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## PORTFOLIO SELECTION WITH MULTIPLE SPECTRAL RISK CONSTRAINTS

CARLOS ABAD\* AND GARUD IYENGAR<sup>†</sup>

**Abstract.** We propose an iterative gradient-based algorithm to efficiently solve the portfolio selection problem with multiple spectral risk constraints. Since the conditional value at risk (CVaR) is a special case of the spectral risk measure, our algorithm solves portfolio selection problems with multiple CVaR constraints. In each step, the algorithm solves very simple separable convex quadratic programs; hence, we show that the spectral risk constrained portfolio selection problem can be solved using the technology developed for solving mean-variance problems. The algorithm extends to the case where the objective is a weighted sum of the mean return and either a weighted combination or the maximum of a set of spectral risk measures. We report numerical results that show that our proposed algorithm is very efficient; it is at least one order of magnitude faster than the state-of-the-art general purpose solver for all practical instances. One can leverage this efficiency to be robust against model risk by including constraints with respect to several different risk models.

Key words. large scale portfolio optimization, coherent risk measures, first-order algorithms

## AMS subject classifications. 90C90, 90B50, 91G10

1. Introduction. Portfolio selection is concerned with distributing a given capital over a finite number of investment opportunities in order to maximize "return" while managing "risk". Although, the benefits of diversification to manage "risk" had been long known, Markowitz [1952] was the first to propose a mathematical model for the portfolio optimization problem, representing "return" by the expected return of the portfolio, and "risk" by the variance in the return of the portfolio. It has been observed that variance is a good measure of risk *only* if the returns are *elliptically* distributed. Moreover, since variance is not sensitive to the tails of the distribution, it is not a good measure of variability when the returns are heavy tailed.

A number of risk measures have been proposed in the literature to accommodate asymmetry and also capture the effects of heavier tails. The Value-at-Risk  $\operatorname{VaR}_{\beta}(\tilde{L})$ at the probability level  $\beta$  for a random loss  $\tilde{L}$  is defined as the  $\beta$  quantile of the loss distribution, i.e. the probability of observing losses larger than  $\operatorname{VaR}_{\beta}(\tilde{L})$  is at most  $1 - \beta$  [Jorion, 2006]. VaR is extensively used in risk management applications, and it is the mandated risk measure in the Basel-II accords. However, it has a number of shortcomings. First, VaR only depends on the probability of tail losses and not their location in the tail. Second, VaR is not a convex risk measure; consequently, portfolio selection with VaR constraints often results in integer programs that are hard to solve.

Conditional Value-at-Risk  $\operatorname{CVaR}_{\beta}(\tilde{L}) = \mathbb{E}[\tilde{L} \mid \tilde{L} \geq \operatorname{VaR}_{\beta}]$  [Rockafellar and Uryasev, 2000] and Expected Shortfall  $\operatorname{ES}_{\beta} = \frac{1}{1-\beta} \int_{\beta}^{1} \operatorname{VaR}_{p}(\tilde{L}) dp$  [Acerbi and Tasche, 2002] are closely related risk functions that address the two shortcomings of VaR listed above. CVaR and ES are both coherent risk measures [Artzner et al., 1999], i.e. they are convex and positively homogeneous. Acerbi and Tasche [2002] showed that the ES of a portfolio can be estimated from samples of the losses on the underlying assets by solving a linear program (LP), and that the estimate converges to the ES of the portfolio with probability 1. Rockafellar and Uryasev [2000] showed a similar result for CVaR assuming that the loss distribution of the portfolio is con-

<sup>\*</sup>IEOR, Columbia University. ca2446@columbia.edu

 $<sup>^\</sup>dagger \rm IEOR,$  Columbia University. Supported in part by NSF grants DMS-1016571, DOE grant DE-FG02-08ER25856 and ONR grant N000140310514

tinuous at the  $\beta$  quantile. Acerbi [2002] extended ES to the spectral risk measure  $M_{\phi}(\tilde{L}) = \int_{0}^{1} \operatorname{VaR}_{p}(\tilde{L})\phi(p)dp$ , where  $\phi(p)$  is a non-increasing probability distribution function. The spectral risk measure  $M_{\phi}(\tilde{L})$  is coherent and, in fact,  $\operatorname{ES}_{\beta}(\tilde{L}) = M_{\hat{\phi}}(\tilde{L})$  with  $\hat{\phi}(p) = \frac{1}{1-\beta} \mathbf{1}_{\beta \leq p \leq 1}$ . Acerbi [2002] also showed that the finite sample estimate  $M_{\phi}^{N} = \sum_{k=1}^{N} \phi(\frac{k}{N}) L_{(N-k)}$ , where  $L_{(k)}$  denotes the k-th order statistic of N independent and identically distributed (IID) samples of the random loss  $\tilde{L}$ , converges to  $M_{\phi}(\tilde{L})$  with probability 1.

From Acerbi [2002], it follows that the portfolio selection problem where the "return" is given by the expected return of the portfolio and the "risk" is given by a spectral risk measure of the portfolio can be approximated by an LP. Rockafellar and Uryasev [2000] established such an LP-based approximation result for the mean-CVaR portfolio selection problem. Agarwal and Naik [2004] showed that the mean-CVaR portfolio selection results in superior portfolios as compared to the mean-variance approach when the risk of the assets is nonlinear in the underlying risk factors, e.g. when the asset is a derivative written on a primary asset. However, the resulting LP is very ill-conditioned, and solving such LP, particularly when the scenario size is large, is very difficult in practice (see, e.g. [Alexander et al., 2006]). Lim et al. [2011] showed that the solution of the mean-CVaR portfolio problem is often very sensitive to estimation errors, i.e. small errors in the estimation of the mean and the return in the scenarios can get amplified in the choice of the optimal portfolio. This sensitivity can be addressed by imposing spectral risk constraints with respect to several different parameter values and also different risk models. Constraints with respect to multiple risk models have become especially important after the 2008 financial crisis (see, e.g. [Ceria et al., 2009]). However, imposing multiple spectral risk constraints increases the size of the LP by such an extent that state-of-the-art solvers are unable to solve most practical instances of the portfolio selection problem.

Our contributions in this paper are as follows:

- (a) We propose a new first-order gradient based algorithm SPECRISKALLOCATE to solve portfolio selection problems with multiple spectral risk constraints that is significantly faster than the naive LP-based approach. We exploit two key features of the portfolio selection problem to construct this algorithm. The first is that the constraints in the LP formulation (2.3) are very loosely coupled in that the samples from a particular risk model only play a role in the corresponding constraint. Thus, one can improve the run time of the algorithm by dualizing these constraints, provided feasibility is maintained. We show in Theorem (3.1) that we are able to recover feasible portfolios for finite values of the dual variables. The second feature we exploit is that, since the LP is in fact a finite sample approximation to the stochastic optimization problem, in practice one is not attempting to solve it to very high accuracy (e.g.  $10^{-12}$  relative error) but rather one is satisfied with moderate accuracy (e.g.  $10^{-3}$  relative error). This allows us to smooth the LP into a smooth convex optimization problem, resulting in significantly faster convergence.
- (b) SPECRISKALLOCATE computes the optimal portfolio by solving a sequence of small separable convex quadratic programs (QPs). Thus, portfolio managers would be able to solve spectral risk constrained portfolio selection problems using existing tools for solving mean-variance problems. The number of variables in each of the convex QPs is equal to the number of assets and, therefore, these problems can be solved very efficiently. In some cases, the optimal solution of the

mean-variance subproblem can be written in closed form or computed by a one dimensional search. SPECRISKALLOCATE is also able to solve portfolio selection problems where the objective is to maximize a weighted sum of the expected return and either a weighted combination or the maximum of a set of spectral risk measures.

- (c) The experimental results in Section 4 clearly show that SPECRISKALLOCATE is able to efficiently solve very large spectral risk constrained portfolio selection problems. For most practical instances, SPECRISKALLOCATE is at least one order of magnitude faster than the state-of-the-art LP solvers. Moreover, we show that, in contrast to the LP-based method, SPECRISKALLOCATE is *not* ill-conditioned. This is a side-benefit of smoothing the problem. "Smoothing" approximates the LP polytope by a convex set without corners; thus, ensuring that the optimal solution is a continuous function of the problem and, therefore, not ill-conditioned.
- (d) A popular method for introducing robustness against model uncertainty is to impose spectral risk constraints with respect to several risk models (see e.g. Brown and Canova [2011] and Renshaw [2012]). In Section 4, we show that SPECRISK-ALLOCATE is able to solve a hedging portfolio selection problem with spectral risk constraints corresponding to multiple risk models in a computationally tractable manner.

SPECRISKALLOCATE is based on the proximal gradient algorithm FISTA proposed by Beck and Teboulle [2009] (see also Nesterov [2005]). The algorithm we propose is similar to the one proposed by Iyengar and Ma [2013] in that both these algorithms use Nesterov smoothing techniques [Nesterov, 2005]. However, there are a number of key differences between the two methods. The algorithm in Iyengar and Ma [2013] is only able to solve a mean-CVaR problem and can be extended to solve a mean-weighted CVaR problem; however, it is not able to compute solutions for portfolio selection problems with CVaR (or, more generally, spectral risk) *constraints*. SPECRICKALLO-CATE uses a different smoothing technique that allows us to scale the algorithm to solve very large portfolio selection problems without encountering any numerical difficulty. Iyengar and Ma [2013] were unable to solve large problem instances because the algorithm proposed therein quickly becomes numerically unstable.

The rest of this paper is organized as follows. In Section 2 we introduce the generalized spectral risk measures and define the generalized spectral risk constrained portfolio selection problem. In Section 3 we construct the SPECRISKALLOCATE algorithm. In Section 4 we discuss the results of our numerical experiments. Finally, in Section 5 we conclude with some final remarks.

2. Single period portfolio selection problem. Suppose there are *n* assets in the market. Let  $\tilde{\mathbf{L}} = (\tilde{L}_1, \ldots, \tilde{L}_n)^\top \in \mathbb{R}^n$  denote the random rate of loss on the assets. Let  $\mathbf{x} \in \mathbb{R}^n$  denote the portfolio of the investor, i.e.,  $\mathbf{1}^\top \mathbf{x} = \sum_{i=1}^n x_i = 1$ . The rate of loss  $\tilde{L}_x$  of portfolio  $\mathbf{x}$  is given by  $\tilde{L}_x = \tilde{\mathbf{L}}^\top \mathbf{x}$ . In this paper, we want to identify portfolios that lie on the Pareto optimal frontier with respect to the expected return  $-\mathbb{E}[\tilde{L}_x]$  and a set of generalized spectral risk measures [Acerbi, 2002].

Except for some special cases –e.g. when the random loss vector  $\tilde{\mathbf{L}}$  is a linear function of the distribution of elliptically distributed risk factors  $\tilde{\mathbf{Z}}$ – the distribution of the random portfolio loss  $\tilde{L}_x$  is hard to characterize explicitly. This is definitely the case if the portfolio  $\mathbf{x}$  contains derivative securities whose distribution is nonlinear in the underlying risk factors. In practice,  $\tilde{\mathbf{L}}$  is approximated by N samples { $\ell_1, \ldots, \ell_N$ } generated by some scenario generator (see, e.g. Koskosidis and Duarte [1997]). Let  $\mathbf{L} = (\boldsymbol{\ell}_1, \dots, \boldsymbol{\ell}_N)^\top \in \mathbb{R}^{N \times n}$  denote the empirical loss matrix, where the *j*-th column represents the vector of N loss realizations of asset *j*. Thus, the random loss  $\tilde{\boldsymbol{L}}_x$  on the portfolio  $\mathbf{x}$  can be approximated by the set of samples  $\{\boldsymbol{\ell}_1^\top \mathbf{x}, \dots, \boldsymbol{\ell}_N^\top \mathbf{x}\}$  or, equivalently, by the vector  $\mathbf{L}\mathbf{x}$ . In the rest of this section, we define the generalized spectral loss function for the vector  $\mathbf{L}\mathbf{x}$  and relate it to the Expected Shortfall measure. This relation will be important for designing our solution algorithm in Section 3.

**2.1. Generalized spectral risk measures.** Let  $\mathbf{y} = (y_1, \ldots, y_N)^{\top}$  denote N samples of a random variable  $\tilde{Y}$ . Let  $\{y_{(\ell)} : \ell = 1, \ldots, N\}$  denote the order statistics of vector  $\mathbf{y}$ .

DEFINITION 2.1 (Expected shortfall (ES) [Acerbi and Tasche, 2002]). The expected shortfall of  $\boldsymbol{y}$  at level  $\beta \in [0,1)$  is the average of the  $\kappa = \lceil (1-\beta)N \rceil$  largest values of  $\boldsymbol{y}$ , i.e.,

$$ES_{\beta}(\boldsymbol{y}) = \frac{1}{\kappa} \sum_{\ell=N-\kappa+1}^{N} y_{(\ell)}$$

It is easy to check that  $\text{ES}_{\beta}(\mathbf{y})$  has the following variational characterization (see, e.g. Artzner et al. [1999], Rockafellar et al. [2002], Lüthi and Doege [2005]) :

$$ES_{\beta}(\mathbf{y}) = \max_{\substack{\text{such that}}} \sum_{\ell=1}^{N} q_{\ell} y_{\ell}, \\ \text{such that} \quad \mathbf{1}^{\top} \mathbf{q} = 1, \\ \mathbf{0} \le \mathbf{q} \le \frac{1}{\kappa} \cdot \mathbf{1}.$$

Using linear programming duality [Bertsimas and Tsitsiklis, 1997] it follows that

(2.1) 
$$\operatorname{ES}_{\beta}(\mathbf{y}) = \min_{z} \left\{ z + \frac{1}{\kappa} \cdot \sum_{\ell=1}^{N} (y_{\ell} - z)^{+} \right\},$$

where  $v^+ = \max\{v, 0\}$ . Acerbi and Tasche [2002] established that  $\mathrm{ES}_{\beta}(\cdot)$  is a coherent risk measure [Artzner et al., 1999] and converges to CVaR [Rockafellar et al., 2002, Lüthi and Doege, 2005] when the cumulative distribution function  $F_Y(\cdot)$  of the random variable  $\tilde{Y}$  is continuous at  $y = \inf\{x : F_Y(x) \ge \beta\}$ .

DEFINITION 2.2 (Spectral risk measure [Acerbi, 2002]). Let  $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_N)^{\top}$ denote a non-decreasing probability mass function, i.e.  $\boldsymbol{\omega} \geq \boldsymbol{0}, \mathbf{1}^{\top} \boldsymbol{\omega} = 1$ , and  $\omega_k \geq \omega_\ell$ whenever  $k \geq \ell$ . The spectral risk measure  $M_{\omega}(\boldsymbol{y})$  generated by  $\boldsymbol{\omega}$  is defined as

$$M_{\omega}(\boldsymbol{y}) = \sum_{\ell=1}^{N} \omega_{\ell} y_{(\ell)}.$$

Let  $\omega_0 = 0$ . Then,

$$M_{\omega}(\mathbf{y}) = \sum_{\ell=1}^{N} \omega_{\ell} y_{(\ell)} = \sum_{\ell=1}^{N} (\omega_{\ell} - \omega_{\ell-1}) \left( \sum_{j=\ell}^{N} y_{(j)} \right) = \sum_{\ell=1}^{N} \gamma_{\ell} \mathrm{ES}_{\beta_{\ell}}(\mathbf{y}),$$

where  $\gamma_{\ell} = (N - \ell + 1)(\omega_{\ell} - \omega_{\ell-1}) \ge 0$  and  $\beta_{\ell} = \frac{\ell-1}{N}$ . Hence, it follows that  $M_{\omega}(\mathbf{y})$  is a coherent risk measure. It is easy to check that  $\sum_{\ell=1}^{N} \gamma_{\ell} = \sum_{\ell=1}^{N} \omega_{\ell} = 1$ , i.e.  $\boldsymbol{\gamma}$  is a probability mass function. This motivates the following definition.

DEFINITION 2.3 (Generalized spectral risk measures). Let  $\boldsymbol{\gamma} \in \mathbb{R}^d$  denote a probability mass function, i.e.  $\boldsymbol{\gamma} \geq \boldsymbol{0}$  and  $\mathbf{1}^\top \boldsymbol{\gamma} = 1$ . Let  $\boldsymbol{\beta} \in [0,1)^d$ . The generalized spectral risk measure  $\rho_{\boldsymbol{\gamma},\boldsymbol{\beta}}(\boldsymbol{y})$  is defined as

$$\rho_{\gamma,\beta}(\boldsymbol{y}) = \sum_{\ell=1}^{d} \gamma_{\ell} ES_{\beta_{\ell}}(\boldsymbol{y})$$

**2.2.** Portfolio selection problem. We measure the risk of portfolio **x** using m different risk models. Let  $\mathbf{L}_k \in \mathbb{R}^{N_k \times n}$  denote the empirical loss matrix corresponding to the k-th risk model, where  $N_k$  denotes the number of samples drawn according to the k-th model. The risk of portfolio **x** according to the k-th model is captured by a generalized spectral risk measure  $\rho_{\gamma_k,\beta_k}(\mathbf{L}_k \mathbf{x}), k = 1, \ldots, m$ . In the remainder of this paper, we will abbreviate  $\rho_{\gamma_k,\beta_k}$  simply as  $\rho_k$ .

The goal of the spectral risk constrained portfolio selection problem is to find the portfolio **x** that maximizes the expected return. Let  $\boldsymbol{\mu} \in \mathbb{R}^n$  be the mean return vector.  $\boldsymbol{\mu}$  is typically set equal to the weighted average  $\boldsymbol{\mu} = -\sum_{k=1}^m q_k \frac{1}{N_k} (\mathbf{L}_k^\top \mathbf{1})$ , where **q** is a probability mass function that assigns weights to the *m* risk models. Hence, the expected return of portfolio **x** is  $\boldsymbol{\mu}^\top \mathbf{x}$ . Given that cardinality constraints are important in practice to control the transaction costs [Chang et al., 2000], we are interested in selecting sparse portfolios, i.e. portfolios whose  $\ell_0$ -norm  $\sum_{i=1}^n \mathbb{1}(|x_i| > 0)$  is small. Unfortunately, the associated cardinality constrained portfolio selection problem is typically NP-hard. Nonetheless, a good approximation is to replace the  $\ell_0$ -norm with the  $\ell_1$ -norm  $\sum_{i=1}^n |x_i|$  [Candes et al., 2008]. Thus, the spectral risk constrained sparse portfolio selection problem we want to solve is of the form:

(2.2) 
$$\begin{aligned} \max \quad \boldsymbol{\mu}^{\top} \mathbf{x} - \lambda \|\mathbf{x}\|_{1} \\ \text{s.t.} \quad \rho_{k}(\mathbf{L}_{k}\mathbf{x}) \leq \alpha_{k}, \quad k = 1, \cdots, m, \\ \mathbf{1}^{\top} \mathbf{x} = 1, \\ \|\mathbf{x}\|_{\infty} \leq B, \end{aligned}$$

where  $\lambda \geq 0$  is the parameter controlling the sparsity of the portfolio,  $\alpha_k$  is the risk budget in the k-th risk model, the  $\ell_{\infty}$ -norm is defined as  $||\mathbf{x}||_{\infty} = \max_{1 \leq i \leq n} |x_i|$ , and the bound B > 0 controls the leverage of the portfolio. There are two additional interpretations for the  $\ell_1$ -norm regularization in (2.2). Since  $\mathbf{1}^\top \mathbf{x} = 1$ , the  $\ell_1$ -norm  $\|\mathbf{x}\|_1 = \sum_{i:x_i > 0} x_i - \sum_{i:x_i < 0} x_i = 1 - 2\sum_{i:x_i < 0} x_i$ , and therefore, penalizing the  $\ell_1$ norm is equivalent to penalizing short positions [DeMiguel et al., 2009]. Penalizing the  $\ell_1$ -norm of the portfolio also helps improve the out-of-sample performance of the portfolio in the presence of parameter estimation errors [DeMiguel et al., 2009] on the particular desired performance. In this paper, we are agnostic to the portfolio manager's reasons for penalizing the  $\ell_1$  norm of the portfolio –controlling transaction costs, constraining short sales, or improving out-of-sample performance of the portfolio. Therefore, we set  $\lambda = 2 |\boldsymbol{\mu}^\top \mathbf{x}^*| / \|\mathbf{x}^*\|$  where  $\mathbf{x}^* \in \operatorname{argmax}\{\boldsymbol{\mu}^\top \mathbf{x} : \mathbf{1}^\top \mathbf{x} = 1, \|\mathbf{x}\|_{\infty} \leq B\}$  to ensure that the two terms in the objective are always comparable. The numerical results reported in Section 4 clearly show that the running time of SPECRISKALLOCATE is not dependent on the value of  $\lambda$ .

The solution method that we develop in Section 3 is also able to solve the following portfolio selection problems:

(a) Sparse weighted mean-spectral risk portfolio selection problem

$$\begin{aligned} \max \quad \boldsymbol{\mu}^{\top} \mathbf{x} &- \lambda \left\| \mathbf{x} \right\|_{1} - \sum_{k=1}^{m} \theta_{k} \rho_{k}(\mathbf{L}_{k} \mathbf{x}) \\ \text{s.t.} \quad \mathbf{1}^{\top} \mathbf{x} &= 1, \\ \left\| \mathbf{x} \right\|_{\infty} &\leq B, \end{aligned}$$

where  $\boldsymbol{\theta} \in \mathbb{R}^m_+$  is a vector of weights.

(b) Sparse mean-max spectral risk portfolio selection problem

$$\max \quad \boldsymbol{\mu}^{\top} \mathbf{x} - \lambda \|\mathbf{x}\|_{1} - \theta \left( \max_{k=1,\cdots,m} \rho_{k}(\mathbf{L}_{k}\mathbf{x}) \right)$$
  
s.t. 
$$\mathbf{1}^{\top} \mathbf{x} = 1,$$
$$\|\mathbf{x}\|_{\infty} \leq B,$$

where  $\theta \ge 0$  is a penalty on the maximum spectral risk measure.

From the dual representation (2.1) of ES, it follows that the portfolio selection problem (2.2) can be reformulated as

$$\max \quad \boldsymbol{\mu}^{\top} \mathbf{x} - \lambda \| \mathbf{x} \|_{1}$$
  
s.t. 
$$\sum_{\ell=1}^{d_{k}} \gamma_{k\ell} \left( z_{k\ell} + \frac{1}{(1-\beta_{k\ell})N_{k}} \sum_{j=1}^{N_{k}} \left( (\mathbf{L}_{k} \mathbf{x})_{j} - z_{k\ell} \right)^{+} \right) \leq \alpha_{k}, \quad k = 1, \cdots, m,$$
$$\mathbf{1}^{\top} \mathbf{x} = 1, \\ \| \mathbf{x} \|_{\infty} \leq B,$$

where  $(\mathbf{L}_k \mathbf{x})_j$  denotes the *j*-th component of the vector  $\mathbf{L}_k \mathbf{x} \in \mathbb{R}^{N_k}$ . By introducing new variables  $y_{jk\ell} = ((\mathbf{L}_k \mathbf{x})_j - z_{k\ell})^+$ , and  $\xi_i = |x_i|$ , the above optimization problem can be reformulated as the LP (2.3)

$$\begin{array}{ll} \max & \boldsymbol{\mu}^{\top} \mathbf{x} - \lambda \mathbf{1}^{\top} \boldsymbol{\xi} \\ \text{s.t.} & \sum_{\ell=1}^{d_k} \gamma_{k\ell} \left( z_{k\ell} + \frac{1}{(1-\beta_{k\ell})N_k} \sum_{j=1}^{N_k} y_{jk\ell} \right) \leq \alpha_k, \quad k = 1, \cdots, m, \\ & y_{jk\ell} \geq (\mathbf{L}_k x)_j - z_{k\ell}, \qquad j = 1, \ldots, N_k, \quad \ell = 1, \ldots, d_k, \quad k = 1, \ldots, m, \\ & \boldsymbol{\xi} \geq \mathbf{x}, \qquad \boldsymbol{\xi} \geq -\mathbf{x}, \\ & \mathbf{1}^{\top} \mathbf{x} = 1, \quad \|\mathbf{x}\|_{\infty} \leq B, \\ & \mathbf{y} \geq 0. \end{array}$$

Unfortunately, this LP is typically very large. For example, when each generalized risk measure  $\rho_k$  has  $d \in S$  components, and the number of samples  $N_k$  is equal to N for each k, the LP (2.3) has  $\mathcal{O}(mdN+n)$  variables and constraints. Thus, with n = 100 assets, m = 5 risk constraints, each with d = 3 ES components, and N = 10,000 samples, the LP has 150,100 variables even though the original portfolio selection problem has only n = 100 variables! In addition, at any optimal solution a very large fraction of the  $y_{jkl}$  variables are zero; consequently, the LP is very ill-conditioned. Large, ill-conditioned LPs are extremely hard to solve in practice. In Section 4 we give empirical evidence supporting this claim.

**3.** Spectral risk constrained portfolio selection algorithm. In this section, we propose a fast iterative algorithm SPECRISKALLOCATE for computing a solution to (2.2) without introducing any new variables. Our goal is to be able to scale SPEC-RISKALLOCATE to solve very large scale portfolio selection problems; therefore, we

restrict ourselves to gradient descent algorithms. SPECRISKALLOCATE is an application of the proximal gradient algorithm FISTA [Beck and Teboulle, 2009] to a suitably defined "smoothed" penalty reformulation of (2.2). In Theorem 3.1 we establish an explicit value for the penalty parameter that guarantees that an  $\varepsilon$ -optimal solution to (2.2) can be reconstructed from the solution to the penalty formulation. The numerical results in Section 4 clearly show that our algorithm, which solves several small convex QPs, is significantly faster than the LP formulation that solves one very large LP. SPECRISKALLOCATE can be viewed as a decomposition algorithm that decomposes the large LP into a number of small QPs by exploiting the fact that its constraints are very loosely coupled, and then smooths the smaller QPs to improve convergence.

**3.1. Smoothed penalty formulation.** The portfolio selection problem (2.2) is clearly equivalent to the problem

$$\max_{\substack{\mathbf{\mu}^{\top} \mathbf{x} - \lambda \| \mathbf{x} \|_{1} \\ \text{s.t.} \quad \max_{1 \le k \le m} \{ \rho_{k}(\mathbf{L}_{k} \mathbf{x}) - \alpha_{k} \} \le 0, \quad k = 1, \cdots, m, \\ \mathbf{1}^{\top} \mathbf{x} = 1, \\ \| \mathbf{x} \|_{\infty} \le B. }$$

An exact penalty formulation of this optimization problem is given by

min 
$$\eta \left( \lambda \| \mathbf{x} \|_{1} - \boldsymbol{\mu}^{\top} \mathbf{x} \right) + \left( \max_{1 \leq k \leq m} \left\{ \rho_{k}(\mathbf{L}_{k} \mathbf{x}) - \alpha_{k} \right\} \right)^{\dagger}$$
  
s.t.  $\mathbf{1}^{\top} \mathbf{x} = 1,$   
 $\| \mathbf{x} \|_{\infty} \leq B,$ 

where  $\eta$  denotes the penalty parameter. We will find it convenient to scale the objective by  $\eta$  instead of scaling the penalty term. Let us express the maximum of m + 1 values,  $t_1, \ldots, t_{m+1}$ , as  $\Psi(t_1, \cdots, t_{m+1}) = \max_{\mathbf{u}} \{ \mathbf{t}^\top \mathbf{u} : \mathbf{1}^\top \mathbf{u} = 1, \mathbf{u} \ge \mathbf{0} \}$ , and define  $g(\mathbf{x}) = \Psi(\rho_1(\mathbf{L}_1\mathbf{x}) - \alpha_1, \ldots, \rho_m(\mathbf{L}_m\mathbf{x}) - \alpha_m, \mathbf{0})$ . Then, the above exact penalty formulation can be written as

(3.1) 
$$G(\eta) = \min_{\mathbf{x}, \mathbf{x}} \eta \left( \lambda \| \mathbf{x} \|_{1} - \boldsymbol{\mu}^{\top} \mathbf{x} \right) + g(\mathbf{x})$$
  
s.t.  $\mathbf{1}^{\top} \mathbf{x} = 1,$   
 $\| \mathbf{x} \|_{\infty} \leq B.$ 

We expect that the solution to (3.1) will converge to a solution to (2.2) as  $\eta \to 0$ . The next result establishes this claim and shows that there exists a lower bound  $\eta^*$  for the penalty parameter that guarantees that one can construct an  $\varepsilon$ -optimal solution for (2.2) from an  $\varepsilon$ -optimal solution to an appropriately smoothed version of  $G(\eta^*)$ .

THEOREM 3.1 (Penalty Representation). Suppose there exists a portfolio z,  $\mathbf{1}^{\top} z = 1$ ,  $\|z\|_{\infty} \leq B$ , such that z strictly satisfies all the generalized spectral risk constraints, i.e.  $\rho_k(\mathbf{L}_k z) < \alpha_k$ , for k = 1, ..., m. Define  $g_{\max}(x) = \max_{1 \leq k \leq m} \{\rho_k(\mathbf{L}_k x) - \alpha_k\}$ . Let  $P_u$  denote any upper bound on the optimal value  $P^*$  of the spectral risk portfolio selection problem (2.2). Suppose  $\overline{x}$  is an  $\varepsilon$ -optimal solution to the penalized problem (3.1) with

$$\eta^* = \frac{|g_{\max}(\boldsymbol{z})|}{P_u - (\boldsymbol{\mu}^\top \boldsymbol{z} - \lambda \|\boldsymbol{z}\|_1)}.$$

Then,

$$\hat{oldsymbol{x}} = rac{1}{1+ heta} \cdot \overline{oldsymbol{x}} + rac{ heta}{1+ heta} \cdot oldsymbol{z}$$

is an  $\varepsilon$ -optimal solution to the spectral risk portfolio selection problem (2.2), where  $\theta = \max \{g_{\max}(\bar{x})/|g_{\max}(z)|, 0\}.$ 

*Proof.* The proof is identical to that of Theorem 2 in Iyengar et al. [2011]. □

We would like to use a gradient-based algorithm to solve problem (3.1). However, both  $\Psi$  and the spectral risk measure  $\rho$  are non-smooth functions of their argument; consequently,  $g(\mathbf{x}) = \Psi(\rho_1(\mathbf{L}_1\mathbf{x}) - \alpha_1, \ldots, \rho_m(\mathbf{L}_m\mathbf{x}) - \alpha_m, 0)$  is a non-smooth function of the portfolio  $\mathbf{x}$ . We use a smooth approximation  $g_{\nu\delta}(\mathbf{x})$  to the function  $g(\mathbf{x})$  such that  $g(\mathbf{x}) - \nu - \delta \leq g_{\nu\delta}(\mathbf{x}) \leq g(\mathbf{x})$ . The details of the construction of  $g_{\nu\delta}$  are given in Appendix A. By replacing  $g(\mathbf{x})$  in (3.1) with  $g_{\nu\delta}(\mathbf{x})$ , we obtain the following smooth optimization problem:

$$G_{\nu\delta}(\eta) = \min_{\substack{ \mathbf{x} \\ \mathbf{x$$

Since the scenario-based spectral risk portfolio selection problem is itself an approximation to the stochastic optimization problem where the distribution of the loss  $\tilde{\mathbf{L}}$ is known, one does not expect to solve these problems to very high accuracy, i.e. a solution error of the order of  $10^{-12}$ . In practice, error of the order of  $10^{-3}$  is sufficient. Therefore, solving the smoothed problem for appropriately chosen values of  $\nu$  and  $\delta$ is sufficient for most practical instances. Moreover, in Section 4 we show that the smoothing significantly improves the computational tractability of this problem.

**3.2. First-order proximal gradient algorithm.** SPECRISKALLOCATE is displayed in Algorithm 1. SpecRISKALLOCATE computes an  $\varepsilon$ -optimal solution for the spectral risk constrained portfolio selection problem (2.2) by approximately solving a sequence of smoothed penalty problems  $G_{\nu\delta}(\eta)$  for a decreasing sequence of  $\eta$ . We begin with  $\eta \leftarrow \eta_0$  and then progressively reduce  $\eta \leftarrow c_n \eta$ , where  $c_n < 1$ . This continuation scheme ensures that SPECRISKALLOCATE is able to take large steps when the iterates are far from optimality. In Theorem 3.1 we showed that there exists  $\eta^* > 0$ such that we can recover an  $\varepsilon$ -optimal solution for (2.2) by solving  $G_{\nu\delta}(\eta^*)$ , i.e. we do not have to drive  $\eta$  all the way to zero. This feature adds stability to SPECRISK-ALLOCATE since the numerical accuracy required to solve  $G_{\nu\delta}(\eta)$  increases as  $\eta \searrow 0$ (see e.g. Nocedal and Wright [1999]). In practice, we stop whenever the relative change in iterate  $\mathbf{x}^{(j)}$  is smaller than the tolerance  $\varsigma$ , and the iterate  $\mathbf{x}^{(j)}$  is  $\varsigma$ -feasible, i.e.  $g_{\max}(\mathbf{x}^{(j)}) \leq \varsigma$ . SpecRiskAllocate calls FISTA to approximately solve  $G_{\nu\delta}(\eta)$ for a fixed value of  $\eta$ . FISTA is a proximal gradient method, i.e. a gradient descent algorithm with an additional proximal term to control the step length. The parameter  $\tau$  controls the accuracy demanded by FISTA. We need  $\tau \searrow 0$  to ensure that the accuracy is increased as  $\eta \searrow 0$ .

Next we describe some of the essential features of FISTA. We refer the reader to Beck and Teboulle [2009] for the details of the algorithm. The particular implementation of FISTA that we employ is displayed in Algorithm 2. FISTA computes an approximate solution to  $G_{\nu\delta}(\eta)$  by iteratively solving a sequence of quadratic optimization problems of the form

(3.2) 
$$\min_{\mathbf{x}, \mathbf{y}, \mathbf{y} \in \mathcal{S}} \|\mathbf{x} - \mathbf{y}\|_{2}^{2} \|\mathbf{x} - \mathbf{y}\|_{2}^{2},$$
$$\sup_{\mathbf{x}, \mathbf{y}, \mathbf{y} \in \mathcal{S}} \|\mathbf{x} - \mathbf{y}\|_{2}^{2},$$
$$\|\mathbf{x}\|_{\infty} \leq B,$$

where  $\boldsymbol{\xi} = \nabla \left( -\eta \boldsymbol{\mu}^{\top} \mathbf{y} + g_{\nu\delta}(\mathbf{y}) \right) = -\eta \boldsymbol{\mu} + \nabla g_{\nu\delta}(\mathbf{y})$ , and *C* is the Lipschitz constant of the gradient  $\boldsymbol{\xi}$ . Although one can explicitly compute its value, it is often the

**Algorithm 1** Algorithm SPECRISKALLOCATE $(\eta_0, c_\eta, \tau_0, c_\tau, \nu, \delta, \varsigma)$ 

1:  $\eta \leftarrow \eta_0$ 2:  $\tau \leftarrow \tau_0$ 3:  $C \leftarrow 1$ 4:  $\mathbf{x} \leftarrow \frac{1}{n} \mathbf{1}$ 5: repeat 6:  $\hat{\mathbf{x}} \leftarrow \mathbf{x}$ 7:  $(\mathbf{x}, C) \leftarrow \text{FISTA}(\hat{\mathbf{x}}, C, \eta, \tau, \nu, \delta)$ 8:  $\eta \leftarrow c_\eta \eta$ 9:  $\tau \leftarrow c_\tau \tau$ 10: until  $(\|\mathbf{x} - \hat{\mathbf{x}}\|_2 / \|\hat{\mathbf{x}}\|_2 < \varsigma)$  and  $\max_{1 \le k \le m} \{\rho_k(\mathbf{L}_k \mathbf{x}) - \alpha_k\} < \varsigma$ 11: return  $\mathbf{x}$ 

**Algorithm 2** Function FISTA( $\mathbf{x}, C, \eta, \tau, \nu, \delta$ )

1:  $\zeta \leftarrow 1.5$ 2:  $t \leftarrow 1$ 3:  $\mathbf{y} \leftarrow \mathbf{x}$ 4: repeat  $\hat{\mathbf{x}} \gets \mathbf{x}$ 5: $t \leftarrow t$ 6:  $\boldsymbol{\xi} \leftarrow \text{COMPUTEGRADIENT}(\mathbf{y}, \nu, \delta)$ 7: repeat 8:  $\mathbf{x} \leftarrow \operatorname{argmin}_{\mathsf{T}} \left\{ \eta \lambda \| \mathbf{z} \|_{1} + \boldsymbol{\xi}^{\mathsf{T}} (\mathbf{z} - \mathbf{y}) + \frac{C}{2} \| \mathbf{z} - \mathbf{y} \|_{2}^{2} : \mathbf{1}^{\mathsf{T}} \mathbf{z} = 1, \| \mathbf{z} \|_{\infty} \le B \right\}$ 9:  $F \leftarrow -\eta \boldsymbol{\mu}^{\top} \mathbf{x} + \eta \lambda \left\| \mathbf{x} \right\|_{1} + g_{\nu \delta}(\mathbf{x})$ 10:  $Q \leftarrow \eta \lambda \| \mathbf{x} \|_1 - \eta \boldsymbol{\mu}^\top \mathbf{y} + g_{\nu \delta}(\mathbf{y}) + \boldsymbol{\xi}^\top (\mathbf{x} - \mathbf{y}) + \frac{C}{2} \| \mathbf{x} - \mathbf{y} \|_2^2$ 11:  $C \leftarrow C\zeta$ 12: until F < Q13:  $C \leftarrow C/\zeta$  $t \leftarrow \frac{1+\sqrt{1+4\hat{t}^2}}{2}$ 14:15:  $\mathbf{y} \leftarrow \mathbf{x} + \frac{\hat{t}-1}{t}(\mathbf{x} - \hat{\mathbf{x}})$ 16: 17: **until**  $\left( \|\mathbf{x} - \hat{\mathbf{x}}\|_{2} / \|\hat{\mathbf{x}}\|_{2} \right) \leq \tau$ 18: return  $(\mathbf{x}, C)$ 

case that the Lipschitz constant C is too large. In practice, it is more efficient to use a backtracking method to compute C. The function FISTA does backtracking in lines 8–13 of Algorithm 2. FISTA is guaranteed to converge to an  $\varepsilon$ -optimal solution in  $\mathcal{O}(1/\varepsilon)$  iterations. However, the worst-case bound is often too conservative in practice. We terminate the FISTA iterations whenever the relative change in the iterates is below a threshold  $\tau$ . We make  $\tau$  progressively tighter as  $\eta$  is decreased.

Let  $\mathbf{y}^{(k)}$  denote the current FISTA iterate. Since  $-\eta \boldsymbol{\mu}^{\top} \mathbf{x} + g_{\nu\delta}(\mathbf{x})$  is a convex function with a Lipschitz continuous derivative, it follows that the quadratic function  $\boldsymbol{\xi}^{\top}(\mathbf{x}-\mathbf{y}) + \frac{C}{2} \|\mathbf{x}-\mathbf{y}\|_2^2$  is an upper bound for  $-\eta \boldsymbol{\mu}^{\top} \mathbf{x} + g_{\nu\delta}(\mathbf{x})$ . This ensures that the improvement in the true objective at the new iterate  $\mathbf{y}^{(k+1)}$  is at least as large as that predicted by the quadratic approximation (3.2). The quadratic approximation (3.2) only uses the first-order gradient information. Therefore, the algorithm used to solve  $G_{\nu\delta}(\eta)$  can be scaled to much larger problem sizes, and is also considerably more stable as the problem size increases, as compared to a full-fledged quadratic approximation that uses all the Hessian information; however, at the cost of a larger iteration count. Finally, note that (3.2) is equivalent to

$$\min_{\substack{\boldsymbol{\lambda} \in \mathcal{A}, \\ \boldsymbol{\lambda} \in \mathcal{A}, \\ \boldsymbol{\lambda} \in \mathcal{A}, \\ \boldsymbol{\lambda} \in \mathcal{A}, \\ \boldsymbol{\lambda} = 1, \\ \boldsymbol{\lambda} = 1,$$

i.e. the FISTA iterates are computed by solving an  $\ell_1$ -penalized separable convex QP with the number of decision variables equal to the number of assets. Thus, this problem can be solved very efficiently if one has access to a mean-variance solver. In Appendix B we show how to solve this problem using a single one-dimensional search. In practical instances, where it is likely that the portfolio selection problem has additional linear constraints, the portfolio manager can use the mean-variance or quadratic solver to compute the FISTA iterates. In Appendix B, we also show how to compute the gradient  $\boldsymbol{\xi}$  using  $\sum_{k=1}^{m} d_k + 1$  one-dimensional searches.

4. Numerical results. In this section we present numerical experiments that show the advantage of SPECRISKALLOCATE over the LP formulation when dealing with large instances of the spectral risk constrained portfolio selection problem. Next, we illustrate the convenience of considering several risk models to overcome the uncertainty in risk parameters when selecting a portfolio to hedge the risk of an existing one.

4.1. Ill-Conditioning and Problem Scaling Results. We tested our algorithm on random instances of the spectral risk constrained portfolio selection problem (2.2). We generated instances with different values for the number of assets n. The number of spectral risk constraints was m = 5 for all instances. For each spectral risk measure, we fixed the number of ES components to d = 3. The number of loss scenarios N was set equal for all risk models. We randomly generated the expected return percentage vector  $\boldsymbol{\mu}$ , the scenario-based loss matrices  $\mathbf{L}_k$ , the ES weight vectors  $\boldsymbol{\gamma}_k$ , and the ES levels  $\boldsymbol{\beta}_k \in [0.9, 1)^d$ . The spectral risk budgets  $\alpha_k$  were set to  $\hat{\alpha}_k - 0.1 |\hat{\alpha}_k|$ , where  $\hat{\alpha}_k$  is the value of the k-th spectral risk measure  $\rho_k(\mathbf{L}_k \hat{\mathbf{x}})$  at portfolio  $\hat{\mathbf{x}} = 1/n\mathbf{1}$ . We set the leverage bound to B = 1, and the parameter controlling the sparsity of the portfolio either to  $\lambda = 0$  or  $\lambda = \lambda^*$ , where  $\lambda^* = 2 |\boldsymbol{\mu}^\top \mathbf{x}^*| / ||\mathbf{x}^*||_1$ , and  $\mathbf{x}^* = \operatorname{argmax}\{\boldsymbol{\mu}^\top \mathbf{x} : \mathbf{1}^\top \mathbf{x} = 1, ||\mathbf{x}||_{\infty} \leq B\}$ . For all the instances generated, the value of  $\lambda^*$  was in the interval [0.01, 0.03]. The SPECRISKALLOCATE parameters were set as follows

$$\eta_0 = 10, c_n = 0.99, \tau_0 = 10^{-4}, c_\tau = 0.95, \nu = 0.01 \min |\alpha_k|, \delta = 0.01, \varsigma = 10^{-2}.$$

We solved each instance of the spectral risk constrained sparse portfolio selection problem using a MATLAB implementation of SPECRISKALLOCATE. For each instance, we also solved the LP formulation (2.3) using the state-of-the-art LP solver Gurobi [Gurobi Optimization, Inc., 2014] with an optimality tolerance of  $\varsigma = 10^{-2}$ . We solved the instances using Gurobi version 5.0.2 and Gurobi version 5.6.0. Our results indicate that, although the performance of Gurobi has improved significantly from one version to the other, our algorithm still offers a significant advantage over this state-of-theart LP solver. We called Gurobi from MATLAB using Gurobi's MATLAB interface. MATLAB was run on a 6-core, 3.07GHz Intel Xeon processor with 66GB of RAM running the Ubuntu OS.

perturbation $t$	solver	$\mu_S$	$\sigma_S$	$\sigma_S/\mu_S$					
0.05	Gurobi 5.0.2	132.53	202.75	1.5298					
0.05	Gurobi 5.6.0	111.77	4.30	0.0385					
0.05	SpecRiskAllocate	82.12	0.41	0.0050					
0.10	Gurobi 5.0.2	107.14	169.25	1.5797					
0.10	Gurobi 5.6.0	118.65	13.59	0.1145					
0.10	SpecRiskAllocate	81.96	0.75	0.0092					
TABLE 1									

Mean, standard deviation, and coefficient of variation of the number of iterations needed to solve 100 perturbed problems. Variance is much higher in the Gurobi case than in the SPECRISK-ALLOCATE case due to ill-conditioning of the LP formulation.

As mentioned in Section 3, the LP formulation (2.3) is very ill-conditioned. This is manifested in a high variance in the number of iterations required to solve similar problems, i.e. with very small perturbations in the parameter values. We now show empirically that one does not face this issue when (2.2) is solved using SPEC-RISKALLOCATE. We generated a base instance with (n, N) = (100, 1000). Next, we created S = 100 perturbed instances by setting each entry  $\ell_{ijk}^s$  of the loss matrix  $\mathbf{L}_k^s$ , corresponding the to the s-th perturbed problem, to  $\ell_{ijk}^s = \ell_{ijk} + t |\ell_{ijk}| \varepsilon_{ijk}^s$ , where  $t \in \{0.05, 0.1\}$  and  $\varepsilon_{ijk}^s$  are I.I.D. standard Normal random variables. Table 1 shows the mean  $\mu_S$  and the standard deviation  $\sigma_S$  of the number of iterations required by Gurobi and by SPECRISKALLOCATE (total FISTA iterations, in this case) to solve the S = 100 perturbed instances. Table 1 also shows the coefficient of variation  $\sigma_S/\mu_S$ of the number of iterations needed to solve the perturbed instances. The number of iterations required by SPECRISKALLOCATE has a coefficient of variation of less than 1%, where the same number for Gurobi 5.0.2 (resp. Gurobi 5.6.0) is approximately 158% (resp. 11%). It is clear that the ill-conditioning is completely resolved by SPECRISKALLOCATE.

In Section 3, we argued that the number of constraints and variables in LP (2.3)is very large. Consequently, we expect the time to solve large instances using the LP formulation to be high. In contrast, we expect SPECRISKALLOCATE to be able to solve large instances in a very reasonable amount of time. To support these claims, we generated 10 random instances for each pair of parameters (n, N) and solved them with the sparsity parameter  $\lambda$  set equal to  $\lambda^*$  or 0. Table 2 reports the results for this problem scaling study. The column labeled "err" lists the mean relative error of the optimal value found by SPECRISKALLOCATE with respect to the one found by Gurobi. For all but the smallest-sized problem, i.e. (n, N) = (10, 100), SPECRISKALLOCATE found a solution with an objective value within 0.5% of the optimal value, and an optimal solution for 7 out of the 11 problems parameterized by (n, N). For each instance, we set a maximum solution time limit of 1 hour. The columns labeled "limit" list the number of instances that could not be solved within the time limit. The columns labeled "time(s)" list the average run time in seconds, where we have included a run time of 3600 seconds for those instances that reached the solution time limit. Note that for three of the largest-sized problems, namely  $(n, N) \in \{(100, 15000), (1000, 10000), (1000, 15000)\},$  Gurobi was unable to solve at least 1 instance and up to 9 out of 10 instances within the time limit. Although the running time of Gurobi 5.6.0 shows a remarkable improvement for smaller problems, it still has trouble solving the instances corresponding to the two largest parameter values. In contrast, SPECRISKALLOCATE is able to solve all the problem instances at least an order of magnitude faster than Gurobi. Note that the run time reported

λ	m N	err	Gurobi 5.0.2		Gurobi 5.6.0		SpecRiskAllocate		
	$\pi$	11	(%)	limit	time(s)	limit	time(s)	limit	time(s)
$\lambda^*$	10	100	3.1	-	0.01	-	0.02	—	0.12
	10	500	-	_	0.51	_	0.12	_	0.24
	10	1000	0.1	-	1.54	-	0.24	-	0.29
	10	1500	-	_	0.5	_	0.48	_	0.6
	100	1000	-	_	4.61	_	2.93	_	3.21
	100	5000	0.1	_	230.37	_	18.96	_	14.19
	100	10000	-	-	497.36	-	54.33	-	15.73
	100	15000	-	-	98.38	-	98.58	-	67.7
	1000	5000	0.1	1	943.61	_	232.74	_	63.6
	1000	10000	-	1	1050.15	1	1199.99	_	247.44
	1000	15000	-	6	2538.93	5	2238.47	-	440.07
0	10	100	0.2	_	0.01	-	0.01	-	0.25
	10	500	0.5	-	0.27	-	0.07	-	0.28
	10	1000	-	-	0.13	-	0.13	-	0.39
	10	1500	0.2	_	0.27	_	0.22	_	0.57
	100	1000	-	_	1.61	_	1.55	_	13.76
	100	5000	-	-	133.78	-	10.61	-	42.77
	100	10000	-	-	26.02	-	25.56	-	94.97
	100	15000	-	_	41.8	_	40.88	_	68.79
	1000	5000	-	_	210.29	-	93.45	-	142.00
	1000	10000	-	9	3274.86	-	286.17	-	408.13
	1000	15000	-	9	3268.33	5	1960.7	-	420.17
					TABLE	2			

Average error (err) of SPECRISKALLOCATE with respect to Gurobi, number of problems (out of 10) that reached a runtime limit of 1 hour before finding a solution and average run times of Gurobi and SPECRISKALLOCATE when solving random instances of the spectral risk constrained portfolio optimization problem.

for Gurobi does not include the time required to set up the LP. Note also that, when the sparsity parameter  $\lambda = 0$ , SPECRISKALLOCATE is slower than Gurobi on the smaller instances, but faster on the largest instances; moreover, in contrast with Gurobi, SPECRISKALLOCATE is able to solve *all* the instances in less than an hour. SPECRISKALLOCATE is slower in this case because the stopping criterion in subroutine FISTA (see Algorithm 2) is harder to achieve when we do not regularize the portfolio by penalizing its  $\ell_1$ -norm. We believe that changing the FISTA stopping criterion to one better suited for the non-regularized problem, will significantly improve the running time.

The run times reported in Table 2 are for the version of SPECRISKALLOCATE that solves the constrained QP subproblems using an iterative line search. In typical applications, the portfolio selection problem is likely to have other side constraints, and it is unlikely that one would be able to solve the QP subproblems in this manner. In order to ensure that the run times are not an artifact of the simple feasible set, we also tested an implementation of SPECRISKALLOCATE where the QP step (and also the gradient computation step) were solved using the quadratic programming solver in Gurobi. The run times for this alternative implementation were similar to those reported in Table 2.

**4.2. Parameter Uncertainty.** Next, we illustrate how the stability and scalability of SPECRISKALLOCATE can be used to overcome parameter uncertainty when hedging the risk of a portfolio of derivatives.

Suppose a portfolio manager wants to hedge the risk of an existing portfolio  $\mathbf{x}_0$  of derivative instruments using a set of n liquid derivative positions. Let  $\tilde{V}_0(\tilde{\mathbf{S}}_t)$ 

and  $\tilde{V}_i(\tilde{\mathbf{S}}_t)$  denote, respectively, the value of the initial portfolio  $\mathbf{x}_0$  and the value of derivative instrument  $i \in \{1, \ldots, n\}$  at time t, as functions of the vector of underlying asset prices  $\tilde{\mathbf{S}}_t \in \mathbb{R}^s$ . Let  $\tilde{\ell}_0(t) = \tilde{V}_0(\tilde{\mathbf{S}}_0) - \tilde{V}_0(\tilde{\mathbf{S}}_t)$  (resp.  $\tilde{\ell}_i(t) = \tilde{V}_i(\tilde{\mathbf{S}}_0) - \tilde{V}_i(\tilde{\mathbf{S}}_t)$ ) denote the loss of the initial portfolio (resp. derivative instrument i) at time t. Then, the loss at time t of a hedging portfolio  $\mathbf{x} \in \mathbb{R}^n$  is given by  $\sum_{i=1}^n \tilde{\ell}_i(t)x_i$ , and the total loss at time t for the portfolio manager is  $\tilde{\ell}_0(t) + \sum_{i=1}^n \tilde{\ell}_i(t)x_i$ . Note that, in contrast with our previous notation,  $x_i$  now denotes the total number of units of derivative i purchased. Therefore, in what follows we drop the portfolio constraint  $\mathbf{1}^\top \mathbf{x} = 1$ .

Suppose the underlying asset prices  $\tilde{\mathbf{S}}_t$  are log-normally distributed with mean vector  $\boldsymbol{\pi}$  and unknown covariance matrix  $\tilde{\boldsymbol{\Sigma}}_t = \tilde{\mathbf{D}}_t \mathbf{R} \tilde{\mathbf{D}}_t$ , where  $\mathbf{R}$  is a constant correlation matrix and  $\tilde{\mathbf{D}}_t = \operatorname{diag}(\tilde{\boldsymbol{\sigma}}_t)$  is a diagonal matrix of unknown volatilities at time t. Suppose the portfolio manager knows the current volatility  $\boldsymbol{\sigma}_0$ , and believes that the volatility at the time horizon T is of the form  $\boldsymbol{\sigma}_T = \boldsymbol{\sigma}_0 + \sum_{p=1}^q \omega_p \boldsymbol{\rho}_p$ , where  $\boldsymbol{\rho}_p \in \mathbb{R}^s$  are known factors and  $\omega_p \in [-1, 1]$  are the corresponding unknown weights. For  $\boldsymbol{\omega} \in \Omega := \{-1, 1\}^q \cup \{\mathbf{0}\}$ , let  $\boldsymbol{\ell}_0(\boldsymbol{\omega}) \in \mathbb{R}^N$  (resp  $\boldsymbol{\ell}_i(\boldsymbol{\omega}) \in \mathbb{R}^N$ ) denote the vector of N samples of the loss  $\tilde{\ell}_0(T)$  on the initial portfolio (resp. the loss  $\tilde{\ell}_i(T)$  on derivative instrument i) when the volatility vector  $\boldsymbol{\sigma}_T = \boldsymbol{\sigma}_0 + \sum_{p=1}^q \omega_p \boldsymbol{\rho}_p$ . For a subset W of  $\Omega$ , consider the following hedging portfolio selection problem:

(4.1) 
$$\Pi(W) := \max_{\mathbf{x}} \min_{\boldsymbol{\omega} \in W} \left\{ \mu_{0} + \boldsymbol{\mu}(\boldsymbol{\omega})^{\top} \mathbf{x} - \lambda \| \mathbf{x} \|_{1} \right\}$$
  
s.t. 
$$\operatorname{ES}_{\beta}(\boldsymbol{\ell}_{0}(\boldsymbol{\omega}) + \mathbf{L}(\boldsymbol{\omega})\mathbf{x}) \leq \alpha \operatorname{ES}_{\beta}(\boldsymbol{\ell}_{0}(\boldsymbol{\omega})), \quad \boldsymbol{\omega} \in W$$
$$\| \mathbf{x} \|_{\infty} \leq B$$
$$= \max_{\mathbf{x},\mu} \quad \mu - \lambda \| \mathbf{x} \|_{1}$$
  
s.t. 
$$\mu \leq \mu_{0} + \boldsymbol{\mu}(\boldsymbol{\omega})^{\top} \mathbf{x}, \qquad \boldsymbol{\omega} \in W$$
$$\operatorname{ES}_{\beta}(\boldsymbol{\ell}_{0}(\boldsymbol{\omega}) + \mathbf{L}(\boldsymbol{\omega})\mathbf{x}) \leq \alpha \operatorname{ES}_{\beta}(\boldsymbol{\ell}_{0}(\boldsymbol{\omega})), \quad \boldsymbol{\omega} \in W$$
$$\| \mathbf{x} \|_{\infty} \leq B,$$

where  $\mathbf{L}(\boldsymbol{\omega}) = [\boldsymbol{\ell}_1(\boldsymbol{\omega}) \dots \boldsymbol{\ell}_n(\boldsymbol{\omega})], \ \boldsymbol{\mu}(\boldsymbol{\omega}) = -\frac{1}{N}\mathbf{L}(\boldsymbol{\omega})^\top \mathbf{1}, \text{ and } \mu_0 = -\frac{1}{N}\boldsymbol{\ell}_0(\boldsymbol{\omega})^\top \mathbf{1}.$  By solving problem (4.1), the portfolio manager is looking to compute an  $\boldsymbol{\ell}_1$ -regularized hedging portfolio  $\mathbf{x}$  that maximizes the worst-case (w.r.t. W) expected return of the total portfolio  $[\mathbf{x}_0^\top, \mathbf{x}^\top]^\top$ , while ensuring that the worst case expected shortfall drops by factor of  $\alpha < 1$ . We define  $\Pi(\{\mathbf{0}\})$  (resp.  $\Pi(\{-1,1\}^q)$ ) as the nominal (resp. robust) portfolio selection problem. Since we allow the hedging portfolio  $\mathbf{x}$  to have both long and short positions, in order to be robust against uncertainty in the parameters  $\omega_p$  we must consider all the possible worst-case risk models  $\boldsymbol{\omega} \in \{-1,1\}^q$ . Problem (4.1) is equivalent to

(4.2) 
$$\max_{\bar{\mathbf{x}}} \quad \bar{\boldsymbol{\mu}}^{\top} \bar{\mathbf{x}} - \lambda \| \bar{\mathbf{x}} \|_{1}$$
  
s.t. 
$$\operatorname{ES}_{0}(\boldsymbol{\ell}_{0}(\boldsymbol{\omega}) + \hat{\mathbf{L}}(\boldsymbol{\omega}) \bar{\mathbf{x}}) \leq 0, \qquad \boldsymbol{\omega} \in W$$
  
$$\operatorname{ES}_{\beta}(\boldsymbol{\ell}_{0}(\boldsymbol{\omega}) + \bar{\mathbf{L}}(\boldsymbol{\omega}) \bar{\mathbf{x}}) \leq \alpha \operatorname{ES}_{\beta}(\boldsymbol{\ell}_{0}(\boldsymbol{\omega})), \quad \boldsymbol{\omega} \in W$$
  
$$\mathbf{l} \leq \bar{\mathbf{x}} \leq \mathbf{u},$$

where  $\bar{\mathbf{x}} = [\mathbf{x}^{\top}, \mu^+, \mu^-]^{\top}$ ,  $\bar{\boldsymbol{\mu}} = [\mathbf{0}^{\top}, \lambda + 1, \lambda - 1]^{\top}$ ,  $\hat{\mathbf{L}}(\boldsymbol{\omega}) = [\mathbf{L}(\boldsymbol{\omega}), \mathbf{1}, -\mathbf{1}]$ ,  $\bar{\mathbf{L}}(\boldsymbol{\omega}) = [\mathbf{L}(\boldsymbol{\omega}), \mathbf{0}, \mathbf{0}]$ ,  $\mathbf{l} = [-B\mathbf{1}^{\top}, 0, 0]^{\top}$ , and  $\mathbf{u} = [B\mathbf{1}^{\top}, \infty, \infty]^{\top}$ . Thus, by slightly modifying SPECRISKALLOCATE to deal with box constraints of the form  $\mathbf{l} \leq \mathbf{x} \leq \mathbf{u}$  instead of the portfolio and leverage constraints  $\mathbf{1}^{\top}\mathbf{x} = 1$  and  $\|\mathbf{x}\|_{\infty} \leq B$ , we are able to exploit its stability and scalability to construct hedging portfolios that are robust against parameter uncertainty.

In what follows, we show that, using SPECRISKALLOCATE, one can construct a portfolio that reduces the risk of the initial portfolio while removing the impact of

the uncertain parameters on the expected return. Following Alexander et al. [2003], we assumed that the initial portfolio consisted of four short positions of European at-the-money binary call options, each on one of four correlated assets, with maturity in 4, 6, 8, and 10 months, respectively. The hedging universe was composed of 20 vanilla European calls on each asset, given by the combination of strike prices  $[0.9, 0.95, 1, 1.05, 1.1]S_0$  and maturities [2, 3, 4, 6] months, and the assets themselves. The time horizon was T = 1 month. We used N = 25,000 Monte Carlo samples to simulate the underlying asset prices. The derivatives were priced using Black-Scholes formulae. The rest of the problem parameters were set as follows: the q = 2 factors affecting the volatility,  $\rho_1 = 0.02[1, 1, 1, 1]^{\top}$  and  $\rho_2 = 0.02[1, -1, 1, -1]^{\top}$ ; the expected shortfall level  $\beta = 0.95$ ; the risk reduction factor  $\alpha = 0.5$ , i.e. the portfolio manager is looking reduce his exposure by half; the leverage bound B = 1; and the parameter controlling the sparsity of the portfolio  $\lambda = \theta \frac{2\mu(0)^{\top} \mathbf{x}^*}{\|\mathbf{x}^*\|_1}$ , where  $\theta \in \{0, 0.5, 1\}$ , and  $\mathbf{x}^*$  is the optimal solution to  $\Pi(\{\mathbf{0}\})$  with  $\lambda = 0$ .

Figures 1 and 2 show the out-of-sample expected shortfall and mean return of the initial, nominal and robust portfolios, as functions of the uncertain parameters  $(\omega_1, \omega_2) \in [-1, 1] \times \{-1, 0, 1\}$ , when the sparsity parameter  $\theta = 0$  and  $\theta = 1$ , respectively. Note that, in all cases, the risk constraint  $\mathrm{ES}_{\beta}(\boldsymbol{\ell}_{0}(\boldsymbol{\omega}) + \bar{\mathbf{L}}(\boldsymbol{\omega})\bar{\mathbf{x}}) \leq \alpha \mathrm{ES}_{\beta}(\boldsymbol{\ell}_{0}(\boldsymbol{\omega}))$ is violated by the nominal portfolio for  $\omega_1 > 0$ . On the other hand, the risk of the final robust portfolio is always less than half of that of the initial portfolio, regardless of the uncertain parameter values. In addition, the expected rate of return of the robust portfolio is virtually independent of the uncertain parameters  $(\omega_1, \omega_2)$ . In contrast, the expected rate of return of the nominal portfolio varies significantly as the uncertain parameters  $\omega_1$  and  $\omega_2$  change. Note that we are able to solve for the robust portfolio only because SPECRISKALLOCATE is computationally much more efficient as compared to the naive LP approach. In fact, SPECRISKALLOCATE is so efficient that one can solve portfolio selection problems with more complicated uncertainty in the covariance matrix  $\Sigma$ , or uncertainty in the mean return vector  $\pi$ , by including more risk constraints in (4.1). Finally, Figure 3 shows the positions  $x_i$  of the optimal nominal and robust portfolios, for  $\theta = 0.5$  and  $\theta = 1$ . Note that the robust portfolio holds position in almost all the instruments that the nominal portfolio does. However, the robust portfolio holds positions in other additional assets. These positions have the desired effect of reducing the out-of-sample risk and reducing the expected return variance. It is also worth noting that the sparsity parameter  $\theta$  seems to have a larger impact on the robust portfolio holdings than on the nominal portfolio ones.

5. Conclusion. In this paper, we propose a simple gradient-based algorithm SPECRISKALLOCATE for solving the portfolio selection problem with multiple spectral risk constraints. This algorithm computes the optimal portfolio by solving a sequence of separable convex QPs over the initial feasible set, i.e. the formulation does not increase the dimension of the problem to represent the risk measures. SPEC-RISKALLOCATE is very efficient both in theory and in practice. Our numerical experiments show that SPECRISKALLOCATE is at least one order of magnitude faster than the state-of-the-art general purpose solver on most instances of the spectral risk constrained portfolio selection problem that are of practical interest. Moreover, our numerical experiments show that SPECRISKALLOCATE allows portfolio managers to impose constraints with respect to multiple risk models as a means of inducing robustness in their portfolios against parameter uncertainty.



FIG. 1. Out-of-sample expected shortfall and mean return of the initial, nominal and robust portfolios, as a function of the uncertain parameters  $(\omega_1, \omega_2) \in [-1, 1] \times \{-1, 0, 1\}$ . The sparsity parameter  $\theta = 0$ .

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FIG. 2. Out-of-sample expected shortfall and mean return of the initial, nominal and robust portfolios, as a function of the uncertain parameters  $(\omega_1, \omega_2) \in [-1, 1] \times \{-1, 0, 1\}$ . The sparsity parameter  $\theta = 1$ .

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FIG. 3. Holdings of the nominal and robust portfolios. The sparsity parameter  $\theta = 0.5$  (top) and  $\theta = 1$  (bottom).

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Appendix A. Smoothing of g(x). Define the function

(A.1) 
$$f_{\beta}^{(\nu)}(\boldsymbol{\zeta}) = \max \quad \boldsymbol{\zeta}^{\top} \mathbf{q} - \frac{\nu}{2} \|\mathbf{q}\|_{2}$$
  
s.t.  $0 \leq \mathbf{q} \leq \frac{1}{(1-\beta)N} \mathbf{1},$   
 $\mathbf{1}^{\top} \mathbf{q} = 1.$ 

Nesterov [2005] establishes that  $f_{\beta}^{(\nu)}(\boldsymbol{\zeta})$  is a differentiable strongly convex function with gradient  $\nabla f_{\beta}^{(\nu)}(\boldsymbol{\zeta}) = \mathbf{q}^*$ , where  $\mathbf{q}^*$  is the unique solution to (A.1). The gradient  $\nabla f_{\beta}^{(\nu)}$  is Lipschitz continuous with Lipschitz constant  $1/\nu$ . Moreover,  $f_{\beta}^{(\nu)}$  satisfies  $\mathrm{ES}_{\beta}(\boldsymbol{\zeta}) - \nu \leq f_{\beta}^{(\nu)}(\boldsymbol{\zeta}) \leq \mathrm{ES}_{\beta}(\boldsymbol{\zeta})$ , i.e.  $f_{\beta}^{(\nu)}(\boldsymbol{\zeta})$  is a  $\nu$ -approximation to  $\mathrm{ES}_{\beta}(\boldsymbol{\zeta})$ .

Let  $\rho(\boldsymbol{\zeta}) = \sum_{\ell=1}^{d} \gamma_{\ell} \text{ES}_{\beta_{\ell}}(\boldsymbol{\zeta})$  denote any generalized spectral risk function. We define the smoothed spectral risk function as

$$\rho^{(\nu)}(\boldsymbol{\zeta}) = \sum_{\ell=1}^{d} \gamma_{\ell} f_{\beta_{\ell}}^{(\nu)}(\boldsymbol{\zeta})$$

Since  $\sum_{\ell=1}^{d} \gamma_{\ell} = 1$  for all generalized spectral risk functions, it follows that  $\rho(\boldsymbol{\zeta}) - \nu \leq \rho^{(\nu)}(\boldsymbol{\zeta}) \leq \rho(\boldsymbol{\zeta})$ . The gradient of  $\rho^{(\nu)}(\boldsymbol{\zeta})$  is given by  $\nabla \rho^{(\nu)}(\boldsymbol{\zeta}) = \sum_{\ell=1}^{d} \gamma_{\ell} \mathbf{q}_{\ell}^{*}$ , where  $\mathbf{q}_{\ell}^{*}$  is the unique optimal solution to (A.1) with  $\beta = \beta_{\ell}$ .

Finally, define

(A.2) 
$$\begin{aligned} \Psi^{(\delta)}(\mathbf{t}) &= \max_{\mathbf{t}} \mathbf{t}^\top \mathbf{u} - \frac{\delta}{2} \|\mathbf{u}\|_2 \\ \text{s.t.} \quad \mathbf{1}^\top \mathbf{u} &= 1 \\ \mathbf{u} \geq \mathbf{0}. \end{aligned}$$

 $\Psi^{(\delta)}$  is a differentiable convex function with Lipschitz continuous gradient  $\nabla \Psi^{(\delta)}(\mathbf{t}) = \mathbf{u}^*$ , where  $\mathbf{u}^*$  is the unique solution to (A.2), and Lipschitz constant  $1/\delta$  [Nesterov, 2005]. In addition, we have that  $\Psi(\mathbf{t}) - \delta \leq \Psi^{(\delta)}(\mathbf{t}) \leq \Psi(\mathbf{t})$ .

We define the smoothing of  $g(\mathbf{x})$  as

$$g_{\nu\delta}(\mathbf{x}) = \Psi^{(\delta)}\left(\rho_1^{(\nu)}(\mathbf{L}_1\mathbf{x}) - \alpha_1, \dots, \rho_m^{(\nu)}(\mathbf{L}_m\mathbf{x}) - \alpha_m, 0\right).$$

Theorem [7] in Iyengar et al. [2011] (see, also Hoda et al. [2010]) guarantees that  $g_{\nu\delta}(\mathbf{x})$  is a convex function with Lipschitz continuous gradient

(A.3) 
$$\nabla g_{\nu\delta}(\mathbf{x}) = \sum_{k=1}^{m} u_k^* \mathbf{L}_k^\top \nabla \rho_k^{(\nu)}(\mathbf{L}_k \mathbf{x}),$$

where

$$\mathbf{u}^* = \operatorname{argmax} \quad \sum_{k=1}^m u_k \left( \rho_k^{(\nu)}(\mathbf{L}_k \mathbf{x}) - \alpha_k \right) - \frac{\delta}{2} \| \mathbf{u} \|_2$$
  
s.t. 
$$\mathbf{1}^\top \mathbf{u} = 1$$
  
$$\mathbf{u} \ge \mathbf{0}.$$

Moreover,  $g_{\nu\delta}(\mathbf{x})$  is a  $(\nu + \delta)$ -approximation to  $g(\mathbf{x})$ , i.e.  $g(\mathbf{x}) - \nu - \delta \leq g_{\nu\delta}(\mathbf{x}) \leq g(\mathbf{x})$ .

Appendix B. Details of SpecRiskAllocate.

Recall that the FISTA iterates are computed by solving an  $\ell_1$ -penalized QP of the form (3.2). Next, we show how to solve this problem using a one-dimensional search. Dualizing the constraint  $\mathbf{1}^{\top}\mathbf{x} = 1$ , we obtain the following optimization problem:

$$\mathcal{L}(\gamma) = \min_{\|\mathbf{x}\|_{\infty} \leq B} \left\{ \eta \lambda \|\mathbf{x}\|_{1} + (\boldsymbol{\xi} - C\mathbf{y} + \gamma \mathbf{1})^{\top} \mathbf{x} + \frac{C}{2} \mathbf{x}^{\top} \mathbf{x} \right\}.$$

Writing  $\mathbf{x} = \mathbf{w} - \mathbf{v}$ , where  $\mathbf{w}, \mathbf{v} \ge \mathbf{0}$ , observe that

$$\begin{aligned} \mathcal{L}(\gamma) &= \min_{\mathbf{0} \leq \mathbf{w} \leq B\mathbf{1}} \left\{ \left( \eta \lambda \mathbf{1} + \boldsymbol{\xi} - C \mathbf{y} + \gamma \mathbf{1} \right)^{\top} \mathbf{w} + \frac{C}{2} \mathbf{w}^{\top} \mathbf{w} \right\} \\ &+ \min_{\mathbf{0} \leq \mathbf{v} \leq B\mathbf{1}} \left\{ \left( \eta \lambda \mathbf{1} - \boldsymbol{\xi} + C \mathbf{y} - \gamma \mathbf{1} \right)^{\top} \mathbf{v} - \frac{C}{2} \mathbf{v}^{\top} \mathbf{v} \right\}, \end{aligned}$$

where we have ignored the cross terms  $\mathbf{w}^{\top}\mathbf{v}$  because they are zero in any optimal solution. The optimal solution to  $\mathcal{L}(\gamma)$  is given by  $x_i^*(\gamma) = \min\{(\bar{c}_i - \gamma)/C, B\}^+ - \min\{(\underline{c}_i + \gamma)/C, B\}^+$ , where  $\bar{c}_i = -\eta\lambda - \xi_i + Cy_i$ , and  $\underline{c}_i = -\eta\lambda + \xi_i - Cy_i$ ,  $i = 1, \ldots, n$ . The optimal solution to (3.2) can be recovered by finding the dual variable  $\gamma^*$  such that  $\mathbf{1}^{\top}\mathbf{x}^*(\gamma^*) = 1$ . Since  $\lim_{\gamma \to \infty} \mathbf{x}^*(\gamma) = -B\mathbf{1}$  and  $\lim_{\gamma \to -\infty} \mathbf{x}^*(\gamma) = B\mathbf{1}$ , it follows that there exists  $\gamma^* \in (-\infty, \infty)$  such that  $\mathbf{1}^{\top}\mathbf{x}^*(\gamma^*) = 1$ . The computational complexity of finding  $\gamma^*$  is dominated by the computational cost of sorting the set  $\cup_{1 \le i \le n} \{\bar{c}_i, \underline{c}_i\}$ .

FISTA (see Algorithm 2) calls subroutine ComputeGradient, displayed in Algorithm 3, to compute the gradient  $\boldsymbol{\xi}$ . Computing gradient  $\boldsymbol{\xi}$  requires computing the gradient  $\nabla g_{\nu\delta}(\mathbf{x})$  (cf. (A.3)), which requires solving one QP of the form (A.2) and  $\sum_{k=1}^{m} d_k$  QPs of the form (A.1). Each of these QPs is of the form

(B.1) 
$$\begin{aligned} \max \quad \mathbf{c}^{\top} \mathbf{x} &- \frac{1}{2} \|\mathbf{x}\|_{2}^{2}, \\ \text{s.t.} \quad \mathbf{1}^{\top} \mathbf{x} &= 1, \\ \mathbf{0} \leq \mathbf{x} \leq \mathbf{b}, \end{aligned}$$

where the bound  $\mathbf{b} \ge 0$  satisfies  $\mathbf{1}^{\top}\mathbf{b} \ge 1$ , and is possibly infinite. Dualizing the constraint  $\mathbf{1}^{\top}\mathbf{x} = 1$ , we obtain the following separable QP:

$$\mathcal{L}(\gamma) = \max_{0 \le \mathbf{x} \le \mathbf{b}} \left\{ \sum_{i=1}^{n} (c_i - \gamma) x_i - \frac{1}{2} x_i^2 \right\}.$$

The optimal solution to  $\mathcal{L}(\gamma)$  is given by  $x_i^*(\gamma) = \min\{c_i - \gamma, b_i\}^+$ , i = 1, ..., n. The optimal solution to (B.1) can be recovered by finding the dual variable  $\gamma^*$  such that  $\mathbf{1}^\top \mathbf{x}^*(\gamma^*) = 1$ . Since  $\lim_{\gamma \to \infty} \mathbf{x}^*(\gamma) = \mathbf{0}$  and  $\lim_{\gamma \to -\infty} \mathbf{x}^*(\gamma) = \mathbf{b}$ , it follows that there exists  $\gamma^* \in (-\infty, \infty)$  such that  $\mathbf{1}^\top \mathbf{x}^*(\gamma^*) = 1$ . The computational complexity of computing  $\gamma^*$  is dominated by the computational cost of sorting the set  $\cup_{1 \le i \le n} \{c_i, c_i - b_i\}$ .

**Algorithm 3** Function COMPUTEGRADIENT( $\mathbf{y}, \nu, \delta$ )

1: for k = 1 to m do 2: for  $\ell = 1$  to  $d_k$  do 3:  $\mathbf{q}_{k\ell} \leftarrow \operatorname{argmax} \left\{ \mathbf{q}^\top \mathbf{L}_k \mathbf{y} - \frac{\nu}{2} \|\mathbf{q}\|_2^2 : \mathbf{1}^\top \mathbf{q} = 1, 0 \le \mathbf{q} \le \frac{1}{(1 - \beta_{k\ell})N_k} \mathbf{1} \right\}$ 4: end for 5: end for 6:  $\mathbf{u} \leftarrow \operatorname{argmax} \left\{ \sum_{k=1}^m v_k(\rho_k^{(\nu)}(\mathbf{L}_k \mathbf{y}) - \alpha_k) - \frac{\delta}{2} \|\mathbf{v}\|_2^2 : \sum_{k=1}^{m+1} v_k = 1, \mathbf{v} \ge \mathbf{0} \right\}$ 7:  $\boldsymbol{\xi} \leftarrow -\eta \boldsymbol{\mu} + \sum_{k=1}^m u_k \left( \sum_{\ell=1}^{d_k} \gamma_{k\ell} \mathbf{L}_k^\top \mathbf{q}_{k\ell} \right)$ 8: return  $\boldsymbol{\xi}$