# Stabilized Finite Element Approximation of the Mean Curvature Vector on Closed Surfaces * 

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#### Abstract

We develop a stabilized discrete Laplace-Beltrami operator that is used to compute an approximate mean curvature vector which enjoys convergence of order one in $L^{2}$. The stabilization is of gradient jump type and we consider both standard meshed surfaces and so called cut surfaces that are level sets of piecewise linear distance functions. We prove a priori error estimates and verify the theoretical results numerically.


## 1 Introduction

Accurate computation of the mean curvature vector on a discrete surface plays an important role in computer graphics and computational geometry, as well as in certain surface evolution problems, see, e.g. [1, 2, 4, 6, 7, 8, 8.

The mean curvature vector is obtained by letting the Laplace-Beltrami operator act on the embedding of the surface in $\mathbf{R}^{3}$ and various formulas has been suggested in the literature, see [13] and the references therein. It is known that the standard mean curvature vector based on the finite element discrete Laplace-Beltrami operator for a piecewise linear triangulated smooth surface is of first order in $H^{-1}$, while no order of convergence can, in general, be expected in $L^{2}$. Convergence will also not occur in other standard methods, for instance of finite difference type, without restrictive assumptions on the mesh, see [15]. In [10] estimates of order $h^{1 / 2}$ in a $H^{-1 / 2}$ type norm, motivated by surface tension

[^0]applications, is derived for an embedded interface defined by a levelset function. Pointwise convergence results, without any factor of the meshsize, was presented in [12].

In this paper we develop a stabilized version of the discrete Laplace-Beltrami operator. The stabilization consists of adding suitably scaled gradient jumps to the $L^{2}$ projection involved in the definition of the standard discrete Laplace-Beltrami operator. The stabilized method produces a mean curvature vector that enjoys first order convergence in $L^{2}$. We consider two different types of piecewise linear approximations of smooth surfaces. The first is the standard unstructured triangulation and the second is a so called cut level set surface, which is the zero level set of a piecewise linear continuous approximation of the distance function defined on a background mesh consisting of tetrahedra. In the cut case an additional stabilization term on the faces of the background mesh plays a crucial role. Such terms were originally proposed and analyzed in 3]. We prove a priori error estimates in the $L^{2}$-norm for both cases and we also illustrate the results with numerical examples.

The outline of the remainder of the paper is as follows: In Section 2 we introduce the discrete surface approximations, in Section 3 we define the stabilized mean curvature vector, in Section 4 we develop the theoretical framework and prove the a priori error estimate, and in Section 5 we present numerical results confirming the theoretical estimates.

## 2 Meshed and Cut Discrete Surfaces

### 2.1 The Exact Surface

Consider a closed smooth surface $\Sigma \subset \mathbf{R}^{3}$ with exterior unit normal $n$. Let $\rho$ be the signed distance function such that $\nabla \rho=n$ on $\Sigma$ and let $p(x)=x-\rho(x) n(p(x))$ be the closest point mapping. Let $U_{\delta}(\Gamma)$ be the open tubular neighborhood $U_{\delta}(\Gamma)=\left\{x \in \mathbf{R}^{3}:|\rho(x)|<\delta\right\}$ for $\delta>0$ of $\Sigma$. Then there is $\delta_{0}>0$ such that the closest point mapping $p(x)$ assigns precisely one point on $\Sigma$ to each $x \in U_{\delta_{0}}(\Sigma)$. More precisely, we may choose $\delta_{0}$ such that

$$
\begin{equation*}
\delta_{0} \max \left(\left|\kappa_{1}(x)\right|,\left|\kappa_{2}(x)\right|\right) \leq C<1 \quad \forall x \in \Sigma \tag{2.1}
\end{equation*}
$$

for some constant $C>0$. Here $\kappa_{1}(x)$ and $\kappa_{2}(x)$ are the principal curvatures at $x \in \Sigma$. See [9], Section 14.6 for further details.

### 2.2 Approximation Properties

We consider families of discrete connected piecewise linear surfaces $\Sigma_{h} \subset U_{\delta_{0}}(\Sigma)$, where $0<h \leq h_{0}$ is a mesh parameter and $h_{0}$ a small enough constant, that satisfy the following approximation properties

$$
\begin{align*}
\|\rho\|_{L^{\infty}\left(\Sigma_{h}\right)} & \lesssim h^{2}  \tag{2.2}\\
\left\|n \circ p-n_{h}\right\|_{L^{\infty}\left(\Sigma_{h}\right)} & \lesssim h \tag{2.3}
\end{align*}
$$

Here and below we use the notation $\lesssim$ to denote less or equal up to a positive constant that is only dependent on given data and, in particular, independent of the mesh parameter $h$.

We will consider two approaches to construct such piecewise linear surfaces:

- Standard meshed surfaces where the surface consists of shape regular triangles.
- Cut surfaces that are piecewise planar iso-levels of a piecewise linear distance function defined on a background mesh consisting of tetrahedra.

We shall treat meshed and cut surfaces in a unified setting but certain concepts such as the mesh and later the interpolation operator will be constructed in different ways. However, the essential properties needed in the construction of the Laplace-Beltrami operator and in the proof of the error estimate are the same.

### 2.3 Meshed Surface Approximation

- Let $\Sigma_{h}=\cup_{K \in \mathcal{K}_{h}} K \subset U_{\delta_{0}}(\Sigma)$, be a quasiuniform triangulated surface with mesh parameter $0<h \leq h_{0}$, i.e.,

$$
\begin{equation*}
\operatorname{Diam}(K) \lesssim h, \quad \operatorname{Diam}(K) / \operatorname{diam}(K) \lesssim 1 \tag{2.4}
\end{equation*}
$$

for all triangles $K$ in the mesh $\mathcal{K}_{h}$. Here $\operatorname{Diam}(K)$ is the diameter of $K$ and $\operatorname{diam}(K)$ is the diameter of the largest inscribed circle in $K$.

- Let $V_{h}$ be the space of piecewise linear continuous functions defined on $\mathcal{K}_{h}$.


### 2.4 Cut Surface Approximation

- Let $\Omega_{0}$ be a polygon that contains $U_{\delta_{0}}(\Sigma)$. Let $\mathcal{T}_{h, 0}$ be a quasiuniform partition of $\Omega_{0}$ into shape regular tetrahedra $T$ with mesh parameter $0<h \leq h_{\Omega_{0}}$, i.e.,

$$
\begin{equation*}
\operatorname{Diam}(T) \lesssim h, \quad \operatorname{Diam}(T) / \operatorname{diam}(T) \lesssim 1 \tag{2.5}
\end{equation*}
$$

for all elements $T \in \mathcal{T}_{h, 0}$. Let $\Sigma_{h} \subset U_{\delta_{0}}(\Sigma)$ be a connected surface such that the intersection $\Sigma_{h} \cap T$ is a subset of a hyperplane (or empty) for all $T \in \mathcal{T}_{h, 0}$. Let $\mathcal{T}_{h}=\left\{T \in \mathcal{T}_{h, 0}: T \cap \Sigma_{h} \neq \emptyset\right\}$ and $\mathcal{K}_{h}=\left\{\Sigma_{h} \cap T: T \in \mathcal{T}_{h}\right\}$ and let $h_{0}$, with $0<h_{0} \leq h_{\Omega_{0}}$, be chosen such that $\cup_{T \in \mathcal{T}_{h}} T \subset U_{\delta_{0}}(\Sigma)$ for $0<h \leq h_{0}$.

- Let $V_{h}$ be the space of piecewise linear continuous functions on $\mathcal{T}_{h}$.

In practice, $\Sigma_{h}$ is constructed by computing an approximation $\rho_{h}$ of the levelset function $\rho$ associated with $\Sigma$ and then defining $\Sigma_{h}$ as the zero levelset. Note that $K \in \mathcal{K}_{h}$ will be a triangle or a planar quadrilateral.

## 3 Stabilized Approximation of the Mean Curvature Vector

### 3.1 The Continuous Mean Curvature Vector

The tangential gradient $\nabla_{\Sigma}$ is defined by $\nabla_{\Sigma}=P_{\Sigma} \nabla$, where $\nabla$ is the $\mathbf{R}^{3}$ gradient and $P_{\Sigma}(x)=I-n(x) \otimes n(x)$ is the projection onto the tangent plane $T_{\Sigma}(x)$ of $\Sigma$ at $x$.

The mean curvature vector $H: \Sigma \rightarrow \mathbf{R}^{3}$ is defined by

$$
\begin{equation*}
H=-\Delta_{\Sigma} x_{\Sigma} \tag{3.1}
\end{equation*}
$$

where $x_{\Sigma}: \Sigma \ni x \mapsto x \in \mathbf{R}^{3}$ is the coordinate map or embedding of $\Sigma$ into $\mathbf{R}^{3}$ and $\Delta_{\Sigma}=$ $\nabla_{\Sigma} \cdot \nabla_{\Sigma}$ is the Laplace-Beltrami operator. Note that for a general vector field $v: \Sigma \rightarrow \mathbf{R}^{3}$ the surface divergence $\operatorname{div}_{\Sigma} v$ is defined by $\operatorname{div}_{\Sigma} v=\operatorname{tr}\left(v \otimes \nabla_{\Sigma}\right)=\operatorname{tr}(v \otimes \nabla)-n \cdot(v \otimes \nabla) \cdot n$, and for tangent vector fields $v$ we have have the identity $\nabla_{\Sigma} \cdot v=\operatorname{div}_{\Sigma} v$.

The relation between the mean curvature vector and mean curvature is given by the identity

$$
\begin{equation*}
H=\left(\kappa_{1}+\kappa_{2}\right) n \tag{3.2}
\end{equation*}
$$

where $\kappa_{1}$ and $\kappa_{2}$ are the two principal curvatures and $\left(\kappa_{1}+\kappa_{2}\right) / 2$ is the mean curvature, see [1].

The mean curvature vector satisfies the following weak problem: find $H \in W=$ $\left[H^{1}(\Sigma)\right]^{3}$ such that

$$
\begin{equation*}
B(H, v)=L(v) \quad \forall v \in W \tag{3.3}
\end{equation*}
$$

The forms are defined by

$$
\begin{equation*}
B(v, w)=(v, w)_{\Sigma}, \quad L(w)=\left(\nabla_{\Sigma} x_{\Sigma}, \nabla_{\Sigma} w\right)_{\Sigma} \tag{3.4}
\end{equation*}
$$

where $\nabla_{\Sigma} w=w \otimes \nabla_{\Sigma}$ for a vector valued function $w$ and $(v, w)_{\omega}=\int_{\omega} v w d x$ is the $L^{2}$-inner product on the set $\omega$ with associated norm $\|v\|_{\omega}^{2}=\int_{\omega} v^{2} d x$.

We let $W_{p}^{s}(\omega)$ denote the standard Sobolev spaces on $\omega \subseteq \Sigma$ or $\omega \subseteq \mathbf{R}^{d}$ with norm $\|\cdot\|_{W_{p}^{s}(\omega)}$, see [14]. We also use the standard notation $W_{p}^{s}(\omega)=H^{s}(\omega)$ for $p=2$ and $W_{p}^{0}(\omega)=L^{p}(\omega)$ for $s=0$. Since the surface is smooth we have the bound

$$
\begin{equation*}
\|H\|_{W_{\infty}^{s}(\Sigma)} \lesssim 1 \tag{3.5}
\end{equation*}
$$

for any choice of $s$ and we will, in particular, use this bound in our analysis with $s=2$.

### 3.2 The Stabilized Discrete Mean Curvature Vector

Given the discrete coordinate map $x_{\Sigma_{h}}: \Sigma_{h} \ni x \mapsto x \in \mathbf{R}^{3}$ we define the stabilized discrete mean curvature vector $H_{h}$ as follows: find $H_{h} \in W_{h}=\left[V_{h}\right]^{3}$ such that

$$
\begin{equation*}
B_{h}\left(H_{h}, v\right)+J_{h}\left(H_{h}, v\right)=L_{h}(v) \quad \forall v \in W_{h} \tag{3.6}
\end{equation*}
$$

where the forms are defined by

$$
\begin{align*}
B_{h}(u, v) & =(u, v)_{\Sigma_{h}}  \tag{3.7}\\
L_{h}(v) & =\left(\nabla_{\Sigma_{h}} x_{\Sigma_{h}}, \nabla_{\Sigma_{h}} v\right)_{\Sigma_{h}}  \tag{3.8}\\
J_{h}(u, v) & = \begin{cases}\tau_{\mathcal{E}_{h}} J_{\mathcal{E}_{h}}(u, v) & \text { Meshed } \\
\tau_{\mathcal{E}_{h}} J_{\mathcal{E}_{h}}(u, v)+\tau_{\mathcal{F}_{h}} J_{\mathcal{F}_{h}}(u, v) & \text { Cut }\end{cases}  \tag{3.9}\\
J_{\mathcal{E}_{h}}(u, v) & =\sum_{E \in \mathcal{E}_{h}} h\left(\left[t_{E} \cdot \nabla_{\Sigma_{h}} u\right],\left[t_{E} \cdot \nabla_{\Sigma_{h}} v\right]\right)_{E}  \tag{3.10}\\
J_{\mathcal{F}_{h}}(u, v) & =\sum_{F \in \mathcal{F}_{h}}\left(\left[n_{F} \cdot \nabla u\right],\left[n_{F} \cdot \nabla v\right]\right)_{F} \tag{3.11}
\end{align*}
$$

Here $\tau_{\mathcal{E}_{h}}, \tau_{\mathcal{F}_{h}} \geq 0$ are parameters, $\mathcal{E}_{h}=\{E\}$ is the set of edges in the partition $\mathcal{K}_{h}$ of $\Sigma_{h}$, $\mathcal{F}_{h}=\{F\}$ is the set of interior faces in the partition $\mathcal{T}_{h}$. The jump in the tangent gradient at an edge $E \in \mathcal{E}_{h}$ shared by elements $K_{1}$ and $K_{2}$ in $\mathcal{K}_{h}$ is defined by

$$
\begin{equation*}
\left[t_{E} \cdot \nabla_{\Sigma_{h}} u\right]=t_{E, K_{1}} \cdot \nabla_{\Sigma_{h}} u_{1}+t_{E, K_{2}} \cdot \nabla_{\Sigma_{h}} u_{2} \tag{3.12}
\end{equation*}
$$

where $u_{i}=\left.u\right|_{K_{i}}, i=1,2$, and $t_{E, K_{i}}$ denotes the unit vector orthogonal to $E$, tangent and exterior to $K_{i}, i=1,2$. In the same way the jump at a face $F \in \mathcal{F}_{h}$ shared by elements $T_{1}$ and $T_{2}$ is defined by

$$
\begin{equation*}
\left[n_{F} \cdot \nabla u\right]=n_{F, T_{1}} \cdot \nabla u_{1}+n_{F, T_{2}} \cdot \nabla u_{2} \tag{3.13}
\end{equation*}
$$

where $n_{F, T_{i}}$ is the unit normal to the face $F$ exterior to element $T_{i}, i=1,2$.
Remark 3.1 The term $J_{\mathcal{F}_{h}}(u, v)$ is crucial in the cut case and enables us to essentially handle the cut case in the same way as the meshed case. It also stabilizes the possibly ill conditioned linear system of equations, see [3]. In Theorem 4.2 we will show that in the cut case it is indeed possible to take $\tau_{\mathcal{E}_{h}}=0$ and thus only add $J_{\mathcal{F}_{h}}(\cdot, \cdot)$. It is however convenient for the analysis to first include both the edge and face stabilization terms and then prove that only the face stabilization term is enough.

## 4 Error Estimates

### 4.1 Extension and Lifting of Functions

Extension. Using the nearest point projection mapping any function $v$ on $\Sigma$ can be extended to $U_{\delta_{0}}(\Sigma)$ using the pull back

$$
\begin{equation*}
v^{e}=v \circ p \quad \text { on } U_{\delta_{0}}(\Sigma) \tag{4.1}
\end{equation*}
$$

Since the surface is smooth we have the stability estimate

$$
\begin{equation*}
\left\|u^{e}\right\|_{W_{\infty}^{s}\left(U_{\delta_{0}}(\Sigma)\right)} \lesssim\|u\|_{W_{\infty}^{s}(\Sigma)} \tag{4.2}
\end{equation*}
$$

for $s>0$. We will, in particular, use $s=2$ in our forthcoming estimates. Using the chain rule we obtain

$$
\begin{equation*}
D v^{e}=D(v \circ p)=D v D p=D v\left(P_{\Sigma}-\rho \mathcal{H}\right) \tag{4.3}
\end{equation*}
$$

Here we used the identity $D p=P_{\Sigma}-\rho \mathcal{H}$, where $\mathcal{H}$ is the Hessian of the distance function $\mathcal{H}=\nabla \otimes \nabla \rho$. For $x \in U_{\delta_{0}}(\Sigma)$ we have

$$
\begin{equation*}
\mathcal{H}(x)=\sum_{i=1}^{2} \frac{\kappa_{i}^{e}}{1+\rho(x) \kappa_{i}^{e}} a_{i}^{e} \otimes a_{i}^{e} \tag{4.4}
\end{equation*}
$$

where $\kappa_{i}$ are the principal curvatures with corresponding principal curvature vectors $a_{i}$, see [9] Lemma 14.7. Thus, using the bound (2.1) for $\delta_{0}$ we obtain

$$
\begin{equation*}
\|\mathcal{H}\|_{L^{\infty}\left(U_{\delta_{0}}(\Sigma)\right)} \lesssim 1 \tag{4.5}
\end{equation*}
$$

Starting from (4.3) we obtain

$$
\begin{equation*}
\nabla_{\Sigma_{h}} v^{e}=\nabla_{\Sigma_{h}}(v \circ p)=\nabla(v \circ p) \cdot P_{\Sigma_{h}}=\nabla v \cdot D p P_{\Sigma_{h}}=\nabla_{\Sigma} v \cdot P_{\Sigma}\left(P_{\Sigma}-\rho \mathcal{H}\right) P_{\Sigma_{h}} \tag{4.6}
\end{equation*}
$$

where we used the fact that $P_{\Sigma}-\rho \mathcal{H}=P_{\Sigma}\left(P_{\Sigma}-\rho \mathcal{H}\right)$, which follows from (4.4). For each element $K \subset \Sigma_{h}$ and $x \in K$ the resulting mapping

$$
\begin{equation*}
B=P_{\Sigma}(I-\rho \mathcal{H}) P_{\Sigma_{h}}: T_{x}(K) \rightarrow T_{p(x)}(\Sigma) \tag{4.7}
\end{equation*}
$$

is invertible and we have the identity

$$
\begin{equation*}
\nabla_{\Sigma_{h}} v^{e}=B^{T} \nabla_{\Sigma} v \tag{4.8}
\end{equation*}
$$

Lifting. The lifting $w^{l}$ of a function $w$ defined on $\Sigma_{h}$ to $\Sigma$ is defined as the push forward

$$
\begin{equation*}
\left(w^{l}\right)^{e}=w^{l} \circ p=w \quad \text { on } \Sigma_{h} \tag{4.9}
\end{equation*}
$$

Using the chain rule we obtain

$$
\begin{equation*}
D w=D\left(w^{l} \circ p\right)=\left(D w^{l}\right) D p=\left(D w^{l}\right)\left(P_{\Sigma}-\rho \mathcal{H}\right) \tag{4.10}
\end{equation*}
$$

and thus

$$
\begin{align*}
\nabla_{\Sigma_{h}} w=\nabla & \left(w^{l} \circ p\right) \cdot P_{\Sigma_{h}}=\nabla\left(w^{l}\right) \cdot D p P_{\Sigma_{h}}  \tag{4.11}\\
& =\left(\nabla w^{l}\right) \cdot\left(P_{\Sigma}-\rho \mathcal{H}\right) P_{\Sigma_{h}}=\left(\nabla_{\Sigma} w^{l}\right) \cdot P_{\Sigma}\left(P_{\Sigma}-\rho \mathcal{H}\right) P_{\Sigma_{h}}=\left(\nabla_{\Sigma} w^{l}\right) \cdot B \tag{4.12}
\end{align*}
$$

where $B$ is defined in (4.7). We obtain

$$
\begin{equation*}
\nabla_{\Sigma} w^{l}=B^{-T} \nabla_{\Sigma_{h}} w \tag{4.13}
\end{equation*}
$$

Estimates Related to $\boldsymbol{B}$. In order to prepare for the proof of the error estimate we collect some estimates related to $B$. First

$$
\begin{equation*}
\left\|P_{\Sigma}-B B^{T}\right\|_{L^{\infty}(\Sigma)} \lesssim h^{2}, \quad\|B\|_{L^{\infty}\left(\Sigma_{h}\right)} \lesssim 1 \quad\left\|B^{-1}\right\|_{L^{\infty}(\Sigma)} \lesssim 1 \tag{4.14}
\end{equation*}
$$

Secondly we note that the surface measure $d \Sigma=|B| d \Sigma_{h}$, where $|B|$ is the absolute value of the determinant of $\left[B \xi_{1} B \xi_{2} n^{e}\right]$ and $\left\{\xi_{1}, \xi_{2}\right\}$ is an orthonormal basis in $T_{x}(K)$, and we have the following estimates

$$
\begin{equation*}
\|1-|B|\|_{L^{\infty}\left(\Sigma_{h}\right)} \lesssim h^{2}, \quad\left\|\left.B\left|\left\|_{L^{\infty}\left(\Sigma_{h}\right)} \lesssim 1, \quad\right\|\right| B\right|^{-1}\right\|_{L^{\infty}\left(\Sigma_{h}\right)} \lesssim 1 \tag{4.15}
\end{equation*}
$$

see [3] and [5]. In view of these bounds we note that we have the following equivalences

$$
\begin{equation*}
\left\|v^{l}\right\|_{L^{p}(\Sigma)} \sim\|v\|_{L^{p}\left(\Sigma_{h}\right)}, \quad\|v\|_{L^{p}(\Sigma)} \sim\left\|v^{e}\right\|_{L^{p}\left(\Sigma_{h}\right)} \tag{4.16}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|\nabla_{\Sigma} v^{l}\right\|_{L^{p}(\Sigma)} \sim\left\|\nabla_{\Sigma_{h}} v\right\|_{L^{p}\left(\Sigma_{h}\right)}, \quad\left\|\nabla_{\Sigma} v\right\|_{L^{p}(\Sigma)} \sim\left\|\nabla_{\Sigma_{h}} v^{e}\right\|_{L^{p}\left(\Sigma_{h}\right)} \tag{4.17}
\end{equation*}
$$

### 4.2 Error Estimate for the Discrete Embedding

Here we formulate an estimate of the difference between the embeddings of the discrete and continuous surfaces.

Lemma 4.1 If the surface approximation assumptions (2.2) and (2.3) hold, then

$$
\begin{equation*}
\left\|x_{\Sigma}^{e}-x_{\Sigma_{h}}\right\|_{L^{\infty}\left(\Sigma_{h}\right)}^{2}+h^{2}\left\|\nabla_{\Sigma_{h}}\left(x_{\Sigma}^{e}-x_{\Sigma_{h}}\right)\right\|_{L^{\infty}\left(\Sigma_{h}\right)}^{2} \lesssim h^{4} \tag{4.18}
\end{equation*}
$$

Proof. For the first term we have

$$
\begin{equation*}
\left\|x_{\Sigma}^{e}-x_{\Sigma_{h}}\right\|_{L^{\infty}\left(\Sigma_{h}\right)}=\|\rho\|_{L^{\infty}\left(\Sigma_{h}\right)} \lesssim h^{2} \tag{4.19}
\end{equation*}
$$

where we used $(2.2)$. For the second term we have the identities

$$
\begin{equation*}
\nabla_{\Sigma_{h}} x_{\Sigma}^{e}=P_{\Sigma_{h}}\left(P_{\Sigma}-\rho \mathcal{H}\right), \quad \nabla_{\Sigma_{h}} x_{\Sigma_{h}}=P_{\Sigma_{h}} \tag{4.20}
\end{equation*}
$$

and thus

$$
\begin{align*}
\left\|\nabla_{\Sigma_{h}} x_{\Sigma}^{e}-\nabla_{\Sigma_{h}} x_{\Sigma_{h}}\right\|_{L^{\infty}\left(\Sigma_{h}\right)} & \leq\left\|P_{\Sigma_{h}}\left(P_{\Sigma}-\rho \mathcal{H}\right)-P_{\Sigma_{h}}\right\|_{L^{\infty}\left(\Sigma_{h}\right)}  \tag{4.21}\\
& \leq\left\|P_{\Sigma_{h}}\left(P_{\Sigma}-P_{\Sigma_{h}}\right)\right\|_{L^{\infty}\left(\Sigma_{h}\right)}+\left\|\rho P_{\Sigma_{h}} \mathcal{H}\right\|_{L^{\infty}\left(\Sigma_{h}\right)}  \tag{4.22}\\
& \lesssim h \tag{4.23}
\end{align*}
$$

where we used (2.2), 2.3), and (4.5).

### 4.3 Some Inequalities

In this section we formulate some useful inequalities. First a trace inequality that allows passage from an edge $E \in \mathcal{E}_{h}$ to a tetrahedron $T \in \mathcal{T}_{h}$ for cut surfaces. Then we prove two inverse inequalities. For convenience we introduce the semi norms

$$
\begin{equation*}
\left\|\|v\|_{\mathcal{E}_{h}}^{2}=J_{\mathcal{E}_{h}}(v, v), \quad\right\| v v \|_{\mathcal{F}_{h}}^{2}=J_{\mathcal{F}_{h}}(v, v) \tag{4.24}
\end{equation*}
$$

Lemma 4.2 In the cut case we have the following trace inequality

$$
\begin{equation*}
\|v\|_{E}^{2} \lesssim h^{-2}\|v\|_{T}^{2}+\|\nabla v\|_{T}^{2}+h^{2}\|\nabla \otimes \nabla v\|_{T}^{2} \quad v \in H^{2}(T) \tag{4.25}
\end{equation*}
$$

where $E \in \mathcal{E}_{h}, T \in \mathcal{T}_{h}$, and $E \subset \partial T \cap \Sigma_{h}$.

Proof. We first apply the trace inequality

$$
\begin{equation*}
\|v\|_{E}^{2} \lesssim h^{-1}\|v\|_{F}^{2}+h\left\|\nabla_{F} v\right\|_{F}^{2} \tag{4.26}
\end{equation*}
$$

see Lemma 4.2 in [11], to pass from the edge $E$ to the face $F=F(E) \in \mathcal{F}_{h}$ such that $E=F \cap \Sigma_{h}$. Then we apply a standard trace inequality to pass from $F$ to an element $T=T(F) \in \mathcal{T}_{h}$ to which $F$ is a face. More precisely

$$
\begin{align*}
\|v\|_{E}^{2} & \lesssim h^{-1}\|v\|_{F}^{2}+h\left\|\nabla_{F} v\right\|_{F}^{2}  \tag{4.27}\\
& \lesssim h^{-2}\|v\|_{T}^{2}+\|\nabla v\|_{T}^{2}+\left\|\nabla_{F} v\right\|_{T}^{2}+h^{2}\left\|\left(\nabla_{F} v\right) \otimes \nabla\right\|_{T}^{2}  \tag{4.28}\\
& \lesssim h^{-2}\|v\|_{T}^{2}+\|\nabla v\|_{T}^{2}+h^{2}\|\nabla \otimes \nabla v\|_{T}^{2} \tag{4.29}
\end{align*}
$$

where $\nabla_{F}=P_{F} \nabla$, with $P_{F}=I-n_{F} \otimes n_{F}$ the constant projection onto the tangent plane of the face $F$, is the tangent gradient to the face $K$. We also used the estimates $\left\|\nabla_{F} v\right\|_{T} \lesssim\|\nabla v\|_{T}$ and $\left\|\left(\nabla_{F} v\right) \otimes \nabla\right\|_{T}=\left\|\left(P_{F} \nabla v\right) \otimes \nabla\right\|_{T}=\left\|P_{F}((\nabla v) \otimes \nabla)\right\|_{T} \leq\|\nabla \otimes \nabla v\|_{T}^{2}$.

Lemma 4.3 The following inverse inequality holds

$$
\begin{equation*}
\sum_{E \in \mathcal{E}_{h}} h\|v\|_{E}^{2} \lesssim\|v\|_{\Sigma_{h}}^{2}+\mid\|v\|_{\mathcal{F}_{h}}^{2} \quad \forall v \in V_{h} \tag{4.30}
\end{equation*}
$$

where $\||v|\|_{\mathcal{F}_{h}}^{2}$ is present only in the cut case.

Proof. In the meshed case we have

$$
\begin{equation*}
\sum_{E \in \mathcal{E}_{h}} h\|v\|_{E}^{2} \lesssim \sum_{K \in \mathcal{K}_{h}}\|v\|_{K}^{2}+h^{2}\left\|\nabla_{\Sigma_{h}} v\right\|_{K}^{2} \lesssim \sum_{K \in \mathcal{K}_{h}}\|v\|_{K}^{2}=\|v\|_{\Sigma_{h}}^{2} \tag{4.31}
\end{equation*}
$$

where we used a standard trace inequality followed by an inverse estimate. In the cut case we use Lemma 4.2 to get

$$
\begin{align*}
\sum_{E \in \mathcal{E}_{h}} h\|v\|_{E}^{2} & \lesssim \sum_{T \in \mathcal{T}_{h}} h^{-1}\|v\|_{T}^{2}+h\|\nabla v\|_{T}^{2}+h^{3}\|\nabla \otimes \nabla v\|_{T}^{2}  \tag{4.32}\\
& \lesssim \sum_{T \in \mathcal{T}_{h}} h^{-1}\|v\|_{T}^{2}  \tag{4.33}\\
& \lesssim\|v\|_{\Sigma_{h}}^{2}+\| \| v \|_{\mathcal{F}_{h}}^{2} \tag{4.34}
\end{align*}
$$

where we used standard inverse inequalities and at last Lemma 4.4 in [3].

Lemma 4.4 The following inverse inequality holds

$$
\begin{equation*}
h^{2}\left\|\nabla_{\Sigma_{h}} v\right\|_{\Sigma_{h}}^{2} \lesssim\|v\|_{\Sigma_{h}}^{2}+\|v\|_{\mathcal{F}_{h}}^{2} \quad \forall v \in V_{h} \tag{4.35}
\end{equation*}
$$

where $\||v|\|_{\mathcal{F}_{h}}^{2}$ is present only in the cut case.

Proof. In the meshed case this estimate follows directly from a standard elementwise inverse inequality. In the cut case we use the fact that $\left.\left(\nabla_{\Sigma_{h}} v\right)\right|_{K}$ is constant

$$
\begin{align*}
h^{2}\left\|\nabla_{\Sigma_{h}} v\right\|_{\Sigma_{h}}^{2} & =\sum_{K \in \mathcal{K}_{h}} h^{2}\left\|\nabla_{\Sigma_{h}} v\right\|_{K}^{2}  \tag{4.36}\\
& =\sum_{K \in \mathcal{K}_{h}} h^{2} \operatorname{meas}(K)(\operatorname{meas}(T(K)))^{-1}\left\|\nabla_{\Sigma_{h}} v\right\|_{T(K)}^{2}  \tag{4.37}\\
& \lesssim \sum_{T \in \mathcal{T}_{h}} h\|\nabla v\|_{T}^{2}  \tag{4.38}\\
& \lesssim \sum_{T \in \mathcal{T}_{h}} h^{-1}\|v\|_{T}^{2}  \tag{4.39}\\
& \lesssim\|v\|_{\Sigma_{h}}^{2}+\| \| v \|_{\mathcal{F}_{h}}^{2} \tag{4.40}
\end{align*}
$$

where $T(K) \in \mathcal{T}_{h}$ is the element such that $T \cap \Sigma_{h}=K$ and we used standard inverse inequalities and at last Lemma 4.4 in [3].

### 4.4 Estimates for the Edge Stabilization Term

In this section we prove two estimates for the edge stabilization term. The first shows that the edge stabilization term acting on an extension of a smooth function is $O\left(h^{2}\right)$. The second lemma is used in the proof of Theorem 4.2 where we show that it is indeed enough to use the simplified stabilization $J_{h}(v, v)=\tau_{\mathcal{F}_{h}} J_{\mathcal{F}_{h}}(\cdot, \cdot)$ in the case of cut surfaces.

Lemma 4.5 If the surface approximation assumptions (2.2) and (2.3) hold, then

$$
\begin{equation*}
\left\|\mid v^{e}\right\|\left\|_{\mathcal{E}_{h}} \lesssim h\right\| v \|_{W_{\infty}^{2}(\Sigma)} \tag{4.41}
\end{equation*}
$$

Proof. Consider the contribution $h\left\|\left[t_{E} \cdot \nabla_{\Sigma_{h}} v^{e}\right]\right\|_{E}^{2}$ to $\left\|\left\|v^{e}\right\|\right\|_{\mathcal{E}_{h}}^{2}$ from edge $E \in \mathcal{E}_{h}$. Let $e_{E}$ be the unit vector parallel with the edge $E$ such that $t_{E, K_{1}}=n_{h, 1} \times e_{E}$ and $t_{E, K_{2}}=-n_{h, 2} \times e_{E}$. Let $t=\left(n^{e} \times e_{E}\right)^{l}$ and $s=t \times n$. Then $t$ and $s$ span the tangent plane $T_{p(x)}(\Sigma)$ for $x \in E$ and

$$
\begin{align*}
& \left\|t_{E, K_{1}}-t^{e}\right\|_{L^{\infty}(E)}+\left\|t^{e}+t_{E, K_{2}}\right\|_{L^{\infty}(E)} \\
& \quad=\left\|\left(n_{h, 1}-n^{e}\right) \times e_{E}\right\|_{L^{\infty}(E)}+\left\|\left(n^{e}-n_{h, 2}\right) \times e_{E}\right\|_{L^{\infty}(E)} \lesssim h \tag{4.42}
\end{align*}
$$

We then have

$$
\begin{align*}
\left\|\left[t_{E} \cdot \nabla_{\Sigma_{h}} v^{e}\right]\right\|_{E} & =\left\|\left(t_{E, K_{1}}+t_{E, K_{2}}\right) \cdot \nabla v^{e}\right\|_{E}  \tag{4.43}\\
& \leq\left(\left\|t_{E, K_{1}}-t^{e}\right\|_{L^{\infty}(E)}+\left\|t^{e}+t_{E, K_{2}}\right\|_{L^{\infty}(E)}\right)\left\|\nabla v^{e}\right\|_{E}  \tag{4.44}\\
& \lesssim h\left\|\nabla v^{e}\right\|_{E}  \tag{4.45}\\
& \lesssim h^{3 / 2}\left\|v^{e}\right\|_{W_{\infty}^{1}(E)}  \tag{4.46}\\
& \lesssim h^{3 / 2}\|v\|_{W_{\infty}^{1}(\Sigma)} \tag{4.47}
\end{align*}
$$

where we used 4.42$)$ and the bound $\left\|v^{e}\right\|_{W_{\infty}^{1}(E)} \lesssim\left\|v^{e}\right\|_{W_{\infty}^{1}\left(U_{\delta_{0}}(\Sigma)\right)} \lesssim\|v\|_{W_{\infty}^{1}(\Sigma)}$, which follows from the stability (4.2) of extensions. Thus we obtain

$$
\begin{equation*}
\mid\left\|u^{e}\right\|\left\|_{\mathcal{E}_{h}}^{2}=\sum_{E \in \mathcal{E}_{h}} h\right\|\left[t_{E} \cdot \nabla_{\Sigma_{h}} u^{e}\right] \|_{E}^{2} \lesssim \sum_{E \in \mathcal{E}_{h}} h^{4} \lesssim h^{2} \tag{4.48}
\end{equation*}
$$

since $\operatorname{card}\left(\mathcal{E}_{h}\right) \lesssim h^{-2}$ both for meshed and cut surfaces.

Lemma 4.6 If the surface approximation assumptions (2.2) and (2.3) hold, then the following bound holds for cut surfaces

$$
\begin{equation*}
\|v\|_{\mathcal{E}_{h}}^{2} \lesssim\|v\|_{\Sigma_{h}}^{2}+\| \| v \|_{\mathcal{F}_{h}}^{2} \quad \forall v \in V_{h} \tag{4.49}
\end{equation*}
$$

Proof. Consider the contribution to $J_{\mathcal{E}_{h}}(v, v)$ from an edge $E \in \mathcal{E}_{h}$. We employ the same notation as in the proof of Lemma 4.5. Adding and subtracting $t^{e}$, using some basic estimates, the trace inequality in Lemma 4.2, the estimate (4.42) for the tangent error, and finally an inverse estimate give

$$
\begin{align*}
& h\left\|t_{E, K_{1}} \cdot \nabla_{\Sigma_{h}} v_{1}+t_{E, K_{2}} \cdot \nabla_{\Sigma_{h}} v_{2}\right\|_{E}^{2} \\
& \quad \lesssim h\left\|t^{e} \cdot[\nabla v]\right\|_{E}^{2}+h\left\|\left(t_{E, K_{1}}-t^{e}\right) \cdot \nabla v_{1}\right\|_{E}^{2}+h\left\|\left(t^{e}+t_{E, K_{2}}\right) \cdot \nabla v_{2}\right\|_{E}^{2}  \tag{4.50}\\
& \quad \lesssim h\left\|t^{e} \cdot n_{F, T_{1}}\left[n_{F} \cdot \nabla v\right]\right\|_{E}^{2}+h^{3}\left\|\nabla v_{1}\right\|_{E}^{2}+h^{3}\left\|\nabla v_{2}\right\|_{E}^{2}  \tag{4.51}\\
& \quad \lesssim\left\|\left[n_{F} \cdot \nabla v\right]\right\|_{F}^{2}+h\left\|\nabla v_{1}\right\|_{T_{1}}^{2}+h\left\|\nabla v_{2}\right\|_{T_{2}}^{2}  \tag{4.52}\\
& \quad \lesssim\left\|\left[n_{F} \cdot \nabla v\right]\right\|_{F}^{2}+h^{-1}\left\|v_{1}\right\|_{T_{1}}^{2}+h^{-1}\left\|v_{2}\right\|_{T_{2}}^{2} \tag{4.53}
\end{align*}
$$

Here $F$ is the face in $\mathcal{F}_{h}$ with $E=F \cap \Sigma_{h}$ and $T_{1}, T_{2}$ are the elements in $\mathcal{T}_{h}$ that share the face $F$. Using this estimate we get

$$
\begin{equation*}
\|v\|_{\mathcal{E}_{h}}^{2} \lesssim\| \| v\left\|_{\mathcal{F}_{h}}^{2}+\sum_{T \in \mathcal{T}_{h}} h^{-1}\right\| v\left\|_{T}^{2} \lesssim\right\| v\left\|_{\Sigma_{h}}^{2}+\right\| v v \|_{\mathcal{F}_{h}}^{2} \tag{4.54}
\end{equation*}
$$

where we used Lemma 4.4 in [3] in the last estimate.

### 4.5 Stability Estimate for the Discrete Mean Curvature Vector

In this section our main result is a stability estimate for the discrete mean curvature vector.
Lemma 4.7 If the surface approximation assumptions (2.2) and (2.3) hold and the stabilization parameters satisfy $0 \leq \tau_{\mathcal{E}_{h}}$ and $0<\tau_{\mathcal{F}_{h}}$ (in the cut case), then the discrete mean curvature vector $H_{h}$ defined by (3.6) satisfies the stability estimate

$$
\begin{equation*}
\left\|H_{h}\right\|_{\Sigma_{h}}^{2}+\tau_{\mathcal{E}_{h}}\left\|\left|H_{h}\| \|_{\mathcal{E}_{h}}^{2}+\tau_{\mathcal{F}_{h}}\left\|\mid H_{h}\right\|_{\mathcal{F}_{h}}^{2} \lesssim 1\right.\right. \tag{4.55}
\end{equation*}
$$

Remark 4.1 We note that in the meshed case the edge stabilization term is not necessary to prove $L^{2}\left(\Sigma_{h}\right)$ stability of $H_{h}$ but with $\tau_{\mathcal{E}_{h}}>0$ we get stability in a stronger norm. However, in the cut case $\tau_{\mathcal{F}_{h}}$ must be strictly positive to establish the stability estimate.

Proof. Setting $v=H_{h}$ in (3.6) we obtain

$$
\begin{align*}
& \left\|H_{h}\right\|_{\Sigma_{h}}^{2}+\tau_{\mathcal{E}_{h}}\left\|H_{h}\right\|\left\|_{\mathcal{E}_{h}}^{2}+\tau_{\mathcal{F}_{h}}\right\|\left\|H_{h}\right\| \|_{\mathcal{F}_{h}}^{2} \\
& \quad=B_{h}\left(H_{h}, H_{h}\right)+J_{h}\left(H_{h}, H_{h}\right)  \tag{4.56}\\
& \quad=L_{h}\left(H_{h}\right)  \tag{4.57}\\
& \quad=\left(\nabla_{\Sigma_{h}} x_{\Sigma_{h}}, \nabla_{\Sigma_{h}} H_{h}\right)_{\Sigma_{h}}  \tag{4.58}\\
& \quad=\left(\nabla_{\Sigma_{h}}\left(x_{\Sigma_{h}}-x_{\Sigma}^{e}\right), \nabla_{\Sigma_{h}} H_{h}\right)_{\Sigma_{h}}+\left(\nabla_{\Sigma_{h}} x_{\Sigma}^{e}, \nabla_{\Sigma_{h}} H_{h}\right)_{\Sigma_{h}}  \tag{4.59}\\
& \quad=I+I I \tag{4.60}
\end{align*}
$$

Term I. Using the geometry approximation Lemma 4.1 followed by the inverse inequality in Lemma 4.4 we obtain

$$
\begin{align*}
|I| & =\left|\left(\nabla_{\Sigma_{h}}\left(x_{\Sigma_{h}}-x_{\Sigma}^{e}\right), \nabla_{\Sigma_{h}} H_{h}\right)_{\Sigma_{h}}\right| \\
& \lesssim\left\|\nabla_{\Sigma_{h}}\left(x_{\Sigma_{h}}-x_{\Sigma}^{e}\right)\right\|_{\Sigma_{h}}\left\|\nabla_{\Sigma_{h}} H_{h}\right\|_{\Sigma_{h}}  \tag{4.61}\\
& \lesssim \delta^{-1} h^{-2}\left\|\nabla_{\Sigma_{h}}\left(x_{\Sigma_{h}}-x_{\Sigma}^{e}\right)\right\|_{\Sigma_{h}}^{2}+\delta h^{2}\left\|\nabla_{\Sigma_{h}} H_{h}\right\|_{\Sigma_{h}}^{2}  \tag{4.62}\\
& \lesssim \delta^{-1}+\delta\left(\left\|H_{h}\right\|_{\Sigma_{h}}^{2}+\| \| H_{h}\| \|_{\mathcal{F}_{h}}^{2}\right) \tag{4.63}
\end{align*}
$$

for any $\delta>0$.

Term II. Element wise partial integration gives

$$
\begin{align*}
|I I| & =\left|\left(\nabla_{\Sigma_{h}} x_{\Sigma}^{e}, \nabla_{\Sigma_{h}} H_{h}\right)_{\Sigma_{h}}\right|  \tag{4.64}\\
& =\left|\sum_{E \in \mathcal{E}_{h}}\left(\left[t_{E} \cdot \nabla_{\Sigma_{h}} x_{\Sigma}^{e}\right], H_{h}\right)_{E}\right|  \tag{4.65}\\
& \leq \sum_{E \in \mathcal{E}_{h}}\left\|\left[t_{E} \cdot \nabla_{\Sigma_{h}} x_{\Sigma}^{e}\right]\right\|_{E}\left\|H_{h}\right\|_{E}  \tag{4.66}\\
& \lesssim \delta^{-1} \sum_{E \in \mathcal{E}_{h}} h^{-1}\left\|\left[t_{E} \cdot \nabla_{\Sigma_{h}} x_{\Sigma}^{e}\right]\right\|_{E}^{2}+\delta \sum_{E \in \mathcal{E}_{h}} h\left\|H_{h}\right\|_{E}^{2}  \tag{4.67}\\
& \lesssim \delta^{-1}+\delta\left(\left\|H_{h}\right\|_{\Sigma_{h}}^{2}+\left\|H_{h}\right\| \|_{\mathcal{F}_{h}}^{2}\right) \tag{4.68}
\end{align*}
$$

Here the first term on the right hand side of (4.67) was estimated using Lemma 4.5 as follows

$$
\begin{equation*}
\sum_{E \in \mathcal{E}_{h}} h^{-1}\left\|\left[t_{E} \cdot \nabla_{\Sigma_{h}} x_{\Sigma}^{e}\right]\right\|_{E}^{2} \lesssim h^{-2}\| \| x_{\Sigma}^{e}\| \|_{\mathcal{E}_{h}}^{2} \lesssim 1 \tag{4.69}
\end{equation*}
$$

and the second term was estimated using Lemma 4.3.
Combining the bounds (4.63) and 4.68) of $I$ and $I I$ we obtain

$$
\begin{equation*}
\left\|H_{h}\right\|_{\Sigma_{h}}^{2}+\tau_{\mathcal{E}_{h}}\| \| H_{h} \mid\left\|_{\mathcal{E}_{h}}^{2}+\tau_{\mathcal{F}_{h}}\right\|\left\|H_{h}\right\| \|_{\mathcal{F}_{h}}^{2} \lesssim \delta^{-1}+\delta\left(\left\|H_{h}\right\|_{\Sigma_{h}}^{2}+\left\|H_{h}\right\| \|_{\mathcal{F}_{h}}^{2}\right) \tag{4.70}
\end{equation*}
$$

The desired bound is finally obtained by, using the fact that $\tau_{\mathcal{F}_{h}}>0$, in the cut case and choosing $\delta$ small enough followed by a kick back argument.

### 4.6 Interpolation

The construction of the interpolation operator is different in the meshed and cut cases but we use the same notation for the operator to get a unified treatment.

Meshed Case: Let $\pi_{h}: C(\Sigma) \rightarrow V_{h}$ be defined by

$$
\begin{equation*}
\pi_{h}: v \mapsto \pi_{L, \mathcal{K}_{h}} v^{e} \tag{4.71}
\end{equation*}
$$

where $\pi_{L, \mathcal{K}_{h}}$ is the Lagrange interpolation operator defined on $\Sigma_{h}$. We have the elementwise error estimate

$$
\begin{equation*}
\left\|v^{e}-\pi_{h} v\right\|_{H^{m}(K)} \lesssim h^{k-m}\left\|v^{e}\right\|_{H^{k}(K)}, \quad 0 \leq m \leq k \leq 2, \quad \forall K \in \mathcal{K}_{h} \tag{4.72}
\end{equation*}
$$

Cut Level Set Surface Case: Let $\pi_{h}: C(\Sigma) \rightarrow V_{h}$ be defined by

$$
\begin{equation*}
\pi_{h}: v \mapsto\left(\pi_{L, \mathcal{T}_{h}} v^{e}\right) \tag{4.73}
\end{equation*}
$$

where $\pi_{L, \mathcal{T}_{h}}$ is the Lagrange interpolation operator defined on the three dimensional mesh $\mathcal{T}_{h}$. We have the elementwise error estimates

$$
\begin{equation*}
\left\|v^{e}-\pi_{h} v\right\|_{H^{m}(T)} \lesssim h^{k-m}\left\|v^{e}\right\|_{H^{k}(T)}, \quad 0 \leq m \leq k \leq 2, \quad \forall T \in \mathcal{T}_{h} \tag{4.74}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|v^{e}-\pi_{h} v\right\|_{H^{m}(F)} \lesssim h^{k-m}\left\|v^{e}\right\|_{H^{k}(F)}, \quad 0 \leq m \leq k \leq 2, \quad \forall F \in \mathcal{F}_{h} \tag{4.75}
\end{equation*}
$$

For convenience we shall use the simplified notation $\pi_{h} u=\pi_{h} u^{e} \in V_{h}$ and $\pi_{h}^{l} u=$ $\left(\left(\pi_{h} u^{e}\right)_{\Sigma_{h}}\right)^{l}$. In both cases we have the following interpolation error estimate

$$
\begin{equation*}
\left\|u-\pi_{h}^{l} u\right\|_{H^{m}(\Sigma)} \lesssim h^{k-m}\|u\|_{H^{k}(\Sigma)}, \quad 0 \leq m \leq k \leq 2 \tag{4.76}
\end{equation*}
$$

See [3] and [5] for a proof of (4.76). We will also need the following interpolation error estimate for the terms emanating from the stabilization.

Lemma 4.8 If the surface approximation assumptions (2.2) and (2.3) hold, then the following interpolation error estimates hold

$$
\begin{array}{r}
\left\|\left\|u^{e}-\pi_{h} u^{e}\right\|\right\|_{\mathcal{E}_{h}} \lesssim h\|u\|_{W_{\infty}^{2}(\Sigma)} \\
\left\|\left\|u^{e}-\pi_{h} u^{e} \mid\right\|_{\mathcal{F}_{h}} \lesssim h\right\| u \|_{W_{\infty}^{2}(\Sigma)} \tag{4.78}
\end{array}
$$

Proof. Estimate (4.77). In the meshed case applying a standard trace inequality elementwise followed by the interpolation estimate 4.72) yields

$$
\begin{align*}
& \sum_{E \in \mathcal{E}_{h}} h\left\|t_{E} \cdot \nabla_{\Sigma_{h}}\left(u^{e}-\pi_{h} u^{e}\right)\right\|_{E}^{2} \\
& \quad \lesssim \sum_{K \in \mathcal{K}_{h}}\left\|\nabla_{\Sigma_{h}}\left(u^{e}-\pi_{h} u^{e}\right)\right\|_{K}^{2}+h^{2}\left\|\nabla_{\Sigma_{h}} \otimes \nabla_{\Sigma_{h}}\left(u^{e}-\pi_{h} u^{e}\right)\right\|_{K}^{2}  \tag{4.79}\\
& \quad \lesssim \sum_{K \in \mathcal{K}_{h}} h^{2}\left\|u^{e}\right\|_{H^{2}(K)}^{2}  \tag{4.80}\\
& \quad \lesssim\left(\sum_{K \in \mathcal{K}_{h}} h^{4}\right)\left\|u^{e}\right\|_{W_{\infty}\left(U_{\delta_{0}}(\Sigma)\right)}^{2}  \tag{4.81}\\
& \quad \lesssim h^{2}\|u\|_{W_{\infty}(\Sigma)}^{2} \tag{4.82}
\end{align*}
$$

where we used the fact that $\operatorname{card}\left(\mathcal{K}_{h}\right) \lesssim h^{-2}$ and the stability (4.2) of the extension $u^{e}$.
In the cut case, we first apply the trace inequality (4.25) to pass from the edge $E$ to the face $F=F(E)$ such that $F \cap \Sigma_{h}=E$. Next we note that second order derivatives of
$\pi_{h} u^{e}$ vanish, then we use a trace inequality to pass from the faces to the tetrahedra and use the interpolation estimate (4.74) as follows

$$
\begin{array}{rl}
\sum_{E \in \mathcal{E}_{h}} & h\left\|t_{E} \cdot \nabla_{\Sigma_{h}}\left(u^{e}-\pi_{h} u^{e}\right)\right\|_{E}^{2} \\
\lesssim & \sum_{F \in \mathcal{F}_{h}}\left\|\nabla_{\Sigma_{h}}\left(u^{e}-\pi_{h} u^{e}\right)\right\|_{F}^{2}+h^{2}\left\|\nabla_{F}\left(\nabla_{\Sigma_{h}}\left(u^{e}-\pi_{h} u^{e}\right)\right)\right\|_{F}^{2} \\
\lesssim & \sum_{T \in \mathcal{T}_{h}} h^{-1}\left\|\nabla_{\Sigma_{h}}\left(u^{e}-\pi_{h} u^{e}\right)\right\|_{T}^{2}+h \| \nabla\left(\nabla_{\Sigma_{h}}\left(u^{e}-\pi_{h} u^{e}\right) \|_{T}^{2}\right. \\
& \quad+\sum_{F \in \mathcal{F}_{h}} h^{2}\left\|\nabla\left(\nabla\left(u^{e}\right)\right)\right\|_{F}^{2} \\
\lesssim & \sum_{T \in \mathcal{T}_{h}} h\left\|u^{e}\right\|_{H^{2}(T)}^{2}+\sum_{F \in \mathcal{F}_{h}} h^{2}\left\|\nabla\left(\nabla\left(u^{e}\right)\right)\right\|_{F}^{2} \\
\lesssim & \left(\sum_{T \in \mathcal{T}_{h}} h^{4}+\sum_{F \in \mathcal{F}_{h}} h^{4}\right)\left\|u^{e}\right\|_{W_{\infty}^{2}\left(U_{\delta_{0}}(\Sigma)\right)}^{2} \\
\lesssim & h^{2}\|u\|_{W_{\infty}^{2}(\Sigma)}^{2} \tag{4.87}
\end{array}
$$

Here we used the fact that $\operatorname{card}\left(\mathcal{F}_{h}\right) \sim \operatorname{card}\left(\mathcal{T}_{h}\right) \lesssim h^{-2}$ and the stability (4.2) of the extension $u^{e}$.
Estimate 4.78). Using a standard trace inequality followed by the interpolation estimate (4.74) we obtain

$$
\begin{align*}
\sum_{F \in \mathcal{F}_{h}}\left\|\left[n_{F} \cdot \nabla\left(u^{e}-\pi_{h} u^{e}\right)\right]\right\|_{F}^{2} & \lesssim \sum_{T \in \mathcal{T}_{h}} h^{-1}\left\|\nabla\left(u^{e}-\pi_{h} u^{e}\right)\right\|_{T}^{2}+h\left\|\nabla\left(\nabla\left(u^{e}-\pi_{h} u^{e}\right)\right)\right\|_{T}^{2}  \tag{4.88}\\
& \lesssim \sum_{T \in \mathcal{T}_{h}} h\left\|u^{e}\right\|_{H^{2}(T)}^{2}  \tag{4.89}\\
& \lesssim\left(\sum_{T \in \mathcal{T}_{h}} h^{4}\right)\left\|u^{e}\right\|_{W_{\infty}^{2}\left(U_{\delta_{0}}(\Sigma)\right)}  \tag{4.90}\\
& \lesssim h^{2}\|u\|_{W_{\infty}^{2}(\Sigma)}^{2} \tag{4.91}
\end{align*}
$$

where again we used the fact that $\operatorname{card}\left(\mathcal{T}_{h}\right) \lesssim h^{-2}$ and the stability (4.2) of the extension $u^{e}$.

### 4.7 Error Estimate for the Discrete Mean Curvature Vector

We are now ready to state and prove our main result.
Theorem 4.1 Let $\Sigma$ be a smooth surface, $\Sigma_{h}$ an approximate surface that is either meshed or cut and satisfies (2.2) and (2.3), then the discrete mean curvature vector $H_{h}$, defined
by (3.6), with parameters $\tau_{\mathcal{E}_{h}}>0$ and $\tau_{\mathcal{F}_{h}}>0$ (in the cut case), satisfies the estimate

$$
\begin{equation*}
\left\|H-H_{h}^{l}\right\|_{\Sigma}^{2}+\tau_{\mathcal{E}_{h}}\| \| H_{h}\| \|_{\mathcal{E}_{h}}^{2}+\tau_{\mathcal{F}_{h}}\| \| H_{h} \|_{\mathcal{F}_{h}}^{2} \lesssim h^{2} \tag{4.92}
\end{equation*}
$$

Proof. We first note that we have the following Galerkin orthogonality property

$$
\begin{align*}
B\left(H-H_{h}^{l}, v^{l}\right) & =L\left(v^{l}\right)-B\left(H_{h}^{l}, v^{l}\right)  \tag{4.93}\\
& =L\left(v^{l}\right)-L_{h}(v)+B_{h}\left(H_{h}, v\right)-B\left(H_{h}^{l}, v^{l}\right)+J_{h}\left(H_{h}, v\right) \tag{4.94}
\end{align*}
$$

for all $v \in W_{h}$. Using this identity we obtain

$$
\begin{align*}
B( & \left.H-H_{h}^{l}, H-H_{h}^{l}\right)+J_{h}\left(H_{h}, H_{h}\right) \\
= & B\left(H-H_{h}^{l}, H-w^{l}\right)+B\left(H-H_{h}^{l}, w^{l}-H_{h}^{l}\right)+J_{h}\left(H_{h}, H_{h}\right)  \tag{4.95}\\
= & B\left(H-H_{h}^{l}, H-w^{l}\right)  \tag{4.96}\\
& +L\left(w^{l}-H_{h}^{l}\right)-L_{h}\left(w-H_{h}\right) \\
& +B_{h}\left(H_{h}, w-H_{h}\right)-B\left(H_{h}^{l}, w^{l}-H_{h}^{l}\right) \\
& +J_{h}\left(H_{h}, w-H_{h}\right)+J_{h}\left(H_{h}, H_{h}\right) \\
= & B\left(H-H_{h}^{l}, H-w^{l}\right)  \tag{4.97}\\
& +\left(L\left(w^{l}-H\right)-L_{h}\left(w-H^{e}\right)\right)+\left(L\left(H-H_{h}^{l}\right)-L_{h}\left(H^{e}-H_{h}\right)\right) \\
& +\left(B_{h}\left(H_{h}, w-H_{h}\right)-B\left(H_{h}^{l}, w^{l}-H_{h}^{l}\right)\right)+J_{h}\left(H_{h}, w\right) \\
=I & +I I+I I I+I V+V \tag{4.98}
\end{align*}
$$

for all $w \in W_{h}$. We choose $w=\pi_{h} H$ and proceed with estimates of terms $I-V$.
Term I. Using Cauchy-Schwarz followed by the interpolation error estimate 4.76), with $k=1$ and $m=0$, we obtain

$$
\begin{align*}
|I| & =\left|B\left(H-H_{h}^{l}, H-w^{l}\right)\right|  \tag{4.99}\\
& \leq\left\|H-H_{h}^{l}\right\|_{\Sigma}\left\|H-w^{l}\right\|_{\Sigma}  \tag{4.100}\\
& \lesssim \delta^{-1}\left\|H-w^{l}\right\|_{\Sigma}^{2}+\delta\left\|H-H_{h}^{l}\right\|_{\Sigma}^{2}  \tag{4.101}\\
& \lesssim \delta^{-1} h^{2}+\delta\left\|H-H_{h}^{l}\right\|_{\Sigma}^{2} \tag{4.102}
\end{align*}
$$

for any $\delta>0$.

Term II. Changing domain of integration from $\Sigma_{h}$ to $\Sigma$ and using Cauchy-Schwarz we obtain

$$
\begin{align*}
I I & =L\left(w^{l}-H\right)-L_{h}\left(w-H^{e}\right)  \tag{4.103}\\
& =\left(\nabla_{\Sigma} x_{\Sigma}, \nabla_{\Sigma}\left(w^{l}-H\right)\right)_{\Sigma}-\left(\nabla_{\Sigma_{h}} x_{\Sigma_{h}}, \nabla_{\Sigma_{h}}\left(w-H^{e}\right)\right)_{\Sigma_{h}}  \tag{4.104}\\
& =\left(\nabla_{\Sigma} x_{\Sigma}, \nabla_{\Sigma}\left(w^{l}-H\right)\right)_{\Sigma}-\left(|B|^{-1} B^{T} \nabla_{\Sigma} x_{\Sigma_{h}}^{l}, B^{T} \nabla_{\Sigma}\left(w^{l}-H\right)\right)_{\Sigma}  \tag{4.105}\\
& =\left(\nabla_{\Sigma} x_{\Sigma}-|B|^{-1} B B^{T} \nabla_{\Sigma} x_{\Sigma_{h}}^{l}, \nabla_{\Sigma}\left(w^{l}-H\right)\right)_{\Sigma}  \tag{4.106}\\
& \leq\left\|\nabla_{\Sigma} x_{\Sigma}-|B|^{-1} B B^{T} \nabla_{\Sigma} x_{\Sigma_{h}}^{l}\right\|_{\Sigma}\left\|\nabla_{\Sigma}\left(w^{l}-H\right)\right\|_{\Sigma}  \tag{4.107}\\
& =I I_{1} \times I I_{2} \tag{4.108}
\end{align*}
$$

Term $I I_{1}$. Adding and subtracting $x_{\Sigma_{h}}^{l}$, using the triangle inequality, and the equivalence of norms (4.17), we obtain

$$
\begin{align*}
I I_{1} & =\left\|\nabla_{\Sigma} x_{\Sigma}-|B|^{-1} B B^{T} \nabla_{\Sigma} x_{\Sigma_{h}}^{l}\right\|_{\Sigma} \\
& \leq\left\|\nabla_{\Sigma}\left(x_{\Sigma}-x_{\Sigma_{h}}^{l}\right)\right\|_{\Sigma}+\left\|\left(P_{\Sigma}-|B|^{-1} B B^{T}\right) \nabla_{\Sigma} x_{\Sigma_{h}}^{l}\right\|_{\Sigma}  \tag{4.109}\\
& \lesssim\left\|\nabla_{\Sigma_{h}}\left(x_{\Sigma}^{e}-x_{\Sigma_{h}}\right)\right\|_{\Sigma_{h}}+\left\|\left(P_{\Sigma}-|B|^{-1} B B^{T}\right)\right\|_{L^{\infty}(\Sigma)}\left\|\nabla_{\Sigma} x_{\Sigma_{h}}^{l}\right\|_{\Sigma}  \tag{4.110}\\
& \lesssim h \tag{4.111}
\end{align*}
$$

Here we used the estimate

$$
\begin{align*}
& \left\|P_{\Sigma}-|B|^{-1} B B^{T}\right\|_{L^{\infty}(\Sigma)}=\left\||B|^{-1}\left(|B| P_{\Sigma}-B B^{T}\right)\right\|_{L^{\infty}(\Sigma)} \\
& \quad \leq\left\||B|^{-1}(1-|B|)\right\|_{L^{\infty}(\Sigma)}+\left\||B|^{-1}\right\|_{L^{\infty}(\Sigma)}\left\|\left(P_{\Sigma}-B B^{T}\right)\right\|_{L^{\infty}(\Sigma)} \lesssim h^{2} \tag{4.112}
\end{align*}
$$

which follows from the bounds (4.14) and (4.15) for $B$ and its determinant. We also used the estimate

$$
\begin{equation*}
\left\|\nabla_{\Sigma} x_{\Sigma_{h}}^{l}\right\|_{\Sigma} \lesssim\left\|\nabla_{\Sigma_{h}}\left(x_{\Sigma_{h}}-x_{\Sigma}^{e}\right)\right\|_{\Sigma_{h}}+\left\|\nabla_{\Sigma} x_{\Sigma}\right\|_{\Sigma} \lesssim h+1 \lesssim 1 \tag{4.113}
\end{equation*}
$$

where we used (4.17) and the first term was estimated using Lemma 4.1.
Term $I I_{2}$. Using the interpolation error estimate 4.76), with $w^{l}=\pi_{h}^{l} H$, we obtain

$$
\begin{equation*}
I I_{2} \lesssim h \tag{4.114}
\end{equation*}
$$

Combining the estimates of $I I_{1}$ and $I I_{2}$ we conclude that

$$
\begin{equation*}
I I \lesssim h^{2} \tag{4.115}
\end{equation*}
$$

Term III. Adding and subtracting a suitable term yields

$$
\begin{align*}
I I I= & L\left(H-H_{h}^{l}\right)-L_{h}\left(H^{e}-H_{h}\right)  \tag{4.116}\\
= & \left(\left(\nabla_{\Sigma} x_{\Sigma}, \nabla_{\Sigma}\left(H-H_{h}^{l}\right)\right)_{\Sigma}-\left(\nabla_{\Sigma_{h}} x_{\Sigma}^{e}, \nabla_{\Sigma_{h}}\left(H^{e}-H_{h}\right)\right)_{\Sigma_{h}}\right)  \tag{4.117}\\
& \quad+\left(\nabla_{\Sigma_{h}}\left(x_{\Sigma}^{e}-x_{\Sigma_{h}}\right), \nabla_{\Sigma_{h}}\left(H^{e}-H_{h}\right)\right)_{\Sigma_{h}} \\
= & I I I_{1}+I I I_{2} \tag{4.118}
\end{align*}
$$

We proceed with estimates of the terms $I I I_{1}$ and $I I I_{2}$.
Term $I I I_{1}$. Changing domain of integration from $\Sigma_{h}$ to $\Sigma$ in the second term and using the bound (4.112) we get

$$
\begin{align*}
I I I_{1} & =\left(\nabla_{\Sigma} x_{\Sigma}, \nabla_{\Sigma}\left(H-H_{h}^{l}\right)\right)_{\Sigma}-\left(|B|^{-1}\left(\nabla_{\Sigma_{h}} x_{\Sigma}^{e}\right)^{l},\left(\nabla_{\Sigma_{h}}\left(H^{e}-H_{h}\right)\right)^{l}\right)_{\Sigma}  \tag{4.119}\\
& =\left(\left(P_{\Sigma}-|B|^{-1} B B^{T}\right) \nabla_{\Sigma} x_{\Sigma}, \nabla_{\Sigma}\left(H-H_{h}^{l}\right)\right)_{\Sigma}  \tag{4.120}\\
& \lesssim\left\|P_{\Sigma}-|B|^{-1} B B^{T}\right\|_{L^{\infty}(\Sigma)}\left\|\nabla_{\Sigma} x_{\Sigma}\right\|_{\Sigma}\left\|\nabla_{\Sigma}\left(H-H_{h}^{l}\right)\right\|_{\Sigma}  \tag{4.121}\\
& \lesssim h^{2}\left\|\nabla_{\Sigma}\left(H-H_{h}^{l}\right)\right\|_{\Sigma} \tag{4.122}
\end{align*}
$$

Next continuing with the estimate, we add and subtract an interpolant and use the interpolation error estimate 4.76) and the inverse inequality in Lemma 4.4 as follows

$$
\begin{align*}
& h^{2}\left\|\nabla_{\Sigma}\left(H-H_{h}^{l}\right)\right\|_{\Sigma} \\
& \quad \lesssim h^{2}\left\|\nabla_{\Sigma}\left(H-\pi_{h}^{l} H\right)\right\|_{\Sigma}+h^{2}\left\|\nabla_{\Sigma}\left(\pi_{h}^{l} H-H_{h}^{l}\right)\right\|_{\Sigma}  \tag{4.123}\\
& \quad \lesssim h^{3}+\delta^{-1} h^{2}+\delta h^{2}\left\|\nabla_{\Sigma}\left(\pi_{h}^{l} H-H_{h}^{l}\right)\right\|_{\Sigma}^{2}  \tag{4.124}\\
& \quad \lesssim h^{3}+\delta^{-1} h^{2}+\delta\left(\left\|\pi_{h}^{l} H-H_{h}^{l}\right\|_{\Sigma}^{2}+\left\|\pi_{h} H^{e}-H_{h}\right\|_{\mathcal{F}_{h}}^{2}\right)  \tag{4.125}\\
& \quad \lesssim h^{3}+\delta^{-1} h^{2}  \tag{4.126}\\
& \quad \quad+\delta\left(\left\|H-H_{h}^{l}\right\|_{\Sigma}^{2}+\left\|H-\pi_{h}^{l} H\right\|_{\Sigma}^{2}+\| \| \pi_{h} H^{e}-H^{e}\left\|_{\mathcal{F}_{h}}^{2}+\right\| H_{h} \|_{\mathcal{F}_{h}}^{2}\right) \\
& \quad \lesssim h^{3}+\delta^{-1} h^{2}+\delta h^{2}+\delta\left(\left\|H-H_{h}^{l}\right\|_{\Sigma}^{2}+\left\|H_{h}\right\|_{\mathcal{F}_{h}}^{2}\right) \tag{4.127}
\end{align*}
$$

where we used the interpolation error estimates (4.76) and (4.78).
Term $\mathrm{III}_{2}$. Element wise partial integration gives

$$
\begin{align*}
I I I_{2}= & \sum_{K \in \mathcal{K}_{h}}\left(\nabla_{\Sigma_{h}}\left(x_{\Sigma}^{e}-x_{\Sigma_{h}}\right), \nabla_{\Sigma_{h}}\left(H^{e}-H_{h}\right)\right)_{K}  \tag{4.128}\\
= & -\sum_{K \in \mathcal{K}_{h}}\left(x_{\Sigma}^{e}-x_{\Sigma_{h}}, \Delta_{K}\left(H^{e}-H_{h}\right)\right)_{K}  \tag{4.129}\\
& +\sum_{E \in \mathcal{E}_{h}}\left(x_{\Sigma}^{e}-x_{\Sigma_{h}},\left[t_{E} \cdot \nabla_{\Sigma_{h}}\left(H^{e}-H_{h}\right)\right]\right)_{E} \\
\lesssim & \sum_{K \in \mathcal{K}_{h}}\left\|x_{\Sigma}^{e}-x_{\Sigma_{h}}\right\|_{K}\left\|\Delta_{K} H^{e}\right\|_{K}  \tag{4.130}\\
& +\delta^{-1}\left(\sum_{E \in \mathcal{E}_{h}} h^{-1}\left\|x_{\Sigma}^{e}-x_{\Sigma_{h}}\right\|_{E}^{2}\right)+\delta\| \| H^{e}-H_{h}\| \|_{\mathcal{E}_{h}}^{2}
\end{align*}
$$

where $\Delta_{K} v=\left.\left(\nabla_{\Sigma_{h}} \cdot \nabla_{\Sigma_{h}} v\right)\right|_{K}$ is the tangent Laplacian on the flat element $K \in \mathcal{K}_{h}$ and therefore $\Delta_{K} H_{h}=0$ since $H_{h}$ is linear on $K$. The first term on the right hand side of (4.130) is estimated using Lemma 4.1 as follows

$$
\begin{equation*}
\sum_{K \in \mathcal{K}_{h}}\left\|x_{\Sigma}^{e}-x_{\Sigma_{h}}\right\|_{K}\left\|\Delta_{K} H^{e}\right\|_{K} \lesssim\left\|x_{\Sigma}^{e}-x_{\Sigma_{h}}\right\|_{\Sigma_{h}}\left\|H^{e}\right\|_{W_{\infty}^{2}\left(U_{\delta_{0}}(\Sigma)\right)} \lesssim h^{2}\|H\|_{W_{\infty}^{2}(\Sigma)} \lesssim h^{2} \tag{4.131}
\end{equation*}
$$

The second term is estimated using Lemma 4.1 as follows

$$
\begin{equation*}
\sum_{E \in \mathcal{E}_{h}} h^{-1}\left\|x_{\Sigma}^{e}-x_{\Sigma_{h}}\right\|_{E}^{2} \lesssim \sum_{E \in \mathcal{E}_{h}}\left\|x_{\Sigma}^{e}-x_{\Sigma_{h}}\right\|_{L^{\infty}(E)}^{2} \lesssim \sum_{E \in \mathcal{E}_{h}} h^{4} \lesssim h^{2} \tag{4.132}
\end{equation*}
$$

since $\operatorname{card}\left(\mathcal{E}_{h}\right) \lesssim h^{-2}$ in both the meshed and cut case. Finally, the third term is estimated using Lemma 4.5 as follows

$$
\begin{equation*}
\left|\| H ^ { e } - H _ { h } \| \left\|_ { \mathcal { E } _ { h } } ^ { 2 } \lesssim \left|\left\|H ^ { e } \left|\left\|_{\mathcal{E}_{h}}^{2}+\left|\left\|H_{h}\right\|\left\|_{\mathcal{E}_{h}}^{2} \lesssim h^{2}+\mid\right\| H_{h} \|_{\mathcal{E}_{h}}^{2}\right.\right.\right.\right.\right.\right.\right. \tag{4.133}
\end{equation*}
$$

Thus we arrive at the bound

$$
\begin{equation*}
I I I_{2} \lesssim h^{2}+\delta^{-1} h^{2}+\delta h^{2}+\delta\| \| H_{h}\| \|_{\mathcal{E}_{h}}^{2} \tag{4.134}
\end{equation*}
$$

Combining the estimates (4.127) and 4.134) of Terms $I I I_{1}$ and $I I I_{2}$ we obtain

$$
\begin{equation*}
I I I \lesssim h^{2}+\delta^{-1} h^{2}+\delta\left(\left\|H-H_{h}^{l}\right\|_{\Sigma}^{2}+\| \| H_{h}\left\|_{\mathcal{E}_{h}}^{2}+\right\|\left\|H_{h}\right\| \|_{\mathcal{F}_{h}}^{2}\right) \tag{4.135}
\end{equation*}
$$

for any $0<\delta \lesssim 1$.
Term IV. Changing domain of integration from $\Sigma$ to $\Sigma_{h}$ we obtain

$$
\begin{align*}
|I V| & =\left|\left(H_{h}, w-H_{h}\right)_{\Sigma_{h}}-\left(H_{h}^{l}, w^{l}-H_{h}^{l}\right)_{\Sigma}\right|  \tag{4.136}\\
& =\left|\left((1-|B|) H_{h}, w-H_{h}\right)_{\Sigma_{h}}\right|  \tag{4.137}\\
& \lesssim h^{2}\left\|H_{h}\right\|_{\Sigma_{h}}\left\|w-H_{h}\right\|_{\Sigma_{h}}  \tag{4.138}\\
& \lesssim h^{2}\left\|H_{h}\right\|_{\Sigma_{h}}\left(\|w\|_{\Sigma_{h}}+\left\|H_{h}\right\|_{\Sigma_{h}}\right)  \tag{4.139}\\
& \lesssim h^{2} \tag{4.140}
\end{align*}
$$

where at last we used the following $L^{\infty}$ stability of the interpolation operator

$$
\begin{equation*}
\|w\|_{\Sigma_{h}}=\left\|\pi_{h} H^{e}\right\|_{\Sigma_{h}} \leq\left\|\pi_{h} H^{e}\right\|_{L^{\infty}\left(\Sigma_{h}\right)} \leq\left\|H^{e}\right\|_{L^{\infty}\left(U_{\delta_{0}}(\Sigma)\right)} \lesssim\|H\|_{L^{\infty}(\Sigma)} \lesssim 1 \tag{4.141}
\end{equation*}
$$

which holds since $\pi_{h}$ is a Lagrange interpolation operator and $\Sigma_{h} \subset U_{\delta_{0}}(\Sigma)$ in the meshed case and $\cup_{T \in \mathcal{T}_{h}} T \subset U_{\delta_{0}}(\Sigma)$ in the cut case, followed by the stability (4.55) of the discrete curvature vector.

Term $\boldsymbol{V}$. Adding and subtracting $H^{e}$ inside the jump we obtain

$$
\begin{align*}
&|V|=\left|\tau_{\mathcal{E}_{h}} J_{\mathcal{E}_{h}}\left(H_{h}, w\right)+\tau_{\mathcal{F}_{h}} J_{\mathcal{F}_{h}}\left(H_{h}, w\right)\right|  \tag{4.142}\\
&=\left|\tau_{\mathcal{E}_{h}} J_{\mathcal{E}_{h}}\left(H_{h}, w-H^{e}\right)+\tau_{\mathcal{E}_{h}} J_{\mathcal{E}_{h}}\left(H_{h}, H^{e}\right)+\tau_{\mathcal{F}_{h}} J_{\mathcal{F}_{h}}\left(H_{h}, w-H^{e}\right)\right|  \tag{4.143}\\
& \lesssim \delta\left(\tau_{\mathcal{E}_{h}}\left|\left\|H_{h}\left|\left\|\tilde{\mathcal{E}}_{h}+\tau_{\mathcal{F}_{h}}\right\|\right| H_{h} \mid\right\|_{\mathcal{F}_{h}}^{2}\right)\right.  \tag{4.144}\\
& \quad+\delta^{-1}\left(\tau_{\mathcal{E}_{h}}\left\|| | w-H^{e}\left|\left\|_{\mathcal{E}_{h}}^{2}+\tau_{\mathcal{E}_{h}}\right\|\right| H^{e}\left|\| \|_{\mathcal{E}_{h}}^{2}+\tau_{\mathcal{F}_{h}}\right|\right\| w-H^{e} \mid \|_{\mathcal{F}_{h}}^{2}\right) \\
& \lesssim \delta \delta\left(\tau_{\mathcal{E}_{h}}\left|\left\|H_{h}\left|\left\|_{\mathcal{E}_{h}}^{2}+\tau_{\mathcal{F}_{h}}\right\| H_{h}\right|\right\|_{\mathcal{F}_{h}}^{2}\right)+\delta^{-1} h^{2}\right. \tag{4.145}
\end{align*}
$$

where we used the fact that $J_{\mathcal{F}_{h}}\left(H_{h}, H^{e}\right)=0$, since $\left[n_{F} \cdot \nabla H^{e}\right]=0$, the interpolation error estimates 4.77) and (4.78), and Lemma 4.5 to estimate $\mid\left\|H^{e}\right\| \|_{\mathcal{E}_{h}}$.

Conclusion of the proof. Collecting the estimates (4.102), (4.115), (4.135), (4.140), and 4.145), of terms $I-V$ we obtain

$$
\begin{align*}
& \left\|H-H_{h}^{l}\right\|_{\Sigma}^{2}+\tau_{\mathcal{E}_{h}}\left\|\mid H_{h}\right\|\left\|_{\mathcal{E}_{h}}^{2}+\tau_{\mathcal{E}_{h}}\right\|\left\|H_{h}\right\|_{\mathcal{F}_{h}}^{2} \\
& \quad \lesssim h^{2}+\delta^{-1} h^{2}+\delta\left(\left\|H-H_{h}^{l}\right\|_{\Sigma}^{2}+\left(1+\tau_{\mathcal{E}_{h}}\right)\left\|H_{h}\right\|\left\|_{\mathcal{E}_{h}}^{2}+\left(1+\tau_{\mathcal{F}_{h}}\right)\right\| H_{h} \mid \|_{\mathcal{F}_{h}}^{2}\right) \tag{4.146}
\end{align*}
$$

for any $0<\delta \lesssim 1$. Since $\tau_{\mathcal{E}_{h}}>0$ and $\tau_{\mathcal{F}_{h}}>0$ we may choose $\delta$ small enough and conclude the proof using kick back argument.

Theorem 4.2 In the cut case we may take to take $\tau_{\mathcal{E}_{h}}=0$ and thus use the simplified stabilization term

$$
\begin{equation*}
J_{h}(v, v)=\tau_{\mathcal{F}_{h}} J_{\mathcal{F}_{h}}(v, v) \tag{4.147}
\end{equation*}
$$

Proof. Using Lemma 4.6 and the interpolation error estimates 4.76) and 4.77) we note that in the case of a cut surface we have the estimate

$$
\begin{align*}
\mid\left\|H_{h}\right\| \|_{\mathcal{E}_{h}}^{2} & \lesssim\left\|\left\|H_{h}-H^{e}\left|\left\|_{\mathcal{E}_{h}}^{2}+\left|\left\|H^{e} \mid\right\|_{\mathcal{E}_{h}}^{2}\right.\right.\right.\right.\right.  \tag{4.148}\\
& \lesssim\left\|\left\|H_{h}-\pi_{h} H^{e}\left|\left\|_{\mathcal{E}_{h}}^{2}+\right\|\right| \pi_{h} H^{e}-H^{e}\left|\left\|_{\mathcal{E}_{h}}^{2}+\right\|\right| H^{e} \mid\right\|_{\mathcal{E}_{h}}^{2}\right.  \tag{4.149}\\
& \lesssim\left\|H_{h}-\pi_{h} H^{e}\right\|_{\Sigma_{h}}^{2}+\left|\left\|H_{h}-\pi_{h} H^{e} \mid\right\|_{\mathcal{F}_{h}}^{2}+h^{2}\right.  \tag{4.150}\\
& \lesssim\left\|H_{h}^{l}-H\right\|_{\Sigma}^{2}+\| \| H_{h} \|_{\mathcal{F}_{h}}^{2}+h^{2} \tag{4.151}
\end{align*}
$$

where we finally used the interpolation estimates (4.76) and (4.78). In view of the final estimate (4.146) in the above proof, we conclude that in the cut case it is enough to use the simplified stabilization term

$$
\begin{equation*}
J_{h}(v, v)=\tau_{\mathcal{F}_{h}} J_{\mathcal{F}_{h}}(v, v) \tag{4.152}
\end{equation*}
$$

since the kick back term may be estimated as follows

$$
\begin{align*}
& \left\|H-H_{h}^{l}\right\|_{\Sigma}^{2}+\left(1+\tau_{\mathcal{E}_{h}}\right)\left\|H_{h}\right\|\left\|_{\mathcal{E}_{h}}+\left(1+\tau_{\mathcal{F}_{h}}\right)\right\| H_{h} \|_{\mathcal{F}_{h}} \\
& \quad \lesssim\left\|H-H_{h}^{l}\right\|_{\Sigma}^{2}+\left(1+\tau_{\mathcal{F}_{h}}\right)\left\|H_{h}\right\| \|_{\mathcal{F}_{h}} \tag{4.153}
\end{align*}
$$

## 5 Numerical Examples

### 5.1 Triangulated Surfaces

We consider a torus with Cartesian coordinates given by a map from a reference coordinate system $(\theta, \varphi)$ representing angles, $0 \leq \varphi<2 \pi, 0 \leq \theta<2 \pi$ :

$$
\begin{cases}x & =(R+r \cos \varphi) \cos \theta  \tag{5.1}\\ y & =(R+r \cos \varphi) \sin \theta \\ z & =r \sin \varphi\end{cases}
$$

where $r$ is the radius of the tube bent into a torus and $R$ is the distance from the center line of the tube to the center of the torus. The mean curvature is then given by

$$
H=-\frac{R+2 r \cos \varphi}{2 r(R+r \cos \varphi)}
$$

and we consider $R=1, r=1 / 2$, in our example.
Our numerical results show that convergence of the mean curvature vector is strongly dependent on stabilization. We compare three different meshes on the torus, one sequence of structured meshes, Figure 1, one where the diagonals are randomly flipped in the structured mesh, Figure 2, and one where the nodes have been moved randomly, creating an unstructured mesh, Figure 3 .

In Figure 4 we show the discrete convergence $\left\|\pi_{h} H-H_{h}\right\|_{\Sigma_{h}}$, where $\pi_{h} H$ is the nodal interpolant, for sequences of meshes of the type just described. The stabilization parameter was chosen as $\tau_{\mathcal{E}_{h}}=1 / 10$ and the mesh size parameter $h=N^{-1 / 2}$ where $N$ denotes the number of nodes in the mesh. We note that the structured mesh does not need stabilization whereas stability is lost even for the minor modification of flipping diagonals. In Figure 56 we show iso-plots of the solution for the structured mesh with flipped diagonals with and without stabilization. The instability of the computed curvature without added stabilization is clearly visible. We also note that the convergence rate is higher than predicted by the theory. This may expected in view of the fact that we have super convergence of second order on the structured mesh and then loss of order is dependent on the perturbations of the mesh.

### 5.2 Cut Level Set Surfaces

We consider the same example as above. A structured mesh $\mathcal{T}_{h}$, consisting of tetrahedra, on the domain $[-1.6,1.6] \times[-1.6,1.6] \times[-0.6,0.6]$ is generated independently of the position of the torus. The mesh size parameter is defined by $h=1 / N^{\frac{1}{3}}$ where $N$ denotes the total number of nodes in the mesh. The signed distance function of the torus $\Sigma$ is given by

$$
\begin{equation*}
\rho=\left(z^{2}+\left(\left(x^{2}+y^{2}\right)^{1 / 2}-R\right)^{2}\right)^{1 / 2}-r \tag{5.2}
\end{equation*}
$$

where we again choose $R=1$ and $r=1 / 2$. We construct an approximate distance function $\rho_{h}$ using the nodal interpolant $\pi_{h} \rho$ on the background mesh and let $\Sigma_{h}$ be the zero levelset of $\rho_{h}$.

We compare our approximation of the mean curvature vector with the exact mean curvature vector $H^{e}=-(\Delta \rho) \nabla \rho$. Also in this case the convergence of the mean curvature vector is strongly dependent upon stabilization. In our example the stabilization parameters were chosen as $\tau_{\mathcal{E}_{h}}=0$ and $\tau_{\mathcal{F}_{h}}=1 / 10$. Recall that for a cut surface we may take $\tau_{\mathcal{E}_{h}}=0$, see Theorem 4.2. The resulting surface mesh $\mathcal{K}_{h}$ on $\Sigma_{h}$ is shown in Figure 7 and in Figure 8 we show the error in the $L^{2}$-norm. We note that we also in this case obtain higher order convergence rate (approximately 1.3) than predicted by the theory.

## References

[1] M. Botsch, L. Kobbelt, M. Pauly, P. Alliez, and B. L'evy. Polygon Mesh Processing. A. K. Peters, Ltd., Natick, MA, 2010.
[2] M. Botsch and O. Sorkine. On linear variational surface deformation methods. IEEE Transactions on Visualization and Computer Graphics, 14(1):213-230, 2008.
[3] E. Burman, P. Hansbo, and M. G. Larson. A well conditioned cut finite element method for second order partial differential equations on surfaces. Part I: The LaplaceBeltrami operator. Technical report, arXiv:1312.1097, 2013.
[4] M. Cenanovic, P. Hansbo, and M. G. Larson. Minimal surface computation using a finite element method on an embedded surface. Technical report, arXiv:1403.3535, 2014.
[5] A. Demlow. Higher order finite element methods and pointwise error estimates for elliptic problems on surfaces. SIAM J. Numer. Anal., 47(2):805-827, 2009.
[6] M. Desbrun, M. Meyer, P. Schröder, and A. H. Barr. Implicit fairing of irregular meshes using diffusion and curvature flow. In Proceedings of the 26th Annual Conference on Computer Graphics and Interactive Techniques, SIGGRAPH '99, pages 317-324, New York, NY, USA, 1999. ACM Press/Addison-Wesley Publishing Co.
[7] G. Dziuk. An algorithm for evolutionary surfaces. Numer. Math., 58(6):603-611, 1991.
[8] G. Dziuk. Computational parametric Willmore flow. Numer. Math., 111(1):55-80, 2008.
[9] David Gilbarg and Neil S. Trudinger. Elliptic partial differential equations of second order. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
[10] S. Gross and A. Reusken. Finite element discretization error analysis of a surface tension force in two-phase incompressible flows. SIAM J. Numer. Anal., 45(4):16791700 (electronic), 2007.
[11] A. Hansbo, P. Hansbo, and M. G. Larson. A finite element method on composite grids based on Nitsche's method. M2AN Math. Model. Numer. Anal., 37(3):495-514, 2003.
[12] K. Hildebrandt, K. Polthier, and M. Wardetzky. On the convergence of metric and geometric properties of polyhedral surfaces. Geom. Dedicata, 123:89-112, 2006.
[13] M. Meyer, M. Desbrun, P. Schröder, and A. H. Barr. Discrete differential-geometry operators for triangulated 2-manifolds. In Visualization and mathematics III, Math. Vis., pages 35-57. Springer, Berlin, 2003.
[14] J. Wloka. Partial differential equations. Cambridge University Press, Cambridge, 1987. Translated from the German by C. B. Thomas and M. J. Thomas.
[15] G. Xu. Consistent approximations of several geometric differential operators and their convergence. Appl. Numer. Math., 69:1-12, 2013.


Figure 1: Structured mesh.


Figure 2: Structured mesh with flipped diagonals.


Figure 3: Unstructured mesh.


Figure 4: Convergence curves and rates of the discrete error.


Figure 5: Isolevels of the computed curvature, stabilized case.


Figure 6: Isolevels of the computed curvature, unstabilized case.


Figure 7: The induced triangulation of $\Sigma_{h}$.


Figure 8: The error in the mean curvature vector for different mesh sizes.


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