ON THE SUMS OF ANY k POINTS IN FINITE FIELDS

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Abstract. For a set $E \subset \mathbb{F}_q^d$, we define the k-resultant magnitude set as $\Delta_k(E) = \{\|\mathbf{x}_1 + \cdots + \mathbf{x}_k\| \in \mathbb{F}_q : \mathbf{x}_1, \ldots, \mathbf{x}_k \in E\}$, where $\|\mathbf{v}\| = v_1^2 + \cdots + v_d^2$ for $\mathbf{v} = (v_1, \ldots, v_d) \in \mathbb{F}_q^d$. In this paper we find a connection between a lower bound of the cardinality of the k-resultant magnitude set and the restriction theorem for spheres in finite fields. As a consequence, it is shown that if $E \subset \mathbb{F}_q^d$ with $|E| \geq Cq^{\frac{d+1}{2} - \frac{1}{6d+2}}$, then $|\Delta_3(E)| \geq cq$ for $d = 4$ or $d = 6$, and $|\Delta_4(E)| \geq cq$ for even dimensions $d \geq 8$. In addition, we prove that if $d \geq 8$ is even, and $|E| \geq C_\varepsilon\, q^{\frac{d+1}{2} - \frac{1}{9d-18} + \varepsilon}$ for $\varepsilon > 0$, then $|\Delta_3(E)| \geq cq$.

1. Introduction

Let $\mathbb{F}_q^d$, $d \geq 2$, be the d-dimensional vector space over a finite field with q elements. Throughout the paper, we assume that the characteristic of $\mathbb{F}_q$ is not equal to two. For $E \subset \mathbb{F}_q^d$, the distance set, denoted by $\Delta_2(E)$, is defined by

$$\Delta_2(E) = \{\|\mathbf{x} - \mathbf{y}\| \in \mathbb{F}_q : \mathbf{x}, \mathbf{y} \in E\},$$

where $\|\mathbf{v}\| = v_1^2 + \cdots + v_d^2$ for $\mathbf{v} = (v_1, \ldots, v_d) \in \mathbb{F}_q^d$. The Erdős-Falconer distance problem in the finite field setting asks for the minimal threshold β such that if $|E| \geq Cq^\beta$ for a sufficiently large constant C, then we have $|\Delta_2(E)| \geq cq$ for some $0 < c \leq 1$. The first distance result was obtained by Bourgain, Katz, and Tao ([1]) when $q \equiv 3 \pmod 4$ is a prime. Iosevich and Rudnev ([14]) studied the general field case, and they obtained the first explicit exponents. Using discrete Fourier machinery, they demonstrated that if $E \subset \mathbb{F}_q^d$ with $|E| \geq Cq^{\frac{d+1}{2}}$, for a sufficiently large constant C, then $|\Delta_2(E)| = q$.

The authors in [8] constructed arithmetic examples which show that the exponent $(d+1)/2$ due to Iosevich and Rudnev is sharp at least in odd dimensions. Thus, the Erdős-Falconer distance problem has been completely resolved in odd dimensions. On the other hand, it has been conjectured for even dimensions $d \geq 2$ that the exponent $(d+1)/2$ can be improved to the exponent $d/2$. While this conjecture is open for all even dimensions, the sharp exponent $(d+1)/2$ for odd dimensions was improved for dimension two by the authors in [2]. More precisely they proved that if $E \subset \mathbb{F}_q^2$ with $|E| \geq Cq^{4/3}$ for a sufficiently large constant C, then $|\Delta_2(E)| \geq cq$ for some $0 < c < 1$. However, the exponent $(d+1)/2$ has not

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been improved for higher even dimensions $d \geq 4$. For further discussion on distance problems in finite fields, readers may refer to [5, 9, 16, 17, 18, 19, 26, 27]. See also [3, 4], and references contained therein for recent results on the distance problems in the ring setting.

The Erdős-Falconer distance problem in finite fields can be extended in various directions. One such direction is as follows. For each integer $k \geq 2$, let us consider a function $M_k : (\mathbb{F}_q^d)^k \to \mathbb{F}_q$. Given this function, determine the minimal value β such that whenever $E \subset \mathbb{F}_q^d$ satisfies $|E| \geq Cq^\beta$ for a sufficiently large constant C, we have $|M_k(E^k)| \geq cq$ for some constant $0 < c \leq 1$ independent of q. Note that when $M_2(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ for $\mathbf{x}, \mathbf{y} \in \mathbb{F}_q^d$, we are reduced the Erdős-Falconer distance problem in the finite field setting as

$$\Delta_2(E) = M_2(E \times E) = \{\|\mathbf{x} - \mathbf{y}\| \in \mathbb{F}_q : \mathbf{x}, \mathbf{y} \in E\}.$$

For $k \geq 2$, we will study the function

$$M_k(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k) = \|\mathbf{x}_1 \pm \mathbf{x}_2 \pm \cdots \pm \mathbf{x}_k\| \text{ for } \mathbf{x}_s \in \mathbb{F}_q^d, s = 1, 2, \dots, k,$$

and we denote $M_k(E^k)$ by $\Delta_k(E)$ for $E \subset \mathbb{F}_q^d$. Namely, for $E \subset \mathbb{F}_q^d$, we define

$$\Delta_k(E) = \{\|\mathbf{x}_1 \pm \mathbf{x}_2 \pm \cdots \pm \mathbf{x}_k\| \in \mathbb{F}_q : \mathbf{x}_s \in E, s = 1, 2, \dots, k\}.$$

As the choice of signs will be independent of our results, we shall simply define

$$\Delta_k(E) = \{\|\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_k\| \in \mathbb{F}_q : \mathbf{x}_s \in E, s = 1, 2, \dots, k\}.$$

Throughout the paper, the set $\Delta_k(E)$ will be referred to as the k-resultant magnitude set. For brevity, we call $\Delta_2(E)$ the distance set, and when $k = 3$, we simply call $\Delta_3(E)$ the magnitude set.

Question 1.1. *Let $E \subset \mathbb{F}_q^d, d \geq 2$, and $k \geq 2$ be an integer. Determine the smallest $\beta > 0$ such that if $|E| \geq Cq^\beta$ with a sufficiently large constant $C > 1$, then $|\Delta_k(E)| \geq cq$ for some $0 < c \leq 1$.*

It is clear that $|\Delta_{k_1}(E)| \leq |\Delta_{k_2}(E)|$ for $2 \leq k_1 \leq k_2$. Therefore, as k becomes larger, one might expect a smaller value β as the answer to Question 1.1. However, we conjecture that the answer to Question 1.1 is independent of k. For example, if $q = p^2$ for prime p and $E = \mathbb{F}_p^d$, then it clearly follows that $|E| = q^{d/2}$ and $|\Delta_k(E)| = \sqrt{q}$ for all $k \geq 2$. This example says that β in Question 1.1 can not be smaller than $d/2$ which is the conjectured exponent for the Erdős-Falconer distance problem in even dimensions. This leads us to the following conjecture.

Conjecture 1.2. *Let $E \subset \mathbb{F}_q^d$. If $d \geq 2$ is even and $|E| \geq Cq^{d/2}$ for a sufficiently large constant C, then for every integer $k \geq 2$, there exists a constant $0 < c \leq 1$ such that*

$$|\Delta_k(E)| \geq cq.$$

1.1. **Statement of results.** The techniques used by Iosevich and Rudnev in [14] show that if $|E| \geq Cq^{\frac{d+1}{2}}$ for a sufficiently large constant C, then $|\Delta_k(E)| = q$. Note that the counterexamples for the Erdős-Falconer distance problem immediately show that the exponent $(d+1)/2$ can not be improved in general for odd dimensions. Thus, we shall only focus on investigating the size of $\Delta_k(E)$ where $E \subset \mathbb{F}_q^d$ is a subset of an even dimensional vector space. In this paper we demonstrate that the exponent $(d+1)/2$ for the magnitude set can be improved for even $d \geq 4$. More precisely, we have the following results.

Theorem 1.3. *Let $E \subset \mathbb{F}_q^d$. Suppose that C is a sufficiently large constant.*
(1) If $d = 4$ or 6, and $|E| \geq Cq^{\frac{d+1}{2} - \frac{1}{6d+2}}$, then $|\Delta_3(E)| \geq cq$ for some $0 < c \leq 1$.
(2) If $d \geq 8$ is even and $|E| \geq Cq^{\frac{d+1}{2} - \frac{1}{6d+2}}$, then $|\Delta_4(E)| \geq cq$ for some $0 < c \leq 1$.

Theorem 1.4. *Suppose that $d \geq 8$ is even and $E \subset \mathbb{F}_q^d$. Then given $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that if $|E| \geq C_\varepsilon q^{\frac{d+1}{2} - \frac{1}{9d-18} + \varepsilon}$, then $|\Delta_3(E)| \geq cq$ for some $0 < c \leq 1$.*

It seems from our results that the exponent $(d+1)/2$ can be improved for the Erdős-Falconer distance problem in even dimensions $d \geq 4$.

Remark 1.5. Aside from thinking of the cardinality of $\Delta_3(E)$ as the number of distinct distances of any three vectors in $E \subset \mathbb{F}_q^d$, we can also consider it as the number of distinct distances between the origin and the centers of mass of triangles determined by $E \subset \mathbb{F}_q^d$ if q has characteristic greater than 3. To see this, notice that if $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{F}_q^d$, then $(\mathbf{x} + \mathbf{y} + \mathbf{z})/3$ can be considered as the center of mass of the triangle with vertices $\mathbf{x}, \mathbf{y}, \mathbf{z}$.

1.2. **Outline of the paper.** In the remaining parts of the paper, we first provide preliminary lemmas in Section 2. In Section 3, we obtain the necessary restriction estimates for spheres. In the final section, we deduce the formula for $|\Delta_k(E)|$ and we provide the link between the set $\Delta_k(E)$ and the restriction estimates for spheres.

2. Discrete Fourier analysis and related lemmas

As a main technical tool, discrete Fourier analysis plays an important role in proving our results. In this section, we review the basic definitions, and we collect preliminary lemmas which are essential for providing a lower bound for $|\Delta_k(E)|$.

2.1. **Discrete Fourier analysis.** Throughout this paper, χ denotes a nontrivial additive character of $\mathbb{F}_q$. The choice of the character χ will be independent of the results in this paper. The orthogonality of the character χ implies

$$\sum_{\mathbf{x} \in \mathbb{F}_q^d} \chi(\mathbf{m} \cdot \mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{m} \neq (0, \dots, 0) \\ q^d & \text{if } \mathbf{m} = (0, \dots, 0), \end{cases}$$

where $\mathbf{m} \cdot \mathbf{x}$ denotes the usual dot-product. Given a function $g : \mathbb{F}_q^d \to \mathbb{C}$, the Fourier transform of g, denoted by $\widetilde{g}$, is defined as

$$\widetilde{g}(\mathbf{x}) = \sum_{\mathbf{m} \in \mathbb{F}_q^d} g(\mathbf{m}) \chi(-\mathbf{x} \cdot \mathbf{m}) \quad \text{for } \mathbf{x} \in \mathbb{F}_q^d. \tag{2.1}$$

On the other hand, if $f : \mathbb{F}_q^d \to \mathbb{C}$, then we denote by $\widehat{f}$ the **normalized** Fourier transform of the function f. Thus, we have

$$\widehat{f}(\mathbf{m}) = \frac{1}{q^d} \sum_{\mathbf{x} \in \mathbb{F}_q^d} f(\mathbf{x}) \chi(-\mathbf{x} \cdot \mathbf{m}) \quad \text{for } \mathbf{m} \in \mathbb{F}_q^d.$$

We also write $f^\vee(\mathbf{m})$ for $\widehat{f}(-\mathbf{m})$. Notice that $\widetilde{(f^\vee)}(\mathbf{x}) = f(\mathbf{x})$ for $\mathbf{x} \in \mathbb{F}_q^d$. Namely, the Fourier inversion theorem in this content is given by the formula

$$f(\mathbf{x}) = \sum_{\mathbf{m} \in \mathbb{F}_q^d} \widehat{f}(\mathbf{m}) \chi(\mathbf{m} \cdot \mathbf{x}) \quad \text{for } \mathbf{x} \in \mathbb{F}_q^d.$$

Remark 2.1. Throughout the rest of the article, we will write $E(\mathbf{x})$ for the characteristic function (or indicator function) of a set $E \subset \mathbb{F}_q^d$.

As a direct application of the orthogonality relation of χ, it follows that

$$\sum_{\mathbf{m}\in\mathbb{F}_q^d} |\widehat{f}(\mathbf{m})|^2 = \frac{1}{q^d} \sum_{\mathbf{x}\in\mathbb{F}_q^d} |f(\mathbf{x})|^2.$$

We refer to this formula as the Plancherel theorem. For example, if we take $f(\mathbf{x}) = E(x)$, then the Plancherel theorem implies that

$$\sum_{\mathbf{m}\in\mathbb{F}_q^d} |\widehat{E}(\mathbf{m})|^2 = \frac{|E|}{q^d}.$$

Furthermore, since $\left|\widehat{E}(\mathbf{m})\right| \leq q^{-d}|E|$, it is clear that for every integer $k \geq 2$,

$$\sum_{\mathbf{m}\in\mathbb{F}_q^d} |\widehat{E}(\mathbf{m})|^k \leq \frac{|E|^{k-2}}{q^{d(k-2)}} \sum_{\mathbf{m}\in\mathbb{F}_q^d} |\widehat{E}(\mathbf{m})|^2 = \frac{|E|^{k-1}}{q^{dk-d}}. \tag{2.2}$$

We now collect information about the normalized Fourier transform on the sphere. For $t \in \mathbb{F}_q$, the sphere $S_t \subset \mathbb{F}_q^d$ is defined by

$$S_t = \{\mathbf{x} \in \mathbb{F}_q^d : x_1^2 + \cdots + x_d^2 = t\}.$$

It is well known from Theorem 6.26 and Theorem 6.27 in [20] that if $d \geq 3$ and $t \in \mathbb{F}_q$, then

$$|S_t| = q^{d-1}(1 + o(1)). \tag{2.3}$$

The following result follows immediately from Lemma 4 in [12].

Proposition 2.2. *Let $d \geq 2$ be even and $t \in \mathbb{F}_q$. Then, for $\boldsymbol{m} \in \mathbb{F}_q^d$,*

$$\widehat{S_t}(\boldsymbol{m}) = q^{-1}\delta_0(\boldsymbol{m}) + q^{-d-1}G^d \sum_{\ell\in\mathbb{F}_q^*} \chi\Big(t\ell + \frac{\|\boldsymbol{m}\|}{4\ell}\Big),$$

where $\delta_0(\boldsymbol{m})$ is the delta-function, so that $\delta_0(\boldsymbol{m}) = 1$ for $\boldsymbol{m} = (0, \ldots, 0)$ and $\delta_0(\boldsymbol{m}) = 0$ otherwise, and G denotes the Gauss sum

$$G = \sum_{s\in\mathbb{F}_q^*} \eta(s)\chi(s),$$

where η is the quadratic character of $\mathbb{F}_q$, and $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. In particular, we have

$$\widehat{S_0}(\boldsymbol{m}) = q^{-1}\delta_0(\boldsymbol{m}) + q^{-d-1}G^d \sum_{\ell\in\mathbb{F}_q^*} \chi(\|\boldsymbol{m}\|\ell) \quad \text{for } \boldsymbol{m} \in \mathbb{F}_q^d.$$

Remark 2.3. Recall that the Gauss sum satisfies $|G| = \sqrt{q}$. For $a, b \in \mathbb{F}_q$, the Kloosterman sum is defined by

$$K(a,b) := \sum_{\ell\in\mathbb{F}_q^*} \chi(a\ell + b/\ell).$$

It is well known that $|K(a,b)| \leq 2\sqrt{q}$ for $ab \neq 0$. For the proof of the Gauss and Kloosterman sum estimation, see [13, 20].

The following result was proved in Proposition 2.2 in [19].

Proposition 2.4. *For $\boldsymbol{m}, \boldsymbol{v} \in \mathbb{F}_q^d$, we have*

$$\sum_{t\in\mathbb{F}_q} \widehat{S}_t(\boldsymbol{m})\ \overline{\widehat{S}_t}(\boldsymbol{v}) = q^{-1}\delta_0(\boldsymbol{m})\ \delta_0(\boldsymbol{v}) + q^{-d-1}\sum_{s\in\mathbb{F}_q^*}\chi(s(\|\boldsymbol{m}\|-\|\boldsymbol{v}\|)).$$

2.2. **Evaluation of the counting function ν_k.** Let $E \subset \mathbb{F}_q^d$ and let $k \geq 2$ be an integer. For $t \in \mathbb{F}_q$, we define the counting function $\nu_k(t)$ by

$$\begin{aligned}\nu_k(t) &:= |\{(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k) \in E^k : \|\mathbf{x}_1+\mathbf{x}_2+\cdots+\mathbf{x}_k\| = t\}|\\ &= \sum_{\mathbf{x}_1,\ldots,\mathbf{x}_k\in E} S_t(\mathbf{x}_1+\mathbf{x}_2+\cdots+\mathbf{x}_k).\end{aligned}$$

Applying the Fourier inversion theorem to $S_t(\mathbf{x}_1+\mathbf{x}_2+\cdots+\mathbf{x}_k)$, it follows from the definition of the normalized Fourier transform that

$$\nu_k(t) = q^{dk}\sum_{\mathbf{m}\in\mathbb{F}_q^d}\widehat{S}_t(\mathbf{m})\left(\overline{\widehat{E}(\mathbf{m})}\right)^k. \tag{2.4}$$

Then an L^2 estimate of ν_k is as follows.

Lemma 2.5. *Let $E \subset \mathbb{F}_q^d, d \geq 2$. Then we have*

$$\sum_{t\in\mathbb{F}_q}\nu_k^2(t) \leq q^{-1}|E|^{2k} + q^{2dk-d}\sum_{r\in\mathbb{F}_q}\left|\sum_{\boldsymbol{v}\in S_r}\left(\widehat{E}(\boldsymbol{v})\right)^k\right|^2.$$

Proof. Since $\nu_k^2(t) = \nu_k(t)\ \overline{\nu_k(t)}$, we see from (2.4) that

$$\sum_{t\in\mathbb{F}_q}\nu_k^2(t) = q^{2dk}\sum_{\mathbf{m},\mathbf{v}\in\mathbb{F}_q^d}\left(\overline{\widehat{E}(\mathbf{m})}\right)^k\left(\widehat{E}(\mathbf{v})\right)^k\left(\sum_{t\in\mathbb{F}_q}\widehat{S}_t(\mathbf{m})\overline{\widehat{S}_t(\mathbf{v})}\right).$$

From Proposition 2.4, we conclude that

$$\begin{aligned}\sum_{t\in\mathbb{F}_q}\nu_k^2(t) &= q^{-1}|E|^{2k} + q^{2dk-d}\sum_{\substack{\mathbf{m},\mathbf{v}\in\mathbb{F}_q^d:\\ \|\mathbf{m}\|=\|\mathbf{v}\|}}\left(\overline{\widehat{E}(\mathbf{m})}\right)^k\left(\widehat{E}(\mathbf{v})\right)^k - q^{2dk-d-1}\left|\sum_{\mathbf{v}\in\mathbb{F}_q^d}\left(\widehat{E}(\mathbf{v})\right)^k\right|^2\\ &\leq q^{-1}|E|^{2k} + q^{2dk-d}\sum_{\substack{\mathbf{m},\mathbf{v}\in\mathbb{F}_q^d:\\ \|\mathbf{m}\|=\|\mathbf{v}\|}}\left(\overline{\widehat{E}(\mathbf{m})}\right)^k\left(\widehat{E}(\mathbf{v})\right)^k\\ &= q^{-1}|E|^{2k} + q^{2dk-d}\sum_{r\in\mathbb{F}_q}\left|\sum_{\mathbf{v}\in S_r}\left(\widehat{E}(\mathbf{v})\right)^k\right|^2.\end{aligned}$$

□

We need the following lemma.

Lemma 2.6. *Suppose that $d \geq 2$ is even and $k \geq 2$ is an integer. If $E \subset \mathbb{F}_q^d$ with $|E| \geq 3q^{d/2}$, then we have*

$$\left(|E|^k - \nu_k(0)\right)^2 \geq \frac{|E|^{2k}}{9}.$$

Proof. Combining (2.4) and Proposition 2.2, we see that

$$\begin{aligned}\nu_k(0) =& q^{dk}\sum_{\mathbf{m}\in\mathbb{F}_q^d}\left(\overline{\widehat{E}(\mathbf{m})}\right)^k\left(q^{-1}\delta_0(\mathbf{m}) + q^{-d-1}G^d\sum_{\ell\in\mathbb{F}_q^*}\chi(\|\mathbf{m}\|\ell)\right)\\ =& q^{dk-1}\left(\overline{\widehat{E}(0,\ldots,0)}\right)^k + q^{dk-d-1}G^d\sum_{\mathbf{m}\in\mathbb{F}_q^d}\left(\overline{\widehat{E}(\mathbf{m})}\right)^k\left(\sum_{\ell\in\mathbb{F}_q^*}\chi(\|\mathbf{m}\|\ell)\right).\end{aligned}$$

Since $\widehat{E}(0,\ldots,0) = q^{-d}|E|$, we have

$$\nu_k(0) = q^{-1}|E|^k + q^{dk-d-1}G^d\sum_{\mathbf{m}\in\mathbb{F}_q^d}\left(\overline{\widehat{E}(\mathbf{m})}\right)^k\left(\sum_{\ell\in\mathbb{F}_q^*}\chi(\|\mathbf{m}\|\ell)\right). \tag{2.5}$$

Since $\nu_k(0)$ is a nonnegative real number, it is clear that

$$\nu_k(0) \le q^{-1}|E|^k + q^{dk-d}|G|^d\sum_{\mathbf{m}\in\mathbb{F}_q^d}|\widehat{E}(\mathbf{m})|^k.$$

As $|G| = q^{1/2}$, it follows from (2.2) that

$$\nu_k(0) \le q^{-1}|E|^k + q^{d/2}|E|^{k-1}.$$

Since $q \ge 3$, this clearly implies that if $|E| \ge 3q^{d/2}$, then

$$\begin{aligned}|E|^k - \nu_k(0) \ge& |E|^k - q^{-1}|E|^k - q^{d/2}|E|^{k-1}\\ \ge& \frac{|E|^k}{3} + \left(\frac{|E|^k}{3} - q^{d/2}|E|^{k-1}\right) \ge \frac{|E|^k}{3},\end{aligned}$$

and the statement of the lemma follows immediately. □

We shall also use the following result.

Lemma 2.7. *Let $E \subset \mathbb{F}_q^d$. Assume that $d \ge 2$ is even and $k \ge 2$ is an integer. If $|E| \ge q^{d/2}$, then we have*

$$q^{2dk-d}\left|\sum_{\boldsymbol{m}\in S_0}\left(\widehat{E}(\boldsymbol{m})\right)^k\right|^2 - \nu_k^2(0) \le 4q^{-1}|E|^{2k}.$$

Proof. Observe from (2.5) that we can write

$$\nu_k(0) = q^{-1}|E|^k + q^{dk-d-1}G^d\sum_{\mathbf{m}\in\mathbb{F}_q^d}\left(\overline{\widehat{E}(\mathbf{m})}\right)^k\left(-1 + \sum_{\ell\in\mathbb{F}_q}\chi(\|\mathbf{m}\|\ell)\right).$$

By the orthogonality relation of χ, it is easy to see that

$$\begin{aligned}\nu_k(0) =& q^{dk-d}G^d\sum_{\mathbf{m}\in S_0}\left(\overline{\widehat{E}(\mathbf{m})}\right)^k + \left(q^{-1}|E|^k - q^{dk-d-1}G^d\sum_{\mathbf{m}\in\mathbb{F}_q^d}\left(\overline{\widehat{E}(\mathbf{m})}\right)^k\right)\\ :=& A + B.\end{aligned}$$

Since $\nu_k(0)\geq 0$, it follows that

$$\begin{aligned}\nu_k^2(0)=&\nu_k(0)\overline{\nu_k(0)}=(A+B)(\overline{A}+\overline{B})\\ =&q^{2dk-d}\left|\sum_{\mathbf{m}\in S_0}\left(\widehat{E}(\mathbf{m})\right)^k\right|^2+A\overline{B}+\overline{A}B+|B|^2.\end{aligned}$$

This observation and the definition of A and B yield that

$$\begin{aligned}&q^{2dk-d}\left|\sum_{\mathbf{m}\in S_0}\left(\widehat{E}(\mathbf{m})\right)^k\right|^2-\nu_k^2(0)\leq 2|A||B|\\ &\leq 2\left(q^{dk-d/2}\sum_{\mathbf{m}\in\mathbb{F}_q^d}|\widehat{E}(\mathbf{m})|^k\right)\left(q^{-1}|E|^k+q^{dk-d/2-1}\sum_{\mathbf{m}\in\mathbb{F}_q^d}|\widehat{E}(\mathbf{m})|^k\right)\\ &\leq 2\left(q^{d/2-1}|E|^{2k-1}+q^{d-1}|E|^{2k-2}\right),\end{aligned}$$

where (2.2) was applied to obtain the last line. We complete the proof by observing that if $|E|\geq q^{d/2}$, then

$$\max\left(q^{d/2-1}|E|^{2k-1},\ q^{d-1}|E|^{2k-2}\right)\leq q^{-1}|E|^{2k}.$$

□

3. Results on the restriction theorem for spheres

In this section we collect lemmas which can be obtained by applying the restriction theorems for spheres in finite fields. To do this, we begin by reviewing the extension problem for spheres. We denote by $(\mathbb{F}_q^d, d\mathbf{x})$ the d-dimensional vector space over $\mathbb{F}_q$ endowed with the normalized counting measure "$d\mathbf{x}$". On the other hand, the dual space of $(\mathbb{F}_q^d, d\mathbf{x})$ will be denoted by $(\mathbb{F}_q^d, d\mathbf{m})$ which we endow with the counting measure "$d\mathbf{m}$." Notice that both spaces are isomorphic as an abstract group but different measures are given between them. For this reason, the norm of a function depends on its domain. For maximum clarity and ease of exposition of norms, we explicitly define the following norms as sums: for $1\leq s<\infty$,

$$\begin{aligned}\|g\|^s_{L^s(\mathbb{F}_q^d,d\mathbf{m})}&=\sum_{\mathbf{m}\in\mathbb{F}_q^d}|g(\mathbf{m})|^s,\\ \|f\|^s_{L^s(\mathbb{F}_q^d,d\mathbf{x})}&=q^{-d}\sum_{\mathbf{x}\in\mathbb{F}_q^d}|f(\mathbf{x})|^s.\end{aligned}$$

In addition, we define

$$\|g\|_{L^\infty(\mathbb{F}_q^d,d\mathbf{m})}=\max_{\mathbf{m}\in\mathbb{F}_q^d}|g(\mathbf{m})|.$$

Next, we introduce the normalized surface measures on spheres in finite fields. For $t\in\mathbb{F}_q^*$, we consider a sphere $S_t\subset(\mathbb{F}_q^d, d\mathbf{x})$. For each $t\in\mathbb{F}_q^*$, we endow the sphere S_t with the normalized surface measure $d\sigma$. Thus, if $f:(\mathbb{F}_q^d, d\mathbf{x})\to\mathbb{C}$, then we define

$$\begin{aligned}\|f\|^s_{L^s(S_t,d\sigma)}&=\frac{1}{|S_t|}\sum_{\mathbf{x}\in S_t}|f(\mathbf{x})|^s\quad\text{for } 1\leq s<\infty,\\ \|f\|_{L^\infty(S_t,d\sigma)}&=\max_{\mathbf{x}\in S_t}|f(\mathbf{x})|.\end{aligned}$$

Also recall that if $f : (S_t, d\sigma) \to \mathbb{C}$, then the inverse Fourier transform of $fd\sigma$ is given by

$$(fd\sigma)^\vee(\mathbf{m}) = \frac{1}{|S_t|}\sum_{\mathbf{x}\in S_t} f(\mathbf{x})\chi(\mathbf{m}\cdot\mathbf{x}) \quad \text{for } \mathbf{m} \in (\mathbb{F}_q^d, d\mathbf{m}).$$

Since $S_t = -S_t := \{\mathbf{x} \in \mathbb{F}_q^d : -\mathbf{x} \in S_t\}$, it follows from the definition of the normalized Fourier transform that

$$(d\sigma)^\vee(\mathbf{m}) = \frac{q^d}{|S_t|}\widehat{S}_t(\mathbf{m}) \quad \text{for } \mathbf{m} \in (\mathbb{F}_q^d, d\mathbf{m}). \tag{3.1}$$

With the above notation, the extension problem for the sphere S_t is to determine $1 \le p, r \le \infty$ such that for some $C > 0$,

$$\|(fd\sigma)^\vee\|_{L^r(\mathbb{F}_q^d, d\mathbf{m})} \le C\|f\|_{L^p(S_t, d\sigma)} \quad \text{for all } f : S_t \to \mathbb{C}, \tag{3.2}$$

where the constant $C > 0$ may depend on p, r, d, S_t, but it must be independent of the functions f and the size of the underlying finite field $\mathbb{F}_q$. By duality, this extension estimate is the same as the following restriction estimate (see [24, 25]) :

$$\|\widetilde{g}\|_{L^{p'}(S_t, d\sigma)} \le C\|g\|_{L^{r'}(\mathbb{F}_q^d, d\mathbf{m})} \quad \text{for all } g : \mathbb{F}_q^d \to \mathbb{C}, \tag{3.3}$$

where $\widetilde{g}$ is defined as in (2.1), and p' and r' denote the Hölder conjugates of p and r, which mean that $1/p + 1/p' = 1$ and $1/r + 1/r' = 1$.

Remark 3.1. In this paper, we will use $X \lesssim Y$ to mean that there exists $C > 0$, independent of q such that $X \le CY$, and we also write $Y \gtrsim X$ for $X \lesssim Y$. We use $X \sim Y$ to indicate that $\lim_{q\to\infty} X/Y = 1$. In addition, $X \lessapprox Y$ means that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $X \le C_\varepsilon q^\epsilon Y$.

By the definition of norms and Fourier transforms, the inequalities in (3.2) and (3.3) are written in terms of the following sums, respectively:

$$\sum_{\mathbf{m}\in\mathbb{F}_q^d}\left|\frac{1}{|S_t|}\sum_{\mathbf{x}\in S_t}\chi(\mathbf{m}\cdot\mathbf{x})f(\mathbf{x})\right|^r \le C^r\frac{1}{|S_t|^{r/p}}\left(\sum_{\mathbf{x}\in S_t}|f(\mathbf{x})|^p\right)^{r/p}$$

and

$$\frac{1}{|S_t|}\sum_{\mathbf{x}\in S_t}\left|\sum_{\mathbf{m}\in\mathbb{F}_q^d}\chi(-\mathbf{m}\cdot\mathbf{x})g(\mathbf{m})\right|^{p'} \le C^{p'}\left(\sum_{\mathbf{m}\in\mathbb{F}_q^d}|g(\mathbf{m})|^{r'}\right)^{p'/r'}. \tag{3.4}$$

In particular, in (3.4) if we take $g(x) = E(x)$ for $E \subset \mathbb{F}_q^d$, and $p' = k$, then we obtain that

$$\frac{1}{|S_t|}\sum_{\mathbf{x}\in S_t}|q^d\widehat{E}(\mathbf{x})|^k \lesssim |E|^{k/r'}.$$

Since $|S_t| \sim q^{d-1}$for $t \ne 0$, if $t \ne 0$, then we can write

$$\sum_{\mathbf{x}\in S_t}|\widehat{E}(\mathbf{x})|^k \lesssim q^{d-dk-1}|E|^{k/r'}.$$

If we put $\ell = r'$ and change the variable $\mathbf{x}$ into $\mathbf{m}$, it follows that

$$\sum_{\mathbf{m}\in S_t}|\widehat{E}(\mathbf{m})|^k \lesssim q^{d-dk-1}|E|^{k/\ell}.$$

In summary, we obtain the following result.

Lemma 3.2. *Assume that the following restriction estimate holds for* $1 \le k, \ell < \infty$

$$\|\widetilde{g}\|_{L^k(S_t, d\sigma)} \lesssim \|g\|_{L^\ell(\mathbb{F}_q^d, d\boldsymbol{m})} \quad \text{for all } g : \mathbb{F}_q^d \to \mathbb{C},\ t \in \mathbb{F}_q^*.$$

Then we have

$$\max_{t \in \mathbb{F}_q^*} \left(\sum_{\boldsymbol{m} \in S_t} |\widehat{E}(\boldsymbol{m})|^k \right) \lesssim q^{d-dk-1} |E|^{k/\ell} \quad \text{for all } \ E \subset \mathbb{F}_q^d.$$

In the finite field setting, the extension problem for various varieties was first posed by Mockenhaupt and Tao ([24]). They mainly obtained good results for paraboloids in lower dimensions. Their results have been recently improved (see, for example, [11, 21, 22, 23]). The extension problem for spheres is more delicate than that of paraboloids, and it was studied by Iosevich and Koh. In [10], they obtained the sharp $L^2 - L^4$ extension result for circles, which the authors of [2] applied to deduce the exponent $4/3$ for the Erdős-Falconer distance problem in dimension two. Recall that if $d = 2$, then the exponent $4/3$ gives a much better result than the exponent $(d+1)/2$ which is optimal for odd dimensions. When $d \ge 3$, the $L^2 - L^{(2d+2)/(d-1)}$ extension result for spheres is also known in [10] and can be also applied to the Erdős-Falconer distance problem but we can only obtain the exponent $(d+1)/2$.

In [12], Iosevich and Koh investigated the $L^p - L^4$ spherical extension problem, and they proved the following result which improves the previous work in [10].

Proposition 3.3 ([12], Theorem 1)**.** *Let $d \ge 4$ be even. Then we have*

$$\|(Ed\sigma)^\vee\|_{L^4(\mathbb{E}_q^d, d\boldsymbol{m})} \lesssim \|E\|_{L^{(12d-8)/(9d-12)}(S_t, d\sigma)} \quad \text{for all } E \subset S_t,\ t \ne 0, \tag{3.5}$$

where we identify the set E with the characteristic function of $E \subset S_t$. In addition, using the pigeonhole principle, (3.5) *implies that*

$$\|(fd\sigma)^\vee\|_{L^4(\mathbb{F}_q^d, d\boldsymbol{m})} \lessapprox \|f\|_{L^{(12d-8)/(9d-12)}(S_t, d\sigma)} \quad \text{for all } f : S_t \to \mathbb{C},\ t \ne 0. \tag{3.6}$$

Remark 3.4. Theorem 1 in [12] is actually the statement (3.6). In order to prove it, the authors in [12] proved the statement (3.5) and then concluded that the statement (3.6) holds by just invoking the pigeonhole principle (a dyadic decomposition). It is well known that (3.5) implies (3.6) (for example, see the proof of Theorem 17, [7]). In fact, if (3.5) is true, then the pigeonhole principle yields that

$$\|(fd\sigma)^\vee\|_{L^4(\mathbb{F}_q^d, d\mathbf{m})} \lesssim \log q \, \|f\|_{L^{(12d-8)/(9d-12)}(S_t, d\sigma)} \quad \text{for all } f : S_t \to \mathbb{C},\ t \ne 0.$$

Proposition 3.3 plays an important role in proving results for the cardinality of $\Delta_3(E)$. For the direct application to the problem, we shall use the following lemma which is actually a corollary of Proposition 3.3 .

Lemma 3.5. *Let $d \ge 4$ be even. If $k > (12d-8)/(3d+4) = 4 - 24/(3d+4)$, then we have*

$$\max_{t \in \mathbb{F}_q^*} \left(\sum_{\boldsymbol{v} \in S_t} \left| \widehat{E}(\boldsymbol{v}) \right|^k \right) \lesssim q^{d-dk-1} |E|^{((3k-3)d+4k+2)/(3d+4)}$$

Proof. By Lemma 3.2, it will be enough to show that

$$\|\widetilde{g}\|_{L^k(S_t,d\sigma)} \lesssim \|g\|_{L^\alpha(\mathbb{F}_q^d,d\mathbf{m})} \quad \text{for all } g:\mathbb{F}_q^d\to\mathbb{C},\ t\neq 0, \tag{3.7}$$

where $\alpha = k(3d+4)/((3k-3)d+4k+2)$. It is clear that

$$\|(fd\sigma)^\vee\|_{L^\infty(\mathbb{F}_q^d,d\mathbf{m})} \le \|f\|_{L^1(S_t,d\sigma)} \quad \text{for all } f:S_t\to\mathbb{C},\ t\neq 0. \tag{3.8}$$

For any even integer $d\ge 4$, recall from (3.5) in Proposition 3.3 that

$$\|(Ed\sigma)^\vee\|_{L^4(\mathbb{F}_q^d,d\mathbf{m})} \lesssim \|E\|_{L^{(12d-8)/(9d-12)}(S_t,d\sigma)} \quad \text{for all } E\subset S_t,\ t\neq 0. \tag{3.9}$$

We need the following proposition which is a direct consequence of Theorem 1.4.19 in [6].

Proposition 3.6. *Let $0<p_0\neq p_1\le\infty$, and $0<r_0\neq r_1\le\infty$. Assume that for some $M_0, M_1>0$, the following estimates hold:*

$$\|(Ed\sigma)^\vee\|_{L^{r_0}(\mathbb{F}_q^d,d\mathbf{m})} \le M_0\|E\|_{L^{p_0}(S_t,d\sigma)}$$
$$\|(Ed\sigma)^\vee\|_{L^{r_1}(\mathbb{F}_q^d,d\mathbf{m})} \le M_1\|E\|_{L^{p_1}(S_t,d\sigma)}$$

for all $E\subset S_t$. Fix $0<\theta<1$ and let

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1},\quad \frac{1}{r}=\frac{1-\theta}{r_0}+\frac{\theta}{r_1},\quad \text{and}\quad p\le r.$$

Then there exists a constant $M>0$ such that

$$\|(fd\sigma)^\vee\|_{L^r(\mathbb{F}_q^d,d\mathbf{m})} \le M\|f\|_{L^p(S_t,d\sigma)} \quad \text{for all } f:S_t\to\mathbb{C},$$

where $M>0$ is independent of the functions f and q, the size of the underlying finite field $\mathbb{F}_q$.

Since we assume that $k>(12d-8)/(3d+4)$ and $d\ge 4$, it is easy to see that $1<k/(k-1)<(12d-8)/(9d-12)$. Therefore, applying Proposition 3.6 with (3.8) and (3.9), we see that

$$\|(fd\sigma)^\vee\|_{L^{k(3d+4)/(3d-2)}(\mathbb{F}_q^d,d\mathbf{m})} \lesssim \|f\|_{L^{k/(k-1)}(S_t,d\sigma)} \quad \text{for all } f:S_t\to\mathbb{C},\ t\neq 0.$$

By duality[1], the statement (3.7) follows immediately and we complete the proof of Lemma 3.5. □

Observe that the hypotheses of Lemma 3.5 are satisfied if $k\ge 4$ and $d\ge 4$ is even or if $k=3$ and $d=4$ or 6. However, in the case when $k=3$ and $d\ge 8$ is even, it is clear that Lemma 3.5 is not applicable. In this case, we shall alternatively use the following result.

Lemma 3.7. *Let $d\ge 8$ be an even integer. If $E\subset\mathbb{F}_q^d$ and $|E|\ge q^{(d-1)/2}$, then we have*

$$\max_{t\in\mathbb{F}_q^*}\left(\sum_{v\in S_t}\left|\widehat{E}(v)\right|^3\right) \lessapprox q^{(-27d^2+75d+12)/(12d-32)}|E|^{(15d-46)/(6d-16)}.$$

[1] This means that the inequality (3.2) is equivalent to the inequality (3.3)

Proof. Since $|S_t| \sim q^{d-1}$ for even $d \geq 8$, and $\widehat{E}(\mathbf{v}) = q^{-d}\widetilde{E}(\mathbf{v})$, it suffices to show from the definition of norms that if $E \subset \mathbb{F}_q^d$ and $|E| \geq q^{(d-1)/2}$, then

$$\|\widetilde{E}\|_{L^3(S_t,d\sigma)} \lessapprox q^{(-3d^2+23d-20)/(36d-96)}|E|^{(15d-46)/(18d-48)} \quad \text{for all } t \neq 0. \tag{3.10}$$

Let us assume for a moment that if $t \in \mathbb{F}_q^*$, then

$$\|\widetilde{E}\|_{L^2(S_t,d\sigma)} \lesssim q^{(-d+1)/4}|E| \quad \text{for all } E \subset \mathbb{F}_q^d \quad \text{with } |E| \geq q^{(d-1)/2}. \tag{3.11}$$

By duality, (3.6) in Proposition 3.3 implies that

$$\|\widetilde{g}\|_{L^{(12d-8)/(3d+4)}(S_t,d\sigma)} \lessapprox \|g\|_{L^{4/3}(\mathbb{F}_q^d,d\mathbf{m})} \quad \text{for all } g : \mathbb{F}_q^d \to \mathbb{C},\ t \neq 0.$$

Taking g as a characteristic function on $E \subset \mathbb{F}_q^d$, we obtain that

$$\|\widetilde{E}\|_{L^{(12d-8)/(3d+4)}(S_t,d\sigma)} \lessapprox \|E\|_{L^{4/3}(\mathbb{F}_q^d,d\mathbf{m})} = |E|^{3/4} \quad \text{for all } E \subset \mathbb{F}_q^d,\ t \neq 0. \tag{3.12}$$

Since $2 < 3 < (12d-8)/(3d+4)$ for $d \geq 8$, we are able to interpolate (3.11) and (3.12) for $E \subset \mathbb{F}_q^d$ with $|E| \geq q^{(d-1)/2}$ so that the inequality (3.10) will be established. For the readers' convenience, we shall show how to deduce the inequality (3.10) from inequalities (3.11) and (3.12). Let $0 < \theta = (6d-4)/(9d-24) < 1$ for even $d \geq 8$. Observe that

$$\frac{1}{3} = \frac{1-\theta}{2} + \frac{(3d+4)\theta}{12d-8}. \tag{3.13}$$

By Hölder's inequality (see [15]) and the definition of norms, it follows

$$\begin{aligned}
\|\widetilde{E}\|_{L^3(S_t,d\sigma)} &= \|\widetilde{E}^{(1-\theta)}\,\widetilde{E}^{\theta}\|_{L^3(S_t,d\sigma)} \\
&\leq \|\widetilde{E}^{(1-\theta)}\|_{L^{2/(1-\theta)}(S_t,d\sigma)}\,\|\widetilde{E}^{\theta}\|_{L^{(12d-8)/[(3d+4)\theta]}(S_t,d\sigma)} \\
&= \|\widetilde{E}\|^{1-\theta}_{L^2(S_t,d\sigma)}\,\|\widetilde{E}\|^{\theta}_{L^{(12d-8)/(3d+4)}(S_t,d\sigma)}
\end{aligned}$$

From (3.11), (3.12), and the definition of θ, we conclude that

$$\begin{aligned}
\|\widetilde{E}\|_{L^3(S_t,d\sigma)} &\lessapprox \left(q^{(-d+1)/4}|E|\right)^{1-\theta}|E|^{3\theta/4} \\
&= q^{(-3d^2+23d-20)/(36d-96)}|E|^{(15d-46)/(18d-48)}.
\end{aligned}$$

To complete the proof of Lemma 3.7, it therefore remains to prove (3.11). Now we prove (3.11). Since $|S_t| \sim q^{d-1}$, by the definition of norms, the proof of (3.11) amounts to showing that if $t \in \mathbb{F}_q^*$ and $E \subset \mathbb{F}_q^d$ with $|E| \geq q^{\frac{d-1}{2}}$, then

$$\sum_{\mathbf{x} \in S_t} |\widetilde{E}(\mathbf{x})|^2 \lesssim q^{\frac{d-1}{2}}|E|^2. \tag{3.14}$$

It follows from the definition of the Fourier transforms that

$$\begin{aligned}\sum_{\mathbf{x}\in S_t}|\widetilde{E}(\mathbf{x})|^2 &= \sum_{\mathbf{x}\in S_t}\sum_{\mathbf{m},\mathbf{m}'\in E}\chi(-\mathbf{x}\cdot(\mathbf{m}-\mathbf{m}')) = \sum_{\mathbf{m},\mathbf{m}'\in E} q^d\widehat{S}_t(\mathbf{m}-\mathbf{m}')\\
&= q^d|E|\widehat{S}_t(0,\ldots,0) + \sum_{\mathbf{m},\mathbf{m}'\in E:\mathbf{m}\neq\mathbf{m}'} q^d\widehat{S}_t(\mathbf{m}-\mathbf{m}')\\
&\leq |E||S_t| + \left(\max_{\mathbf{n}\in\mathbb{F}_q^d\setminus\{(0,\ldots,0)\}}|\widehat{S}_t(\mathbf{n})|\right)\sum_{\mathbf{m},\mathbf{m}'\in E:\mathbf{m}\neq\mathbf{m}'} q^d\\
&\lesssim |E|q^{d-1} + |E|^2q^d\left(\max_{\mathbf{n}\in\mathbb{F}_q^d\setminus\{(0,\ldots,0)\}}|\widehat{S}_t(\mathbf{n})|\right).\end{aligned}$$

Now, observe from Proposition 2.2 that if $t\neq 0$, then

$$\left(\max_{\mathbf{n}\in\mathbb{F}_q^d\setminus\{(0,\ldots,0)\}}|\widehat{S}_t(\mathbf{n})|\right)\lesssim q^{-\frac{d+1}{2}}.$$

Thus, we conclude that

$$\sum_{\mathbf{x}\in S_t}|\widetilde{E}(\mathbf{x})|^2 \lesssim |E|q^{d-1} + q^{\frac{d-1}{2}}|E|^2 \lesssim q^{\frac{d-1}{2}}|E|^2,$$

where the last inequality follows by our assumption that $|E|\geq q^{\frac{d-1}{2}}$.

□

4. Proofs of main theorems (Theorems 1.3 and 1.4)

We begin by deriving the formula for a lower bound of $|\Delta_k(E)|$. Let $E\subset\mathbb{F}_q^d$ and let $k\geq 2$ be an integer. For $t\in\mathbb{F}_q$, recall that the counting function $\nu_k(t)$ is defined by

$$\nu_k(t) = |\{(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_k)\in E^k : \|\mathbf{x}_1+\mathbf{x}_2+\cdots+\mathbf{x}_k\| = t\}|.$$

Also recall that the k-resultant magnitude set $\Delta_k(E)$ is given by

$$\Delta_k(E) = \{\|\mathbf{x}_1+\mathbf{x}_2+\cdots+\mathbf{x}_k\|\in\mathbb{F}_q : \mathbf{x}_s\in E, s=1,2,\ldots,k\}.$$

Notice that $\nu_k(t)\neq 0 \iff t\in\Delta_k(E)$. It is clear that

$$|E|^k - \nu_k(0) = \sum_{t\in\mathbb{F}_q^*}\nu_k(t).$$

Squaring both sizes and using the Cauchy-Schwarz inequality, we see that

$$(|E|^k-\nu_k(0))^2 \leq |\Delta_k(E)|\sum_{t\in\mathbb{F}_q^*}\nu_k^2(t).$$

Namely, we obtain that

$$|\Delta_k(E)| \geq \frac{(|E|^k-\nu_k(0))^2}{\sum_{t\in\mathbb{F}_q^*}\nu_k^2(t)}. \tag{4.1}$$

Lemma 4.1. *Let $E\subset\mathbb{F}_q^d$. Suppose that $d\geq 2$ is even and $k\geq 2$ is an integer. If $|E|\geq 3q^{d/2}$, then we have*

$$|\Delta_k(E)| \gtrsim \min\left(q,\ \frac{|E|^{k+1}}{q^{dk}\max_{r\in\mathbb{F}_q^*}\left(\sum_{\boldsymbol{v}\in S_r}|\widehat{E}(\boldsymbol{v})|^k\right)}\right).$$

Proof. First, we find an upper bound for $\sum_{t\in\mathbb{F}_q^*}\nu_k^2(E)$. Write

$$\sum_{t\in\mathbb{F}_q^*}\nu_k^2(t)=\left(\sum_{t\in\mathbb{F}_q}\nu_k^2(t)\right)-\nu_k^2(0).$$

From Lemma 2.5 and Lemma 2.7, we see that

$$\begin{aligned}\sum_{t\in\mathbb{F}_q^*}\nu_k^2(E)&\le q^{-1}|E|^{2k}+q^{2dk-d}\sum_{r\in\mathbb{F}_q}\left|\sum_{\mathbf{v}\in S_r}\left(\widehat{E}(\mathbf{v})\right)^k\right|^2-\nu_k^2(0)\\&\le 5q^{-1}|E|^{2k}+q^{2dk-d}\sum_{r\in\mathbb{F}_q^*}\left|\sum_{\mathbf{v}\in S_r}\left(\widehat{E}(\mathbf{v})\right)^k\right|^2\\&\lesssim q^{-1}|E|^{2k}+q^{2dk-d}\sum_{r\in\mathbb{F}_q^*}\left(\sum_{\mathbf{v}\in S_r}\left|\widehat{E}(\mathbf{v})\right|^k\right)^2.\end{aligned}$$

Since S_i and S_j are disjoint for $i\neq j$, and $\bigcup\limits_{r\in\mathbb{F}_q} S_r=\mathbb{F}_q^d$, it follows that

$$\sum_{t\in\mathbb{F}_q^*}\nu_k^2(E)\lesssim q^{-1}|E|^{2k}+q^{2dk-d}\left[\max_{r\in\mathbb{F}_q^*}\left(\sum_{\mathbf{v}\in S_r}\left|\widehat{E}(\mathbf{v})\right|^k\right)\right]\sum_{\mathbf{v}\in\mathbb{F}_q^d}\left|\widehat{E}(\mathbf{v})\right|^k$$

Now, using (2.2), we obtain that

$$\sum_{t\in\mathbb{F}_q^*}\nu_k^2(E)\lesssim q^{-1}|E|^{2k}+q^{dk}|E|^{k-1}\left[\max_{r\in\mathbb{F}_q^*}\left(\sum_{\mathbf{v}\in S_r}\left|\widehat{E}(\mathbf{v})\right|^k\right)\right].\tag{4.2}$$

Since it follows from Lemma 2.6 that $\left(|E|^k-\nu_k(0)\right)^2\ge\frac{|E|^{2k}}{9}$, combining (4.1) with (4.2) yields that

$$|\Delta_k(E)|\gtrsim\frac{|E|^{2k}}{q^{-1}|E|^{2k}+q^{dk}|E|^{k-1}\left[\max_{r\in\mathbb{F}_q^*}\left(\sum_{\mathbf{v}\in S_r}\left|\widehat{E}(\mathbf{v})\right|^k\right)\right]}.$$

This implies the conclusion of Lemma 4.1 and completes the proof. □

We are ready to prove our main results.

4.1. **Proof of Theorem 1.3.** In this subsection, we restate Theorem 1.3 and provide a complete proof. The statement of Theorem 1.3 will be a direct consequence from Lemma 4.1 and Lemma 3.5.

Theorem 1.3. *Let $E\subset\mathbb{F}_q^d$. Suppose that $C>1$ is a sufficiently large constant independent of q.*
(1) If $d=4$ or 6, and $|E|\ge Cq^{\frac{d+1}{2}-\frac{1}{6d+2}}$, then $|\Delta_3(E)|\gtrsim q$.
(2) If $d\ge 8$ is even and $|E|\ge Cq^{\frac{d+1}{2}-\frac{1}{6d+2}}$, then $|\Delta_4(E)|\gtrsim q$.

Proof. We shall prove the statements *(1)* and *(2)* of Theorem 1.3 at one time. To the end, notice that if we take $k=3$ for $d=4$ or 6, or if we choose $k=4$ for $d\ge 8$ even, then $k>(12d-8)/(3d+4)$ which is the hypothesis of Lemma 3.5. In either case, we can therefore use the conclusion of Lemma 3.5. Thus, combining Lemma 4.1 and Lemma 3.5 yields that

$$|\Delta_k(E)| \gtrsim \min\left(q, \frac{|E|^{k+1}}{q^{d-1}|E|^{((3k-3)d+4k+2)/(3d+4)}}\right). \tag{4.3}$$

By a direct computation, this implies that there exists a large constant $C > 1$ such that if $|E| \ge Cq^{(3d^2+4d)/(6d+2)} = Cq^{\frac{d+1}{2}-\frac{1}{6d+2}}$, then $|\Delta_k(E)| \gtrsim q$. Thus, the proof is complete. □

4.2. **Proof of Theorem 1.4.** The proof of Theorem 1.4 can be completed by applying Lemma 4.1 and Lemma 3.7. Here, we restate Theorem 1.4 and provide a complete proof.

Theorem 1.4 *Suppose that $d \ge 8$ is even and $E \subset \mathbb{F}_q^d$. Then given $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that if $|E| \ge C_\varepsilon q^{\frac{d+1}{2}-\frac{1}{9d-18}+\varepsilon}$, then $|\Delta_3(E)| \gtrsim q$.*

Proof. Suppose that $d \ge 8$ is even and $E \subset \mathbb{F}_q^d$ with $|E| \ge 3q^{d/2}$. Then Lemma 4.1 with $k = 3$ yields

$$|\Delta_3(E)| \gtrsim \min\left(q, \frac{|E|^4}{q^{3d}\max_{t\in\mathbb{F}_q^*}\left(\sum_{\mathbf{v}\in S_t}|\widehat{E}(\mathbf{v})|^3\right)}\right). \tag{4.4}$$

Recall from Lemma 3.7 that

$$\max_{t\in\mathbb{F}_q^*}\left(\sum_{\mathbf{v}\in S_t}\left|\widehat{E}(\mathbf{v})\right|^3\right) \lessapprox q^{(-27d^2+75d+12)/(12d-32)}|E|^{(15d-46)/(6d-16)}.$$

Given $\varepsilon > 0$, let $\delta = \varepsilon(9d-18)/(6d-16) > 0$. Choose $C_\delta > 0$ such that

$$\max_{t\in\mathbb{F}_q^*}\left(\sum_{\mathbf{v}\in S_t}\left|\widehat{E}(\mathbf{v})\right|^3\right) \le C_\delta q^\delta q^{(-27d^2+75d+12)/(12d-32)}|E|^{(15d-46)/(6d-16)}.$$

It follows from this inequality and (4.4) that if $|E| \ge 3q^{d/2}$, then

$$|\Delta_3(E)| \gtrsim \min\left(q, \frac{|E|^4}{q^{3d}C_\delta q^\delta q^{(-27d^2+75d+12)/(12d-32)}|E|^{(15d-46)/(6d-16)}}\right).$$

We may assume that $C_\delta > 0$ is a sufficiently large constant. Thus, a direction calculation shows that if

$$|E| \ge C_\delta^{(6d-16)/(9d-18)} q^{\delta(6d-16)/(9d-18)} q^{(9d^2-9d-20)/(18d-36)},$$

then we have $|\Delta_3(E)| \gtrsim q$. Letting $C_\epsilon = C_\delta^{(6d-16)/(9d-18)}$, we conclude that if

$$|E| \ge C_\varepsilon q^\varepsilon q^{(9d^2-9d-20)/(18d-36)} = C_\varepsilon q^{\frac{d+1}{2}-\frac{1}{9d-18}+\varepsilon},$$

then $|\Delta_3(E)| \gtrsim q$. This completes the proof.

□

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