# DYNAMIC PROGRAMMING FOR GENERAL LINEAR QUADRATIC OPTIMAL STOCHASTIC CONTROL WITH RANDOM COEFFICIENTS * 

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#### Abstract

We are concerned with the linear-quadratic optimal stochastic control problem where all the coefficients of the control system and the running weighting matrices in the cost functional are allowed to be predictable (but essentially bounded) processes and the terminal state-weighting matrix in the cost functional is allowed to be random. Under suitable conditions, we prove that the value field $V(t, x, \omega),(t, x, \omega) \in[0, T] \times R^{n} \times \Omega$, is quadratic in $x$, and has the following form: $V(t, x)=\left\langle K_{t} x, x\right\rangle$ where $K$ is an essentially bounded nonnegative symmetric matrix-valued adapted processes. Using the dynamic programming principle (DPP), we prove that $K$ is a continuous semimartingale of the form


$$
K_{t}=K_{0}+\int_{0}^{t} d k_{s}+\sum_{i=1}^{d} \int_{0}^{t} L_{s}^{i} d W_{s}^{i}, \quad t \in[0, T]
$$

with $k$ being a continuous process of bounded variation and

$$
E\left[\left(\int_{0}^{T}\left|L_{s}\right|^{2} d s\right)^{p}\right]<\infty, \quad \forall p \geq 2 ;
$$

and that $(K, L)$ with $L:=\left(L^{1}, \cdots, L^{d}\right)$ is a solution to the associated backward stochastic Riccati equation (BSRE), whose generator is highly nonlinear in the unknown pair of processes. The uniqueness is also proved via a localized completion of squares in a self-contained manner for a general BSRE. The existence and uniqueness of adapted solution to a general BSRE was initially proposed by the French mathematician J. M. Bismut [in SIAM J. Control $\mathcal{G}$ Optim., 14(1976), pp. 419-444, and in Séminaire de Probabilités XII, Lecture Notes in Math. 649, C. Dellacherie, P. A. Meyer, and M. Weil, eds., Springer-Verlag, Berlin, 1978, pp. 180-264], and subsequently listed by Peng [in Control of Distributed Parameter and Stochastic Systems (Hangzhou, 1998), S. Chen, et al., eds., Kluwer Academic Publishers, Boston, 1999, pp. 265-273] as the first open problem for backward stochastic differential equations. It had remained to be open until a general solution by the author [in SIAM J. Control 8 Optim., 42(2003), pp. 53-75] via the stochastic maximum principle with a viewpoint of stochastic flow for the associated stochastic Hamiltonian system. The present paper is its companion, and gives the second but more comprehensive (seemingly much simpler, but appealing to the advanced tool of Doob-Meyer decomposition theorem, in addition to the DDP) adapted solution to a general BSRE via the DDP. Further extensions to the jump-diffusion control system and to the general nonlinear control system are possible.

Key words. linear quadratic optimal stochastic control, random coefficients, Riccati equation, backward stochastic differential equations, dynamic programming, semi-martingale

AMS subject classifications. 93E20, 49K45, 49N10, 60H10

1. Formulation of the problem and basic assumptions. Consider the following linear quadratic optimal stochastic control (SLQ in short form) problem: minimize over $u \in \mathscr{L}_{\mathscr{F}}^{2}\left(0, T ; \mathbb{R}^{m}\right)$ the following quadratic cost functional:

$$
\begin{equation*}
J(u ; 0, x):=E^{0, x ; u}\left[\left\langle M X_{T}, X_{T}\right\rangle+\int_{0}^{T}\left(\left\langle Q_{s} X_{s}, X_{s}\right\rangle+\left\langle N_{s} u_{s}, u_{s}\right\rangle\right) d s\right] \tag{1.1}
\end{equation*}
$$

[^0]where $X$ is the solution of the following linear stochastic control system:
\[

\left\{$$
\begin{align*}
d X_{t} & =\left(A_{t} X_{t}+B_{t} u_{t}\right) d t+\sum_{i=1}^{d}\left(C_{t}^{i} X_{t}+D_{t}^{i} u_{t}\right) d W_{t}^{i}  \tag{1.2}\\
X_{0} & =x \in \mathbb{R}^{n}
\end{align*}
$$\right.
\]

Here, $\left\{W_{t}:=\left(W_{t}^{1}, \ldots, W_{t}^{d}\right)^{\prime}, 0 \leq t \leq T\right\}$ is a $d$-dimensional standard Brownian motion defined on some probability space $(\Omega, \mathscr{F}, P)$. Denote by $\left\{\mathscr{F}_{t}, 0 \leq t \leq T\right\}$ the augmented natural filtration of the standard Brownian motion $W$. The control $u$ belongs to the Banach space $\mathscr{L}_{\mathscr{F}}^{2}\left(0, T ; \mathbb{R}^{m}\right)$, which consists of all $\mathbb{R}^{m}$-valued square integrable $\left\{\mathscr{F}_{t}, 0 \leq t \leq T\right\}$-adapted processes. Denote by $\mathbb{S}^{n}$ the totality of $n \times n$ symmetric matrices, and by $\mathbb{S}_{+}^{n}$ the totality of $n \times n$ nonnegative matrices.

Throughout this paper, we make the following two assumptions on the coefficients of the above problem.
(A1) Assume that the matrix processes $A:[0, T] \times \Omega \rightarrow \mathbb{R}^{n \times n}, B:[0, T] \times \Omega \rightarrow$ $\mathbb{R}^{n \times m} ; C^{i}:[0, T] \times \Omega \rightarrow \mathbb{R}^{n \times n}, D^{i}:[0, T] \times \Omega \rightarrow \mathbb{R}^{n \times m}, i=1, \ldots, d ; Q:[0, T] \times \Omega \rightarrow$ $\mathbb{S}_{+}^{n}, N:[0, T] \times \Omega \rightarrow \mathbb{S}_{+}^{m}$ and the random matrix $M: \Omega \rightarrow \mathbb{S}_{+}^{n}$ are uniformly bounded and $\left\{\mathcal{F}_{t}, 0 \leq t \leq T\right\}$-adapted or $\mathcal{F}_{T}$-measurable.
(A2) Assume that the control weighting matrix process $N$ is uniformly positive.
Define for $(t, K, L) \in[0, T] \times \mathbb{S}^{n} \times\left(\mathbb{S}^{n}\right)^{d}$,

$$
\begin{align*}
\mathscr{N}_{t}(K) & :=N_{t}+\sum_{i=1}^{d}\left(D_{t}^{i}\right)^{\prime} K D_{t}^{i}  \tag{1.3}\\
\mathscr{M}_{t}(K, L) & :=K B_{t}+\sum_{i=1}^{d}\left(C_{t}^{i}\right)^{\prime} K D_{t}^{i}+\sum_{i=1}^{d} L^{i} D_{t}^{i}
\end{align*}
$$

For $(t, K) \in[0, T] \times \mathbb{S}_{+}^{n}$ and $L=\left(L^{1}, \ldots, L^{d}\right) \in\left(\mathbb{S}^{n}\right)^{d}$, define

$$
\begin{align*}
& G(t, K, L):=A_{t}^{\prime} K+K A_{t}+Q_{t}+\sum_{i=1}^{d}\left(C_{t}^{i}\right)^{\prime} K C_{t}^{i}+\sum_{i=1}^{d}\left[\left(C_{t}^{i}\right)^{\prime} L^{i}+L^{i} C_{t}^{i}\right]  \tag{1.4}\\
&-\mathscr{M}_{t}(K, L) \mathscr{N}_{t}^{-1}(K) \mathscr{M}_{t}^{\prime}(K, L)
\end{align*}
$$

Here, we use the prime to denote the transpose of a vector or a matrix. Associated to the above SLQ problem is the following backward stochastic Riccati equation (BSRE):

$$
\left\{\begin{array}{l}
d K_{t}=-G\left(t, K_{t}, L_{t}\right) d t+\sum_{i=1}^{d} L_{t}^{i} d W_{t}^{i}, \quad t \in[0, T)  \tag{1.5}\\
K_{T}=M, \quad L_{t}:=\left(L_{t}^{1}, \ldots, L_{t}^{d}\right)
\end{array}\right.
$$

The generator is highly nonlinear in the unknown pair of variables $(K, L)$.
Definition 1.1. A solution of BSRDE (1.5) is defined as a pair $(K, L)$ of matrixvalued adapted processes such that
(i) $\int_{0}^{T}\left|L_{t}\right|^{2} d t+\int_{0}^{T}\left|G\left(t, K_{t}, L_{t}\right)\right| d t<\infty$, a.s.;
(ii) The $m \times m$ matrix-valued process $\left\{\mathscr{N}_{t}\left(K_{t}\right), t \in[0, T]\right\}$ is a.s.a.e. positive; and
(iii) $K_{t}=M+\int_{t}^{T} G\left(s, K_{s}, L_{s}\right) d s-\int_{t}^{T} \sum_{i=1}^{d} L_{s}^{i} d W_{s}^{i}$ a.s. for all $t \in[0, T]$.

The adapted solution to a general BSRE (1.5) was initially proposed by the French mathematician J. M. Bismut [1, 2], and subsequently listed by Peng [17] as the first open problem for backward stochastic differential equations. It had remained to be open until a general solution by the author [20] via the stochastic maximum principle and using a viewpoint of stochastic flow for the associated stochastic Hamiltonian system. For more details on the historical studies on BSRE (1.5) and the progress, see the author's previous paper [20, Section 4, pages 60-61] and the plenary lecture by Peng [18] at the International Congress of Mathematicians in 2010. In the paper, we shall give a novel proof to the existence for BSRE (1.5) via dynamic programming principle. A crucial point is that we can show the value field is a semi-martingale of both "sufficiently good" parts of bounded variation and martingale.

The rest of our paper is organized as follows. Section 2 gives preliminaries. In Section 3, we prove that the value field $V(t, x, \omega)$ is quadratic in $x$. In Section 4, we prove that the value field is a semi-martingale and that BSRE (1.5) has an adapted solution. Section 5 is concerned with a verification theorem for the SLQ problem, and the uniqueness of solution to BSRE (1.5). Finally, in Section 6, we give some comments and possible extensions.
2. Preliminaries. For each $u \in \mathscr{L}_{\mathscr{F}}^{2}\left(0, T ; \mathbb{R}^{m}\right)$, the following linear stochastic differential equation

$$
\left\{\begin{align*}
d X_{t} & =\left(A_{t} X_{t}+B_{t} u_{t}\right) d t+\sum_{i=1}^{d}\left(C_{t}^{i} X_{t}+D_{t}^{i} u_{t}\right) d W_{t}^{i}, \quad \tau \leq t \leq T  \tag{2.1}\\
X_{s} & =x \in \mathbb{R}^{n}
\end{align*}\right.
$$

has a unique strong solution (see Bismut [2]), denoted by $X^{s, x ; u}$ with the superscripts indicating the dependence on the initial data $(s, x)$ and the control action. We have the following well-known quantitative dependence of the solution $X^{s, x ; u}$ on the initial data $(s, x)$ and the control action $u$.

Lemma 2.1. Let assumption (A1) be satisfied. For any $p \geq 1$, there is a positive constant $C_{p}$ such that for any initial state $\xi \in L^{p}\left(\Omega, \mathscr{F}_{s}, P ; \mathbb{R}^{n}\right)$ and predictable control u with

$$
E\left[\left(\int_{s}^{T}\left|u_{r}\right|^{2} d r\right)^{p / 2}\right]<\infty
$$

we have

$$
\begin{equation*}
E\left[\max _{t \in[s, T]}\left|X^{s, \xi ; u}\right|^{p} \mid \mathscr{F}_{s}\right] \leq C_{p}\left(|\xi|^{p}+E\left[\left(\int_{s}^{T}\left|u_{r}\right|^{2} d r\right)^{p / 2} \mid \mathscr{F}_{s}\right]\right) \tag{2.2}
\end{equation*}
$$

Consider the initial-data-parameterized SLQ problem: minimize over $u \in \mathscr{L}_{\mathscr{F}}^{2}\left(0, T ; \mathbb{R}^{m}\right)$ the quadratic cost functional

$$
\begin{equation*}
J(u ; s, x):=E^{s, x ; u}\left[\left\langle M X_{T}, X_{T}\right\rangle+\int_{s}^{T}\left(\left\langle Q_{r} X_{r}, X_{r}\right\rangle+\left\langle N_{r} u_{r}, u_{r}\right\rangle\right) d r \mid \mathscr{F}_{s}\right] \tag{2.3}
\end{equation*}
$$

Define the value field

$$
\begin{equation*}
V(s, x):=\underset{u \in \mathscr{L}_{\mathscr{F}_{t}}^{2}\left(s, T ; \mathbb{R}^{m}\right)}{\operatorname{ess} . \inf } J(u ; s, x), \quad(s, x) \in[0, T] \times \mathbb{R}^{n} \tag{2.4}
\end{equation*}
$$

Assumptions (A1) and (A2) imply that the above SLQ problem has a unique optimal control for any $\xi \in L^{2}\left(\Omega, \mathscr{F}_{s}, P ; \mathbb{R}^{n}\right)$, that is, there is unique $\bar{u} \in \mathscr{U}_{s}$ such that

$$
V(s, \xi)=J(\bar{u} ; s, \xi)
$$

See Bismut [2] for the proof of such a result. A further step is to characterize the optimal control.

We easily prove the following
Lemma 2.2. Let Assumptions (A1) and (A2) be satisfied. There is a positive constant $\lambda$ such that

$$
0 \leq V(s, \xi) \leq J(0 ; s, \xi) \leq \lambda|\xi|^{2}, \quad \forall(s, \xi) \in[0, T] \times L^{2}\left(\Omega, \mathscr{F}_{s}, P ; \mathbb{R}^{n}\right)
$$

Proof. In view of assumption (A1) and the definition of the value field $V$, it is sufficient to show $J(0 ; s, \xi) \leq \lambda|\xi|^{2}$, which is an immediate consequence of Lemma 2.2 and the following estimate:

$$
\begin{aligned}
J(0 ; s, \xi) & \leq \lambda E\left[\left|X_{T}^{0, \xi ; 0}\right|^{2}+\int_{0}^{T}\left|X_{t}^{0, \xi ; 0}\right|^{2} d t \mid \mathscr{F}_{s}\right] \\
& \leq \lambda(1+T) E\left[\max _{t \in[0, T]}\left|X_{t}^{0, \xi ; 0}\right|^{2} \mid \mathscr{F}_{s}\right]
\end{aligned}
$$

3. The value field $V$ is quadratic in the space variable. This section is an adaptation of Faurre [4] to our SLQ problem with random coefficients.

We have
Theorem 3.1. Let Assumptions (A1) and (A2) be satisfied. The value field $V(s, x)$ is quadratic in $x$. Moreover, there is an essentially bounded continuous nonnegative matrix-valued process $K$ such that

$$
\begin{equation*}
V(s, x)=\left\langle K_{s} x, x\right\rangle, \quad \forall(s, x) \in[0, T] \times \mathbb{R}^{n} \tag{3.1}
\end{equation*}
$$

The state-quadratic property follows from the following lemma.
Lemma 3.2. Let Assumptions (A1) and (A2) be satisfied. The value field has the following two laws in the state variable $x$ of (i) square homogeneity

$$
V(s, \xi x)=\xi^{2} V(s, x), \quad \forall(s, x, \xi) \in[0, T] \times \mathbb{R}^{n} \times L^{\infty}\left(\Omega, \mathscr{F}_{s}, P\right)
$$

and (ii) parallelogram

$$
V(s, x+y)+V(s, x-y)=2 V(s, x)+2 V(s, y), \quad \forall(s, x, y) \in[0, T] \times \mathbb{R}^{n} \times \mathbb{R}^{n}
$$

Proof. It is easy to derive from the linearity of the control system and the quadratic structure of the cost functional the following two identities for any $u \in \mathscr{U}_{s}$,

$$
\xi X^{s, x ; u}=X^{s, \xi x ; \xi u}, \quad \xi^{2} J(u ; s, x)=J(\xi u ; s, \xi x)
$$

Therefore, we have

$$
\xi^{2} V(s, x)=\xi^{2} \underset{u \in \mathscr{\mathscr { U }}_{s}}{\operatorname{ess} . \inf } J(u ; s, x)=\underset{u \in \mathscr{\mathscr { U }}_{s}}{\operatorname{ess} . \inf } \xi^{2} J(u ; s, x)=\underset{u \in \mathscr{\mathscr { U }}_{s}}{\operatorname{ess} . \inf } J(\xi u ; s, \xi x)
$$

which is equal to $V(s, \xi x)$ by definition, immediately giving assertion (i).
Let us show assertion (ii). It is easy to see (see Bismut [2]) that there are $\alpha, \beta \in \mathscr{U}_{s}$ such that

$$
V(s, x+y)=J(\alpha ; s, x+y), \quad V(s, x-y)=J(\beta ; s, x-y)
$$

Then, we easily see that

$$
V(s,(x+y) \pm(x-y)) \leq J(\alpha \pm \beta ; s,(x+y) \pm(x-y))
$$

and therefore,

$$
V(s, 2 x)+V(s, 2 y) \leq J(\alpha+\beta ; s, 2 x)+J(\alpha-\beta ; s, 2 y)
$$

Since $J(u ; s, x)$ is quadratic in the pair $(u, x)$ and satisfies the parallelogram

$$
2 J(\alpha+\beta ; s, 2 x)+2 J(\alpha-\beta ; s, 2 y)=J(2 \alpha ; s, 2(x+y))+J(2 \beta ; s, 2(x-y))
$$

we have

$$
V(s, 2 x)+V(s, 2 y) \leq \frac{1}{2}[J(2 \alpha ; s, 2(x+y))+J(2 \beta ; s, 2(x-y))]
$$

and therefore by the square homogeneity of $J(u ; s, x)$ in the pair $(u, x)$

$$
V(s, x+y)+V(s, x-y) \leq 2 J(\alpha ; s, x+y)+2 J(\beta ; s, x-y)=2 V(s, x)+2 V(s, y)
$$

By symmetry, it holds for $x^{\prime}:=x+y$ and $y^{\prime}:=x-y$ :

$$
V(s,(x+y)+(x-y))+V(s,(x+y)-(x-y)) \leq 2 V(s, x+y)+2 V(s, x-y)
$$

which leads by assertion (i) to the following desired reverse inequality

$$
4 V(s, x)+4 V(s, y)=V(s, 2 x)+V(s, 2 y) \leq 2 V(s, x+y)+2 V(s, x-y)
$$

The proof is then complete.
The nonnegativity and the essential bound of the process $K$ are immediate consequences of Lemma 2.2.
4. Dynamic programming principle and the semi-martingale property of the value field. For simplicity, define the function

$$
\begin{equation*}
l(t, x, u):=\left\langle Q_{t} x, x\right\rangle+\left\langle N_{t} u, u\right\rangle, \quad(t, x, u) \in[0, T] \times \mathbb{R}^{n} \times \mathbb{R}^{m} \tag{4.1}
\end{equation*}
$$

and the set

$$
\begin{equation*}
\mathscr{U}_{s}:=\mathscr{L}_{\mathscr{F}}^{2}\left(s, T ; \mathbb{R}^{m}\right) . \tag{4.2}
\end{equation*}
$$

We denote by $\mathbb{V}(t, \cdot)$ the restriction of $V(t, \cdot)$ to $\mathbb{R}^{n}$. By definition, we have almost surely

$$
V(t, x)=\mathbb{V}(t, x), \quad \forall x \in \mathbb{R}^{n}
$$

For any $\xi \in L^{2}\left(\Omega, \mathscr{F}_{t}, P ; \mathbb{R}^{n}\right)$, in an analogous way to the proof of Peng [16, Lemma 6.5 , page 122], we also have almost surely

$$
V(t, \xi)=\mathbb{V}(t, \xi)
$$

We have
Theorem 4.1. (Bellman's Principle). Let Assumptions (A1) and (A2) be satisfied. We have
(i) For $s \leq t \leq T$ and $\xi \in L^{2}\left(\Omega, \mathscr{F}_{s}, P ; \mathbb{R}^{n}\right)$,

$$
\mathbb{V}(s, \xi)=\text { ess. } \inf _{u \in \mathscr{U}_{s}} E^{s, \xi ; u}\left\{\int_{s}^{t} l\left(r, X_{r}, u_{r}\right) d r+\mathbb{V}\left(t, X_{t}\right) \mid \mathscr{F}_{s}\right\}
$$

For the optimal control $\bar{u} \in \mathscr{U}_{s}$, we have

$$
\mathbb{V}(s, \xi)=E^{s, \xi ; \bar{u}}\left\{\int_{s}^{t} l\left(r, X_{r}, \bar{u}_{r}\right) d r+\mathbb{V}\left(t, X_{t}\right) \mid \mathscr{F}_{s}\right\} .
$$

(ii) For $(s, x, u) \in[0, T] \times \mathbb{R}^{n} \times \mathscr{U}_{s}$, the process

$$
\kappa_{t}^{s, x ; u}:=\mathbb{V}\left(t, X_{t}^{s, x ; u}\right)+\int_{s}^{t} l\left(r, X_{r}^{s, x ; u}, u_{r}\right) d r
$$

defined for $t \in[s, T]$, is a submartingale w.r.t. $\left\{\mathscr{F}_{t}\right\}$; and for the optimal control $\bar{u} \in \mathscr{U}_{s}$, the process $\kappa_{t}^{s, x ; \bar{u}}, t \in[s, T]$, is a martingale w.r.t. $\left\{\mathscr{F}_{t}\right\}$.

Proof. It is easy to check that Assertion (ii) is an immediate consequence of Assertion (i). Assertion (i) is more or less standard, and the proof is similar to that of Krylov [12, Theorem 6, Section 3, Chapter 3, page 150] or Peng [16, Theorem 6.6, page 123].

From assertion (i), we have
Corollary 4.2. We have the following time continuity of $\mathbb{V}$ and $K$ : for any $(s, x) \in[0, T] \times \mathbb{R}^{n}$,

$$
\lim _{t \rightarrow s} E\left[\mathbb{V}(t, x)-\mathbb{V}(s, x) \mid \mathscr{F}_{s}\right]=0, \quad \lim _{t \rightarrow s} E\left[K_{t}-K_{s} \mid \mathscr{F}_{s}\right]=0, \quad \text { a.s.. }
$$

Proof. In view of Theorem 3.1, the second limit easily follows from the first one. It remains to prove the first limit.

Assume without loss of generality that $s \leq t$. We have

$$
\mathbb{V}(s, x)=E^{s, x ; \bar{u}}\left\{\int_{s}^{t} l\left(r, X_{r}, \bar{u}_{r}\right) d r+\mathbb{V}\left(t, X_{t}\right) \mid \mathscr{F}_{s}\right\}
$$

where $\bar{u} \in \mathscr{U}_{s}$ is the optimal control. Therefore,

$$
\left|E\left[\mathbb{V}(t, x)-\mathbb{V}(s, x) \mid \mathscr{F}_{s}\right]\right| \leq E^{s, x ; \bar{u}}\left\{\int_{s}^{t} l\left(r, X_{r}, \bar{u}_{r}\right) d r+\left|\mathbb{V}\left(t, X_{t}\right)-\mathbb{V}(t, x)\right| \mid \mathscr{F}_{s}\right\}
$$

Since

$$
\left|\mathbb{V}\left(t, X_{t}^{s, x ; \bar{u}}\right)-\mathbb{V}(t, x)\right| \leq \lambda\left(|x|+\left|X_{t}^{s, x ; \bar{u}}\right|\right)\left|X_{t}^{s, x ; \bar{u}}-x\right|,
$$

using estimate (2.2), we have

$$
\begin{aligned}
& \left|E\left[\mathbb{V}(t, x)-\mathbb{V}(s, x) \mid \mathscr{F}_{s}\right]\right| \leq \lambda E^{s, x ; \bar{u}}\left\{\int_{s}^{t}\left(\left|X_{r}\right|^{2}+\left|\bar{u}_{r}\right|^{2}\right) d r \mid \mathscr{F}_{s}\right\} \\
& \quad+\lambda\left\{|x|+E^{s, x ; \bar{u}}\left[\left(\int_{s}^{t}\left|\bar{u}_{r}\right|^{2} d r\right)^{1 / 2} \mid \mathscr{F}_{s}\right]\right\} E^{s, x ; \bar{u}}\left[\left(\int_{s}^{t}\left|\bar{u}_{r}\right|^{2} d r\right)^{1 / 2} \mid \mathscr{F}_{s}\right]
\end{aligned}
$$

which implies the desired limit.
Using Theorems 3.1 and 4.1, we can prove the following
ThEOREM 4.3. The value field $V$ is a semi-martingale of the following representation:

$$
\begin{equation*}
\mathbb{V}(t, x)=\left\langle K_{t} x, x\right\rangle \tag{4.3}
\end{equation*}
$$

where $K$ is an essentially bounded nonnegative symmetric matrix-valued continuous semi-martingale of the form

$$
\begin{equation*}
K_{t}=K_{0}-\int_{0}^{t} d k_{s}+\sum_{i=1}^{d} \int_{0}^{t} L_{s}^{i} d W_{s}^{i}, \quad t \in[0, T] ; \quad K_{T}=M \tag{4.4}
\end{equation*}
$$

with $k$ being an $n \times n$ atrix-valued continuous process of bounded variation such that

$$
\begin{equation*}
d k_{s}=G\left(s, K_{s}, L_{s}\right) d s, \quad \text { almost everywhere }(s, \omega) \in[0, T] \times \Omega \tag{4.5}
\end{equation*}
$$

and

$$
\begin{equation*}
E\left[\left(\int_{0}^{T}\left|L_{s}\right|^{2} d s\right)^{p}\right]<\infty, \quad \forall p \geq 2 \tag{4.6}
\end{equation*}
$$

Proof. Theorem 3.1 states that there is an essentially bounded nonnegative symmetric matrix-valued process $K$ such that (4.3) holds true. The rest of the proof is divided into the following three steps.

Step 1. $K$ is a semi-martingale of form (4.4) in the Doob-Meyer decomposition. Let $e_{i}$ be the unit column vector of $\mathbb{R}^{n}$ whose $i$-th component is the number 1 for $i=1, \ldots, n$. In view of Assertion (ii) of Theorem 4.1, we see that for $x=e_{i}, e_{i}+e_{j}, e_{i}-e_{j}, i, j=1, \ldots, n,\left\{\kappa_{t}^{0, x ; 0}, t \in[0, T]\right\}$ is a sub-martingale, and since

$$
\left|\kappa_{t}^{0, x ; 0}\right| \leq \lambda\left|X_{t}^{0, x ; 0}\right|^{2}+\int_{0}^{t}\left|X_{s}^{0, x ; 0}\right|^{2} d s \leq \lambda \max _{t \in[0, T]}\left|X_{t}^{0, x ; 0}\right|^{2} \in L^{1}\left(\Omega, \mathscr{F}_{T}, P\right)
$$

it is of class $D$. Since $V(t, x)$ is continuous in the sense of conditional mean in $t$ (see corollary 4.2), $\left\{\kappa_{t}^{0, x ; 0}, t \in[0, T]\right\}$ is continuous in the sense of conditional mean in $s$. In view of Doob-Meyer decomposition (see Protter [19, Theorem 11, page 112]), its bounded variational process is continuous and increasing in time, and $\left\{\kappa_{t}^{0, x ; 0}, t \in\right.$ $[0, T]\}$ is sample continuous. Define the $n \times n$ symmetric matrix-valued process

$$
\begin{equation*}
\Gamma_{t}:=\left(\kappa_{t}(i, j)\right)_{1 \leq i, j \leq n} \tag{4.7}
\end{equation*}
$$

where

$$
\begin{equation*}
\kappa_{t}(i, i):=\kappa_{t}^{0, e_{i} ; 0}, \quad \kappa_{t}(i, j):=\frac{1}{4}\left[\kappa_{t}^{0, e_{i}+e_{j} ; 0}-\kappa_{t}^{0, e_{i}-e_{j} ; 0}\right], \quad 1 \leq i \neq j \leq n . \tag{4.8}
\end{equation*}
$$

It is a $n \times n$ matrix-valued semi-martingale and the bounded variational process in the Doob-Meyer decomposition is continuous in time. Define

$$
\Phi_{t}:=\left(X_{t}^{0, e_{1} ; 0}, \cdots, X_{t}^{0, e_{n} ; 0}\right), \quad t \in[0, T]
$$

Then, we have

$$
\begin{equation*}
\Gamma_{t}=\Phi_{t}^{\prime} K_{t} \Phi_{t}+\int_{0}^{t} \Phi_{r}^{\prime} Q_{r} \Phi_{r} d r, \quad t \in[0, T] \tag{4.9}
\end{equation*}
$$

and $\Phi$ satisfies the following matrix-valued stochastic differential equation (SDE):

$$
\begin{equation*}
d \Phi_{t}=A_{t} \Phi_{t} d t+C_{t}^{i} \Phi_{t} d W_{t}^{i}, \quad t \in(0, T] ; \quad \Phi_{0}=I_{n} \tag{4.10}
\end{equation*}
$$

It is well-known that $\Phi_{t}$ has an inverse $\Psi_{t}:=\Phi_{t}^{-1}$, satisfying the following SDE:

$$
\begin{equation*}
d \Psi_{t}=\Psi_{t}\left(-A_{t}+C_{t}^{i} C_{t}^{i}\right) d t-\Psi_{t} C_{t}^{i} d W_{t}^{i}, \quad t \in(0, T] ; \quad \Psi_{0}=I_{n} \tag{4.11}
\end{equation*}
$$

Therefore, we have

$$
\begin{equation*}
K_{t}=\Psi_{t}^{\prime}\left(\Gamma_{t}-\int_{0}^{t} \Phi_{r}^{\prime} Q_{r} \Phi_{r} d r\right) \Psi_{t}, \quad t \in[0, T] \tag{4.12}
\end{equation*}
$$

Since $\Gamma$ is a semi-martingale, using Itô-Wentzell formula, we see that $K$ is a semimartingale of form (4.4) from the Doob-Meyer decomposition, with the bounded variational process $k$ being continuous in time. It remains to derive the formula (4.5) for $k$ and the estimate (4.6) for $L$.

Step 2. Formula for the bounded variational process $k$. Define the function:

$$
\begin{align*}
F(t, x, v ; K, L)= & 2\left\langle K x, A_{t} x+B_{t} v\right\rangle+2\left\langle L^{i} x, C_{t}^{i} x+D_{t}^{i} v\right\rangle  \tag{4.13}\\
& +\left\langle L^{i}\left(C_{t}^{i} x+D_{t}^{i} v\right), C_{t}^{i} x+D_{t}^{i} v\right\rangle
\end{align*}
$$

for $(t, x, v, K, L) \in[0, T] \times \mathbb{R}^{n} \times \mathbb{R}^{m} \times \mathbb{S}^{n} \times\left(\mathbb{S}^{n}\right)^{m}$. Using Itô-Wentzell formula, we have

$$
\left\{\begin{align*}
d V\left(t, X_{t}^{0, x ; v}\right)= & {\left[-\left\langle d k_{t} X_{t}^{0, x ; v}, X_{t}^{0, x ; v}\right\rangle+F\left(t, X_{t}^{0, x ; v}, v ; K_{t}, L_{t}\right) d t\right] } \\
& +\left[\left\langle K_{t}\left(C_{t}^{i} X_{t}^{0, x ; v}+D_{t}^{i} v\right), X_{t}^{0, x ; v}\right\rangle\right. \\
& +\left\langle K_{t} X_{t}^{0, x ; v},\left(C_{t}^{i} X_{t}^{0, x ; v}+D_{t}^{i} v\right)\right\rangle  \tag{4.14}\\
& \left.\quad+\left\langle L_{t}^{i} X_{t}^{0, x ; v}, X_{t}^{0, x ; v}\right\rangle\right] d W_{t}^{i}, \quad t \in[0, T) \\
V\left(T, X_{T}^{0, x ; v}\right)= & \left\langle M X_{T}^{0, x ; v}, X_{T}^{0, x ; v}\right\rangle
\end{align*}\right.
$$

and

$$
\begin{align*}
\kappa_{t}^{0, x ; v}= & \left\langle K_{0} x, x\right\rangle+\int_{0}^{t}\left[-\left\langle d k_{s} X_{s}^{0, x ; v}, X_{s}^{0, x ; v}\right\rangle+F\left(s, X_{s}^{0, x ; v}, v ; K_{s}, L_{s}\right) d s\right. \\
& \left.+l\left(s, X_{s}^{0, x ; v}, v\right) d s\right]+\int_{0}^{t}\left[\left\langle K_{s}\left(C_{s}^{i} X_{s}^{0, x ; v}+D_{s}^{i} v\right), X_{s}^{0, x ; v}\right\rangle\right.  \tag{4.15}\\
& \left.+\left\langle K_{s} X_{s}^{0, x ; v},\left(C_{s}^{i} X_{s}^{0, x ; v}+D_{s}^{i} v\right)\right\rangle+\left\langle L_{s}^{i} X_{s}^{0, x ; v}, X_{s}^{0, x ; v}\right\rangle\right] d W_{s}^{i}, \quad t \in[0, T] .
\end{align*}
$$

Assertion (ii) of Theorem 4.1 states that $\left\{\kappa_{t}^{0, x ; v}, t \in[0, T]\right\}$ is a sub-martingale for any $(v, x) \in \mathbb{R}^{m} \times \mathbb{R}^{n}$, yielding the following fact: for any $(x, v) \in \mathbb{R}^{n} \times \mathbb{R}^{m}$, we have $E \int_{0}^{T} \eta(s, x) \gamma(d s, x ; v) \leq 0$ for any essentially bounded nonnegative predictable process $\eta$ on $[0, T] \times \Omega$, where

$$
\begin{gather*}
\gamma(d s, x ; v):=-\left\langle d k_{s} X_{s}^{0, x ; v}, X_{s}^{0, x ; v}\right\rangle+F\left(s, X_{s}^{0, x ; v}, v ; K_{s}, L_{s}\right) d s  \tag{4.16}\\
+l\left(s, X_{s}^{0, x ; v}, v\right) d s
\end{gather*}
$$

and for the optimal control $\bar{u} \in \mathscr{U}_{0}$, the process $\kappa_{t}^{s, x ; \bar{u}}, t \in[s, T]$, is a martingale w.r.t. $\left\{\mathscr{F}_{t}\right\}$, yielding the following fact: for any $x \in \mathbb{R}^{n}$, we have $E \int_{0}^{T} \eta(s, x) \gamma(d s, x ; \bar{u})=0$ for any essentially bounded nonnegative predictable process $\eta$ on $[0, T] \times \Omega$, where

$$
\begin{gather*}
\gamma(d s, x ; \bar{u}):=-\left\langle d k_{s} X_{s}^{0, x ; \bar{u}}, X_{s}^{0, x ; \bar{u}}\right\rangle+F\left(s, X_{s}^{0, x ; \bar{u}}, \bar{u}_{s} ; K_{s}, L_{s}\right) d s  \tag{4.17}\\
+l\left(s, X_{s}^{0, x ; \bar{u}}, \bar{u}_{s}\right) d s
\end{gather*}
$$

It is well-known that the stochastic flow $X_{s}^{0, x ; v}, x \in \mathbb{R}^{n}$ has an inverse $Y_{s}^{0, x ; v}, x \in$ $\mathbb{R}^{n}$. Since (see Yong and Zhou [21, Theorem 6.14, page 47])

$$
\begin{equation*}
X_{s}^{0, x ; v}=\Phi_{t} x+\Phi_{t} \int_{0}^{t} \Psi_{s}\left(B_{s} v-C_{s}^{i} D_{s}^{i} v\right) d s+\Phi_{t} \int_{0}^{t} \Psi_{s} D_{s}^{i} v d W_{s}^{i} \tag{4.18}
\end{equation*}
$$

for $t \in[0, T]$, we have

$$
\begin{equation*}
Y_{s}^{0, x ; v}=\Psi_{t} x-\int_{0}^{t} \Psi_{s}\left(B_{s} v-C_{s}^{i} D_{s}^{i} v\right) d s-\int_{0}^{t} \Psi_{s} D_{s}^{i} v d W_{s}^{i}, \quad t \in[0, T] \tag{4.19}
\end{equation*}
$$

More generally, we define for any $u \in \mathscr{U}_{0}$ and $t \in[0, T]$,

$$
\begin{equation*}
Y_{s}^{0, x ; u}=\Psi_{t} x-\int_{0}^{t} \Psi_{s}\left(B_{s} u_{s}-C_{s}^{i} D_{s}^{i} u_{s}\right) d s-\int_{0}^{t} \Psi_{s} D_{s}^{i} u_{s} d W_{s}^{i} \tag{4.20}
\end{equation*}
$$

We have

$$
\begin{equation*}
\left.X_{s}^{0, y ; u}\right|_{y=Y_{s}^{0, x ; u}}=x, \quad \forall x \in \mathbb{R}^{n} \tag{4.21}
\end{equation*}
$$

Incorporating the composition of $\gamma(s, \cdot ; v)$ with the inverse flow $Y_{s}^{0, x ; v}, x \in \mathbb{R}^{n}$, we have

$$
\begin{align*}
0 & \leq \gamma\left(d s, Y_{s}^{0, x ; v} ; v\right) \\
& =-\left\langle d k_{s} x, x\right\rangle+\left[F\left(s, x, v ; K_{s}, L_{s}\right)+l(s, x, v)\right] d s \tag{4.22}
\end{align*}
$$

and in a similar way, we have for almost everywhere $(s, \omega) \in[0, T] \times \Omega$,

$$
\begin{align*}
0 & =\gamma\left(d s, Y_{s}^{0, x ; \bar{u}} ; \bar{u}\right) \\
& =-\left\langle d k_{s} x, x\right\rangle+\left[F\left(s, x, \bar{u}_{s} ; K_{s}, L_{s}\right)+l\left(s, x, \bar{u}_{s}\right)\right] d s \tag{4.23}
\end{align*}
$$

Therefore, we have

$$
\begin{equation*}
\left\langle d k_{s} x, x\right\rangle=\min _{v \in \mathbb{R}^{m}}\left[F\left(s, x, v ; K_{s}, L_{s}\right)+l(s, x, v)\right] d s, \quad \forall x \in \mathbb{R}^{n} \tag{4.24}
\end{equation*}
$$

which implies formula (4.5).

## Step 3. Estimate for $L$.

From the theory of BSDEs, we have from BSDE (4.14)

$$
\begin{align*}
& \int_{0}^{T}\left|\left\langle K_{t} X_{t}^{0, x ; v},\left(C_{t}^{i} X_{t}^{0, x ; v}+D_{t}^{i} v\right)\right\rangle+\left\langle L_{t}^{i} X_{t}^{0, x ; v}, X_{t}^{0, x ; v}\right\rangle\right|^{2} d t \\
= & \left|\left\langle M X_{T}^{0, x ; v}, X_{T}^{0, x ; v}\right\rangle\right|^{2}-\left|V\left(t, X_{t}^{0, x ; v}\right)\right|^{2} \\
.25) & +2 \int_{0}^{T} V\left(t, X_{t}^{0, x ; v}\right)\left[\left\langle k_{t} X_{t}^{0, x ; v}, X_{t}^{0, x ; v}\right\rangle-F\left(t, X_{t}^{0, x ; v}, v ; K_{t}, L_{t}\right)\right] d t \\
\quad & -\int_{0}^{T} V\left(t, X_{t}^{0, x ; v}\right)\left[2\left\langle K_{t}\left(C_{t}^{i} X_{t}^{0, x ; v}+D_{t}^{i} v\right), X_{t}^{0, x ; v}\right\rangle-\left\langle L_{t}^{i} X_{t}^{0, x ; v}, X_{t}^{0, x ; v}\right\rangle\right] d W_{t}^{i} .
\end{align*}
$$

Since $V\left(t, X_{t}^{0, x ; v}\right) \geq 0$, taking $v=0$ and using the inequality (4.16), we have

$$
\begin{align*}
& \int_{0}^{T}\left|\left\langle K_{t} X_{t}^{0, x ; 0}, C_{t}^{i} X_{t}^{0, x ; 0}\right\rangle+\left\langle L_{t}^{i} X_{t}^{0, x ; 0}, X_{t}^{0, x ; 0}\right\rangle\right|^{2} d t \\
\leq & \left|M \| X_{T}^{0, x ; 0}\right|^{4}+2 \int_{0}^{T} V\left(t, X_{t}^{0, x ; 0}\right) l\left(t, X_{t}^{0, x ; 0}, 0\right) d t  \tag{4.26}\\
& -\int_{0}^{T} V\left(t, X_{t}^{0, x ; 0}\right)\left[2\left\langle K_{t} C_{t}^{i} X_{t}^{0, x ; 0}, X_{t}^{0, x ; 0}\right\rangle-\left\langle L_{t}^{i} X_{t}^{0, x ; 0}, X_{t}^{0, x ; 0}\right\rangle\right] d W_{t}^{i} .
\end{align*}
$$

Since $V\left(t, X_{t}^{0, x ; 0}\right)=\left\langle K_{t} X_{t}^{0, x ; 0}, X_{t}^{0, x ; 0}\right\rangle$ and $K$ is uniformly bounded, there is a positive constant $\lambda$ such that

$$
\begin{align*}
& \int_{0}^{T}\left|\left\langle L_{t}^{i} X_{t}^{0, x ; 0}, X_{t}^{0, x ; 0}\right\rangle\right|^{2} d t \\
\leq & 2 \int_{0}^{T}\left|\left\langle K_{t} X_{t}^{0, x ; 0}, C_{t}^{i} X_{t}^{0, x ; 0}\right\rangle\right|^{2} d t+2\left|M \| X_{T}^{0, x ; 0}\right|^{4} \\
& +4 \int_{0}^{T} V\left(t, X_{t}^{0, x ; 0}\right) l\left(t, X_{t}^{0, x ; 0}, 0\right) d t  \tag{4.27}\\
& -2 \int_{0}^{T} V\left(t, X_{t}^{0, x ; 0}\right)\left[2\left\langle K_{t} C_{t}^{i} X_{t}^{0, x ; 0}, X_{t}^{0, x ; 0}\right\rangle-\left\langle L_{t}^{i} X_{t}^{0, x ; 0}, X_{t}^{0, x ; 0}\right\rangle\right] d W_{t}^{i} \\
\leq & \lambda \max _{t \in[0, T]}\left|X_{t}^{0, x ; 0}\right|^{4}-4 \int_{0}^{T} V\left(t, X_{t}^{0, x ; 0}\right)\left\langle K_{t} C_{t}^{i} X_{t}^{0, x ; 0}, X_{t}^{0, x ; 0}\right\rangle d W_{t}^{i} \\
& +2 \int_{0}^{T} V\left(t, X_{t}^{0, x ; 0}\right)\left\langle L_{t}^{i} X_{t}^{0, x ; 0}, X_{t}^{0, x ; 0}\right\rangle d W_{t}^{i} .
\end{align*}
$$

Therefore, for $p \geq 1$, we have

$$
\begin{aligned}
& E\left(\int_{0}^{T}\left|\left\langle L_{t}^{i} X_{t}^{0, x ; 0}, X_{t}^{0, x ; 0}\right\rangle\right|^{2} d t\right)^{p} \\
\leq & \lambda_{p} E\left[\max _{t \in[0, T]}\left|X_{t}^{0, x ; 0}\right|^{4 p}\right] \\
& +\lambda_{p} E\left|\int_{0}^{T} V\left(t, X_{t}^{0, x ; 0}\right)\left\langle K_{t} C_{t}^{i} X_{t}^{0, x ; 0}, X_{t}^{0, x ; 0}\right\rangle d W_{t}^{i}\right|^{p} \\
& +\lambda_{p} E\left|\int_{0}^{T} V\left(t, X_{t}^{0, x ; 0}\right)\left\langle L_{t}^{i} X_{t}^{0, x ; 0}, X_{t}^{0, x ; 0}\right\rangle d W_{t}^{i}\right|^{p} \\
\leq & \lambda_{p} E\left[\max _{t \in[0, T]}\left|X_{t}^{0, x ; 0}\right|^{4 p}\right] \\
& +\lambda_{p} E\left[\int_{0}^{T}\left|V\left(t, X_{t}^{0, x ; 0}\right)\left\langle K_{t} C_{t}^{i} X_{t}^{0, x ; 0}, X_{t}^{0, x ; 0}\right\rangle\right|^{2} d t\right]^{p / 2} \\
& +\lambda_{p} E\left[\int_{0}^{T}\left|V\left(t, X_{t}^{0, x ; 0}\right)\left\langle L_{t}^{i} X_{t}^{0, x ; 0}, X_{t}^{0, x ; 0}\right\rangle\right|^{2} d t\right]^{p / 2} \\
\leq & \lambda_{p} E\left[\max _{t \in[0, T]}\left|X_{t}^{0, x ; 0}\right|^{4 p}\right]+\lambda_{p} E\left[\int_{0}^{T}\left|\left\langle L_{t}^{i} X_{t}^{0, x ; 0}, X_{t}^{0, x ; 0}\right\rangle\right|^{2}\left|X_{t}^{0, x ; 0}\right|^{4} d t\right]^{p / 2} \\
\leq & \lambda_{p} E\left[\max _{t \in[0, T]}\left|X_{t}^{0, x ; 0}\right|^{4 p}\right] \\
& +\lambda_{p} E\left[\left(\int_{0}^{T}\left|\left\langle L_{t}^{i} X_{t}^{0, x ; 0}, X_{t}^{0, x ; 0}\right\rangle\right|^{2} d t\right)^{p / 2} \max _{t \in[0, T]}\left|X_{t}^{0, x ; 0}\right|^{2 p}\right] \\
\leq & \lambda_{p} E\left[\max _{t \in[0, T]}\left|X_{t}^{0, x ; 0}\right|^{4 p}\right]+\frac{1}{2} E\left[\left(\int_{0}^{T}\left|\left\langle L_{t}^{i} X_{t}^{0, x ; 0}, X_{t}^{0, x ; 0}\right\rangle\right|^{2} d t\right)^{p}\right]
\end{aligned}
$$

Consequently, we have for any $x \in \mathbb{R}^{n}$,
(4.29) $E\left(\int_{0}^{T}\left|\left\langle L_{t}^{i} X_{t}^{0, x ; 0}, X_{t}^{0, x ; 0}\right\rangle\right|^{2} d t\right)^{p} \leq 2 \lambda_{p} E\left[\max _{t \in[0, T]}\left|X_{t}^{0, x ; 0}\right|^{4 p}\right] \leq \lambda_{p}^{\prime}|x|^{4 p}$,
which implies the following inequality

$$
\begin{equation*}
E\left(\int_{0}^{T}\left|\Phi_{t}^{\prime} L_{t}^{i} \Phi_{t}\right|^{2} d t\right)^{p} \leq \lambda_{p} \tag{4.30}
\end{equation*}
$$

Hence,

$$
\begin{align*}
& E\left(\int_{0}^{T}\left|L_{t}^{i}\right|^{2} d t\right)^{p} \leq E\left(\int_{0}^{T}\left|\Psi_{t}^{\prime} \Phi_{t}^{\prime} L_{t}^{i} \Phi_{t} \Psi_{t}\right|^{2} d t\right)^{p} \\
\leq & E\left(\int_{0}^{T}\left|\Psi_{t}^{\prime}\right|^{2}\left|\Psi_{t}\right|^{2}\left|\Phi_{t}^{\prime} L_{t}^{i} \Phi_{t}\right|^{2} d t\right)^{p} \\
\leq & E\left[\left(\int_{0}^{T}\left|\Phi_{t}^{\prime} L_{t}^{i} \Phi_{t}\right|^{2} d t\right)^{p} \max _{t \in[0, T]}\left|\Psi_{t}\right|^{4 p}\right]  \tag{4.31}\\
\leq & \left\{E\left[\left(\int_{0}^{T}\left|\Phi_{t}^{\prime} L_{t}^{i} \Phi_{t}\right|^{2} d t\right)^{2 p}\right] E\left[\max _{t \in[0, T]}\left|\Psi_{t}\right|^{8 p}\right]\right\}^{1 / 2} \leq \lambda_{p}
\end{align*}
$$

The proof is complete.
Remark 4.1. We have shown in Steps 1 and 2 that ( $K, L$ ) solves BSRE (1.5) with $K$ being nonnegative and uniformly bounded. Then from Tang [20, Theorem 5.1, page 62], we have the desired estimate. Here we have given a different proof to the estimate (4.6).

Immediately, we have the following existence of adapted solution to BSRE (1.5).
Corollary 4.4. (Existence result for BSRE). Let assumptions (A1) and (A2) be satisfied. Then $(K, L)$ is an adapted solution to BSDE (1.5).
5. Verification theorem and uniqueness result for BSRE. In the theory of linear quadratic optimal stochastic control, the Riccati equation as a nonlinear system of backward (stochastic) differential equations is an equivalent form of the underlying Bellman equation as a nonlinear backward (stochastic) partial differential equations, and both the optimal control and the value function are expected to be given in terms of the solution to the Riccati equation. The following verification theorem illustrates such a philosophy, which, however, has more or less been addressed in the author's work [20, Theorem 3.2, page 60].

THEOREM 5.1. (Verification Theorem). Let assumptions (A1) and (A2) be satisfied. Let $(K, L)$ be an adapted solution to BSDE (1.5) such that $K$ is essentially bounded and nonnegative (and consequently L satisfies estimate (4.6) in view of Tang [20, Theorem 5.1, page 62]). Then, (i) the following linear SDE

$$
\left\{\begin{align*}
d \bar{X}_{t}= & {\left[A_{t}-B_{t} \mathscr{N}_{t}^{-1}\left(K_{t}\right) \mathscr{M}_{t}^{\prime}\left(K_{t}, L_{t}\right)\right] \bar{X}_{t} d t }  \tag{5.1}\\
& +\sum_{i=1}^{d}\left[C_{t}^{i}-D_{t}^{i} \mathscr{N}_{t}^{-1}\left(K_{t}\right) \mathscr{M}_{t}^{\prime}\left(K_{t}, L_{t}\right)\right] \bar{X}_{t} d W_{t}^{i}, \quad t \in[0, T] \\
\bar{X}_{0}= & x
\end{align*}\right.
$$

has a unique strong solution $\bar{X}$ such that

$$
\begin{equation*}
E\left[\max _{t \in[0, T]}\left|\bar{X}_{t}\right|^{2}\right]<\infty \tag{5.2}
\end{equation*}
$$

(ii) the following given process

$$
\begin{equation*}
\bar{u}_{t}=-\mathscr{N}_{t}^{-1}\left(K_{t}\right) \mathscr{M}_{t}^{\prime}\left(K_{t}, L_{t}\right) \bar{X}_{t}, \quad t \in[0, T] \tag{5.3}
\end{equation*}
$$

belongs to $\mathscr{L}_{\mathscr{F}}^{2}\left(0, T ; \mathbb{R}^{m}\right)$, and is the optimal control for the $S L Q$; and (iii) the value field $V$ is given by

$$
\begin{equation*}
V(t, x)=\left\langle K_{t} x, x\right\rangle, \quad(t, x) \in[0, T] \times \mathbb{R}^{n} \tag{5.4}
\end{equation*}
$$

Remark 5.1. A proof using the stochastic maximum principle (the so-called stochastic Hamilton system) is given in Tang [20, Section 3, pages 58-60]. The main difficulty of the proof comes from the appearance of $L$ in the optimal feedback law (5.3) since $L$ is in general not expected to be essentially bounded. Since the coefficients of the optimal closed system (5.1) contain L, we could directly have neither the integrability (5.2) nor the square integrability of $\bar{u}$, which prevent us from going through the conventional method of "completion of squares" in a straightforward way. In what follows, we get around the difficulty via the technique of localization by stopping times, and develop a localized version of the conventional method of "completion of squares", which give a different self-contained proof.

Proof. Since the coefficients of the optimal closed system (5.1) is square integrable on $[0, T]$ almost surely, $\operatorname{SDE}(5.1)$ has a unique strong solution $\bar{X}$ (see Gal'chuk [5]). Define for sufficiently large integer $j$, the stopping time $\tau_{j}$ as follows:

$$
\begin{equation*}
\tau_{j}:=T \wedge \min \left\{t \geq 0:\left|\bar{X}_{t}\right| \geq j\right\} \tag{5.5}
\end{equation*}
$$

with the convention that $\min \emptyset=\infty$. It is obvious that $\tau_{j} \uparrow T$ almost surely as $j \uparrow \infty$. Then, we have

$$
\begin{equation*}
\left\langle K_{0} x, x\right\rangle=E\left\langle K_{\tau_{j}} \bar{X}_{\tau_{j}}, \bar{X}_{\tau_{j}}\right\rangle+E \int_{0}^{\tau_{j}} l\left(t, \bar{X}_{t}, \bar{u}_{t}\right) d t \tag{5.6}
\end{equation*}
$$

which together with assumption (A2) implies the following (with the constant $\delta>0$ )

$$
\begin{equation*}
E \int_{0}^{\tau_{j}}\left|\bar{u}_{t}\right|^{2} d t \leq \delta^{-1} E \int_{0}^{\tau_{j}}\left\langle N_{t} \bar{u}_{t}, \bar{u}_{t}\right\rangle d t \leq \delta^{-1}\left\langle K_{0} x, x\right\rangle \tag{5.7}
\end{equation*}
$$

Using Fatou's lemma, we have $\bar{u} \in \mathscr{L}_{\mathscr{F}}^{2}\left(0, T ; \mathbb{R}^{m}\right)$. Since $\bar{X}=X^{0, x ; \bar{u}}$, we have from estimate (2.2) the integrability (5.2). Assertion (i) has been proved.

From Assertion (i), we see that

$$
0 \leq\left\langle K_{\tau_{j}} \bar{X}_{\tau_{j}}, \bar{X}_{\tau_{j}}\right\rangle \leq \lambda \max _{t \in[0, T]}\left|\bar{X}_{t}\right|^{2} \in L^{1}\left(\Omega, \mathscr{F}_{T}, P\right)
$$

and

$$
0 \leq \int_{0}^{\tau_{j}} l\left(t, \bar{X}_{t}, \bar{u}_{t}\right) d t \leq(\text { and } \uparrow) \int_{0}^{T} l\left(t, \bar{X}_{t}, \bar{u}_{t}\right) d t \in L^{1}\left(\Omega, \mathscr{F}_{T}, P\right)
$$

Using Lebesgue's dominant convergence theorem, we have

$$
\begin{align*}
\lim _{j \rightarrow \infty} E\left\langle K_{\tau_{j}} \bar{X}_{\tau_{j}}, \bar{X}_{\tau_{j}}\right\rangle & =E\left\langle K_{T} \bar{X}_{T}, \bar{X}_{T}\right\rangle \\
\lim _{j \rightarrow \infty} E \int_{0}^{\tau_{j}} L\left(t, \bar{X}_{t}, \bar{u}_{t}\right) d t & =E \int_{0}^{T} l\left(t, \bar{X}_{t}, \bar{u}_{t}\right) d t \tag{5.8}
\end{align*}
$$

In view of the equality (5.6), we have

$$
\begin{equation*}
\left\langle K_{0} x, x\right\rangle=E\left\langle K_{T} \bar{X}_{T}, \bar{X}_{T}\right\rangle+E \int_{0}^{T} l\left(t, \bar{X}_{t}, \bar{u}_{t}\right) d t=J(\bar{u} ; 0, x) \tag{5.9}
\end{equation*}
$$

It remains to prove that for any $u \in \mathscr{L}_{\mathscr{F}}^{2}\left(0, T ; \mathbb{R}^{m}\right)$, we have $J(u ; 0, x) \geq\left\langle K_{0} x, x\right\rangle$.
For given $u \in \mathscr{L}_{\mathscr{F}}^{2}\left(0, T ; \mathbb{R}^{m}\right)$ and sufficiently large integer $j$, define the stopping time $\tau_{j}^{u}$ as follows:

$$
\begin{equation*}
\tau_{j}^{u}:=T \wedge \min \left\{t \geq 0:\left|X_{t}^{u}\right| \geq j\right\} \tag{5.10}
\end{equation*}
$$

with the notation $X^{u}:=X^{0, x ; u}$. It is obvious that $\tau_{j}^{u} \uparrow T$ almost surely as $j \uparrow \infty$. Define

$$
\begin{equation*}
\widetilde{u}_{t}:=-\mathscr{N}_{t}^{-1}\left(K_{t}\right) \mathscr{M}_{t}^{\prime}\left(K_{t}, L_{t}\right) X_{t}^{u}, \quad t \in[0, T] . \tag{5.11}
\end{equation*}
$$

Then, the restriction of $\widetilde{u}$ to the random time interval $\left[0, \tau_{j}^{u}\right]$ lies in $\mathscr{L}_{\mathscr{F}}^{2}\left(0, \tau_{j}^{u} ; \mathbb{R}^{m}\right)$ for any $j$. Using BSRE (1.5) to complete the square in a straightforward manner, we have

$$
\begin{align*}
& E\left\langle K_{\tau_{j}^{u}} X_{\tau_{j}^{u}}^{u}, X_{\tau_{j}^{u}}^{u}\right\rangle+E \int_{0}^{\tau_{j}^{u}} l\left(t, X_{t}^{u}, u_{t}\right) d t  \tag{5.12}\\
& \quad=\left\langle K_{0} x, x\right\rangle+E \int_{0}^{\tau_{j}^{u}}\left\langle\mathscr{N}_{t}^{-1}\left(K_{t}\right)\left(u_{t}-\widetilde{u}_{t}\right), u_{t}-\widetilde{u}_{t}\right\rangle d t .
\end{align*}
$$

Therefore, we have

$$
\begin{equation*}
E\left\langle K_{\tau_{j}^{u}} X_{\tau_{j}^{u}}^{u}, X_{\tau_{j}^{u}}^{u}\right\rangle+E \int_{0}^{\tau_{j}^{u}} l\left(t, X_{t}^{u}, u_{t}\right) d t \geq\left\langle K_{0} x, x\right\rangle \tag{5.13}
\end{equation*}
$$

In view of estimate (2.2) in Lemma 2.1, we see that

$$
0 \leq\left\langle K_{\tau_{j}^{u}} X_{\tau_{j}^{u}}^{u}, X_{\tau_{j}^{u}}^{u}\right\rangle \leq \lambda \max _{t \in[0, T]}\left|X_{t}^{u}\right|^{2} \in L^{1}\left(\Omega, \mathscr{F}_{T}, P\right)
$$

and

$$
0 \leq \int_{0}^{\tau_{j}^{u}} l\left(t, X_{t}^{u}, u_{t}\right) d t \leq(\text { and } \uparrow) \int_{0}^{T} l\left(t, X_{t}^{u}, u_{t}\right) d t \in L^{1}\left(\Omega, \mathscr{F}_{T}, P\right)
$$

Passage to the limit in inequality (5.13), again using Lebesgue's dominant convergence theorem, we have

$$
\begin{equation*}
J(u ; s, x)=E\left\langle K_{T} X_{T}^{u}, X_{T}^{u}\right\rangle+E \int_{0}^{T} l\left(t, X_{t}^{u}, u_{t}\right) d t \geq\left\langle K_{0} x, x\right\rangle \tag{5.14}
\end{equation*}
$$

The proof is then complete.
Immediately, we have the following uniqueness of adapted solution to BSRE (1.5).
Corollary 5.2. (Uniqueness result for BSRE). Let assumptions (A1) and (A2) be satisfied. Let $(\widetilde{K}, \widetilde{L})$ be an adapted solution to $B S D E(1.5)$ such that $\widetilde{K}$ is essentially bounded and nonnegative and $\widetilde{L}$ satisfies estimate (4.6). Then, $\widetilde{K}=K$ and $\widetilde{L}=L$.

The corollary and its proof can be found in Tang [20, the beginning paragraph of Section 8, page 70].
6. Comments and possible extensions. The results of this paper can be adapted to the singular case ( $N$ is allowed to be only nonnegative) but with suitable additional conditions such as the following:
(A3) Assume that the matrix process $\sum_{i=1}^{d}\left(D^{i}\right)^{\prime} D^{i}$ and the terminal state weighting random matrix $M$ are uniformly positive.

This subject will be detailed elsewhere.
The singular case has received much recent interests because of its appearance in financial mean-variance problems. More generally, $N$ can also be possibly negativethis is the so-called indefinite case. On these features, the interested reader is referred to Chen and Yong [3], Hu and Zhou [6], Kohlmann and Tang [7, 10], Yong and Zhou [21], and the references therein.

Finally, the main results of the paper can also be adapted to the quadratic optimal control problem for linear stochastic differential system driven by jump-diffusion processes under suitable assumptions. The details will be presented elsewhere.

Consider a general non-Markovian nonlinear optimal stochastic control problem. Let $A$ be a separable metric space, and $\mathscr{U}_{s}$ be the set of $A$-valued predictable processes on $[s, T]$.

For any triplet $(u, s, \xi) \in \mathscr{U}_{s} \times[0, T] \times L^{2}\left(\Omega, \mathscr{F}_{s}, P ; \mathbb{R}^{n}\right)$, consider the following SDE:

$$
X_{t}=\xi+\int_{s}^{t} \sigma\left(r, X_{r}, u_{r}\right) d W_{r}+\int_{s}^{t} b\left(r, X_{r}, u_{r}\right) d r, \quad t \in[s, T] .
$$

Assume that the following functions

$$
\begin{aligned}
\sigma(t, x, \alpha) \in \mathbb{R}^{n \times d}, & b(t, x, \alpha) \in \mathbb{R}^{n}, \\
l(t, x, \alpha) \in \mathbb{R}, & g(x) \in \mathbb{R} ;
\end{aligned} \quad(t, x, \alpha) \in[0, T] \times \mathbb{R}^{n} \times A
$$

are continuous in $(x, \alpha)$ and continuous in $x$ uniformly over $\alpha$ for each $(t, \omega)$. Also, assume thatthere is positive constant $\lambda$ such that

$$
\begin{aligned}
\|\sigma(t, x, \alpha)-\sigma(t, y, \alpha)\|+|b(t, x, \alpha)-(t, y, \alpha)| & \leq \lambda|x-y|, \\
\|\sigma(t, x, \alpha)\|+|b(t, x, \alpha)| & \leq \lambda(1+|x|), \\
|l(t, x, \alpha)|+|g(x)| & \leq \lambda(1+|x|)^{m} .
\end{aligned}
$$

For $(s, \xi, u) \in[s, T] \times L^{2}\left(\Omega, \mathscr{F}_{s}, P ; \mathbb{R}^{n}\right) \times \mathscr{U}_{s}$, define

$$
\begin{aligned}
J(u ; s, \xi) & =E^{s, \xi ; u}\left[\int_{s}^{T} l\left(t, X_{t}, u_{t}\right) d t+g\left(X_{T}\right) \mid \mathscr{\mathscr { F }}_{s}\right], \\
V(s, \xi) & :=\underset{u \in \mathscr{\mathscr { U }}_{s}}{\operatorname{ess}} \inf J(u ; s, \xi) .
\end{aligned}
$$

Denote by $\mathbb{V}(s, \cdot)$ the restriction of $V(s, \cdot)$ to $\mathbb{R}^{n}$. In the nonlinear context, the restricted value field $\mathbb{V}$ can be proved to satisfy the stochastic dynamic programming principle:
(i) For $s \leq t \leq T$ and $\xi \in L^{2}\left(\Omega, \mathscr{F}_{s}, P ; \mathbb{R}^{n}\right)$,

$$
\mathbb{V}(s, \xi)=\text { ess. } \inf _{u \in \mathscr{\mathscr { U }}_{s}} E^{s, \xi ; u}\left\{\int_{s}^{t} l\left(r, X_{r}, u_{r}\right) d r+\mathbb{V}\left(t, X_{t}\right) \mid \mathscr{F}_{s}\right\} .
$$

(ii) For $(s, x, u) \in[0, T] \times R^{n} \times \mathscr{U}_{s}$, the process

$$
\kappa_{t}^{s, x ; u}:=\mathbb{V}\left(t, X_{t}^{s, x ; u}\right)+\int_{s}^{t} l\left(r, X_{r}^{s, x ; u}, u_{r}\right) d r
$$

defined for $t \in[s, T]$, is a submartingale w.r.t. $\left\{\mathscr{F}_{t}\right\}$.
Using the above dynamic programming principle and Kunita's stochastic calculus [13], we can still show that $\mathbb{V}$ is a Sobolev space valued semi-martingale and satisfy the associated backward Bellman equation in the strong sense. All the details shall be given in our forthcoming paper to extend Krylov [11] to the non-Markovian framework for optimal stochastic control problem.

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