CONSTRUCTION OF LYAPUNOV FUNCTIONS FOR INTERCONNECTED PARABOLIC SYSTEMS: AN IISS APPROACH *

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Abstract. This paper is devoted to two issues. One is to provide Lyapunov-based tools to establish integral input-to-state stability (iISS) and input-to-state stability (ISS) for some classes of nonlinear parabolic equations. The other is to provide a stability criterion for interconnections of iISS parabolic systems. The results addressing the former problem allow us to overcome obstacles arising in tackling the latter one. The results for the latter problem are a small-gain condition and a formula of Lyapunov functions which can be constructed for interconnections whenever the small-gain condition holds. It is demonstrated that for interconnections of partial differential equations, the choice of a right state and input spaces is crucial, in particular for iISS subsystems which are not ISS. As illustrative examples, stability of two highly nonlinear reaction-diffusion systems is established by the the proposed small-gain criterion.

Key words. nonlinear control systems, infinite-dimensional systems, integral input-to-state stability, Lyapunov methods

AMS subject classifications. 93C20, 93C25, 37C75, 93D30, 93C10.

1. Introduction. During more than two decades, input-to-state stability (ISS) has been used widely in the study of robust stabilizability [8], detectability [30, 21] and other branches of nonlinear control theory [20, 29]. ISS unified into one framework two different types of stable behavior: asymptotic stability and input-output stability [29], and allowed us to use the small-gain argument to study ISS of interconnected systems [18, 17], which is sometimes referred to as the ISS small-gain theorem. In spite of these advantages, for many practical systems ISS is still far too restrictive. This is because ISS excludes systems whose state stave bounded as long as the magnitude of the applied inputs remains below a specific threshold, but becomes unbounded when the input magnitude exceeds the threshold. Such behavior is frequently caused by saturation and limitations in actuation and processing rate. The idea of integral inputto-state stability (iISS) is to capture those nonlinearities [28, 2]. Serious obstacles were encountered in addressing interconnections of iISS systems [11]. In contrast to ISS subsystems, iISS subsystems which are not ISS usher the issue of incompatibility of spaces in time domain for trajectory-based approaches, as well as insufficiency of maxtype Lyapunov functions popular in ISS Lyapunov-based approaches (e.g. [17, 6]). Breakthroughs made in [10, 13, 1, 19] allowed us to use small-gain criteria as in the ISS small-gain theorem in spite of the inevitable and considerable difference between their proofs and Lyapunov constructions.

In contrast to the extensive literature on ISS and iISS of ordinary differential equations, for partial differential equations (PDEs) these theories are making their first steps. In [16], [4], [5], [23], ISS of infinite-dimensional systems

(1.1)
$$\dot{x}(t) = Ax(t) + f(x(t), u(t)), \ x(t) \in X, u(t) \in U$$

has been studied via methods of semigroup theory [15], [3]. Here the state space X

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and the space of input values U are Banach spaces, $A: D(A) \to X$ is the generator of a C_0 -semigroup over X and $f: X \times U \to X$ is Lipschitz with respect to the first argument. Many classes of evolution PDEs, such as parabolic and hyperbolic equations are of this kind [9], [3]. As in the case of finite-dimensional systems [29], the notion of an ISS Lyapunov function can be defined for (1.1) so that the existence of an ISS Lyapunov function is sufficient for ISS of (1.1) (see [4]). This motivated the results in [4] on constructions of ISS Lyapunov functions for a class of parabolic systems belonging to (1.1). More direct approach to the construction of Lyapunov functions for some classes of nonlinear parabolic and linear time-varying hyperbolic systems has been proposed in [24, 27]. In [16] and [23], systems (1.1) with a linear function f have been investigated via frequency-domain methods.

To the best of the authors' knowledge, with some exceptions of time-delay systems, the study of Lyapunov functions for iISS of infinite-dimensional systems has begun in [25], where iISS of bilinear distributed parameter systems was investigated. It was shown that bilinear systems in the form of (1.1) which are uniformly globally asymptotically stable without inputs are always iISS. The second result in [25] is an extension to bilinear systems over Hilbert spaces of a method for construction of iISS Lyapunov functions for bilinear ODE systems, introduced by Sontag [28]. In this paper we use similar method in Section 4 to construct iISS Lyapunov functions for some classes of nonlinear parabolic systems.

In [4, 5], ISS of large scale systems whose subsystems are in the form of (1.1) has been studied and the ISS small gain theorem, already available for finite-dimensional systems (see [17, 6]) has been extended to the infinite-dimensional systems. However, the method does not accommodate iISS subsystems which are not ISS.

This paper studies stability of interconnections of two parabolic systems, each of which is of the form

(1.2)
$$\frac{\partial x}{\partial t} = c \frac{\partial^2 x}{\partial l^2} + f(x(l,t), \frac{\partial x}{\partial l}(l,t), u(l,t)), \quad \forall t > 0,$$

where $l \in (0, L)$, $x(l, t) \in \mathbb{R}$. This class of systems (1.2) allows more general functions f than the class considered in [4, 25], and possesses systems, which are not ISS. The primary goal of this paper is to accomplish an iISS small gain theorem [13, 11], originally proved for finite-dimensional systems, in the infinite-dimensional setting. In contrast to the small-gain theorem from [4], we require ISS property only from one subsystem and not from both of them; the other subsystem may be only iISS.

Interestingly, this extension is much more involved than it seems on the first glance. When working with PDEs which are not ISS, an obstacle is not only in a higher complexity in dealing with Lyapunov functions in infinite dimensions, but also in the necessity to choose the state space in a right way. In particular, it is quite hard to find an iISS parabolic system whose state and input spaces are both L_p -spaces, while we do not encounter such difficulties considering ISS systems. To address this issue, this paper reexamines tools developed in [4, 25] for constructing iISS and ISS Lyapunov functions for some classes of nonlinear parabolic systems, and actively exploits Sobolev spaces as state spaces. For interconnections of PDE systems additional difficulties arise since we need not only choose right spaces for every subsystem, but also match them with the state and input spaces for another subsystems. Last but not least, incompatibility of spaces in the time domain, which is crucial for interconnections of ODE systems, is as important for PDE systems. These issues make an investigation of interconnections of iISS infinite-dimensional systems a challenging problem, which we solve here for some classes of parabolic systems in one-dimensional spatial domain.

The rest of this paper is organized as follows. Having defined the stability notions in Section 2, a construction of an ISS Lyapunov function for nonlinear parabolic systems (1.2) with Sobolev state space is proposed in Section 3, in which the role of the input space is also highlighted. In Section 4 we consider another class of nonlinear parabolic systems, and provide a construction of an iISS Lyapunov function with L_p state space. In Section 5, a construction of an iISS Lyapunov function for (1.2) is developed with Sobolev state space in order to go beyond systems dealt in Sections 3. 4. Next in Section 6 we state an iISS small-gain theorem which is a sufficient condition for iISS of the interconnection of two iISS distributed parameter systems, and an iISS Lyapunov function is constructed explicitly for the interconnection. As it was proved to be necessary for finite-dimensional systems [13], one subsystem is required to be ISS if the other subsystem is iISS and does not admit an ISS Lyapunov function. Finally, in Section 7 we use all of the obtained results to prove stability of two types of highly nonlinear parabolic systems. The examples illustrate how fruitful the use of Sobolev spaces is in dealing with interconnections involving iISS systems. Finally, conclusions are drawn in Section 8.

Notation. We define $\mathbb{R}_+ := [0, \infty)$, and the symbol \mathbb{N} denotes the set of natural numbers. By $C(\mathbb{R}_+, Y)$ we denote the space of continuous functions from \mathbb{R}_+ to Y, equipped with the standard sup-norm. We exploit throughout the paper the following function spaces:

- $C_0^k(0, L)$ is a space of k times continuously differentiable functions $f: (0, L) \to \mathbb{R}$ with a support, compact in (0, L).
- $L_p(0,L), p \ge 1$ is a space of *p*-th power integrable functions $f: (0,L) \to \mathbb{R}$ with the norm $\|f\|_{L_p(0,L)} = \left(\int_0^L |f(l)|^p dl\right)^{\frac{1}{p}}$.
- $W^{k,p}(0,L)$ is a Sobolev space of functions $f \in L_p(0,L)$, which have weak derivatives of order $\leq k$, all of which belong to $L_p(0,L)$. Norm in $W^{k,p}(0,L)$ is defined
- by $||f||_{W^{k,p}(0,L)} = \left(\int_0^L \sum_{1 \le s \le k} \left|\frac{\partial^s f}{\partial l^s}(l)\right|^p dl\right)^{\frac{1}{p}}$. • $W_0^{k,p}(0,L)$ is a closure of $C_0^k(0,L)$ in the norm of $W^{k,p}(0,L)$. We endow $W_0^{k,p}(0,L)$ with a norm $||f||_{W_0^{k,p}(0,L)} = \left(\int_0^L \left|\frac{\partial^k f}{\partial l^k}(l)\right|^p dl\right)^{\frac{1}{p}}$, equivalent to the norm $||\cdot||_{W^{k,p}(0,L)}$ on $W^{k,p}(0,L)$ see [0, p, 8]
- on $W_0^{k,p}(0,L)$, see, [9, p.8]. • $H^k(0,L) = W^{k,2}(0,L), H_0^k(0,L) = W_0^{k,2}(0,L).$

To define and analyze stability properties we use so-called comparison functions

 $\begin{array}{ll} \mathcal{P} & := \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous, } \gamma(0) = 0 \text{ and } \gamma(r) > 0 \text{ for } r > 0 \} \\ \mathcal{K} & := \{\gamma \in \mathcal{P} \mid \gamma \text{ is strictly increasing} \} \\ \mathcal{K}_{\infty} & := \{\gamma \in \mathcal{K} \mid \gamma \text{ is unbounded} \} \\ \mathcal{L} & := \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous and strictly decreasing with } \lim_{t \to \infty} \gamma(t) = 0 \} \\ \mathcal{K}\mathcal{L} & := \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \mid \beta \text{ is continuous, } \beta(\cdot, t) \in \mathcal{K}, \ \forall t \ge 0, \ \beta(r, \cdot) \in \mathcal{L}, \ \forall r > 0 \} \end{array}$

2. Problem formulation. Consider the system (1.1) and assume throughout the paper that X and U are Banach spaces and f(0,0) = 0, i.e., $x = 0 \in X$ is an equilibrium point of (1.1). Let also T be a semigroup generated by A from (1.1).

Some useful spaces for interconnections anowing for 1155 subsystems.		
(a) Choice #1		
	State values X_i	Input values U_i
iISS subsystem $(i = 1)$	$L_2(0, L)$	$H_0^1(0,L)$
ISS subsystem $(i = 2)$	$H_0^1(0, L)$	$L_2(0,L)$
(b) Choice $\#2$		
	State values X_i	Input values U_i
iISS subsystem $(i = 1)$	$H_0^1(0,L)$	$H_0^1(0,L)$
ISS subsystem $(i = 2)$	$H_0^1(0,L)$	$H_0^1(0,L)$
	TABLE 2	
A typical choice of spaces for interconnections of ISS subsystems.		
	State values X_i	Input values U_i
ISS subsystem $(i = 1)$	$L_p(0,L)$	$L_q(0,L)$
ISS subsystem $(i = 2)$	$L_q(0,L)$	$L_p(0,L)$

TABLE 1 Some useful spaces for interconnections allowing for iISS subsystems. (a) Choice #1

Under (weak) solutions of (1.1) we understand solutions of the integral equation

(2.1)
$$x(t) = T(t)x(0) + \int_0^t T(t-s)f(x(s), u(s))ds,$$

where $\forall t \in [0, \tau]$, belonging to $C([0, \tau], X)$ for all $\tau > 0$.

DEFINITION 1. We call $f: X \times U \to X$ Lipschitz continuous on bounded subsets of X, uniformly with respect to the second argument if $\forall C > 0 \exists K(C) > 0$, such that $\forall x, y: ||x||_X \leq C, ||y||_X \leq C, \forall v \in U$,

(2.2)
$$||f(y,v) - f(x,v)||_X \le K(C)||y - x||_X.$$

We will use the following assumption concerning nonlinearity f throughout the paper

ASSUMPTION 1. $f: X \times U \to X$ is Lipschitz continuous on bounded subsets of X, uniformly with respect to the second argument and $f(x, \cdot)$ is continuous for all $x \in X$.

Assumption 1 ensures that the weak solution of (1.1) exists and is unique, according to a variation of a classical existence and uniqueness theorem [3, Proposition 4.3.3]. Let $\phi(t, \phi_0, u)$ denote the state of a system (1.1), i.e. the solution to (1.1), at moment $t \in \mathbb{R}_+$ associated with an initial condition $\phi_0 \in X$ at t = 0, and input $u \in U_c$, where U_c is a linear normed space of admissible functions mapping \mathbb{R}_+ into U, equipped with a norm $\|\cdot\|_{U_c}$.

Next we introduce stability properties for the system (1.1).

DEFINITION 2. System (1.1) is globally asymptotically stable at zero uniformly with respect to state (0-UGASs), if $\exists \beta \in \mathcal{KL}$, such that $\forall \phi_0 \in X$, $\forall t \ge 0$ it holds

(2.3)
$$\|\phi(t,\phi_0,0)\|_X \le \beta(\|\phi_0\|_X,t).$$

To study stability properties of (1.1) with respect to external inputs, we use the notion of input-to-state stability [4]:

DEFINITION 3. System (1.1) is called input-to-state stable (ISS) with respect to space of inputs U_c , if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that the inequality

(2.4)
$$\|\phi(t,\phi_0,u)\|_X \le \beta(\|\phi_0\|_X,t) + \gamma(\|u\|_{U_c})$$

holds $\forall \phi_0 \in X, \forall u \in U_c \text{ and } \forall t \geq 0.$

We emphasize that the above definition does not yet exactly correspond to ISS of finite dimensional systems [29] since Definition 3 allows the flexibility of the choice of U_c . A system (1.1) is called ISS, without expressing the normed space of inputs explicitly, if it is ISS with respect to $U_c = C(\mathbb{R}_+, U)$ endowed with a usual supremum norm. This terminology follows that of ISS for finite dimensional systems.

If the system is not ISS, it may still have some sort of robustness. Thus we introduce another stability property

DEFINITION 4. System (1.1) is called integral input-to-state stable (iISS) if there exist $\alpha \in \mathcal{K}_{\infty}$, $\mu \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ such that the inequality

(2.5)
$$\alpha(\|\phi(t,\phi_0,u)\|_X) \le \beta(\|\phi_0\|_X,t) + \int_0^t \mu(\|u(s)\|_U) ds$$

holds $\forall \phi_0 \in X$, $\forall u \in U_c = C(\mathbb{R}_+, U)$ and $\forall t \ge 0$.

The following defines a useful notion for studying iISS.

DEFINITION 5. A continuous function $V : X \to \mathbb{R}_+$ is called an iISS Lyapunov function, if there exist $\psi_1, \psi_2 \in \mathcal{K}_{\infty}, \alpha \in \mathcal{P}$ and $\sigma \in \mathcal{K}$ such that

(2.6)
$$\psi_1(||x||_X) \le V(x) \le \psi_2(||x||_X), \quad \forall x \in X$$

and system (1.1) satisfies

(2.7)
$$\dot{V}_u(x) \le -\alpha(\|x\|_X) + \sigma(\|u(0)\|_U)$$

for all $x \in X$ and $u \in U_c$, where the Lie derivative of V corresponding to the input u is defined by

(2.8)
$$\dot{V}_u(x) = \lim_{t \to +0} \frac{1}{t} (V(\phi(t, x, u)) - V(x)).$$

Furthermore, if

(2.9)
$$\liminf_{\tau \to \infty} \alpha(\tau) = \infty \quad or \quad \liminf_{\tau \to \infty} \alpha(\tau) \ge \lim_{\tau \to \infty} \sigma(\tau)$$

holds, system V is called an ISS Lyapunov function.

PROPOSITION 2.1 (Proposition 1, [25]). If there exist an iISS (resp. ISS) Lyapunov function for (1.1), then (1.1) is iISS (resp. ISS).

As a rule a construction of a Lyapunov function is the only realistic way to prove ISS/iISS of control systems. This makes the construction of an ISS/iISS Lyapunov functions a fundamental problem in stability theory. In the next sections we propose a method for constructing iISS and ISS Lyapunov functions for certain equations (1.2). Then we show how to construct the Lyapunov functions for systems of PDEs from the information about Lyapunov functions of subsystems by means of an small-gain approach.

In this paper, for simplicity we write \dot{V} instead of $\dot{V}_u(x)$ when solutions along which the derivative is taken are clear from the context.

REMARK 1. For finite-dimensional systems iISS notion is strictly weaker than ISS, in sense that all ISS systems are iISS [2]. In addition, there are iISS systems which are not ISS [28]. This strict inclusive relationship has not yet been proved completely for PDEs of the form (1.2). However, in terms of Lyapunov functions defined above, ISS Lyapunov functions establishing ISS are always iISS Lyapunov functions establishing iISS. Furthermore, iISS PDEs of the form (1.2) are not necessarily ISS, see [25].

3. ISS Lyapunov functions for a class of nonlinear parabolic systems: Sobolev state space. The purpose of this section is to develop a Lyapunov-type characterization of ISS for PDEs in (1.2). There is a number of papers, where such characterizations for parabolic systems whose state space is an L_p space have been provided [24, 4]. However, as we will see in Section 7 the iISS systems in many cases cannot have the L_p space both as an input and state space. Since our final goal is to consider interconnections of iISS and ISS systems, we need to have the constructions of ISS Lyapunov functions with Sobolev state spaces. This section provides one of such constructions.

Consider a system

(3.1)
$$\frac{\partial x}{\partial t} = c \frac{\partial^2 x}{\partial l^2} + f\left(x(l,t), \frac{\partial x}{\partial l}(l,t)\right) + u(l,t), \quad \forall t > 0$$

defined on the spatial domain (0, L) with the Dirichlet boundary conditions

(3.2)
$$x(0,t) = x(L,t) = 0, \quad \forall t \ge 0.$$

The next theorem gives a sufficient condition for ISS of (3.1) with respect to the state space $X = W_0^{1,2q}(0,L), q \in \mathbb{N}$ and two types of spaces U of input values by construction of a Lyapunov function.

THEOREM 2. Suppose

(3.3)
$$\int_{0}^{L} \left(\frac{\partial x}{\partial l}\right)^{2q-2} \frac{\partial^{2} x}{\partial l^{2}} f\left(x, \frac{\partial x}{\partial l}\right) dl \ge \int_{0}^{L} \eta\left(\left(\frac{\partial x}{\partial l}\right)^{2q}\right) dl$$

holds all $x \in X$ with some convex continuous function $\eta : \mathbb{R}_+ \to \mathbb{R}$ and some $\epsilon \in \mathbb{R}_+$ such that

(3.4)
$$\hat{\alpha}(s) := \frac{\pi^2}{q^2 L^2} (c - \epsilon) s + L\eta\left(\frac{s}{L}\right) \ge 0, \ \forall s \in \mathbb{R}_+.$$

Then the following statements hold:

1. If $\epsilon > 0$, then the function

(3.5)
$$V(x) = \int_0^L \left(\frac{\partial x}{\partial l}\right)^{2q} dl = \|x\|_{W_0^{1,2q}(0,L_0^2)}^{2q}$$

is an ISS Lyapunov function of (3.1)-(3.2) with respect to the space $U = L_{2q}(0,L)$ of input values and $U = W_0^{1,2q}(0,L) \cap W^{2,2q}(0,L)$ as well. 2. If there exists $g \in \mathcal{K}_{\infty}$ so that

(3.6)
$$Lsg(s) = \hat{\alpha}\left(Ls^{\frac{2q}{2q-1}}\right), \quad \forall s \in \mathbb{R}_+$$

holds, then the function V given in (3.5) is an ISS Lyapunov function of (3.1)-(3.2) with respect to the space of input values $U = U_g$, consisting of $u \in L_1(0,L)$: u(0) = u(L) = 0, $\frac{\partial u}{\partial l}$ exists and $\int_0^L \left|\frac{\partial u}{\partial l}\right| g^{-1}\left(\left|\frac{\partial u}{\partial l}\right|\right) dl$ is finite.

Proof. Along the solution of (3.1)-(3.2), the function V given as in (3.5) satisfies

$$\dot{V} = 2q \int_0^L \left(\frac{\partial x}{\partial l}\right)^{2q-1} \frac{\partial^2 x}{\partial l \partial t} dl$$

= $-2q(2q-1) \int_0^L \frac{\partial x}{\partial t} \left(\frac{\partial x}{\partial l}\right)^{2q-2} \frac{\partial^2 x}{\partial l^2} dl + 2q \left(\frac{\partial x}{\partial l}\right)^{2q-1} \frac{\partial x}{\partial t}\Big|_{l=0}^L.$

Due to (3.2) we have $\frac{\partial x}{\partial t}\Big|_{l=0}^{L} = 0$ and consequently

$$\dot{V} = -2q(2q-1)\int_0^L \left(\frac{\partial x}{\partial l}\right)^{2q-2} \frac{\partial^2 x}{\partial l^2} \cdot \left(c\frac{\partial^2 x}{\partial l^2} + f(x(l,t),\frac{\partial x}{\partial l}(l,t)) + u(l,t)\right) dl$$

Next we utilize (3.3) to obtain

(3.7)
$$\frac{1}{2q(2q-1)}\dot{V} \leq -c \int_{0}^{L} \left(\frac{\partial x}{\partial l}\right)^{2q-2} \left(\frac{\partial^{2} x}{\partial l^{2}}\right)^{2} dl - \int_{0}^{L} \eta\left(\left(\frac{\partial x}{\partial l}\right)^{2q}\right) dl - \int_{0}^{L} \left(\frac{\partial x}{\partial l}\right)^{2q-2} \frac{\partial^{2} x}{\partial l^{2}} u dl.$$

Using Young's inequality $\frac{\partial^2 x}{\partial l^2} u \leq \frac{\omega}{2} \left(\frac{\partial^2 x}{\partial l^2} \right)^2 + \frac{1}{2\omega} u^2$, which holds for any $\omega > 0$, we get

(3.8)
$$\frac{1}{2q(2q-1)}\dot{V} \leq \left(\frac{\omega}{2} - c\right) \int_0^L \left(\frac{\partial x}{\partial l}\right)^{2q-2} \left(\frac{\partial^2 x}{\partial l^2}\right)^2 dl - \int_0^L \eta\left(\left(\frac{\partial x}{\partial l}\right)^{2q}\right) dl + \frac{1}{2\omega} \int_0^L \left(\frac{\partial x}{\partial l}\right)^{2q-2} u^2 dl.$$

It is easy to see that

$$\int_0^L \left(\frac{\partial x}{\partial l}\right)^{2q-2} \left(\frac{\partial^2 x}{\partial l^2}\right)^2 dl = \frac{1}{q^2} \int_0^L \left(\frac{\partial}{\partial l} \left(\left(\frac{\partial x}{\partial l}\right)^q\right)\right)^2 dl.$$

Due to Friedrichs' inequality (8.5) we proceed to

(3.9)
$$\int_0^L \left(\frac{\partial x}{\partial l}\right)^{2q-2} \left(\frac{\partial^2 x}{\partial l^2}\right)^2 dl \ge \frac{\pi^2}{q^2 L^2} \int_0^L \left(\frac{\partial x}{\partial l}\right)^{2q} dl = \frac{\pi^2}{q^2 L^2} V(x).$$

Define

(3.10)
$$\xi(s) := \frac{1}{L}\hat{\alpha}(Ls) = \frac{\pi^2}{q^2 L^2}(c-\epsilon)s + \eta(s), \quad \forall s \in \mathbb{R}_+.$$

The convexity of η implies the convexity of ξ . Due to the definition (3.5) of V, Jensen's inequality (8.3) yields

(3.11)
$$\int_{0}^{L} \xi\left(\left(\frac{\partial x}{\partial l}\right)^{2q}\right) dl \ge \hat{\alpha}(V(x)), \quad \forall x \in \mathbb{R}$$

Now we continue the estimates of \dot{V} in the cases of q = 1 and q > 1 separately.

In the case of q = 1, inequality (3.8) implies

$$\begin{aligned} \dot{V} &\leq 2\left(\frac{\omega}{2} - c\right)\frac{\pi^2}{L^2}V(x) - 2\int_0^L \left(\left(\frac{\partial x}{\partial l}\right)^2\right) dl + \frac{1}{\omega} \|u\|_{L_2(0,L)}^2 \\ &= 2\left(\frac{\omega}{2} - c\right)\frac{\pi^2}{L^2}V(x) - 2\int_0^L \left(\left(\frac{\partial x}{\partial l}\right)^2\right) dl \\ &+ 2\left(c - \epsilon\right)\frac{\pi^2}{L^2}\int_0^L \left(\frac{\partial x}{\partial l}\right)^2 dl + \frac{1}{\omega} \|u\|_{L_2(0,L)}^2 \\ &\leq 2\left(\frac{\omega}{2} - \epsilon\right)\frac{\pi^2}{L^2}V(x) - 2\hat{\alpha}(V(x)) + \frac{1}{\omega} \|u\|_{L_2(0,L)}^2.\end{aligned}$$

$$(3.12)$$

Here, the last inequality uses (3.11). Recall that $\epsilon > 0$. Pick $\omega \in (0, 2\epsilon)$. Property (3.4) ensures that V is an ISS Lyapunov function of (3.1) with respect to input space $U = L_{2q}(0, L)$. Application of Friedrichs' inequality to $||u||^2_{L_2(0,L)}$ proves that V is an ISS Lyapunov function with respect to $U = W_0^{1,2}(0,L) \cap W^{2,2}(0,L) = H_0^1(0,L) \cap H^2(0,L)$.

Next, assume that q > 1. We apply Young's inequality (8.1) to the last term in (3.8) with $\omega_2 > 0$ as follows

$$\left(\frac{\partial x}{\partial l}\right)^{2q-2} u^2 \le \frac{1}{q\omega_2} u^{2q} + \omega_2^{\frac{1}{q-1}} \frac{q-1}{q} \left(\frac{\partial x}{\partial l}\right)^{2q}$$

Putting this expression into (3.8) and using (3.10) we obtain finally

(3.13)

$$\frac{1}{2q(2q-1)}\dot{V} \leq \left(\left(\frac{\omega}{2}-c\right)\frac{\pi^{2}}{q^{2}L^{2}}+\omega_{2}^{\frac{1}{q-1}}\frac{q-1}{2\omega q}\right)V(x) \\
-\int_{0}^{L}\eta\left(\left(\frac{\partial x}{\partial l}\right)^{2q}\right)dl+\frac{1}{2\omega\omega_{2}q}\|u\|_{L_{2q}(0,L)}^{2q} \\
\leq \left(\left(\frac{\omega}{2}-\epsilon\right)\frac{\pi^{2}}{q^{2}L^{2}}+\omega_{2}^{\frac{1}{q-1}}\frac{q-1}{2\omega q}\right)V(x) \\
-\hat{\alpha}(V(x))+\frac{1}{2\omega\omega_{2}q}\|u\|_{L_{2q}(0,L)}^{2q}.$$

It is easy to see that choosing $\omega > 0$ and $\omega_2 > 0$ small enough, we can ensure $(\frac{\omega}{2} - \epsilon)\frac{\pi^2}{q^2L^2} + \omega_2^{\frac{1}{q-1}}\frac{q-1}{2\omega q} < 0$. Hence, due to (3.4), the function V is an ISS Lyapunov function of (3.1) with respect to input space $U = L_{2q}(0, L)$. Application of Friedrichs' inequality to $\|u\|_{L_{2q}(0,L)}^{2q}$ proves that V is an ISS Lyapunov function with respect to $U = W_0^{1,2q}(0,L) \cap W^{2,2q}(0,L)$.

Finally, to prove Item 2 of the theorem, we are going to estimate the last term in (3.7) with the help of

$$(3.14) - \int_{0}^{L} \left(\frac{\partial x}{\partial l}\right)^{2q-2} \frac{\partial^{2} x}{\partial l^{2}} u dl = -\frac{1}{2q-1} \int_{0}^{L} \frac{\partial}{\partial l} \left(\left(\frac{\partial x}{\partial l}\right)^{2q-1}\right) u dl$$
$$= \frac{1}{2q-1} \int_{0}^{L} \left(\frac{\partial x}{\partial l}\right)^{2q-1} \frac{\partial u}{\partial l} dl$$
$$\leq \frac{1}{2q-1} \int_{0}^{L} \left|\frac{\partial x}{\partial l}\right|^{2q-1} \left|\frac{\partial u}{\partial l}\right| dl.$$

Here, u(0) = u(L) = 0 is used in integration by parts. By virtue of (3.6) it holds that $sg(s) = \xi(s^{\frac{2q}{2q-1}})$ and applying inequality (8.2) to the term $\left|\frac{\partial x}{\partial l}\right|^{2q-1}\left|\frac{\partial u}{\partial l}\right|$ with g yields

$$(3.15) \qquad \int_{0}^{L} \left| \frac{\partial x}{\partial l} \right|^{2q-1} \left| \frac{\partial u}{\partial l} \right| dl \le \omega \int_{0}^{L} \xi \left(\left| \frac{\partial x}{\partial l} \right|^{2q} \right) dl + \int_{0}^{L} \left| \frac{\partial u}{\partial l} \right| g^{-1} \left(\frac{1}{\omega} \left| \frac{\partial u}{\partial l} \right| \right) dl$$

for $\omega > 0$. From (3.7), (3.9) and the definition of ξ it follows that

$$(3.16) \qquad \frac{1}{2q(2q-1)}\dot{V} \leq -\frac{c\pi^2}{q^2L^2}V(x) - \int_0^L \eta\left(\left(\frac{\partial x}{\partial l}\right)^{2q}\right)dl - \int_0^L \left(\frac{\partial x}{\partial l}\right)^{2q-2}\frac{\partial^2 x}{\partial l^2}udl = -\frac{\epsilon\pi^2}{q^2L^2}V(x) - \int_0^L \xi\left(\left(\frac{\partial x}{\partial l}\right)^{2q}\right)dl - \int_0^L \left(\frac{\partial x}{\partial l}\right)^{2q-2}\frac{\partial^2 x}{\partial l^2}udl.$$

Substituting (3.14) and (3.15) into (3.16) we obtain

$$\frac{1}{2q(2q-1)}\dot{V}(x) \leq -\frac{\epsilon\pi^2}{q^2L^2}V(x) - \left(1 - \frac{\omega}{2q-1}\right)\int_0^L \xi\left(\left(\frac{\partial x}{\partial l}\right)^{2q}\right)dl + \frac{1}{2q-1}\int_0^L \left|\frac{\partial u}{\partial l}\right|g^{-1}\left(2\left|\frac{\partial u}{\partial l}\right|\right)dl.$$

Let $\omega < 2q - 1$. In view of (3.11) this implies

(3.17)
$$\frac{1}{2q(2q-1)}\dot{V}(x) \leq -\frac{\epsilon\pi^2}{q^2L^2}V(x) - \left(1 - \frac{\omega}{2q-1}\right)\hat{\alpha}(V(x)) + \frac{1}{2q-1}\int_0^L \left|\frac{\partial u}{\partial l}\right|g^{-1}\left(\frac{1}{\omega}\left|\frac{\partial u}{\partial l}\right|\right)dl.$$

Recall that $\epsilon \geq 0$. Property (3.6) satisfied with $g \in \mathcal{K}_{\infty}$ implies $\hat{\alpha} \in \mathcal{K}_{\infty}$. Consequently (3.17) means that V is an ISS Lyapunov function of (3.1) with respect to input space $U = U_g$. \square

REMARK 3. In the above proof, several times we have used integration by parts as well as partial derivatives of x and u and thus the derivations are justified if the functions are smooth enough. Notice that having established estimates of \dot{V} for the spaces of smooth functions, which are dense subspaces of X and U respectively, the density argument as in [25, Propositions 2,3, proof of Theorem 6] ensures the result on the whole spaces X and U.

Items 1 and 2 of Theorem 2 demonstrate that different choices of input spaces result in different properties of a single system even if the state space is the same. Item 2 of Theorem 2 also illustrates that ISS does not necessarily imply an exponential decay rate. In contrast, for Item 1 of Theorem 2, the constructed function V is guaranteed to exhibit an exponential or faster decay rate globally. As this is the case for finite dimensional systems, it is observed for infinite dimensional systems that according to (2.7) with $\alpha \in \mathcal{P}$ which can be bounded, iISS allows the decay rate of V to be much slower for large magnitude of state variables than ISS can allow. This indicates that significantly different constructions for iISS Lyapunov functions are needed. Next section is devoted to this question.

4. iISS of a class of nonlinear parabolic systems: L_p state space. Consider a system

(4.1)
$$\frac{\partial x}{\partial t} = c \frac{\partial^2 x}{\partial l^2} + f(x(l,t), u(l,t)), \quad \forall t > 0$$

defined on the spatial domain (0, L) with

(4.2)
$$x(0,t)\frac{\partial x}{\partial l}(0,t) = x(L,t)\frac{\partial x}{\partial l}(L,t) = 0, \quad \forall t \ge 0$$

which represents boundary conditions of Dirichlet, Neumann or mixed type. The state space for (4.1) we choose as $X = L_{2q}(0, L)$ for some $q \in \mathbb{N}$ and input space we take as $U = L_{\infty}(0, L)$ and $H_0^1(0, L)$.

Define the following ODE associated with (4.1) given by

(4.3)
$$\dot{y}(t) = f(y(t), u(t)), \quad y(t), u(t) \in \mathbb{R}$$

The next theorem provides a construction of an iISS Lyapunov function for a class of nonlinear systems of the form (4.1).

THEOREM 4. Suppose that $W: y \mapsto y^{2q}$ satisfies

(4.4)
$$\dot{W}(y) := 2qy^{2q-1}f(y,u) \le -\alpha(W(y)) + W(y)\sigma(|u|)$$

for some $\alpha \in \mathcal{K}_{\infty} \cup \{0\}, \sigma \in \mathcal{K}$. Let any of the following conditions hold:

1. x(0,t) = 0 for all $t \ge 0$ or x(L,t) = 0 for all $t \ge 0$.

2. α is convex and \mathcal{K}_{∞} .

Then an iISS Lyapunov function of (4.1) with (4.2) with respect to the spaces of input values $U = L_{\infty}(0, L)$ as well as $U = H_0^1(0, L)$ is given by

(4.5)
$$V(x) = \ln(1 + Z(x)),$$

where Z is defined as

(4.6)
$$Z(x) = \int_0^L W(x(l))dl = ||x||_{L_{2q}(0,L)}^{2q}$$

Furthermore, if α is convex and satisfies

(4.7)
$$\liminf_{s \to \infty} \frac{\alpha(s)}{s} = \infty,$$

then V given above is an ISS Lyapunov system of (4.1) with (4.2) with respect to $U = L_{\infty}(0, L)$ as well as $U = H_0^1(0, L)$.

Proof. Consider Z given by (4.6) and let $U = L_{\infty}(0, L)$. Using (4.4) we have

$$\begin{split} \dot{Z}(x) =& 2q \int_{0}^{L} x^{2q-1}(l,t) \cdot \left(c \frac{\partial^{2} x}{\partial l^{2}}(l,t) + f(x(l,t),u(l,t)) \right) dl \\ \leq & - 2q(2q-1)c \int_{0}^{L} x^{2q-2} \left(\frac{dx}{dl} \right)^{2} dl \\ & + \int_{0}^{L} \left\{ -\alpha \left(W(x(l,t)) \right) + W(x(l,t))\sigma(|u(l,t)|) \right\} dl \\ (4.8) \quad \leq & - \frac{2(2q-1)c}{q} \int_{0}^{L} \left(\frac{d}{dl}(x^{q}) \right)^{2} dl - \int_{0}^{L} \alpha \left(W(x(l,t)) \right) dl + Z(x)\sigma(||u(\cdot,t)||_{U}). \end{split}$$

In the last estimate we have used boundary conditions (4.2).

First, suppose that Item 1 holds. Since $x \in W^{2q,1}(0,L)$ then $x^q \in L_2(0,L)$ and $\frac{d}{dl}(x^q) = qx^{q-1}\frac{dx}{dl} \in L_2(0,L)$ due to Hölder's inequality (since $\frac{dx}{dl} \in L_{2q}(0,L)$). Overall, we have $x^q \in W^{2,1}(0,L)$. Applying Poincare's inequality to the first term in (4.8), we obtain

$$\dot{Z}(x) \le -\frac{2(2q-1)c}{q} \frac{\pi^2}{4L^2} Z(x) + Z(x)\sigma(\|u(\cdot,t)\|_U)$$

with the help of x(0,t) = 0 for all $t \ge 0$ or x(L,t) = 0 for all $t \ge 0$. Defining V as in (4.5) results in

(4.9)
$$\dot{V}(x) \le -\frac{2(2q-1)c}{q} \frac{\pi^2}{4L^2} \frac{\|x\|_X^{2q}}{1+\|x\|_X^{2q}} + \sigma(\|u(\cdot,t)\|_U).$$

Recall that $X = L_{2q}(0, L)$. Hence, Definition 5 indicates that V is an iISS Lyapunov function of (4.1) with boundary conditions (4.2) for the space $U = L_{\infty}(0, \infty)$.

Next, assume that Item 2 is satisfied. Due to the convexity of α , Jensen's inequality in (4.8) allows us to obtain

$$\dot{Z}(x) \le -L\alpha \left(\frac{1}{L}Z(x(l,t))\right) + Z(x)\sigma(\|u(\cdot,t)\|_U).$$

For V in (4.5) we have

(4.10)
$$\dot{V}(x) \le -\frac{L\alpha\left(\frac{1}{L}\|x\|_X^{2q}\right)}{1+\|x\|_X^{2q}} + \sigma(\|u(\cdot,t)\|_U)$$

According to Definition 5, the function V is an iISS Lyapunov function of (4.1) with boundary conditions (4.2) for the space of input values $U = L_{\infty}(0, \infty)$. Since (4.7) implies $\lim \inf_{s\to\infty} \alpha(s)/(1+Ls) = \infty$, the above inequality guarantees that V is an ISS Lyapunov function in the case of (4.7).

Finally, to deal with the space $H_0^1(0, \pi)$ for the input values, we recall Agmon's inequality (8.6), which implies for $u \in H_0^1(0, \pi)$

$$\|u\|_{L_{\infty}(0,L)}^{2} \leq \|u\|_{L_{2}(0,L)}^{2} + \left\|\frac{\partial u}{\partial l}\right\|_{L_{2}(0,L)}^{2}.$$

This inequality yields

(4.11)
$$\|u\|_{L_{\infty}(0,L)}^{2} \leq \left(\frac{L^{2}}{\pi^{2}}+1\right) \|u\|_{H_{0}^{1}(0,L)}^{2}$$

with the help of Friedrichs' inequality (8.5). Substitution of (4.11) into (4.9) and (4.10), proves that V is an iISS Lyapunov function of (4.1)-(4.2) with respect to $U = H_0^1(0, \pi)$ under either Item 1 or Item 2. \Box

REMARK 5. We want to stress a reader's attention on the choice of an input space. First we have proved iISS of the system (5.1) for the input space $L_{\infty}(0, L)$. For many applications this choice of input space is reasonable and sufficient. However, when considering interconnections of control systems, the input to one system is a state of another system. Thus, having $L_{\infty}(0, L)$ as an input space of the first subsystem automatically means that it is a state space of another subsystem, which complicates the proof its ISS, since the constructions of Lyapunov functions for this choice of state space are hard to find (e.g. how to differentiate such Lyapunov functions?), if possible. As we have seen in Section 3, this is not the case if we choose $H_0^1(0, L)$ as a state space. This underlines the role of the Agmon's inequality in our constructions, which made possible the transition from the space $L_\infty(0, L)$ to $H_0^1(0, L)$ in the previous theorem.

Note that the term $W(y)\sigma(|u|)$ in (4.4) allows to analyze PDEs (4.1) with bilinear or generalized bilinear terms which do not possess ISS property.

5. iISS of a class of nonlinear parabolic systems: Sobolev state space. Instead of the L_2 state space we used for characterizing iISS in Section 4, for a class of parabolic systems, this section demonstrates that iISS can be established with Sobolev state space. We consider

(5.1)
$$\frac{\partial x}{\partial t} = c \frac{\partial^2 x}{\partial l^2} + f\left(x(l,t), \frac{\partial x}{\partial l}(l,t)\right) + \frac{\partial x}{\partial l}(l,t)u(l,t)$$

defined for $(l,t) \in (0,L) \times (0,\infty)$ with the Dirichlet boundary conditions

(5.2)
$$x(0,t) = x(L,t) = 0, \quad \forall t \ge 0$$

We take $X = W_0^{1,2q}(0,L), q \in \mathbb{N}$. Modifying Item 1 of Theorem 2, we can verify the following.

THEOREM 6. Suppose that (3.3) holds for all $x \in X$ with some convex continuous function $\eta : \mathbb{R}_+ \to \mathbb{R}$ and some $\epsilon \in \mathbb{R}_+$ such that (3.4) holds. If $\epsilon > 0$, then the function V given by

(5.3)
$$V(x) = \ln(1 + Z(x)),$$

(5.4)
$$Z(x) = \int_0^L \left(\frac{\partial x}{\partial l}\right)^{2q} dl = \|x\|_{W_0^{1,2q}(0,L)}^{2q}$$

is an iISS Lyapunov function of (5.1)-(5.2) with respect to the space $U = L_{\infty}(0, L)$ of input values and $U = H_0^1(0, L)$ as well.

Proof. As in the proof of Theorem 2, in the case of q = 1, along solutions of (5.1)-(5.2), the function Z in (3.5) satisfies

(5.5)
$$\dot{Z} \leq 2\left(\frac{\omega}{2} - \epsilon\right) \frac{\pi^2}{L^2} Z(x) - 2\hat{\alpha}(Z(x)) + \frac{1}{\omega} \int_0^L \left(\frac{\partial x}{\partial l}\right)^2 u^2 dl.$$
$$\leq 2\left(\frac{\omega}{2} - \epsilon\right) \frac{\pi^2}{L^2} Z(x) - 2\hat{\alpha}(Z(x)) + \frac{Z(x)}{\omega} \|u\|_{L_{\infty}(0,L)}^2$$

for any $\omega > 0$. Due to (5.3) we have

(5.6)
$$\dot{V} \leq 2\left(\frac{\omega}{2} - \epsilon\right) \frac{\pi^2}{L^2} \frac{\|x\|_X^2}{1 + \|x\|_X^2} - \frac{2\hat{\alpha}(\|x\|_X^2)}{1 + \|x\|_X^2} + \frac{1}{\omega} \|u\|_{L_{\infty}(0,L)}^2$$

Pick $\omega \in (0, 2\epsilon)$. Then $\epsilon > 0$ and property (5.6) imply that V is an iISS Lyapunov function with respect to the space $U = L_{\infty}(0, L)$ of input values.

Next consider q > 1. Again, following the argument used in the proof of Theorem 2, we obtain

(5.7)
$$\frac{\frac{1}{2q(2q-1)}\dot{Z}}{\hat{Z}} \leq \left(\left(\frac{\omega}{2} - \epsilon\right)\frac{\pi^2}{q^2L^2} + \omega_2^{\frac{1}{q-1}}\frac{q-1}{2\omega q}\right)Z(x) - \hat{\alpha}(Z(x)) + \frac{Z(x)}{2\omega\omega_2 q}\|u\|_{L_{\infty}(0,L)}^{2q}.$$

for any $\omega, \omega_2 > 0$. From (5.3) it follows that

(5.8)
$$\frac{\frac{1}{2q(2q-1)}\dot{V} \leq \left(\left(\frac{\omega}{2}-\epsilon\right)\frac{\pi^2}{q^2L^2} + \omega_2^{\frac{1}{q-1}}\frac{q-1}{2\omega q}\right)\frac{\|x\|_X^{2q}}{1+\|x\|_X^{2q}} - \frac{\hat{\alpha}(\|x\|_X^{2q})}{1+\|x\|_X^{2q}} + \frac{1}{2\omega\omega_2 q}\|u\|_{L_{\infty}(0,L)}^{2q}.$$

This inequality with sufficiently small $\omega, \omega_2 > 0$ implies that V is an iISS Lyapunov function with respect to the space $U = L_{\infty}(0, L)$ of input values.

To deal with the space $H_0^1(0,\pi)$ for the input values, substitute (4.11) into (5.6) and (5.8). \Box

6. Interconnections of iISS systems. Consider the following interconnected system:

(6.1)
$$\begin{aligned} \dot{x}_i(t) &= A_i x_i(t) + f_i(x_1, x_2, u), \quad i = 1, 2\\ x_i(t) &\in X_i, \quad u \in U_c, \end{aligned}$$

where X_i is a state space of the *i*-th subsystem, $A_i : D(A_i) \to X_i$ is a generator of a strongly continuous semigroup over X_i . Let $X = X_1 \times X_2$ which is the space of $x = (x_1, x_2)$, and the norm in X is defined as $\|\cdot\|_X = \|\cdot\|_{X_1} + \|\cdot\|_{X_2}$. In this section, we assume that there exist continuous functions $V_i : X_i \to \mathbb{R}_+, \psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty, \alpha_i \in \mathcal{P},$ $\sigma_i \in \mathcal{K}$ and $\kappa_i \in \mathcal{K} \cup \{0\}$ for i = 1, 2 such that

(6.2)
$$\psi_{i1}(\|x_i\|_{X_i}) \le V_i(x_i) \le \psi_{i2}(\|x_i\|_{X_i}), \quad \forall x_i \in X_i$$

and system (6.1) satisfies

(6.3)
$$\dot{V}_i(x_i) \le -\alpha_i(\|x_i\|_{X_i}) + \sigma_i(\|x_{3-i}\|_{X_{3-i}}) + \kappa_i(\|u(0)\|_U)$$

for all $x_i \in X_i$, $x_{3-i} \in X_{3-i}$ and $u \in U_c$, where the Lie derivative of V_i corresponding to the inputs $u \in U_c$ and $v \in PC(\mathbb{R}_+, X_{3-i})$ with $v(0) = x_{3-i}$ is defined by

(6.4)
$$\dot{V}_i(x_i) = \overline{\lim_{t \to +0} \frac{1}{t}} (V_i(\phi_i(t, x_i, v, u)) - V_i(x_i)).$$

To present a small-gain criterion for the interconnected system (6.1) whose components are not necessarily ISS, we make use of an generalized expression of inverse mappings on the set of extended non-negative numbers $\overline{\mathbb{R}}_+ = [0, \infty]$. For $\omega \in \mathcal{K}$, define the function ω^{\ominus} : $\overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ as $\omega^{\ominus}(s) = \sup\{v \in \mathbb{R}_+ : s \ge \omega(v)\}$. Notice that $\omega^{\ominus}(s) = \infty$ holds for $s \ge \lim_{\tau \to \infty} \omega(\tau)$, and $\omega^{\ominus}(s) = \omega^{-1}(s)$ holds elsewhere. A function $\omega \in \mathcal{K}$ is extended to ω : $\overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ as $\omega(s) := \sup_{v \in \{y \in \mathbb{R}_+ : y \le s\}} \omega(v)$. These notations are useful for presenting the following result succinctly.

THEOREM 7. Suppose that

(6.5)
$$\lim_{s \to \infty} \alpha_i(s) = \infty \text{ or } \lim_{s \to \infty} \sigma_{3-i}(s) \kappa_i(1) < \infty$$

is satisfied for i = 1, 2. If there exists c > 1 such that

(6.6)
$$\psi_{11}^{-1} \circ \psi_{12} \circ \alpha_1^{\ominus} \circ c\sigma_1 \circ \psi_{21}^{-1} \circ \psi_{22} \circ \alpha_2^{\ominus} \circ c\sigma_2(s) \le s$$

holds for all $s \in \mathbb{R}_+$, then system (6.1) is iISS. Moreover, if additionally $\alpha_i \in \mathcal{K}_{\infty}$ for i = 1, 2, then system (6.1) is ISS. Furthermore,

(6.7)
$$V(x) = \int_0^{V_1(x_1)} \lambda_1(s) ds + \int_0^{V_2(x_2)} \lambda_2(s) ds$$

is an iISS (ISS) Lyapunov function for (6.1), where $\lambda_i \in \mathcal{K}$ is given for i = 1, 2 by

(6.8)
$$\lambda_i(s) = [\alpha_i(\psi_{i2}^{-1}(s))]^{\psi} [\sigma_{3-i}(\psi_{3-i1}^{-1}(s))]^{\psi+1}, \forall s \in \mathbb{R}_+$$

with an arbitrary $\psi \geq 0$ satisfying

(6.9)
$$\begin{aligned} \psi &= 0 &, \text{ if } c > 2\\ \psi^{-\frac{\psi}{\psi+1}} < \frac{c}{\psi+1} \le 1 &, \text{ otherwise.} \end{aligned}$$

Proof. For the continuous function $V: X \to \mathbb{R}_+$ given by (6.7), we have

$$\dot{V} \leq \sum_{i=1}^{2} \lambda_{i}(V_{i}) \left\{ -\alpha_{i}(\psi_{i2}^{-1}(V_{i}(x_{i}))) + \sigma_{i}(\psi_{3-i1}^{-1}(V_{3-i}(x_{3-i}))) + \kappa_{i}(||u||_{U}) \right\}$$

along the solution of (6.1), due to (6.2) and (6.3). Following the arguments used in [13, 14], we can verify that with (6.8) and (6.9), the property

$$\dot{V} \leq \sum_{i=1}^{2} \left\{ -\delta_i \lambda_i (\psi_{i1}(V_i(x_i))) \alpha_i (\psi_{i2}^{-1}(\psi_{i1}(V_i(x_i))) + \hat{\kappa}_i (\|u\|_U) \right\}$$

holds for some $\hat{\kappa_1}, \hat{\kappa_2} \in \mathcal{K} \cup \{0\}$ and constants $\delta_2, \delta_1 > 0$ if (6.6) and (6.5) are satisfied. In addition, we have $\hat{\kappa_i} = 0$ if $\kappa_i = 0$. Hence, Proposition 2.1 completes the proof with the help of the definition of $\|\cdot\|_X$ and $\lambda_i \in \mathcal{K}$. \Box

It is straightforward to see that there always exists $\psi \ge 0$ satisfying (6.9). It is also worth mentioning that the Lyapunov function (6.7) is not in the maximization form, employed in [4] for establishing ISS. The use of the summation form (6.7) for systems which are not necessarily ISS is motivated by the limitation of the maximization form and clarified in [12] for finite-dimensional systems.

7. Examples. This section puts the results presented in the preceding sections together to analyze two reaction-diffusion systems.

7.1. Example 1. Consider

(7.1)
$$\begin{cases} \frac{\partial x_1}{\partial t}(l,t) = \frac{\partial^2 x_1}{\partial l^2}(l,t) + x_1(l,t)x_2^4(l,t), \\ x_1(0,t) = x_1(\pi,t) = 0; \\ \frac{\partial x_2}{\partial t} = \frac{\partial^2 x_2}{\partial l^2} + ax_2 - bx_2\left(\frac{\partial x_2}{\partial l}\right)^2 + \left(\frac{x_1^2}{1+x_1^2}\right)^{\frac{1}{2}}, \\ x_2(0,t) = x_2(\pi,t) = 0. \end{cases}$$

defined on the region $(l, t) \in (0, \pi) \times (0, \infty)$. To fully define the system we should choose the state spaces of subsystems. We take $X_1 := L_2(0, \pi)$ for $x_1(\cdot, t)$ and $X_2 :=$ $H_0^1(0,\pi)$ for $x_2(\cdot,t)$ as in Table 1 (a). We divide the analysis into three parts. First we prove that x_1 -subsystem is iISS based on the development in Sections 4. Next we prove that the x_2 -subsystem is ISS using the result in Section 3. In the last part we exploit the small-gain theorem presented in Section 6 to prove UGASs of x = 0 of the overall system (7.1).

7.1.1. The first subsystem is iISS. First we invoke Item 1 of Theorem 4 with q = 1 for $X_1 = L_2(0, \pi)$. Then $W(y) = y^2$ and due to $\dot{W}(y) \leq 2W(y)|x_2|^4$ and $x_1(0,t) = x_1(\pi,t) = 0$ for all $t \in \mathbb{R}_+$, one can choose

(7.2)
$$V_1(x_1) := \ln\left(1 + \|x_1\|_{L_2(0,\pi)}^2\right)$$

as an iISS Lyapunov function for x_1 -subsystem. Its Lie derivative according to (4.9) satisfies

(7.3)
$$\dot{V}_1 \le -\frac{2\|x_1\|_{L_2(0,\pi)}^2}{1+\|x_1\|_{L_2(0,\pi)}^2} + 2\|x_2\|_{L_\infty(0,\pi)}^4.$$

Note, that we have put $1 = \frac{\pi^2}{\pi^2}$ instead of $\frac{\pi^2}{4\pi^2}$ in formula (4.9) because $x_1 = 0$ holds at both ends of the interval $[0, \pi]$, and thus less conservative Friedrichs' inequality instead of Poincare's inequality can be used in getting (4.9). To replace $L_{\infty}(0, \pi)$ with $X_2 = H_0^1(0, \pi)$ for the input space used in (7.3), we recall (4.11), which results in

(7.4)
$$\dot{V}_1(x_1) \le -\frac{2\|x_1\|_{L_2(0,\pi)}^2}{1+\|x_1\|_{L_2(0,\pi)}^2} + 8\|x_2\|_{H_0^1(0,\pi)}^4.$$

Thus, we arrive at (6.3) for i = 1, and x_1 -subsystem is iISS with respect to the state space $X_1 = L_2(0, \pi)$ and the input space $X_2 = H_0^1(0, \pi)$.

7.1.2. The second subsystem is ISS. We invoke Item 1 of Theorem 2 with q = 1. To simplify notation we denote $u_2 := (x_1^2/(1+x_1^2))^{1/2}$. As in (3.5), we take

(7.5)
$$V_2(x_2) = \int_0^{\pi} \left(\frac{\partial x_2}{\partial l}\right)^2 dl = \|x_2\|_{H_0^1(0,\pi)}^2.$$

Notice that x_2 -subsystem is of the form (3.1) with c = 1, $f(x_2, \frac{\partial x_2}{\partial l}) = ax_2 - bx_2(\frac{\partial x_2}{\partial l})^2$. To arrive at (3.3), we obtain

(7.6)
$$\int_0^L \frac{\partial^2 x_2}{\partial l^2} \left(ax_2 - bx_2 \left(\frac{\partial x_2}{\partial l} \right)^2 \right) dl = -aV_2(x_2) - b \int_0^L \frac{\partial^2 x_2}{\partial l^2} x_2 \left(\frac{\partial x_2}{\partial l} \right)^2 dl$$

by integration by parts with $x_2(0,t) = x_2(\pi,t) = 0$ for all $t \in \mathbb{R}_+$. Due to $x_2(0,t) = x_2(\pi,t) = 0$ for all $t \in \mathbb{R}_+$, we have

$$\int_0^\pi \frac{\partial^2 x_2}{\partial l^2} x_2 \left(\frac{\partial x_2}{\partial l}\right)^2 dl = -\int_0^\pi \frac{\partial x_2}{\partial l} \left(2x_2 \frac{\partial x_2}{\partial l} \frac{\partial^2 x_2}{\partial l^2} + \left(\frac{\partial x_2}{\partial l}\right)^3\right) dl$$
$$= -\int_0^\pi 2x_2 \left(\frac{\partial x_2}{\partial l}\right)^2 \frac{\partial^2 x_2}{\partial l^2} dl - \int_0^\pi \left(\frac{\partial x_2}{\partial l}\right)^4 dl,$$

which implies that

(7.7)
$$\int_0^\pi \frac{\partial^2 x_2}{\partial l^2} x_2 \left(\frac{\partial x_2}{\partial l}\right)^2 dl = -\frac{1}{3} \int_0^\pi \left(\frac{\partial x_2}{\partial l}\right)^4 dl.$$

Thus, we arrive at

(7.8)
$$\eta(s) = -as + \frac{b}{3}s^2, \quad \forall s \in \mathbb{R}_+$$

which is convex if $b \ge 0$. We also obtain

(7.9)
$$\hat{\alpha}(s) = (1 - a - \epsilon)s + \frac{b}{3\pi}s^2$$

for (3.4). The inequality in (3.4) is achieved for $\epsilon = 1 - a > 0$ if a < 1. Hence, if a < 1 and $b \ge 0$ hold, Item 1 of Theorem 2 with q = 1 proves that for $\omega \in (0, 2(1 - a)]$, the function V_2 satisfies (with $\epsilon = 1 - a$)

(7.10)
$$\dot{V}_2 \le -2(1-a-\frac{\omega}{2})V_2(x_2) - \frac{2b}{3\pi}V_2(x_2)^2 + \frac{1}{\omega}\|u\|_{L_2(0,\pi)}^2$$

as in (3.12), and V_2 is an ISS Lyapunov function of x_2 -subsystem with respect to the state space $X_2 = H_0^1(0,\pi)$ for $x_2(\cdot,t)$ and the input space $U_1 = L_2(0,\pi)$ for $u_2(\cdot,t)$. Since $s \mapsto s/(1+s)$ is a concave function of $s \in \mathbb{R}_+$, Jensen's inequality yields

$$\int_0^{\pi} \frac{x_1^2}{1+x_1^2} dl \le \pi \frac{(1/\pi) \|x_1\|_{L_2(0,\pi)}^2}{1+(1/\pi) \|x_1\|_{L_2(0,\pi)}^2} \le \frac{\pi \|x_1\|_{L_2(0,\pi)}^2}{1+\|x_1\|_{L_2(0,\pi)}^2}$$

Using this property in (7.10) we have

$$(7.11) \quad \dot{V}_2 \le -2(1-a-\frac{\omega}{2})\|x_2\|_{H_0^1(0,\pi)}^2 - \frac{2b}{3\pi}\|x_2\|_{H_0^1(0,\pi)}^4 + \frac{\pi}{\omega} \left(\frac{\|x_1\|_{L_2(0,\pi)}^2}{1+\|x_1\|_{L_2(0,\pi)}^2}\right).$$

Therefore, V_2 is an ISS Lyapunov function of x_2 -subsystem with respect to the state space $X_2 = H_0^1(0,\pi)$ for $x_2(\cdot,t)$ and the input space $X_1 = L_2(0,\pi)$ for $x_1(\cdot,t)$.

Although property (7.11) is satisfactory for establishing UGASs of the overall system (7.1), it may be worth mentioning that we can obtain a different estimate for V_2 using Item 2 of Theorem 2. Pick $\epsilon = 1 - a$. We obtain $\hat{\alpha}(s) = (b/3\pi)s^2$ from (7.9). Then inequality (3.4) and $\epsilon \ge 0$ hold if $a \le 1$ and $b \ge 0$. Furthermore, if b > 0, $g: s \mapsto (b/3)s^3$ satisfies (3.6) with q = 1 and it is of class \mathcal{K}_{∞} . Hence using $\omega = 1/2$ and (3.17), we arrive at

(7.12)
$$\dot{V}_2 \le -2(1-a) \|x_2\|_{H^1_0(0,\pi)}^2 - \frac{b}{3\pi} \|x_2\|_{H^1_0(0,\pi)}^4 + 2\left(\frac{6}{b}\right)^{\frac{1}{3}} \int_0^\pi \left|\frac{\partial u}{\partial l}\right|^{4/3} dl,$$

which implies that V_2 is an ISS Lyapunov function of x_2 -subsystem with respect to the input space $U_2 = W_0^{1,\frac{4}{3}}(0,\pi)$ if $a \leq 1$ and b > 0.

Finally, it is worth noting that if a > 1, then x = 0 of the linearization of x_2 -subsystem for $x_1 \equiv 0$ is not UGASs (see [9, Theorem 5.1.3]). Thus x = 0 of the nonlinear x_2 -subsystem for $x_1 \equiv 0$ also cannot be UGASs if a > 1.

7.1.3. Interconnection is UGASs. Now we collect the findings of two previous subsections. Assume that a < 1 and $b \ge 0$. For the space $X = L_2(0, \pi) \times H_0^1(0, \pi)$, the Lyapunov functions defined as (7.2) and (7.5) for the two subsystems satisfy (6.2) with the class \mathcal{K}_{∞} functions $\psi_{11} = \psi_{12} : s \mapsto \ln(1 + s^2)$ and $\psi_{21} = \psi_{22} : s \mapsto s^2$. Due

to (7.4) and (7.11), we have (6.3) for

(7.13)
$$\alpha_1(s) = \frac{2s^2}{1+s^2}, \quad \sigma_1(s) = 8s^4, \quad \kappa_1(s) = 0$$

(7.14)
$$\alpha_2(s) = 2\left(1 - a - \frac{\omega}{2}\right)s^2 + \frac{2b}{3\pi}s^4, \ \sigma_2(s) = \frac{\pi}{\omega}\left(\frac{s^2}{1 + s^2}\right), \ \kappa_2(s) = 0$$

defined with $\omega \in (0, 2(1-a)]$. For these functions, condition (6.6) holds for all $s \in \mathbb{R}_+$ if and only if

(7.15)
$$\frac{12c^2\pi^2}{b\omega}\left(\frac{s^2}{1+s^2}\right) \le \frac{2s^2}{1+s^2}, \quad \forall s \in \mathbb{R}_+$$

is satisfied. Thus, there exists c > 1 such that (6.6) holds if and only if $6\pi^2/b < \omega$ holds. Combining this with $\omega \in (0, 2(1-a)]$, a < 1 and $b \ge 0$, Theorem 7 establishes UGASs of x = 0 for the whole system (7.1) when

(7.16)
$$a + \frac{3\pi^2}{b} < 1, \quad b \ge 0.$$

Note that (6.5) is satisfied. Due to the boundary conditions of x_2 , Friedrichs' inequality ensures $||x_2(\cdot,t)||_{L_2(0,\pi)} \leq ||x_2(\cdot,t)||_{H_0^1(0,\pi)}$. Thus, the UGASs guarantees the existence of $\beta \in \mathcal{KL}$ such that

(7.17)
$$\|\phi(t,\phi_0,0)\|_{L_2(0,\pi)\times L_2(0,\pi)} \le \|\phi(t,\phi_0,0)\|_X \le \beta(\|\phi_0\|_X,t)$$

holds for all $\phi_0 \in X$ and all $t \in \mathbb{R}_+$, where $X = L_2(0, \pi) \times H_0^1(0, \pi)$.

7.2. Example 2. Consider

(7.18)
$$\begin{cases} \frac{\partial x_1}{\partial t}(l,t) = \frac{\partial^2 x_1}{\partial l^2}(l,t) + \frac{\partial x_1}{\partial t}(l,t)x_2^4(l,t), \\ x_1(0,t) = x_1(\pi,t) = 0; \\ \frac{\partial x_2}{\partial t} = \frac{\partial^2 x_2}{\partial l^2} + ax_2 - bx_2\left(\frac{\partial x_2}{\partial l}\right)^2 + \left(\frac{x_1^2}{1+x_1^2}\right)^{\frac{1}{2}}, \\ x_2(0,t) = x_2(\pi,t) = 0. \end{cases}$$

defined on the region $(l, t) \in (0, \pi) \times (0, \infty)$. For (7.18), we take $X_1 := H_0^1(0, \pi)$ and $X_2 := H_0^1(0, \pi)$ as in Table 1 (b).

7.2.1. The first subsystem is iISS. We apply Theorem 6 to x_1 -subsystem on $X_1 = H_0^1(0, \pi)$ by taking q = 1. Let $V_1(x_1)$ be

(7.19)
$$V_1(x_1) := \ln\left(1 + \|x_1\|_{H_0^1(0,\pi)}^2\right)$$

We can use $\eta = 0$ for (3.3), which is convex on \mathbb{R}_+ . Let $\epsilon = c = 1$. Then Property (3.4) holds with $\hat{\alpha} = 0$. From (5.6) with $\omega = 1$ and (4.11) it follows that

(7.20)
$$\dot{V}_{1} \leq -\frac{\|x_{1}\|_{H_{0}^{1}(0,\pi)}^{2}}{1+\|x_{1}\|_{H_{0}^{1}(0,\pi)}^{2}} + 4\|x_{2}\|_{H_{0}^{1}(0,\pi)}^{4}.$$

Hence, inequality (6.3) is obtained for i = 1, and the V_1 in (7.19) is an iISS Lyapunov function of x_1 -subsystem with respect to the state space $X_1 = H_0^1(0, \pi)$ and the input space $X_2 = H_0^1(0, \pi)$.

7.2.2. The second subsystem is ISS. Since x_2 -subsystem of (7.18) is identical with that of (7.1), If a < 1 and $b \ge 0$ holds, the function V_1 in (7.5) is an ISS Lyapunov function and satisfies (7.11) with respect to the state space $X_2 = H_0^1(0, \pi)$ and the input space $X_1 = H_0^1(0, \pi)$.

7.2.3. Interconnection is UGASs. The above analysis for system (7.18) yields (6.2) and (6.3) for i = 1, 2, with functions which are the same as those for (7.1) except the following change:

(7.21)
$$\alpha_1(s) = \frac{s^2}{1+s^2}, \quad \sigma_1(s) = 4s^4$$

Again, condition (6.6) holds for all $s \in \mathbb{R}_+$ if and only if (7.15) is satisfied. Hence, Theorem 7 establishes UGASs of x = 0 for the whole system (7.18) if (7.16) holds. The UGASs ensures the existence of $\beta \in \mathcal{KL}$ satisfying (7.17) for all $\phi_0 \in X$ and all $t \in \mathbb{R}_+$ in terms of $X = H_0^1(0, \pi) \times H_0^1(0, \pi)$. Interestingly, in addition, for system (7.18), Agmon's and Friedrichs' inequalities yield

$$\|\phi(t,\phi_0,0)\|_{L_{\infty}(0,\pi)\times L_{\infty}(0,\pi)} \le \sqrt{2\beta}(\|\phi_0\|_X,t).$$

for all $\phi_0 \in X$ and all $t \in \mathbb{R}_+$.

8. Conclusion. We addressed the problem of stability of interconnected nonlinear parabolic systems. A small-gain criterion has been proposed together with a method to construct Lyapunov functions of interconnected systems. We emphasized the importance of a correct choice of state spaces in accordance of iISS subsystems which are not ISS. In ISS literature about parabolic systems [4, 24] the systems over L_p -spaces (which is the simplest possible case) have been studied most extensively. However, as indicated in [25], the presence of a bilinear term in a PDE makes the L_p setting break down. Indeed, pointwise multiplication of state and input variables in PDEs defined on L_2 state space cannot be bounded by the product of the spatial L_2 -norm of the state and the spatial L_2 -norm of the input, while this is true for norms in Euclidean space in case of ODEs. Importantly, when two system are connected to each other, a choice of state and input and spaces of one system affects the pair of the other systems. Thus, the bilinearity makes the choice in Table 2 useless, while the choice is often satisfactory for interconnections of ISS subsystems. In this paper, tools to construct Lyapunov functions characterizing iISS of infinite-dimensional systems have been developed, which are not covered by ISS Lyapunov functions. In addition, new methods for construction of ISS Lyapunov functions for parabolic systems over Sobolev spaces have been proposed as well. These new developments allowed one to formulate interconnections involving iISS subsystems in the setting as in Table 1, and they have led successfully to a small-gain theorem by which stability and robustness can be established for a class of nonlinear parabolic systems without requiring ISS properties.

Appendix. For L > 0, let $W^{2,1}(0, L)$ denote a Sobolev space of functions $x \in L_2(0, L)$ which have the first order weak derivatives, all of which belong to $L_2(0, L)$. PROPOSITION 8.1 (Young's inequality). For all $a, b \ge 0$ and all $\omega, p > 0$ it holds

(8.1)
$$ab \le \frac{\omega}{p}a^p + \frac{1}{\omega^{\frac{1}{p-1}}}\frac{p-1}{p}b^{\frac{p}{p-1}}.$$

Proof. See [26, p. 20]. □

PROPOSITION 8.2 (\mathcal{K}_{∞} -inequality). For all $a, b \geq 0$, for all $g \in \mathcal{K}_{\infty}$ and all $\omega > 0$ it holds

(8.2)
$$ab \le \omega ag(a) + bg^{-1}(\frac{b}{\omega}).$$

Proof. Follows from estimate of ab for $b \leq \omega g(a)$ and $b \geq \omega g(a)$. **D** PROPOSITION 8.3 (Jensen's inequality). For any convex $f : \mathbb{R} \to \mathbb{R}$ and any summable x

(8.3)
$$\int_0^L f(x(l,t))dl \ge Lf\Big(\frac{1}{L}\int_0^L x(l,t)dl\Big).$$

Proof. See [7, p. 705]. □

PROPOSITION 8.4 (Poincare's inequality). For every $x \in W^{2,1}(0,L)$ it holds that

(8.4)
$$\frac{4L^2}{\pi^2} \int_0^L \left(\frac{\partial x(l)}{\partial l}\right)^2 dl \ge \int_0^L x^2(l) dl$$

PROPOSITION 8.5 (Friedrichs' inequality). For all $f \in H^1_0(0,L) \cap H^2(0,L)$ it holds that

(8.5)
$$\frac{L^2}{\pi^2} \int_0^L \left(\frac{\partial x(l)}{\partial l}\right)^2 dl \ge \int_0^L x^2(l) dl$$

PROPOSITION 8.6 (Agmon's inequality). For all $f \in H^1(0, L)$ it holds that

(8.6)
$$\|f\|_{L_{\infty}(0,L)}^{2} \leq |f(0)|^{2} + 2\|f\|_{L_{2}(0,L)} \left\|\frac{df}{dl}\right\|_{L_{2}(0,L)}$$

Proof. See [22, Lemma 2.4., p. 20]. □

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