By the constraints of (L.P.1), S must be an s-t cut. Note that  $\mathbb{E}(\mathcal{X}^J) = x$  and  $|J \cap \delta(S)| \ge 1$  since J is a random spanning tree. This implies  $|J \cap \delta(S)| = 1$  always holds. However, since S is an s-t cut and also a T-odd cut, we have  $|\delta(S) \cap J|$  is even by Lemma 2.3. This is a contradiction.

**Theorem 4.5** If the input is a half-integral solution x of (L.P.1), then the

randomized Christofides' algorithm outputs a Hamiltonian s-t path with cost at most  $\frac{3}{2}c(x)$ .

**Proof.** By Lemma 4.4,  $\frac{1}{2}x$  is a feasible solution of the T-join polyhedron (L.P.3). This means  $\mathbb{E}(c(F)) \leq \frac{1}{2}c(x)$ . Therefore  $\mathbb{E}(c(J)) + \mathbb{E}(c(F)) \leq \frac{3}{2}c(x)$ .

Now we are ready to prove the ratio for the optimal integral values of the two LPs.

**Theorem 4.6**  $Opt_{int}(LP_4) \leq Opt_{int}(LP_1) \leq \frac{3}{2}Opt_{int}(LP_4)$ . Moreover, the bounds are tight.

**Proof.** The lower bound is due to (7). Now consider the upper bound. Let x be an optimal integral solution of (L.P.4). By Lemma 4.2, there exists a half-integral solution x' of (L.P.1) such that  $c(x') \leq c'(x)$ . By Theorem 4.5, we can get an s-t Hamiltonian path with cost at most  $\frac{3}{2}c(x')$ . This means  $Opt_{int}(LP_1) \leq \frac{3}{2}c(x') \leq \frac{3}{2}c'(x) = \frac{3}{2}Opt_{int}(LP_4)$ .

The tight example for the upper bound is shown in Figure 1. For the tightness of the lower bound, consider the graph H consisting of one path connecting s and t where every edge has unit cost.

# 5 Counterexample to two approaches

For s-t path TSP, the main question is whether there exists a  $\frac{3}{2}$ -approximation algorithm. When addressing this problem, two natural questions arise:

- [Gao13] presented a simple  $\frac{3}{2}$ -approximation algorithm for the s-t path TSP in the graphic case. Does it extend to give the same approximation factor for the general metric case?
- Does every spanning tree in a given convex decomposition of an optimal solution x of (L.P.1) achieve a  $\frac{3}{2}$ -approximation factor by adding a minimum-cost T-join to fix the wrong degree vertices?

The first question concerns the extension of the algorithm for the graphic case. The second question focuses on the role of randomness and probabilistic methods in the analysis of the recent LP-based approximation algorithms. We answer these questions negatively by providing a counterexample. In the following, we make the questions more precise and then show how our counterexample serves as a negative answer.

Let G be the metric completion of some connected graph H with unit edge costs c'. The s-t path TSP defined on G is called s-t path graph-TSP. In this important special case, the gap between the upper bound and lower bound of the LP integrality ratio has been closed. The first  $\frac{3}{2}$ -approximation algorithm for the s-t path graph-TSP was given by [SV14] using sophisticated techniques. [Gao13] presented another  $\frac{3}{2}$ -approximation algorithm which was conceptually simpler than that in [SV14].

Let  $x^*$  be an optimal solution of the (L.P.4) defined on H. Note that  $c'_e = 1$  for  $e \in E(H)$  in this case. Let Q be an s-t cut. If  $x^*(\delta(Q)) < 2$ , we call it a narrow cut, which is exactly a 1-narrow cut as defined in Section 3. Note that the narrow cuts containing s still have the nice structural property of Lemma 3.1 even when  $x^*$  is an optimal solution of (L.P.4). We recall some notation from Section 3. The cuts  $Q_1, Q_2, \ldots, Q_k$  are all the narrow cuts containing s such that  $s \in Q_1 \subsetneq Q_2 \subsetneq Q_3 \cdots \subsetneq Q_k \subsetneq V$ . Define  $L_i = Q_i \backslash Q_{i-1}$  for  $i = 1, 2, \ldots, k, k+1$  where  $Q_0 = \emptyset$  and  $Q_{k+1} = V$ . Note that each  $L_i$  is nonempty and  $\bigcup_{1 \leq i \leq k+1} L_i = V$ . It is shown in [Gao13] that H restricted on each  $L_i$  is connected and also there exists at least one edge between each two consecutive  $L_i$  and  $L_{i+1}$  in H.

We sketch the  $\frac{3}{2}$ -approximation algorithm in [Gao13]. The algorithm constructs a minimal spanning tree on each  $L_i$  and then connects them together by a unit cost edge between each two consecutive  $L_i$  and  $L_{i+1}$ . This results in a spanning tree on H, which is called a good spanning tree. Then a minimum-cost T-join  $F_{good}$  is added to correct the wrong degree vertices of the good spanning tree. Since every edge in H has unit cost, the good spanning tree has minimum cost, which is at most  $Opt(LP_4)$ . Furthermore, it is shown in [Gao13] that the minimum-cost T-join  $F_{good}$  has cost at most  $\frac{1}{2}Opt(LP_4)$ . This gives a  $\frac{3}{2}$ -approximation factor in total.

The only part in the analysis using the graphic property is that the good spanning tree has cost at most  $Opt(LP_4)$ . A natural extension of the definition of a good spanning tree would be as follows:

• In the general metric case, a good spanning tree is constructed by connecting the minimum-cost spanning tree in each  $L_i$  with a minimum-cost edge from  $L_i$  to  $L_{i+1}$ .

If the cost of this "extended" good spanning tree is bounded above by  $Opt(LP_4)$  in the general metric case, then it gives us a  $\frac{3}{2}$ -approximation factor for s-t path TSP. Unfortunately, this is not true. To show this, we present our counterexample, a complete graph  $G = H = H_b$  with metric edge costs  $c^{H_b}$  and vertex set  $\{0, 1, \ldots, 7\}$  where s = 0, t = 7. The metric edge costs  $c^{H_b}$  are given by the metric completion of the costs indicated in

Figure 2 below. Note that for every edge e in Figure 2,  $c_e^{H_b}$  is exactly the edge cost value shown in that figure.

Figure 2 shows the support graph of a feasible solution  $x^{H_b}$  of (L.P.4), where the first number on each edge denotes the  $x^{H_b}$  value and the second number denotes the cost of the edge.

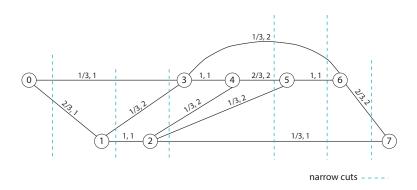


Figure 2: Support graph of  $x^{H_b}$  with edge  $x^{H_b}$  values and edge costs

**Lemma 5.1**  $x^{H_b}$  is an optimal solution for (L.P.4) with respect to  $c^{H_b}$ . Furthermore,  $x^{H_b}$  is an extreme point of the polyhedron of (L.P.4) on  $H_b$ .

**Proof.** To show the optimality of  $x^{H_b}$  for (L.P.4), it is sufficient to prove that  $x^{H_b}$  is an optimal solution of (L.P.1) by Corollary 4.3. We use complementary slackness conditions to prove the optimality of  $x^{H_b}$  for (L.P.1). Let  $S_1$  be the set of all s-t cuts and  $S_2$  be the set of all  $\{s,t\}$ -even cuts. Let  $S = S_1 \cup S_2$ .

(Dual of (L.P.1))

maximize : 
$$y_s + y_t + 2\sum_{v \notin \{s,t\}} y_v + \sum_{S \in S_1} d_S + 2\sum_{S \in S_2} d_S - \sum_e u_e$$

subject to:

$$y_w + y_v - u_{(w,v)} + \sum_{(w,v)\in\delta(S),S\in\mathcal{S}} d_S \le c_{(w,v)}, (w,v) \in E$$

$$u, d \ge 0$$

The following dual solution y, d, u witnesses the optimality of  $x^{H_b}$  to (L.P.1) by the complementary slackness conditions:

• 
$$u_{(1,2)} = u_{(3,4)} = \frac{2}{3}$$
,  $u_{(5,6)} = \frac{4}{3}$ , and  $u_e = 0$  for any other edge  $e$ 

- $d_{\{3,4,5,6\}} = \frac{1}{3}$  and  $d_S = 0$  for any other S
- $y_0 = 0, y_2 = y_3 = \frac{2}{3}, y_1 = y_4 = y_5 = 1, y_6 = \frac{4}{3}, y_7 = \frac{1}{3}$

Hence  $x^{H_b}$  is also an optimal solution of (L.P.4).

Denote the polyhedron of (L.P.4) on  $H_b$  by K. We now show that  $x^{H_b}$  is an extreme point of K. Otherwise, there exists  $x^{H_b} \neq z \in K$  and  $z' \in K$  such that  $x^{H_b} = \lambda z + (1 - \lambda)z'$  for some  $0 < \lambda < 1$ .

Clearly, for any edge e not in the support graph of  $x^{H_b}$ , we have  $z_e=0$  by Lemma 2.4. We also apply Lemma 2.4 to  $\delta(v)$  for each vertex v, and the cuts  $S_1=\{3,4\}, S_2=\{1,2\}, S_3=\{5,6\}, S_4=\{3,4,5,6\}$ . Then,  $z(\delta(v))=1$  for v=0,7 and  $z(\delta(v))=2$  for other vertices, and  $z(\delta(S_j))=2$  for  $1\leq j\leq 4$ . Hence,  $z_e=1$  for each  $e\in E_1=\{(3,4),(1,2),(5,6)\}$ . Let  $a=z_{(0,3)}, b=z_{(4,5)}$ . By the z-values on the edges in  $E_1$  and the values  $z(\delta(v))$  for  $v\in V(H_b)$ , we have  $z_{(0,1)}=1-a,z_{(1,3)}=a,z_{(3,6)}=1-2a,z_{(6,7)}=2a,z_{(2,7)}=1-2a,z_{(2,5)}=1-b,z_{(2,4)}=1-b$ . Now consider  $\delta(2)$  and  $\delta(S_4)$ . Then

$$2(1-b) + (1-2a) + 1 = 2$$
,  $4a + 2(1-b) = 2$ .

Hence,  $a = \frac{1}{3}, b = \frac{2}{3}$ . By checking each edge,  $z = x^{H_b}$ . This is a contradiction. Therefore,  $x^{H_b}$  is an extreme point of K. Note that the analysis above also shows that  $x^{H_b}$  is an extreme point of the polytope of (L.P.1) on  $H_b$ .

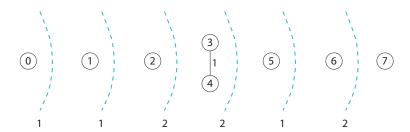


Figure 3: Cost of the good spanning tree

The cost of the corresponding good spanning tree is 10 and is shown in Figure 3. The number on the edge between 3 and 4 in Figure 3 is the edge cost. The numbers below the dashed narrow cuts are the minimum costs

of the edges crossing the narrow cuts to connect two consecutive parts. By Lemma 5.1, we know the optimal value of (L.P.4) is  $c^{H_b}(x^{H_b}) = 9\frac{2}{3}$ . So, we can see that the cost of the good spanning tree is strictly larger than the optimal value of (L.P.4). This refutes the statement that the cost of the "extended" good spanning tree can be upper bounded by  $Opt(LP_4)$ .

Interestingly, this instance also illustrates that probabilistic methods are important for the analyses of improved LP-based approximation algorithms such as the "randomized Christofides' algorithm" or its deterministic version the "best-of-many Christofides' algorithm" (see [AKS12]). The randomized Christofides' algorithm obtains a better approximation factor by sampling a spanning tree J from the convex decomposition of  $x^*$ . However, is it true that for an arbitrary spanning tree in the support of a given convex decomposition, the cost of the spanning tree plus a minimum-cost T-join is at most  $\frac{3}{2}Opt(LP_1)$ ? In the rest of this section, via the instance  $H_b$ , we show this statement is false in general.

We recall the optimal solution  $x^{H_b}$  of (L.P.1) on  $H_b$  with metric costs  $c^{H_b}$ . We know that  $x^{H_b}$  is in the spanning tree polytope (L.P.2). The tight constraints of  $x^{H_b}$  for the inequality constraints of (L.P.2) are illustrated as dashed circles in the Figure 4 except the tight constraints for  $V\setminus\{s\}$ ,  $V\setminus\{s,t\}$ .

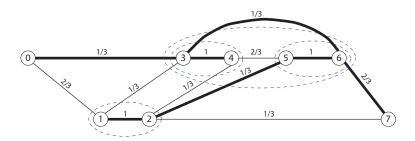


Figure 4: Tree  $J_b$ 

By Lemma 2.4, the tree  $J_b$  with the dark edges in the graph of Figure 4 is in some convex decomposition of  $x^{H_b}$  in (L.P.2), i.e.,  $J_b$  is a spanning tree in the support of some convex decomposition of  $x^{H_b}$ . Let  $T_b$  be the set of wrong degree vertices of  $J_b$ , i.e.,  $T_b = \{1, 3, 4, 6\}$ .  $F_b = \{(3, 6), (1, 4)\}$  is a minimum-cost  $T_b$ -join with cost 5. Hence, the total cost of the disjoint

union of  $J_b$  and  $F_b$  is 15, which is larger than  $\frac{3}{2}$  times the optimal value  $c^{H_b}(x^{H_b}) = 9\frac{2}{3}$  of (L.P.1). This shows the importance of the probabilistic techniques in the analysis of the "randomized Christofides' algorithm" or its deterministic version the "best-of-many Christofides' algorithm". Note that the minimum-cost  $T_b$ -join  $F_b$  to fix the wrong degree vertices of  $J_b$  is also larger than half of the optimal value  $9\frac{2}{3}$  of (L.P.1).

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