CONVERGENCE OF A HIGHER-ORDER SCHEME FOR KORTEWEG-DE VRIES EQUATION

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ABSTRACT. We are study the convergence of higher order schemes for the Cauchy problem associated to the KdV equation. More precisely, we design a Galerkin type implicit scheme which has higher order accuracy in space and first order accuracy in time. The convergence is established for initial data in L^2 , and we show that the scheme converges strongly in $L^2(0,T;L^2_{loc}(\mathbb{R}))$ to a weak solution. Finally, the convergence is illustrated by several examples.

1. INTRODUCTION

In this paper, we consider a higher order finite element Galerkin type scheme for computing approximate solutions of the Cauchy problem for Korteweg-de Vries (KdV) equation

(1)
$$\begin{cases} u_t + (\frac{u^2}{2})_x + u_{xxx} = 0, & x \in \mathbb{R} \times (0, T) \\ u(x, 0) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where T > 0 is fixed, $u : \mathbb{R} \times [0, T) \to \mathbb{R}$ is the unknown, and u_0 the initial data.

It is well known that the KdV equation models the propagation of waves of small amplitude in dispersive systems (e.g., magneto-acoustic waves in plasmas, shallow water waves, lattice waves and so on). Also, the KdV equation has localized solutions, i.e., solutions whose value approach a constant for |x| large, called solitons. These have the property that their speed increases with their amplitude, and that such solitary waves interact in a particle like manner.

The first mathematical proof of existence and uniqueness of solutions of the KdV equation was accomplished by Sjöberg [13] in 1970, using a semi-discrete finite difference approximation, where one discretizes the spatial variable, thereby reducing the equation to a system of ordinary differential equations.

Well posedness for the KdV equation has been studied extensively in the last three decades, see [14, 10] and the references therein. We will not discuss the vast literature regarding the mathematical properties of the KdV equation here, but mention that local well posedness local is proved in the Sobolev spaces H^s for s > -3/4 in [8].

On the other hand, numerical computations for the KdV equation has also been of great interest, since the landmark work by Zabusky and Kruskal [16], where they discovered the permanence of solitons for KdV equation using numerical techniques. In fact, the numerical computation of solutions of the KdV equation is rather capricious. Two competing effects are involved, namely the nonlinear convective term uu_x , which in the context of the Burgers equation $u_t + uu_x = 0$ yields infinite

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gradients in finite time even for smooth data, and the linear dispersive term u_{xxx} , which in the Airy equation $u_t + u_{xxx} = 0$ produces hard-to-compute dispersive waves, and these two effects combined makes it difficult to obtain accurate and fast numerical methods.

There are number of numerical schemes available to analyze the behaviour of solutions to the KdV equation numerically. We will discuss the full literature here, but only refer to those results which are relevant to this paper.

Spectral methods have been studied extensively, see [11, 6] and references therein. Multi-symplectic schemes have been studied in [2] (see also references therein). Standard Galerkin type approximations, using smooth splines on a uniform mesh, to periodic solutions of KdV equation are analyzed in [1, 3, 15]. All these work aimed at deriving optimal rate of convergence estimate for Galerkin approximations. The discontinuous Galerkin method has been used to approximate the solution of (1) and rate of convergence analysis has been presented for both periodic and full line case in [12].

All the above mentioned references use the well posedness theory for the KdV equation to prove convergence, and convergence rates. Therefore, by themselves, they do not yield the existence of a solution by furnishing constructive existence proofs.

There are however a few results regarding proof of convergence of numerical methods for the KdV equation, which also give a direct and constructive existence theorem. Indeed, the first proof of existence and uniqueness of solutions to the KdV equation for initial data in $H^3(\mathbb{R}/\mathbb{Z})$ is based on a semi-discrete difference approximation [13]. The corresponding fully discrete scheme, which incidentally coincides with a fully discrete splitting scheme, was analyzed in [5], and it was shown that the scheme converges to the classical solution if the initial data is in $H^3(\mathbb{R})$, and to the weak solution if the initial data lies in $L^2(\mathbb{R})$. The proof assumes the CFL condition $\Delta t = \mathcal{O}(\Delta x^2)$ where Δt and Δx are the temporal and the spatial discretizations respectively. Laumer proved the direct convergence of a similar scheme, but under the improved CFL condition $\Delta t = \mathcal{O}(\Delta x)$. The results in this paper can be seen as a generalization of the above in the context of higher order approximation methods.

Our main tool is an observation due to Kato. In [7] it was proved that the solution operator of the KdV equation has a smoothing effect due to the dispersion. This smoothing permits a proof of existence of solutions if the initial data are only in $L^2(\mathbb{R})$. The smoothing effect inherent in the KdV equation is not as strong as for parabolic equations, and is of course absent in the case of hyperbolic conservation laws. Precisely, solutions of (1) satisfy

(2)
$$\int_{-T}^{T} \int_{-R}^{R} |u_x|^2 \, dx dt \le C(T, R).$$

An analogue of this estimate is the main ingredient in our proof of the convergence of our approximate solutions $u_{\Delta x}$.

The approximation $u_{\Delta x}$ is generated by an implicit Euler discretization of a Galerkin scheme with approximations in a subspace of $H^2(\mathbb{R})$ consisting of piecewise polynomial functions. Inspired by the proof of (2) we define the Galerkin approximations using a weight function φ , which is positive and constant outside an interval (-Q, Q). Using this in our scheme enables us to prove that the collection

 $\{u_{\Delta x}\}_{\Delta x > 0}$ lies in the set

$$W = \left\{ w \in L^2(0,T; H^1([-R,R])) \mid w_t \in L^2(0,T; H^{-2}([-R,R])) \right\},\$$

which is compact in $L^2(0,T;L^2([-R,R]))$ by the Aubin–Simon compactness lemma.

The rest of the paper is organized as follows: In Section 2, we present the necessary notation and define the fully-discrete finite element Galerkin type numerical scheme. Since the fully-discrete scheme is implicit in nature, the solvability of the scheme cannot be taken for granted and this is addressed in Section 2.2. In Section 3, we show the convergence to a weak solution if the initial data is in $L^2(\mathbb{R})$ and finally in Section 4, we exhibit some numerical experiments showing the convergence.

2. Numerical Scheme

We start by introducing some notation needed to define the Galerkin finite element scheme. Throughout this paper we reserve Δx and Δt to denote two small positive numbers that are the spatial and temporal discretization parameters respectively, of the numerical scheme.

For $j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we set $x_j = j\Delta x$, and for $n = 0, 1, \dots, N$, where $N\Delta t = T$ for some fixed time horizon T > 0, we set $t_n = n\Delta t$. Furthermore, we introduce the spatial grid cells $I_j = [x_{j-1}, x_j]$.

Moreover given R > 0, we define the cut off function φ as $\varphi(x) = \overline{\varphi} * w(x)$ where $\overline{\varphi}(x) = \max\{1, \min\{1 + x + R, 1 + 2R\}\}$ and w is a symmetric positive function with integral one and support in [-1, 1]. Let C_R be defined as

(3)
$$C_R = \max\left\{ \|\varphi\|_{L^{\infty}(\mathbb{R})}, \|\varphi_x\|_{L^{\infty}(\mathbb{R})}, \|\varphi_{xx}\|_{L^{\infty}(\mathbb{R})}, \|\varphi_{xxx}\|_{L^{\infty}(\mathbb{R})} \right\}.$$

We define the weighted L^2 inner product as

$$\langle u, v \rangle_{\varphi} = (u, v\varphi)$$

where (\cdot, \cdot) denotes the usual L^2 inner product, and the associated weighted norm by $||u||_{2,\varphi}^2 = \langle u, u \rangle_{\varphi}$.

2.1. Variational Formulation. We assume that r is a fixed integer ≥ 2 and let $\mathbb{P}_r(I)$ denotes the space of polynomials on the interval I of degree $\leq r$. We seek an approximation u to the solution of (1) such that for each $t \in [0, T]$, u belongs to the finite dimensional space

$$S_{\Delta x} = \left\{ v \in H^2(\mathbb{R}) \mid v \in \mathbb{P}_r(I_j) \text{ for all } j \right\}.$$

The variational form is derived by multiplying the strong form (1) with test functions φv , with $v \in S_{\Delta x}$ and φ specified above, and integrating over each element separately. After integrating by parts twice, we obtain

$$(u_t, \varphi v) - \left(\frac{u^2}{2}, (\varphi v)_x\right) + (u_x, (\varphi v)_{xx}) = 0, \quad \forall v \in S_{\Delta x}$$

This is the semi-discrete form of the variational formulation. However, in order to have a practical numerical method, we must use a numerical method to integrate in time. We use the implicit Euler method for this. This scheme reads as follows: Find $u^n \in S_{\Delta x}$ such that

(4)
$$(u^{n+1},\varphi v) - (u^n,\varphi v) - \Delta t \left(\frac{(u^{n+1})^2}{2},(\varphi v)_x\right) + \Delta t \left(u_x^{n+1},(\varphi v)_{xx}\right) = 0,$$

for all $v \in S_{\Delta x}$ and for $n = 0, 1, \ldots$, with initial data given by $u^0 = Pu_0$, where P is the $L^2(\mathbb{R})$ orthogonal projection onto $S_{\Delta x}$. Observe that, this is an implicit scheme, and in order to calculate u^{n+1} given u^n one must solve a non-linear equation.

2.2. Solvability for one time step. To solve (4), we use a simple fixpoint iteration, and define the sequence $\{w^{\ell}\}_{\ell \geq 0}$ by letting $w^{\ell+1}$ be the solution of the linear equation

(5)
$$\begin{cases} \left(w^{\ell+1},\varphi v\right) + \Delta t \left(w^{\ell} w_x^{\ell}, \varphi v\right) + \Delta t \left(w_x^{\ell+1}, (\varphi v)_{xx}\right) = (u,\varphi v), \\ w^0 = u^n, \end{cases}$$

this is to hold for all test functions $v \in S_{\Delta x}$. The following lemma guarantee the solvability of the implicit scheme (4).

Lemma 2.1. Choose a constant L such that 0 < L < 1 and set

$$K = \frac{7 - L}{1 - L} > 7.$$

We consider the iteration (5) with $w^0 = u^n$, and assume that the following CFL condition holds

(6)
$$\lambda \le \frac{L}{\sqrt{C_R} 2\sqrt{2}K \|u^n\|_{2,\varphi}}$$

where where C_R is defined by (3) and λ is given by

(7)
$$\lambda^2 = \frac{\Delta t^2}{\Delta x^3}$$

Then there exists a function u^{n+1} which solves (4), and $\lim_{\ell\to\infty} w^{\ell} = u^{n+1}$. Furthermore

(8)
$$||u^{n+1}||_{2,\varphi} \le K ||u^n||_{2,\varphi}$$

Proof. From (5) we have

(9) $(w^{\ell+1} - w^{\ell}, \varphi v) + \Delta t (w^{\ell} w_x^{\ell} - w^{\ell-1} w_x^{\ell-1}, \varphi v) + \Delta t (w_x^{\ell+1} - w_x^{\ell}, (\varphi v)_{xx}) = 0$ for any $v \in S_{\Delta x}$. We choose $v = w^{\ell+1} - w^{\ell}$ in (9) to get

$$\begin{aligned} \langle v, v \rangle_{\varphi} + \Delta t \left(v_x, (\varphi v)_{xx} \right) &= -\Delta t \left(w^{\ell} w_x^{\ell} - w^{\ell-1} w_x^{\ell-1}, \varphi(w^{\ell+1} - w^{\ell}) \right) \\ &\leq \frac{1}{2} \langle v, v \rangle_{\varphi} + \frac{\Delta t^2}{2} \left(w^{\ell} w_x^{\ell} - w^{\ell-1} w_x^{\ell-1}, \varphi\left(w^{\ell} w_x^{\ell} - w^{\ell-1} w_x^{\ell-1} \right) \right), \end{aligned}$$

by Young's inequality. Therefore

$$\begin{split} &\frac{1}{2} \left\| v \right\|_{2,\varphi}^{2} + \Delta t \left(v_{x}, \left(v\varphi \right)_{xx} \right) \leq \frac{\Delta t^{2}}{2} \int_{\mathbb{R}} \left(w^{\ell} w_{x}^{\ell} - w^{\ell-1} w_{x}^{\ell-1} \right)^{2} \varphi \, dx \\ &= \frac{\Delta t^{2}}{2} \int_{\mathbb{R}} \left(\left(w^{\ell} - w^{\ell-1} \right) w_{x}^{\ell} - w^{\ell-1} \left(w^{\ell} - w^{\ell-1} \right)_{x} \right)^{2} \varphi \, dx \\ &\leq \Delta t^{2} \int_{\mathbb{R}} \left(w^{\ell} - w^{\ell-1} \right)^{2} \left(w_{x}^{\ell} \right)^{2} \varphi \, dx + \Delta t \int_{\mathbb{R}} \left(w^{\ell-1} \right)^{2} \left(w_{x}^{\ell} - w_{x}^{\ell-1} \right)^{2} \varphi \, dx \\ &\leq \Delta t^{2} \left\| w_{x}^{\ell} \right\|_{L^{\infty}(\mathbb{R})}^{2} \int_{\mathbb{R}} \left(w^{\ell} - w^{\ell-1} \right)^{2} \varphi \, dx + \Delta t \left\| w_{x}^{\ell} - w_{x}^{\ell-1} \right\|_{L^{\infty}(\mathbb{R})}^{2} \int_{\mathbb{R}} \left(w^{\ell-1} \right)^{2} \varphi \, dx \\ &\leq \frac{C\Delta t^{2}}{\Delta x^{3}} \left\| w^{\ell} \right\|_{L^{2}(\mathbb{R})}^{2} \int_{\mathbb{R}} \left(w^{\ell} - w^{\ell-1} \right)^{2} \varphi \, dx + \frac{C\Delta t}{\Delta x^{3}} \left\| w^{\ell} - w^{\ell-1} \right\|_{L^{2}(\mathbb{R})}^{2} \int_{\mathbb{R}} \left(w^{\ell-1} \right)^{2} \varphi \, dx \end{split}$$

$$\leq C_R \lambda^2 \max\left\{ \left\| w^{\ell} \sqrt{\varphi} \right\|_{L^2(\mathbb{R})}^2, \left\| w^{\ell-1} \sqrt{\varphi} \right\|_{L^2(\mathbb{R})}^2 \right\} \int_{\mathbb{R}} \left(w^{\ell} - w^{\ell-1} \right)^2 \varphi \, dx,$$

We have used the following "inverse inequalities"; for any function $z \in S_{\Delta x}$

(10)
$$||z_x||_{L^{\infty}(\mathbb{R})} \le \frac{C}{\Delta x^{1/2}} ||z_x||_{L^2(\mathbb{R})} \le \frac{C}{\Delta x^{3/2}} ||z||_{L^2(\mathbb{R})},$$

where the constant C is independent of z and Δx . To take care the second term on the left, we make an use of the following identity

(11)
$$\int_{\mathbb{R}} w_x (\varphi w)_{xx} dx = \frac{3}{2} \int_{\mathbb{R}} w_x^2 \varphi_x dx - \frac{1}{2} \int_{\mathbb{R}} w^2 \varphi_{xxx} dx,$$

which is established by repeated use of integration by parts. Thus

$$\int_{\mathbb{R}} \left(w^{\ell+1} - w^{\ell} \right)_{x} \left((w^{\ell+1} - w^{\ell}) \varphi \right)_{xx} dx$$
$$= \frac{3}{2} \int_{\mathbb{R}} \left(w^{\ell+1}_{x} - w^{\ell}_{x} \right)^{2} \varphi_{x} dx - \frac{1}{2} \int_{\mathbb{R}} \left(w^{\ell+1} - w^{\ell} \right)^{2} \varphi_{xxx} dx$$

Since $\varphi_x \ge 0$, the second term after the above inequality is non-negative, and we have

$$\Delta t\left(v_{x}, \left(v\varphi\right)_{xx}\right) \geq -C_{R}\Delta t \left\|v\right\|_{L^{2}(\mathbb{R})}^{2} \geq -C_{R}\Delta t \left\|v\right\|_{2,\varphi}^{2}$$

where the constant C_R depends on the φ_{xxx} . Collecting these bounds we get

(12)
$$(1 - \Delta t C_R) \int_{\mathbb{R}} (w^{\ell+1} - w^{\ell})^2 \varphi \, dx$$

 $\leq C_R \lambda^2 \max \left\{ \|w^{\ell}\|_{2,\varphi}^2, \|w^{\ell-1}\|_{2,\varphi}^2 \right\} \int_{\mathbb{R}} (w^{\ell} - w^{\ell-1})^2 \varphi \, dx$

We can always assume that $C_R \Delta t < 1/2$, thus

(13)
$$\begin{aligned} \left\| \left(w^{\ell+1} - w^{\ell} \right) \right\|_{2,\varphi}^{2} \\ &\leq 2C_{R}\lambda^{2} \max\left\{ \left\| w^{\ell} \right\|_{2,\varphi}^{2}, \left\| w^{\ell-1} \right\|_{2,\varphi}^{2} \right\} \left\| \left(w^{\ell} - w^{\ell-1} \right) \right\|_{2,\varphi}^{2} \end{aligned}$$

For w^1 , setting $v = w^1$ in (5), we have

$$\int_{\mathbb{R}} (w^{1})^{2} \varphi \, dx + \Delta t \int_{\mathbb{R}} w^{1}_{x} (\varphi w^{1})_{xx} \, dx$$

$$= \int_{\mathbb{R}} w^{1} (u^{n} - \Delta t u^{n} u^{n}_{x}) \varphi \, dx$$

$$\leq \frac{1}{2} \int_{\mathbb{R}} (w^{1})^{2} \varphi \, dx + \frac{1}{2} \int_{\mathbb{R}} (u^{n} - \Delta t u^{n} u^{n}_{x})^{2} \varphi \, dx$$

$$\leq \frac{1}{2} \int_{\mathbb{R}} (w^{1})^{2} \varphi \, dx + \int_{\mathbb{R}} (u^{n})^{2} \varphi \, dx + \Delta t^{2} \int_{\mathbb{R}} (u^{n} u^{n}_{x})^{2} \varphi \, dx.$$

Therefore, using the inverse inequality (10) and the identity (11), we have

$$\frac{1}{2} \left\| w^{1} \right\|_{2,\varphi}^{2} \leq \left\| u^{n} \right\|_{2,\varphi}^{2} + \frac{\Delta t}{2} \int_{\mathbb{R}} \left(w^{1} \right)^{2} \varphi_{xxx} \, dx + \Delta t^{2} \left\| u_{x}^{n} \right\|_{L^{\infty}(\mathbb{R})}^{2} \int_{\mathbb{R}} \left(u^{n} \right)^{2} \varphi \, dx$$
$$\leq \left\| u^{n} \right\|_{2,\varphi}^{2} + \frac{C_{R} \Delta t}{2} \left\| w^{1} \right\|_{2,\varphi}^{2} + C_{R} \frac{\Delta t^{2}}{\Delta x^{3}} \left\| u^{n} \right\|_{2,\varphi}^{4},$$

and thus

(14)
$$\|w^1\|_{2,\varphi}^2 \le 4\left(1 + C_R\lambda^2 \|u^n\|_{2,\varphi}^2\right) \|u^n\|_{2,\varphi}^2$$

Then we claim that the following holds for $\ell \geq 1$

(15a)
$$\|w^{\ell+1} - w^{\ell}\|_{2,\varphi} \le L \|w^{\ell} - w^{\ell-1}\|_{2,\varphi},$$

(15b)
$$\|w^{\ell}\|_{2,\varphi} \le K \|u^n\|_{2,\varphi},$$

(15c)
$$\left\|w^{1}\right\|_{2,\varphi} \leq 5 \left\|u^{n}\right\|_{2,\varphi}$$

for $\ell = 1, 2, 3, \ldots$ To prove these claims, we argue by induction. Setting $\ell = 1$ in (13) and using (14) gives

$$\begin{split} \left\| w^{2} - w^{1} \right\|_{2,\varphi} &\leq 2\sqrt{C_{R}}\lambda \max\left\{ \left\| w^{1} \right\|_{2,\varphi}, \left\| u^{n} \right\|_{2,\varphi} \right\} \left\| w^{1} - u^{n} \right\|_{2,\varphi} \\ &\leq \frac{2\sqrt{C_{R}}L}{2\sqrt{2}\sqrt{C_{R}}K \left\| u^{n} \right\|_{2,\varphi}} \left\| u^{n} \right\|_{2,\varphi} \\ &\quad \times 4\left(1 + \frac{\sqrt{C_{R}}L}{\sqrt{C_{R}}2\sqrt{2}K \left\| u^{n} \right\|_{2,\varphi}} \left\| u^{n} \right\|_{2,\varphi} \right) \left\| w^{1} - u^{n} \right\|_{2,\varphi} \\ &\leq \frac{4L}{7\sqrt{2}} \left(1 + \frac{L}{14\sqrt{2}} \right) \left\| w^{1} - u^{n} \right\|_{2,\varphi} \\ &\leq L \frac{4}{7\sqrt{2}} \left(1 + \frac{1}{14\sqrt{2}} \right) \left\| w^{1} - u^{n} \right\|_{2,\varphi} \\ &\leq 0.85L \left\| w^{1} - u^{n} \right\|_{2,\varphi}, \end{split}$$

which shows (15a) for $\ell = 1$. To show (15c) note that

$$4\left(1+\sqrt{C_R}\lambda \left\|u^n\right\|_{2,\varphi}\right) \le 4\left(1+\frac{1}{14\sqrt{2}}\right) < 5.$$

Next assume that (15a) and (15b) hold for $\ell = 1, \ldots, m$, then

$$\begin{split} \left\| w^{m+1} \right\|_{2,\varphi} &\leq \sum_{\ell=0}^{m} \left\| w^{\ell+1} - w^{\ell} \right\|_{2,\varphi} + \left\| w^{0} \right\|_{2,\varphi} \\ &\leq \left\| w^{1} - w^{0} \right\|_{2,\varphi} \sum_{\ell=0}^{m} L^{\ell} + \left\| w^{0} \right\|_{2,\varphi} \\ &\leq 6 \left\| u^{n} \right\|_{2,\varphi} \frac{1}{1-L} + \left\| u^{n} \right\|_{2,\varphi} \\ &= \frac{7-L}{1-L} \left\| u^{n} \right\|_{2,\varphi} = K \left\| u^{n} \right\|_{2,\varphi}. \end{split}$$

Hence, (15b) holds for all ℓ . Using (13), this implies that (15a) holds as well. Using (13), one can show that $\{w^{\ell}\}$ is Cauchy, hence $\{w^{\ell}\}$ converges. This completes the proof.

Remark 2.1. Note that, we aim to prove that the iteration scheme (5) converges for all times $t_n = n\Delta t$. We have already shown in previous section that the iteration scheme converges for one time step. However, we had to impose a CFL condition where the ratio between temporal and spatial mesh sizes must be smaller than an upper bound that depends on the computed solution at that time, i.e., u^n . Having said this, since we want the CFL-condition to only depend on the initial data u_0 , we have to derive local a-priori bounds for the computed solution u^n . This will be done in the next section (cf. bound (16)). This bound finally implies that the iteration scheme (5) converges for sufficiently small Δt .

3. Convergence

As we mentioned earlier, the convergence analysis exploits the fact that the solution of the KdV equation possesses an inherent smoothing effect due to its dispersive character. In particular, we need $H^1_{\text{loc}}(\mathbb{R})$ estimate of the approximate solution generated by the scheme (4). We proceed with the following Lemma.

Lemma 3.1. We assume that K and L are given as in the hypothesis of Lemma 2.1. We assume that the initial data $u_0 \in L^2(\mathbb{R})$. Let u^n be the solution of the scheme (4). Then there exists a finite time T and a constant C, depending only on $\|u_0\|_{L^2(\mathbb{R})}$, such that for all n satisfying $n\Delta t \leq T$, the following estimate holds

(16)
$$||u_n||_{L^2(\mathbb{R})} \le C\left(||u_0||_{L^2(\mathbb{R})}\right)$$

provided the following assumption holds

(17)
$$\lambda \le \frac{L}{\sqrt{C_R 2\sqrt{2}K}\sqrt{y_T}}$$

for some y_T which depends only on $||u_0||_{L^2(\mathbb{R})}$. Furthermore, the approximation u^n satisfies the following H^1 estimate

(18)
$$\Delta t \sum_{n \Delta t \leq T} \left\| u_x^{n+1} \right\|_{L^2([-R,R])}^2 \leq C\left(\left\| u^0 \right\|_{L^2(\mathbb{R})}, R \right), \quad \text{for } n \Delta t < T$$

where the constant C depends only on R and $||u_0||_{L^2(\mathbb{R})}$.

Proof. We choose $v = u^{n+1}$ in (4) to obtain

$$\int_{\mathbb{R}} (u^{n+1})^2 \varphi \, dx + \Delta t \int_{\mathbb{R}} u_x^{n+1} (\varphi u^{n+1})_{xx} \, dx = \int_{\mathbb{R}} u^n \varphi u^{n+1} \, dx - \Delta t \int_{\mathbb{R}} (u^{n+1})^2 u_x^{n+1} \varphi \, dx$$

Using Cauchy-Schwartz inequality and the identity (11), we have

(19)
$$\frac{1}{2} \int_{\mathbb{R}} (u^{n+1})^2 \varphi dx + \Delta t \int_{\mathbb{R}} u_x^{n+1} (\varphi u^{n+1})_{xx} dx$$
$$\leq \frac{1}{2} \int_{\mathbb{R}} (u^n)^2 \varphi dx + \frac{\Delta t}{3} \int_{\mathbb{R}} (u^{n+1})^3 \varphi_x dx.$$

Next we estimate $\frac{\Delta t}{3} \int_{\mathbb{R}} (u^{n+1})^3 \varphi_x dx$. To do that, we make an use of the following identity

$$\sup_{x \in \mathbb{R}} v^2(x) \le \frac{1}{2} \int_{\mathbb{R}} |v(x)| |v_x(x)| dx$$

valid for $v \in H^1(\mathbb{R})$. Taking $v = u\sqrt{\varphi_x}$ in we obtain

$$\sup_{x} |u\sqrt{\varphi_x}| \le \frac{1}{\sqrt{2}} \left(\int |u\sqrt{\varphi_x}| |(u\sqrt{\varphi_x})_x| dx \right)^{\frac{1}{2}}$$
$$\le \frac{1}{\sqrt{2}} \left(\int_{\mathbb{R}} |uu_x\varphi_x dx| \right)^{\frac{1}{2}} + \frac{1}{2} \left(\int_{\mathbb{R}} |u^2\varphi_{xx}| dx \right)^{\frac{1}{2}}$$

$$\leq \frac{1}{\sqrt{2}} \left(\int_{\mathbb{R}} u_x^2 \varphi_x \, dx \right)^{\frac{1}{4}} \left(\int_{\mathbb{R}} u^2 \varphi_x \, dx \right)^{\frac{1}{4}} + \frac{1}{2} \left(\int_{\mathbb{R}} u^2 \left| \varphi_{xx} \right| \, dx \right)^{\frac{1}{2}}$$

by the Cauchy-Schwartz inequality. Therefore

$$\begin{split} \int_{\mathbb{R}} u^{3} \varphi_{x} \, dx &\leq \left(\sup |u\sqrt{\varphi_{x}}| \right) \int_{\mathbb{R}} u^{2} \sqrt{\phi_{x}} \, dx \\ &\leq \frac{1}{\sqrt{2}} \left(\int_{\mathbb{R}} u^{2} \varphi_{x} \, dx \right)^{\frac{1}{4}} \left(\int_{\mathbb{R}} u^{2} \varphi_{x} \, dx \right)^{\frac{1}{4}} \left(\int_{\mathbb{R}} u^{2} \sqrt{\phi_{x}} \, dx \right) \\ &\quad + \frac{1}{2} \left(\int_{\mathbb{R}} u^{2} \sqrt{\phi_{x}} \, dx \right) \left(\int_{\mathbb{R}} u^{2} |\varphi_{xx}| \, dx \right)^{\frac{1}{2}}. \end{split}$$

Applying Young's inequality $ab \leq \frac{1}{4}a^4 + \frac{3}{4}b^{4/3}$ for non-negative numbers a and b, we obtain

$$(20) \quad \frac{1}{3} \int_{\mathbb{R}} u^3 \varphi_x \, dx \le \frac{1}{12\sqrt{2}} \int_{\mathbb{R}} u_x^2 \varphi_x \, dx + \frac{1}{4\sqrt{2}} \left(\int_{\mathbb{R}} u^2 \varphi_x \, dx \right)^{\frac{1}{3}} \left(\int_{\mathbb{R}} u^2 \sqrt{\phi_x} \, dx \right)^{\frac{4}{3}} \\ \quad + \frac{1}{6} \left(\int_{\mathbb{R}} u^2 \sqrt{\phi_x} \, dx \right) \left(\int_{\mathbb{R}} u^2 \left| \varphi_{xx} \right| \, dx \right)^{\frac{1}{2}}.$$

Replacing the last term of (19) by the above inequality (20), and using the identity (11) for the second term of (19) gives

$$\frac{1}{2} \int_{\mathbb{R}} (u^{n+1})^2 \varphi \, dx + \left(\frac{3}{2} - \frac{1}{12\sqrt{2}}\right) \Delta t \int_{\mathbb{R}} (u^{n+1})^2 \varphi_x \, dx$$

$$\leq \frac{1}{2} \int_{\mathbb{R}} (u^n)^2 \varphi \, dx + \frac{\Delta t}{4\sqrt{2}} \left(\int_{\mathbb{R}} (u^{n+1})^2 \varphi_x \, dx\right)^{\frac{1}{3}} \left(\int_{\mathbb{R}} (u^{n+1})^2 \sqrt{\phi_x} \, dx\right)^{\frac{4}{3}}$$

$$+ \frac{\Delta t}{6} \left(\int_{\mathbb{R}} (u^{n+1})^2 \sqrt{\phi_x} \, dx\right) \left(\int_{\mathbb{R}} (u^{n+1})^2 |\varphi_{xx}| \, dx\right)^{\frac{1}{2}}$$

$$+ \frac{\Delta t}{2} \int_{\mathbb{R}} (u^{n+1})^2 |\varphi_{xxx}| \, dx.$$

As the derivatives of φ are bounded by the constant C_R , the derivatives of $\varphi^{(j)}(x) \leq \varphi(x)$ for j = 1, 2, 3. Thus, from (21), we obtain

(22)
$$\int_{\mathbb{R}} \left(u^{n+1} \right)^2 \varphi \, dx \leq \int_{\mathbb{R}} \left(u^n \right)^2 \varphi \, dx + \Delta t \, C_R \Big[\left(\int_{\mathbb{R}} \left(u^{n+1} \right)^2 \varphi \, dx \right)^{\frac{5}{3}} + \left(\int_{\mathbb{R}} \left(u^{n+1} \right)^2 \varphi \, dx \right)^{\frac{3}{2}} + \left(\int_{\mathbb{R}} \left(u^{n+1} \right)^2 \varphi \, dx \right) \Big].$$

We have ignored the second term in the inequality (21) since the coefficient $\Delta t(\frac{3}{2} - \frac{1}{12\sqrt{2}})$ is positive. Setting $a_n = \int (u^n)^2 \varphi \, dx$ in (22) gives

(23) $a_{n+1} \le a_n + \Delta t f(a_{n+1})$

where the function f is given by

$$f(a) = C_R \left[a^{\frac{5}{3}} + a^{\frac{3}{2}} + a \right].$$

Therefore, $\{a_n\}$ solves the implicit Backward Euler method for the following differential inequality

$$\frac{da}{dt} \le f(a).$$

Thus we consider the following ordinary differential equation

$$\begin{cases} \frac{dy}{dt} = f(K^2 y), & t > 0, \\ y(0) = a_0. \end{cases}$$

Since the function f is locally Lipschitz continuous for positive arguments, this differential equation has a unique solution which blows up at some finite time, say at $t = T^{\infty}$. We choose $T = T^{\infty}/2$. Also, note that the solution y(t) of the above differential equation is *strictly-increasing* and *convex*. Next we compare the solution of this ODE with (23) under the assumption that (17) holds.

Next we claim that $a_n \leq y(t_n)$ for all $n \geq 0$. We argue by induction. Since $y(0) = a_0$, the claim follows for n = 0. We assume that the claim holds for n = 0, 1, 2, ..., m. As $0 < a_m \leq y(T)$, (17) implies that λ satisfies the CFL condition (6). So, from Lemma 2.1, we have $a_{m+1} \leq K^2 a_m$.

Then, using the convexity of f we have

$$a_{m+1} \le a_m + \Delta t f(K^2 a_m)$$

$$\le y(t_m) + \Delta t f(K^2 y(t_m))$$

$$\le y(t_m) + \Delta t \frac{dy}{dt} \Big|_{t=t_m} \le y(t_{m+1})$$

This proves the claim. Therefore, as $\varphi \geq 1$, we have the required L^2 -stability estimate

$$\left\|u^{n}\right\|_{L^{2}(\mathbb{R})} \leq \sqrt{y(T)} \leq C\left(\left\|u^{0}\right\|_{L^{2}(\mathbb{R})}, R\right).$$

Therefore, summing (21) over n, we obtain

$$\Delta t \sum_{n \Delta t \leq T} \int_{-R}^{R} \left| u_x^{n+1} \right|^2 dx \leq C(R, \|u_0\|_{L^2(\mathbb{R})}).$$

This proves (18) and completes the proof of Lemma 3.1.

3.1. Bounds on temporal derivative. Next, we estimate the temporal derivative of the approximate solution. In doing so, we need the following lemma which some bounds on a weighted-
$$L^2$$
 projection on the space of $S_{\Delta x}$ corresponding to the weight function φ .

Lemma 3.2. Let $\psi \in C_c^{\infty}(-R, R)$. Then there exists a projection $P : C_c^{\infty}(-R, R) \to S_{\Delta x} \cap C_c(-R, R)$ such that

$$\int_{\mathbb{R}} u P(\psi) \varphi \, dx = \int_{\mathbb{R}} u \psi \varphi \, dx \qquad \text{for all } u \in S_{\Delta x}.$$

In addition, P satisfies the following bounds

(24)
$$\begin{cases} \|P(\psi)\|_{L^{2}(\mathbb{R})} \leq C \|\psi\|_{L^{2}(\mathbb{R})}, \\ \|P(\psi)\|_{H^{1}(\mathbb{R})} \leq C \|\psi\|_{H^{1}(\mathbb{R})}, \\ \|P(\psi)\|_{H^{2}(\mathbb{R})} \leq C \|\psi\|_{H^{2}(\mathbb{R})} \end{cases}$$

where the constant C is independent of Δx .

Proof. This proof is an easy adaptation of the classical L^2 projection results, see the monograph of Ciarlet [4].

Lemma 3.3. Let $\{u_n\}$ be the solution of the scheme (4). We also assume that the hypothesis of Lemma 3.1 hold. Then the following estimate holds

(25)
$$\left\| D_t^+(u^n \varphi) \right\|_{H^{-2}([-R,R])} \le C(\|u_0\|_{L^2(\mathbb{R})}, R) \left(\left\| u_x^{n+1} \right\|_{L^2([-R,R])} + 1 \right),$$

where $D_t^+ u^n$ is the forward time difference given by

$$D_t^+ u^n = \frac{u^{n+1} - u^n}{\Delta t}$$

Proof. Using the definition of $D_t^+ u^n$, we rewrite the scheme (4) as

(26)
$$\left(D_t^+ u^n, \varphi v\right) - \left(\frac{(u^{n+1})^2}{2}, (\varphi v)_x\right) + \left(u_x^{n+1}, (\varphi v)_{xx}\right) = 0,$$

which holds for all $v \in S_{\Delta x}$. Let $\psi \in C_c^{\infty}(-R, R)$ and choose $v = P(\psi)$ in (26), to obtain

$$(D_t^+ u^n, \varphi P(\psi)) + (u^{n+1} u_x^{n+1}, \varphi P(\psi)) + (u_x^{n+1}, (\varphi P(\psi))_{xx}) = 0.$$

The second and third terms of the above identity can be estimated as follows. Using the bounds (24) and the Sobolev inequality we get

$$\begin{split} &-\int_{\mathbb{R}} \left((u^{n+1})^2 \right)_x \varphi P(\psi) \, dx = \int_{\mathbb{R}} \left(u^{n+1} \right)^2 \varphi_x P(\psi) \, dx + \int_{\mathbb{R}} \left(u^{n+1} \right)^2 \varphi P(\psi)_x \, dx \\ &\leq \left(\|P(\psi)\|_{L^{\infty}([-R,R])} + \|P(\psi)_x\|_{L^{\infty}([-R,R])} \left(2R+1 \right) \right) \int_{-R}^{R} \left(u^{n+1} \right)^2 \, dx \\ &\leq \left(\|P(\psi)\|_{H^1([-R,R])} + \|P(\psi)_x\|_{H^1([-R,R])} \left(2R+1 \right) \right) \left\| u^{n+1} \right\|_{L^2(\mathbb{R})}^2 \\ &\leq C \left(\|u_0\|_{L^2(\mathbb{R})} , R \right) \|\psi\|_{H^2([-R,R])} , \end{split}$$

and

$$- \int_{\mathbb{R}} u_x^{n+1}(\varphi P(\psi))_{xx} \, dx \le \left\| u_x^{n+1} \right\|_{L^2([-R,R])} \left\| (\varphi P(\psi))_{xx} \right\|_{L^2(\mathbb{R})} \\ \le C(\left\| u_0 \right\|_{L^2(\mathbb{R})}, R) \left\| u_x^{n+1} \right\|_{L^2([-R,R])} \left\| \psi \right\|_{H^2(\mathbb{R})}.$$

Therefore

$$\begin{split} \left| \int_{\mathbb{R}} D_t^+ u^n \varphi \psi \, dx \right| &= \left| \int_{\mathbb{R}} D_t^+ u^n \varphi P(\psi) \, dx \right| \\ &\leq C(\|u_0\|_{L^2(\mathbb{R})}, R) \left(\left\| u_x^{n+1} \right\|_{L^2([-R,R])} + 1 \right) \|\psi\|_{H^2(\mathbb{R})} \,, \end{split}$$

which completes the proof.

Before stating the theorem of convergence, we define the weak solution of the Cauchy problem (1) as follows.

Definition 3.1. Let Q be a given positive number. Then $u \in L^2(0,T; H^1(-Q,Q))$ is said to be a weak solution of (1) in the interval (-Q,Q) if

(27)
$$\int_0^T \int_{-\infty}^\infty \left(\phi_t u + \phi_x \frac{u^2}{2} - \phi_{xx} u_x \right) dx dt + \int_{-\infty}^\infty \phi(x, 0) u_0(x) dx = 0.$$

for all $\phi \in C_c^{\infty}((-Q,Q) \times [0,T)).$

Next we define the approximation $u^{\Delta x}$ as,

(28)
$$u^{\Delta x}(x,t) = u^n(x) + (t-t_n)D_t^+ u^n, \ t_n \le t < t_{n+1}.$$

Then we have the following theorem for convergence.

Theorem 3.1. Let $\{u^n\}_{n\in\mathbb{N}}$ be a sequence of functions defined by the scheme (4), and assume that $\|u_0\|_{L^2(\mathbb{R})}$ is finite. Assume furthermore that $\Delta t = \mathcal{O}(\Delta x^2)$, then there exists a constant C (depends only on R and $\|u_0\|_{L^2(\mathbb{R})}$) such that

(29)
$$\|u^{\Delta x}\|_{L^{\infty}(0,T;L^{2}([-R,R]))} \leq C(R, \|u_{0}\|_{L^{2}(\mathbb{R})}),$$

(30)
$$\|u^{\Delta x}\|_{L^2(0,T;H^1([-R,R]))} \le C(R, \|u_0\|_{L^2(\mathbb{R})}),$$

(21) $\|\partial_{\Omega}(x^{\Delta x}, x)\|_{L^2(0,T;H^1([-R,R]))} \le C(R, \|u_0\|_{L^2(\mathbb{R})}),$

(31)
$$\|\partial_t(u^{\Delta x}\varphi)\|_{L^2(0,T;H^{-2}([-R,R]))} \le C(R, \|u_0\|_{L^2(\mathbb{R})})$$

where $u^{\Delta x}$ is given by (28). Moreover, there exists a sequence of $\{\Delta x_j\}_{j=1}^{\infty}$ with $\lim_{j\to\infty}$ and a function $u \in L^2(0,T; L^2([-R,R]))$ such that

(32)
$$u^{\Delta x_j} \to u \text{ strongly in } L^2(0,T;L^2([-R,R])),$$

as j goes to infinity. The function u is a weak solution of the Cauchy problem (1), that is, it satisfies (27) with Q = R - 1.

Proof. We write the approximation $u^{\Delta x}$ as, for $t_n \leq t < t_{n+1}$

$$u^{\Delta x}(x,t) = (1 - \alpha_n(t))u^n(x) + \alpha_n(t)u^{n+1}(x),$$

where $\alpha_n(t) = (t - t_n)/\Delta t \in [0, 1]$. Therefore, we have

$$\|u^{\Delta x}\|_{L^{2}(\mathbb{R})} \leq \|u^{n}\|_{L^{2}(\mathbb{R})} + \|u^{n+1}\|_{L^{2}(\mathbb{R})}$$

Thus, using (16), we conclude that (29) holds.

To prove (30), we calculate, for $t \in [t_n, t_{n+1})$

$$\left\|u_x^{\Delta x}\right\|_{L^2([-R,R])}^2 \le 2(1-\alpha_n(t))^2 \left\|u_x^n\right\|_{L^2([-R,R])}^2 + 2\alpha_n(t)^2 \left\|u_x^{n+1}\right\|_{L^2([-R,R])}^2$$
Thus

Thus,

$$\begin{split} \int_{0}^{T} \left\| u_{x}^{\Delta x} \right\|_{L^{2}([-R,R])}^{2} dt &\leq 2 \int_{0}^{T} (1 - \alpha_{n}(t))^{2} \left\| u_{x}^{n} \right\|_{L^{2}([-R,R])}^{2} dt \\ &+ 2 \int_{0}^{T} \alpha_{n}(t)^{2} \left\| u_{x}^{n+1} \right\|_{L^{2}([-R,R])}^{2} dt \\ &= 2 \sum_{n=0}^{N-1} \left\| u_{x}^{n} \right\|_{L^{2}([-R,R])}^{2} \int_{t_{n}}^{t_{n+1}} (1 - \alpha_{n}(t))^{2} dt \\ &+ 2 \sum_{n=0}^{N-1} \left\| u_{x}^{n+1} \right\|_{L^{2}([-R,R])}^{2} \int_{t_{n}}^{t_{n+1}} \alpha_{n}(t)^{2} dt \\ &\leq \Delta t \sum_{n=0}^{N-1} \left\| u_{x}^{n} \right\|_{L^{2}([-R,R])} + \Delta t \sum_{n=0}^{N-1} \left\| u_{x}^{n+1} \right\|_{L^{2}([-R,R])} \\ &\leq \Delta t \left\| u_{x}^{0} \right\|_{L^{2}([-R,R])}^{2} + 2\Delta t \sum_{n=0}^{N-1} \left\| u_{x}^{n+1} \right\|_{L^{2}([-R,R])} \\ &\leq C(\left\| u_{0} \right\|_{L^{2}(\mathbb{R})}, R) \end{split}$$

where N satisfies $N\Delta t = T$. Here we have used the inverse inequality (10) for the first term and the estimate (18) for the second term. This concludes the proof of (30).

Next we prove (31). We first note that, for $t \in [t_n, t_{n+1})$

$$\partial_t u^{\Delta x}(x,t) = D_t^+ u^n.$$

Thus, using Lemma 3.3 and Lemma 3.1, we have

$$\int_{0}^{T} \left\| \partial_{t} u^{\Delta x} \right\|_{H^{-2}([-R,R])}^{2} dt \leq C \int_{0}^{T} \left\| u_{x}^{n+1} \right\|_{L^{2}([-R,R])}^{2} dt$$
$$\leq C \Delta t \sum_{n=0}^{N-1} \left\| u_{x}^{n+1} \right\|_{L^{2}([-R,R])}^{2} \leq C(\|u_{0}\|_{L^{2}(\mathbb{R})}, R).$$

This shows that (31) holds.

Since φ is a positive and bounded smooth function, using (29), (30) we have

(33a)
$$\left\|\varphi u^{\Delta x}\right\|_{L^{\infty}(0,T;L^{2}([-R,R]))} \leq C(\left\|u_{0}\right\|_{L^{2}(\mathbb{R})}, R),$$

(33b)
$$\|\varphi u^{\Delta x}\|_{L^2(0,T;H^1([-R,R]))} \le C(\|u_0\|_{L^2(\mathbb{R})}, R).$$

Using (33) and (31) we can apply the Aubin-Simon compactness lemma (see [5]) applied to the set $\{\varphi u^{\Delta x}\}_{\Delta x>0}$ to conclude that there exist a sequence $\{\Delta x_j\}_{j\in\mathbb{N}}$ such that $\Delta x_j \to 0$, and a function \tilde{u} such that

(34)
$$u^{\Delta x_j} \varphi \to \tilde{u}$$
 strongly in $L^2(0,T;L^2([-R,R])),$

as j goes to infinity. As $\varphi \ge 1$, (34) implies that there exists a u such that (32) holds.

This strong convergence allows passage to the limit in nonlinearity. However, it remains to prove that u is a weak solution of (1). In what follows, we will consider the standard L^2 -projection of a function ψ with k + 1 continuous derivatives into space $S_{\Delta x}$, denoted by \mathcal{P} , i.e.,

$$\int_{\mathbb{R}} \left(\mathcal{P}\psi(x) - \psi(x) \right) v(x) = 0, \quad \forall v \in S_{\Delta x}.$$

For the projection mentioned above we have that (for a proof, see the monograph of Ciarlet [4])

$$\left\|\psi(x) - \mathcal{P}\psi(x)\right\|_{H^{k}(\mathbb{R})} \le C\Delta x \left\|\psi\right\|_{H^{k+1}(\mathbb{R})}$$

where C is a constant independent of Δx .

We also need the following inequality:

(35)
$$\|u^n\|_{L^{\infty}[-R+1,R-1]} \le C(R) \|u^n\|_{H^1(-R,R)}$$

where C_R is some positive constant depends only on R. To show this inequality, we consider the the smooth function η such that $\eta = 1$ on [-R+1, -R-1] and $\eta = 0$ on the set $\{|x| > R - \frac{1}{2}\}$. Then, it is easy to see that

$$|u^{n}(x)\eta(x)| \leq \left(\|\eta\|_{L^{\infty}(\mathbb{R})} + \|\eta_{x}\|_{L^{\infty}(\mathbb{R})} \right) (2R)^{1/2} \|u^{n}\|_{H^{1}([-R,R])}$$

As $\eta = 1$ on [-R+1, R-1], we conclude that (35) holds. We first show that

We first show that

(36)
$$\int_0^T \int_{\mathbb{R}} u_t^{\Delta x} \varphi v - \frac{(u^{\Delta x})^2}{2} (\varphi v)_x + (u^{\Delta x})_x (\varphi v)_{xx} \, dx \, dt = \mathcal{O}(\Delta x),$$

for any test function $v \in C_c^{\infty}((-R+1, R-1) \times [0, T))$, where φ is specified in the beginning of Section 2. Let $v^{\Delta x} = \mathcal{P}v$, then from the definition of $u_{\Delta x}$ (c.f. (28)), it is evident that

$$\begin{split} \int_{0}^{T} \int_{\mathbb{R}} u_{t}^{\Delta x} \varphi v - \frac{(u^{\Delta x})^{2}}{2} (\varphi v)_{x} + (u^{\Delta x})_{x} (\varphi v)_{xx} \, dx dt \\ &= \underbrace{\sum_{n} \int_{\mathbb{R}} \int_{t_{n}}^{t_{n+1}} D_{t}^{+} u^{n} \varphi v^{\Delta x} - \frac{(u^{n+1})^{2}}{2} (\varphi v^{\Delta x})_{x} + (u^{n+1})_{x} (\varphi v^{\Delta x})_{xx} \, dt dx \\ &= \underbrace{\sum_{n} \int_{\mathbb{R}} \int_{t_{n}}^{t_{n+1}} D_{t}^{+} u^{n} (\varphi v - \varphi v^{\Delta x})}_{\mathcal{E}_{\Delta x}^{1,n}} - \underbrace{\underbrace{(u^{n+1})^{2}}{2} (\varphi v - \varphi v^{\Delta x})_{x}}_{\mathcal{E}_{\Delta x}^{2,n}} \, dt dx \\ &+ \sum_{n} \int_{\mathbb{R}} \int_{t_{n}}^{t_{n+1}} \underbrace{(u^{n+1})_{x} (\varphi v - \varphi v^{\Delta x})_{xx}}_{\mathcal{E}_{\Delta x}^{3,n}} + \underbrace{(u^{\Delta x} - u^{n+1})_{x} (\varphi v)_{xx}}_{\mathcal{E}_{\Delta x}^{4,n}} \, dt dx \\ &+ \sum_{n} \int_{\mathbb{R}} \int_{t_{n}}^{t_{n+1}} \underbrace{\left(-\frac{(u^{\Delta x})^{2}}{2} + \frac{(u^{n+1})^{2}}{2}\right) (\varphi v)_{x}}_{\mathcal{E}_{\Delta x}^{5,n}} \, dt dx \end{split}$$

We proceed with

$$\begin{split} & \left| \sum_{n} \int_{\mathbb{R}} \int_{t_{n}}^{t_{n+1}} \mathcal{E}_{\Delta x}^{1,n} \, dx \, dt \right| = \left| \sum_{n} \int_{\mathbb{R}} \int_{t_{n}}^{t_{n+1}} D_{t}^{+} u^{n} \left(\varphi v - \varphi v^{\Delta x} \right) \, dt dx \right| \\ & \leq C(R) \sum_{n} \int_{t_{n}}^{t_{n+1}} \left\| D_{t}^{+} \left(u^{n} \varphi \right) \right\|_{H^{-2}([-R,R])} \left\| v - v^{\Delta x} \right\|_{H^{2}([-R+1,R-1])} \, dt \\ & \leq \Delta x \, C(\|u_{0}\|_{L^{2}(\mathbb{R})} \, , R) \, \|v\|_{L^{2}((0,T);H^{3}([-R+1,R-1]))} \to 0, \text{ as } \Delta x \downarrow 0. \end{split}$$

Next, using (35), we obtain

$$\begin{split} \sum_{n} \int_{\mathbb{R}} \int_{t_{n}}^{t_{n+1}} \mathcal{E}_{\Delta x}^{2,n} dt dx \Big| &= \Big| \sum_{n} \int_{\mathbb{R}} \int_{t_{n}}^{t_{n+1}} \left(\frac{u^{n+1}}{2} \right)^{2} \left(\varphi v - \varphi v^{\Delta x} \right)_{x} dt dx \Big| \\ &= \Big| \sum_{n} \int_{-R+1}^{R-1} \int_{t_{n}}^{t_{n+1}} \left(\frac{u^{n+1}}{2} \right)^{2} \left(\varphi \left(v - v^{\Delta x} \right)_{x} + \left(v - v^{\Delta x} \right) \varphi_{x} \right) dt dx \Big| \\ &\leq C(R) \left(\sum_{n} \int_{t_{n}}^{t_{n+1}} \Big\| u^{n+1} \Big\|_{L^{\infty}([-R+1,R-1])} \int_{-R+1}^{R-1} |u^{n+1}| \left| \left(v - v^{\Delta x} \right)_{x} \right| dt dx \\ &+ \sum_{n} \int_{t_{n}}^{t_{n+1}} \Big\| u^{n+1} \Big\|_{L^{\infty}([-R+1,R-1])} \int_{-R+1}^{R-1} |u^{n+1}| \left| \left(v - v^{\Delta x} \right) \right| dt dx \Big) \\ &\leq C(R) \sum_{n} \int_{t_{n}}^{t_{n+1}} \left\| u^{n+1} \right\|_{H^{1}([-R,R])}^{2} \left\| v - v^{\Delta x} \right\|_{H^{1}([-R+1,R-1])} dt \\ &\leq C(\|u_{0}\|_{L^{2}(\mathbb{R})}, R) \Delta x \|v\|_{L^{\infty}((0,T);H^{2}([-R+1,R-1]))} \to 0, \text{ as } \Delta x \downarrow 0. \end{split}$$

Using the Cauchy-Schwartz inequality

$$\begin{split} & \left| \sum_{n} \int_{\mathbb{R}} \int_{t_{n}}^{t_{n+1}} \mathcal{E}_{\Delta x}^{3,n} dt dx \right| = \left| \sum_{n} \int_{\mathbb{R}} \int_{t_{n}}^{t_{n+1}} (u^{n+1})_{x} \left(\varphi v - \varphi v^{\Delta x} \right)_{xx} dt dx \right| \\ & \leq \left\| u^{\Delta x} \right\|_{L^{2}(0,T;H^{1}([-R,R]))} \left(\int_{0}^{T} \left\| \varphi v(t) - \varphi \mathcal{P} v(t) \right\|_{H^{2}([-R+1,R-1])}^{2} dt \right)^{1/2} \\ & \leq \Delta x \, C(\left\| u_{0} \right\|_{L^{2}(\mathbb{R})}, R) \left\| v \right\|_{L^{2}((0,T);H^{3}([-R+1,R-1]))} \to 0, \text{ as } \Delta x \downarrow 0, \end{split}$$

and integration by parts

$$\begin{split} \left|\sum_{n} \int_{\mathbb{R}} \int_{t_{n}}^{t_{n+1}} \mathcal{E}_{\Delta x}^{4,n} dt dx\right| &= \left|\sum_{n} \int_{\mathbb{R}} \int_{t_{n}}^{t_{n+1}} \left(u^{\Delta x} - u^{n+1}\right)_{x} (\varphi v)_{xx} dt dx\right| \\ &= \left|\sum_{n} \int_{\mathbb{R}} \int_{t_{n}}^{t_{n+1}} \left(-\Delta t D_{t}^{+} u^{n} + (t-t_{n}) D_{t}^{+} u^{n}\right) (\varphi v)_{xxx} dt dx\right| \\ &\leq \Delta t \sum_{n} \int_{-R+1}^{R-1} \int_{t_{n}}^{t_{n+1}} \left|D_{t}^{+} u^{n}\right| \left|(\varphi v)_{xxx}\right| dt dx \\ &\leq \Delta t C(\|u_{0}\|_{L^{2}(\mathbb{R})}, R) \|\varphi v\|_{L^{2}((0,T); H^{5}([-R+1,R-1]))} \to 0, \text{ as } \Delta t \downarrow 0. \end{split}$$

Next, we estimate the term containing $\mathcal{E}^{5,n}_{\Delta x}$,

$$\begin{split} \sum_{n} \int_{\mathbb{R}} \int_{t_{n}}^{t_{n+1}} \mathcal{E}_{\Delta x}^{5,n} \, dx \, dt &= \sum_{n} \int_{\mathbb{R}} \int_{t_{n}}^{t_{n+1}} \left(-\frac{(u^{\Delta x})^{2}}{2} + \frac{(u^{n+1})^{2}}{2} \right) (\varphi v)_{x} \, dt dx \\ &= \sum_{n} \int_{\mathbb{R}} \int_{t_{n}}^{t_{n+1}} \left(-\frac{(u^{n})^{2}}{2} + \frac{(u^{n+1})^{2}}{2} \right) (\varphi v)_{x} \, dt dx \\ &- \sum_{n} \int_{\mathbb{R}} \int_{t_{n}}^{t_{n+1}} \left(u^{n}(t-t^{n})D_{t}^{+}u^{n} + \frac{1}{2}(t-t^{n})^{2}(D_{t}^{+}u^{n})^{2} \right) (\varphi v)_{x} \, dt dx \\ &= \Delta t \sum_{n} \int_{\mathbb{R}} \int_{t_{n}}^{t_{n+1}} \left(\frac{1}{2}(u^{n+1}+u^{n})D_{t}^{+}u^{n} \right) (\varphi v)_{x} \, dt dx \\ &- \sum_{n} \int_{\mathbb{R}} \int_{t_{n}}^{t_{n+1}} \left(u^{n}(t-t_{n})D_{t}^{+}u^{n} + \frac{1}{2}(t-t_{n})^{2}(D_{t}^{+}u^{n})^{2} \right) (\varphi v)_{x} \, dt dx. \end{split}$$

We claim that all the terms in the above expression converges to zero as Δt converges to zero since;

$$\begin{split} & \Big| \sum_{n} \int_{t_{n}}^{t_{n+1}} \int_{-R+1}^{R-1} u^{n} D_{t}^{+} u^{n} (\varphi v)_{x} \, dx dt \Big| \\ & \leq \sum_{n} \int_{t_{n}}^{t_{n+1}} \| u^{n} \|_{L^{\infty}([-R+1,R-1])} \, \left\| D_{t}^{+} u^{n} \varphi \right\|_{H^{-2}([-R,R])} \| \varphi v \|_{H^{3}([-R+1,R-1])} \, dt \\ & \leq C \sum_{n} \int_{t_{n}}^{t_{n+1}} \| u^{n} \|_{H^{1}([-R,R])} \left(\left\| u_{x}^{n+1} \right\|_{L^{2}([-R,R])} + 1 \right) \| \varphi v \|_{H^{3}([-R+1,R-1])} \, dt \\ & \leq C (\| u_{0} \|_{L^{2}(\mathbb{R})} \, , R) \, \| v \|_{L^{\infty}((0,T);H^{3}([-R+1,R-1]))} \end{split}$$

and similarly,

$$\left| \sum_{n} \int_{t_{n}}^{t_{n+1}} \int_{-R+1}^{R-1} u^{n+1} D_{t}^{+} u^{n} (\varphi v)_{x} \, dx dt \right|$$

$$\leq C(\|u_{0}\|_{L^{2}(\mathbb{R})}, R) \, \|v\|_{L^{\infty}((0,T); H^{3}([-R+1,R-1]))}.$$

Furthermore

$$\begin{split} \left| \int_{\mathbb{R}} \int_{t_n}^{t_{n+1}} u^n \, (t-t_n) D_t^+ u^n \, (\varphi v)_x \, dt dx \right| \\ &\leq \| u^n \|_{L^{\infty}([-R+1,R-1])} \, \Delta t \int_{-R+1}^{R-1} \int_{t_n}^{t_{n+1}} \left| D_t^+ u^n \right| \, |(\varphi v)_x| \, dt dx, \end{split}$$

and

$$\begin{split} \left| \int_{\mathbb{R}} \int_{t_n}^{t_{n+1}} (t-t_n)^2 \left(D_t^+ u^n \right)^2 (\varphi v)_x \, dt dx \right| \\ & \leq \left(\left\| u^{n+1} - u^n \right\|_{L^{\infty}([-R+1,R-1])} \right) \Delta t \int_{-R+1}^{R-1} \int_{t_n}^{t_{n+1}} \left| D_t^+ u^n \right| \, |(\varphi v)_x| \, dt dx. \end{split}$$

Therefore, these two terms can be estimated in the same manner as the preceding two term. Hence

$$\sum_{n} \int_{\mathbb{R}} \int_{t_n}^{t_{n+1}} \mathcal{E}_{\Delta x}^{5,n} dt dx \to 0, \text{ as } \Delta t \downarrow 0.$$

Combining all these above estimates, we conclude that (36) holds. Furthermore, passing limit as $\Delta x \to 0$, we conclude that

(37)
$$\int_0^T \int_{\mathbb{R}} u_t \varphi v - \frac{u^2}{2} (\varphi v)_x + u_x (\varphi v)_{xx} \, dx \, dt = 0,$$

for any test function $v \in C_c^{\infty}([-R+1, R-1] \times [0, T))$. Finally, we choose $v = \phi/\varphi$ in (37) with $\phi \in C_c^{\infty}([-R+1, R-1] \times [0, T))$ and integrate-by-parts to conclude that (27) holds, i.e. that

$$\int_0^T \int_{-\infty}^\infty \left(\phi_t u + \phi_x \frac{u^2}{2} - \phi_{xx} u_x \right) dx dt + \int_{-\infty}^\infty \phi(x, 0) u_0(x) dx = 0.$$

This finishes the proof of the Theorem 3.1.

4. Numerical experiments

The fully-discrete scheme given by (4) has been tested on several numerical experiments in order to test how well this method works in practice.

We let $S_{\Delta x}$ consist of piecewise cubic splines defined as follows: Let f and g be the functions

$$f(y) = 1 + y^{2} (2 |y| - 3),$$

$$g(y) = \begin{cases} y(y+1)^{2} & y \leq 0, \\ -y(y-1)^{2} & y > 0, \end{cases}$$

and we define f(y) = g(y) = 0 for |y| > 1. For $j \in \mathbb{Z}$ we define

$$v_{2j}(x) = f\left(\frac{x-x_j}{\Delta x}\right), \qquad v_{2j+1}(x) = g\left(\frac{x-x_j}{\Delta x}\right),$$

where $x_j = j\Delta x$. The space spanned by $\{v_j\}_{j=-M}^M$ is a 4M+2 dimensional subspace of $H^2(\mathbb{R})$. In our numerical examples, we used periodic boundary conditions. In the examples computing solitary waves, the exact solution, as well as the numerical approximations are all very close to zero at the boundary. Regarding the weight function, we chose this to be $\varphi(x) = 50 + x$ in the intervals under consideration in all our examples. In the Newton iteration to obtain u^{n+1} , (5), we terminated the iteration if $||w^{\ell+1} - w^{\ell}|| \leq \Delta x^2$.

For $t = n\Delta t$, we set $u_{\Delta x}(x,t) = u^n(x,t) = \sum_{j=-M}^M u_j^n v_j(x)$. In all our examples, we measured the percentage L^2 error, defined as

$$E = 100 \, \frac{\|u - u_{\Delta x}\|_{L^2}}{\|u\|_{L^2}}.$$

4.1. **One-soliton solution.** The equation (1) has an exact solution

(38)
$$w_1(x,t) = 9\left(1 - \tanh^2\left(\sqrt{3/2}(x-3t)\right)\right).$$

This represents a single "bump" moving to the right with speed 3. We have tested our scheme with initial data $u_0(x) = w_1(x, -1)$ in order to check how fast this scheme converges. Recall that we are using $w_1(x, -1)$ as initial data, so that $w_1(x, 1)$ represents the solution at t = 2. The solution was calculated on a uniform grid with $\Delta x = 20/(2M)$ in the interval [-10, 10]. In Table 1 we show the relative errors as well as the numerical convergence rates for this example. From Table 1 we see that

M	E	rate
	$ \begin{array}{c} L \\ 61.5 \\ 33.6 \\ 5.8 \\ 3.2 \\ 3.1 \\ 1.9 \\ $	$\begin{array}{c} 0.87\\ 2.52\\ 0.86\\ 0.03\\ 0.69\\ 0.87\end{array}$
$512 \\ 1024$	$\begin{array}{c} 1.1 \\ 0.6 \end{array}$	0.94

TABLE 1. Relative percentage L^2 errors for the one-soliton solution, $w_1(x, 2)$

the scheme converges, that the rate is a bit erratic, but seems to converge to one.

4.2. **Two-soliton solution.** Physically, two solitons which have different shapes move with different velocities, which is a result of the dependence between the height of the soliton and the velocity. A higher soliton travels faster than a lower soliton. If the two solitons travel along a surface, the higher soliton will overtake the lower soliton, and after the collision, both solitons will emerge unchanged. We use the following test problem for the two-soliton interaction, where $u(x, 0) = w_2(x, -10)$, with

(39)

$$w_{2}(x,t) = 6(b-a) \frac{b \operatorname{csch}^{2} \left(\sqrt{b/2}(x-2bt)\right) + a \operatorname{sech}^{2} \left(\sqrt{a/2}(x-2at)\right)}{\left(\sqrt{a} \tanh\left(\sqrt{a/2}(x-2at)\right) - \sqrt{b} \coth\left(\sqrt{b/2}(x-2bt)\right)\right)^{2}},$$

for any real numbers a and b. We have used a = 0.5 and b = 1. This solution represents two waves that "collide" at t = 0 and separate for t > 0. For large |t|, $w_2(\cdot, t)$ is close to a sum of two one-solitons at different locations.

Computationally, this is a much harder problem than the one-soliton solution. We computed the approximate solution at t = 20. The exact solution in this case is $w_2(x, 10)$. Figure 1 we show the exact and numerical solutions at t = 20. The

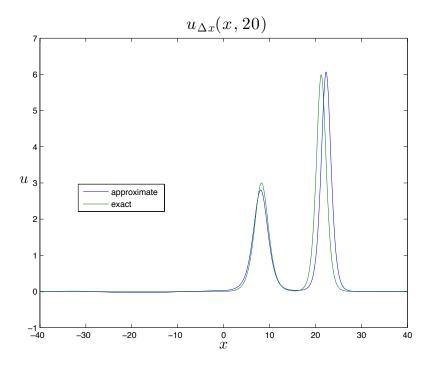


FIGURE 1. The exact and numerical solutions at t = 20 with initial data $w_2(x, -10)$ with M = 256.

computed solution in Figure 1 looks "right", in the sense that the two bumps in the solution have separated well and passed through each other. Nevertheless, the error is more than 50%. This is due to an error in the position of the larger bump, which again is due to a much smaller error in the height of the bump. This error causes the speed of the wave to be slightly larger than the speed of the corresponding wave in the exact solution. Since the wave is quite narrow, this causes the L^2 error to be large. In Table 2 we show the percentage errors for the two-soliton simulation.

4.3. Initial data in L^2 . We have also applied our scheme on an example where the initial data are in L^2 , but not in any Sobolev space with positive index. To this end we have chosen initial data

(40)
$$u_0(x) = \begin{cases} 0 & x \le 0, \\ x^{-1/3} & 0 < x < 1, \\ 0 & x \ge 1, \end{cases}$$

M	E	rate
64	108	1.3
128	41	-0.3
256	54	-0.3
512	49	0.1
1024	30	0.9
2046	16	0.9

TABLE 2. Relative percentage L^2 errors for the two-soliton solution.

if x is in [-5, 5] and extended periodically outside this interval. An exact solution is not available in this case, so we used a third-order discontinuous Galerkin approximation with 386 degrees of freedom as a reference solution, see [12, 5]. There is no proof that this reference solution is close to the exact solution, but lacking other alternatives, we choose to compare the approximate solutions generated by our finite element scheme with this solution.

In Table 3 we show the relative errors for our element method. The large errors

M	E	rate
$\begin{array}{c} M \\ 16 \\ 32 \\ 64 \\ 128 \\ 256 \\ 512 \\ 1024 \\ 2048 \\ 4096 \end{array}$	$\begin{array}{c} E \\ 65 \\ 61 \\ 55 \\ 50 \\ 46 \\ 42 \\ 40 \\ 39 \\ 37 \end{array}$	$\begin{array}{c} 0.09\\ 0.13\\ 0.14\\ 0.14\\ 0.11\\ 0.08\\ 0.05\\ 0.07\\ \end{array}$
8192	34	0.12

TABLE 3. Relative percentage L^2 error between a reference solution using the discontinuous Galerkin method and our element method with initial data (40) and t = 0.5.

and the slow convergence rate both indicate that we are not yet in asymptotic regime. In Figure 2 we show the approximate solution at with the finest resolution (32768 degrees of freedom) and the reference solution. There is however some doubt about the accuracy of the reference solution. Our approximate solution is very close to an approximate solution found by a simple difference scheme, see [5], using $\Delta x = 10/512000$.

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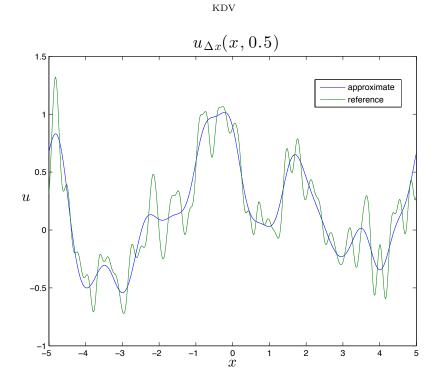


FIGURE 2. The numerical solution $u_{\Delta x}(x, 0.5)$ with initial data (40) with M = 8192, and the reference solution found by the third-order discontinuous Galerkin method.

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