

Stability Estimates for an Inverse Hyperbolic Initial Boundary Value Problem with Unknown Boundaries

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Abstract

In this paper we prove stability estimates of logarithmic type for an inverse problem consisting in the determination of unknown portions of the boundary of a domain in \mathbb{R}^n , from a knowledge, in a finite time observation, of overdetermined boundary data for initial boundary value problem for anisotropic wave equation.

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1 Introduction

The reconstruction of obstacles from scattering waves has been widely investigated [Col-Kr]. This approach requires enough information on the scattered amplitude and generally infinitely many boundary measurements. In many practical situations these data are not available, for example in physical situations where only transient waves are detectable and one measurement in a finite time observation is obtainable. A typical situation is described by the wave equation in a domain Ω in \mathbb{R}^n ($n \geq 2$) whose boundary, assumed sufficiently smooth, consists of two non overlapping portion $\Gamma^{(a)}$ (accessible portion) and $\Gamma^{(i)}$ (inaccessible portion) where $\Gamma^{(i)}$ is a not known obstacle.

In the case in which $\Gamma^{(i)}$ is a soft obstacle the mathematical problem is represented by the following initial boundary value problem (the direct

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problem). Given a nontrivial function ψ on $\partial\Omega \times [0, T]$, $0 < T < +\infty$, such that

$$\psi = 0, \text{ on } \Gamma^{(i)} \times (0, T),$$

let u be the (weak) solution to the following problem

$$(1.1) \quad \begin{cases} \partial_t^2 u - \operatorname{div}(A(x)\nabla_x u) = 0, & \text{in } \Omega \times [0, T], \\ u|_{\partial\Omega \times [0, T]} = \psi, & \text{on } \partial\Omega \times [0, T], \\ u(\cdot, 0) = \partial_t u(\cdot, 0) = 0, & \text{in } \Omega, \end{cases}$$

($\operatorname{div} := \sum_{j=1}^n \partial_{x_j}$) where $A(x) = \{a^{ij}(x)\}_{i,j=1}^n$ denotes a known symmetric matrix which satisfies a hypothesis of uniform ellipticity and some smoothness conditions that we shall specify in the sequel of the paper. The inverse problem, we are interested in, is to determine $\Gamma^{(i)}$ from the knowledge of

$$(1.2) \quad A(x)\nabla_x u(x, t) \cdot \nu, \text{ on } \Sigma \times (0, T),$$

where $\Sigma \subset \Gamma^{(a)}$ and ν denotes the exterior unit normal to Ω .

The uniqueness for the above inverse problem has been proved in [Is2], however, in contrast to the analogues problems for elliptic equations or systems [Al-B-Ro-Ve], [Be-Ve], [Che-H-Y], [M-R1], [M-R2], [M-R-V2] and parabolic equations [C-Ro-Ve1], [C-Ro-Ve2], [Dc-R-Ve], [Ve1], [Ve2], the stability issue in the hyperbolic context is much less studied. In this paper we are interested in the stability issue for the inverse problem above. More precisely we are interested in the continuous dependence of $\Gamma^{(i)}$ from the Cauchy data u , $A\nabla_x u \cdot \nu$ on $\Sigma \times (0, T)$. Here we prove a logarithmic stability estimate under some a priori information on the domain Ω , on $\Gamma^{(i)}$, on ψ and whenever T is large enough, but *finite and independent by the errors on the Cauchy data*. In view of John counterexample [Jo] it is reasonable to expect that the logarithmic rate of stability is the optimal one. We are currently work on that topic

Now we describe briefly the main tools that we use to prove the stability result.

(a) *Stability Estimates for Cauchy Problem and Smallness Propagation Estimates.* In order to determine the unknown portion of boundary $\Gamma^{(i)}$ it seems necessary to determine the values of u from Cauchy data on $\Sigma \times (0, T)$ up to $\Gamma^{(i)} \times (0, T')$ for suitable $T' < T$. More precisely, let Ω_1 and Ω_2 be two domains whose boundary agree on $\Gamma^{(a)}$ and let u_j be the solutions of (1.1) for $\Omega = \Omega_j$, $j = 1, 2$. Denote by G the connected component of $\Omega_1 \cap \Omega_2$ that contains $\Gamma^{(a)}$, we need to estimate $u_1 - u_2$ in $G \times (0, T')$ in terms of the error on the Cauchy data on $\Sigma \times (0, T)$. In order to gain such estimates

we use the method introduced by Robbiano in [Ro1] and [Ro2] based on the Fourier Bros Iagolnitzer (FBI) transform defined by

$$U(x, y) := \sqrt{\frac{\mu}{2\pi}} \int_0^T e^{-\frac{\mu}{2}(iy+\tau-t)^2} (u_1 - u_2) dt, \quad \text{for every } (x, y) \in G \times \mathbb{R},$$

where μ be a positive number and $\tau \in (0, T)$.

By applying such a FBI transform the wave equation is transformed in a second order elliptic equation in $G \times \mathbb{R}$ with a nonhomogeneous term f depending on the final values $(u_1 - u_2)(\cdot, T)$, $\partial_t(u_1 - u_2)(\cdot, T)$ and on μ . Since, roughly speaking, f is small whether μ and T are large and $U(\cdot, 0)$ is close to $(u_1 - u_2)(\cdot, \tau)$ for large μ , we can apply the estimates for the Cauchy problem for elliptic equations proved in [Al-R-Ro-Ve] and we can obtain useful estimates of $u_1 - u_2$ in $G \times (0, T')$. We wish to stress that here we have, with respect to [Ro1] and [Ro2], the additional difficulty that the boundary of G might be irregular.

(b) *Quantitative estimates of strong unique continuation for wave equations.* For our proof it is crucial to know that the vanishing rate of u near the unknown boundary $\Gamma^{(i)}$ is of polynomial type. Namely we need quantitative estimates of strong unique continuation at the interior and at the boundary. Such estimates have been proved in [Ve3] (in the present paper, Theorems 4.1 and 4.2). It is exactly this property that allows us to obtain a sharp estimate of the Hausdorff distance, $d_{\mathcal{H}}(\overline{\Omega}_1, \overline{\Omega}_2)$, of the unknown domains Ω_1, Ω_2 in terms of the error on the Cauchy data (Corollary 5.5). The use of quantitative estimate of strong unique continuation is not new in inverse problem with unknown boundaries. The first paper in which such quantitative estimates was successfully used is, in the elliptic context, [Al-B-Ro-Ve]. Afterwards, quantitative estimates of strong unique continuation have been used and proved also for the parabolic problems, we refer to the papers mentioned above and the review paper [Ve2]. To the authors knowledge the quantitative estimate of strong unique continuation was never used before in the context of hyperbolic inverse problems.

(c) *Lemma of relative graphs and sharp three sphere inequality.* At a first stage the estimate of $d_{\mathcal{H}}(\overline{\Omega}_1, \overline{\Omega}_2)$ is worse than logarithmic and, in addition, the observation time T for which such estimate is available may depend on the error on the Cauchy data. In order to obtain the logarithmic stability estimate for T finite and independent by the errors on the Cauchy data we combine the geometric Lemma of relative graphs (Lemma 5.3) and a three sphere inequality for elliptic equations whose exponent is sharply evaluated when the radii of the three balls are close to each other (Theorem 4.6). This point is the most delicate part of the proof and is developed in Section 5.3.

The plan of the paper is as follows.

In Section 2 we will introduce the main notation and definition.

In Section 3 we will state the main Theorem 3.2.

The Section 4 contains some preliminary results concerning the quantitative estimates of strong unique continuation (Subsection 4.1), a regularity result for hyperbolic equation (Subsection 4.2), some elementary estimates for the FBI transform (Subsection 4.3) and a sharp form of the three sphere inequality for elliptic equations (Subsection 4.4).

In Section 5 we prove the main Theorem 3.2.

In the Appendix (Section 6) we prove some results of Section 4.

2 Notation and Definition

Let $n \in \mathbb{N}$, $n \geq 2$. For any $x \in \mathbb{R}^n$, we will denote $x = (x', x_n)$, where $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$ and $|x| = \left(\sum_{j=1}^n x_j^2\right)^{1/2}$. Given $x \in \mathbb{R}^n$, $r > 0$, we will use the following notation for balls and cylinders.

$$B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}, \quad B_r = B_r(0),$$

$$B'_r(x') = \{y' \in \mathbb{R}^{n-1} : |y' - x'| < r\}, \quad B'_r = B'_r(0),$$

$$Q_{a,b}(x) = \{y = (y', y_n) \in \mathbb{R}^n : |y' - x'| < a, |y_n - x_n| < b\}, \quad Q_{a,b} = Q_{a,b}(0).$$

For any $x \in \mathbb{R}^n$ $x = (x_1, \dots, x_n)$ and any $r > 0$ we denote by $\tilde{x} \in \mathbb{R}^{n+1}$ the point $\tilde{x} = (x_1, \dots, x_n, 0)$, or shortly $\tilde{x} = (x, 0)$ and by $\tilde{B}_r(\tilde{x})$ the ball of \mathbb{R}^{n+1} of radius r centered at \tilde{x} . For any open set $\Omega \subset \mathbb{R}^n$ and any function (smooth enough) u we denote by $\nabla_x u = (\partial_{x_1} u, \dots, \partial_{x_n} u)$ the gradient of u . Also, for the gradient of u we use the notation D_x . If $j = 0, 1, 2$ we denote by $D_x^j u$ the set of the derivatives of u of order j , so $D_x^0 u = u$, $D_x^1 u = \nabla_x u$ and D_x^2 is the hessian matrix $\{\partial_{x_i x_j} u\}_{i,j=1}^n$. Similar notation are used whenever other variables occur and Ω is an open subset of \mathbb{R}^{n-1} or a subset \mathbb{R}^{n+1} . By $H^\ell(\Omega)$, $\ell = 0, 1, 2$ we denote the usual Sobolev spaces of order ℓ , in particular we have $H^0(\Omega) = L^2(\Omega)$.

For any interval $J \subset \mathbb{R}$ and Ω as above we denote by

$$\mathcal{W}(J; \Omega) = \{u \in C^0(J; H^2(\Omega)) : \partial_t^\ell u \in C^0(J; H^{2-\ell}(\Omega)), \ell = 1, 2\}.$$

Definition 2.1 ($C^{k,1}$ **regularity of a domain**). Let Ω be a bounded domain in \mathbb{R}^n . Given $k \in \mathbb{N} \cup 0$, we say that a portion S of $\partial\Omega$ is of *class* $C^{k,1}$ *with*

constants $\rho_0, E > 0$, if, for any $P \in S$, there exists a rigid transformation of coordinates under which we have $P = 0$ and

$$\Omega \cap Q_{\frac{\rho_0}{E}, \rho_0} = \{x = (x', x_n) \in Q_{\frac{\rho_0}{E}, \rho_0} \mid x_n > \varphi(x')\},$$

where φ is a $C^{k,1}$ function on $B'_{\frac{\rho_0}{E}}$ satisfying

$$\|\varphi\|_{C^{k,1}(B'_{\rho_0/E})} \leq E\rho_0,$$

$$\varphi(0) = 0,$$

and, whenever $k \geq 1$,

$$\nabla_{x'}\varphi(0) = 0.$$

When $\partial\Omega$ is of class $C^{k,1}$ with constants $\rho_0, E > 0$ we also say that Ω is of class $C^{k,1}$ with constants $\rho_0, E > 0$. Moreover, when $k = 0$ we also say that S is of *Lipschitz class with constants ρ_0, E* .

Remark 2.2. We use the convention of normalizing all norms in such a way that all their terms are dimensionally homogeneous. For example:

$$\|\varphi\|_{C^{0,1}(B'_{r_0})} = \|\varphi\|_{L^\infty(B'_{r_0})} + r_0 \|\nabla_{x'}\varphi\|_{L^\infty(B'_{r_0})}.$$

Similarly, if $u \in H^m(\Omega)$, where Ω is a domain of \mathbb{R}^n of class $C^{k,1}$ with constants ρ_0, E denoting by $D^j u$ the vector which components are the derivatives of order j of the function u ,

$$\|u\|_{H^m(\Omega)} = \rho_0^{-n/2} \left(\sum_{j=0}^m \rho_0^{2j} \int_{\Omega} |D_x^j u|^2 \right)^{\frac{1}{2}},$$

$$\|u\|_{C^k(\Omega)} = \sum_{i=0}^k r_0^i \|D_x^i u\|_{L^\infty(\Omega)}.$$

Definition 2.3. (relative graphs). We shall say that two bounded domains Ω_1 and Ω_2 in \mathbb{R}^n of class $C^{1,1}$ with constants ρ_0, E are *relative graphs* if for any $P \in \partial\Omega_1$ there exists a rigid transformation of coordinates under which we have $P \equiv 0$ and there exist $\varphi_{P,1}, \varphi_{P,2} \in C^{1,1}(B'_{r_0}(0))$, where $\frac{r_0}{\rho_0} \leq 1$ depends on E only, satisfying the following conditions

$$(2.1a) \quad \varphi_{P,1}(0) = |\nabla_{x'} \varphi_{P,1}(0)| = 0 \quad , \quad |\varphi_{P,2}(0)| \leq \frac{r_0}{2},$$

$$(2.1b) \quad \|\varphi_{P,i}\|_{C^{1,1}(B'_{r_0}(0))} \leq E\rho_0,$$

$$(2.1c) \quad \Omega_i \cap B_{r_0}(0) = \{x \in B_{r_0}(0) : x_n > \varphi_{P,i}(x')\}, \quad i = 1, 2.$$

We shall denote

$$(2.2) \quad \gamma_0(\Omega_1, \Omega_2) = \sup_{P \in \partial\Omega_1} \|\varphi_{P,1} - \varphi_{P,2}\|_{L^\infty(B'_{r_0}(0))}$$

and, for any $\alpha \in (0, 1]$,

$$(2.3) \quad \gamma_{1,\alpha}(\Omega_1, \Omega_2) = \sup_{P \in \partial\Omega_1} \|\varphi_{P,1} - \varphi_{P,2}\|_{C^{1,\alpha}(B'_{r_0}(0))}.$$

Definition 2.4. (Hausdorff distance). Let Ω_1 and Ω_2 be bounded domains in \mathbb{R}^n . We call Hausdorff distance between Ω_1 and Ω_2 the number

$$(2.4) \quad d_{\mathcal{H}}(\overline{\Omega}_1, \overline{\Omega}_2) = \max \left\{ \sup_{x \in \overline{\Omega}_1} \text{dist}(x, \overline{\Omega}_2), \sup_{x \in \overline{\Omega}_2} \text{dist}(x, \overline{\Omega}_1) \right\}.$$

Definition 2.5. (modified distance). Let Ω_1 and Ω_2 be bounded domains in \mathbb{R}^n . We call modified distance between Ω_1 and Ω_2 the number

$$(2.5) \quad d_m(\overline{\Omega}_1, \overline{\Omega}_2) = \max \left\{ \sup_{x \in \partial\Omega_1} \text{dist}(x, \overline{\Omega}_2), \sup_{x \in \partial\Omega_2} \text{dist}(x, \overline{\Omega}_1) \right\}.$$

For any open set $\Omega \subset \mathbb{R}^n$ and $r > 0$, we shall denote

$$\Omega_r = \{x \in \Omega : \text{dist}(x, \partial\Omega) > r\}.$$

We shall use the the letters C to denote constants larger or equal than 1. Sometime, for special constants or to emphasize the role that it have in the proof we will use the notation C_0, C_1, \dots . The value of the constants may change from line to line, but we shall specified their dependence everywhere they appear.

3 The Inverse Problem: The Main Theorem

i) *A priori information on the domain.*

Given $\rho_0, M > 0, E \geq 1$ we assume

$$(3.1a) \quad |\Omega| \leq M\rho_0^n,$$

$$(3.1b) \quad \partial\Omega \text{ of class } C^{1,1} \text{ with constants } \rho_0 \text{ and } E,$$

here, and in the sequel, $|\Omega|$ denotes the Lebesgue measure of Ω .

Let $\Gamma^{(a)}$ be a nonempty closed proper subset of $\partial\Omega$ and assume that the closure of the interior part of $\Gamma^{(a)}$ in the relative topology in $\partial\Omega$ is equal to $\Gamma^{(a)}$. In addition we assume that

$$(3.2) \quad \text{Int}_{\partial\Omega}(\Gamma^{(a)}) \text{ is connected,}$$

and we set

$$(3.3) \quad \Gamma^{(i)} = \partial\Omega \setminus \text{Int}_{\partial\Omega}(\Gamma^{(a)}),$$

here and in the sequel, $\text{Int}_{\partial\Omega}(\Gamma^{(a)})$ is the interior part of $\Gamma^{(a)}$ in the relative topology in $\partial\Omega$. In the sequel we will refer to $\Gamma^{(a)}$ and $\Gamma^{(i)}$ as the *accessible* and *inaccessible* part of $\partial\Omega$ respectively.

Moreover denoting

$$\Gamma_\rho^{(a)} = \{x \in \Gamma^{(a)} : \text{dist}(x, \Gamma^{(i)}) \geq \rho\},$$

we assume that, for any $\rho \in (0, \rho_0]$, $\Gamma_\rho^{(a)}$ is a nonempty and connected set and we assume that we can select a portion Σ satisfying for some $P_0 \in \Sigma$

$$(3.4) \quad \partial\Omega \cap B_{\rho_0}(P_0) \subset \Sigma \subset \Gamma_{\rho_0}^{(a)}.$$

Remark 3.1. Observe that (3.1b) automatically implies a lower bound on the diameter of every connected component of $\partial\Omega$. Moreover, by combining (3.1a) with (3.1b), an upper bound on the diameter of Ω can also be obtained. Note also that (3.1a), (3.1a) implicitly comprise an a priori upper bound on the number of connected components of $\partial\Omega$. Finally observe that the hypotheses (3.1)-(3.4) are satisfied in the case $\Omega = \widehat{\Omega} \setminus \overline{D}$, where $\widehat{\Omega}$ and D are two open domains in \mathbb{R}^n whose boundaries, $\partial\widehat{\Omega}$ and ∂D , are connected, $D \subset \widehat{\Omega}$, $\text{dist}(D, \partial\Omega) \geq 2\rho_0$ and $\widehat{\Omega}, D$ satisfy condition (3.1). In addition $\Gamma^a = \partial\widehat{\Omega}$, $\Gamma^i = \partial D$ and Σ is a portion of $\partial\widehat{\Omega}$ satisfying, for some $P_0 \in \Sigma$, the condition $\partial\widehat{\Omega} \cap B_{\rho_0}(P_0) \subset \Sigma$.

ii) *Assumptions about the boundary data.*

Let $m := \lfloor \frac{n+2}{4} \rfloor$. Assume that ψ is a function on $\partial\Omega \times [0, +\infty)$ which satisfies the following conditions

$$(3.5a) \quad \partial_t^j \psi(\cdot, t) \in C^{1,1}(\partial\Omega) \quad , \text{ for } j \in \{0, \dots, 2m+4\}, \text{ and } t \in [0, +\infty),$$

$$(3.5b) \quad \partial_t^j \psi(\cdot, 0) = 0 \quad , \text{ for } j \in \{0, \dots, 2m+4\}, \text{ and } t \in [0, +\infty).$$

Denote, for $t \in [0, +\infty)$

$$(3.6) \quad H(t) = \sum_{j=0}^{2m+4} \rho_0^j \sup_{\xi \in [0, t]} \|\partial_\xi^j \psi(\cdot, \xi)\|_{C^{1,1}(\partial\Omega)}.$$

Let $t_1 \geq \rho_0$ and assume

$$(3.7) \quad \frac{H(t_1)}{\|\psi\|_{L^\infty(\Gamma^{(a)} \times [0, t_1])}} \leq F.$$

iii) *Assumptions about the matrix A .*

$A(x) = \{a^{ij}(x)\}_{i,j=1}^n$ is assumed to be a real-valued symmetric $n \times n$ matrix whose the entries are measurable function and satisfying the following conditions for given constants $\lambda \in (0, 1]$, $\Lambda > 0$,

$$(3.8a) \quad \lambda |\xi|^2 \leq A(x) \xi \cdot \xi \leq \lambda^{-1} |\xi|^2, \quad \text{for every } x, \xi \in \mathbb{R}^n,$$

$$(3.8b) \quad |A(x) - A(y)| \leq \frac{\Lambda}{\rho_0} |x - y|, \quad \text{for every } x, y \in \mathbb{R}^n.$$

Theorem 3.2. *Let Ω_1, Ω_2 be two domains satisfying (3.1). Let $\Gamma_j^{(a)}, \Gamma_j^{(i)} = \partial\Omega_j \setminus \text{Int}_{\partial\Omega_j}(\Gamma_j^{(a)})$, $j = 1, 2$, be the corresponding accessible and inaccessible parts of their boundaries. Let us assume $\Gamma_1^{(a)} = \Gamma_2^{(a)} = \Gamma^{(a)}$, Ω_1, Ω_2 lie on the same side of $\Gamma^{(a)}$ and that (3.2), (3.3) and (3.4) are satisfied.*

Then there exists a constant C depending on λ, Λ, E, M and F only such that if $T = \max\{C\rho_0, 2t_1\}$ then the following holds true.

Let $u_j \in \mathcal{W}([0, T]; \Omega)$ be the solution to (1.1) when $\Omega = \Omega_j$, $j = 1, 2$, and if, for a given $\varepsilon \in (0, e^{-1})$, we have

$$(3.9) \quad \int_0^T \int_\Sigma |A(x) \nabla u_1 \cdot \nu - A(x) \nabla u_2 \cdot \nu|^2 dS dt \leq T \rho_0^{n-3} \varepsilon^2,$$

where dS is the surface element in dimension $n - 1$, then we have

$$(3.10) \quad d_{\mathcal{H}}(\overline{\Omega}_1, \overline{\Omega}_2) \leq C_{\star} \rho_0 |\log \varepsilon|^{-1/C_{\star}},$$

where C_{\star} depends on $\lambda, \Lambda, E, M, F$ and the ratio $\frac{H(T)}{H(t_1)}$.

We prove this Theorem in Section 5.

4 Preliminary results

4.1 Quantitative estimates of strong unique continuation.

Theorems presented in this subsection are crucial to prove Theorem 3.2. They are analogs of the quantitative estimates of strong unique continuation (doubling inequalities, three sphere inequality, three cylinders inequality, two-sphere one cylinder inequality at the interior and at the boundary) which are well known in the elliptic [Ga-Li], [La], [A-E] and in the parabolic context [Es-Fe-Ve], [Es-Ve]. Theorem 4.1 is basically the quantitative version of the strong unique continuation property for the self-adjoint hyperbolic equation proved by Lebeau in [Le]. Theorems 4.1 and 4.2 have been proved in [Ve3].

Let $u \in \mathcal{W}([-\lambda\rho_0, \lambda\rho_0]; B_{\rho_0})$ be a weak solution to

$$(4.1) \quad \partial_t^2 u - \operatorname{div}(A(x)\nabla_x u) = 0, \quad \text{in } B_{\rho_0} \times (-\lambda\rho_0, \lambda\rho_0).$$

Let $r_0 \in (0, \rho_0]$ and denote by

$$(4.2) \quad \varepsilon_0 := \sup_{t \in (-\lambda\rho_0, \lambda\rho_0)} \left(\rho_0^{-n} \int_{B_{r_0}} u^2(x, t) dx \right)^{1/2}$$

and

$$(4.3) \quad H_0 := \left(\sum_{j=0}^2 \rho_0^{j-n} \int_{B_{\rho_0}} |D_x^j u(x, 0)|^2 dx \right)^{1/2}.$$

Theorem 4.1. *Let $A(x)$ be a real-valued symmetric $n \times n$ matrix satisfying (3.8) and let $u \in \mathcal{W}([-\lambda\rho_0, \lambda\rho_0]; B_{\rho_0})$ be a weak solution to (4.1). Then there*

exist constants $s_0 \in (0, 1)$ and $C \geq 1$ depending on λ and Λ only such that for every r_0 and ρ satisfying $0 < r_0 \leq \rho \leq s_0 \rho_0$ the following inequality holds true

$$(4.4) \quad \|u(\cdot, 0)\|_{L^2(B_\rho)} \leq \frac{C (\rho_0 \rho^{-1})^C (H_0 + e\varepsilon_0)}{\left(\theta \log \left(\frac{H_0 + e\varepsilon_0}{\varepsilon_0}\right)\right)^{1/6}},$$

where

$$(4.5) \quad \theta = \frac{\log(\rho_0/C\rho)}{\log(\rho_0/r_0)}.$$

In order to state Theorem 4.2 below let us introduce some notation. Let φ be a function belonging to $C^{1,1}(B'_{\rho_0})$ that satisfies

$$(4.6) \quad \varphi(0) = |\nabla_{x'} \varphi(0)| = 0$$

and

$$(4.7) \quad \|\varphi\|_{C^{1,1}(B'_{\rho_0})} \leq E\rho_0,$$

where

$$\|\varphi\|_{C^{1,1}(B'_{\rho_0})} = \|\varphi\|_{L^\infty(B'_{\rho_0})} + \rho_0 \|\nabla_{x'} \varphi\|_{L^\infty(B'_{\rho_0})} + \rho_0^2 \|D_{x'}^2 \varphi\|_{L^\infty(B'_{\rho_0})}.$$

For any $r \in (0, \rho_0]$ denote by

$$K_r := \{(x', x_n) \in B_r : x_n > \varphi(x')\}$$

and

$$S_{\rho_0} := \{(x', \varphi(x')) : x' \in B'_{\rho_0}\}.$$

Let $u \in \mathcal{W}([-\lambda\rho_0, \lambda\rho_0]; K_{\rho_0})$ be a solution to

$$(4.8) \quad \partial_t^2 u - \operatorname{div}(A(x)\nabla_x u) = 0, \quad \text{in } K_{\rho_0} \times (-\lambda\rho_0, \lambda\rho_0),$$

satisfying one of the following conditions

$$(4.9) \quad u = 0, \quad \text{on } S_{\rho_0} \times (-\lambda\rho_0, \lambda\rho_0),$$

$$(4.10) \quad A \nabla_x u \cdot \nu = 0, \quad \text{on } S_{\rho_0} \times (-\lambda \rho_0, \lambda \rho_0),$$

where ν denotes the outer unit normal to S_{ρ_0} .

Let $r_0 \in (0, \rho_0]$ and denote by

$$(4.11) \quad \varepsilon_0 := \sup_{t \in (-\lambda \rho_0, \lambda \rho_0)} \left(\rho_0^{-n} \int_{K_{r_0}} u^2(x, t) dx \right)^{1/2}$$

and

$$(4.12) \quad H_0 := \left(\sum_{j=0}^2 \rho_0^{j-n} \int_{K_{\rho_0}} |D_x^j u(x, 0)|^2 dx \right)^{1/2}.$$

Theorem 4.2. *Let (3.8) be satisfied. Let $u \in \mathcal{W}([-\lambda \rho_0, \lambda \rho_0]; K_{\rho_0})$ be a solution to (4.8) satisfying (4.11) and (4.12). Assume that u satisfies either (4.9) or (4.10). There exist constants $\bar{s}_0 \in (0, 1)$ and $C \geq 1$ depending on λ , Λ and E only such that for every r_0 and ρ satisfying $0 < r_0 \leq \rho \leq \bar{s}_0 \rho_0$ the following inequality holds true*

$$(4.13) \quad \|u(\cdot, 0)\|_{L^2(K_\rho)} \leq \frac{C (\rho_0 \rho^{-1})^C (H_0 + e \varepsilon_0)}{\left(\tilde{\theta} \log \left(\frac{H_0 + e \varepsilon_0}{\varepsilon_0} \right) \right)^{1/6}},$$

where

$$(4.14) \quad \tilde{\theta} = \frac{\log(\rho_0 / C \rho)}{\log(\rho_0 / r_0)}.$$

4.2 A regularity result for hyperbolic equation

The next Theorem is a mere simplified version of a regularity result proved in [Co]. For the reader convenience we give a sketch of the proof of such a result in the Appendix, Subsection 6.1.

Theorem 4.3. *Let Ω be a bounded domain of \mathbb{R}^n that satisfies (3.1). Let $A(x)$ be a real-valued symmetric $n \times n$ matrix satisfying (3.8). Let $m := \lceil \frac{n+2}{4} \rceil$. Assume that ψ is a function on $\partial\Omega \times [0, T]$ which satisfies the condition (3.5).*

Let $u \in \mathcal{W}([0, T]; \Omega)$ be the solution to the problem

$$(4.15) \quad \begin{cases} \partial_t^2 u - \operatorname{div}(A(x)\nabla_x u) = 0, & \text{in } \Omega \times [0, T], \\ u = \psi, & \text{on } \partial\Omega \times [0, T], \\ u(\cdot, 0) = \partial_t u(\cdot, 0) = 0, & \text{in } \Omega. \end{cases}$$

Then for every $\alpha \in (0, 1)$ and $t \in [0, T]$ we have $\partial_t^2 u(\cdot, t) \in L^\infty(\Omega)$, $u(\cdot, t) \in C^{1,\alpha}(\Omega)$ and the following inequalities hold true

$$(4.16a) \quad \sup_{t \in [0, T]} \|\partial_t^2 u(\cdot, t)\|_{L^\infty(\Omega)} \leq C\rho_0^{-2}(T\rho_0^{-1} + 1)H(T),$$

$$(4.16b) \quad \sup_{t \in [0, T]} \|u(\cdot, t)\|_{H^2(\Omega)} \leq C(T\rho_0^{-1} + 1)H(T),$$

$$(4.16c) \quad \sup_{t \in [0, T]} \|u(\cdot, t)\|_{C^{1,\alpha}(\Omega)} \leq C(T\rho_0^{-1} + 1)H(T),$$

where $H(T)$ is defined by (3.6) and C depends on α, n, E, M, λ and Λ only.

4.3 Elementary estimates for the FBI transform

For the convenience of the the reader, we collect in this section some well known elementary properties of the FBI transform see also [Che-D-Y], [Che-P-Y], [Ro1], [Ro2], [Ro-Zu]. Let Ω be a domain of \mathbb{R}^n and T a positive number. Let $u \in \mathcal{W}([0, T]; \Omega)$ satisfy

$$(4.17) \quad \begin{cases} \partial_t^2 u - \operatorname{div}(A(x)\nabla_x u) = 0, & \text{in } \Omega \times [0, T], \\ u(\cdot, 0) = 0, & \text{in } \Omega, \\ \partial_t u(\cdot, 0) = 0, & \text{in } \Omega. \end{cases}$$

Let μ be a positive number. For a fixed $\tau \in (0, T/2]$ we denote by $U_\mu^{(\tau)}$ the FBI transform of u defined by

$$(4.18) \quad U_\mu^{(\tau)}(x, y) := \sqrt{\frac{\mu}{2\pi}} \int_0^T e^{-\frac{\mu}{2}(iy+\tau-t)^2} u(x, t) dt, \quad \text{for every } (x, y) \in \Omega \times \mathbb{R}.$$

Observe that $U_\mu^{(\tau)}$ as a function of y is a $C^\infty(\mathbb{R})$ with values in $H^2(\Omega)$.

The following propositions holds true.

Proposition 4.4. *We have*

$$(4.19) \quad \begin{aligned} & |D_x^j U_\mu^{(\tau)}(x, y)| \leq \\ & \leq c\mu^{1/4} e^{\frac{\mu}{2}y^2} \left(\int_0^T |D_x^j u(x, t)|^2 dt \right)^{1/2}, \quad \text{for a.e. } x \in \Omega, \quad \text{and } 0 \leq j \leq 2, \end{aligned}$$

and

$$(4.20) \quad |U_\mu^{(\tau)}(x, 0) - u(x, \tau)| \leq c\mu^{-1/2} \|\partial_t u(x, \cdot)\|_{L^\infty[0, T]}$$

where c is an absolute constant.

Proof. See Subsection 6.2. □

Proposition 4.5. *Let $u \in \mathcal{W}([0, T]; \Omega)$ satisfy (4.17) and let $U_\mu^{(\tau)}$ be defined by (4.18). Then U_μ satisfies the equation*

$$(4.21) \quad \partial_y^2 U_\mu^{(\tau)} + \operatorname{div}(A(x) \nabla_x U_\mu^{(\tau)}) = f_\mu^{(\tau)}(x, y), \quad \text{in } \Omega \times \mathbb{R},$$

where

$$(4.22) \quad f_\mu(x, y) = \sqrt{\frac{\mu}{2\pi}} e^{-\frac{\mu}{2}(iy + \tau - T)^2} (\partial_t u(x, T) - \mu(iy + \tau - T)u(x, T)).$$

Proof. See Subsection 6.2. □

4.4 A sharp three sphere inequality for elliptic equations

In the following theorem we give a three sphere inequality for elliptic equations in which we take care to evaluate the exponent of the inequality when the radii of the three balls are close to each other. Except for this feature the following Theorem is quite standard and, for the convenience of the reader, we will prove it in the Appendix (Subsection 6.3).

Let $\tilde{A}(X) = \{\tilde{a}^{ij}(x)\}_{i,j=1}^N$, $N \geq 2$ be a real-valued symmetric $N \times N$ matrix. Assume that the entries of matrix \tilde{A} are measurable function and it satisfies

$$(4.23) \quad \lambda_0 |\xi|^2 \leq \tilde{A}(X) \xi \cdot \xi \leq \lambda_0^{-1} |\xi|^2, \quad \text{for every } X, \xi \in \mathbb{R}^N,$$

where $\lambda_0 \in (0, 1]$.

Theorem 4.6 (Three sphere inequality). *Let \tilde{r}_3 and Λ_0 be positive numbers. Assume that \tilde{A} satisfies (4.23) and*

$$(4.24) \quad \left| \tilde{A}(X) - \tilde{A}(Y) \right| \leq \frac{\Lambda_0}{\tilde{r}_3} |X - Y|, \quad \text{for every } X, Y \in B_{\tilde{r}_3}.$$

Let $\tilde{f} \in L^2(B_{\tilde{r}_3})$ and let $u \in H^1(B_{\tilde{r}_3})$ be a solution to

$$(4.25) \quad Pu := \operatorname{div}(\tilde{A} \nabla u) = \tilde{f}, \quad \text{in } B_{\tilde{r}_3}.$$

Let $\tilde{r}_1, \tilde{r}_2, \tilde{r}_3$ be such that $0 < \tilde{r}_1 \leq \tilde{r}_2 < \tilde{r}_3$. Let δ be such that

$$(4.26) \quad 0 < \delta \leq \frac{\tilde{r}_3 - \tilde{r}_2}{2\tilde{r}_3}.$$

Denote by

$$(4.27) \quad \vartheta_0 = \frac{\tilde{r}_2^{-\beta} - [(1 - \delta)\tilde{r}_3]^{-\beta}}{[(1 - 2\delta)\tilde{r}_1]^{-\beta} - [(1 - \delta)\tilde{r}_3]^{-\beta}}.$$

and

$$(4.28) \quad C_0 = \frac{e^{C[(\tilde{r}_2\tilde{r}_3^{-1})^{-\beta} - (1 - \delta)^{-\beta}]}}{\delta^4},$$

where C depends on λ_0, Λ_0 .

There exists $\beta_1 \geq 1$ depending on λ_0, Λ_0 only such that if $\beta \geq \beta_1$ then the following inequality holds true

$$(4.29) \quad \int_{B_{\tilde{r}_2}} |u|^2 \leq \leq C_0 \left(\int_{B_{\tilde{r}_1}} |u|^2 + \tilde{r}_3^2 \int_{B_{\tilde{r}_3}} |\tilde{f}|^2 \right)^{\vartheta_0} \left(\int_{B_{\tilde{r}_3}} |u|^2 + \tilde{r}_3^2 \int_{B_{\tilde{r}_3}} |\tilde{f}|^2 \right)^{1 - \vartheta_0}.$$

5 Proof of the Main Theorem

In order to prove Theorem 3.2 we proceed in the following way.

Set

G the connected component of $\Omega_1 \cap \Omega_2$ whose closure contains $\Gamma^{(a)}$.

First step In Proposition 5.1 we prove that for a given $t_0 > 0$ there exists $T(\varepsilon) > 2t_0$ such that if (3.9) is satisfied for $T = T(\varepsilon)$ and $u_j \in \mathcal{W}([0, T(\varepsilon)]; \Omega)$ are the solutions to (1.1) when $\Omega = \Omega_j$, $j = 1, 2$ then

$$\sup_{t \in [0, t_0]} \left(\rho_0^{-n} \int_{\Omega_j \setminus G} u_j^2(x, t) dx \right) \leq C\omega(\varepsilon, t_0) \quad , \text{ for } j = 1, 2,$$

where

$$\lim_{\varepsilon \rightarrow 0} \omega(\varepsilon, t_0) = 0 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} T(\varepsilon) = +\infty.$$

Second step First we prove (Proposition 5.2) an estimate from below, in terms of the a priori information and boundary data, of the quantity $\sup \|u(\cdot, t)\|_{L^2(B_{\bar{\varrho}}(y_0))}$ where the sup is taken for $t \in [0, \bar{t}]$, \bar{t} is large enough, $B_{\bar{\varrho}}(y_0) \subset \Omega$ and $\bar{\varrho} \in (0, \rho_0/2E]$. Afterwards (Proposition 5.4) we prove that if t_0 is large enough and

$$\sup_{t \in [0, t_0]} \left(\rho_0^{-n} \int_{\Omega_j \setminus G} u_j^2(x, t) dx \right) \leq \eta^2$$

then

$$d_{\mathcal{H}}(\bar{\Omega}_1, \bar{\Omega}_2) \leq C\rho_0\eta^\alpha,$$

for suitable constant $C \geq 1$ and $\alpha \in (0, 1)$.

Third step We conclude the proof of Theorem 3.2.

5.1 Step 1

Proposition 5.1. *There exist $C \geq 1$ and $\bar{\varepsilon}, \bar{\sigma}, \vartheta_2 \in (0, 1]$ depending on E, M, λ and Λ only such that the following holds true.*

Denoting

$$(5.1) \quad T_\sigma := \max \left\{ 2t_0, \sqrt{10}\rho_0\vartheta_2^{-\frac{1}{2}\sigma^{-(n+1)}} \right\},$$

$$(5.2) \quad \Phi(\sigma) = \sigma^{-\left(\frac{n+1}{4}\right)} (T_\sigma \rho_0^{-1})^{11/2} (H(T_\sigma) + 1)^2.$$

Let us define for any $\varepsilon \in (0, \bar{\varepsilon}]$

$$(5.3) \quad T(\varepsilon) := T_{\sigma(\varepsilon)},$$

where

$$(5.4) \quad \sigma(\varepsilon) := \inf \{ \sigma \in (0, \bar{\sigma}] : \Phi(\sigma) \leq |\log \varepsilon|^{\frac{1}{8}} \},$$

Let $u_j \in \mathcal{W}([0, T(\varepsilon)]; \Omega)$ be the solution to (1.1) (when $T = T(\varepsilon)$) and $\Omega = \Omega_j$, $j = 1, 2$.

If, for a given $\varepsilon \in (0, \bar{\varepsilon}]$, we have

$$(5.5) \quad \frac{1}{T(\varepsilon)\rho_0^{n-3}} \int_0^{T(\varepsilon)} \int_{\Sigma} |A(x)\nabla u_1 \cdot \nu - A(x)\nabla u_2 \cdot \nu|^2 dS dt \leq \varepsilon^2$$

then for every $t_0 \in (0, T(\varepsilon)/2]$ we have

$$(5.6) \quad \sup_{t \in [0, t_0]} \left(\rho_0^{-n} \int_{\Omega_j \setminus G} u_j^2(x, t) dx \right) \leq C\omega(\varepsilon, t_0) \quad , \text{ for } j = 1, 2,$$

where

$$(5.7) \quad \omega(\varepsilon, t_0) = (t_0 \rho_0^{-1})^6 (H(t_0))^2 (\sigma(\varepsilon))^{1/4} + |\log \varepsilon|^{-1/8}.$$

Proof of Proposition 5.1. Let $t_0 > 0$. We begin by assuming only that $T \geq 2t_0$. Let $u_j \in \mathcal{W}([0, T]; \Omega)$ be the solution to (1.1) when $\Omega = \Omega_j$, $j = 1, 2$. Let $u = u_1 - u_2$, in $G \times [0, T]$ and for any positive number μ such that $\mu T^2 \geq 1$ and $\tau \in (0, T/2]$ denote by $U_\mu^{(\tau)}$ the FBI transform of u defined by

$$(5.8) \quad \begin{aligned} U_\mu^{(\tau)}(x, y) &= \\ &= \sqrt{\frac{\mu}{2\pi}} \int_0^T e^{-\frac{\mu}{2}(iy+\tau-t)^2} u(x, t) dt, \quad \text{for every } (x, y) \in G \times \mathbb{R}. \end{aligned}$$

By (1.1), (3.9) and Proposition 4.5 we have

$$(5.9) \quad \begin{cases} \partial_y^2 U_\mu^{(\tau)} + \operatorname{div} \left(A(x) \nabla U_\mu^{(\tau)} \right) = f_\mu^{(\tau)}(x, y), & \text{in } G \times \mathbb{R}, \\ U_\mu^{(\tau)}(x, y) = 0, & \text{for } (x, y) \in \Sigma \times \mathbb{R}, \\ \int_{\Sigma} \left| A(x) \nabla U_\mu^{(\tau)}(x, y) \cdot \nu \right|^2 dS dt \leq C \mu^{1/2} T \rho_0^{n-3} e^{\mu y^2} \varepsilon^2, \end{cases}$$

and C is an absolute constant.

By (3.5), Proposition 4.4, Proposition 4.5, by Theorem 4.3 and by the elementary inequality $s^{3/2} e^{-s^2/8} \leq c e^{-s^2/10}$ we have, for every $R > 0$

$$(5.10) \quad \|f_\mu\|_{L^\infty(G \times (-R, R))} \leq C T \rho_0^{-3} H(T) e^{\mu(R^2/2 - T^2/10)},$$

and

$$(5.11) \quad \|U_\mu^{(\tau)}\|_{L^\infty(G \times (-R, R))} \leq CT\rho_0^{-1}H(T)e^{\mu R^2/2},$$

where C depends on E, M, λ and Λ only. Here and in the sequel, we fix $\alpha = \frac{1}{2}$ in Theorem 4.3.

Now denote by $P_1 = P_0 - \frac{\rho_0}{2E}\nu$, $\tilde{P}_1 = (P_1, 0)$, $\rho_1 = \sigma_1\rho_0$, where $\sigma_1 = \frac{1}{4E\sqrt{1+E^2}}$ and denote by

$$(5.12) \quad \varepsilon_1 = \frac{(\mu T^2)^{1/4}\varepsilon}{(H(T) + 1)T\rho_0^{-1}}.$$

By (5.9), (5.10) and by applying [Al-R-Ro-Ve, Theorem 1.7] we have

$$(5.13) \quad \|U_\mu^{(\tau)}\|_{L^2(\tilde{B}_{\rho_1}(\tilde{P}_1))} \leq CT\rho_0^{-1}H(T)e^{\mu\rho_0^2/2} \left(e^{-\mu T^2/10} + \varepsilon_1\right)^{\vartheta_1},$$

where $\vartheta_1, \vartheta_1 \in (0, 1)$, and C depend on E, M, λ and Λ only.

Let $\sigma \in (0, \sigma_1]$ and denote by $r = \rho_0\sigma$. Let V_r be the connected component of $\Omega_{1,r} \cap \Omega_{2,r}$ whose closure contains $B_{\rho_1}(P_1)$. Moreover denote by $\omega_r = \Omega_{1,r} \setminus V_r$. We have

$$(5.14a) \quad \Omega_1 \setminus G \subset [(\Omega_1 \setminus \Omega_{1,r}) \setminus G] \cup \omega_r,$$

$$(5.14b) \quad \partial\omega_r = \Gamma_{1,r} \cup \Gamma_{2,r},$$

where

$$\Gamma_{1,r} \subset \partial\Omega_{1,r}, \quad \Gamma_{2,r} \subset \partial\Omega_{2,r} \cap \partial V_r.$$

Let $z \in \Gamma_{2,r}$ be fixed. Since V_r is connected, $\Gamma_{2,r} \subset \partial V_r$ and $P_1 \in V_r$, there exists a continuous path $\gamma : [0, 1] \rightarrow V_r$ such that $\gamma(0) = P_1$, $\gamma(1) = z$. Let us define $0 = s_0 < s_1 < \dots < s_N = 1$, according to the following rule. We set $s_{k+1} = \max\{s \mid |\gamma(s) - x_k| = \frac{r}{2}\}$ if $|x_k - z| > \frac{r}{2}$, otherwise we stop the process and set $N = k + 1$, $s_N = 1$. By (3.1a) we have $N \leq c_n M \sigma^{-n}$ where c_n depends on n only. Let $x_k = \gamma(s_k)$ and $\tilde{x}_k = (x_k, 0)$. The balls (of \mathbb{R}^{n+1}) $\tilde{B}_{r/4}(\tilde{x}_k)$ are pairwise disjoint for $k = 0, \dots, N - 1$ and $|\tilde{x}_{k+1} - \tilde{x}_k| = \frac{r}{2}$. We have that $\tilde{B}_{r/4}(\tilde{x}_{k+1}) \subset \tilde{B}_{3r/4}(\tilde{x}_k)$ and $\tilde{B}_r(\tilde{x}_k) \subset G \times (-r, r)$ and therefore, by the three sphere inequality (4.29) we have

$$(5.15) \quad \|U_\mu^{(\tau)}\|_{L^2(\tilde{B}_{r/4}(\tilde{x}_{k+1}))} \leq \|U_\mu^{(\tau)}\|_{L^2(\tilde{B}_{3r/4}(\tilde{x}_k))} \leq \\ \leq C \left(\|U_\mu^{(\tau)}\|_{L^2(\tilde{B}_{r/4}(\tilde{x}_k))} + \|f_\mu^{(\tau)}\|_{L^2(\tilde{B}_r(\tilde{x}_k))} \right)^{\vartheta_*} \left(\|U_\mu^{(\tau)}\|_{L^2(\tilde{B}_r(\tilde{x}_k))} + \|f_\mu^{(\tau)}\|_{L^2(\tilde{B}_r(\tilde{x}_k))} \right)^{1-\vartheta_*},$$

where C and ϑ_* , $0 < \vartheta_* < 1$, depend on E, λ and Λ only.

Now, we denote by

$$(5.16) \quad \alpha_k = \frac{\left\| U_\mu^{(\tau)} \right\|_{L^2(\tilde{B}_{r/4}(\tilde{x}_k))} e^{-\mu r^2/2}}{T \rho_0^{-1}(H(T) + 1)} + e^{-\mu T^2/10} \quad \text{for } k = 0, \dots, N,$$

and by (5.15), (5.10) and (5.11) we have

$$(5.17) \quad \alpha_{k+1} \leq C \alpha_k^{\vartheta_*} \quad \text{for } k = 0, \dots, N-1,$$

where C and ϑ_* , $0 < \vartheta_* < 1$, depend on E, λ and Λ only. By iterating (5.17) we get

$$(5.18) \quad \alpha_N \leq C^{1/1-\vartheta_*} \alpha_0^{\vartheta_*^N}.$$

Now let us denote by $\vartheta_2 = \min \{ \vartheta_1, \vartheta_*^{c_n^M} \}$. By (5.16) and (5.18) we have

$$(5.19) \quad \left\| U_\mu^{(\tau)} \right\|_{L^2(\tilde{B}_{r/4}(\tilde{z}))} \leq C T \rho_0^{-1} H(T) e^{\mu r^2/2} \times \\ \times \left(\frac{\left\| U_\mu^{(\tau)} \right\|_{L^2(\tilde{B}_{r/4}(\tilde{P}_1))} e^{-\mu r^2/2}}{T \rho_0^{-1}(H(T) + 1)} + e^{-\mu T^2/10} \right)^{\vartheta_2^{\sigma^{-n}}},$$

where C depends on E, M, λ and Λ only. Moreover, by applying [G-T, Theorem 8.17] and by using (5.10), (5.13) and (5.19) we have

$$(5.20) \quad |U_\mu^{(\tau)}(z, 0)| \leq C T \rho_0^{-1}(H(T) + 1) e^{\mu r^2/2} \varepsilon_2,$$

where

$$(5.21) \quad \varepsilon_2 = \sigma^{-\left(\frac{n+1}{2}\right)} \left(e^{-\mu T^2/10} + e^{\mu \rho_0^2/2} \left(e^{-\mu T^2/10} + \varepsilon_1 \right)^{\vartheta_2} \right)^{\vartheta_2^{\sigma^{-n}}}$$

and C depends on E, M, λ and Λ only.

By (5.20), (4.16) and (4.20) we have

$$(5.22) \quad \|u\|_{L^\infty(\Gamma_{2,r} \times [0, t_0])} \leq C (\mu T^2)^{-1/2} (\rho_0^{-1} T)^2 H(T) + \\ + \sup_{\tau \in [0, t_0]} \|U_\mu^{(\tau)}(\cdot, 0)\|_{L^\infty(\Gamma_{2,r})} \leq C (T \rho_0^{-1})^3 (H(T) + 1) \varepsilon_3,$$

where

$$(5.23) \quad \varepsilon_3 = (\mu T^2)^{-1/2} + e^{\mu r^2/2} \varepsilon_2$$

and C depends on E, M, λ and Λ only.

By (5.14a) and Schwarz inequality we have, for any $t \in (0, t_0]$,

$$(5.24) \quad \begin{aligned} & \int_{\Omega_1 \setminus G} u_1^2(x, t) dx = \\ & = \int_{\Omega_1 \setminus G} \left(\int_0^t \partial_\xi u_1(x, \xi) d\xi \right)^2 \leq t_0 \int_0^{t_0} \int_{\Omega_1 \setminus G} |\partial_\xi u_1(x, \xi)|^2 dx d\xi \leq \\ & \leq t_0 \int_0^{t_0} \int_{\omega_r} |\partial_\xi u_1(x, \xi)|^2 dx d\xi + t_0 \int_0^{t_0} \int_{\Omega_1 \setminus \Omega_{1,r}} |\partial_\xi u_1(x, \xi)|^2 dx d\xi. \end{aligned}$$

Now by (3.1) we have

$$(5.25) \quad |\Omega_1 \setminus \Omega_{1,r}| \leq C \rho_0^n \sigma,$$

where C depends on E and M only.

By (3.1a) (5.24), (5.25) and (4.16a) we have, for any $t \in (0, t_0]$,

$$(5.26) \quad \begin{aligned} & \rho_0^{-n} \int_{\Omega_1 \setminus G} u_1^2(x, t) dx \leq \\ & \leq t_0 \rho_0^{-n} \int_0^{t_0} \int_{\omega_r} |\partial_\xi u_1(x, \xi)|^2 dx d\xi + C (t_0 \rho_0^{-1})^6 H(t_0)^2 \sigma, \end{aligned}$$

where C depends on E, M, λ and Λ only.

Now, in order to estimate from above the integral on the right hand side of (5.26) we multiply both the side of the equation $\partial_t^2 u_1 - \operatorname{div}(A(x) \nabla u_1) = 0$ by $\partial_t u_1$ and integrate over ω_r and by integration by parts we have, for every $\xi \in [0, t_0]$,

$$(5.27) \quad \begin{aligned} & \frac{1}{2} \int_{\omega_r} (|\partial_\xi u_1(x, \xi)|^2 + A(x) \nabla u_1(x, \xi) \cdot \nabla u_1(x, \xi)) dx = \\ & = \int_0^\xi \int_{\Gamma_{1,r}} (A(x) \nabla u_1(x, t)) \partial_t u_1(x, t) dS dt + \int_0^\xi \int_{\Gamma_{2,r}} (A(x) \nabla u_1(x, t)) \partial_t u_1(x, t) dS dt := J_1 + J_2. \end{aligned}$$

Estimate of J_1 .

By Schwarz inequality and by (3.8a) we have

$$(5.28) \quad |J_1| \leq \lambda^{-1} \left(\int_0^\xi \int_{\Gamma_{1,r}} |\nabla u_1|^2 dS dt \right)^{1/2} \left(\int_0^\xi \int_{\Gamma_{1,r}} |\partial_t u_1(x, t)|^2 dS dt \right)^{1/2}.$$

By (5.28) and (4.16c) we have, for every $\xi \in [0, t_0]$,

$$(5.29) \quad |J_1| \leq C \left((t_0 \rho_0^{-1} + 1) t_0 \rho_0^{n-3} \right)^{1/2} H(t_0) \left(\int_0^\xi \int_{\Gamma_{1,r}} |\partial_t u_1(x, t)|^2 dS dt \right)^{1/2},$$

where C depends on E, M, λ and Λ only.

Now, by interpolation inequality we have

$$\|\partial_t u_1\|_{L^\infty(\Gamma_{1,r} \times [0, t_0])} \leq C \|u_1\|_{L^\infty(\Gamma_{1,r} \times [0, t_0])}^{1/2} \|\partial_t^2 u_1\|_{L^\infty(\Gamma_{1,r} \times [0, t_0])}^{1/2},$$

where C is an absolute constant, hence by using (5.29), (4.16) and recalling that $u_1 = 0$ on $\Gamma_1 \times [0, T]$ we obtain

$$(5.30) \quad |J_1| \leq C (t_0 \rho_0^{-1})^{5/2} \rho_0^{n-2} (H(t_0))^2 \sigma^{1/4},$$

where C depends on E, M, λ and Λ only.

Estimate of J_2 .

By Schwarz inequality, (3.8a) and (4.16c) we have, for every $\xi \in [0, t_0]$,

$$(5.31) \quad |J_2| \leq C (t_0^2 \rho_0^{n-4})^{1/2} H(t_0) \left(\int_0^\xi \int_{\Gamma_{2,r}} |\partial_t u_1(x, t)|^2 dS dt \right)^{1/2},$$

where C depends on E, M, λ and Λ only.

By the triangle inequality and taking into account that $u = u_1 - u_2$ on $\Gamma_{2,r} \times [0, T]$ we have, for every $\xi \in [0, t_0]$,

$$(5.32) \quad \left(\int_0^\xi \int_{\Gamma_{2,r}} |\partial_t u_1(x, t)|^2 dS dt \right)^{1/2} \leq \\ \leq C (t_0 \rho_0^{n-1})^{1/2} \left(\|\partial_t u\|_{L^\infty(\Gamma_{2,r} \times [0, t_0])} + \|\partial_t u_2\|_{L^\infty(\Gamma_{2,r} \times [0, t_0])} \right),$$

where C depends on E and M only.

Arguing as in the estimate of J_1 , by (5.22), (5.31), (5.32) and (4.16a) we have

$$(5.33) \quad |J_2| \leq C\rho_0^{n-2}(t_0\rho_0^{-1})^{5/2}(H(t_0))^2\sigma^{1/4} + \\ + C\rho_0^{n-2}(T\rho_0^{-1})^3(H(T)+1)^2\varepsilon_3^{1/2},$$

where C depends on E, M, λ and Λ only.

By (5.26), (5.27), (5.30) and (5.33) we have, for every $t \in (0, t_0]$,

$$(5.34) \quad \rho_0^{-n} \int_{\Omega_1 \setminus G} u_1^2(x, t) dx \leq C(t_0\rho_0^{-1})^6(H(t_0))^2\sigma^{1/4} + \\ + C(T\rho_0^{-1})^5(H(T)+1)^2\varepsilon_3^{1/2},$$

where C depends on α, E, M, λ and Λ only. In order estimate from above the right hand side of (5.34) first we assume

$$\varepsilon \leq e^{-5}.$$

Let μ and T be such that

$$(5.35) \quad \mu T^2 = \frac{1}{5} |\log \varepsilon|.$$

By (5.12), (5.35) we have trivially

$$(5.36) \quad e^{-\mu T^2/10} + \varepsilon_1 \leq c\varepsilon^{1/2},$$

where c is an absolute constant. Hence, taking into account (5.21) and (5.23) we have

$$(5.37) \quad \varepsilon_3^{1/2} \leq (\mu T^2)^{-1/4} + e^{\mu T^2(\rho_0 T^{-1})^2 \sigma^2/4} \varepsilon_2^{1/2} \leq \\ \leq (\mu T^2)^{-1/4} + C\sigma^{-\left(\frac{n+1}{4}\right)} e^{\frac{1}{4}\mu K(\sigma, T)},$$

where

$$(5.38) \quad K(\sigma, T) = 2\rho_0^2\sigma^2 - \frac{T^2}{5}\vartheta_2^{\sigma^{-n-1}}.$$

Let us choose

$$(5.39) \quad T = T_\sigma = \max \left\{ 2t_0, \sqrt{10}\rho_0\vartheta_2^{-\frac{1}{2}\sigma^{-(n+1)}} \right\},$$

and we have $K(\sigma, T_\sigma) \leq -3\rho_0^2$. Hence by (5.37) and (5.39) we have

$$(5.40) \quad \varepsilon_3^{1/2} \leq C\sigma^{-(\frac{n+1}{4})} (T_\sigma\rho_0^{-1})^{1/2} |\log \varepsilon|^{-1/4},$$

where C depends on E, M, λ and Λ only. By (5.34) and (5.40) we have, for every $t \in (0, t_0]$ and $\sigma \in (0, \sigma_1]$,

$$(5.41) \quad \begin{aligned} \rho_0^{-n} \int_{\Omega_1 \setminus G} u_1^2(x, t) dx &\leq \\ &\leq C \left((t_0\rho_0^{-1})^6 (H(t_0))^2 \sigma^{1/4} + \Phi(\sigma) |\log \varepsilon|^{-1/4} \right), \end{aligned}$$

where C depends on E, M, λ and Λ only and $\Phi(\sigma)$ is defined by (5.2).

Let $\bar{\sigma} = \min\{\sigma_1, (2n|\log \vartheta_2|)^{1/(n+1)}\}$, by (5.39) we have that Φ is a decreasing function in $(0, \bar{\sigma}]$, so that $\min_{(0, \bar{\sigma}]} \Phi = \Phi(\bar{\sigma})$. Now, let us denote by $\bar{\varepsilon} = \min\{e^{-5}, e^{-(\Phi(\bar{\sigma}))^8}\}$ and for any $\varepsilon \in (0, \bar{\varepsilon}]$ let us choose $\sigma = \sigma(\varepsilon)$ where $\sigma = \sigma(\varepsilon)$ is defined by (5.4). By (5.41) we have

$$(5.42) \quad \rho_0^{-n} \int_{\Omega_1 \setminus G} u_1^2(x, t) dx \leq C\omega(\varepsilon, t_0),$$

where C depends on E, M, λ and Λ only and $\omega(\varepsilon, t_0)$ is defined by (5.7). \square

5.2 Step 2

Proposition 5.2. *Let $\bar{\varrho} \in (0, \rho_0/2E]$ and let $y_0 \in \Omega$ be such that $B_{\bar{\varrho}}(y_0) \subset \Omega$. Assume that u is solution to (1.1). Then there exists a constant C_F , $C_F \geq 2$, depending on $E, M, \lambda, \Lambda, \bar{\varrho}\rho_0^{-1}$ and F only such that if $\bar{t} \geq t_* := \max\{C_F\rho_0, 2t_1\}$ then the following inequality holds true*

$$(5.43) \quad \bar{t}\rho_0^{-1}H(\bar{t})e^{-\mathcal{F}(\bar{t})} \leq \sup_{t \in [0, \bar{t}]} \|u(\cdot, t)\|_{L^2(B_{\bar{\varrho}}(y_0))},$$

where

$$(5.44) \quad \mathcal{F}(\bar{t}) = \left(\frac{C_F(\bar{t}\rho_0^{-1})^3 H(\bar{t})}{H(t_1)} \right)^2.$$

Proof. For any number \bar{t} such that $\bar{t} \geq 2t_1$ let us denote

$$(5.45) \quad \eta = \sup_{t \in [0, \bar{t}]} \|u(\cdot, t)\|_{L^2(B_{\bar{\varrho}}(y_0))}.$$

Let $(x_0, \bar{\tau}) \in \Gamma^{(a)} \times [0, t_1]$ be such that

$$(5.46) \quad |\psi(x_0, \bar{\tau})| = \|\psi\|_{L^\infty(\Gamma^{(a)} \times [0, t_1])}.$$

Let $\delta \in (0, \frac{1}{4}]$ be a number that we will choose later and let $x_\delta = x_0 - 4\delta\bar{\nu}(x_0)$. By Proposition 4.3 and by (5.46) we have

$$(5.47) \quad \begin{aligned} \|\psi\|_{L^\infty(\Gamma^{(a)} \times [0, t_1])} &\leq |u(x_0, \bar{\tau}) - u(x_\delta, \bar{\tau})| + |u(x_\delta, \bar{\tau})| \leq \\ &\leq C_0 t_1 \bar{\nu} \rho_0^{-2} H(t_1) \delta + |u(x_\delta, \bar{\tau})| \end{aligned}$$

where C_0 depends on E, M, λ and Λ only. Now let us choose

$$\bar{\delta} = \min \left\{ \frac{1}{4}, \frac{\rho_0}{2C_0 t_1 F} \right\}$$

and by (5.47) we have

$$(5.48) \quad \frac{1}{2} \|\psi\|_{L^\infty(\Gamma^{(a)} \times [0, t_1])} \leq \left| u(x^{(\bar{\delta})}, \bar{\tau}) \right|.$$

Now we estimate from above in terms of η the right hand side of (5.48). In order to get such an estimate we proceed similarly to Proposition 5.1. For any positive number μ such that $\mu \bar{t}^2 \geq 1$ and $\tau \in (0, \bar{t}/2]$ denote by $U_\mu^{(\tau)}$ the FBI transform of u defined by

$$(5.49) \quad U_\mu^{(\tau)}(x, y) := \sqrt{\frac{\mu}{2\pi}} \int_0^{\bar{t}} e^{-\frac{\mu}{2}(iy + \tau - t)^2} u(x, t) dt, \quad \text{for every } (x, y) \in \Omega \times \mathbb{R}.$$

Denote by $x_1 = x_0 - \bar{\nu}(x_0)$ where $\nu(x_0)$ is the exterior unit normal to $\partial\Omega$ in x_0 . Since

$$\begin{cases} \partial_y^2 U_\mu^{(\tau)} + \operatorname{div} \left(A(x) \nabla U_\mu^{(\tau)} \right) = f_\mu^{(\tau)}(x, y), & \text{in } \Omega \times \mathbb{R}, \\ \int_{B_{\bar{\nu}}(y_0)} \left| U_\mu^{(\tau)}(x, y) \right|^2 dx \leq c \mu^{1/2} \bar{t} \rho_0^n e^{\mu y^2} \eta^2, \end{cases}$$

by arguing as in Proposition 5.1 we get

$$(5.50) \quad \begin{aligned} \|U_\mu^{(\tau)}\|_{L^2(\tilde{B}_{\bar{\nu}/4}(\tilde{x}_1))} &\leq C \bar{t} \rho_0^{-1} H(\bar{t}) e^{\mu \bar{\nu}^2/2} \times \\ &\times \left(\frac{\|U_\mu^{(\tau)}\|_{L^2(\tilde{B}_{\bar{\nu}/4}(\tilde{y}_0))} e^{-\mu \bar{\nu}^2/2}}{\bar{t} \rho_0^{-1} (H(\bar{t}) + 1)} + e^{-\mu \bar{t}^2/10} \right)^{\vartheta_1^*}, \end{aligned}$$

and

(5.51)

$$\|U_\mu^{(\tau)}\|_{L^2(\tilde{B}_{\bar{\varrho}/4}(\tilde{x}_1))} \leq C(\bar{t}\rho_0^{-1}H(\bar{t}) + 2\eta)e^{\mu\bar{\varrho}^2/2} \left((\mu\bar{t}^2)^{3/2}e^{-\mu\bar{t}^2/8} + \eta_1 \right)^{\vartheta_1^*},$$

where

$$(5.52) \quad \eta_1 = \frac{(\mu\bar{t}^2)^{1/4}\eta}{H(\bar{t})\bar{t}\rho_0^{-1} + 2\eta},$$

$\vartheta_1^* \in (0, 1)$ depends on E, M, λ, Λ and $\bar{\varrho}$ only and C depends on E, M, λ and Λ only.

By (4.16), (4.20), (4.22), (5.48) and by applying [G-T, Theorem 8.17] we have

$$(5.53) \quad \begin{aligned} \frac{1}{2} \|\psi\|_{L^\infty(\Gamma^{(a)} \times [0, t_1])} &\leq |u(x_{\bar{\delta}}, \bar{\tau}) - U_\mu^{(\bar{\tau})}(x_{\bar{\delta}}, 0)| + |U_\mu^{(\bar{\tau})}(x_{\bar{\delta}}, 0)| \leq \\ &\leq C(\mu\bar{t}^2)^{-1/2}(\bar{t}\rho_0^{-1})^2\bar{t}H(\bar{t}) + \|U_\mu^{(\bar{\tau})}\|_{L^\infty(\tilde{B}_{\bar{\varrho}(1-3\bar{\delta})}(\tilde{x}_1))} \leq \\ &\leq C \left((\mu\bar{t}^2)^{-1/2} + e^{\mu(\bar{\varrho}^2/2 - \bar{t}^2/10)} \right) (\bar{t}\rho_0^{-1})^3 H(\bar{t}) + \frac{C'}{\bar{\delta}^{(n+1)/2}} \|U_\mu^{(\bar{\tau})}\|_{L^2(\tilde{B}_{\bar{\varrho}(1-2\bar{\delta})}(\tilde{x}_1))}, \end{aligned}$$

where C depends on E, M, λ and Λ only and where C' depends on λ only.

Now let us apply the three sphere inequality (4.29) with $r_1 = \frac{\bar{\varrho}}{4}$, $r_2 = \bar{\varrho}(1 - 2\bar{\delta})$ and $r_3 = \bar{\varrho}$. By (4.22) and (5.51) we have

$$(5.54) \quad \|U_\mu^{(\bar{\tau})}\|_{L^2(\tilde{B}_{\bar{\varrho}(1-2\bar{\delta})}(\tilde{x}_1))} \leq C((\bar{t}\rho_0^{-1})^3 H(\bar{t}) + 2\eta) e^{\mu\bar{\varrho}^2/2} \left(e^{-\mu\bar{t}^2/10} + \eta_1 \right)^{\vartheta_2^*}$$

where $\vartheta_2^*, \vartheta_2^* \in (0, 1)$, and C depend on $E, M, \lambda, \Lambda, \bar{\varrho}\rho_0^{-1}$ and F only.

By (5.53) and (5.54) we have

(5.55)

$$\|\psi\|_{L^\infty(\Gamma^{(a)} \times [0, t_1])} \leq C(\bar{t}\rho_0^{-1}H(\bar{t}) + 2\eta) \left((\mu\bar{t}^2)^{-1/2} + e^{\mu\bar{\varrho}^2/2} \left(e^{-\mu\bar{t}^2/10} + \eta_1 \right)^{\vartheta_2^*} \right),$$

where C depends on $E, M, \lambda, \Lambda, \bar{\varrho}$ and F only. Now if $\bar{t} \geq \max \{ \sqrt{10}(\vartheta_2^*)^{-1/2}\bar{\varrho}, 2\rho_0, 2t_1 \}$ then (5.52) and (5.55) give

(5.56)

$$\begin{aligned} \|\psi\|_{L^\infty(\Gamma^{(a)} \times [0, t_1])} &\leq C((\bar{t}\rho_0^{-1})^3 H(\bar{t}) + 2\eta) \times \\ &\times \left((\mu\bar{t}^2)^{-1/2} + (\mu\bar{t}^2)^{\vartheta_2^*/4} e^{\mu\vartheta_2^*\bar{t}^2/20} \left(\frac{\eta}{\bar{t}\rho_0^{-1}H(\bar{t}) + 2\eta} \right)^{\vartheta_2^*} \right), \end{aligned}$$

where C depends on $E, M, \lambda, \Lambda, \bar{\rho}_0^{-1}$ and F only.

Now let us choose

$$\mu = \frac{10}{\bar{t}^2} \left| \log \left(\frac{\eta}{\bar{t}\rho_0^{-1}H(\bar{t}) + 2\eta} \right) \right|$$

and by (5.56), taking into account that $\eta \leq CH(\bar{t})$, we get

$$(5.57) \quad \|\psi\|_{L^\infty(\Gamma^{(a)} \times [0, t_1])} \leq C(\bar{t}\rho_0^{-1})^2 H(\bar{t}) \left| \log \left(\frac{\eta}{CT_1\rho_0^{-1}H(\bar{t})} \right) \right|^{-1/2},$$

where C depends on $E, M, \lambda, \Lambda, \bar{\rho}_0^{-1}$ and F only. By (5.57) the thesis follows. \square

Now we recall the following Lemma that was proved in [Al-B-Ro-Ve, Lemma 8.1].

Lemma 5.3 (relative graphs). *Let Ω_1 and Ω_2 be bounded domains in \mathbb{R}^n of class $C^{1,1}$ with constants ρ_0, E and satisfying $|\Omega_j| \leq M\rho_0^n, j = 1, 2$. There exist numbers $d_0, \bar{\rho}_0 \in (0, \rho_0]$ such that $\frac{d_0}{\rho_0}$ and $\frac{\bar{\rho}_0}{\rho_0}$ depend on E only, and such that if we have*

$$(5.58) \quad d_{\mathcal{H}}(\bar{\Omega}_1, \bar{\Omega}_2) \leq d_0,$$

then the following facts hold true

i) Ω_1 and Ω_2 are relative graphs and

$$(5.59) \quad \gamma_0(\Omega_1, \Omega_2) \leq Cd_{\mathcal{H}}(\bar{\Omega}_1, \bar{\Omega}_2),$$

where C depends on E only,

$$(5.60) \quad \gamma_{1,\alpha}(\Omega_1, \Omega_2) \leq C\rho_0^{\frac{1+\alpha}{2}} (d_{\mathcal{H}}(\bar{\Omega}_1, \bar{\Omega}_2))^{\frac{1-\alpha}{2}}, \quad \text{for every } \alpha \in (0, 1),$$

where C depends on E and α only,

ii) there exists an absolute positive constant c such that

$$(5.61) \quad d_{\mathcal{H}}(\bar{\Omega}_1, \bar{\Omega}_2) \leq cd_m(\bar{\Omega}_1, \bar{\Omega}_2),$$

iii) $\Omega_1 \cap \Omega_2$ is a domain of Lipschitz class with constants $\bar{\rho}_0, L$, where $\bar{\rho}_0$ is as above and $L > 0$ depends on E only.

Proposition 5.4. *There exist constants C_F and C depending on E, M, λ, Λ and F only and on E, M, λ, Λ only respectively, such that if $t_0 \geq t_* + \lambda\rho_0$ where t_* is introduced in Proposition 5.2 and if*

$$(5.62) \quad \sup_{t \in [0, t_0]} \left(\rho_0^{-n} \int_{\Omega_j \setminus G} u_j^2(x, t) dx \right) \leq \eta^2$$

then

$$(5.63) \quad d_{\mathcal{H}}(\overline{\Omega}_1, \overline{\Omega}_2) \leq C\rho_0 \left(\frac{\eta}{\overline{t}_0 \rho_0^{-1} H(\overline{t}_0)} \right)^{1/K_0},$$

where

$$(5.64) \quad K_0 = e^{C\mathcal{F}(\overline{t}_0)}.$$

$$\overline{t}_0 = t_0 - \lambda\rho_0$$

and $\mathcal{F}(\overline{t}_0)$ is defined by (5.44).

Proof. First we prove the following inequality

$$(5.65) \quad d_m(\overline{\Omega}_1, \overline{\Omega}_2) \leq C\rho_0 \left(\frac{\eta}{\overline{t}_0 \rho_0^{-1} H(\overline{t}_0)} \right)^{1/K_0},$$

where $d_m(\overline{\Omega}_1, \overline{\Omega}_2)$ is the quantity introduced in Definition 2.5.

For the sake of brevity let us denote $d_m = d_m(\overline{\Omega}_1, \overline{\Omega}_2)$. Let us assume, with no loss of generality, that there exists $x_0 \in \Gamma_1^{(i)} \subset \partial\Omega_1$ such that $\text{dist}(x_0, \Omega_2) = d_m$.

By (5.62) we have trivially

$$(5.66) \quad \sup_{t \in [0, t_0]} \left(\rho_0^{-n} \int_{\Omega_1 \cap B_{d_m}(x_0)} u_1^2(x, t) dx \right) \leq \eta^2$$

let us distinguish the following two cases

i) $d_m \leq \frac{1}{2}\overline{s}_0\rho_0$,

ii) $d_m > \frac{1}{2}\overline{s}_0\rho_0$,

where $\overline{s}_0, \overline{s}_0 \in (0, 1)$, is defined in Theorem 4.2 and depends on E, λ , and Λ only.

In case i), by applying Theorem 4.2 with $r_0 = d_m$ and $\rho = \frac{\overline{s}_0\rho_0}{2}$ we have

$$\begin{aligned}
(5.67) \quad & \sup_{t \in [0, t_0 - \lambda \rho_0]} \|u_1(\cdot, t)\|_{L^2(B_{\bar{s}_0 \rho_0/2}(x_0) \cap \Omega_1)} \leq \\
& \leq C (\rho_0^{-1} t_0 H(t_0)) \left(\theta_1 \log \left(\frac{\rho_0^{-1} t_0 H(t_0)}{\eta} \right) \right)^{-1/6},
\end{aligned}$$

where

$$(5.68) \quad \theta_1 = \frac{1}{C \log(\rho_0/d_m)}.$$

and C depends on E, M, λ and Λ only.

Now let us introduce the following notation: $s^* = \min \left\{ \frac{\bar{s}_0}{4}, \frac{1}{2E} \right\}$ and $y_0 = x_0 - s^* \rho_0 \nu(x_0)$, $\bar{t}_0 = t_0 - \lambda \rho_0$. We have $B_{s^* \rho_0/2}(y_0) \subset B_{s_0 \rho_0/2}(x_0) \cap \Omega_1$. Let us assume that $t_0 \geq \max\{2C_F \rho_0, 2t_1\}$ where C_F is defined in Proposition 5.2. By (5.67) and Proposition 5.2 we have

$$\begin{aligned}
(5.69) \quad & \bar{t}_0 \rho_0^{-1} H(\bar{t}_0) e^{-\mathcal{F}(\bar{t}_0)} \leq \\
& \leq C (\rho_0^{-1} t_0 H(t_0)) \left(\theta_1 \log \left(\frac{\rho_0^{-1} t_0 H(t_0)}{\eta} \right) \right)^{-1/6},
\end{aligned}$$

where C depends on E, M, λ and Λ only and $\mathcal{F}(\bar{t}_0)$ is defined by (5.44).

By (5.2) and (5.69) it is easy to get

$$(5.70) \quad d_m \leq \rho_0 \left(\frac{\eta}{\rho_0^{-1} t_0 H(t_0)} \right)^{1/K_0},$$

where K_0 is defined in (5.64).

Consider now case ii). Since we have $B_{s^* \rho_0/2}(y_0) \subset B_{s_0 \rho_0/2}(x_0) \cap \Omega_1$ we get

$$\begin{aligned}
(5.71) \quad & \bar{t}_0 \rho_0^{-1} H(\bar{t}_0) e^{-\mathcal{F}(\bar{t}_0)} \leq \sup_{t \in [0, t_0 - \lambda \rho_0]} \|u_1(\cdot, t)\|_{L^2(B_{\vartheta_1^* \rho_0/2}(y_0))} \leq \\
& \leq \sup_{t \in [0, t_0 - \lambda \rho_0]} \|u_1(\cdot, t)\|_{L^2(B_{s_0 \rho_0/2}(x_0) \cap \Omega_1)} \leq \eta.
\end{aligned}$$

Hence

$$(5.72) \quad 1 \leq \frac{e^{\mathcal{F}(\bar{t}_0)} \eta}{\bar{t}_0 \rho_0^{-1} H(\bar{t}_0)}.$$

Now by a priori information we have $d_m \leq C \rho_0$ where C depends on E and M only. Therefore by (5.72) we have trivially

$$(5.73) \quad d_m \leq C\rho_0 \leq C\rho_0 \left(\frac{e^{\mathcal{F}(\bar{t}_0)}\eta}{\bar{t}_0\rho_0^{-1}H(\bar{t}_0)} \right)^{1/K_0}.$$

Therefore in both the cases we have (5.65).

Now we prove (5.63). Let us denote by $d = d_{\mathcal{H}}(\overline{\Omega}_1, \overline{\Omega}_2)$. With no loss of generality, let $\bar{y} \in \overline{\Omega}_1 \setminus \overline{\Omega}_2$ be such that $\text{dist}(\bar{y}, \overline{\Omega}_2) = d$. Since in general \bar{y} needs not to belong to $\partial\Omega_1$, [Al-B-Ro-Ve] it is necessary to analyze various different cases separately. Denoting by $h = \text{dist}(\bar{y}, \partial\Omega_1)$, let us distinguish the following three cases:

- i) $h \leq \frac{d}{2}$,
- ii) $h > \frac{d}{2}$, $h > \frac{d_0}{2}$,
- iii) $h > \frac{d}{2}$, $h \leq \frac{d_0}{2}$,

where d_0 is the number introduced in Proposition 5.3.

If case i) occurs, taking $\bar{z} \in \partial\Omega_1$ such that $|\bar{y} - \bar{z}| = h$, we have that $\text{dist}(\bar{z}, \overline{\Omega}_2) \geq d - h \geq \frac{d}{2}$, so that $d \leq 2d_m$ and (5.63) follows by (5.65).

Let us now consider case ii). Let us denote

$$(5.74) \quad d_1 = \min \left\{ \frac{d}{2}, \frac{s_0 d_0}{4} \right\}.$$

where $s_0, s_0 \in (0, 1)$, is defined in Theorem 4.1 and depends on λ and Λ only. We have that

$$(5.75) \quad B_{d_0/2}(\bar{y}) \subset \Omega_1 \quad \text{and} \quad B_{d_1}(\bar{y}) \subset \Omega_1 \setminus \overline{\Omega}_2.$$

Now by applying Theorem 4.1 with $r_0 = d_1$ and $\rho = \frac{s_0 \rho_0}{2}$ we have

$$\begin{aligned} & \sup_{t \in [0, t_0 - \lambda \rho_0]} \|u_1(\cdot, t)\|_{L^2(B_{d_1}(\bar{y}))} \leq \\ & \leq C (\rho_0^{-1} t_0 H(t_0)) \left(\theta_2 \log \left(\frac{\rho_0^{-1} t_0 H(t_0)}{\eta} \right) \right)^{-1/6}, \end{aligned}$$

where

$$\theta_2 = \frac{1}{C \log(\rho_0/d_m)},$$

and C depends on λ and Λ only.

Now proceeding exactly as in the proof of (5.70) we have

$$(5.76) \quad d_1 \leq \rho_0 \left(\frac{\eta}{\rho_0^{-1} t_0 H(t_0)} \right)^{1/K_0},$$

where K_0 is defined by (5.64) (perhaps with a different value of constant C).

Now, if

$$\rho_0 \left(\frac{\eta}{\rho_0^{-1} t_0 H(t_0)} \right)^{1/K_0} < \frac{s_0 d_0}{4}$$

then by (5.76) we have $d_1 < \frac{s_0 d_0}{4}$, hence $d_1 = \frac{d}{2}$. Therefore we get

$$(5.77) \quad d = 2d_1 \leq 2\rho_0 \left(\frac{\eta}{\rho_0^{-1} t_0 H(t_0)} \right)^{1/K_0}.$$

if, instead, we have

$$\rho_0 \left(\frac{\eta}{\rho_0^{-1} t_0 H(t_0)} \right)^{1/K_0} \geq \frac{s_0 d_0}{4},$$

we have trivially

$$(5.78) \quad d \leq C\rho_0 \leq \frac{4C\rho_0^2}{s_0 d_0} \left(\frac{\eta}{\rho_0^{-1} t_0 H(t_0)} \right)^{1/K_0},$$

where C depends on E and M only.

If case iii) occurs we have in particular that $d < d_0$, hence by Proposition 5.3 we have $d \leq cd_m$ and by (5.65) the thesis follows again. \square

Corollary 5.5. *Let t_* be defined in Proposition 5.4 and let $t_0 \geq t_*$ be fixed. We have for every $\varepsilon \in (0, \bar{\varepsilon}]$, $\bar{\varepsilon}$ is defined in Proposition 5.1,*

$$(5.79) \quad d_{\mathcal{H}}(\bar{\Omega}_1, \bar{\Omega}_2) \leq \rho_0 \omega_1(\varepsilon, t_0),$$

where

$$(5.80) \quad \omega_1(\varepsilon, t_0) := C \left(\frac{\omega(\varepsilon, t_0)}{\bar{t}_0 \rho_0^{-1} H(\bar{t}_0)} \right)^{1/K_0},$$

$\omega(\varepsilon, t_0)$ is defined by (5.7) and C on E, M, λ, Λ and F only and K_0 is defined in (5.64).

Proof. Inequality (5.79) is an immediate consequence of Proposition 5.1 and Proposition 5.4 \square

5.3 Step 3

Now we conclude the proof of the main Theorem.

Let $t_0 \geq t_*$ be fixed and let d_0 be defined in Proposition 5.3 and let $s \in (0, \frac{d_0}{\rho_0}]$ be a number that we shall choose later. Denote by

$$\epsilon(s) = \sup \{ \varepsilon \in (0, \bar{\varepsilon}] : \omega_1(\varepsilon, t_0) \leq s \}.$$

By Proposition 5.3 we have that, for every $s \in (0, \frac{d_0}{\rho_0}]$ and every $\varepsilon \in (0, \epsilon(s)]$, $\partial\Omega_1$ and $\partial\Omega_2$ are relative graphs, moreover G is equal to $\Omega_1 \cap \Omega_2$ and is a domain of Lipschitz class with constants CE and ρ_0/C where $C \geq 1$ depends on E only.

We have

$$\partial(\Omega_1 \setminus G) \subset \Gamma_1^{(i)} \cup \left(\Gamma_2^{(i)} \cap \partial G \right).$$

Denote by $u = u_1 - u_2$. By Schwarz inequality, energy inequality, (4.16a), (4.16c) and recalling that $u_2 = 0$ on $\Gamma_2^{(i)}$ we have, for any $t \in (0, t_0]$,

$$\begin{aligned} (5.81) \quad \rho_0^{-n} \int_{\Omega_1 \setminus G} u_1^2(x, t) dx &\leq t_0 \rho_0^{-n} \int_0^{t_0} \int_{\Omega_1 \setminus G} |\partial_\xi u_1(x, \xi)|^2 dx d\xi \leq \\ &\leq C(t_0 \rho_0^{-1})^{5/2} (H(t_0))^{3/2} \|u\|_{L^\infty((\Gamma_2^{(i)} \cap \partial G) \times [0, t_0])}^{1/2}, \end{aligned}$$

where C depends on α, E, M, λ and Λ only.

Let $P \in \partial G$, without restriction we may assume that $P \equiv 0$. By (5.79) and Proposition 5.3 we have that if $s \in (0, \frac{d_0}{\rho_0}]$ and $\varepsilon \in (0, \epsilon(s)]$ then there exist $\varphi_1, \varphi_2 \in C^{1,1}(B'_{r_0}(0))$, where $\frac{r_0}{\rho_0} \leq 1$ depends on E only, satisfying the following conditions

$$(5.82a) \quad \|\varphi_i\|_{C^{1,1}(B'_{r_0}(0))} \leq E\rho_0,$$

$$(5.82b) \quad \Omega_i \cap B_{r_0}(0) = \{x \in B_{r_0}(0) : x_n > \varphi_i(x')\}, \quad i = 1, 2.$$

It is not restrictive to assume that

$$(5.83) \quad \varphi_1(0) = |\nabla_{x'} \varphi_1(0)| = 0, \quad \varphi_2(0) \leq 0.$$

Now, Let us denote by $\varphi = \max\{\varphi_1, \varphi_2\}$ and by $d_1 = \min\{d_0, r_0\}$ By (5.79) and (5.60) we have (we fix $\alpha = 1/2$), for every $s \in (0, \frac{d_1}{\rho_0}]$ and every $\varepsilon \in (0, \epsilon(s)]$,

$$(5.84) \quad \|\nabla_{x'} \varphi\|_{L^\infty(B'_{s\rho_0})} \leq L_s := C_* s^{1/4}$$

where C_* depends on E only.

For any $s \in (0, \frac{d_1}{\rho_0}]$ let us introduce the following notation

$$(5.85) \quad T_s := \max \{T(\epsilon(s)), 2t_0\},$$

where $T(\epsilon)$ is defined in (5.3),

$$(5.86) \quad \gamma = \arctan \frac{1}{L_s}.$$

Moreover let γ_1, γ_2 two numbers such that $0 < \gamma_1 < \gamma_2 < \gamma < \frac{\pi}{2}$ that we shall choose later and let

$$(5.87a) \quad \chi = \frac{1 - \sin \gamma_2}{1 - \sin \gamma_1},$$

$$(5.87b) \quad l_1 = \frac{sL_s\rho_0/2}{1 + \sin \gamma},$$

$$(5.87c) \quad l_k = \chi^{k-1} l_1, \quad k \in \mathbb{N},$$

$$(5.87d) \quad w_k = l_k e_n, \quad k \in \mathbb{N},$$

$$(5.87e) \quad R_k = l_k \sin \gamma, \quad \rho_k = l_k \sin \gamma_2, \quad r_k = l_k \sin \gamma_1, \quad k \in \mathbb{N}.$$

It is easy to check that denoting by \mathcal{C} the cone

$$\mathcal{C} = \left\{ x \in \mathbb{R}^n : L_s |x'| \leq x_n \leq \frac{sL_s\rho_0}{2} \right\}$$

we have

$$(5.88) \quad B_{r_{k+1}}(w_{k+1}) \subset B_{\rho_k}(w_k) \subset B_{R_k}(w_k) \subset \mathcal{C} \subset G, \quad \text{for every } k \in \mathbb{N}$$

and

$$(5.89) \quad \text{dist}(B_{r_1}(w_1), \partial G) \geq \frac{1}{2}\rho_0 sh,$$

where

$$(5.90) \quad h = \frac{\sin \gamma - \sin \gamma_1}{1 + \sin \gamma}.$$

Let $T \geq T_s$ be a number that we will choose. For any positive number μ such that $\mu T^2 \geq 1$ and $\tau \in (0, T/2]$ denote by $U_\mu^{(\tau)}$ the FBI transform of u defined by

$$(5.91) \quad U_\mu^{(\tau)}(x, y) = \sqrt{\frac{\mu}{2\pi}} \int_0^T e^{-\frac{\mu}{2}(iy+\tau-t)^2} u(x, t) dt, \quad \text{for } (x, y) \in G \times \mathbb{R}.$$

Let $\kappa_0 \leq 1$ such that G_r is connected for every $r \in (0, \kappa_0 \rho_0]$ [Al-R-Ro-Ve]. Let $\kappa_1 = \min\{\frac{d_1}{\rho_0}, \kappa_0\}$. Arguing as in Proposition 5.1 we have by (3.9), for every $s \in (0, \kappa_1]$ and every $\varepsilon \in (0, \epsilon(s)]$,

$$(5.92) \quad \begin{aligned} & \|U_\mu^{(\tau)}\|_{L^2(\tilde{B}_{r_1}(\tilde{w}_1))} \leq \\ & \leq CT\rho_0^{-1} (H(T) + 1) e^{\mu(s\rho_0)^2/2} \left(e^{2\mu\rho_0^2} \left(e^{-\mu T^2/10} + \varepsilon_1 \right)^{\vartheta_2} \right)^{\vartheta_2^{(hs/2)^{-n}}}, \end{aligned}$$

where $\vartheta_2 \in (0, 1)$ is the same exponent of inequality (5.19), ϑ_2, C depend on E, M, λ and Λ only and

$$(5.93) \quad \varepsilon_1 = \frac{(\mu T^2)^{1/4} \varepsilon}{(H(T) + 1) T \rho_0^{-1}}.$$

Now we apply inequality (4.29) when $\tilde{r}_1 = r_k, \tilde{r}_2 = \rho_k, \tilde{r}_3 = R_k$ and $x_0 = w_k, k \in \mathbb{N}$.

Let us denote by

$$(5.94) \quad \alpha_k = e^{-\mu T^2/10} + \frac{e^{-\mu R_k^2/2} \|U_\mu^{(\tau)}\|_{L^2(\tilde{B}_{r_k}(\tilde{w}_k))}}{(H(T) + 1) T \rho_0^{-1}}.$$

Taking into account (5.88) we have

$$(5.95) \quad \alpha_{k+1} \leq \tilde{C}_0 e^{\frac{\mu}{2}(R_k^2 - R_{k+1}^2)} \alpha_k^{\tilde{\vartheta}_0}, \text{ for every } k \in \mathbb{N},$$

where

$$(5.96) \quad \tilde{\vartheta}_0 = \frac{\rho_1^{-\beta_1} - [(1 - \delta)R_1]^{-\beta_1}}{[(1 - 2\delta)r_1]^{-\beta_1} - [(1 - \delta)R_1]^{-\beta_1}},$$

$$(5.97) \quad 0 < \delta \leq \frac{R_1 - \rho_1}{2R_1},$$

$$(5.98) \quad \tilde{C}_0 = C \frac{e^{C[(\rho_1 R_1^{-1})^{-\beta_1} - (1 - \delta)^{-\beta_1}]}}{\delta^4},$$

β_1 has been introduced in Theorem 4.6 and C depends on E, M, λ and Λ only.

Notice that

$$R_k^2 - R_{k+1}^2 = \chi^{2k} R_1^2 (\chi^{-2} - 1).$$

By iterating (5.95) we get

$$(5.99) \quad \alpha_{k+1} \leq (C\tilde{C}_0)^{1/(1-\tilde{\vartheta}_0)} \left(e^{\mu R_1^2 A_k/2} \alpha_1 \right)^{\tilde{\vartheta}_0^k}, \quad k \in \mathbb{N},$$

where

$$(5.100) \quad A_k = (\chi^{-2} - 1)(\chi^2 \tilde{\vartheta}_0^{-1}) \frac{1 - (\chi^2 \tilde{\vartheta}_0^{-1})^k}{1 - (\chi^2 \tilde{\vartheta}_0^{-1})}, \quad k \in \mathbb{N}.$$

Let $\kappa_2 = \min\{\kappa_1, 2(|\log_4 \vartheta_2|)^{1/n}\}$ and taking into account that, by (5.90), $h \leq 1$, from (5.92) and (5.99) we get that for every $s \leq \kappa_2$ the following inequality holds true

$$(5.101) \quad \frac{\|U_\mu^{(\tau)}\|_{L^2(\tilde{B}_{r_{k+1}}(\tilde{w}_{k+1}))}}{(H(T) + 1)T\rho_0^{-1}} \leq \\ \leq (C\tilde{C}_0)^{1/(1-\tilde{\vartheta}_0)} \left(e^{\mu A_{s,k}^{(1)} \varepsilon_1^{\vartheta_1} \vartheta_2^{(sh/2)^{-n}}} + e^{\mu A_{s,k}^{(2)}} \right)^{\tilde{\vartheta}_0^k}, \quad k \in \mathbb{N},$$

where

$$(5.102) \quad A_{s,k}^{(1)} = \frac{1}{2} \left(A_k + (\chi^2 \tilde{\vartheta}_0^{-1}) \right) R_1^2 + \rho_0^2 \quad , \quad k \in \mathbb{N},$$

$$(5.103) \quad A_{s,k}^{(2)} = A_{s,k}^{(1)} - \frac{1}{10} T^2 \vartheta_2^{1+(sh/2)^{-n}}, \quad k \in \mathbb{N},$$

and C depends on E, M, λ and Λ only.

Since we need that $A_{s,k}^{(1)}$ is bounded for $k \in \mathbb{N}$ we search for which $s \in (0, \kappa_2]$ we have

$$(5.104) \quad \chi^2 \tilde{\vartheta}_0^{-1} < 1,$$

Let $\varsigma, a, b, q \in (0, 1)$ three numbers that we will fix later on and let

$$(5.105) \quad \sin \gamma_1 = 1 - \varsigma, \quad \sin \gamma_2 = 1 - a\varsigma, \quad \sin \gamma = 1 - ab\varsigma$$

and

$$(5.106) \quad \delta = q \left(\frac{R_1 - \rho_1}{2R_1} \right) = \frac{q}{2} \frac{a(1-b)\varsigma}{1-ab\varsigma},$$

by (5.87) and (5.96) we have respectively

$$(5.107) \quad \chi = a,$$

and

$$(5.108) \quad \tilde{\vartheta}_0 = \frac{a(1-b)(1-q/2)}{1-ab+qa(1-b)/2} + o(1) \quad , \quad \text{as } \varsigma \rightarrow 0.$$

In order that (5.104) is satisfied it is enough that

$$(5.109) \quad a^2 < \frac{a(1-b)(1-q/2)}{1-ab+qa(1-b)/2},$$

for instance if we choose

$$(5.110) \quad q = \frac{1}{2}, \quad a = \frac{1}{4}, \quad b = \frac{1}{3}$$

then (5.109) is satisfied and we have

$$(5.111) \quad \chi^2 \tilde{\vartheta}_0^{-1} = \frac{23}{48} + o(1) \quad , \text{ as } \varsigma \rightarrow 0.$$

By (5.111) we have that there exists $\varsigma_0 > 0$ such that if $0 < \varsigma \leq \varsigma_0$ then

$$(5.112) \quad \chi^2 \tilde{\vartheta}_0^{-1} \leq \frac{1}{2}.$$

Let

$$\varsigma_1 = \left(1 - \left(1 + C_* \kappa_2^{1/2} \right)^{-1/2} \right)^{1/2},$$

where C_* is defined in (5.84) and depends on E only.

Now, let us fix $\varsigma = \bar{\varsigma} := \min \left\{ \varsigma_0, \varsigma_1, \frac{1}{4} \right\}$ and denote by $\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}$ the numbers belonging to $(0, \frac{\pi}{2})$ such that

$$(5.113) \quad \sin \bar{\gamma}_1 = 1 - \bar{\varsigma}, \quad \sin \bar{\gamma}_2 = 1 - \frac{1}{4} \bar{\varsigma}, \quad \sin \bar{\gamma} = 1 - \frac{1}{12} \bar{\varsigma}$$

and denote by

$$(5.114) \quad \bar{s} = \frac{1}{C_*^4} \left((1 - \bar{\varsigma}/12)^{-4} - 1 \right).$$

Notice that (5.84), (5.114) and the third equality of (5.113) imply that equality (5.86) is satisfied. Namely we have

$$\bar{\gamma} = \arctan \frac{1}{L_{\bar{s}}}.$$

Now for any quantity g introduced in (5.87), (5.90), (5.96) and (5.106) we denote by \bar{g} the value of such a quantity when $s = \bar{s}$ or, equivalently, when $\varsigma = \bar{\varsigma}$. In particular we have

$$(5.115) \quad \bar{\delta} = \frac{\bar{\varsigma}}{24 - 2\bar{\varsigma}},$$

$$(5.116) \quad \bar{h} = \frac{\sin \bar{\gamma} - \sin \bar{\gamma}_1}{1 + \sin \bar{\gamma}} = \frac{2\bar{s}}{24 - 3\bar{s}},$$

$$(5.117) \quad \bar{\vartheta}_0 = \frac{\left(\frac{\sin \bar{\gamma}_2}{\sin \bar{\gamma}} \right)^{-\beta_1} - (1 - \bar{\delta})^{-\beta_1}}{\left((1 - 2\bar{\delta}) \frac{\sin \bar{\gamma}_1}{\sin \bar{\gamma}} \right)^{-\beta_1} - (1 - \bar{\delta})^{-\beta_1}}.$$

By (5.100), (5.112), (5.102) and (5.114) we have

$$(5.118) \quad A_{\bar{s},k}^{(1)} \leq 2\rho_0^2, \quad k \in \mathbb{N}.$$

Let

$$(5.119) \quad \tilde{T} = \max \left\{ \left(40\vartheta_2^{-1-(\bar{s}\bar{h}/2)^{-n}} \right)^{1/2} \rho_0, T_{\bar{s}} \right\}.$$

By (5.103), (5.118) and (5.119) we have, for every $T \geq \tilde{T}$,

$$(5.120) \quad A_{\bar{s},k}^{(2)} \leq -\frac{1}{20} T^2 \vartheta_2^{1+(\bar{s}\bar{h}/2)^{-n}}, \quad k \in \mathbb{N},$$

and C depends on E, M, λ and Λ only.

By (5.118), (5.120) we have, for every $T \geq \tilde{T}$,

$$(5.121) \quad \begin{aligned} & \|U_{\mu}^{(\tau)}\|_{L^2(\tilde{B}_{\bar{\tau}_{k+1}}(\tilde{w}_{k+1}))} \leq \\ & \leq \bar{C}_0(H(T) + 1)T\rho_0^{-1} \left(e^{2\mu\rho_0^2\tilde{\varepsilon}_1^{\delta_3}} + e^{-\frac{1}{20}\mu T^2\delta_3} \right)^{\bar{\vartheta}_0^k}, \quad k \in \mathbb{N}, \end{aligned}$$

where

$$\begin{aligned} \bar{C}_0 &= (C\tilde{C}_0)^{1/(1-\bar{\vartheta}_0)}, \\ \tilde{\varepsilon}_1 &= \frac{(\mu T^2)^{1/4}\varepsilon}{(H(T) + 1)(T\rho_0^{-1} + 1)} \\ \delta_3 &= \vartheta_2^{1+(\bar{s}\bar{h}/2)^{-n}} \end{aligned}$$

and C depends on E, M, λ and Λ only.

Denote by

$$d_1 = \bar{l}_1 (1 - \sin \bar{\gamma}_1), \quad d_k = \chi^{k-1} d_1, \quad \text{for every } k \in \mathbb{N}$$

here, we recall that by (5.107) and (5.110) we have $\chi = \frac{1}{4}$ and, by (5.87b) $\bar{l}_1 = \frac{C_* \bar{s}^{3/2} \rho_0/2}{1 + \sin \bar{\gamma}}$.

Let $r \in (0, d_1]$ be a number that we will choose later on. Let us denote by $\varrho = r d_1^{-1}$,

$$k_0 = \min \{k \in \mathbb{N} : d_k \leq r\}$$

and

$$\delta_4 = \left| \log_4 \bar{\vartheta}_0 \right|.$$

We have

$$(5.122) \quad |\log_4(\varrho/4)| \leq k_0 < |\log_4(\varrho/16)|$$

and

$$(5.123) \quad \overline{\vartheta}_0^2 \varrho^{\delta_4} \leq \overline{\vartheta}_0^{k_0} \leq \overline{\vartheta}_0^2 \varrho^{\delta_4}.$$

Now by applying [G-T, Theorem 8.17] (4.16) (4.20), (4.22) we have, for every $\tau \in (0, t_0]$ and every $T \geq \tilde{T}$,

$$(5.124) \quad \begin{aligned} |u(0, \tau)| &\leq \left| u(0, \tau) - u\left(\tilde{w}_{k_0+1}, \tau\right) \right| + \left| u\left(\tilde{w}_{k_0+1}, \tau\right) - U_\mu^{(\tau)}\left(\tilde{w}_{k_0+1}\right) \right| + \\ &+ \left| U_\mu^{(\tau)}\left(\tilde{w}_{k_0+1}\right) \right| \leq C t_0 \rho_0^{-1} H(t_0) \varrho + C (T \rho_0^{-1})^2 H(T) (T^2 \mu)^{-1/2} + \\ &+ C \varrho^{-\left(\frac{n+1}{2}\right)} (H(T) + 1) T \rho_0^{-1} \left(e^{2\mu \rho_0^2 \tilde{\varepsilon}_1^{\delta_3}} + e^{-\frac{1}{20} \mu T^2 \delta_3} \right)^{\overline{\vartheta}_0^2 \varrho^{\delta_4}} \end{aligned}$$

where C depends on E, M, λ and Λ only. Now we have trivially

$$(5.125) \quad e^{2\mu \rho_0^2 \tilde{\varepsilon}_1^{\delta_3}} + e^{-\frac{1}{20} \mu T^2 \delta_3} \leq e^{2\mu T^2 \varepsilon^{\delta_3}} + e^{-\frac{1}{20} \mu T^2 \delta_3}.$$

Hence, if

$$\varepsilon \leq e^{-(2/\delta_3 + 1/20)}$$

then we choose

$$\mu = \frac{1}{T^2} \frac{\delta_3 |\log \varepsilon|}{2 + \delta_3/20}$$

and by (5.124) and (5.125) we have

$$(5.126) \quad \begin{aligned} |u(0, \tau)| &\leq C t_0 \rho_0^{-1} H(t_0) \varrho + C (T \rho_0^{-1})^2 H(T) |\log \varepsilon|^{-1/2} + \\ &+ C \varrho^{-\left(\frac{n+1}{2}\right)} (H(T) + 1) T \rho_0^{-1} \varepsilon^{\delta_5 \varrho^{\delta_4}} \end{aligned}$$

where C depends on E, M, λ and Λ only and

$$\delta_5 = \frac{\overline{\vartheta}_0^2}{40 + \delta_3}.$$

Now let us choose

$$\varrho = |\log \varepsilon|^{-1/(2\delta_4)}$$

and by (5.126) we have

$$(5.127) \quad |u(0, \tau)| \leq C (T \rho_0^{-1})^2 (H(T) + 1) |\log \varepsilon|^{-1/2}$$

where C depends on E, M, λ and Λ only.

Otherwise, if

$$\varepsilon \geq e^{-(2/\delta_3 + 1/20)}$$

then by (4.16c) we have trivially

$$(5.128) \quad |u(0, \tau)| \leq C t_0 \rho_0^{-1} H(t_0) \leq C e^{(2/\delta_3 + 1/20)} t_0 \rho_0^{-1} H(t_0) \varepsilon.$$

where C depends on E, M, λ and Λ only. By (5.127) and (5.128) we have, for $0 < \varepsilon < e^{-1}$ and every $T \geq \tilde{T}$

$$(5.129) \quad \|u\|_{L^\infty((\Gamma_2^{(i)} \cap \partial G) \times [0, t_0])} \leq C (T \rho_0^{-1})^2 (H(T) + 1) |\log \varepsilon|^{-1/2}$$

where C depends on E, M, λ and Λ only. By (5.129) and (5.81) we have

$$(5.130) \quad \sup_{t \in [0, t_0]} \left(\rho_0^{-n} \int_{\Omega_j \setminus G} u_j^2(x, t) dx \right) \leq \\ \leq C (t_0 \rho_0^{-1} + 1)^{5/2} (H(t_0))^{3/2} (T \rho_0^{-1}) (H(T) + 1)^{1/2} |\log \varepsilon|^{-1/4}$$

where C depends on E, M, λ and Λ only.

Now we fix $t_0 = t_* + \lambda \rho_0$ and $T = \tilde{T}$ and by (5.130) and Proposition 5.4 we have

$$(5.131) \quad d_{\mathcal{H}}(\overline{\Omega}_1, \overline{\Omega}_2) \leq K_1 \rho_0 |\log \varepsilon|^{-1/(8K_0)},$$

where

$$K_0 = e^{\mathcal{F}(t_*)},$$

$$\bar{t}_0 = t_0 - \lambda \rho_0,$$

$$K_1 = C \left(\frac{H(\tilde{T})}{H(t_*)} \right)^{1/(8K_0)},$$

where $\mathcal{F}(t_*)$ is defined by (5.44) and C depends on E, M, λ and Λ only. \square

6 Appendix

6.1 Proof of Theorem 4.3

Theorem 4.3 is a straightforward consequence of Theorem 6.1 below and of standard results concerning the extension of function

Theorem 6.1. *Let Ω be a bounded domain of \mathbb{R}^n that satisfies (3.1). Let $A(x)$ be a real-valued symmetric $n \times n$ matrix satisfying (3.8). Let $m := \lfloor \frac{n+2}{4} \rfloor$. Assume that $\partial_t^k F \in L^\infty(\Omega \times (0, T))$ for every $k \in \{0, \dots, 2m+2\}$ and let $u \in \mathcal{W}([0, T]; \Omega)$ be the solution to the problem*

$$(6.1) \quad \begin{cases} \partial_t^2 u - \operatorname{div}(A(x) \nabla_x u) = F(x, t), & \text{in } \Omega \times [0, T], \\ u = 0 & \text{on } \partial\Omega \times [0, T] \\ u(\cdot, 0) = \partial_t u(\cdot, 0) = 0, & \text{in } \Omega. \end{cases}$$

Let $\alpha \in (0, 1)$. Then for every $t \in [0, T]$ we have $u(\cdot, t) \in C^{1, \alpha}(\Omega)$ and the following inequalities hold true

$$(6.2a) \quad \|\partial_t^2 u(\cdot, t)\|_{L^\infty(\Omega)} \leq C \rho_0^{-2} \left(\rho_0^{2m+3} T F_{2m+2} + \sum_{j=0}^m \rho_0^{2j+2} F_{2j} \right),$$

$$(6.2b) \quad \|u(\cdot, t)\|_{C^{1, \alpha}(\Omega)} \leq C \left(\rho_0^{2m+3} T F_{2m+2} + \sum_{j=0}^m \rho_0^{2j+2} F_{2j} \right),$$

where $F_j := \|\partial_t^j F\|_{L^\infty(\Omega \times [0, T])}$ for every $j \in \mathbb{N} \cup \{0\}$ and C depends on α, n, E, M, λ and Λ only.

In order to prove Theorem 6.1 we use propositions 6.2, 6.3 given below.

Proposition 6.2. *Assume that Ω and $A(x)$ are as in Theorem 6.1. Let Ω be a bounded domain of \mathbb{R}^n that satisfies (3.1a) and whose boundary is of class $C^{1,1}$. Let $A(x)$ be a real-valued symmetric $n \times n$ matrix satisfying (3.8). If $f \in L^p(\Omega)$, $p \in (1, \infty)$, then the solution v to the Dirichlet problem*

$$(6.3) \quad \begin{cases} \operatorname{div}(A(x) \nabla v) = f, & \text{in } \Omega, \\ v \in H_0^1(\Omega), \end{cases}$$

belongs to $W^{2,p}(\Omega)$ and the following estimate holds true

$$(6.4) \quad \|v\|_{W^{2,p}(\Omega)} \leq C \rho_0^2 \|f\|_{L^p(\Omega)},$$

where C depends on λ, Λ, E, M and p only.

Proof. The Proposition is an immediate consequence of [G-T, Theorem 9.15] and [G-T, Lemma 9.17]. \square

Proposition 6.3. *Assume that Ω and $A(x)$ are as in Theorem 6.1. Let $F \in L^2(\Omega \times (0, T))$ and let $u \in \mathcal{W}([0, T]; \Omega)$ be the solution to the problem*

$$(6.5) \quad \begin{cases} \partial_t^2 u - \operatorname{div}(A(x) \nabla_x u) = F, & \text{in } \Omega \times [0, T], \\ u = 0 & \text{on } \partial\Omega \times [0, T], \\ u(\cdot, 0) = \partial_t u(\cdot, 0) = 0, & \text{in } \Omega. \end{cases}$$

Then the following inequality holds true

$$(6.6) \quad \|u(\cdot, t)\|_{L^{p_0}(\Omega)} \leq C \rho_0 T \|F\|_{L^\infty(\Omega \times (0, T))}, \quad \text{for every } t \in (0, T).$$

where p_0 is the Sobolev imbedding exponent, namely

$$(6.7a) \quad p_0 = \frac{2n}{n-2}, \quad \text{for } n > 2,$$

$$(6.7b) \quad p_0 \text{ is an arbitrary number of } [2, +\infty) \quad , \text{ for } n = 2$$

and C depends on n, E, M and λ only.

Proof. Let $\tau \in (0, T]$. By multiplying both the sides of first equation in (6.5) by $\partial_t u$ and by integrating over $\Omega \times (0, \tau)$ we get

$$\begin{aligned} \int_0^\tau \int_\Omega F \partial_t u dx dt &= -\frac{1}{2} \int_0^\tau \int_\Omega \partial_t (A(x) \nabla u \cdot \nabla u + (\partial_t u)^2) dx dt = \\ &= -\frac{1}{2} \int_\Omega (A(x) \nabla u \cdot \nabla u + (\partial_t u)^2) dx. \end{aligned}$$

Hence, denoting by

$$K(\tau) = \int_\Omega (A(x) \nabla u(x, \tau) \cdot \nabla u(x, \tau) + (\partial_t u(x, \tau))^2) dx,$$

we get

$$\begin{aligned} K(\tau) &\leq 2 \int_0^\tau \int_\Omega |F| |\partial_t u| dx dt \leq T \int_0^\tau \int_\Omega F^2 dx dt + \frac{1}{T} \int_0^\tau \int_\Omega (\partial_t u)^2 dx dt \leq \\ &\leq T \int_0^\tau \int_\Omega F^2 dx dt + \frac{1}{T} \int_0^\tau K(t) dt. \end{aligned}$$

By Gronwall inequality we derive the energy inequality

$$(6.8) \quad K(\tau) \leq eT \int_0^T \int_{\Omega} F^2 dx dt.$$

In particular (6.8) gives

$$(6.9) \quad \int_{\Omega} |\nabla u(x, t)|^2 dx \leq e\lambda^{-1}T \int_0^T \int_{\Omega} F^2 dx dt.$$

Since $u(\cdot, t) \in H_0^1(\Omega)$, by (6.9) and the Poincaré inequality we have

$$(6.10) \quad \|u(\cdot, t)\|_{H^1(\Omega)} \leq C\rho_0 T \|F\|_{L^\infty(\Omega \times (0, T))}.$$

Finally by the imbedding Sobolev theorem the thesis follows. \square

Sketch of the proof of Theorem 6.1.

In this sketch of the proof we skip on the question of regularity of the solution for which we refer to [Co] and we focus on the proof of inequality (6.2).

In order to estimate $\|\partial_t^2 u(\cdot, t)\|_{L^\infty(\Omega)}$, for every $t \in (0, T)$ we distinguish two cases: (a) n is not of the type $4h + 2$, $h \in \mathbb{N} \cup \{0\}$, (b) n is of the type $4h + 2$, $h \in \mathbb{N} \cup \{0\}$.

Case (a). Denote by p_k , $k \in \mathbb{N} \cup \{0\}$, the sequence such that

$$\frac{1}{p_k} = \frac{1}{p_0} - \frac{2k}{n}, \text{ for } k \in \mathbb{N} \cup \{0\}.$$

Notice that

$$\frac{1}{p_k} = \frac{1}{p_{k-1}} - \frac{2}{n}, \text{ for } k \in \mathbb{N}$$

and that

$$\frac{1}{p_{m-1}} - \frac{2}{n} > 0, \text{ and } \frac{1}{p_m} - \frac{2}{n} < 0.$$

Let us denote

$$u^{(j)} := \partial_t^j u, \text{ for every } j \in \{0, \dots, 2m + 2\}.$$

By (6.1) we have, for every $j \in \{0, \dots, 2m + 2\}$,

$$(6.11) \quad \begin{cases} \partial_t^2 u^{(j)} - \operatorname{div} (A(x) \nabla_x u^{(j)}) = \partial_t^j F, & \text{in } \Omega \times [0, T], \\ u^{(j)} = 0, & \text{on } \partial\Omega \times [0, T], \\ u^{(j)}(\cdot, 0) = \partial_t u^{(j)}(\cdot, 0) = 0, & \text{in } \Omega. \end{cases}$$

Observe that since $u^{(2j+2)} = \partial_t^2 u^{(2j)}$ by (6.11) we have that $u^{(2j)}$ is the solution to the following Dirichlet elliptic problem

$$(6.12) \quad \begin{cases} \operatorname{div} (A(x) \nabla_x u^{(2j)}) = u^{(2j+2)} - \partial_t^{2j} F & , \text{ in } \Omega, \\ u^{(2j)} \in H_0^1(\Omega), \end{cases}$$

hence by Proposition 6.2 we have, for every $j \in \{1, \dots, m\}$ and $t \in (0, T)$,

$$(6.13) \quad \|u^{(2j)}(\cdot, t)\|_{W^{2,p_{m-j}}(\Omega)} \leq C\rho_0^2 \left(\|u^{(2j+2)}(\cdot, t)\|_{L^{p_{m-j}}(\Omega)} + F_{2j} \right),$$

where C depends on λ, Λ, E and M only. Hence by Sobolev imbedding theorem we get, for every $j \in \{2, \dots, m\}$ and $t \in (0, T)$,

$$(6.14) \quad \|u^{(2j)}(\cdot, t)\|_{L^{p_{m-j+1}}(\Omega)} \leq C_0\rho_0^2 \left(\|u^{(2j+2)}(\cdot, t)\|_{L^{p_{m-j}}(\Omega)} + F_{2j} \right),$$

and

$$(6.15) \quad \|u^{(2)}(\cdot, t)\|_{L^\infty(\Omega)} \leq C_0\rho_0^2 \left(\|u^{(4)}(\cdot, t)\|_{L^{p_{m-2}}(\Omega)} + F_2 \right),$$

where $C_0 \geq 1$ depends on n, λ, Λ, E and M only. Now by applying Proposition 6.3 to $u^{(2m+2)}$ we have, for every $t \in (0, T)$,

$$(6.16) \quad \|u^{(2m+2)}(\cdot, t)\|_{L^{p_0}(\Omega)} \leq C_1\rho_0 T F_{2m+2},$$

where $C_1 \geq 1$ depends on n, λ, E and M only. Therefore, by iterating (6.14) and by (6.15) and (6.16) we get, for every $t \in (0, T)$,

$$(6.17) \quad \|u^{(2)}(\cdot, t)\|_{L^\infty(\Omega)} \leq C_1 C_0^m \left(\rho_0^{2m+1} T F_{2m+2} + \sum_{j=0}^m \rho_0^{2j} F_{2j} \right),$$

Now since $u = u^{(0)}$, by (6.12) and by [G-T, Theorem 8.33] we get, for every $t \in (0, T)$,

$$(6.18) \quad \|u(\cdot, t)\|_{C^{1,\alpha}(\Omega)} \leq C\rho_0^2 \left(\|u^{(2)}(\cdot, t)\|_{L^\infty(\Omega)} + F_0 \right),$$

where C depends on α, n, E, M, λ and Λ only. By (6.18) and (6.17) we obtain (6.2) in the case a.

Case (b) We consider only the case $n > 2$, because if $n = 2$ we can proceed similarly.

If n is of the type $4h + 2$, $h \in \mathbb{N} \cup \{0\}$ then inequality (6.13) continues to hold, but by Sobolev imbedding Theorem, instead of inequality (6.15) we have, for every $q \in [2, \infty)$,

$$(6.19) \quad \|u^{(4)}(\cdot, t)\|_{L^q(\Omega)} \leq C_2 \rho_0^2 \left(\|u^{(6)}(\cdot, t)\|_{L^{p_{m-1}}(\Omega)} + F_4 \right),$$

where $C_2 \geq 1$ depends on $n, \lambda, \Lambda, E, M$ and q only.

Let us choose $q > \frac{n}{2}$, by applying [G-T, Theorem 8.29] to $u^{(2)}(\cdot, t)$ we have

$$(6.20) \quad \|u^{(2)}(\cdot, t)\|_{L^\infty(\Omega)} \leq C_3 \rho_0^2 \left(\|u^{(4)}(\cdot, t)\|_{L^q(\Omega)} + F_2 \right),$$

where $C_3 \geq 1$ depends on n, λ, E, M and q .

Now, by iterating (6.13) and by using (6.19) and (6.20) we get

$$(6.21) \quad \|u^{(2)}(\cdot, t)\|_{L^\infty(\Omega)} \leq C_2 C_3 C_0^{m-1} \left(\rho_0^{2m+1} T F_{2m+2} + \sum_{j=0}^m \rho_0^{2j} F_{2j} \right)$$

and arguing as in the case (a) the thesis follows. \square

6.2 Proof of Propositions 4.4, 4.5

Proof of Proposition 4.4

We prove (4.19) for $j = 0$, the proof for $j > 0$ being the same.

By (4.18) we have

$$\begin{aligned} \sqrt{2\pi} |U_\mu^{(\tau)}(x, y)| &= \left| \sqrt{\mu} \int_0^T e^{-\frac{\mu}{2}(iy+\tau-t)^2} u(x, t) dt \right| \leq \\ &\leq \sqrt{\mu} e^{\frac{\mu}{2}y^2} \int_0^T e^{-\frac{\mu}{2}(\tau-t)^2} |u(x, t)| dt, \end{aligned}$$

hence, the Schwarz inequality yields

$$\begin{aligned} \sqrt{2\pi} |U_\mu^{(\tau)}(x, y)| &\leq \sqrt{\mu} e^{\frac{\mu}{2}y^2} \left(\int_0^T e^{-\mu(\tau-t)^2} dt \right)^{1/2} \left(\int_0^T |u(x, t)|^2 dt \right)^{1/2} \leq \\ &\leq \sqrt{\mu} e^{\frac{\mu}{2}y^2} \left(\int_0^{+\infty} e^{-\mu t^2} dt \right)^{1/2} \left(\int_0^T |u(x, t)|^2 dt \right)^{1/2} \leq c \mu^{1/4} e^{\frac{\mu}{2}y^2} \left(\int_0^T |u(x, t)|^2 dt \right)^{1/2}. \end{aligned}$$

Now we prove (4.20). By the change of variable $\eta = \sqrt{\mu}(t - \tau)$ we have

$$\begin{aligned}
(6.22) \quad & \sqrt{2\pi} (U_\mu^{(\tau)}(x, 0) - u(x, \tau)) = \\
& = \sqrt{\mu} \int_0^T e^{-\frac{\mu}{2}(\tau-t)^2} u(x, t) dt - u(x, \tau) \int_{-\infty}^{+\infty} e^{-\frac{\eta^2}{2}} d\eta = \\
& = \int_{-\sqrt{\mu}\tau}^{\sqrt{\mu}(T-\tau)} e^{-\frac{\eta^2}{2}} u\left(x, \tau + \frac{\eta}{\sqrt{\mu}}\right) d\eta - u(x, \tau) \int_{-\infty}^{+\infty} e^{-\frac{\eta^2}{2}} d\eta = \\
& = \int_{-\sqrt{\mu}\tau}^{\sqrt{\mu}(T-\tau)} e^{-\frac{\eta^2}{2}} \left(u\left(x, \tau + \frac{\eta}{\sqrt{\mu}}\right) - u(x, \tau) \right) d\eta - \\
& - u(x, \tau) \left(\int_{\sqrt{\mu}(T-\tau)}^{+\infty} e^{-\frac{\eta^2}{2}} d\eta + \int_{-\infty}^{-\sqrt{\mu}\tau} e^{-\frac{\eta^2}{2}} d\eta \right) := I_1 + I_2.
\end{aligned}$$

We begin to estimate $|I_1|$. We have

$$\begin{aligned}
(6.23) \quad |I_1| & \leq \mu^{-1/2} \|\partial_t u(x, \cdot)\|_{L^\infty[0, T]} \int_{-\infty}^{+\infty} |\eta| e^{-\frac{\eta^2}{2}} d\eta \leq \\
& \leq c \mu^{-1/2} \|\partial_t u(x, \cdot)\|_{L^\infty[0, T]},
\end{aligned}$$

where c is an absolute constant.

Now we estimate $|I_2|$. Taking into account that $\tau \in (0, T/2)$ we have

$$\begin{aligned}
(6.24) \quad |I_2| & \leq 2 |u(x, \tau)| \int_{\sqrt{\mu}\tau}^{+\infty} e^{-\frac{\eta^2}{2}} d\eta \leq \\
& \leq 2 |u(x, \tau)| e^{-\frac{\mu}{4}\tau^2} \int_{\sqrt{\mu}\tau}^{+\infty} e^{-\frac{\eta^2}{4}} d\eta \leq c e^{-\frac{\mu}{4}\tau^2} |u(x, \tau)|,
\end{aligned}$$

where c is an absolute constant.

Now, since $u(x, 0) = 0$ we have

$$\begin{aligned}
(6.25) \quad & e^{-\frac{\mu}{4}\tau^2} |u(x, \tau)| \leq \\
& \leq e^{-\frac{\mu}{4}\tau^2} \tau \|\partial_t u(x, \cdot)\|_{L^\infty[0, T]} \leq 2e^{-1} \mu^{-1/2} \|\partial_t u(x, \cdot)\|_{L^\infty[0, T]}.
\end{aligned}$$

By (6.22), (6.23), (6.24) and (6.25) we get (4.20). \square

Proof of Proposition 4.5. We have

$$\begin{aligned}
\partial_y U_\mu(x, y) &= \sqrt{\frac{\mu}{2\pi}} \int_0^T -i\mu(iy + \tau - t) e^{-\frac{\mu}{2}(iy + \tau - t)^2} u(x, t) dt = \\
&= \sqrt{\frac{\mu}{2\pi}} \int_0^T -i\partial_t \left(e^{-\frac{\mu}{2}(iy + \tau - t)^2} \right) u(x, t) dt = \\
&= -i\sqrt{\frac{\mu}{2\pi}} \left(e^{-\frac{\mu}{2}(iy + \tau - T)^2} u(x, T) - \int_0^T e^{-\frac{\mu}{2}(iy + \tau - t)^2} \partial_t u(x, t) dt \right)
\end{aligned}$$

and similarly

$$\begin{aligned}
(6.26) \quad \partial_y^2 U_\mu(x, y) &= \sqrt{\frac{\mu}{2\pi}} e^{-\frac{\mu}{2}(iy + \tau - T)^2} (\partial_t u(x, T) - \mu(iy + \tau - T)u(x, T)) - \\
&\quad - \sqrt{\frac{\mu}{2\pi}} \int_0^T e^{-\frac{\mu}{2}(iy + \tau - t)^2} \partial_t^2 u(x, t) dt.
\end{aligned}$$

On the other side by (4.17) we have

$$\begin{aligned}
(6.27) \quad \operatorname{div} (A(x) \nabla U_\mu) &= \sqrt{\frac{\mu}{2\pi}} \int_0^T e^{-\frac{\mu}{2}(iy + \tau - t)^2} \operatorname{div} (A(x) \nabla u) dt = \\
&= \sqrt{\frac{\mu}{2\pi}} \int_0^T e^{-\frac{\mu}{2}(iy + \tau - t)^2} \partial_t^2 u(x, t) dt.
\end{aligned}$$

By (6.26) and (6.27) the thesis follows. \square

6.3 Proof of Theorem 4.6

In the sequel, for seek of brevity we omit the tilde over r_j , $j = 1, 2, 3$.

First we consider the homogeneous case in which $\tilde{f} = 0$ and we assume that $r_3 = 1$. In [M-R-V1, Theorem 4.5] it has been proved that there exists a positive number $\bar{\beta}$ depending on λ_0, Λ_0 only such that if $\beta > \bar{\beta}$, then there exist constants C , τ_1 and r_0 , ($C \geq 1$, $\tau_1 \geq 1$, $0 < r_0 \leq 1$) depending only on λ_0, Λ_0 and β such that the following estimate holds true

$$\begin{aligned}
(6.28) \quad \tau \int |X|^\beta e^{2\tau|X|^{-\beta}} |\nabla v|^2 + \tau^3 \int |X|^{-\beta-2} e^{2\tau|X|^{-\beta}} |v|^2 &\leq \\
&\leq C \int |X|^{2\beta+2} e^{|X|^{-\beta}} |Pv|^2,
\end{aligned}$$

for every $v \in C_0^\infty(B_{r_0} \setminus \{0\})$ and for every $\tau \geq \tau_1$.

On the other hand it is simple to check that there exists $\tilde{\beta}$ depending on λ_0, Λ_0 only such that if $\beta \geq \tilde{\beta}$ then $|X|^{-\beta}$ satisfies the pseudoconvexity

conditions of [Hö, Theorem 8.3.1] in $B_1 \setminus \overline{B}_{r_0/2}$. Therefore there exist $\tau_2 \geq \tau_1$ and C depending on λ_0, Λ_0 and β only such that

$$(6.29) \quad \begin{aligned} \tau \int e^{2\tau|X|^{-\beta}} |\nabla v|^2 + \tau^3 \int e^{2\tau|X|^{-\beta}} |v|^2 &\leq \\ &\leq C \int e^{|X|^{-\beta}} |Pv|^2, \end{aligned}$$

for every $v \in C_0^\infty(B_1 \setminus \overline{B}_{r_0/2})$ and for every $\tau \geq \tau_2$.

Now we have trivially

$$(6.30a) \quad \int e^{2\tau|X|^{-\beta}} |\nabla v|^2 \geq \int |X|^\beta e^{2\tau|X|^{-\beta}} |\nabla v|^2,$$

$$(6.30b) \quad \int e^{2\tau|X|^{-\beta}} |v|^2 \geq (r_0/2)^{\beta+2} \int |X|^{-\beta-2} e^{2\tau|X|^{-\beta}} |v|^2,$$

$$(6.30c) \quad \int e^{|X|^{-\beta}} |Pv|^2 \leq \frac{1}{(r_0/2)^{2\beta+2}} \int |X|^{2\beta+2} e^{|X|^{-\beta}} |Pv|^2,$$

for every $v \in C_0^\infty(B_1 \setminus \overline{B}_{r_0/2})$. Let $\zeta \in C_0^\infty(B_{r_0})$ such that $0 \leq \zeta \leq 1$, $|\nabla \zeta|, |D^2 \zeta| \leq C$ and $\zeta(X) = 1$ for every $X \in B_{r_0/2}$.

Now let us denote $\beta_1 := \max\{\bar{\beta}, \tilde{\beta}, 1\}$ and let $v \in C_0^\infty(B_1 \setminus \{0\})$. By applying (6.31) and (6.29) to ζv and $(1 - \zeta)v$ respectively and taking into account (6.30) we have, for $\beta \geq \beta_1$

$$(6.31) \quad \begin{aligned} \tau \int |X|^\beta e^{2\tau|X|^{-\beta}} |\nabla v|^2 + \tau^3 \int |X|^{-\beta-2} e^{2\tau|X|^{-\beta}} |v|^2 &\leq \\ &\leq C \int |X|^{2\beta+2} e^{|X|^{-\beta}} |Pv|^2 + C \int |X|^{2\beta+2} e^{|X|^{-\beta}} (|D^2 \zeta|^2 v^2 + |\nabla \zeta|^2 |\nabla v|^2). \end{aligned}$$

Now the second term at the right hand side can be absorbed by the left hand side. Hence there exists $\tau_3 \geq \tau_2$ and C depending on λ_0, Λ_0 and β only such that for every $v \in C_0^\infty(B_1 \setminus \{0\})$ and every $\tau \geq \tau_3$ the following inequality holds true

$$(6.32) \quad \begin{aligned} \tau \int |X|^\beta e^{2\tau|X|^{-\beta}} |\nabla v|^2 + \tau^3 \int |X|^{-\beta-2} e^{2\tau|X|^{-\beta}} |v|^2 &\leq \\ &\leq C \int |X|^{2\beta+2} e^{|X|^{-\beta}} |Pv|^2. \end{aligned}$$

Now we use a standard argument to derive by (6.32) the desired three sphere inequality.

First we observe that, by density, estimate (6.32) holds true for every $v \in H_0^2(B_1 \setminus \{0\})$. Now let $u \in H^1(B_1)$ a solution to equation $Pu = 0$. By L^2 regularity theorem we have that $u \in H_{loc}^2(B_1)$. Let $0 < r_1 \leq r_2 < 1$, $0 < \delta \leq \min\{\frac{1-r_2}{2}, \frac{1}{2}\}$ and let us consider a cutoff function $\eta \in C_0^2(B_{1-\delta} \setminus \overline{B_{r_1(1-\delta)}})$ such that $0 \leq \eta \leq 1$ and satisfying the following conditions

$$\begin{aligned} \eta &= 1 \quad , \text{ in } B_{1-2\delta} \setminus B_{r_1(1-\delta)}, \\ |\nabla \eta| &\leq \frac{c}{\delta r_1} \quad , \quad |D^2 \eta| \leq \frac{c}{\delta^2 r_1^2} \quad , \text{ in } B_{r_1(1-\delta)} \setminus B_{r_1(1-2\delta)} \end{aligned}$$

and

$$|\nabla \eta| \leq \frac{c}{\delta} \quad , \quad |D^2 \eta| \leq \frac{c}{\delta^2} \quad , \text{ in } B_{1-\delta} \setminus B_{1-2\delta},$$

where c is an absolute constant.

By (4.25), since $\tilde{f} = 0$ we have

$$|P(\eta u)| \leq C(|\nabla \eta| |\nabla u| + |P\eta| |u|),$$

$$\begin{aligned} (6.33) \quad & \int |X|^{2\beta+2} e^{|X|^{-\beta}} |P(\eta u)|^2 \leq C e^{2\tau((1-2\delta)r_1)^{-\beta}} r_1^{2\beta+2} \times \\ & \times \left[\int_{B_{r_1(1-\delta)} \setminus B_{r_1(1-2\delta)}} ((\delta r_1)^{-2} |\nabla u|^2 + (\delta r_1)^{-4} |u|^2) \right] + \\ & + C e^{2\tau(1-2\delta)^{-\beta}} \left[\int_{B_{1-\delta} \setminus B_{1-2\delta}} (\delta^{-2} |\nabla u|^2 + \delta^{-4} |u|^2) \right], \end{aligned}$$

where C depends on λ_0, Λ_0 and β only.

By applying the Caccioppoli inequality to the right hand side of (6.33) and by (6.32) we have, for every $\tau \geq \tau_3$

$$\begin{aligned} (6.34) \quad & \int_{B_{r_2}} |X|^{-\beta-2} e^{2\tau|X|^{-\beta}} |u\eta|^2 \leq C e^{2\tau((1-2\delta)r_1)^{-\beta}} r_1^{2\beta-2} \delta^{-4} \int_{B_{r_1}} |u|^2 + \\ & + C e^{2\tau(1-2\delta)^{-\beta}} \delta^{-4} \int_{B_1} |u|^2, \end{aligned}$$

where C depends on λ_0, Λ_0 and β only.

On the other hand we have trivially

$$(6.35) \quad \int_{B_{r_2}} |X|^{-\beta-2} e^{2\tau|X|^{-\beta}} |u\eta|^2 \geq r_2^{-\beta-2} e^{2\tau r_2^{-\beta}} \int_{B_{r_2} \setminus B_{r_1}} |u|^2.$$

Now let us denote

$$(6.36) \quad \epsilon := \left(\int_{B_{r_1}} |u|^2 \right)^{1/2}, \text{ and } K := \left(\int_{B_1} |u|^2 \right)^{1/2}.$$

By (6.34) and (6.35) we have for every $\tau \geq \tau_3$

$$(6.37) \quad \int_{B_{r_2} \setminus B_{r_1}} |u|^2 \leq C\delta^{-4} \left\{ e^{2\tau[(1-2\delta)r_1]^{-\beta} - r_2^{-\beta}} \epsilon^2 + e^{2\tau[(1-\delta)]^{-\beta} - r_2^{-\beta}} K^2 \right\},$$

where C depends on λ_0, Λ_0 and β only.

Now we add to both the side of (6.37) the integral $\int_{B_{r_1}} u^2 dX$ and we get

$$(6.38) \quad \int_{B_{r_2}} |u|^2 \leq C\delta^{-4} \left\{ e^{2\tau[(1-2\delta)r_1]^{-\beta} - r_2^{-\beta}} \epsilon^2 + e^{2\tau[(1-\delta)]^{-\beta} - r_2^{-\beta}} K^2 \right\},$$

where C depends on λ_0, Λ_0 and β only.

Now denote by $\bar{\tau}$ the number

$$(6.39) \quad \bar{\tau} = \frac{\log(\epsilon^{-1}K)}{((1-2\delta)r_1)^{-\beta} - (1-\delta)^{-\beta}}$$

such a number satisfies the equality

$$e^{2\bar{\tau}[(1-2\delta)r_1]^{-\beta} - r_2^{-\beta}} \epsilon^2 = e^{2\bar{\tau}[(1-\delta)]^{-\beta} - r_2^{-\beta}} K^2.$$

If $\bar{\tau} \geq \tau_3$ then we choose $\tau = \bar{\tau}$ in (6.38) and we obtain

$$(6.40) \quad \int_{B_{r_2}} |u|^2 \leq C\delta^{-4} K^{2(1-\vartheta)} \epsilon^{\vartheta},$$

where

$$\vartheta = \frac{r_2^{-\beta} - (1-\delta)^{-\beta}}{[(1-2\delta)r_1]^{-\beta} - (1-\delta)^{-\beta}}.$$

On the other side if $\bar{\tau} < \tau_3$ then by (6.39) we have

$$(6.41) \quad (\epsilon^{-1}K)^{2\vartheta} < e^{2\tau_3[r_2^{-\beta} - (1-\delta)^{-\beta}]}$$

hence we have trivially

$$(6.42) \quad \int_{B_{r_2}} |u|^2 \leq \int_{B_1} |u|^2 = K^2 = K^{2(1-\vartheta)} K^{2\vartheta} \leq e^{2\tau_3[r_2^{-\beta} - (1-\delta)^{-\beta}]} K^{2(1-\vartheta)} \epsilon^{2\vartheta}.$$

Therefore by (6.43) and (6.42) we have

$$(6.43) \quad \int_{B_{r_2}} |u|^2 \leq C \delta^{-4} e^{2\tau_3[r_2^{-\beta} - (1-\delta)^{-\beta}]} K^{2(1-\vartheta)} \epsilon^{\vartheta}.$$

In the nonhomogeneous case, let $u \in H^1(B_1)$ a solution to $Pu = \tilde{f}$ and let w be the solution to the Dirichlet problem

$$\begin{cases} Pw = \tilde{f}, & \text{in } B_1, \\ w \in H_0^1(B_1), \end{cases}$$

we have that

$$(6.44) \quad \int_{B_1} |w|^2 \leq C \int_{B_1} |\tilde{f}|^2,$$

where C depends on λ_0 . By applying (6.43) to the function $u - w$ and by (6.44) we obtain inequality (4.29) when $\tilde{r}_3 = 1$. Finally, by using the dilation $X \rightarrow r_3 X$ the thesis follows easily. \square

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