A CMV-BASED EIGENSOLVER FOR COMPANION MATRICES

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Abstract.

In this paper we present a novel matrix method for polynomial root finding. By exploiting the properties of the QR eigenvalue algorithm applied to a suitable CMV–like form of a companion matrix we design a fast and computationally simple structured QR iteration. $AMS\ classification:\ 65F15$

Key words. CMV-like matrix, companion matrix, QR eigenvalue algorithm, rank structure.

1. Introduction. This paper stems from two research lines which blend in the effective solution of certain eigenproblems for companion—like matrices arising in polynomial rootfinding. The first one begins with the exploitation of the structure of companion-like matrices under the QR eigenvalue algorithm. In the recent years based on the concept of rank structure many authors have provided fast adaptations of the QR iteration applied to small rank modifications of Hermitian or unitary matrices. However, despite the common framework, there are several significant differences between the Hermitian and the unitary case which makes the latter much more involved computationally. The second line originates from the treatment of the unitary eigenproblem. It has been observed in the seminal paper [11] that the CMV-like banded form of a unitary matrix rather than its Hessenberg reduction leads to a QRtype algorithm which is ideally close to the Hermitian tridiagonal QR algorithm as it maintains the band shape of the initial matrix at any step. The present work lies at the intersection of these two strands and is specifically aimed to incorporate the CMV technology for the unitary eigenproblem in the design of fast QR-based eigensolvers for companion-like matrices.

The first fast structured variant of the QR iteration for companion matrices was proposed in [5]. The invariance of the rank properties of the matrices generated by the QR scheme is captured by means of three rank—one matrices which are easily updated under the iterative process. Since the representation breaks down for reducible Hessenberg matrices the price paid to keep the algorithm simple is a progressive deterioration in the limit of the accuracy of computed eigenvalues. Overcoming this drawback is the main subject of many subsequents papers [4, 6, 7, 14, 25], where more refined parametrizations of the rank structure are employed. While this leads to numerically stable methods, it also opens the way to involved algorithms which exhibit worse timing performance and are difficult to generalize to the block matrix/pencil case. This is astonishingly unpleasant when compared with the simplicity and the effectiveness of adjusting the QR scheme for perturbed Hermitian matrices [16, 26].

The approach pursued here moves away from the classical scenario where nonsymmetric matrices are converted in Hessenberg form for eigenvalue computation, focusing instead on a preliminary reduction of a companion matrix $A \in \mathbb{C}^{n \times n}$ into a different staircase form. More specifically, recall that $A \in \mathbb{C}^{n \times n}$ can be expressed as a rank-one correction of a unitary matrix U generating the circulant matrix algebra. The transformation of U by unitary congruence into a CMV-like form [12, 22] induces a corresponding reduction of the matrix A into an upper block Hessenberg form with

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a certain specified staircase pattern. The CMV-like form of a unitary matrix is particularly suited for the application of the QR eigenvalue algorithm [11]. The staircase shape also reveals invariance properties under the same algorithm [2].

From these properties it follows that the matrices generated by the shifted QR method applied to the transformed companion matrix inherit a simplified rank structure which can be expressed in terms of two rank—one matrices. This yields a data sparse parametrization of each matrix which at the same time is able to capture the structural properties of the matrix and yet to be very easy to manipulate and update for computations. We shall develop a fast adaptation of the QR eigenvalue algorithm for companion matrices that exploits this parametrization and requires O(n) arithmetic operations per step. The main complexity of the algorithm lies in updating the narrow diagonal staircase of each matrix. The results from numerical experiments indicate that the proposed approach works stable and efficient.

The paper is organized as follows. In Section 2, we first recall some preliminaries about CMV–like representations of unitary matrices and then introduce the considered reduction of a companion matrix. The structural properties of the modified matrix under the shifted QR iteration are analyzed in Section 3. In Section 4 we present our fast adaptation of the shifted QR algorithm for companion matrices and report the results of numerical experiments. Finally, in Section 5 the conclusion and further developments are drawn.

2. Preliminaries. For a given pair $(\gamma, k) \in \mathbb{D} \times \mathbb{I}_n$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{I}_n = \{1, 2, \dots, n-1\}$, we set

$$\mathcal{G}_k(\gamma) = I_{k-1} \oplus \begin{bmatrix} \bar{\gamma} & \sigma \\ \sigma & -\gamma \end{bmatrix} \oplus I_{n-k-1} \in \mathbb{C}^{n \times n},$$

where $\sigma \in \mathbb{R}, \sigma > 0$ and $|\gamma|^2 + \sigma^2 = 1$. Similarly, if $\gamma \in \mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ then denote

$$\mathcal{G}_n(\gamma) = I_{n-1} \oplus \gamma \in \mathbb{C}^{n \times n}$$
.

Observe that $\mathcal{G}_k(\gamma)$, $1 \leq k \leq n$, is a unitary matrix. Given coefficients $\gamma_1, \ldots, \gamma_{n-1} \in \mathbb{D}$ and $\gamma_n \in \mathbb{S}^1$ we introduce the unitary block diagonal matrices

$$\mathcal{L} = \mathcal{G}_1(\gamma_1) \cdot \mathcal{G}_3(\gamma_3) \cdots \mathcal{G}_{2 \lfloor \frac{n+1}{2} \rfloor - 1}(\gamma_{2 \lfloor \frac{n+1}{2} \rfloor - 1}), \quad \mathcal{M} = \mathcal{G}_2(\gamma_2) \cdot \mathcal{G}_4(\gamma_4) \cdots \mathcal{G}_{2 \lfloor \frac{n}{2} \rfloor}(\gamma_{2 \lfloor \frac{n}{2} \rfloor}),$$
 and define

$$C = \mathcal{L} \cdot \mathcal{M} \tag{2.1}$$

as the CMV matrix associated with the prescribed coefficient list [12]. The decomposition (2.1) of a unitary matrix was first investigated for eigenvalue computation in [11]. The staircase shape of CMV matrices is analyzed in [22] where the next definition is given. A matrix $A \in \mathbb{C}^{n \times n}$ has CMV shape if the possibly nonzero entries exhibit the following pattern where + denotes a positive entry:

Obviously, CMV matrices have a CMV shape and, conversely, a unitary matrix with CMV shape is CMV [13]. By skipping the positivity condition in [3] the fairly more general class of CMV-like shaped matrices is considered. There it is shown that the block Lanczos method can be used to reduce a unitary matrix into the direct sum of CMV-like shaped matrices.

Staircase matrix patterns can be exploited for eigenvalue computation [2]. The shifted QR algorithm

$$\begin{cases}
A_s - \rho_s I_n = Q_s R_s \\
A_{s+1} = Q_s^H A_s Q_s, \quad s \ge 0,
\end{cases}$$
(2.2)

is the standard algorithm for computing the Schur form of a general matrix $A = A_0 \in \mathbb{C}^{n \times n}$ [20]. The matrix A is said to be staircase if $m_j(A) \geq m_{j-1}(A)$, $2 \leq j \leq n$, where

$$m_j(A) = \max\{j, \max_{i>j}\{i : a_{i,j} \neq 0\}\}.$$

The staircase form is preserved under the QR iteration (2.2) in the sense that [2]

$$m_j(A_{s+1}) \le m_j(A_s), \quad 1 \le j \le n.$$

For Hermitian and unitary matrices the staircase form also implies a zero pattern or a rank structure in the upper triangular part. The invariance of this pattern by the QR algorithm is proved in [2] for Hermitian matrices and in [11] for unitary CMV-shaped matrices. An alternative proof for the unitary case that is suitable for generalizations is given in [3] by relying upon the classical nullity theorem [18].

THEOREM 2.1. Suppose $A \in \mathbb{C}^{n \times n}$ is a nonsingular matrix and α and β to be nonempty proper subsets of \mathbb{I}_{n+1} : = $\{1, \ldots, n\}$. Then

$$\operatorname{rank}(A^{-1}(\boldsymbol{\alpha};\boldsymbol{\beta})) = \operatorname{rank}(A(\mathbb{I}_{n+1} \setminus \boldsymbol{\beta}; \mathbb{I}_{n+1} \setminus \boldsymbol{\alpha})) + |\boldsymbol{\alpha}| + |\boldsymbol{\beta}| - n,$$

where, as usual, |J| denotes the cardinality of the set J.

The design of efficient numerical methods for eigenvalue computation of almost Hermitian and unitary matrices has recently attracted a lot of attention (see [17, 19, 28] and the references given therein). A motivating application is given by matrix methods for polynomial rootfinding. From a given n-th degree polynomial

$$p(z) = p_0 + p_1 z + \ldots + p_n z^n, \quad (p_n \neq 0),$$

we can set up the associated companion matrix $C \in \mathbb{C}^{n \times n}$ in upper Hessenberg form,

$$C = C(p) = \begin{bmatrix} -\frac{p_{n-1}}{p_n} & -\frac{p_{n-2}}{p_n} & \dots & -\frac{p_0}{p_n} \\ 1 & 0 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ & & 1 & 0 \end{bmatrix}.$$

As $p_n \det(zI - C) = p(z)$ is satisfied, thus we can obtain approximations of the zeros of p(z) by applying a standard eigenvalue method to the associated companion matrix C. This is exactly the approach taken by the MATLAB function roots.

In the recent years many fast adaptations of the QR iteration (2.2) applied to an initial companion matrix $A_0 = C$ have been proposed [4, 6, 7, 14, 25], based on the decomposition of C as a rank—one correction of a unitary matrix, that is,

$$C = U - e_1 p^H = \begin{bmatrix} 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ & & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \left[\frac{p_{n-1}}{p_n}, \frac{p_{n-2}}{p_n}, \dots, \frac{p_0}{p_n} + 1 \right].$$

In this paper we further elaborate on this decomposition by developing a different structured representation. Let $P \in \mathbb{R}^{n \times n}$, $P = (\delta_{i,\pi(j)})$ be the permutation matrix associated with the permutation given by

$$\pi: \mathbb{I}_{n+1} \to \mathbb{I}_{n+1}, \quad \pi(1) = 1; \ \pi(j) = \begin{cases} k+1, & \text{if } j = 2k; \\ n-k+1, & \text{if } j = 2k+1. \end{cases}$$

Then it can be easily verified that the matrix $\widehat{U} = P^T \cdot U \cdot P$ is a CMV-like shaped matrix. Indeed, we have that the nonzero entries of \widehat{U} are precisely (2,1), (n-1,n) and those of the form (2j-1,2j+1) and (2j,2j+2) for $j \geq 1$. For instance in the case n=8 the nonzero pattern looks as follows:

Moreover, since $P^T e_1 = e_1$ it follows that, denoting by $\hat{p} = P^T p$ we have

$$\widehat{C} = P^T \cdot C \cdot P = \widehat{U} - e_1 \,\widehat{p}^H, \tag{2.3}$$

is a rank—one correction of a unitary CMV-like shaped matrix in staircase form. In the next section we investigate the properties of the shifted QR iteration (2.2) applied to $A_0 = \hat{C}$ for the computation of the zeros of p(z).

3. Structural Properties under the QR iteration. To put our derivation on a firm theoretical ground, in this section we perform a thorough analysis of the structural properties of $A_0 = \widehat{C}$ which are maintained under the shifted QR iteration (2.2).

Remark 3.1. Several different properties can easily be checked by assuming that the matrix $A_s - \sigma_s I_n$ in (2.2) and, hence, a fortiori R_s is invertible. Clearly, this might not always be the case but, however, it is well known that the one-parameter matrix function $A_s - \lambda I_n$ is analytic in λ and an analytic QR decomposition $A_s - \lambda I_n = Q_s(\lambda)R_s(\lambda)$ of this analytic matrix function exists [15]. For any given fixed initial pair

 $(Q_s(\sigma_s), R_s(\sigma_s))$ we can find a branch of the analytic QR decomposition of $A_s - \lambda I_n$ that passes through $(Q_s(\sigma_s), R_s(\sigma_s))$. Following this path it makes it possible to extend the proof of the properties that are closed in the limit. This is for instance the case of the rank properties.

It has already been noticed above that the staircase form of $A_0 = C$ is preserved under the shifted QR iteration (2.2). This means that each unitary matrix Q_s is also in staircase form. In particular, if

$$\mathcal{G}_k(\gamma,\sigma) = I_{k-1} \oplus \begin{bmatrix} \bar{\gamma} & \sigma \\ \bar{\sigma} & -\gamma \end{bmatrix} \oplus I_{n-k-1} \in \mathbb{C}^{n \times n}, \quad 1 \le k \le n-1,$$

where $\gamma, \sigma \in \mathbb{D} \cup \mathbb{S}^1$ and $|\gamma|^2 + |\sigma|^2 = 1$, denote generalized Givens reflectors then the matrix $Q = Q_s$ can be expressed as

$$Q = \mathcal{G}_1(\bar{\gamma}_1, \bar{\sigma}_1) \cdot \mathcal{G}_{2,3} \cdot \mathcal{G}_{4,5} \cdots \mathcal{G}_{2(\lfloor \frac{n+1}{2} \rfloor - 2), 2(\lfloor \frac{n+1}{2} \rfloor - 2) + 1} \cdot \mathcal{G}_{n-1}, \tag{3.1}$$

where

$$\mathcal{G}_{\ell,\ell+1} = \mathcal{G}_{\ell+1}(\bar{\gamma}_{\ell+1,1}, \bar{\sigma}_{\ell+1,1}) \cdot \mathcal{G}_{\ell}(\bar{\gamma}_{\ell}, \bar{\sigma}_{\ell}) \cdot \mathcal{G}_{\ell+1}(\bar{\gamma}_{\ell+1,2}, \bar{\sigma}_{\ell+1,2}),$$

and

$$\mathcal{G}_{n-1} = \mathcal{G}_{n-1}(\bar{\gamma}_{n-1,1}, \bar{\sigma}_{n-1,1}) \cdot \mathcal{G}_{n-2}(\bar{\gamma}_{n-2}, \bar{\sigma}_{n-2}) \cdot \mathcal{G}_{n-1}(\bar{\gamma}_{n-1,2}, \bar{\sigma}_{n-1,2})$$

for an even n and

$$G_{n-1} = G_{n-1}(\bar{\gamma}_{n-1,1}, \bar{\sigma}_{n-1,1})$$

if, otherwise, n is odd.

Since from (2.3)

$$A_0 = \widehat{U} - \boldsymbol{e}_1 \, \widehat{\boldsymbol{p}}^H = U_0 - \boldsymbol{z}_0 \boldsymbol{w}_0^H$$

we find that

$$A_{s+1} = Q_s^H A_s Q_s = Q_s^H (U_s - \boldsymbol{z}_s \boldsymbol{w}_s^H) Q_s = U_{s+1} - \boldsymbol{z}_{s+1} \boldsymbol{w}_{s+1}^H, \quad s \ge 0, \tag{3.2}$$

where

$$U_{s+1} := Q_s^H U_s Q_s, \quad \boldsymbol{z}_{s+1} := Q_s^H \boldsymbol{z}_s, \quad \boldsymbol{w}_{s+1} := Q_s^H \boldsymbol{w}_s.$$
 (3.3)

Theorem (3.3) describes the structure of the unitary matrix U_s , for any $s \geq 0$. We need the following result characterizing the structure of the Q factor appearing in the QR factorization of A_s , $s \geq 0$.

Lemma 3.2. The unitary factor Q generated by means of a QR factorization of A_s , $s \ge 0$, has both a lower and upper staircase profile. Specifically, it holds

$$Q(1:2j,2(j+1)+1:n)=0, \quad 1 \le j \le \lfloor \frac{n+1}{2} \rfloor -2.$$

Proof. It has already been observed that since the staircase form of A_0 is preserved under the shifted QR iteration (2.2), the unitary factor Q, corresponding to the unitary matrix involved in a QR iteration without shift has a lower staircase profile.

To prove that Q has also an upper staircase profile, observe that the matrix A_0 is such that $\operatorname{rank}(A_0(2j+1:2(j+1),2j:2j+1))=1,\ 1\leq j\leq \lfloor\frac{n+1}{2}\rfloor-1$. From the argument stated in Remark 3.1 it follows that this rank constraint is preserved under the QR iteration and, specifically, we have $\operatorname{rank}(A_s(2j+1:2(j+1),2j:2j+1))=1,\ 1\leq j\leq \lfloor\frac{n+1}{2}\rfloor-1$, for any $s\geq 0$. The same property is also inherited from the unitary factor $Q=Q_s$ generated by means of the QR factorization of A_s , i.e., $A_s=QR$. From Theorem 2.1 we obtain that

$$rank(Q(1:2j,2(j+1)+1:n)) = rank(Q^H(2(j+1)+1:n,1:2j)) = rank(Q(2j+1:n,1:2(j+1)) + (n-2) - n = rank(Q(2j+1:n,1:2(j+1)) - 2.$$

Hence, by combining the constraint $\operatorname{rank}(Q(2j+1:2(j+1),2j:2j+1))=1$ with the staircase shape of Q one deduces that $\operatorname{rank}(Q(2j+1:n,1:2(j+1))=2$ which implies

$$rank(Q(1:2j,2(j+1)+1:n)) = 0, \quad 1 \le j \le \lfloor \frac{n+1}{2} \rfloor - 2.$$

Equivalently, the relation says that Q(1:2j,2(j+1)+1:n) is a zero matrix and this concludes the proof. \square

Lemma 3.2 can be used to exploit the rank properties of the unitary matrices U_s , $s \ge 0$.

Theorem 3.3. We have

$$\operatorname{rank}(U_s(1:2j,2(j+1)+1:n)) \le 1, \quad 1 \le j \le \lfloor \frac{n+1}{2} \rfloor -2, \ s \ge 0.$$

Moreover, if A_0 is invertible then

$$U_s(1:2j,2(j+1)+1:n) = B_s(1:2j,2(j+1)+1:n), \quad 1 \le j \le \lfloor \frac{n+1}{2} \rfloor -2, \ s \ge 0,$$

where

$$B_{s} = \frac{U_{s} \boldsymbol{w}_{s} \boldsymbol{z}_{s}^{H} U_{s}}{\boldsymbol{z}^{H} U^{H} \boldsymbol{w}_{s} - 1} = Q_{s}^{H} B_{s-1} Q_{s}, \quad s \ge 1,$$
(3.4)

is a rank one matrix.

Proof. Let $A_s = QR$ be a QR factorization of the matrix A_s assumed invertible. From

$$Q^{H}A = Q^{H}(U_{s} - \boldsymbol{z}_{s}\boldsymbol{w}_{s}^{H}) = Q^{H}U_{s} - Q^{H}\boldsymbol{z}_{s}\boldsymbol{w}_{s}^{H} = R$$

we obtain that

$$(Q^H A)^{-H} = Q^H (U_s - z_s w_s^H)^{-H} = R^{-H}.$$

Using the Sherman–Morrison formula [20] yields

$$Q^{H}(U_s + \frac{U_s \boldsymbol{w}_s \boldsymbol{z}_s U_s}{1 - \boldsymbol{z}_s^{H} U_s^{H} \boldsymbol{w}_s}) = R^{-H},$$

which gives

$$U_s = QR + \boldsymbol{z}_s \boldsymbol{w}_s^H = QR^{-H} - \frac{U_s \boldsymbol{w}_s \boldsymbol{z}_s U_s}{1 - \boldsymbol{z}_s^H U_s^H \boldsymbol{w}_s}.$$

Since R^{-H} is upper triangular we have that QR^{-H} has the same upper staircase shape as Q and, therefore, from Lemma 3.2 we conclude that

$$\operatorname{rank}(U_s(1:2j,2(j+1)+1:n)) \le 1, \quad 1 \le j \le \lfloor \frac{n+1}{2} \rfloor - 2, \ s \ge 0.$$

The argument stated in Remark 3.1 extends this property to a possibly singular A_0 and a fortiori A_s , $s \ge 0$. \square

Remark 3.4. It is worth pointing out that although the rank structure of U_s is closed in the limit its parametrization via generators is not [27]. This means that the rank one representation of the entries of U_s located in the upper triangular portion does not hold in the general case where the starting matrix A_0 can be singular.

From the previous theorem we derive a structural representation of each matrix A_s , $s \ge 0$, generated under the QR process (2.2) applied to $A_0 = \widehat{C}$ given as in (2.3). In the next section we provide a fast adaptation of this process based on the relations (3.2),(3.3), and (3.4).

- **4. Fast Algorithms and Numerical Results.** In this section we devise a fast adaptation of the QR iteration (2.2) applied to a starting invertible matrix $A_0 = \hat{C} \in \mathbb{C}^{n \times n}$ given as in (2.3) by using the structural properties described above. Let us first observe that each matrix A_s , $s \geq 0$, generated by (2.2) can be represented by means of the following sparse data set of size O(n):
 - 1. the nonzero entries of the banded matrix $\widehat{A}_s \in \mathbb{C}^{n \times n}$ obtained from A_s according to

$$\widehat{A}_{s} = (\widehat{a}_{i,j}^{(s)}), \quad \widehat{a}_{i,j}^{(s)} = \begin{cases} 0, & \text{if } j \ge 2 \lfloor \frac{i+1}{2} \rfloor + 3, \ 1 \le i \le 2 \lfloor \frac{n+1}{2} \rfloor - 4; \\ a_{i,j}^{(s)}, & \text{elsewhere;} \end{cases}$$

2. the vectors $\boldsymbol{z}_s = (z_i^{(s)}), \boldsymbol{w}_s = (w_i^{(s)}) \in \mathbb{C}^n$ and $\boldsymbol{f}_s \colon = U_s \boldsymbol{w}_s, \boldsymbol{f}_s = (f_i^{(s)})$, and $\boldsymbol{g}_s \colon = U_s^H \boldsymbol{z}_s, \boldsymbol{g}_s = (g_i^{(s)})$.

The nonzero pattern of the matrix \hat{A}_s looks as below:

or

From (3.2) and (3.4) we find that the entries of the matrix $A_s = (a_{i,j}^{(s)})$ can be expressed in terms of elements of this data set as follows:

$$a_{i,j}^{(s)} = \begin{cases} -\sigma^{-1} f_i^{(s)} \bar{g_j}^{(s)} - z_i^{(s)} \bar{w_j}^{(s)}, & \text{if } j \ge 2\lfloor \frac{i+1}{2} \rfloor + 3, \ 1 \le i \le 2\lfloor \frac{n+1}{2} \rfloor - 4; \\ \widehat{a}_{i,j}^{(s)}, & \text{elsewhere;} \end{cases}$$
(4.1)

where $\sigma = 1 - \boldsymbol{z}_s^H \boldsymbol{U}_s^H \boldsymbol{w}_s = 1 - \boldsymbol{z}_0^H \boldsymbol{U}_0^H \boldsymbol{w}_0 = 1 - \boldsymbol{e}_3^H \boldsymbol{p} = 1 - 1 - \frac{\bar{p}_0}{\bar{p}_n} = -\frac{\bar{p}_0}{\bar{p}_n}$. The next procedure performs a structured variant of the QR iteration (2.2) applied to an initial matrix $A_0 = \widehat{C} \in \mathbb{C}^{n \times n}$ given as in (2.3).

Procedure $Fast_QR$

Input: \widehat{A}_s , σ , \boldsymbol{z}_s , \boldsymbol{w}_s , \boldsymbol{f}_s , \boldsymbol{g}_s ;
Output: \widehat{A}_{s+1} , σ , \boldsymbol{z}_{s+1} , \boldsymbol{w}_{s+1} , \boldsymbol{f}_{s+1} , \boldsymbol{g}_{s+1} ;
1. Compute the shift ρ_s .

- 2. Find the factored form (3.1) of the matrix Q_s such that $Q_s^H(A_s-\rho_s I)=R_s, \quad R_s$ upper triangular, where A_s is represented via (4.1).
- 3. Determine \widehat{A}_{s+1} from the entries of $A_{s+1} = Q_s^H A_s Q_s$. 4. Evaluate $\boldsymbol{z}_{s+1} = Q_s^H \boldsymbol{z}_s$, $\boldsymbol{w}_{s+1} = Q_s^H \boldsymbol{w}_s$, $\boldsymbol{f}_{s+1} = Q_s^H \boldsymbol{f}_s$, $\boldsymbol{g}_{s+1} = Q_s^H \boldsymbol{g}_s$.

The factored form of Q_s makes it possible to execute the steps 2, 3 and 4 simultaneously by improving the efficiency of computation. The matrix A_s is represented by means of four vectors and a diagonally structured matrix A_s encompassing the band profile of A_s . This matrix could be stored in a rectangular array but for the sake of simplicity in our implementation we adopt the MatLab* sparse matrix format. Due to the occurrences of deflations the QR process is applied to a principal submatrix of A_s starting at position pst + 1 and ending at position n - qst, where pst = qst = 0at beginning. At the core of Fast_QR there is a structured adaptation of the QR iteration applied to $B = A_s(pst + 1 : n - qst, pst + 1 : n - qs) - \rho_s I_{n-pst-qst}$. In particular, we compute the Givens reflector $\mathcal{G}_1(\bar{\gamma}_1,\bar{\sigma}_1)$ of equation (3.1) based on the shit ρ_s computed in step 1, and we perform the similarity transformation

$$B_1 = \mathcal{G}_1(\bar{\gamma}_1, \bar{\sigma}_1)^H B \, \mathcal{G}_1(\bar{\gamma}_1, \bar{\sigma}_1).$$

This is done by using only the representation of B, that is the portion of the four vectors f, g, z, w with indices between pst + 1 and n - qst and $A_s(pst + 1 : n$ qst, pst + 1: n - qs), and acting only on the first two rows and columns of them.

Then, defining ndim = n - pst - qst the dimension of B and for $\ell = 2:2:$ 2*(|(ndim+1)/2|-2) we compute the matrices $\mathcal{G}_{\ell,\ell+1}$ as the unitary factor of a QR factorization of the 3×3 diagonal blocks $B_1(\ell : \ell + 2, \ell : \ell + 2)$, and updating B_1 as follows

$$B_1 = \mathcal{G}_{\ell,\ell+1}^H B_1 \, \mathcal{G}_{\ell,\ell+1}.$$

As in equation (3.1), the last unitary transformation \mathcal{G}_{n-1} is computed in a different way in the odd and in the even case.

Despite the simplicity of this scheme we have to deal carefully with the representation of B in order to update the banded matrix and the four generators.

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The computation of the shift ρ_s at the first step of Fast_QR can be carried out by several strategies [20]. In our implementation we employ the Wilkinson idea by choosing as a shift one of the roots of the trailing 2-by-2 submatrix of $A_s(pst+1:n-qst,pst+1:n-qs)$ (the one closest to the final entry). For an input companion matrix expressed as a rank-one correction of a unitary CMV-like shaped matrix this technique ensures zero shifting at the early iterations. It has been observed experimentally that this fact is important for the correct fill in both in the rank-two structure in the upper triangular part and in the band profile of A_s . Incorporating the Wilkinson shifting within the explicit shifted QR method Fast_QR and implementing a step of QR iteration on the representation as just described, yields our proposed fast CMV-based eigensolver for companion matrices. The algorithm has been implemented in MatLab and tested on several examples. This implementation can be obtained from the authors upon request.

In order to check the accuracy of the output we compare the computed approximations with the ones returned by the internal function eig applied to the initial companion matrix $C = C(p) \in \mathbb{C}^{n \times n}$ without the balance option. Specifically, we match the two lists of approximations and then find the average absolute error $err = \sum_{j=1}^{n} err_j/n$.

For a backward stable algorithm in the light of the classical perturbation results for eigenvalue computation [20] we know that this error would be of the order of $\|\Delta C\|_{\infty} \mathcal{K}_{\infty}(V) \varepsilon$, where $\|\Delta C\|_{\infty}$ is the backward error, $\mathcal{K}_{\infty}(V) = \|V\|_{\infty} \cdot \|V^{-1}\|_{\infty}$ is the condition number of V, the eigenvector matrix of C and ε denotes the machine precision. A backward stability analysis of the customary QR eigenvalue algorithm is performed in [23] by showing that $\|\Delta C\|_F \leq cn^3 \|C\|_F$ for a small integer constant c. A partial extension of this result to certain fast adaptations of the QR algorithm for rank–structured matrices is provided in [16] by replacing $\|C\|_F$ with a measure of the magnitude of the generators. The numerical experience reported in [9] further support this extension. In the present case we find that

$$||C||_{\infty} = ||A_0||_{\infty} \le ||\widehat{A}_0||_{\infty} + ||\sigma^{-1} f_0||_{\infty} ||g_0||_{\infty} + ||w_0||_{\infty} ||z_0||_{\infty}$$
$$= ||\widehat{A}_0||_{\infty} + ||\sigma^{-1} f_0||_{\infty} + ||w_0||_{\infty}.$$

The parameter $\sigma^{-1} = -\bar{p}_n/\bar{p}_0$ in the starting representation via generators should be incorporated into the vector \mathbf{f}_0 , leading to a vector whose entries depend on the ratios $\pm p_j/p_0$. Viceversa, the entries of vector \mathbf{w}_0 , depend on the ratios $\pm p_j/p_n$. When the coefficients of the polynomial p(z) are unbalanced, to keep trace of the possible unbalanced entries of both \mathbf{f}_0 or \mathbf{w}_0 , we may consider the maximum expected error as $nne = \left(\|\widehat{A}_0\|_{\infty} + \|\sigma^{-1}\mathbf{f}_0\|_{\infty} + \|\mathbf{w}_0\|_{\infty}\right) \mathcal{K}_{\infty}(V) \varepsilon$. Our implementation reports as output the value of werr = err/nne. In accordance with our claim this quantity would be bounded by a small multiple of n^3 .

As a measure of efficiency of the algorithm we also determine the average number averit of QR steps per eigenvalue.

We have performed many numerical experiments with real polynomials of both small and large degree. Moreover, to support our expectation about roundoff errors we consider several cases where the input polynomial is (anti)palindromic in such a way that $\|\sigma^{-1}\mathbf{f}_0\|_{\infty} = \|\mathbf{w}_0\|_{\infty}$. Our test suite consists of the following polynomials:

- way that $\|\sigma^{-1} f_0\|_{\infty} = \|w_0\|_{\infty}$. Our test suite consists of the following polynomials: • (P1) $p(z) = 1 + (\frac{n}{n+1} + \frac{n+1}{n})z^n + z^{2n}$ [8]. The zeros can be explicitly determined and lie on two circles centered at the origin that are poorly separated.
 - (P2) $p(z) = \frac{1}{n} \left(\sum_{j=0}^{n-1} (n+j)z^j + (n+1)z^n + \sum_{j=0}^{n-1} (n+j)z^{2n-j} \right)$ [10]. This

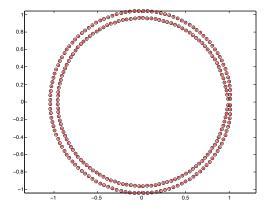


Figure 4.1: Distribution of the zeros computed by our routine (red plus) and eig (black circles) for the polynomial in the class P2 of degree n=128.

is another test problem for spectral factorization algorithms.

- (P3) $p(z) = (1 \lambda)z^{n+1} (\lambda + 1)z^n + (\lambda + 1)z (1 \lambda)$ [1]. This family of antipalindromic polynomials arises in the context of a boundary-value problem whose eigenvalues coincides with the zeros of an entire function related with p(z).
- (P4) A collection of small-degree polynomials [24]:
 - 1. the Bernoulli polynomial $p(z) = \sum_{j=0}^{n} \binom{n}{j} b_{n-j} z^{j}$, where b_{j} are the Bernoulli numbers;
 - 2. the Chebyshev polynomial of first kind;
- 3. the partial sum of the exponential $p(z) = \sum_{j=0}^{n} (2z)^j/j!$.

 (P5) Polynomials $p(z) = \sum_{j=0}^{n} p_j z^j$ with coefficients of the form $p_j = a_j \times 10^{e_j}$, where a_j and e_j are drawn from the uniform distribution in [-1,1]and [-3,3], respectively. These polynomials were proposed in [21] for testing purposes.
- (P6) The symmetrized version of the previous polynomials, that is, p(z) = $s(z)s(z^{-1})z^n$ where $s(z) = \sum_{j=0}^n s_j z^j$ with coefficients of the form $s_j = a_j \times 10^{e_j}$ and $a_j \in [-1,1]$ and $e_j \in [-3,3]$.

Table 4.1 shows the numerical results for the first three sets of symmetric polynomials. For the sake of illustration in Figure 4.1 and 4.2 we also display the distribution of the zeros computed by our routine and the MatLab function eig applied to polynomials in the class P2 and P3, respectively.

A certain degeneration of the accuracy of computed results can be observed in example P2 in Table 4.1, but this is within the bounds provided by the backward error analysis.

Table 4.2 shows the numerical results for the small degree polynomials P4. For the sake of illustration in Figure 4.3 and 4.4 we also display the distribution of the zeros computed by our routine and the MatLab function eig applied to polynomials in the class P4(1-2) and P4(3), respectively.

It is worth pointing out the loss of information in the Chebyshev case due to the usage of generators depending on the normalization for both the leading and the

Test Set Number	n	nne/ε	err	werr	averit
P1	64	4.14e+04	4.12e-14	7.47e-03	4.55
	128	1.65e + 05	1.16e-13	5.29 e-03	4.53
	256	6.57e + 05	2.84e-13	3.24 e-03	4.51
	512	2.62e + 06	8.87e-13	2.54e-03	4.51
	1024	1.05e + 07	2.61e-12	1.87e-03	4.51
P2	64	2.36e + 05	3.94e-12	7.52e-02	3.66
	128	1.62e + 06	1.03e-10	2.86e-01	3.44
	256	1.13e + 07	1.21e-09	4.84e-01	3.23
	512	8.01e + 07	2.73e-08	1.53e + 00	3.06
	1024	5.77e + 08	5.45 e-06	4.26e + 01	2.97
	64	1.10e+04	4.12e-15	1.69e-03	2.94
	128	2.20e + 04	1.07e-14	2.18e-03	2.67
$P3(\lambda = 0.9)$	256	4.41e + 04	2.83e-14	2.88e-03	2.57
	512	8.83e + 04	3.83e-14	1.96e-03	2.53
	1024	1.77e + 05	4.19e-14	1.07e-03	2.51
$P3(\lambda = 0.999)$	64	1.08e + 06	6.48e-15	2.71e-05	3.03
	128	2.16e + 06	9.97e-15	2.08e-05	2.71
	256	4.34e + 06	2.50e-14	2.59e-05	2.58
	512	8.68e + 06	3.66e-14	1.90e-05	2.54
	1024	1.74e + 07	4.25e-14	1.10e-05	2.52

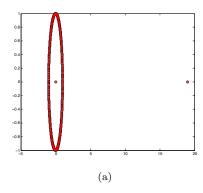
Table 4.1: Numerical results for the sets P1, P2 and P3 of (anti)palindromic polynomials

Test Set Number	n	nne/ε	err	werr	averit
	10	5.01e + 05	2.75e-14	2.47e-04	3.50
	20	1.34e + 13	2.47e-13	8.31e-11	3.50
P4(1)	30	5.94e + 25	2.02e-12	1.53e-22	3.77
	10	4.82e + 06	5.34e-12	4.99e-03	3.40
	20	1.69e + 14	3.52 e-05	9.41e-04	3.40
P4(2)	30	6.27e + 21	1.89e-01	1.36e-07	4.03
	10	2.93e + 08	3.12e-14	4.79e-07	3.20
	20	9.83e + 25	1.27e-11	5.81e-22	3.35
P4(3)	30	8.49e + 47	3.77e-08	2.00e-40	3.30

Table 4.2: Numerical results for the sets P4(1-3).

trailing coefficient of the polynomial. This is a potential drawback of our approach.

Table 4.3 finally gives the numerical results for the polynomials P5 and P6. Here we report for nne/ε the min/max range and for the other columns the maximum value of the data output variables over fifty experiments.



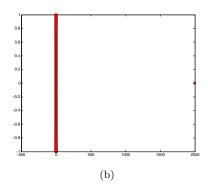
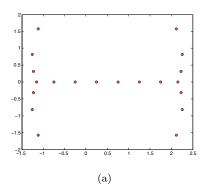


Figure 4.2: Distribution of the zeros computed by our routine (red plus) and eig (black circles) for the polynomials in the class P3 of degree n=128 with $\lambda \in \{0.9, 0.999\}$.



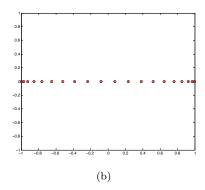
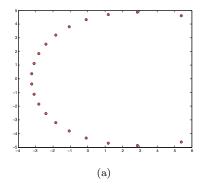


Figure 4.3: Distribution of the zeros of Bernoulli and Chebyshev polynomial of degree 20 computed by our routine (green diamonds) and eig (red circles).



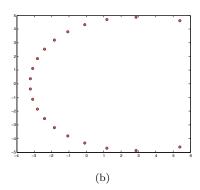


Figure 4.4: Distribution of the zeros of truncated Taylor series of e^{2z} of degree 20 and 30 computed by our routine (green diamonds) and eig (red circles).

Test Set Number	n	nne/arepsilon	err	werr	averit
	32	4.76e+05 - 1.49e+20	7.50e-03	1.94e-01	3.67
	64	2.87e + 03 - 3.66e + 19	5.33e-04	2.40e-03	3.65
P5	128	9.90e+04 - 7.43e+19	4.48e-03	1.47e-01	3.41
	16	2.55e+03 - 5.47e+19	1.71e-02	6.16e-03	3.53
	32	7.64e+04 - 2.49e+22	1.34e-02	9.48e-03	3.61
P6	64	1.08e+06 - 2.13e+20	4.76e-02	1.51e-02	3.42
	128	1.46e+07 - 6.96e+23	1.40 e-01	8.71e + 00	3.33

Table 4.3: Numerical results for the sets P5, P6.

5. Conclusion and Future Work. In this paper we have presented a novel fast QR-based eigensolver for companion matrices exploiting the structured technology for CMV-like representations. To our knowledge this is the first numerically reliable fast adaptation of the QR algorithm for perturbed unitary matrices which makes use of only four vectors to express the rank structure of the matrices generated under the iterative process. As a result, we obtain a data sparse parametrization of these matrices which at the same time is able to capture the structural properties of the matrices and yet to be sufficiently easy to manipulate and update for computations. Although very promising, some numerical issues associated with the proposed approach are still under investigation. The first one is a certain sensibility of the algorithm in the initial steps where the band profile of the matrix is filled using the information propagated from the polynomial coefficients. The second issue is concerned with the magnitude of the generator vectors depending on the normalization for both the leading and the trailing coefficient of the polynomial. Both these problems can be circumvented by using different representations of the rank-two structure. Finding the right balance between robustness and efficiency is the main subject of our current research.

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