Turán problems and shadows III: expansions of graphs

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Abstract

The expansion G^+ of a graph G is the 3-uniform hypergraph obtained from G by enlarging each edge of G with a new vertex disjoint from $V(G)$ such that distinct edges are enlarged by distinct vertices. Let $\mathrm{ex}_3(n, F)$ denote the maximum number of edges in a 3-uniform hypergraph with n vertices not containing any copy of a 3-uniform hypergraph F. The study of $\mathrm{ex}_3(n, G^+)$ includes some well-researched problems, including the case that F consists of k disjoint edges [6], G is a triangle [5, 9, 18], G is a path or cycle [12, 13], and G is a tree [7, 8, 10, 11, 14]. In this paper we initiate a broader study of the behavior of $\mathrm{ex}_3(n, G^+)$. Specifically, we show

$$\mathrm{ex}_3(n, K_{s,t}^+) = \Theta(n^{3-3/s})$$

whenever $t > (s-1)!$ and $s \geq 3$. One of the main open problems is to determine for which graphs G the quantity $\mathrm{ex}_3(n, G^+)$ is quadratic in n. We show that this occurs when G is any bipartite graph with Turán number $o(n^{\varphi})$ where $\varphi = \frac{1+\sqrt{5}}{2}$, and in particular, this shows $\mathrm{ex}_3(n, G^+) = O(n^2)$ when G is the three-dimensional cube graph.

1 Introduction

An r-uniform hypergraph F, or simply *r-graph*, is a family of r-element subsets of a finite set. We associate an r-graph F with its edge set and call its vertex set $V(F)$. Given an r-graph F, let $\mathrm{ex}_r(n, F)$ denote the maximum number of edges in an r-graph on n vertices that does not contain F. The *expansion* of a graph G is the 3-graph G^+ with edge set $\{e \cup \{v_e\} : e \in G\}$ where v_e are distinct vertices not in $V(G)$. By definition, the expansion of G has exactly $|G|$ edges. Note that Füredi and Jiang [10, 11] used a notion of expansion to r-graphs for general r, but this paper considers only 3-graphs.

Expansions include many important hypergraphs who extremal functions have been investigated, for instance the celebrated Erdős-Ko-Rado Theorem [6] for 3-graphs is the case of expansions of a matching. A well-known result is that $\mathrm{ex}_3(n, K_3^+) = \binom{n-1}{2}$ [5, 9, 18]. If a graph is not 3-colorable

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then its expansion has positive Turán density and this case is fairly well understood [16, 19], so we focus on the case of expansions of 3-colorable graphs. It is easy to see that $\mathrm{ex}_3(n, G^+) = \Omega(n^2)$ unless G is a star (the case that G is a star is interesting in itself, and for $G = P_2$ determining $\mathrm{ex}_3(n, G^+)$ constituted a conjecture of Erdős and Sós [7] which was solved by Frankl [8]). The authors [13] had previously determined $\mathrm{ex}_3(n, G^+)$ exactly (for large n) when G is a path or cycle of fixed length $k \geq 3$, thereby answering questions of Füredi-Jiang-Siever [12] and Füredi-Jiang [11]. The case when G is a forest is solved asymptotically in [14], thus settling a conjecture of Füredi [10]. The following straightforward result provides general bounds for $\mathrm{ex}_3(n, G^+)$ in terms of the number of edges of G.

Proposition 1.1. *If G is any graph with v vertices and $f \geq 4$ edges, then for some $a > 0$,*

$$an^{3-\frac{3v-9}{f-3}} \leq \mathrm{ex}_3(n, G^+) \leq (n-1)\mathrm{ex}_2(n, G) + (f + v - 1)\binom{n}{2}.$$

The proof of Proposition 1.1 is given in Section 3. Some key remarks are that $\mathrm{ex}_3(n, G^+)$ is not quadratic in n if $f > 3v - 6$, and if G is not bipartite then the upper bound in Proposition 1.1 is cubic in n. This suggests the question of identifying the graphs G for which $\mathrm{ex}_3(n, G^+) = O(n^2)$, and in particular evaluation of $\mathrm{ex}_3(n, G^+)$ for planar G.

1.1 Expansions of planar graphs

We give a straightforward proof of the following proposition, which is a special case of a more general result of Füredi [10] for a larger class of triple systems.

Proposition 1.2. *Let G be a graph with treewidth at most two. Then* $\mathrm{ex}_3(n, G^+) = O(n^2)$.

On the other hand, there are 3-colorable planar graphs G for which $\mathrm{ex}_3(n, G^+)$ is not quadratic in n. To state this result, we need a definition. A proper k-coloring $\chi : V(G) \to \{1, \ldots, k\}$ is *acyclic* if every pair of color classes induces a forest in G. We pose the following question:

Question 1. *Does every planar graph G with an acyclic 3-coloring have* $\mathrm{ex}_3(n, G^+) = O(n^2)$*?*

Let $g(n, k)$ denote the maximum number of edges in an n-vertex graph of girth larger than k.

Proposition 1.3. *Let G be a planar graph such that in every proper 3-coloring of G, every pair of color classes induces a subgraph containing a cycle of length at most k. Then* $\mathrm{ex}_3(n, G^+) = \Omega(ng(n, k)) = \Omega(n^{2+\Theta(\frac{1}{k})})$.

The last statement follows from the known fact that $g(n, k) \geq n^{1+\Theta(\frac{1}{k})}$. The octahedron graph O is an example of a planar graph where in every proper 3-coloring, each pair of color classes induces a cycle of length four, and so $\mathrm{ex}_3(n, O^+) = \Omega(n^{5/2})$. Even wheels do not have acyclic 3-colorings, and we do not know whether their expansions have quadratic Turán numbers.

Question 2. *Does every even wheel G have* $\mathrm{ex}_3(n, G^+) = O(n^2)$*?*

1.2 Expansions of bipartite graphs

The behavior of $\mathrm{ex}_3(n, G^+)$ when G is a dense bipartite graph is somewhat related to the behavior of $\mathrm{ex}_2(n, G)$ according to Proposition 1.1. In particular, Proposition 1.1 shows for $t \geq s \geq 2$ and some constants $a, c > 0$ that

$$an^{3-\frac{3s+3t-9}{st-3}} \leq \mathrm{ex}_3(n, K_{s,t}^+) \leq cn^{3-\frac{1}{s}}.$$

We show that both the upper and lower bound can be improved to determine the order of magnitude of $\mathrm{ex}_3(n, K_{s,t}^+)$ when good constructions of $K_{s,t}$-free graphs are available (see Alon, Rónyai and Szabo [2]):

Theorem 1.4. *Fix $3 \leq s \leq t$. Then $\mathrm{ex}_3(n, K_{s,t}^+) = O(n^{3-\frac{3}{s}})$ and, if $t > (s-1)! \geq 2$, then $\mathrm{ex}_3(n, K_{s,t}^+) = \Theta(n^{3-\frac{3}{s}})$.*

The following closely related problem was recently investigated by Alon and Shikhelman [3]. For a graph F, let $g(n, F)$ denote the maximum number of triangles in an n-vertex graph that contains no copy of F as a subgraph. From a graph G achieving this maximum, we can form a 3-graph H with $V(H) = V(G)$ and H consists of the triangles in G. Then a copy K of F^+ in H would yield a copy of F in G as $\partial K \supset F$. Consequently, we have

$$g(n, F) \leq \mathrm{ex}_3(n, F^+).$$

Alon and Shikhelman [3] independently proved that for fixed $3 \leq s \leq t$ and $t > (s-1)!$ we have $g(n, K_{s,t}) = \Theta(n^{3-3/s})$. Their lower bound construction is exactly the same as ours, though the proofs are different.

The case of $K_{3,t}$ is interesting since $\mathrm{ex}_3(n, K_{3,t}^+) = O(n^2)$, and perhaps it is possible to determine a constant c such that $\mathrm{ex}_3(n, K_{3,3}^+) \sim cn^2$, since the asymptotic behavior of $\mathrm{ex}_2(n, K_{3,3})$ is known, due to a construction of Brown [4] and the upper bounds of Füredi [10]. In general, the following bounds hold for expansions of $K_{3,t}$:

Theorem 1.5. *For fixed $r \geq 1$ and $t = 2r^2+1$, we have $(1-o(1))\frac{t-1}{12}n^2 \leq \mathrm{ex}_3(n, K_{3,t}^+) = O(n^2)$.*

The upper bound in this theorem is a special case of a general upper bound for all graphs G with $\sigma(G^+) = 3$ (see Theorem 1.7). Finally, we prove a general result that applies to expansions of a large class of bipartite graphs.

Theorem 1.6. *Let G be a graph with $\mathrm{ex}_2(n, G) = o(n^{\varphi})$, where $\varphi = (1+\sqrt{5})/2$ is the golden ratio. Then $\mathrm{ex}_3(n, G^+) = O(n^2)$.*

Let $\mathbb{Q}$ be the graph of the 3-dimensional cube (with 8 vertices and 12 edges). Erdős and Simonovits [7] proved $\mathrm{ex}_2(n, \mathbb{Q}) = O(n^{1.6}) = o(n^{\varphi})$, so a corollary to Theorem 1.6 is that

$$\mathrm{ex}_3(n, \mathbb{Q}^+) = \Theta(n^2).$$

Determining the growth rate of $\mathrm{ex}_2(n, \mathbb{Q})$ is a longstanding open problem. Since it is known that for any graph G the 1-subdivision of G has Turán Number $O(n^{3/2})$ – see Alon, Krivelevich and Sudakov [1] – Theorem 1.6 also shows that for such graphs G, $\mathrm{ex}_3(n,G^+)=\Theta(n^2)$. Erdős conjectured that $\mathrm{ex}_2(n,G)=O(n^{3/2})$ for each 2-degenerate bipartite graph G. If this conjecture is true, then by Theorem 1.6, $\mathrm{ex}_3(n,G^+)=O(n^2)$ for any 2-degenerate bipartite graph G.

1.3 Crosscuts

A set of vertices in a hypergraph containing exactly one vertex from every edge of a hypergraph is called a *crosscut* of the hypergraph, following Frankl and Füredi [9]. For a 3-uniform hypergraph F, let $\sigma(F)$ be the minimum size of a crosscut of F if it exists, i.e.,

$$\sigma(F):=\min\{|X|:\forall e\in F, |e\cap X|=1\}$$

if such an X exists. Since the triple system consisting of all edges containing exactly one vertex from a set of size $\sigma(F)-1$ does not contain F, we have

$$\mathrm{ex}_3(n,F)\geq(\sigma(F)-1+o(1))\binom{n}{2}. \tag{1}$$

An intriguing open question is: For which F an asymptotic equality is attained in (1)? Recall that a graph has tree-width at most two if and only if it has no subdivision of K_4. Informally, these are subgraphs of a planar graph obtained by starting with a triangle, and then picking some edge uv of the current graph, adding a new vertex w, and then adding the edges uw and vw.

Question 3. *Is it true that*

$$\mathrm{ex}_3(n,G^+)\sim(\sigma(G^+)-1)\binom{n}{2} \tag{2}$$

for every graph G with tree-width two?

If G is a forest or a cycle, then (2) holds [13, 14] (corresponding results for $r>3$ were given by Füredi [10]). If G is a graph with $\sigma(G^+)=2$, then again (2) holds [14]. Proposition 1.1 and Theorem 1.4 give examples of graphs G with $\sigma(G^+)=4$ and $\mathrm{ex}_3(n,G^+)$ superquadratic in n. This leaves the case $\sigma(G^+)=3$, and in this case, Theorem 1.5 shows that $\mathrm{ex}_3(n,K^+_{3,t})/n^2\to\infty$ as $t\to\infty$, even though $\sigma(K^+_{3,t})=3$ for all $t\geq 3$. A quadratic upper bound for $\mathrm{ex}_3(n,K^+_{3,t})$ in Theorem 1.5 is a special case of the following theorem:

Theorem 1.7. *For every G with $\sigma(G^+)=3$, $\mathrm{ex}_3(n,G^+)=O(n^2)$.*

2 Preliminaries

Notation and terminology. A 3-graph is called a *triple system*. The edges will be written as unordered lists, for instance, xyz represents $\{x,y,z\}$. For a set X of vertices of a hypergraph H, let $H-X=\{e\in H:e\cap X=\emptyset\}$. If $X=\{x\}$, then we write $H-x$ instead of $H-X$. For a set S of two vertices in a 3-graph H, $N_H(S) = \{x \in V(H) : S \cup \{x\} \in H\}$. The *codegree* of a pair $S = \{x, y\}$ of vertices in a 3-graph H is $d_H(x, y) = |N_H(S)|$. The *shadow* of H is the graph $\partial H = \{xy : \exists e \in H, \{x, y\} \subset e\}$. The edges of ∂H will be called the *sub-edges* of H. As usual, for a graph G and $v \in V(G)$, $N_G(v)$ is the set of neighbors of v in G and $d_G(v) = |N_G(v)|$.

A 3-graph H is *d-full* if every sub-edge of H has codegree at least d.

Thus H is d-full is equivalent to the fact that the minimum non-zero codegree in H is at least d. The following lemma from [14] extends the well-known fact that any graph G has a subgraph of minimum degree at least d with at least $|G| - (d-1)|V(G)|$ edges.

Lemma 2.1. *For $d \geq 1$, every n-vertex 3-graph H has a $(d+1)$-full subgraph F with*

$$|F| \geq |H| - d|\partial H|.$$

Proof. A *d-sparse sequence* is a maximal sequence $e_1, e_2, \ldots, e_m \in \partial H$ such that $d_H(e_1) \leq d$, and for all $i > 1$, e_i is contained in at most d edges of H which contain none of $e_1, e_2, \ldots, e_{i-1}$. The 3-graph F obtained by deleting all edges of H containing at least one of the e_i is $(d+1)$-full. Since a d-sparse sequence has length at most $|\partial H|$, we have $|F| \geq |H| - d|\partial H|$. $\square$

3 Proofs of Propositions

Proof of Proposition 1.1. The proof of the lower bound in Proposition 1.1 is via a random triple system. The idea is to take a random graph not containing a particular graph G, and then observe that the triple system of triangles in the random graph does not contain G^+. Consider the random graph on n vertices, whose edges are placed independently with probability p, to be chosen later. If X is the number of triangles and Y is the number of copies of G in the random graph, then

$$\mathbb{E}(X) = p^3 \binom{n}{3} \qquad \mathbb{E}(Y) \leq p^f n^v.$$

Therefore choosing $p = 0.1 n^{-(v-3)/(f-3)}$, since $f \geq 4$, we find

$$\mathbb{E}(X - Y) \geq 0.0001 n^{3-3(v-3)/(f-3)}.$$

Now let H be the triple system of vertex-sets of triangles in the graph obtained by removing one edge from each copy of G in the random graph. Then $\mathbb{E}(|H|) \geq \mathbb{E}(X - Y)$, and $G^+ \not\subset H$. Select an H so that $|H| \geq 0.0001 n^{3-3(v-3)/(f-3)}$. This proves the lower bound in Proposition 1.1 with $a = 0.0001$.

Now suppose G is a bipartite graph with f edges $e_1, e_2, \ldots, e_f$ and v vertices. If a triple system H on n vertices has more than $(n-1)\mathrm{ex}_2(n, G) + (f+v-1)\binom{n}{2}$ triples, then by deleting at most $(f+v-1)\binom{n}{2}$ triples we arrive at a triple system $H' \subset H$ which is $(f+v)$-full, by Lemma 2.1 and $|H'| > (n-1)\mathrm{ex}_2(n, G)$. There exists $x \in V(H')$ such that more than $\mathrm{ex}_2(n, G)$ triples of H' contain x. So the graph of all pairs $\{w, y\}$ such that $\{w, x, y\} \in H'$ contains G. Since every pair $\{w, y\}$ has codegree at least $f+v$, we find vertices $z_1, z_2, \ldots, z_f \notin V(G)$ such $e_i \cup \{z_i\} \in H'$ for all $i = 1, 2, \ldots, f$, and this forms a copy of G^+ in H'. $\square$

Proof of Proposition 1.2. Let G be a graph of tree-width two. Then $G \subset F$, where F is a graph obtained from a triangle by repeatedly adding a new vertex and joining it to two adjacent vertices of the current graph. It is enough to show $\mathrm{ex}_3(n, F^+) = O(n^2)$. Suppose F has v vertices and f edges. By definition, F has a vertex x of degree two such that the neighbors x' and x'' of x are adjacent. Then $F' := F - x$ has $v-1$ vertices and $f-2$ edges. Let H be an n-vertex triple system with more than $(v+f-1)\binom{n}{2}$ edges. By Lemma 2.1, H has a $(v+f)$-full subgraph H'. We claim H' contains F^+. Inductively, H' contains a copy H'' of the expansion of F'. By the definition of H', $\{x', x''\}$ has codegree at least $v+f$ in H'. Therefore we may select a new vertex z that is not in H'' such that $\{z, x', x''\}$ is an edge of H', and now F is embedded in H' by mapping x to z. $\square$

Proof of Proposition 1.3. Let G be a 3-colorable planar graph with the given conditions. To show $\mathrm{ex}_3(n, G^+) = \Omega(ng(n,k))$, form a triple system H on n vertices as follows. Let F be a bipartite $\lfloor \frac{n}{2} \rfloor$-vertex graph of girth $k+1$ with at least $\frac{1}{2}g(\lfloor \frac{n}{2} \rfloor, k)$ edges. Let U and V be the partite sets of F. Let X be a set of $\lceil \frac{n}{2} \rceil$ vertices disjoint from $U \cup V$. Then set $V(H) = U \cup V \cup X$ and let the edges of H consist of all triples $e \cup \{x\}$ such that $e \in F$ and $x \in X$. Then

$$|H| \geq |X| \cdot g(\lfloor \frac{n}{2} \rfloor, k) = \Omega(ng(n,k)).$$

Now ∂H has a natural 3-coloring given by U, V, X. If $G^+ \subset H$, then $G \subset \partial H$ and therefore G is properly colored, with color classes $V(G) \cap U$, $V(G) \cap V$ and $V(G) \cap X$. By the assumptions on G, $V(G) \cap (U \cup V)$ induces a subgraph of G which contains a cycle of length at most k. However, that cycle is then a subgraph of F, by the definition of H, which is a contradiction. Therefore $G^+ \not\subset H$. $\square$

4 Proof of Theorem 1.6

Proof of Theorem 1.6. Suppose $\mathrm{ex}_2(n, G) = o(n^{\varphi})$ and $|G| = k$, and H is an G^+-free 3-graph with $|H| \geq (k+1)\binom{n}{2}$. By Lemma 2.1, H has a k-full-subgraph H_1 with at least $n^2/3$ edges. If $G \subset \partial H_1$, then we can expand G to $G^+ \subset H_1$ using that H_1 is k-full. Therefore $|\partial H_1| \leq \mathrm{ex}_2(n, G) = o(n^{\varphi})$. By Lemma 2.1, and since $|H_1| \geq \delta n^2$, H_1 has a non-empty $n^{2-\varphi}$-full subgraph H_2 if n is large enough. Let H_3 be obtained by removing all isolated vertices of H_2 and let $m = |V(H_3)|$. Since H_3 is $n^{2-\varphi}$-full, $m > n^{2-\varphi}$. Since H_1 is G^+-free, $H_3 \subset H_1$ is also G^+-free, and therefore if $F = \partial H_3$, $|V(F)| = |V(H_3)| = m$ and $|F| \leq \mathrm{ex}_2(m, G) = o(m^{\varphi})$. So some vertex v of the graph $F = \partial H_3$ has degree $o(m^{\varphi - 1})$. Now the number of edges of F between the vertices of $N_F(v)$ is at least the number of edges of H_3 containing v. Since H_3 is

$n^{2-\varphi}$-full, there are at least $\frac{1}{2}n^{2-\varphi}|N_F(v)|$ such edges. On the other hand, since the subgraph of F induced by $N_F(v)$ does not contain G, the number of such edges is $o(|N_F(v)|^{\varphi})$. It follows that $n^{2-\varphi} = o(|N_F(v)|^{\varphi-1})$. Since $|N_F(v)| = o(m^{\varphi-1}) = o(n^{\varphi-1})$, we get $2-\varphi < (\varphi-1)^2$, contradicting the fact that φ is the golden ratio. $\square$

5 Proof of Theorem 1.4

Proof of Theorem 1.4. For the upper bound, we repeat the proof of Theorem 1.6 when $F = K_{s,t}$, using the bounds $\mathrm{ex}_2(n, K_{s,t}) = O(n^{2-1/s})$ provided by the Kövari-Sós-Turán Theorem [15], except at the stage of the proof where we use the bound on $\mathrm{ex}_2(|N_G(v)|, F)$, we may now use

$$\mathrm{ex}_2(|N_G(v)|, K_{s-1,t}) = O(|N_G(v)|^{2-1/(s-1)})$$

for if the subgraph of G of edges between $N_G(v)$ contains $K_{s-1,t}$, then by adding v we see G contains $K_{s,t}$. A calculation gives $|H| = O(n^{3-3/s})$.

For the lower bound we must show that $\mathrm{ex}_3(n, K_{s,t}^+) = \Omega(n^{3-3/s})$ if $t > (s-1)!$. We will use the *projective norm graphs* defined by Alon, Rónyai and Szabo [2]. Given a finite field $\mathbb{F}_q$ and an integer $s \geq 2$, the norm is the map $N : \mathbb{F}_{q^{s-1}}^* \to \mathbb{F}_q^*$ given by $N(X) = X^{1+q+\cdots+q^{s-2}}$. The norm is a (multiplicative) group homomorphism and is the identity map on elements of $\mathbb{F}_q^*$. This implies that for each $x \in \mathbb{F}_q^*$, the number of preimages of x is exactly

$$\frac{q^{s-1}-1}{q-1} = 1 + q + \cdots + q^{s-2}. \tag{3}$$

Definition 5.1. *Let q be a prime power and $s \geq 2$ be an integer. The projective norm graph $PG(q,s)$ has vertex set $V = \mathbb{F}_{q^{s-1}} \times \mathbb{F}_q^*$ and edge set*

$$\{(A,b)(B,b) : N(A+B) = ab\}.$$

Lemma 5.2. *Fix an integer $s \geq 3$ and a prime power q. Let $x \in \mathbb{F}_q^*$, and $A, B \in \mathbb{F}_{q^{s-1}}$ with $A \neq B$. Then the number of $C \in \mathbb{F}_{q^{s-1}}$ with*

$$N\left(\frac{A+C}{B+C}\right) = x \tag{4}$$

is at least q^{s-2}.

Proof. By (3) there exist distinct $X_1, \ldots, X_{q^{s-2}+1} \in \mathbb{F}_{q^{s-1}}^*$ such that $N(X_i) = x$ for each i. As long as $X_i \neq 1$, define

$$C_i = \frac{BX_i - A}{1 - X_i}.$$

Then $(A + C_i)/(B + C_i) = X_i$, and $C_i \neq C_j$ for $i \neq j$ since $A \neq B$. $\square$

Lemma 5.3. *Fix an integer $s \geq 3$ and a prime power q. The number of triangles in $PG(q,s)$ is at least $(1-o(1))q^{3s-3}/6$ as $q \to \infty$.*

Proof. Pick a vertex (A,a) and then one of its neighbors (B,b). The number of ways to do this is at least $q^{s-1}(q-1)(q^{s-1}-1)$. Let $x = a/b$ and apply Lemma 5.2 to obtain at least $q^{s-2}-2$ distinct $C \notin \{-A,-B\}$ satisfying (4). For each such C, define

$$c = \frac{N(A+C)}{a} = \frac{N(B+C)}{b}.$$

Then (C,c) is adjacent to both (A,a) and (B,b). Each triangle is counted six times in this way and the result follows. $\square$

For appropriate n the n-vertex norm graphs $PG(q,s)$ (for fixed s and large q) have $\Theta(n^{2-1/s})$ edges and no $K_{s,t}$. By Lemma 5.3 the number of triangles in $PG(q,s)$ is $\Theta(n^{3-3/s})$. The hypergraph H whose edges are the vertex sets of triangles in $PG(q,s)$ is a 3-graph with $\Theta(n^{3-3/s})$ edges and no $K^+_{s,t}$. This completes the proof of Theorem 1.4. $\square$

6 Proof of Theorems 1.5 and 1.7

We need the following result.

Theorem 6.1. *Let F be a 3-uniform hypergraph with v vertices and $\mathrm{ex}_3(n,F) < c\binom{n}{2}$. Then $ex_3(n,(\partial F)^+) < (c+v+|F|)\binom{n}{2}$.*

Proof. Suppose we have an n vertex 3-uniform hypergraph H with $|H| > (c+v+|H|)\binom{n}{2}$. Apply Lemma 2.1 to obtain a subhypergraph $H' \subset H$ that is $(v+|F|)$-full with $|H'| > c\binom{n}{2}$. By definition, we may find a copy of $F \subset H'$ and hence a copy of $\partial F \subset \partial H'$. Because H' is $(v+|F|)$-full, we may expand this copy of ∂F to a copy of $(\partial F)^+ \subset H' \subset H$ as desired. $\square$

Define H_t to be the 3-uniform hypergraph with vertex set $\{a,b,x_1,y_1,\ldots,x_t,y_t\}$ and $2t$ edges x_iy_ia and x_iy_ib for all $i \in [t]$. It is convenient (though not necessary) for us to use the following theorem of the authors [17].

Theorem 6.2. ([17]) *For each $t \geq 2$, we have $\mathrm{ex}_3(n,H_t) < t^4\binom{n}{2}$.*

Proof of Theorems 1.5 and 1.7. First we prove the upper bound in Theorem 1.7. Suppose $\sigma(G^+) \leq 3$. This means that G has an independent set I and set R of edges such that I intersects each edge in $G-R$, and $|I|+|R| \leq 3$. It follows that G is a subgraph of one of the following graphs (Cases (i) and (ii) correspond to $|I|=1$, Case (iii) corresponds to $|I|=2$, and Case (iv) corresponds to $|I|=3$):

(i) $K_4 - e$ together with a star centered at one of the degree 3 (in $K_4 - e$) vertices,

(ii) two triangles sharing a vertex x and a star centered at x,

(iii) the graph obtained from $K_{2,t}$ by adding an edge joining two vertices in the part of size t,

(iv) $K_{3,t}$.

Now suppose we have a 3-uniform H with $|H| > cn^2$ for some $c > |G| + |V(G)|$. Applying Lemma 2.1, we find a c-full $H' \subset H$ with $|H'| > c\binom{n}{2}$. As in the proof of Theorem 6.1, it is enough to find G in $\partial H'$. Since $|H'| > c\binom{n}{2}$, the codegree of some pair $\{x, y\}$ is at least $c + 1$. Then the shadow of the set of triples in H' containing $\{x, y\}$ contains the graph of the form i). Similarly, H' contains two edges sharing exactly one vertex, say x, and the shadow of the set of triples in H' containing x contains the graph of the form ii). If G is of the form in iii), we apply Theorems 6.1 and 6.2 and observe that $\partial H_t \supset G$. Finally, if $G \subseteq K_{3,t}$ then we apply Theorem 1.4.

For the lower bound in Theorem 1.5, we use a slight modification of the construction in Theorem 1.4. Set $s = 3$ and let $r|q - 1$. Let Q_r denote a subgroup of $\mathbb{F}_q^*$ of order r. Define the graph $H = H_r(q)$ with $V(H) = \mathbb{F}_{q^2} \times \mathbb{F}_q^*/Q_r$ and two vertices (A, aQ_r) and (B, bQ_r) are adjacent in H if $N(A + B) \in abQ_r$. Then H has $n = (q^3 - q^2)/r$ vertices and each vertex has degree $q^2 - 1$. It also follows from [2] that H has no $K_{3,t}$ where $t = 2r^2 + 1$. Now we construct a 3-uniform hypergraph H' with $V(H') = V(H)$ and whose edges are the triangles of H. We must count the number of triangles in H to determine $|H'|$. For every choice of $(A, a), (B, b)$ in $\mathbb{F}_{q^2} \times \mathbb{F}_q^*$, the number of $(C, c) \in \mathbb{F}_{q^2} \times \mathbb{F}_q^*$ with $C \neq A, B$, $N(A + C) = ac$ and $N(B + C) = bc$ is at least $q - 2$ by (the proof of) Lemma 5.3. Consequently, the number of (C, c) such that $N(A+C) \in acQ_r$ and $N(B+C) \in bcQ_r$ is at least $r^2(q-2)$. Since (C, c) satisfies these equations iff (C, cq) satisfies these equations for all $q \in Q_r$ (i.e. the solutions come in equivalence classes of size r), the number of common neighbors of (A, aQ_r) and (B, bQ_r) is at least $r(q - 2)$. The number of edges in H is at least $(1 - o(1))q^5/2r$, so the number of triangles in H is at least $(1 - o(1))q^6/6 = (1 - o(1))(r^2/6)n^2$. $\square$

7 Concluding remarks

• In this paper we studied $\mathrm{ex}_3(n, G^+)$ where G is a 3-colorable graph. If G has treewidth two, then we believe $\mathrm{ex}_3(n, G^+) \sim (\sigma(G^+) - 1)\binom{n}{2}$ (Question 3), and if a planar graph G has an acyclic 3-coloring, then we believe $\mathrm{ex}_3(n, G^+) = O(n^2)$ (Question 1). In fact, we also do not know any nonplanar acyclically 3-colorable graph G with superquadratic $\mathrm{ex}_3(n, G^+)$. We are also not able to prove or disprove $\mathrm{ex}_3(n, G^+) = O(n^2)$ when G is an even wheel (Question 2). This is equivalent to showing that if F is an n-vertex graph with a superquadratic number of triangles, then F contains every even wheel with a bounded number of vertices.

• A number of examples of 3-colorable G with superquadratic $\mathrm{ex}_3(n, G^+)$ were given. In particular we determined the order of magnitude of $\mathrm{ex}_3(n, K_{s,t}^+)$ when near-extremal constructions of

$K_{s,t}$-free bipartite graphs are known. One may ask for the asymptotic behavior of $\mathrm{ex}_3(n, K_{3,t}^+)$ for each $t \geq 3$, since in that case we have shown $\mathrm{ex}_3(n, K_{3,t}^+) = \Theta(n^2)$. Finally, we gave a general upper bound on $\mathrm{ex}_3(n, G^+)$ when G is a bipartite graph, and showed that if G has Turán number much smaller than n^{φ} where φ is the golden ratio, then $\mathrm{ex}_3(n, G^+) = O(n^2)$. Determining exactly when $\mathrm{ex}_3(n, G^+)$ is quadratic in n remains an open problem for further research.

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