# A UNIQUENESS RESULT FOR AN INVERSE PROBLEM OF THE STEADY STATE CONVECTION-DIFFUSION EQUATION

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ABSTRACT. We consider the inverse boundary value problem for the steady state convection diffusion equation. We prove that a velocity field V, is uniquely determined by the Dirichlet-to-Neumann map, when  $V \in C^{0,\gamma}(\Omega)$ ,  $2/3 < \gamma \leq 1$ , i.e. when V is a Hölder continuous vector field with  $2/3 < \gamma \leq 1$ .

### 1. INTRODUCTION

The steady state convection-diffusion equation

(1.1) 
$$(-\Delta + V \cdot \nabla)u = 0, \quad \text{in} \quad \Omega,$$
$$u|_{\partial\Omega} = f,$$

can be seen as a time independent model for transport phenomena in a fluid due to a diffusion process and convection caused by the fluid velocity V. One specific model is heat transfer in a fluid, in which case u is taken as the temperature. In the following we will consider this problem assuming that<sup>1</sup>  $f \in H^{1/2}(\partial \Omega)$  and  $V \in C^{0,\gamma}(\Omega, \mathbb{R}^n)$ , with  $2/3 < \gamma \leq 1$  and where the set  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 3$  will be a bounded open set with Lipschitz boundary. Recall that the space of Hölder continuous functions,  $C^{0,\gamma}(\Omega)$ ,  $0 < \gamma \leq 1$  is defined as

$$C^{0,\gamma}(\Omega) = \Big\{g \in C(\overline{\Omega}) : |g|_{C^{0,\gamma}(\Omega)} := \sup_{x,y \in \Omega, x \neq y} \frac{|g(x) - g(y)|}{|x - y|^{\gamma}} < \infty \Big\},$$

equipped with the norm

$$||g||_{C^{0,\gamma}(\Omega)} := ||g||_{L^{\infty}(\Omega)} + |g|_{C^{0,\gamma}(\Omega)}.$$

A physical formulation of the inverse problem we are about to consider, is to think of u as the temperature in the region  $\Omega$ , we then ask if it is possible to determine the velocity field V in the region  $\Omega$ by controlling the temperature on the boundary and by measuring the heat flux on the boundary.

The boundary measurements are mathematically modeled by the so called Dirichlet to Neumann map (DN-map for short). This is the map

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<sup>&</sup>lt;sup>1</sup>Here  $H^{s}(\Omega)$  refers to the  $L^{2}$  based Sobolev space with smoothness index s.

 $\Lambda_V$  taking f to  $\partial_n u := (n \cdot \nabla u)|_{\partial\Omega}$ , where n is the outward pointing unit normal to  $\partial\Omega$ . The unique solvability of the Dirichlet problem (1.1) in  $H^1(\Omega)$  (see Theorems 8.1 and 8.3 in [4]) shows that the DNmap well defined. The normal derivative  $\partial_n u$  needs, however in this case to be understood in a distributional sense, because of the nonsmooth solutions we consider. The DN-map can then be defined in a weak sense, as the operator  $\Lambda_V : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$  given by

$$\langle \Lambda_V f, \varphi \rangle := \int_{\Omega} (\nabla u \cdot \nabla \phi + V \cdot \nabla u \, \phi) dx,$$

where  $L_V u := (-\Delta + V \cdot \nabla)u = 0$ , in  $\Omega$ ,  $u|_{\Omega} = f$  and  $\varphi \in H^{1/2}(\partial \Omega)$ ,  $\phi \in H^1(\Omega)$ , with  $\phi|_{\partial\Omega} = \varphi$ . Here  $\langle \cdot, \cdot \rangle$  denotes the distribution duality on  $\partial \Omega$ . Notice also that the definition is independent of the choice of an extension  $\phi$  of  $\varphi$ .

The mathematical form of the inverse problem is then the question, if the DN-map of the Dirichlet problem (1.1) determines the velocity field V. The main result of this paper is the following theorem.

**Theorem 1.1.** Let  $V_j \in C^{0,\gamma}(\Omega, \mathbb{R}^n)$ , j = 1, 2 with  $2/3 < \gamma \leq 1$ . Assume that  $\Lambda_{V_1} = \Lambda_{V_2}$ , then  $V_1 = V_2$  in  $\Omega$ .

The first uniqueness result for the above inverse problem was given by Cheng, Nakamura and Sommersalo in [1], where they prove the unique determination of the velocity field V, for  $V \in C^{\infty}(\overline{\Omega})$ , and  $\partial \Omega \in C^{\infty}$ . Salo improved this in [8], where it is shown that the result also holds when V is Lipschitz continuous, i.e.  $V \in C^{0,1}(\Omega)$ . This was in turn improved by Knudsen and Salo in [6] where they prove that V can be any Hölder continuous function provided that  $\nabla \cdot V \in L^{\infty}$ . Theorem 1.1 improves on this by showing that the restriction  $\nabla \cdot V \in L^{\infty}$ , is unnecessary for Hölder continuous vector fields  $V \in C^{0,\gamma}(\Omega)$ , when  $2/3 < \gamma \leq 1$ .

The inverse problem of the closely related magnetic Schrödinger equation, was first studied by Sun in [11]. There have been several improvements of this result by various authors. The sharpest and most recent result is given by Krupchyk and Uhlmann in [7] where they prove that the inverse problem is solvable for an electric potential  $q \in L^{\infty}$ and a magnetic potential  $A \in L^{\infty}$ .

A first remark on Theorem 1.1 concerns its relations to the celebrated Calderon problem (see e.g. [12]). The Calderon problem asks if one can determine the conductivity in the interior of an object by measuring the current on the boundary, when one controls the voltage on the boundary (or vice versa), or in more mathematical terms if the DNmap corresponding to a Dirichlet problem of the conductivity equation  $\nabla \cdot (\sigma \nabla u) = 0$ , where  $\sigma$  is the conductivity, determines the conductivity. Writing the conductivity equation in non-divergence form we get that

$$\Delta u + \nabla \log(\sigma) \cdot \nabla u = 0.$$

This shows that the (1.1) is a more general and therefore a more difficult problem then the Calderon problem.

As a second remark on Theorem 1.1 we point out that the over all method of proving Theorem 1.1 is to reduce it to an inverse problem for the magnetic Schrödinger equation, which is a self-adjoint first order perturbation of the Laplacian. We will more specifically be utilizing the method of proving uniqueness for the inverse problem of the magnetic Schrödinger equation given in [7]. One of the main ideas is that one can still use the methods of [7] for electric potentials with worse regularity of a specific distributional form, provided one assumes that the magnetic potentials are more regular.

The paper is organized as follows. In section 2 we reduce Theorem 1.1 to a claim about the magnetic Schrödinger operator. Section 3 is devoted to constructing complex geometric optics solutions. In section 4 we prove the unique determination of the magnetic field and in section 5 we prove the unique determination of the electric potential.

### 2. Reduction to the Magnetic Schrödinger case

The purpose of this section is to reduce Theorem 1.1 to a similar statement concerning the magnetic Schrödinger operator. The argument is formulated by Cheng, Nakamura and Sommersalo in [1] and by Salo in [8]. The magnetic Schrödinger operator is formally given by

$$L_{A,q}u = -\Delta u - iA \cdot \nabla u - i\nabla \cdot (Au) + (A^2 + q)u.$$

We are going to consider the case where  $A \in C^{0,\gamma}(\Omega, \mathbb{R}^n)$  and  $q = \nabla \cdot F + p$ , with  $F \in C^{0,\gamma}(\Omega, \mathbb{R}^n)$  and  $p \in L^{\infty}(\Omega, \mathbb{C})$ . Hence we need to understand  $L_{A,q}$  in a distributional sense, as an operator  $L_{A,q}$ :  $H^1(\Omega) \to H^{-1}(\Omega)$ , given by

$$\langle L_{A,q}\phi,\psi\rangle := \int_{\Omega} \nabla\phi \cdot \nabla\psi + iA \cdot (\phi\nabla\psi - \psi\nabla\phi) + (A^2 + p)\phi\psi - F \cdot \nabla(\phi\psi) \, dx,$$

where  $\phi \in H^1(\Omega)$  and  $\psi \in H^1_0(\Omega)$ .

The inverse problem for the magnetic Schrödinger operator we are about to consider comes from the Dirichlet Problem

$$L_{A,q}u = 0, \quad \text{in} \quad \Omega,$$
$$u|_{\partial\Omega} = f,$$

where f is in the Sobolev space  $H^{1/2}(\partial \Omega)$ . The normal component of the magnetic gradient on the boundary,  $(\partial_n + in \cdot A)u|_{\partial \Omega}$ , here n denotes the outward pointing unit normal vector on  $\partial \Omega$ , is in our case defined, following [7], as the bounded linear map  $N_{A,q}: H^1(\Omega) \to H^{-1/2}(\partial \Omega)$ given by

$$\langle N_{A,q}u,\varphi\rangle = \int_{\Omega} \nabla u \cdot \nabla \phi + iA \cdot (u\nabla \phi - \phi\nabla u) + (A^2 + p)u\phi - F \cdot \nabla (u\phi) \, dx$$

for any  $u \in H^1(\Omega)$  such that  $L_{A,q}u = 0$  and any  $\varphi \in H^{1/2}(\partial \Omega)$ , such that  $\phi|_{\partial\Omega} = \varphi$ . The definition is independent of the choice of an extension  $\phi$  of  $\varphi$ .

We shall consider the more general notion of a Cauchy data set, instead of the DN-map when dealing with the magnetic Schrödinger equation. The Cauchy data sets are the sets of boundary data of solutions, i.e.

 $C_{A,q} := \{ (u|_{\partial\Omega}, N_{A,q}u) : u \in H^1(\Omega) \text{ and } L_{A,q}u = 0 \text{ in } \Omega \}.$ 

The magnetic field corresponding to a potential A is given by the 2-form dA, which is defined as

(2.1) 
$$dA = \sum_{1 \le j < k \le n} (\partial_j A_k - \partial_k A_j) dx_j \wedge dx_k,$$

this definition should be understood in the sense of non-smooth differential forms (a.k.a. currents).

Our aim is now to reduce Theorem 1.1 to the following Proposition, after which the rest of the paper is devoted to proving this Proposition.

**Proposition 2.1.** Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain with Lipschitz boundary. Assume that  $A_1, A_2, F_1, F_2 \in C^{0,\gamma}(\Omega, \mathbb{R}^n)$ ,  $2/3 < \gamma \leq 1$ , with  $A_1 = A_2$  and  $F_1 = F_2$  on  $\partial \Omega$ , and let  $p_1, p_2 \in L^{\infty}(\Omega, \mathbb{C})$ . Assume that  $C_{A_1,q_1} = C_{A_2,q_2}$ , then  $dA_1 = dA_2$  and  $\nabla \cdot F_1 + p_1 = \nabla \cdot F_2 + p_2$  in  $\Omega$ .

The above result is a variation of the main result in [7]. It differs from this by being applicable to lower regularity electric potentials (i.e. of the special distributional form), but it also by requires more regularity on the magnetic potentials.

Another more general point concerning the above result is that, we cannot in general hope to recover the magnetic potential A. This is because of the gauge invariance of the Cauchy data sets. If  $\psi \in C^{1,\gamma}(\Omega)$  and  $\psi|_{\partial\Omega} = 0$ , then  $C_{A,q} = C_{A+\nabla\psi,q}$ , i.e. it is possible to change the magnetic potentials without disturbing the boundary data (see Proposition 6.1 in the appendix).

At several points we will need extensions of Hölder continuous functions to a larger set containing  $\Omega$ . The following basic extension result on Hölder continuous functions will be used for this (see Theorem 3 on page 174 in [10] and Theorem 16.11 on page 342 in [2]).

**Lemma 2.2.** Let  $\Omega \subset \mathbb{R}^n$  be open set with Lipschitz boundary. Then there exists a continuous linear extension operator E,

$$E: C^{0,\gamma}(\Omega) \to C_0^{0,\gamma}(\mathbb{R}^n),$$

for  $0 \leq \gamma \leq 1$ . More precisely there exists a constant  $C = C(\Omega) > 0$ , such that for every  $f \in C^{0,\gamma}(\Omega)$ ,  $\operatorname{supp}(E(f))$  is compact,

$$E(f)|_{\Omega} = f$$

and one has the norm estimate

$$||E(f)||_{C^{0,\gamma}(\mathbb{R}^n)} \le C ||f||_{C^{0,\gamma}(\Omega)}$$

We will also need the following boundary reconstruction result from [8] (see Theorem 1.9 in [8]).

**Theorem 2.3.** Let  $\Omega \subset \mathbb{R}^n$  be open set with Lipschitz boundary and  $n \geq 3$ . Assume  $V_1, V_2 \in C^{0,\gamma}(\Omega, \mathbb{R}^n)$ ,  $0 < \gamma \leq 1$ . If  $\Lambda_{V_1} = \Lambda_{V_2}$ , then  $V_1|_{\partial\Omega} = V_2|_{\partial\Omega}$ .

Next we show how Theorem 1.1 follows from Proposition 2.1. We follow the argument given in [8]. The rest of the paper will focus on proving Proposition 2.1.

Proof of Theorem 1.1. By Theorem 2.3 we know that  $V_1 = V_2$  on  $\partial \Omega$ . Lemma 2.2 allows us then to extend  $V_j$  to a ball  $B, \Omega \subset B$  so that  $V_j \in C^{0,\gamma}(B, \mathbb{R}^n), V_j|_{\partial B} = 0$  and  $V_1 = V_2$  on  $B \setminus \Omega$ . Lemma 6.2 below shows that the above extension does not alter the DN-maps, i.e.  $\Lambda_{V_1}^B = \Lambda_{V_2}^B$ . We may thus assume that that  $\Omega = B$  and that  $V_1 = V_2 = 0$  on  $\partial \Omega = \partial B$ .

We now consider the magnetic Schrödinger operators  $L_{A_j,q_j}$ , j = 1, 2 that coincide with  $L_{V_i}$ . That is we choose

$$A_j := iV_j/2$$
 and  $q_j := V_j^2/4 - \nabla \cdot V_j/2$ ,

which gives that  $L_{A_i,q_i} = L_{V_i}$ .

Next we want to show that  $C_{A_j,q_j} = \{(f, \Lambda_{V_j}f) \mid f \in H^{1/2}(\partial B)\}$ . We need only to show that  $N_{A_j,q_j}u_j = \Lambda_{V_j}u_j$ , j = 1, 2. Let  $u_j \in H^1(B)$ be such that  $L_{A_j,q_j}u_j = 0$  and assume that  $\varphi \in H^{1/2}(\partial B)$  and that  $\phi \in H^1(B)$  is an extension of  $\varphi$ , i.e.  $\phi|_{\partial B} = \varphi$ . Then by definition and because  $V_j = 0$  on  $\partial B$ 

$$\langle N_{A_j,q_j} u_j, \varphi \rangle = \int_B (\nabla u_j \cdot \nabla \phi - \frac{1}{2} V_j \cdot (u_j \nabla \phi - \phi \nabla u_j) + \frac{1}{2} V_j \cdot \nabla (u_j \phi)) \, dx$$
  
= 
$$\int_B (\nabla u_j \cdot \nabla \phi + V_j \cdot \nabla u_j \phi) \, dx$$
  
= 
$$\langle \Lambda_{V_j} u_j, \varphi \rangle.$$

The assumption that  $\Lambda_{V_1} = \Lambda_{V_2}$ , implies therefore that  $C_{A_1,q_1} = C_{A_2,q_2}$ .

We can now apply Proposition 2.1, which gives that  $dV_1 = dV_2$ . By the Poincaré Lemma (see Theorem 8.3 in [2]), there exists an  $\psi \in C^{1,\gamma}(B)$ , s.t.  $V_1 - V_2 = \nabla \psi$ , since  $\nabla \psi = 0$  outside  $\operatorname{supp}(V_1) \cup \operatorname{supp}(V_2)$ , we have that  $\psi$  is constant near  $\partial B$ . We may hence add a constant to  $\psi$ , so that  $\psi = 0$  near  $\partial B$ . The second consequence of Proposition 2.1 is that  $q_1 = q_2$ , so that  $V_1^2/2 - \nabla \cdot V_1 = V_2^2/2 - \nabla \cdot V_2$ . This together with the fact that  $V_2 = \nabla \psi - V_1$ , gives the equation

(2.2) 
$$\Delta \psi - V_1 \cdot \nabla \psi + \frac{1}{2} (\nabla \psi)^2 = 0 \text{ in } B,$$

Next we prove that  $\psi \in C^2(B)$ . Because of (2.2) we have that  $\psi \in C^0(\overline{B})$  satisfies

$$\Delta \psi = f \text{ in } B,$$

with  $f = V_1 \cdot \nabla \psi - \frac{1}{2} (\nabla \psi)^2 \in C^{0,\gamma}(B)$ . By interior Schauder estimates (see Theorem 7.18 in [13]) we know that  $\psi \in C^{2,\gamma}(\overline{V})$ , for every open  $V \subset C B$ . It follows that  $\psi \in C^2(B)$ .

We may now apply the maximum principle to  $\psi$  (see Theorem 10.1 in [4]). From this it follows that  $\psi = 0$  in B, since  $\psi|_{\partial B} = 0$ . We may thus conclude that  $V_1 = V_2$ .

## 3. Complex geometric optics solutions and remainder estimates

In this section we shortly review the construction of complex geometric optics (CGO for short) solutions and then derive some remainder estimates related to these. We follow by large the construction given in [7]. We are however dealing with more regular magnetic potentials, which allows us to get the better remainder estimates that are needed. This and the more irregular electric potentials require us to make some modifications to the argument in [7].

Smooth approximations of the potentials will be an important tool in the following. Our smoothing procedure will consist of an extension followed by a convolution with a mollifier. More specifically, given an  $A \in C^{0,\gamma}(\Omega, \mathbb{C}^n)$ , we consider an open bounded set  $\Omega'$ , s.t.  $\Omega \subset \subset \Omega'$ . By Lemma 2.2 there is an extension of A to  $\mathbb{R}^n$ ,  $A' \in C^{0,\gamma}(\mathbb{R}^n, \mathbb{C}^n)$ , s.t. A = A' in  $\Omega$ ,  $A'|_{\mathbb{R}^n \setminus \Omega'} = 0$  and

(3.1) 
$$\|A'\|_{C^{0,\gamma}(\mathbb{R}^n,\mathbb{C}^n)} \le C \|A\|_{C^{0,\gamma}(\Omega,\mathbb{C}^n)}.$$

Moreover let  $\Psi$  belong to  $C_0^{\infty}(\mathbb{R}^n)$  with  $0 \leq \Psi(x) \leq 1$  for all  $x \in \mathbb{R}^n$ , supp  $\Psi \subset \{x \in \mathbb{R}^n : |x| \leq 1\}$  and  $\int_{\mathbb{R}^n} \Psi dx = 1$ . Define  $\Psi_{\theta}(x) = \theta^n \Psi(\theta x)$  for  $\theta \in (0, \infty)$  and  $x \in \mathbb{R}^n$ . We define  $A^{\sharp}$  for any  $A' \in C_0^{0,\gamma}(\mathbb{R}^n, \mathbb{C}^n)$ , as

$$A^{\sharp} := \Psi_{\theta} * A'.$$

Notice also that (3.1) implies that  $||A^{\sharp}||_{C^{0,\gamma}(\mathbb{R}^n,\mathbb{C}^n)} \leq C||A||_{C^{0,\gamma}(\Omega,\mathbb{C}^n)}$ , where C is independent of  $\theta$ .

The following Lemma gives some basic and well known estimates for the above approximation scheme (see [5]).

**Lemma 3.1.** Assume that  $A \in C^{0,\gamma}(\Omega, \mathbb{C}^n)$ , with  $0 < \gamma \leq 1$  and let A' be the above extension of A to  $\mathbb{R}^n$ . Then

(3.2) 
$$\|A' - A^{\sharp}\|_{L^{\infty}(\mathbb{R}^{n},\mathbb{C}^{n})} \leq C\theta^{-\gamma},$$

(3.3) 
$$\|\partial^{\alpha} A^{\sharp}\|_{L^{\infty}(\mathbb{R}^{n},\mathbb{C}^{n})} \leq C\theta^{|\alpha|-\gamma},$$

as  $\theta \to \infty$ , for any multi-index  $\alpha$ , with  $|\alpha| \ge 1$ .

*Proof.* Let  $\Psi$  be as above. Assume that  $x \in \mathbb{R}^n$ . For the first estimate we use (3.1) and have that

$$\begin{aligned} |A'(x) - A^{\sharp}(x)| &= \left| \int_{\mathbb{R}^n} A'(x)\Psi(y) \, dy - \int_{\mathbb{R}^n} A'(x-y)\theta^n \Psi(\theta y) \, dy \right| \\ &\leq \int_{\mathbb{R}^n} |A'(x)\Psi(y) - A'(x-y/\theta)\Psi(y)| \, dy \\ &\leq C ||A||_{C^{0,\gamma}(\Omega,\mathbb{C}^n)} \theta^{-\gamma} \int_{\mathbb{R}^n} |y|^{\gamma} |\Psi(y)| \, dy \\ &\leq C \theta^{-\gamma}. \end{aligned}$$

To derive the second estimate (3.3) notice firstly that

$$\int_{\mathbb{R}^n} \partial^\alpha \Psi(y) dy = 0,$$

for all multi indexes  $\alpha$ , with  $|\alpha| \ge 1$ . Let  $x \in \mathbb{R}^n$ , then using the above observation, we have that

$$\begin{split} |\partial^{\alpha} A^{\sharp}(x)| &= \Big| \int_{\mathbb{R}^{n}} A'(y) \theta^{n+|\alpha|} (\partial^{\alpha} \Psi) \big( \theta(x-y) \big) \, dy \Big| \\ &= \Big| \int_{\mathbb{R}^{n}} A'(x-y/\theta) \theta^{|\alpha|} (\partial^{\alpha} \Psi)(y) \, dy \Big| \\ &= \Big| \int_{\mathbb{R}^{n}} \big( A'(x-y/\theta) - A'(x) \big) \theta^{|\alpha|} (\partial^{\alpha} \Psi)(y) \, dy \Big| \\ &\leq \|A\|_{C^{0,\gamma}(\Omega,\mathbb{C}^{n})} \theta^{|\alpha|} \int_{\mathbb{R}^{n}} |y/\theta|^{\gamma} \big| (\partial^{\alpha} \Psi)(y) \Big| \, dy \\ &\leq C \theta^{|\alpha|-\gamma}. \end{split}$$

**Remark.** In the rest of this section we will consider A to be extended as A' outside  $\Omega$ , i.e. we use A to denote the extension A'.

We will now show how to construct so called complex geometric optics solutions following the argument in [7]. It is natural to formulate this in terms of certain semiclassical norms that are defined as follows

$$\begin{aligned} \|u\|_{H^{-1}_{\mathrm{scl}}(\Omega)}^{2} &:= \|u\|_{L^{2}(\Omega)}^{2} + \|h\nabla u\|_{L^{2}(\Omega)}^{2}, \\ \|v\|_{H^{-1}_{\mathrm{scl}}(\Omega)} &:= \sup_{0 \neq \psi \in C_{0}^{\infty}(\Omega)} \frac{|\langle v, \psi \rangle_{\Omega}|}{\|\psi\|_{H^{1}_{\mathrm{scl}}(\Omega)}}. \end{aligned}$$

The construction of CGO solutions is based on the solvability result below. The solvability result is in turn a consequence of a perturbed Carleman estimate, Proposition 7.2 in the appendix. The argument that shows how to obtain the solvability result from the Carleman estimate is standard and we refer to the proof of Proposition 2.3 in [7].

**Proposition 3.2.** Let  $A, F \in L^{\infty}(\Omega, \mathbb{C}^n)$ ,  $p \in L^{\infty}(\Omega, \mathbb{C})$  and  $q = \nabla \cdot F + p$ . Furthermore let  $\varphi(x) = \alpha \cdot x$ ,  $\alpha \in \mathbb{R}^n$  with  $|\alpha| = 1$ . If h > 0 is small enough, then for any  $v \in H^{-1}(\Omega)$ , there is a solution of the equation

$$e^{\varphi/h}h^2 L_{A,q}(e^{-\varphi/h}u) = v, \text{ in } \Omega,$$

which satisfies

(3.4) 
$$||u||_{H^1_{\mathrm{scl}}(\Omega)} \le \frac{C}{h} ||v||_{H^{-1}_{\mathrm{scl}}(\Omega)}$$

The CGO solutions  $u \in H^1(\Omega)$  considered here solve

$$L_{A,q}u = 0,$$

with  $A, F \in C^{0,\gamma}(\Omega, \mathbb{C}^n), 0 < \gamma \leq 1, p \in L^{\infty}(\Omega, \mathbb{C})$  and have the form

(3.5) 
$$u(x;\zeta,h) = e^{x\cdot\zeta/h}(a(x;\zeta,h) + r(x;\zeta,h)),$$

where  $\zeta \in \mathbb{C}^n$  with  $\zeta \cdot \zeta = 0$  and  $|\zeta| \sim 1$ ; *h* is a small semiclassical parameter; *a* is a smooth amplitude and *r* is a reminder term.

We begin by assuming that  $\zeta \in \mathbb{C}^n$ ,  $\zeta = \zeta_0 + \zeta_1$  is such that

(3.6) 
$$\zeta \cdot \zeta = 0, \ \zeta_0 \text{ is constant with respect to } h, \ \zeta_1 = \mathcal{O}(h),$$
  
as  $h \to 0$  and  $|\operatorname{Re} \zeta_0| = |\operatorname{Im} \zeta_0| = 1.$ 

Abbreviate the conjugated operator multiplied by  $h^2$ , with

$$L_{\zeta} := e^{-\zeta \cdot x/h} h^2 L_{A,q}(e^{\zeta \cdot x/h}).$$

Then in order to construct  $u(\cdot; \zeta, h)$  of the form (3.5), it is enough to prove the existence of a  $r(\cdot; \zeta, h) \in H^1(\Omega)$  solving

$$(3.7) L_{\zeta}r = -L_{\zeta}a,$$

in  $\Omega$  for a suitable a. The  $a \in C^{\infty}(\mathbb{R}^n)$  is picked as the solution to

(3.8) 
$$\zeta_0 \cdot \nabla a + i\zeta_0 \cdot A^{\sharp} a = 0, \quad \text{in} \quad \mathbb{R}^n,$$

so that left hand side of (3.7) becomes, using (3.6), (3.8) and (3.10) given below,

$$-L_{\zeta}a = h^2 \Delta a + ih^2 A \cdot \nabla a - h^2 m_A(a) - h^2 (A^2 + p)a + 2h\zeta_1 \cdot \nabla a + 2hi\zeta_0 \cdot (A - A^{\sharp})a + 2hi\zeta_1 \cdot Aa - h^2 m_{\nabla \cdot F}(a).$$

Here  $m_A$  and  $m_{\nabla \cdot F}$  are the bounded linear operators from  $H^1(\Omega)$  to  $H^{-1}(\Omega)$  defined by

$$\langle m_A(\phi), \psi \rangle := \int_{\Omega} i\phi A \cdot \nabla \psi \, dx,$$
$$\langle m_{\nabla \cdot F}(\phi), \psi \rangle := -\int_{\Omega} F \cdot \nabla(\phi\psi) \, dx,$$

for all  $\phi \in H^1(\Omega)$  and all  $\psi \in H^1_0(\Omega)$ . It easy to see that

(3.10) 
$$e^{-\zeta \cdot x/h} \circ h^2 m_A \circ e^{\zeta \cdot x/h} = -hi\zeta \cdot A + h^2 m_A,$$
$$e^{-\zeta \cdot x/h} \circ h^2 m_{\nabla \cdot F} \circ e^{\zeta \cdot x/h} = h^2 m_{\nabla \cdot F}.$$

If we look for solutions to (3.8) in the form  $a = e^{\Phi^{\sharp}}$ , it will be enough that  $\Phi^{\sharp}(\cdot; \zeta_0, \theta)$  satisfies

(3.11) 
$$\zeta_0 \cdot \nabla \Phi^{\sharp} + i\zeta_0 \cdot A^{\sharp} = 0$$

in  $\mathbb{R}^n$ . The fact that  $\operatorname{Re} \zeta_0 \cdot \operatorname{Im} \zeta_0 = 0$  and  $|\operatorname{Re} \zeta_0| = |\operatorname{Im} \zeta_0| = 1$ , implies that  $N_{\zeta_0} := \zeta_0 \cdot \nabla$  is a  $\overline{\partial}$ -operator in suitable coordinates. The Cauchy operator  $N_{\zeta_0}^{-1}$ , defined by

$$(N_{\zeta_0}^{-1}f)(x) := \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{f(x - y_1 \operatorname{Re} \zeta_0 - y_2 \operatorname{Im} \zeta_0)}{y_1 + iy_2} \, dy_1 dy_2$$

for  $f \in C_0(\mathbb{R}^n)$ , is the inverse of the  $\overline{\partial}$ -operator and gives thus that

$$\Phi^{\sharp} = N_{\zeta_0}^{-1}(-i\zeta_0 \cdot A^{\sharp}) \in C^{\infty}(\mathbb{R}^n).$$

We will also use the following basic continuity result for the Cauchy operator (see [8], Lemma 7.4).

**Lemma 3.3.** Let  $f \in W^{k,\infty}(\mathbb{R}^n)$ ,  $k \geq 0$ , with  $\operatorname{supp}(f) \subset B(0, R)$ . Then we have that

(3.12) 
$$\|N_{\zeta_0}^{-1}f\|_{W^{k,\infty}(\mathbb{R}^n)} \le C \|f\|_{W^{k,\infty}(\mathbb{R}^n)},$$

where C = C(R).

Using now Lemma 3.1 and Lemma 3.3, we have that

(3.13) 
$$\|\partial^{\alpha} \Phi^{\sharp}\|_{L^{\infty}(\mathbb{R}^{n})} \leq C \theta^{|\alpha|-\gamma}$$

for  $\theta \in (1,\infty)$  and a multi-indexes  $\alpha$ ,  $|\alpha| \geq 1$ . Moreover, defining  $\Phi(\cdot;\zeta_0) := (\zeta_0 \cdot \nabla)^{-1} (-i\zeta_0 \cdot A) \in L^{\infty}(\mathbb{R}^n)$ , solves analogously

(3.14) 
$$\zeta_0 \cdot \nabla \Phi + i\zeta_0 \cdot A = 0$$

and satisfies

(3.15) 
$$\|\Phi(\cdot;\zeta_0)\|_{L^{\infty}(\mathbb{R}^n)} \le C \|A\|_{L^{\infty}(\mathbb{R}^n)}.$$

Lemma 3.3 and estimate (3.2) imply that the functions  $\Phi^{\sharp}$  converge to  $\Phi$  in  $L^{\infty}(\Omega)$  or more explicitly that

$$\left\| \Phi^{\sharp}(\cdot,\zeta_{0},\theta) - \Phi(\cdot;\zeta_{0}) \right\|_{L^{\infty}(\mathbb{R}^{n})} \leq C\theta^{-\gamma}.$$

With the a at hand the solvability result, Proposition 3.2 guarantees the existence of a solution r, to equation (3.7), such that

(3.16) 
$$||r||_{H^1_{\rm scl}(\Omega)} \le \frac{C}{h} ||L_{\zeta}a||_{H^{-1}_{\rm scl}(\Omega)}.$$

Now we determine how the left hand side of the above estimate depends on h, i.e. we estimate the  $H_{\rm scl}^{-1}(\Omega)$ -norm of the terms in equation (3.9). This gives us the behaviour of the  $H_{\rm scl}^1(\Omega)$ -norm of the remainder term r in the parameter h.

Let  $0 \neq \psi \in C_0^{\infty}(\Omega)$ . Then using (3.13), the fact that  $\zeta_1 = \mathcal{O}(h)$  and the Cauchy–Schwarz inequality we get that

$$\begin{split} |\langle h^2 \Delta a, \psi \rangle_{\Omega}| &\leq \mathcal{O}(h^2 \theta^{2-\gamma}) \|\psi\|_{L^2(\Omega)} \leq \mathcal{O}(h^2 \theta^{2-\gamma}) \|\psi\|_{H^1_{\mathrm{scl}}(\Omega)}, \\ |\langle ih^2 A \cdot \nabla a, \psi \rangle_{\Omega}| &\leq \mathcal{O}(h^2 \theta^{1-\gamma}) \|\psi\|_{H^1_{\mathrm{scl}}(\Omega)}, \\ |\langle 2h\zeta_1 \cdot \nabla a, \psi \rangle_{\Omega}| &\leq \mathcal{O}(h^2 \theta^{1-\gamma}) \|\psi\|_{H^1_{\mathrm{scl}}(\Omega)}, \\ |\langle 2hi\zeta_1 \cdot Aa, \psi \rangle_{\Omega}| &\leq \mathcal{O}(h^2) \|\psi\|_{H^1_{\mathrm{scl}}(\Omega)}, \\ |\langle h^2(A^2+p)a, \psi \rangle_{\Omega}| &\leq \mathcal{O}(h^2) \|\psi\|_{H^1_{\mathrm{scl}}(\Omega)}. \end{split}$$

By Lemma 3.1 we have on the other hand that

$$\begin{aligned} |\langle 2hi\zeta_0 \cdot (A - A^{\sharp})a, \psi \rangle_{\Omega}| &\leq \mathcal{O}(h) \|a\|_{L^{\infty}(\Omega)} \|A - A^{\sharp}\|_{L^2(\Omega)} \|\psi\|_{L^2(\Omega)} \\ &\leq \mathcal{O}(h)\theta^{-\gamma} \|\psi\|_{H^1_{\mathrm{scl}}(\Omega)}. \end{aligned}$$

Again by Lemma 3.1 and estimate (3.13) we have that

$$\begin{aligned} |\langle h^2 m_A(a), \psi \rangle_{\Omega}| &\leq \left| \int_{\Omega} i h^2 A^{\sharp} a \cdot \nabla \psi dx \right| + \left| \int_{\Omega} i h^2 (A - A^{\sharp}) a \cdot \nabla \psi dx \right| \\ &\leq \left| \int_{\Omega} i h^2 (\nabla \cdot (A^{\sharp} a)) \psi dx \right| + \mathcal{O}(h) \|A - A^{\sharp}\|_{L^2(\Omega)} \|h \nabla \psi\|_{L^2(\Omega)} \\ &\leq (\mathcal{O}(h^2 \theta^{1-\gamma}) + \mathcal{O}(h) \theta^{-\gamma}) \|\psi\|_{H^1_{\mathrm{scl}}(\Omega)}. \end{aligned}$$

Similarly with the help of Lemma 3.1 and estimate (3.13) we have that

$$\begin{split} |\langle h^2 m_{\nabla \cdot F}(a), \psi \rangle_{\Omega}| &\leq \left| \int_{\Omega} ih^2 F^{\sharp} \cdot \nabla(a\psi) dx \right| + \left| \int_{\Omega} ih^2 (F - F^{\sharp}) \cdot \nabla(a\psi) dx \right| \\ &\leq \left| \int_{\Omega} ih^2 \nabla F^{\sharp} \cdot a\psi dx \right| + \left| \int_{\Omega} ih^2 (F - F^{\sharp}) \cdot \nabla(a\psi) dx \right| \\ &\leq Ch^2 \theta^{1-\gamma} \|\psi\|_{H^1_{\mathrm{scl}}(\Omega)} + Ch^2 \|F - F^{\sharp}\|_{L^2(\Omega)} \|\nabla a\|_{L^{\infty}(\Omega)} \|\psi\|_{L^2(\Omega)} \\ &\quad + Ch \|F - F^{\sharp}\|_{L^2(\Omega)} \|a\|_{L^{\infty}(\Omega)} \|h \nabla \psi\|_{L^2(\Omega)} \\ &\leq C(h^2 \theta^{1-\gamma} + h^2 \theta^{1-2\gamma} + h\theta^{-\gamma}) \|\psi\|_{H^1_{\mathrm{scl}}(\Omega)}. \end{split}$$

Combining the above estimates gives that

$$\|L_{\zeta}a\|_{H^{-1}_{scl}(\Omega)} \le C(h^2\theta^{2-\gamma} + h\theta^{-\gamma})$$

By choosing  $\theta = h^{-1/2}$ , we get hence by estimate (3.16) that

$$||r||_{H^1_{scl}(\Omega)} \le Ch^{\gamma/2}$$

We have thus derived the following Proposition.

**Proposition 3.4.** Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 3$ , be a bounded open set with Lipschitz boundary. Let  $A, F \in C^{0,\gamma}(\Omega, \mathbb{R}^n)$ ,  $0 < \gamma \leq 1$ ,  $p \in L^{\infty}(\Omega, \mathbb{C}^n)$ , with  $q := \nabla \cdot F + p$  and let  $\zeta \in \mathbb{C}^n$  satisfy (3.6). Then for all h > 0 small enough, there exists a solution  $u(x, \zeta; h) \in H^1(\Omega)$  of

$$L_{A,q}u = 0, in \Omega$$

of the form  $u(x,\zeta;h) = e^{x\cdot\zeta/h}(e^{\Phi^{\sharp}(x,\zeta_0;h)} + r(x,\zeta;h))$ . The function  $\Phi^{\sharp}(\cdot,\zeta_0;h) \in C^{\infty}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$  satisfies

(3.17) 
$$\|\partial^{\alpha} \Phi^{\sharp}\|_{L^{\infty}(\mathbb{R}^{n})} \leq C_{\alpha} h^{\frac{\gamma-|\alpha|}{2}},$$

for all  $\alpha$ ,  $|\alpha| \ge 1$ , and  $\Phi^{\sharp}(\cdot, \zeta_0; h)$  converges in the  $L^{\infty}$ -norm to  $\Phi(\cdot, \zeta_0) := N_{\zeta_0}^{-1}(-i\zeta_0 \cdot A) \in L^{\infty}(\mathbb{R}^n)$ . More precisely

(3.18) 
$$\left\| \Phi^{\sharp}(\cdot,\zeta_{0},h) - \Phi(\cdot;\zeta_{0}) \right\|_{L^{\infty}(\mathbb{R}^{n})} \leq Ch^{\gamma/2}$$

The remainder r is such that

(3.19) 
$$||r||_{H^1_{rel}(\Omega)} \le Ch^{\gamma/2},$$

as  $h \to 0$ .

### 4. Uniqueness of the magnetic field

This section contains a proof of the first part of Proposition 2.1, i.e. we show that  $dA_1 = dA_2$ . We begin by stating an integral identity, which readily follows from the assumption that  $C_{A_1,q_1} = C_{A_2,q_2}$ . The proof can be found in [7] and only minor modifications are needed to make it work with electric potentials used here.

**Proposition 4.1.** Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 3$ , be a bounded open set with Lipschitz boundary. Assume that  $p_1, p_2 \in L^{\infty}(\Omega, \mathbb{C})$  and  $A_1, A_2, F_1, F_2 \in C^{0,\gamma}(\Omega, \mathbb{C}^n)$ , with  $0 < \gamma \leq 1$ . If  $C_{A_1,q_1} = C_{A_2,q_2}$ , then the following integral identity

$$\int_{\Omega} i(A_1 - A_2) \cdot (u_1 \nabla \overline{u_2} - \overline{u_2} \nabla u_1) + (A_1^2 - A_2^2 + p_1 - p_2) u_1 \overline{u_2}$$
(4.1) 
$$-(F_1 - F_2) \cdot (u_1 \nabla \overline{u_2} + \overline{u_2} \nabla u_1) \, dx = 0$$

holds for any  $u_1, u_2 \in H^1(\Omega)$  satisfying  $L_{A_1,q_1}u_1 = 0$  in  $\Omega$  and  $L_{\overline{A_2},\overline{q_2}}u_2 = 0$  in  $\Omega$ , respectively.

The idea is then to choose specific CGO solutions and insert them into the integral identity and then show that this reduces, in the limit  $h \rightarrow 0$  to a specific Fourier transform. The CGO will be chosen as follows. Let  $\xi, \mu_1, \mu_2 \in \mathbb{R}^n$  be such that  $|\mu_1| = |\mu_2| = 1$  and  $\mu_1 \cdot \mu_2 = \mu_1 \cdot \xi = \mu_2 \cdot \xi = 0$ . Define

(4.2) 
$$\zeta_1 = \frac{ih\xi}{2} + \mu_1 + i\sqrt{1 - h^2 \frac{|\xi|^2}{4}}\mu_2,$$
$$\zeta_2 = -\frac{ih\xi}{2} - \mu_1 + i\sqrt{1 - h^2 \frac{|\xi|^2}{4}}\mu_2,$$

so that  $\zeta_j \cdot \zeta_j = 0, \ j = 1, 2$ , and

(4.3) 
$$(\zeta_1 + \overline{\zeta_2})/h = i\xi.$$

Here h > 0 is a small enough. Moreover,  $\zeta_1 = \mu_1 + i\mu_2 + \mathcal{O}(h)$  and  $\zeta_2 = -\mu_1 + i\mu_2 + \mathcal{O}(h)$  as  $h \to 0$ .

For all h > 0, that are small enough there exists, by Proposition 3.4 a solution  $u_1(x, \zeta_1; h) \in H^1(\Omega)$  to the equation  $L_{A_1,q_1}u_1 = 0$  in  $\Omega$ , of the form

(4.4) 
$$u_1(x,\zeta_1;h) = e^{x \cdot \zeta_1/h} (e^{\Phi_1^{\sharp}(x,\mu_1+i\mu_2;h)} + r_1(x,\zeta_1;h)),$$

where  $\Phi_1^{\sharp}(\cdot, \mu_1 + i\mu_2; h) \in C^{\infty}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$  is given by

(4.5) 
$$\Phi_1^{\sharp}(\cdot,\mu_1+i\mu_2;h) := N_{\mu_1+i\mu_2}^{-1} \left( -i(\mu_1+i\mu_2) \cdot A_1^{\sharp} \right)$$

and  $\Phi_1^{\sharp}(\cdot, \mu_1 + i\mu_2; h) \to \Phi_1(\cdot, \mu_1 + i\mu_2)$  in  $L^{\infty}(\mathbb{R}^n)$  as  $h \to 0$ , where  $\Phi_1$  is given by Proposition 3.4.

Similarly, for all h > 0 small enough, there exists a solution  $u_2(x, \zeta_2; h) \in H^1(\Omega)$  to the equation  $L_{\overline{A_2}, \overline{q_2}} u_2 = 0$  in  $\Omega$ , of the form

(4.6) 
$$u_2(x,\zeta_2;h) = e^{x\cdot\zeta_2/h} (e^{\Phi_2^{\sharp}(x,-\mu_1+i\mu_2;h)} + r_2(x,\zeta_2;h)),$$

where  $\Phi_2^{\sharp}(\cdot, -\mu_1 + i\mu_2; h) \in C^{\infty}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$  is given by

(4.7) 
$$\Phi_2^{\sharp}(\cdot, -\mu_1 + i\mu_2; h) := N_{-\mu_1 + i\mu_2}^{-1} \left( -i(-\mu_1 + i\mu_2) \cdot A_2^{\sharp} \right)$$

and  $\Phi_2^{\sharp}(\cdot, -\mu_1 + i\mu_2; h) \to \Phi_2(\cdot, -\mu_1 + i\mu_2)$  in  $L^{\infty}(\mathbb{R}^n)$  as  $h \to 0$ , where  $\Phi_2$  is given by Proposition 3.4.

Notice also that we have by estimates (3.17) and (3.19), of Proposition 3.4, that

(4.8) 
$$\|\nabla \Phi_j^{\sharp}\|_{L^{\infty}(\mathbb{R}^n)} \le Ch^{\frac{\gamma-1}{2}},$$

(4.9) 
$$\|r_j\|_{H^1_{scl}(\Omega)} \le Ch^{\gamma/2},$$

for j = 1, 2.

The next step is to insert the  $u_1$  and  $u_2$  specified above into (4.1), multiply by h and let  $h \to 0$ , in an attempt to obtain a Fourier transform of the magnetic field. This is done in the next Lemma. The proof is based on the argument found in [7]. The difference is however in how the electric potential is estimated. The crucial observation is that the last term in (4.1) containing the electric potentials, goes to zero, in h when multiplied with an extra factor of h, even though it closely resembles the first term with the magnetic potentials, for which this does not happen.

**Lemma 4.2.** For  $A_1, A_2, \mu_1, \mu_2$  and  $\xi$  as above we have that

(4.10) 
$$(\mu_1 + i\mu_2) \cdot \int_{\mathbb{R}^n} (A_1 - A_2) e^{ix \cdot \xi} e^{\Phi_1 + \overline{\Phi_2}} dx = 0$$

*Proof.* We use the abbreviations  $A := A_1 - A_2$ ,  $F := F_1 - F_2$  and  $p := p_1 - p_2$ . First we multiply (4.1) by h. For the non-gradient terms in (4.1) we have by (4.9) that

$$\begin{aligned} \left| h \int_{\Omega} (A_1^2 - A_2^2 + p) u_1 \overline{u_2} \, dx \right| \\ &= \left| h \int_{\Omega} (A_1^2 - A_2^2 + p) e^{ix \cdot \xi} (e^{\Phi_1^{\sharp} + \overline{\Phi_2^{\sharp}}} + e^{\Phi_1^{\sharp}} \overline{r_2} + r_1 e^{\overline{\Phi_2^{\sharp}}} + r_1 \overline{r_2}) \, dx \right| \\ &\leq Ch \|A_1^2 - A_2^2 + p\|_{L^{\infty}} \Big( \|e^{\Phi_1^{\sharp} + \overline{\Phi_2^{\sharp}}}\|_{L^{\infty}} + \|e^{\Phi_1^{\sharp}}\|_{L^{\infty}} \|\overline{r_2}\|_{L^2} \\ &+ \|r_1\|_{L^2} \|e^{\overline{\Phi_2^{\sharp}}}\|_{L^{\infty}} + \|r_1\|_{L^2} \|\overline{r_2}\|_{L^2} \Big) \\ &\leq Ch \to 0, \end{aligned}$$

as  $h \to 0$ . For our specific CGO solutions,  $u_1$  and  $u_2$ , we hence have that

(4.11)  

$$h \Big| \int_{\Omega} iA \cdot (u_1 \nabla \overline{u_2} - \overline{u_2} \nabla u_1) \, dx - h \int_{\Omega} F \cdot (u_1 \nabla \overline{u_2} + \overline{u_2} \nabla u_1) \, dx \Big| = \mathcal{O}(h),$$
as  $h \to 0$ .

We continue by estimating the first integral in (4.11). Since the solutions  $u_1$  and  $u_2$  are of the CGO form one gets the following by expanding

$$(4.12) hu_1 \nabla \overline{u_2} = \overline{\zeta_2} e^{ix \cdot \xi} \left( e^{\Phi_1^{\sharp} + \overline{\Phi_2^{\sharp}}} + e^{\Phi_1^{\sharp}} \overline{r_2} + r_1 e^{\overline{\Phi_2^{\sharp}}} + r_1 \overline{r_2} \right) \\ + h e^{ix \cdot \xi} \left( e^{\Phi_1^{\sharp}} \nabla e^{\overline{\Phi_2^{\sharp}}} + e^{\Phi_1^{\sharp}} \nabla \overline{r_2} + r_1 \nabla e^{\overline{\Phi_2^{\sharp}}} + r_1 \nabla \overline{r_2} \right)$$

The first term in the first parantheses in (4.12) gives

(4.13) 
$$\overline{\zeta_2} \cdot \int_{\Omega} iAe^{ix \cdot \xi} e^{\Phi_1^{\sharp} + \overline{\Phi_2^{\sharp}}} dx \to -(\mu_1 + i\mu_2) \cdot \int_{\Omega} iAe^{ix \cdot \xi} e^{\Phi_1 + \overline{\Phi_2}} dx.$$

as  $h \to 0$ . This is because  $\overline{\zeta_2} = -\mu_1 - i\mu_2 + \mathcal{O}(h)$  and by (3.18) we have that

$$\left| (\mu_1 + i\mu_2) \cdot \int_{\Omega} A e^{ix \cdot \xi} \left( e^{\Phi_1^{\sharp} + \overline{\Phi_2^{\sharp}}} - e^{\Phi_1 + \overline{\Phi_2}} \right) dx \right| \le C \left\| e^{\Phi_1^{\sharp} + \overline{\Phi_2^{\sharp}}} - e^{\Phi_1 + \overline{\Phi_2}} \right\|_{L^{\infty}(\Omega)}$$
$$\le C h^{\gamma/2} \to 0$$

as  $h \to 0$ . For the next three terms in (4.12), we can use estimate (4.9) and Cauchy–Schwarz to conclude that

$$\begin{aligned} \left| \int_{\Omega} iA \cdot \overline{\zeta_2} e^{ix \cdot \xi} (e^{\Phi_1^{\sharp}} \overline{r_2} + r_1 e^{\overline{\Phi_2^{\sharp}}} + r_1 \overline{r_2}) dx \right| \\ (4.14) &\leq C \|A\|_{L^{\infty}} (\|e^{\Phi_1^{\sharp}}\|_{L^2} \|\overline{r_2}\|_{L^2} + \|r_1\|_{L^2} \|e^{\overline{\Phi_2^{\sharp}}}\|_{L^2} + \|r_1\|_{L^2} \|\overline{r_2}\|_{L^2}) \\ &\leq C h^{\gamma/2} \to 0, \end{aligned}$$

as  $h \to 0$ . For the last part of (4.12) containing the factor h, we have using estimates (4.9) and (4.8) that

(4.15) 
$$\left| \int_{\Omega} hiA \cdot e^{ix \cdot \xi} (e^{\Phi_1^{\sharp}} \nabla e^{\overline{\Phi_2^{\sharp}}} + e^{\Phi_1^{\sharp}} \nabla \overline{r_2} + r_1 \nabla e^{\overline{\Phi_2^{\sharp}}} + r_1 \nabla \overline{r_2}) dx \right|$$
$$\leq Ch \left( h^{(\gamma-1)/2} + h^{-1} h^{\gamma/2} + h^{\gamma/2} h^{(\gamma-1)/2} + h^{\gamma} h^{-1} \right) \to 0,$$

as  $h \to 0$ . Expanding the  $\overline{u_2} \nabla u_1$  term in (4.11) gives

(4.16) 
$$h\overline{u_2}\nabla u_1 = \zeta_1 e^{ix\cdot\xi} \left( e^{\Phi_1^{\sharp} + \overline{\Phi_2^{\sharp}}} + e^{\Phi_1^{\sharp}} \overline{r_2} + r_1 e^{\overline{\Phi_2^{\sharp}}} + r_1 \overline{r_2} \right) \\ + h e^{ix\cdot\xi} \left( \nabla e^{\Phi_1^{\sharp}} e^{\overline{\Phi_2^{\sharp}}} + \nabla e^{\Phi_1^{\sharp}} \overline{r_2} + \nabla r_1 e^{\overline{\Phi_2^{\sharp}}} + \nabla r_1 \overline{r_2} \right).$$

Again  $-\zeta_1 = -\mu_1 - i\mu_2 + \mathcal{O}(h)$ . The terms in (4.12) and (4.16) are of the same form. Doing the analogous estimates for (4.16) gives then that

$$h \int_{\Omega} iA \cdot (u_1 \nabla \overline{u_2} - \overline{u_2} \nabla u_1) \, dx \to -2i(\mu_1 + i\mu_2) \cdot \int_{\mathbb{R}^n} Ae^{ix \cdot \xi} e^{\Phi_1 + \overline{\Phi_2}} dx,$$

as  $h \to 0$ .

We end the proof by showing that

(4.17) 
$$h \int_{\Omega} F \cdot (u_1 \nabla \overline{u_2} + \overline{u_2} \nabla u_1) dx \to 0,$$

as  $h \to 0$ . Using (4.12) and (4.16) gives that

$$h(u_1 \nabla \overline{u_2} + \overline{u_2} \nabla u_1) = (\overline{\zeta_2} + \zeta_1) e^{ix \cdot \xi} \left( e^{\Phi_1^{\sharp} + \overline{\Phi_2^{\sharp}}} + e^{\Phi_1^{\sharp}} \overline{r_2} + r_1 e^{\overline{\Phi_2^{\sharp}}} + r_1 \overline{r_2} \right)$$

$$(4.18) + h e^{ix \cdot \xi} \left( e^{\Phi_1^{\sharp}} \nabla e^{\overline{\Phi_2^{\sharp}}} + e^{\Phi_1^{\sharp}} \nabla \overline{r_2} + r_1 \nabla e^{\overline{\Phi_2^{\sharp}}} + r_1 \nabla \overline{r_2} \right)$$

$$+ \nabla e^{\Phi_1^{\sharp}} e^{\overline{\Phi_2^{\sharp}}} + \nabla e^{\Phi_1^{\sharp}} \overline{r_2} + \nabla r_1 e^{\overline{\Phi_2^{\sharp}}} + \nabla r_1 \overline{r_2} \right)$$

The second term on the right hand side is of the same form as the second term on the right hand side of (4.12) and (4.16). The contribution of these terms are therefore zero in the limit  $h \to 0$ .

For the first term on the right hand side of (4.18) we get, using (4.3) and (4.9), the estimate

$$\left| h \int_{\Omega} i\xi \cdot F(e^{ix \cdot \xi}(e^{\Phi_1^{\sharp} + \overline{\Phi_2^{\sharp}}} + e^{\Phi_1^{\sharp}} \overline{r_2} + r_1 e^{\overline{\Phi_2^{\sharp}}} + r_1 \overline{r_2})) dx \right|$$
  
$$\leq \mathcal{O}(h)(1 + h^{\gamma/2} + h^{\gamma/2} + h^{\gamma}) \to 0,$$

as  $h \to 0$ . This shows that (4.17) holds.

It turns out that the  $e^{\Phi_1 + \overline{\Phi_2}}$  term can be dropped from (4.10). This is guaranteed by Proposition 3.3 in [7] (see also [3] and [11]). Using the abbreviation  $A := A_1 - A_2$  we thus obtain

$$(\mu_1 + i\mu_2) \cdot \int_{\mathbb{R}^n} Ae^{ix\cdot\xi} dx = (\mu_1 + i\mu_2) \cdot \widehat{A}(-\xi) = 0,$$

where  $\widehat{A}$  stands for the Fourier transform of A. Moreover for any  $\mu \in \mathbb{R}^n$ , with  $\mu \cdot \xi = 0$ , we have therefore that  $\mu \cdot \widehat{A} = 0$ . It follows that the Fourier transform of the component functions of (2.1) are zero. To see this notice that the above implies that

$$\xi_j \widehat{A_k} - \xi_k \widehat{A_j} = (\xi_j e_k - \xi_k e_j) \cdot \widehat{A} = 0,$$

since  $\xi \cdot (\xi_j e_k - \xi_k e_j) = 0$ , where  $e_k$  denote the standard basis vectors of  $\mathbb{R}^n$ . We have thus proved that  $dA_1 = dA_2$ .

**Remark.** Notice that, we only need the condition  $0 < \gamma \leq 1$  in recovering the magnetic potentials, instead of  $2/3 < \gamma \leq 1$ .

### 5. Uniqueness of the electric potential

To finish the proof of Proposition 2.1, we need to show that  $q_1 = \nabla \cdot F_1 + p_1 = \nabla \cdot F_2 + p_2 = q_2$ . Lemma 2.2 and the assumption that  $A_1 = A_2, F_1 = F_2$  on  $\partial \Omega$  and that  $\partial \Omega$  is Lipschitz, allows us to extend  $A_j$  and  $F_j, j = 1, 2$  to a ball B, with  $\overline{\Omega} \subset B$ , so that  $A_1 = A_2$  and  $F_1 = F_2$  in  $B \setminus \Omega, F_j = A_j = 0$  on  $\partial B$  and  $A_j, F_j \in C^{0,\gamma}(B)$ , for j = 1, 2.

In the previous section we proved that  $d(A_1 - A_2) = 0$ . The Poincaré Lemma implies now that there is a  $\psi \in C^{1,\gamma}(B)$  s.t.  $A_1 - A_2 = \nabla \psi$ in B (see [2]). We can moreover choose  $\psi$  so that  $\psi|_{\partial B} = 0$ , since  $A_1 = A_2 = 0$  in  $B \setminus \Omega$ . By Lemma 6.3 and Proposition 6.1 below, we have that

$$C^B_{A_1,q_1} = C^B_{A_2,q_2} = C^B_{A_2 + \nabla \psi,q_2} = C^B_{A_1,q_2}.$$

Proposition 4.1 gives then that

(5.1) 
$$\int_{B} (-F \cdot \nabla(u_1 \overline{u_2}) + p u_1 \overline{u_2}) \, dx = 0,$$

for any  $u_1, u_2 \in H^1(B)$ , satisfying  $L_{A_1,q_1}u_1 = 0$ ,  $L_{\overline{A_2},\overline{q_2}}u_2 = 0$  in B and where  $F := F_1 - F_2$  and  $p := p_1 - p_2$ .

We now suppose, as in section 4 that  $u_1$  and  $u_2$  are given by (4.4) and (4.6) (when  $\Omega = B$ ), with  $A_1 = A_2$  and consider the limit of (5.1)

as  $h \to 0$ . Expanding (5.1), using (4.3) gives

(5.2) 
$$\begin{aligned} \int_{B} -F \cdot i\xi e^{ix \cdot \xi} (e^{\Phi_{1}^{\sharp} + \overline{\Phi_{2}^{\sharp}}} + e^{\Phi_{1}^{\sharp}} \overline{r_{2}} + r_{1}e^{\overline{\Phi_{2}^{\sharp}}} + r_{1}\overline{r_{2}}) dx \\ + \int_{B} -F \cdot e^{ix \cdot \xi} \nabla (e^{\Phi_{1}^{\sharp} + \overline{\Phi_{2}^{\sharp}}} + e^{\Phi_{1}^{\sharp}} \overline{r_{2}} + r_{1}e^{\overline{\Phi_{2}^{\sharp}}} + r_{1}\overline{r_{2}}) dx \\ + \int_{B} pu_{1}\overline{u_{2}} dx = 0. \end{aligned}$$

We begin by showing that the second integral in (5.2) tends to zero, in the limit  $h \to 0$ .

We simplify (5.2) firstly by writing  $\tilde{F} := Fe^{ix\cdot\xi}$ . Notice also that  $\tilde{F} \in C^{0,\gamma}(B)$ . The second simplification comes from the fact that  $e^{\Phi_1^{\sharp} + \Phi_2^{\sharp}} = 1$ . To show this notice first that the Cauchy operator has the following properties

$$\overline{N_{\zeta}^{-1}f} = N_{\overline{\zeta}}^{-1}\overline{f}, \quad N_{-\zeta}^{-1}f = -N_{\zeta}^{-1}f.$$

Applying these to the definitions (4.5) and (4.7) together with the fact that we are now considering the case with  $A_1 = A_2$  yields

$$\Phi_1^{\sharp} + \overline{\Phi_2^{\sharp}} = N_{\mu_1 + i\mu_2}^{-1} \left( -i(\mu_1 + i\mu_2) \cdot (A_1^{\sharp} - A_2^{\sharp}) \right) = 0,$$

so that

(5.3) 
$$e^{\Phi_1^{\sharp} + \Phi_2^{\sharp}} = 1$$

Split the second integral in (5.2) into pieces by taking the absolute value and applying the triangle inequality. Consider first the first term of the second integral in (5.2). By (5.3) we have immediately that

(5.4) 
$$\left| \int_{B} \widetilde{F} \cdot \nabla e^{\Phi_{1}^{\sharp} + \overline{\Phi_{2}^{\sharp}}} dx \right| = 0.$$

Next we consider the terms  $\nabla(e^{\Phi_1^{\sharp}}\overline{r_2})$  and  $\nabla(r_1e^{\overline{\Phi_2^{\sharp}}})$ , coming from the second integral in (5.2). Notice firstly that  $\widetilde{F}|_{\partial B} = 0$ , since  $F|_{\partial B} = 0$ . Letting  $\widetilde{F}^{\sharp} := \Psi_{\theta} * \widetilde{F}$ , where  $\Psi_{\theta}$  is defined as in the beginning of section

3 and using the estimates of Proposition 3.4 and Lemma 3.1 we get that

$$\begin{aligned} \left| \int_{B} \widetilde{F} \cdot \nabla \left( e^{\Phi_{1}^{\sharp}} \overline{r_{2}} \right) dx \right| &= \left| \int_{B} \widetilde{F} \cdot \left( \nabla e^{\Phi_{1}^{\sharp}} \overline{r_{2}} + e^{\Phi_{1}^{\sharp}} \nabla \overline{r_{2}} \right) dx \right| \\ &\lesssim \|\widetilde{F} \cdot \nabla e^{\Phi_{1}^{\sharp}}\|_{\infty} \|\overline{r_{2}}\|_{2} + \left| \int_{B} \widetilde{F} \cdot e^{\Phi_{1}^{\sharp}} \nabla \overline{r_{2}} \, dx \right| \\ &\lesssim h^{(\gamma-1)/2} h^{\gamma/2} + \left| \int_{B} \widetilde{F} \cdot e^{\Phi_{1}^{\sharp}} \nabla \overline{r_{2}} \, dx \right| \\ &\lesssim h^{\gamma-1/2} + \left| \int_{B} \nabla \cdot \left( \widetilde{F}^{\sharp} e^{\Phi_{1}^{\sharp}} \right) \overline{r_{2}} \, dx \right| \\ &+ \left| \int_{B} \left( \widetilde{F} - \widetilde{F}^{\sharp} \right) \cdot e^{\Phi_{1}^{\sharp}} \nabla \overline{r_{2}} \, dx \right| \\ &\lesssim h^{\gamma-1/2} + \theta^{1-\gamma} h^{\gamma/2} + \|\widetilde{F} - \widetilde{F}^{\sharp}\|_{\infty} \|\nabla \overline{r_{2}}\|_{2} \\ &\lesssim h^{\gamma-1/2} + \theta^{1-\gamma} h^{\gamma/2} + \theta^{-\gamma} h^{\gamma/2-1}. \end{aligned}$$

The last term from the second integral in (5.2) is handled as follows

$$\begin{split} \left| \int_{B} \widetilde{F} \cdot \nabla(r_{1}\overline{r_{2}}) \, dx \right| \lesssim \left| \int_{B} \nabla \cdot \widetilde{F}^{\sharp} r_{1}\overline{r_{2}} \, dx \right| + \left| \int_{B} \left( \widetilde{F} - \widetilde{F}^{\sharp} \right) \cdot \nabla(r_{1}\overline{r_{2}}) \, dx \right| \\ \lesssim \| \nabla \cdot \widetilde{F}^{\sharp} \|_{\infty} \| r_{1} \|_{2} \| \overline{r_{2}} \|_{2} \\ + \| \widetilde{F} - \widetilde{F}^{\sharp} \|_{\infty} \left( \| \nabla r_{1} \|_{2} \| \overline{r_{2}} \|_{2} + \| r_{1} \|_{2} \| \nabla \overline{r_{2}} \|_{2} \right) \\ \lesssim \theta^{1 - \gamma} h^{\gamma/2} h^{\gamma/2} + \theta^{-\gamma} h^{-1} h^{\gamma/2} h^{\gamma/2} \\ \lesssim \theta^{1 - \gamma} h^{\gamma} + \theta^{-\gamma} h^{\gamma - 1}. \end{split}$$

Combining (5.4), (5.5) and (5.6) and then choosing  $\theta = h^{-1}$ , gives for the second integral in (5.2) that

$$\begin{split} \left| \int_{B} F \cdot e^{ix \cdot \xi} \nabla \left( e^{\Phi_{1}^{\sharp} + \overline{\Phi_{2}^{\sharp}}} + e^{\Phi_{1}^{\sharp}} \overline{r_{2}} + r_{1} e^{\overline{\Phi_{2}^{\sharp}}} + r_{1} \overline{r_{2}} \right) dx \\ &\lesssim \theta^{1 - \gamma} h^{\gamma/2} + \theta^{-\gamma} h^{\gamma/2 - 1} + h^{\gamma - 1/2} \\ &= 2h^{(3\gamma - 2)/2} + h^{\gamma - 1/2} \to 0, \end{split} \right.$$

as  $h \to 0$ , since we require that  $\gamma > 2/3$ .

We now return to the first integral in (5.2). It can be estimated using (4.9) and the Cauchy–Schwarz inequality as follows

$$\begin{split} \left| \int_{B} -F \cdot i\xi e^{ix \cdot \xi} (e^{\Phi_{1}^{\sharp}} \overline{r_{2}} + r_{1} e^{\overline{\Phi_{2}^{\sharp}}} + r_{1} \overline{r_{2}}) dx \right| \\ & \lesssim \left\| e^{\Phi_{1}^{\sharp}} \right\|_{\infty} \left\| \overline{r_{2}} \right\|_{2} + \left\| r_{1} \right\|_{2} \left\| e^{\overline{\Phi_{2}^{\sharp}}} \right\|_{\infty} + \left\| r_{1} \right\|_{2} \left\| \overline{r_{2}} \right\|_{2} \\ & \lesssim h^{\gamma/2} \to 0, \end{split}$$

as  $h \to 0$ . Estimating the third integral in (5.2) in a similar fashion and using (5.3) we thus conclude that (5.2) reduces to

$$\int_{B} (-F \cdot i\xi e^{ix \cdot \xi} + p e^{ix \cdot \xi}) \, dx = 0,$$

in the limit  $h \to 0$ . This implies that  $\mathcal{F}(\nabla \cdot F + p)(-\xi) = 0$  in the distributional sense, which in turn implies that  $0 = \nabla \cdot F + p = (\nabla \cdot F_1 + p_1) - (\nabla \cdot F_2 + p_2)$ , finishing the proof of Proposition 2.1.

### 6. Appendix A – Gauge invariance and Boundary data

Gauge invariance plays an important role when working with the magnetic Schrödinger equation. Here we state the basic result concerning the gauge invariance of the Cauchy data sets. This section also includes two results on when the equality of the boundary data on a smaller set implies the equality of the boundary data on a bigger set.

**Proposition 6.1.** Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 3$ , be a bounded open set with Lipschitz boundary. Assume  $A, F \in C^{0,\gamma}(\Omega, \mathbb{C}^n)$ ,  $0 < \gamma \leq 1$ ,  $p \in L^{\infty}(\Omega, \mathbb{C})$ ,  $\psi \in C^{1,\gamma}(\Omega, \mathbb{C})$  and let  $q = \nabla \cdot F + p$ . Then we have

(6.1) 
$$e^{-i\psi} \circ L_{A,q} \circ e^{i\psi} = L_{A+\nabla\psi,q}.$$

If furthermore,  $\psi|_{\partial\Omega} = 0$  then

(6.2) 
$$C_{A,q} = C_{A+\nabla\psi,q}.$$

*Proof.* Let  $\psi \in C^{1,\gamma}(\Omega)$ . By direct computation we know that for  $L_{A,p}$ , we have

$$e^{-i\psi} \circ L_{A,p} \circ e^{i\psi} = L_{A+\nabla\psi,p}.$$

Furthermore we have that

$$e^{-i\psi} \circ (\nabla \cdot F) \circ e^{i\psi} = \nabla \cdot F,$$

since for  $u, v \in C_0^{\infty}(\Omega)$ , we have that

$$\begin{split} \left\langle e^{-i\psi} \circ (\nabla \cdot F) \circ e^{i\psi} u, v \right\rangle &= -\int_{\Omega} F \cdot \nabla (e^{-i\psi} u e^{i\psi} v) \, dx \\ &= -\int_{\Omega} F \cdot \nabla (uv) \, dx, \end{split}$$

where  $\langle \cdot, \cdot \rangle$  stands for the distributional duality. Thus recalling that  $q = \nabla \cdot F + p$  it follows that

$$e^{-i\psi} \circ L_{A,q} \circ e^{i\psi} = L_{A+\nabla\psi,q},$$

which proves (6.1).

In order to prove (6.2), assume that  $\psi|_{\partial\Omega} = 0$ . Let  $u \in H^1(\Omega)$  be a solution to

$$L_{A,q}u = 0$$
, in  $\Omega$ .

By (6.1) we know that  $e^{-i\psi}u \in H^1(\Omega)$  satisfies

$$L_{A+\nabla\psi,q}(e^{-i\psi}u)=0, \text{ in } \Omega$$

Moreover we have that  $e^{-i\psi}u|_{\partial\Omega} = u|_{\partial\Omega}$ . It remains hence to show that

$$N_{A+\nabla\psi,q}(e^{-i\psi}u) = N_{A,q}u, \text{ on } \partial\Omega.$$

To that end let  $\varphi \in H^{1/2}(\partial \Omega)$  and let  $\phi \in H^1(\Omega)$  be such that  $\phi|_{\partial \Omega} = \varphi$ . Then

$$\begin{split} \left\langle N_{A+\nabla\psi,q}(e^{-i\psi}u),\varphi\right\rangle &= \left\langle N_{A+\nabla\psi,q}(e^{-i\psi}u),e^{i\psi}\varphi\right\rangle \\ &= \int_{\Omega} \nabla(e^{-i\psi}u)\cdot\nabla(e^{i\psi}\phi) + i(A+\nabla\psi)\cdot(e^{-i\psi}u\nabla(e^{i\psi}\phi)) \\ &- \nabla(e^{-i\psi}u)e^{i\psi}\phi) + ((A+\nabla\psi)^2 + p)u\phi - F\cdot\nabla(u\phi)\,dx \\ &= \int_{\Omega} \nabla u\cdot\nabla\phi + iA\cdot(u\nabla\phi - \nabla u\phi) + (A^2 + p)u\phi \\ &- F\cdot\nabla(u\phi)\,dx \\ &= \left\langle N_{A,q}u,\varphi\right\rangle. \end{split}$$

The next Lemma is a slight modification of Lemma 4.2 in [8], we include the proof for the convenience of the reader. The Lemma shows that two DN-maps that coincide on small set, give two DN-maps that coincide on a bigger if we extend the potentials so that they are identical outside the smaller set.

**Lemma 6.2.** Assume that  $\Omega, \Omega' \subset \mathbb{R}^n$  be bounded open sets with Lipschitz boundaries, such that  $\overline{\Omega} \subset \Omega'$  and let  $V_1, V_2 \in L^{\infty}(\Omega', \mathbb{C}^n)$ . Denote by  $\Lambda_{V_j}^{\Omega}$  the DN-map corresponding to the Dirichlet problem on the set  $\Omega$ . Assume that  $V_1 = V_2$  in  $\Omega' \setminus \Omega$ . If  $\Lambda_{V_1}^{\Omega} = \Lambda_{V_2}^{\Omega}$  then  $\Lambda_{V_1}^{\Omega'} = \Lambda_{V_2}^{\Omega'}$ .

*Proof.* Given  $u'_1 \in H^1(\Omega')$ , solving  $L_{V_1}u'_1 = 0$ , in  $\Omega'$  we need to find an  $u'_2 \in H^1(\Omega')$  solving  $L_{V_2}u'_2 = 0$ , in  $\Omega'$  with  $u'_2|_{\partial \Omega'} = u'_1|_{\partial \Omega'}$  and  $\partial_n u'_2|_{\partial \Omega'} = \partial_n u'_1|_{\partial \Omega'}$ .

The function  $u_1 := u'_1|_{\Omega}$  solves  $L_{V_1}u_1 = 0$ , in  $\Omega$ . Let  $u_2 \in H^1(\Omega)$ be such that  $L_{V_2}u_2 = 0$  in  $\Omega$  and  $u_2|_{\partial\Omega} = u_1|_{\partial\Omega}$ . We know that  $\partial_n u_2|_{\partial\Omega} = \partial_n u_1|_{\partial\Omega}$ , since  $\Lambda_{V_1}^{\Omega} = \Lambda_{V_2}^{\Omega}$ . Thus  $u_1 - u_2 \in H^1_0(\Omega)$ . Define

$$u'_2 := u'_1 - (u_1 - u_2), \text{ in } \Omega',$$

where we extended by  $u_1 - u_2$  by zero into  $\Omega'$ . Clearly  $u'_2 \in H^1(\Omega')$ ,  $u'_2|_{\partial \Omega'} = u'_1|_{\partial \Omega'}$  and  $\partial_n u'_2|_{\partial \Omega'} = \partial_n u'_1|_{\partial \Omega'}$ .

It remains to check that  $L_{V_2}u'_2 = 0$ , in  $\Omega'$  in a weak sense. Let  $\varphi \in C_0^{\infty}(\Omega')$ , then

$$\begin{split} \langle L_{V_2} u'_2, \varphi \rangle_{\Omega'} &= \int_{\Omega'} \nabla u'_2 \cdot \nabla \varphi + V_2 \cdot \nabla u'_2 \varphi \\ &= \int_{\Omega} \nabla u'_2 \cdot \nabla \varphi + V_2 \cdot \nabla u'_2 \varphi + \int_{\Omega' \setminus \Omega} \nabla u'_2 \cdot \nabla \varphi + V_2 \cdot \nabla u'_2 \varphi \\ &= \int_{\Omega} \nabla u_2 \cdot \nabla \varphi + V_2 \cdot \nabla u_2 \varphi + \int_{\Omega' \setminus \Omega} \nabla u'_1 \cdot \nabla \varphi + V_1 \cdot \nabla u'_1 \varphi \\ &= \int_{\Omega'} \nabla u'_1 \cdot \nabla \varphi + V_1 \cdot \nabla u'_1 \varphi \\ &= \langle L_{V_1} u'_1, \varphi \rangle_{\Omega'} \\ &= 0. \end{split}$$

where we use the fact that  $u_2|_{\partial\Omega} = u_1|_{\partial\Omega}$ ,  $u_1 = u'_1|_{\Omega}$  and  $\Lambda^{\Omega}_{V_1} = \Lambda^{\Omega}_{V_2}$  to get the fourth equality.

We need a similar result concerning the magnetic Schrödinger operator.

**Lemma 6.3.** Let  $\Omega, \Omega' \subset \mathbb{R}^n$  be bounded open sets with Lipschitz boundaries, such that  $\overline{\Omega} \subset \Omega'$ . Let  $A_1, A_2, F_1, F_2 \in C^{0,\gamma}(\Omega', \mathbb{C}^n), 0 < \gamma \leq 1, p_1, p_2 \in L^{\infty}(\Omega', \mathbb{C}^n)$  and let  $q_j := \nabla \cdot F_j + p_j$ . Denote by  $C^{\Omega}_{A_j,q_j}$ the Cauchy data for  $L_{A_j,q_j}$  in the set  $\Omega, j = 1, 2$ . Assume that

(6.3) 
$$A_1 = A_2, F_1 = F_2 \text{ and } p_1 = p_2, \text{ in } \Omega' \setminus \Omega.$$

If 
$$C^{\Omega}_{A_1,q_1} = C^{\Omega}_{A_2,q_2}$$
 then  $C^{\Omega'}_{A_1,q_1} = C^{\Omega'}_{A_2,q_2}$ 

*Proof.* Given  $u'_1 \in H^1(\Omega')$ , solving  $L_{A_1,q_1}u'_1 = 0$ , in  $\Omega'$  we need to find an  $u'_2 \in H^1(\Omega')$  solving  $L_{A_2,q_2}u'_2 = 0$ , in  $\Omega'$  with  $u'_2|_{\partial\Omega'} = u'_1|_{\partial\Omega'}$  and  $N_{A_2,q_2}u'_2 = N_{A_1,q_1}u'_1$ . This implies that  $C^{\Omega'}_{A_1,q_1} \subset C^{\Omega'}_{A_2,q_2}$ , from which the claim follows.

Let  $u_1 := u'_1|_{\Omega}$ . Then  $L_{A_1,q_1}u_1 = 0$ , in  $\Omega$ . Let  $u_2 \in H^1(\Omega)$  be such that  $L_{A_2,q_2}u_2 = 0$ , in  $\Omega$  and  $u_2|_{\partial\Omega} = u_1|_{\partial\Omega}$ . Because  $C^{\Omega}_{A_1,q_1} = C^{\Omega}_{A_2,q_2}$ , we know that  $N_{A_2,q_2}u_2 = N_{A_1,q_1}u_1$ , on  $\partial\Omega$ .

In particular we have that  $\varphi := u_2 - u_1 \in H_0^1(\Omega) \subset H_0^1(\Omega')$ . Define

$$u'_2 := u'_1 + \varphi, \text{ in } \Omega',$$

where we extended by  $u_1 - u_2$  by zero into  $\Omega'$ . Clearly  $u'_2 \in H^1(\Omega')$ ,  $u'_2|_{\partial\Omega'} = u'_1|_{\partial\Omega'}$ . We need thus to check that  $L_{A_2,q_2}u'_2 = 0$ , in  $\Omega'$  and that  $N_{A_2,q_2}u'_2 = N_{A_1,q_1}u'_1$ .

Let  $\psi \in C_0^{\infty}(\Omega')$ , then

$$\langle L_{A_2,q_2} u'_2, \psi \rangle_{\Omega'} = \int_{\Omega'} \nabla (u'_1 + \varphi) \cdot \nabla \psi + iA_2 \cdot ((u'_1 + \varphi)\nabla \psi - \psi \nabla (u'_1 + \varphi))$$
  
 
$$+ (A_2^2 + p_2)(u'_1 + \varphi)\psi - F_2 \cdot \nabla ((u'_1 + \varphi)\psi) \, dx.$$

Since  $u'_1 + \varphi = u_2$  on  $\Omega$ , we have that

$$\begin{split} \langle L_{A_2,q_2} u'_2, \psi \rangle_{\Omega'} &= \int_{\Omega} \nabla u_2 \cdot \nabla \psi + iA_2 \cdot (u_2 \nabla \psi - \psi \nabla u_2) \\ &+ (A_2^2 + p_2) u_2 \psi - F_2 \cdot \nabla (u_2 \psi) \, dx \\ &+ \int_{\Omega' \setminus \Omega} \nabla u'_1 \cdot \nabla \psi + iA_1 \cdot (u'_1 \nabla \psi - \psi \nabla u'_1) \\ &+ (A_1^2 + p_1) u'_1 \psi - F_1 \cdot \nabla (u'_1 \psi) \, dx \\ &+ \int_{\Omega' \setminus \Omega} \nabla \varphi \cdot \nabla \psi + iA_1 \cdot (\varphi \nabla \psi - \psi \nabla \varphi) \\ &+ (A_1^2 + p_1) \varphi \psi - F_1 \cdot \nabla (\varphi \psi) \, dx \end{split}$$

The last integral is zero, since  $\operatorname{supp}(\varphi) \subset \Omega$ . Hence using the assumption that  $N_{A_2,q_2}u_2 = N_{A_1,q_1}u_1$ , on  $\partial \Omega$  gives

$$\begin{split} \langle L_{A_2,q_2} u'_2, \psi \rangle_{\Omega'} &= \langle N_{A_2,q_2} u_2, \psi |_{\Omega} \rangle_{\partial \Omega} \\ &+ \int_{\Omega' \setminus \Omega} \nabla u'_1 \cdot \nabla \psi + iA_1 \cdot (u'_1 \nabla \psi - \psi \nabla u'_1) \\ &+ (A_1^2 + p_1) u'_1 \psi - F_1 \cdot \nabla (u'_1 \psi) \, dx \\ &= \langle L_{A_1,q_1} u'_1, \psi \rangle_{\Omega'} = 0. \end{split}$$

Thus we see that  $L_{A_2,q_2}u'_2 = 0$ , in  $\Omega'$ . A similar deduction shows that  $N_{A_2,q_2}u'_2 = N_{A_1,q_1}u'_1$ . Hence we have that  $C_{A_1,q_1}^{\Omega'} \subset C_{A_2,q_2}^{\Omega'}$ .

# 7. Appendix B – A Carleman estimate

In this section we prove a Carleman estimate that implies the solvability result Proposition 3.2, in section 3. The proof is a straight forward extension of the one in [7], and we give it here for the convenience of the reader. The main concern is how to incorporate the  $\nabla \cdot F$ term into the result in [7].

The estimate we are about to prove is a perturbation of the Carleman estimate for the Laplacian, given in [9] (see also [7]). We state this result as follows.

**Proposition 7.1.** Let  $\varphi(x) = \alpha \cdot x$ ,  $\alpha \in \mathbb{R}^n$ ,  $|\alpha| = 1$  and let  $\varphi_{\varepsilon} =$  $\varphi + \frac{h}{2\varepsilon} \varphi^2$ . Then for  $0 < h \ll \varepsilon \ll 1$  and  $s \in \mathbb{R}$ , we have

(7.1) 
$$\frac{h}{\sqrt{\varepsilon}} \|u\|_{H^{s+2}_{scl}(\mathbb{R}^n)} \le C \|e^{\varphi_{\varepsilon}/h} h^2 \Delta(e^{-\varphi_{\varepsilon}/h} u)\|_{H^s_{scl}(\mathbb{R}^n)}, \quad C > 0,$$

for all  $u \in C_0^{\infty}(\Omega)$ .

We now apply this result in the case s = -1 and a fixed  $\varepsilon > 0$  that is sufficiently small.

**Proposition 7.2.** Let  $\varphi(x) = \alpha \cdot x$ ,  $\alpha \in \mathbb{R}^n$  with  $|\alpha| = 1$ . Assume  $A, F \in L^{\infty}(\Omega, \mathbb{C}^n)$ ,  $p \in L^{\infty}(\Omega, \mathbb{C})$  and  $q = \nabla \cdot F + p$ . Then for  $0 < h \ll 1$ , we have

(7.2) 
$$h \|u\|_{H^{1}_{scl}(\mathbb{R}^{n})} \leq C \|e^{\varphi/h}h^{2}L_{A,q}(e^{-\varphi/h}u)\|_{H^{-1}_{scl}(\mathbb{R}^{n})},$$

for all  $u \in C_0^{\infty}(\Omega)$ .

*Proof.* Let  $\varphi_{\varepsilon} = \varphi + \frac{h}{2\varepsilon}\varphi^2$  be the convexified weight, with  $\varepsilon > 0$  and  $0 < h \ll \varepsilon \ll 1$ . Then in the proof Proposition 2.2 in [7], it is shown that

$$\|e^{\varphi_{\varepsilon}/h}h^{2}A \cdot D(e^{-\varphi_{\varepsilon}/h}u) + e^{\varphi_{\varepsilon}/h}h^{2}D \cdot (Ae^{-\varphi_{\varepsilon}/h}u)\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^{n})} \leq \mathcal{O}(h)\|u\|_{H^{1}_{\mathrm{scl}}(\mathbb{R}^{n})},$$

where  $D := i^{-1} \nabla$ . Here the implicit constant depends on  $||A||_{L^{\infty}(\Omega)}$ ,  $||\varphi||_{L^{\infty}(\Omega)}$  and  $||D\varphi||_{L^{\infty}(\Omega)}$  (see (2.4) in [7]).

Furthermore, we have for all  $0 \neq \psi \in C_0^{\infty}(\Omega)$  that

$$\begin{aligned} \left| \langle e^{\varphi_{\varepsilon}/h} h^{2} \nabla \cdot F(e^{-\varphi_{\varepsilon}/h}u), \psi \rangle \right| &\leq h^{2} \int_{\mathbb{R}^{n}} \left| F \nabla \cdot (e^{\varphi_{\varepsilon}/h}ue^{-\varphi_{\varepsilon}/h}\psi) \right| \\ &\leq h \|F\|_{L^{\infty}(\mathbb{R}^{n})} \left( \|h \nabla u\|_{L^{2}(\mathbb{R}^{n})} \|\psi\|_{L^{2}(\mathbb{R}^{n})} \\ &\quad + \|u\|_{L^{2}(\mathbb{R}^{n})} \|h \nabla \psi\|_{L^{2}(\mathbb{R}^{n})} \right) \\ &\leq \mathcal{O}(h) \|u\|_{H^{1}_{scl}(\mathbb{R}^{n})} \|\psi\|_{H^{1}_{scl}(\mathbb{R}^{n})}. \end{aligned}$$

It follows from the definition of the  $H_{\rm scl}^{-1}$ -norm that

(7.4) 
$$\|e^{\varphi_{\varepsilon}/h}h^2\nabla \cdot F(e^{-\varphi_{\varepsilon}/h}u)\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^n)} \leq \mathcal{O}(h)\|u\|_{H^{1}_{\mathrm{scl}}(\mathbb{R}^n)}.$$

By choosing a small fixed  $\varepsilon > 0$  that is independent of h, we conclude from estimates (7.1),(7.3) and (7.4) that

$$\begin{split} \left\| e^{\varphi_{\varepsilon}/h} (-h^{2} \Delta) (e^{-\varphi_{\varepsilon}/h} u) + e^{\varphi_{\varepsilon}/h} h^{2} A \cdot D(e^{-\varphi_{\varepsilon}/h} u) \right. \\ \left. + e^{\varphi_{\varepsilon}/h} h^{2} D \cdot (A e^{-\varphi_{\varepsilon}/h} u) + e^{\varphi_{\varepsilon}/h} h^{2} \nabla \cdot F(e^{-\varphi_{\varepsilon}/h} u) \right\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^{n})} \\ &\geq \frac{h}{C} \| u \|_{H^{1}_{\mathrm{scl}}(\mathbb{R}^{n})}. \end{split}$$

Moreover we have that

$$\|h^{2}(A^{2}+p)u\|_{H^{-1}_{\rm scl}(\mathbb{R}^{n})} \leq \mathcal{O}(h^{2})\|u\|_{H^{1}_{\rm scl}(\mathbb{R}^{n})}$$

Combining the two previous estimates gives then that

$$C \| e^{\varphi_{\varepsilon}/h} h^2 L_{A,q}(e^{-\varphi_{\varepsilon}/h}u) \|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^n)} \ge \frac{h}{C} \| u \|_{H^{1}_{\mathrm{scl}}(\mathbb{R}^n)},$$

where C > 0. By using  $e^{-\varphi_{\varepsilon}/h}u = e^{-\varphi_{\varepsilon}/h}e^{-\varphi^{2}/(2\varepsilon)}u$ , we obtain (7.2).  $\Box$ 

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