

A computer-assisted proof of Barnette-Goodey conjecture: Not only fullerene graphs are Hamiltonian.

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Abstract

Fullerene graphs, i.e., 3-connected planar cubic graphs with pentagonal and hexagonal faces, are conjectured to be Hamiltonian. This is a special case of a conjecture of Barnette and Goodey, stating that 3-connected planar graphs with faces of size at most 6 are Hamiltonian. We prove the conjecture.

1 Introduction

Tait conjectured in 1880 that cubic polyhedral graphs (i.e., 3-connected planar cubic graphs) are Hamiltonian. The first counterexample to Tait's conjecture was found by Tutte in 1946; later many others were found, see Figure 1. Had the conjecture been true, it would have implied the Four-Color Theorem.

However, each known non-Hamiltonian cubic polyhedral graph has at least one face of size 7 or more [1, 17]. It was conjectured that all cubic polyhedral graphs with maximum face size at most 6 are Hamiltonian. In the literature, the conjecture is usually attributed to Barnette (see, e.g., [13]), however, Goodey [6] stated it in an informal way as well.

This conjecture covers in particular the class of fullerene graphs, 3-connected cubic planar graphs with pentagonal and hexagonal faces only. Hamiltonicity was verified for all fullerene graphs with up to 176 vertices [1]. Later on, the conjecture in the general form was verified for all graphs with up to 316 vertices [2]. On the other hand, cubic polyhedral graphs having only faces of sizes 3 and 6 or 4 and 6 are known to be Hamiltonian [6, 7].

Jendroľ and Owens proved that the longest cycle of a fullerene graph of order n covers at least $4n/5$ vertices [9], the bound was later improved to $5n/6 - 2/3$ by Král' et al. [11] and to $6n/7 + 2/7$ by Erman et al. [5]. Marušič [14] proved that the fullerene graph obtained from another fullerene graph with an odd number of faces by the so-called leapfrog operation (truncation of the dual; replacing each vertex by a hexagonal face) is Hamiltonian. In fact, Hamiltonian cycle in the derived graph corresponds to a decomposition of the original graph into an induced forest and a stable set. We will use similar technique to prove the conjecture in the general case.

In this paper we prove

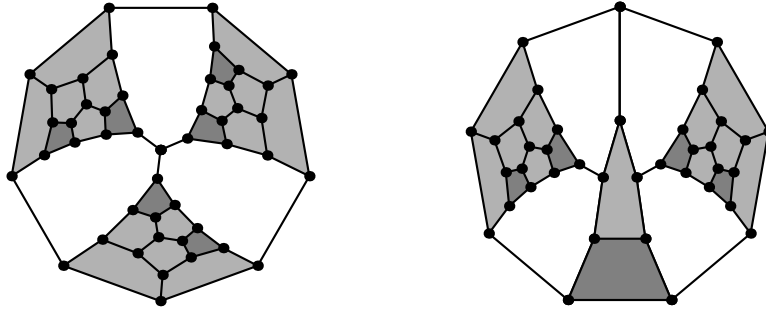


Figure 1: Tutte's first example of a non-Hamiltonian cubic polyhedral graph (left); one of minimal examples on 38 vertices (right).

Theorem 1 *Let G be a 3-connected planar cubic graph with faces of size at most 6. Then G is Hamiltonian.*

In the next sections, we reduce the main theorem to Theorem 2 and further to Theorem 4 and we introduce terminology and techniques used in the proof of Theorem 4.

2 Preliminaries

2.1 First reduction

A *Barnette graph* is a 3-connected planar cubic graph with faces of size at most 6, having no triangles and no two adjacent quadrangles.

We reduce Theorem 1 to the case of Barnette graphs:

Theorem 2 *Let G be a Barnette graph on at least 318 vertices. Then G is Hamiltonian.*

Lemma 1 *Theorem 2 implies Theorem 1.*

Proof. Suppose Theorem 2 true. Let G be a smallest counterexample to Theorem 1. We know that G has at least 318 vertices, since Theorem 1 has already been verified for all cubic planar graphs with faces of size at most 6 on at most 316 vertices [2]. (The number of vertices of a cubic graph is always even.)

Assume $f = v_1v_2v_3$ is a triangle in G . If one of the faces adjacent to f is a triangle, then, by 3-connectivity, G is (isomorphic to) K_4 , a Hamiltonian graph. Therefore, all the three faces adjacent to f are of size at least 4. Let G_1 be a graph obtained from G by replacing $v_1v_2v_3$ by a single vertex v . It is easy to see that G_1 is a 3-connected cubic planar graph with faces of size at most 6, moreover, every Hamiltonian cycle of G_1 can be extended to a Hamiltonian cycle of G , see Figure 2 for illustration.

From this point on we may assume that G contains no triangles. Let f_1 and f_2 be two adjacent faces of size 4 in G . Let v_1 and v_2 be the vertices they share; let $f_1 = v_1v_2u_3u_4$, let $f_2 = v_1v_2w_3w_4$. We denote by f_3 (resp. f_4) the face incident to u_3 and w_3 (u_4 and w_4 , respectively). If both f_3 and f_4 are quadrangles, then, by 3-connectivity, G is the graph of a

cube, which is Hamiltonian. Suppose $d(f_4) \geq 5$ and $d(f_3) = 4$. Let G_2 be a graph obtained from G by collapsing the faces f_1, f_2, f_3 to a single vertex. Again, G_2 is a 3-connected cubic planar graph with faces of size at most 6, moreover, every Hamiltonian cycle of G_2 can be extended to a Hamiltonian cycle of G , see Figure 2.

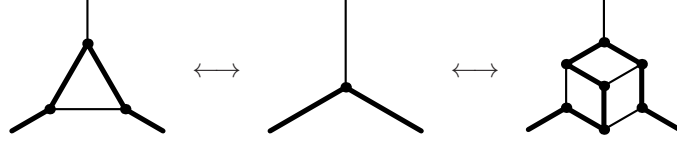


Figure 2: A triangle, as well as three quadrangles sharing a vertex, can be reduced to a single vertex.

Finally, suppose that both f_3 and f_4 are of size at most 5. We remove the vertices v_1 and v_2 , identify u_3 with w_3 and u_4 with w_4 ; in this way we obtain a graph G_3 . It can be verified that G_3 is a 3-connected cubic planar graph with all the faces of size at most 6, unless G is the 12-vertex graph obtained from the cube by replacing two adjacent vertices by triangles, which is impossible since G has no triangles. Again, every Hamiltonian cycle of G_3 can be extended to a Hamiltonian cycle of G , as seen on Figure 3. \square

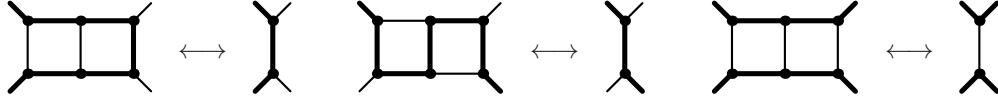


Figure 3: A pair of adjacent quadrangles can be reduced to a single edge.

2.2 Cyclic edge-connectivity of Barnette graphs

Let G be a graph. For a set of vertices X , we denote $G[X]$ the subgraph of G induced by X . For a set of vertices X , $\emptyset \neq X \neq V(G)$, the set of edges of G having exactly one end-vertex in X form a cut-set of G . An edge-cut (X, Y) , where $Y = V(G) \setminus X$, is *cyclic* if both $G[X]$ and $G[Y]$ contain a cycle. Finally, a graph is cyclically k -edge-connected if it has no cyclic edge-cuts of size smaller than k .

Lemma 2 *Let G be a Barnette graph. Then G is cyclically 4-edge-connected.*

Proof. Suppose that G contains a cyclic 3-edge-cut (X, Y) . Choose X inclusion-wise minimal. It is easy to see that the cut-edges are pairwise non-adjacent. Let x_1, x_2, x_3 be the vertices of X incident to the cut-edges. We prove that they are pairwise non-adjacent: Suppose that two of them, say x_1 and x_2 , are adjacent. Then, by minimality of X , $X' = X \setminus \{x_1, x_2\}$ is acyclic with $(X', V(G) \setminus X')$ being a 3-edge-cut, and hence, $|X'| = 1$, $X' = \{x_3\}$, so thus $G[X]$ is a triangle, which is impossible in a Barnette graph.

Let y_i be the other endvertex of the cut-edge incident to x_i , $i = 1, 2, 3$. We prove that these three vertices are also pairwise non-adjacent: Since G has no triangles, $G[\{y_1, y_2, y_3\}]$ has at most two edges. If it had exactly two edges, then G would contain a 2-edge-cut, which

is impossible. Suppose now that y_1 and y_2 are adjacent, but y_3 is not adjacent to any of them. Each of the two faces incident to the edge x_3y_3 has at least three incident vertices in both X and Y , therefore, it is a hexagon, and there are exactly three incident vertices in both X and Y . Let z_i be the common neighbor of y_3 and y_i , $i = 1, 2$. Then z_1 and z_2 are adjacent, otherwise there would be a 2-edge-cut in G . But then $y_3z_1w_2$ is a triangle in G , a contradiction.

As y_1, y_2, y_3 are pairwise non-adjacent, for each face incident to any cut-edge, there are at least three incident vertices in both X and Y , therefore, each such face is a hexagon having three incident vertices in both X and Y . Let x_{ij} be the common neighbor of x_i and x_j , $1 \leq i < j \leq 3$. By minimality of X , $X' = X \setminus \{x_1, x_2, x_3, x_{12}, x_{13}, x_{23}\}$ is a single vertex, and so $G[X]$ is the union of three 4-faces pairwise adjacent to each other, which is impossible in a Barnette graph. \square

2.3 Goldberg vectors, Coxeter coordinates, and nanotubes

Let f_1 and f_2 be two faces of an infinite hexagonal grid H . Then there is a (unique) translation ϕ of H that maps f_1 to f_2 . The vector \vec{u} defining ϕ can be expressed as an integer combination of two *unit vectors* – those that define translations mapping a hexagon to an adjacent one. Out of the six possible unit vectors, we choose a pair \vec{u}_1, \vec{u}_2 making a 60° angle such that f_2 is inside this angle starting from f_1 . Then the coordinates (c_1, c_2) of $\vec{u} = c_1\vec{u}_1 + c_2\vec{u}_2$ are non-negative integers, called the *Coxeter coordinates* of ϕ [3].

We may always assume that $c_1 \geq c_2$. The pair (c_1, c_2) determines the mutual position of a pair of hexagons in a hexagonal grid, it is also called a *Goldberg vector*. Observe that, for example, $(1, 0)$ corresponds to a pair of adjacent faces, $(1, 1)$ corresponds to a pair of non-adjacent faces with an edge connecting them (and thus having two distinct common neighboring faces), whereas $(2, 0)$ corresponds to a pair of non-adjacent faces with two paths of length 2 connecting them (and thus sharing a single common neighboring face), etc.

The Coxeter coordinates are used to define nanotubical graphs in the following way:

Let (c_1, c_2) be a pair of integers with $c_1 \geq c_2$. Fix a pair of unit vectors \vec{u}_1 and \vec{u}_2 making a 60° angle. A graph obtained from an infinite hexagonal grid by identifying objects (vertices, edges, and faces) whose mutual position is (an integer multiple of) the vector $c_1\vec{u}_1 + c_2\vec{u}_2$ is the *infinite nanotube* of type (c_1, c_2) .

If $c_1 + c_2 \leq 2$ then the infinite nanotube is not 3-connected. Since nanotubes with $c_1 + c_2 = 3$ contain cyclic 3-edge-cuts and Barnette graphs are cyclically 4-edge-connected, we will only be interested in nanotubes with $c_1 + c_2 \geq 4$.

Let N be an infinite nanotube of type (c_1, c_2) . Let f_1 and f_2 be two hexagons of the hexagonal grid H at mutual position (c_1, c_2) corresponding to the same hexagon f of N . Let P be a dual path of length $c_1 + c_2$ connecting the vertices f_1^* and f_2^* in H^* . Then the edges corresponding to the edges of P form a cyclic edge-cut in H of cardinality $c_1 + c_2$. A cyclic sequence of hexagonal faces of N corresponding to the vertices of P is called a *ring* in N .

A finite 2-connected subgraph of an infinite nanotube is an *open-ended nanotube* if it contains at least one ring. A Barnette graph is a *nanotube* if it contains an open-ended nanotube of some type as a subgraph. Observe that the same graph may be considered as a nanotube of more than one type.

Let G be a nanotube. We call a *cap* any of the two inclusion-wise minimal 2-connected subgraphs of G that can be obtained as a component of a cyclic edge-cut defined by a set of

edges intersecting a line perpendicular to the vector defining the corresponding open-ended nanotube. See Figures 4, 5, and 7 for illustration.

Lemma 3 *Let G be a Barnette graph which is a nanotube of type (p_1, p_2) with $p_1 + p_2 = 4$. Then $(p_1, p_2) = (4, 0)$.*

We omit the details of the proof, as it is similar to the proof of Lemma 2: It suffices to prove that every (potential) cap of a nanotube of type $(3, 1)$ or $(2, 2)$ contains a triangle or a pair of adjacent quadrangles.

Lemma 4 *Let G be a Barnette graph which is a nanotube of type (p_1, p_2) with $(p_1, p_2) \in \{(4, 0), (5, 0), (4, 1), (5, 1), (3, 2), (4, 2), (3, 3), (4, 3)\}$. Then G is Hamiltonian.*

Proof. We may suppose that G has at least 318 vertices (at least 161 faces). Since the caps of the tube are of bounded size (at most 5, 10, 6, 11, 5, 10, 10, 14 faces, respectively, each), the tubical part of G contains a large number of disjoint rings.

We provide a construction of a Hamilton cycle in such graphs: First, we find a pair of paths covering the vertices of the tubical part of G ; then, we verify that for each possible cap it is always possible to connect the two paths in a way that all the vertices of the cap are covered as well.

In a nanotube of type $(p, 0)$, $p \geq 4$, for each p -edge-cut corresponding to a ring, we construct the two paths transversing the tube in a way that each path contains one cut-edge incident to the same hexagonal face. Let us call this hexagon a *transition face*. For two consecutive rings, the transition faces are adjacent and once the transition face is fixed for one ring, we are free to choose any of the two adjacent hexagons in the next one to be the transition face, see Figure 4 for illustration.

To complete the proof for $(4, 0)$ - and for $(5, 0)$ -nanotubes, it suffices to verify that for every possible cap, there exists a path covering all the vertices of the cap leaving the cap by two edges adjacent to the same hexagonal face of the first ring of the tube. Since the tubical part of G is sufficiently long, we can choose a transition face in the first and the last ring of hexagons regardless of the relative position of the two caps.

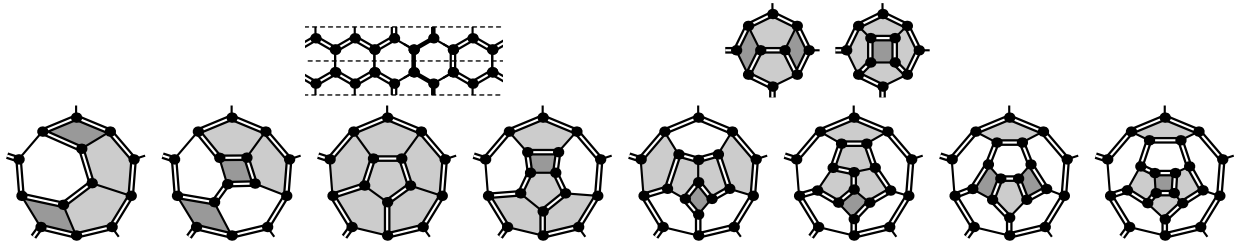


Figure 4: Two ways to cover the $2p$ vertices separated by two consecutive cyclic p -edge-cuts in a $(p, 0)$ -nanotube by two paths (top left for $p = 5$). A path joining two consecutive pending edges covering all the vertices, for every possible cap of $(p, 0)$ -nanotubes for $p = 4$ (top right line) and for $p = 5$ (bottom line).

For nanotubes of type $(3, 3)$, the construction is described in Figure 5.

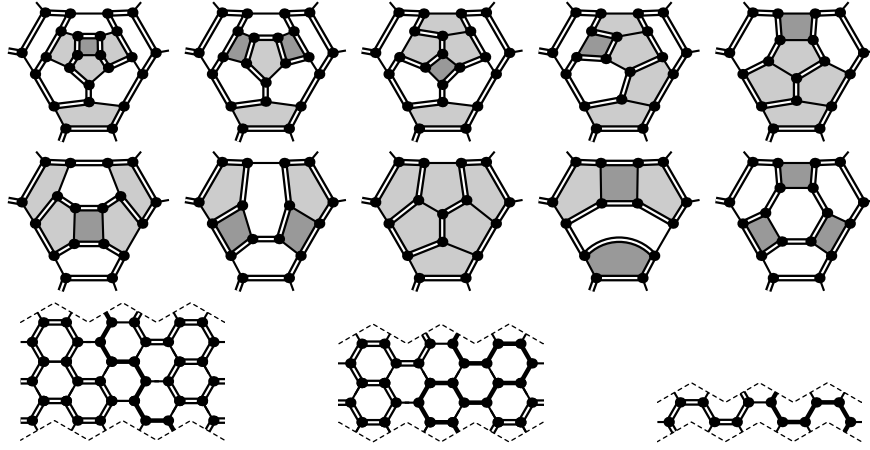


Figure 5: For each possible cap of a $(3, 3)$ -nanotube, a path leaving the cap by a prescribed pair of edges is given (first two rows). For the last cap, we added three hexagons of the tube to make the construction work. To connect the two caps and to cover the tubical part of the graph, it suffices to combine an appropriate number of the first two patterns of the last row (and/or their mirror images) and finish by the third one.

For nanotubes of type (p_1, p_2) with $p_1 > p_2 > 0$, we provide a repetitive pattern to cover the tubical part (see Figure 6) and, for every cap and for every position of the cap with respect to the pattern, a path covering the vertices of the cap (see Figure 7 for the first two types of nanotubes; we omit the details for the remaining three types). \square

2.4 Second reduction

Let H be a plane cubic graph. We denote H^\parallel the 6-regular multigraph obtained from H by replacing each edge by a pair of parallel edges, equipped with the following black-and-white face-coloring: We color the 2-gons between pairs of parallel edges white and we color the faces of H^\parallel corresponding to the faces of H black. It is easy to see that this is a proper face-coloring of H^\parallel .

Let G be a Barnette graph and let M be a perfect matching of G . Then $F = E(G) \setminus M$ is a 2-factor of G . A hexagonal face of G incident to three edges of M is called *resonant*.

There is a canonical face-coloring of G with two colors, say black and white, such that each edge of F is incident to one black and one white face. Let h be a white resonant hexagon. Since it is incident to three edges from M , the colors of its neighboring faces are alternating black and white.

We transform F into a 6-regular plane pseudograph in the following way: First, inside each white resonant hexagon h we introduce a new vertex v_h . We remove the three edges incident to h from F and we replace them by six new edges, joining v_h to all the six vertices incident to h . Each of the newly created triangles receives the color of the corresponding face adjacent to h . This way we obtain a black-and-white face-colored plane graph with two types of vertices: vertices of degree 2 are the vertices of the underlying Barnette graph, vertices of

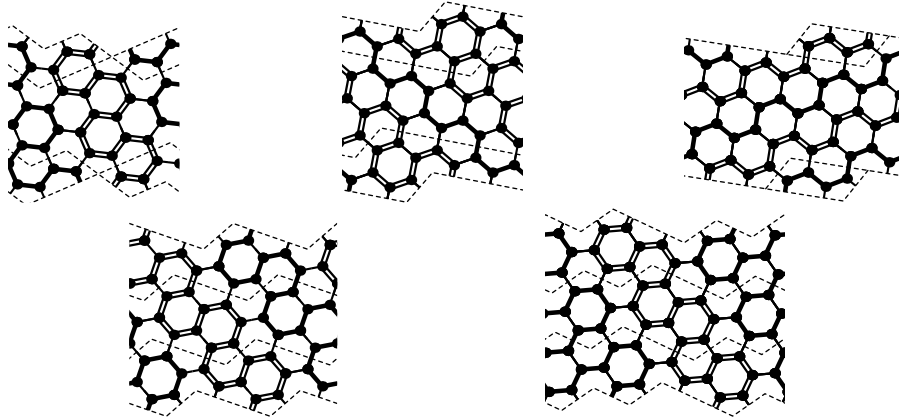


Figure 6: Two paths covering all the vertices of a (potentially infinite) open-ended nanotube of type $(3, 2)$, $(4, 1)$, $(5, 1)$, $(4, 2)$, and $(4, 3)$, respectively. For each end of the tube, the two dashed lines separate the smallest period of the covering.

degree 6 correspond to white resonant hexagons.

Finally, we suppress all vertices of degree 2. This operation may create loops, parallel edges, and even circular edges incident to no vertex, see Figure 8 for illustration. Let G^M be the resulting black-and-white face-colored plane 6-regular pseudograph.

A 2-factor F is called *odd* if it consists of an even number of (disjoint) cycles; otherwise it is *even*. The same applies to the corresponding perfect matching.

A 2-factor F (as well as the corresponding perfect matching $M = E(G) \setminus F$) is called *simple* if G^M has no circular edges and $G^M \cong H^\parallel$ for some cubic planar graph H . If this is the case, H is called the *residual graph*.

Lemma 5 *Let F be a simple 2-factor of a Barnette graph G . Let n be the number of vertices of the corresponding residual graph H . If F is odd, then $n = 4k + 2$ for some $k \geq 1$; otherwise $n = 4k$ for some $k \geq 1$.*

Proof. The number of vertices of a residual graph is always even, since it is a cubic graph. Moreover, the number of cycles in F , say c , is equal to the number of faces of the residual graph. By Euler's formula,

$$c = 2 + |E(H)| - |V(H)| = 2 + \frac{3n}{2} - n = \frac{n+4}{2},$$

so the claim follows immediately.

We will make use of the following classical result:

Theorem 3 (Payan and Sakarovitch [16]) *Let H be a cubic graph on $n = 4k + 2$ vertices ($k \geq 1$). If H is cyclically 4-edge-connected, then $V(H)$ admits a partition into two sets, say B and W , such that $H[B]$ is a stable set and $H[W]$ is a tree.*

Observe (by double-counting white-white and black-white edges) that the divisibility condition is a necessary condition for such a partition to exist. That's why we will only be interested in odd 2-factors.

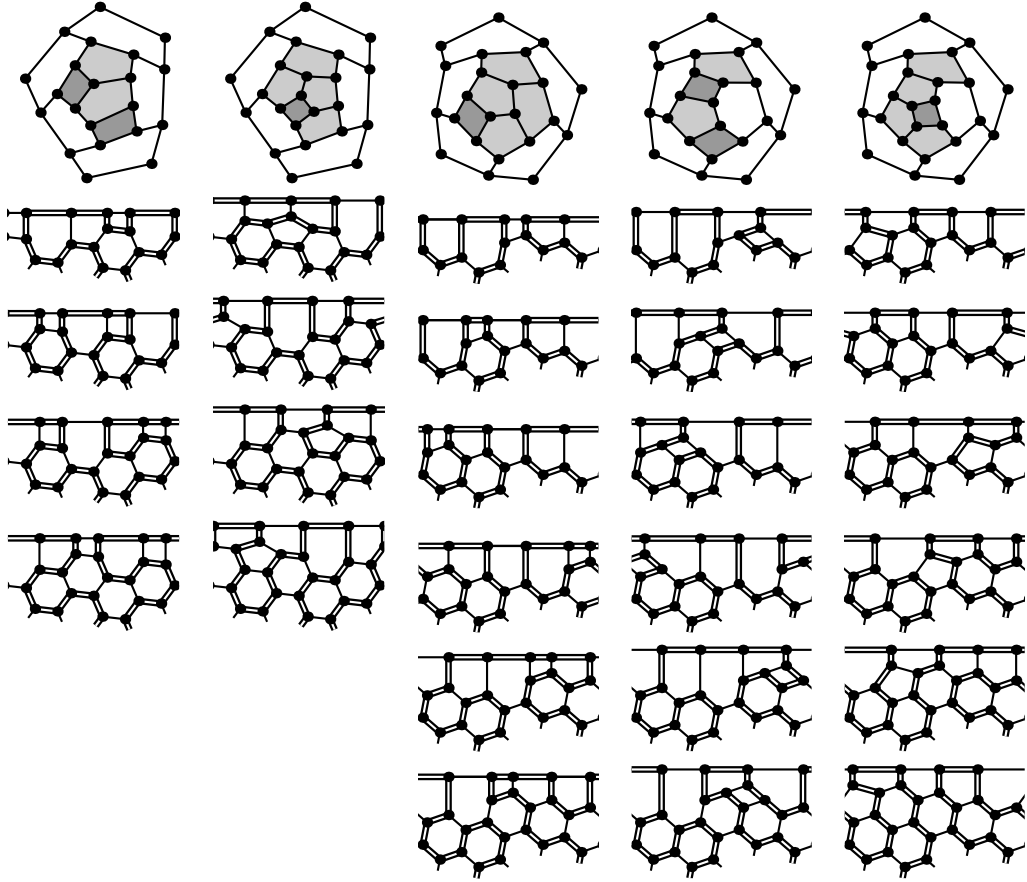


Figure 7: For every cap of a nanotube of types $(3, 2)$ (first two columns) and $(4, 1)$ (the rest), and for every position of the cap relative to the two paths covering the tubical part of the graph, a completion of the Hamilton cycle in the cap is given. In the first row, the caps are drawn together with the first ring of the tube.

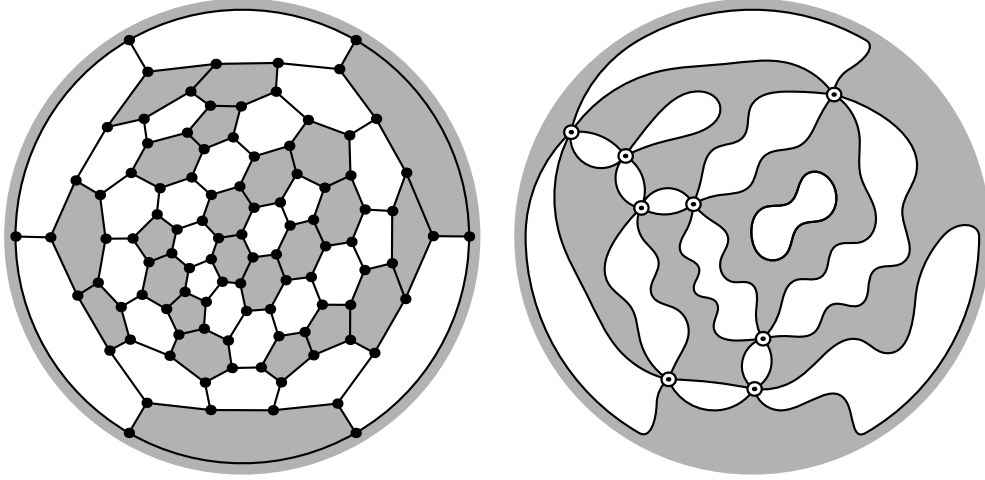


Figure 8: An example of a black-and-white face-colored 6-regular pseudograph (right) corresponding to a 2-factor of a Barnette graph (left).

Lemma 6 *Let G be a Barnette graph and let M be an odd simple perfect matching of G . If the residual graph is a cyclically 4-edge-connected, then G is Hamiltonian.*

Proof. Let H be a cyclically 4-edge-connected cubic planar graph on $4k + 2$ vertices ($k \geq 1$) such that $G^M = H^{\parallel}$. Let $F = E(G) \setminus M$. Recall that vertices of H correspond to white resonant hexagons in G with respect to a fixed canonical face-coloring of F . Let (B, W) be a partition of $V(H)$ into an induced (black) stable set B and an induced (white) tree W given by Theorem 3.

We transform the 2-factor F and the black-and-white face-coloring of G in the following way: For each resonant hexagon h corresponding to a black vertex b of H , replace the three edges from F incident to h in G by the other three edges; recolor the hexagon h black. Since B induces a stable set in H , this operation can be carried out independently for all black vertices of H at once. For each such vertex, the number of edges from F incident to any vertex of G remains unchanged, therefore, F becomes a 2-factor of G , say F' .

We claim that it consists of a single cycle. To prove that, it suffices to observe that the graph $(V(G), F')$ has a single white face (as $H[W]$ is connected) and a single black face (as $H[B]$ is acyclic). See Figure 9 for illustration. \square

It remains to prove that such a situation occurs for at least one perfect matching for any Barnette graph not known to be Hamiltonian yet.

Theorem 4 *Let G be a Barnette graph on at least 318 vertices. Then there exists an odd simple perfect matching M of G such that the residual graph H is cyclically 4-edge-connected, unless G is a nanotube of type $(4, 0)$, $(5, 0)$, $(4, 1)$, $(5, 1)$, $(3, 2)$, $(4, 2)$, $(3, 3)$, or $(4, 3)$.*

In the rest of the paper, we prove Theorem 4. We describe the general approach in Section 3, and we specify the computer-assisted part in Section 4.

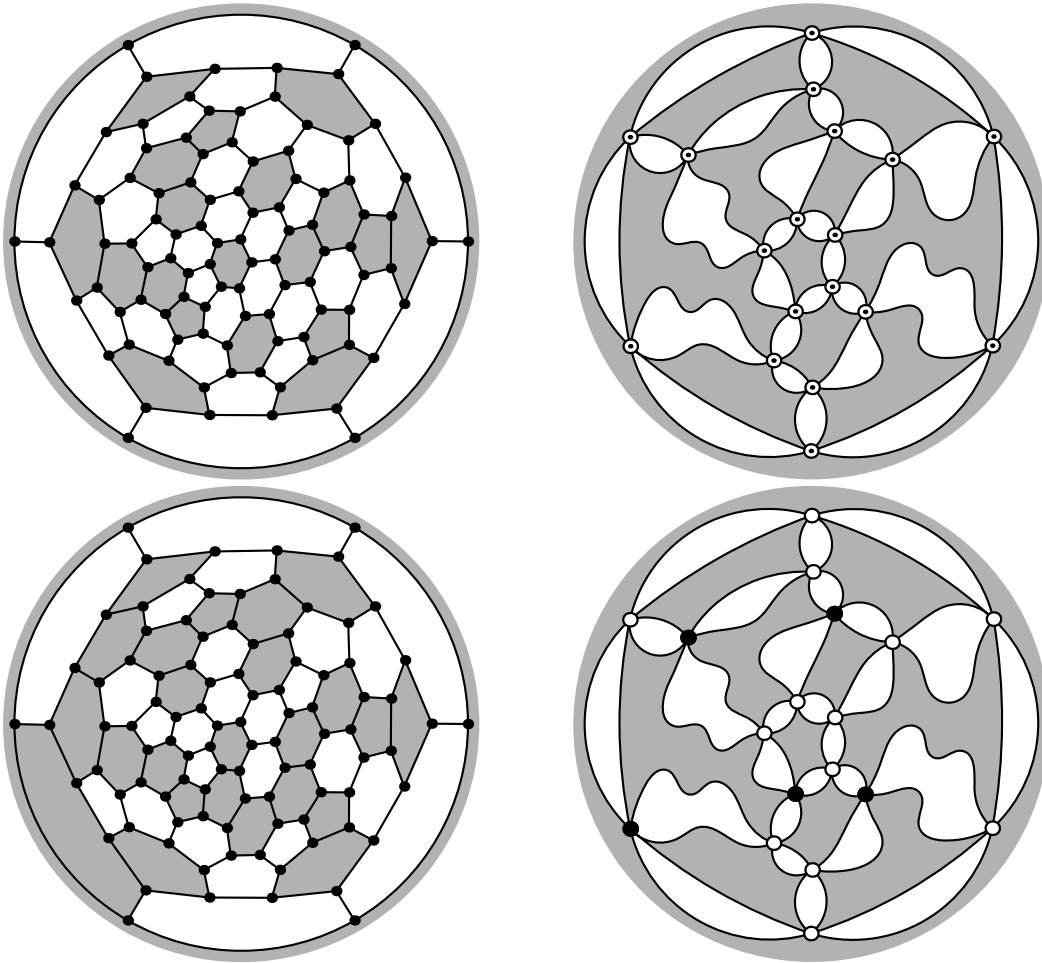


Figure 9: Clockwise, starting from upper left: An example of a simple 2-factor F of a Barnette graph G ; the corresponding planar 6-regular pseudograph G^M which is a double of a cyclically 4-edge connected cubic graph H ; A decomposition of H into a (black) stable set and a (white) induced tree; the corresponding Hamilton cycle in G .

We claim (without proof) that in order to prove Theorem 4 it suffices to consider a simple odd 2-factor maximizing the number of white resonant hexagons.

2.5 Generalized 2-factors

We will call a 2*-factor of a Barnette graph G any spanning subgraph F of G such that each component of F is a connected regular graph of degree 1 or 2 – an isolated edge or a cycle. For a 2*-factor F of a Barnette graph G , let $F^{(0)}$ be the set of isolated edges of F ; let $G^{(2)}$ be a plane graph obtained from G by replacing each edge of $F^{(0)}$ by a 2-gon; let $F^{(2)}$ be the set of edges of $G^{(2)}$ corresponding to those from F . Then $F^{(2)}$ is a 2-factor of $G^{(2)}$ in the common (strict) sense.

Given a 2*-factor F of a Barnette graph G , there are two canonical black-and-white face-colorings of $G^{(2)}$ (complementary to each other) with the following property: an edge e of $G^{(2)}$ is incident to a white and a black face if and only if e belongs to $F^{(2)}$ (otherwise e is incident to two faces of the same color).

A 2*-factor F of a Barnette graph G is called *quite good* if for each of the two canonical black-and-white face-colorings of $G^{(2)}$ induced by $F^{(2)}$ the 2-gons corresponding to the edges of $F^{(0)}$ have all the same color. Given a quite good 2*-factor of a Barnette graph G , we will always assume that a canonical coloring of $G^{(2)}$ such that all the 2-gons of $G^{(2)}$ are black is given along.

A quite good 2*-factor F of a Barnette graph G is called *good* if, after having fixed a planar embedding of G such that the outer face is a white one, no cycle of F is inside another.

Observe that given a good 2*-factor F of G , for any planar embedding of G with a white outer face, the set of faces inside a fixed cycle C of F is always the same and these faces correspond to a sub-tree of the dual graph G^* (empty if C is a 2-cycle).

Lemma 7 *Let F be a good 2*-factor of a Barnette graph G . Let f be the number of all the faces of G , let q_k be the number of non-resonant white faces of size k in G ($k = 4, 5, 6$); let c be the number of components of F . Then q_5 is even, moreover, $f + q_4 + q_5/2 + c \equiv 0 \pmod{2}$.*

Proof. Let n be the number of vertices of G , let f_k be the number of all faces of size k in G , let x_k be the number of black faces of size k in G . Euler's formula yields $n = 8 + f_5 + 2f_6$. If a cycle covers $c_4 \geq 0$ quadrangles, $c_5 \geq 0$ pentagons, and $c_6 \geq 0$ hexagons, its length is $2 + 2c_4 + 3c_5 + 4c_6$.

Clearly, each vertex is covered by exactly one cycle, thus we have

$$8 + f_5 + 2f_6 = n = 2c + 2x_4 + 3x_5 + 4x_6 = 2c + 2(f_4 - q_4) + 3(f_5 - q_5) + 4x_6,$$

since only hexagons can be resonant, and thus $f_k = x_k + q_k$ for $k = 4, 5$. Therefore,

$$8 + q_5 + 2f_6 = 2c + 2(f_4 - q_4) + 2(f_5 - q_5) + 4x_6,$$

so q_5 is even. By dividing by two and rearranging the terms we obtain

$$4 + f_4 + f_5 + f_6 + q_4 + q_5/2 + c = 2c + 2f_4 + 2f_5 - q_5 + 2x_6,$$

the claim immediately follows. \square

Let F be a good 2^* -factor in a Barnette graph G . Let us consider the structure of the graph $G^{(2)}$. We introduce an auxiliary graph $\Gamma = \Gamma_G(F)$, defined in the following way: $V(\Gamma)$ is the set of the white non-resonant faces of G (as of $G^{(2)}$). The edges of Γ are defined in the next two paragraphs.

Let C be the facial cycle of a (black) 2-gon f_0 in $G^{(2)}$. Let f_0 be incident to vertices u and v and adjacent to two (white) faces f and f' . Then each of u and v is incident to one more face (which has to be white), say f_u and f_v , respectively. Since f_0 only shares a vertex with f_u and with f_v , the faces f and f' are two consecutive white neighbors of f_u (f_v). Therefore, the faces f_u and f_v cannot be resonant. We add the edge $f_u f_v$ to $E(\Gamma)$; we call this type of edge of Γ *white*.

Let C be a cycle of F (and of $F^{(2)}$) which is not a facial cycle of a face of $G^{(2)}$. It means that C is a boundary of a union of at least two faces of G . We consider every pair of adjacent faces inside C . Let f and f' be such a pair of faces. Let u and v be the endvertices of the edge incident to both f and f' . Then each of u and v is incident to a third face (which has to be white), say f_u and f_v , respectively. The faces f and f' are two consecutive black neighbors of f_u (f_v). Therefore, the faces f_u and f_v cannot be resonant. We add the edge $f_u f_v$ to $E(\Gamma)$; we call this type of edge of Γ *black*.

Observe that for each edge of Γ , its endvertices are two faces of G at mutual position $(1, 1)$. Each edge of Γ covers two vertices of G and these pairs of vertices are pairwise disjoint. Therefore, Γ is a planar graph.

Let f be a white pentagon of $G^{(2)}$. It cannot be resonant, so f is a vertex of Γ . Let f_1, \dots, f_5 be the faces adjacent to f (sharing an edge with f) in $G^{(2)}$. (Observe that some f_i can be a 2-face: if it is the case, then there is another face f'_i adjacent to f in G , and adjacent to f_i in $G^{(2)}$.) Since the size of f is odd, the number of pairs (f_i, f_{i+1}) (with $f_6 = f_1$) of the same color (both black or both white) has to be odd. If both f_i and f_{i+1} are black, then none of them can be a 2-face, and thus there is a black edge incident to f in Γ . If both f_i and f_{i+1} are white, then again none of them can be a 2-face, and the vertex incident to f , f_i , and f_{i+1} is (in $G^{(2)}$) covered by a 2-cycle adjacent both to f_i and f_{i+1} , and thus there is a white edge incident to f in Γ . Altogether, f is a vertex of odd degree in Γ .

Similarly, for each non-resonant white hexagon f , there is an even number of pairs of consecutive adjacent faces of the same color, hence f is a vertex of non-zero even degree in Γ .

A white quadrangle f is always considered non-resonant. Its degree in Γ is also always even, however, it can be equal to 0 if the neighboring faces are colored alternatively black and white.

As a result of these local observations, the graph Γ can always be edge-decomposed into a set of paths with endvertices at the white pentagons of G , a set of cycles, and, eventually, a set of isolated vertices (corresponding to white quadrangles). The number of paths in the decomposition is equal to $q_5/2$, where q_5 is the number of white pentagons.

2.6 Structure of Barnette graphs

Let G be a Barnette graph and let p_1 and p_2 be two small faces of G . Suppose that there exists an induced dual path P^* connecting p_1 and p_2 passing only through hexagons. Then if we consider only faces of G corresponding to P^* , and if we replace the two small faces by hexagons, we obtain a graph with a canonical embedding into an infinite hexagonal grid.

The Goldberg vector (c_1, c_2) joining the first and the last hexagon is uniquely determined. We will use this vector to characterize the mutual position of p_1 and p_2 in G . Observe that the vector of two small faces may depend on the choice of the path joining them, see Figure 10 for illustration.

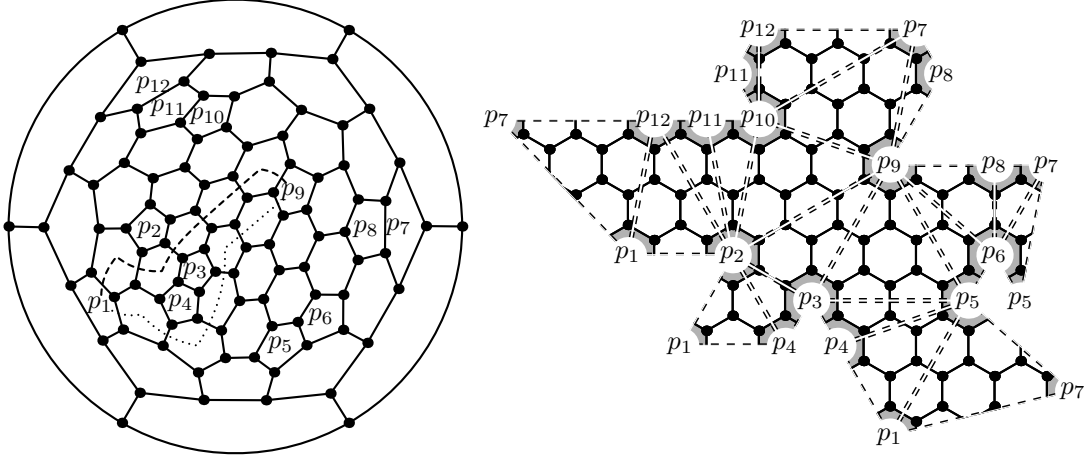


Figure 10: An example of a Barnette graph (left). Pentagonal faces are denoted p_1, \dots, p_{12} . The mutual position of p_1 and p_9 is characterized by vectors $(3, 3)$ (dotted line) or $(4, 2)$ (dashed line). The same graph embedded into a hexagonal grid after being cut along a spanning tree of a triangulation capturing the mutual position of all the small faces (right).

Graver [8] used the Coxeter coordinates to describe the structure of fullerene graphs. His technique may be extended to a full description of Barnette graphs as well in the following way: A given Barnette/fullerene graph G is represented by a planar triangulation T , whose vertices represent the small faces of G , and each edge uv is labelled with a Goldberg vector representing the mutual position of the faces represented by u and v . The angle between face-adjacent edges (incident to the same triangle of T) is well defined and is determined by the labels of the three edges forming the triangle. For a vertex of T representing a pentagon (a quadrangle) the angles around it sum up to $5/3\pi = 300^\circ$ ($4/3\pi = 240^\circ$, respectively).

The existence of a triangulation T is guaranteed by a structural theorem of Alexandrov (see e.g. [4], Theorem 23.3.1, or [15], Theorem 37.1), which states (in a more general setting) that any Barnette graph can be embedded onto the surface of a convex (possibly degenerate) polyhedron so that every face is isometric to a regular polygon with unit edge length; it suffice then to triangulate the faces of this polyhedron. Any spanning tree of T may be used to cut the graph G in order to obtain a graph embeddable into the infinite hexagonal grid, see Figure 10 for illustration.

We say that a Goldberg vector $\vec{u} = (c_1, c_2)$ is *shorter* than $\vec{u}' = (c'_1, c'_2)$ if and only if the Euclidean length of a segment determined by \vec{u} is shorter than the Euclidean length of a segment determined by \vec{u}' when both embedded into the same hexagonal grid.

Observe that the triangulation representing a Barnette graph is not unique: wherever two adjacent triangles form a convex quadrilateral (once embedded into the hexagonal grid), we

may choose the other diagonal of the quadrilateral instead of the existing one as an edge of the triangulation. For example, in the graph depicted in Figure 10 we could have chosen the edge p_3p_{10} instead of the edge p_2p_9 , etc.

However, for a triangulation T representing a Barnette graph G , the operation switching the diagonals of a convex quadrilateral eventually leads to a triangulation minimal with respect to the sum of lengths of its edges. For example, the triangulation depicted in Figure 10 is already minimal.

Lemma 8 *Let G be a Barnette graph, let T be a minimal triangulation representing G . Then T has a Hamiltonian path.*

Proof. Suppose that T has no Hamiltonian path. Then there exists a set X of vertices such that $T \setminus X$ has at least $|X| + 2$ connected components. Since T has at most 12 vertices, $|X| \leq 5$.

For each component C , the set of vertices in $G \setminus C$ having a neighbor in C contains a cycle in T (as T is a triangulation). Therefore, $G \setminus X$ is a plane graph with $|X| \leq 5$ vertices and $|V(T) \setminus X| \geq 7$ faces.

However, a planar graph on at most 5 vertices can have at most 6 faces. (Adding edges increases the number of faces, and (the) planar triangulation on 5 vertices (the triangular bipyramid) only has 6 faces.) \square

Note that the smallest planar graph with desired properties is a bipyramid over a square (which has 6 vertices and 8 faces).

Lemma 9 *Let G be a Barnette graph, let T be a minimal triangulation representing G . Then either T is Hamiltonian, or T can be transformed to a Hamiltonian triangulation by a single diagonal switch.*

Proof. Suppose that T has no Hamiltonian cycle. Then there exists a set X of vertices such that $T \setminus X$ has at least $|X| + 1$ connected components. Since T has at most 12 vertices, $|X| \leq 5$.

For each component C , the set of vertices in $G \setminus C$ having a neighbor in C form a cycle in T (as T is a triangulation). Therefore, $G \setminus X$ is a plane graph with $|X| \leq 5$ vertices and $|V(T) \setminus X| \geq 6$ faces.

There is only one such graph: the triangular bipyramid B , which has 5 vertices and 6 triangular faces. Out of the six components of $T \setminus B$, at least five are singletons, the sixth may eventually be an isolated edge. It means T has five vertices of degree at least 6, six vertices of degree 3, and eventually a vertex of degree 4.

Let $e = uv$ be an edge of B . It is incident to two triangles, each incident to a different component of $T \setminus B$. Let x and y be the vertices of $T \setminus B$ such that uvx and uvy are triangles of T . If the quadrilateral $uxvy$ is convex, then the triangulation T' obtained from T by switching uv to xy has at most five vertices of degree 3, so T' has to be Hamiltonian.

It remains to consider the case when for each edge e of B , the union of the two incident triangles is a non-convex quadrilateral, meaning that at one of its endvertices, the sum of the angles in the incident triangles is greater than 180° . Since B has five vertices and nine edges, there is at least one vertex of B with two (disjoint) pairs of incident triangles whose union gives a non-convex angle. But then the sum of the angles around this vertex is greater than 360° , a contradiction. \square

In Figure 11, an example of a Barnette graph on 322 vertices is depicted, along with the corresponding triangulation and a shortest Hamilton cycle in it.

3 Proof of Theorem 4: Finding a 2-factor

In this section we explain the general procedure in the case when the small faces of G are far from each other. We will deal with the case when some small faces of G are close to each other in Section 4.

3.1 Phase 1: Cut the graph and fix a coloring

Let G be a Barnette graph, let T be a Hamiltonian triangulation capturing the mutual position of the small faces of G , whose existence is given by Lemma 9. Let C_T be a Hamiltonian cycle in T such that the sum of the lengths of the corresponding Goldberg vectors is minimal. Then there exists a cycle C^* in G^* including all the small vertices of G^* in the same order as the corresponding vertices of C_T .

A cycle in G^* corresponds to an edge-cut in G . We cut the graph G along C^* . We obtain two graphs, say G_1 and G_2 , containing only hexagons as internal faces, and with semi-edges and partial faces on the boundary.

Both G_1 and G_2 are subgraphs of the hexagonal grid, hence there is a canonical face coloring using three colors for each of them. We will use colors 1, 2, 3 for one and colors A , B , C for the other. We color the partial faces in both graphs too.

We choose one color in each graph, say 1 and A (there are 9 color combinations in total), and recolor black all the faces of G_1 and G_2 colored 1 or A ; we color white the other faces. (Later we will inspect all the nine colorings.) This gives a black-and-white face-coloring ϕ_i inducing a 2-factor F_i in G_i , $i = 1, 2$.

Observe that for any choice of a color in G_i ($i = 1, 2$), the edges incident to one face of the other two colors each form a matching M_i such that $G_i^{M_i} = H_i^\parallel$, where H_i is the graph whose vertices are the centers of the faces of the other two colors.

We merge the two black-and-white face-colorings ϕ_1 and ϕ_2 of G_1 and G_2 , respectively, into an intermediate black-and-white (multi-)face-coloring $\phi^{(i)}$ of G in the natural way: A face not corresponding to a vertex of C^* inherits a color from either G_1 or G_2 ; A face which is cut by the cycle C^* is divided into two partial faces, one inheriting a color from G_1 and the other from G_2 , see Figure 12 for illustration.

3.1.1 Active and inactive segments

The cycle C^* can always be decomposed into a sequence of $\ell \leq 12$ subpaths P_1^*, \dots, P_ℓ^* joining consecutive pairs of small vertices. Let us call these subpaths *segments*.

We may suppose that a segment only contains hexagons with a non-empty intersection with the straight line joining the end-vertices of the segment.

For each segment P_i^* , the two face-colorings of G_1 and G_2 meet along P_i^* , and there is a unique canonical bijection $\varphi_i : \{1, 2, 3\} \rightarrow \{A, B, C\}$ between the two sets of colors.

If $\varphi_i(1) = A$ then the two black-and-white colorings coincide along P_i , we say that the segment P_i is *inactive*; otherwise it is *active*. Out of the nine colorings, each segment is active

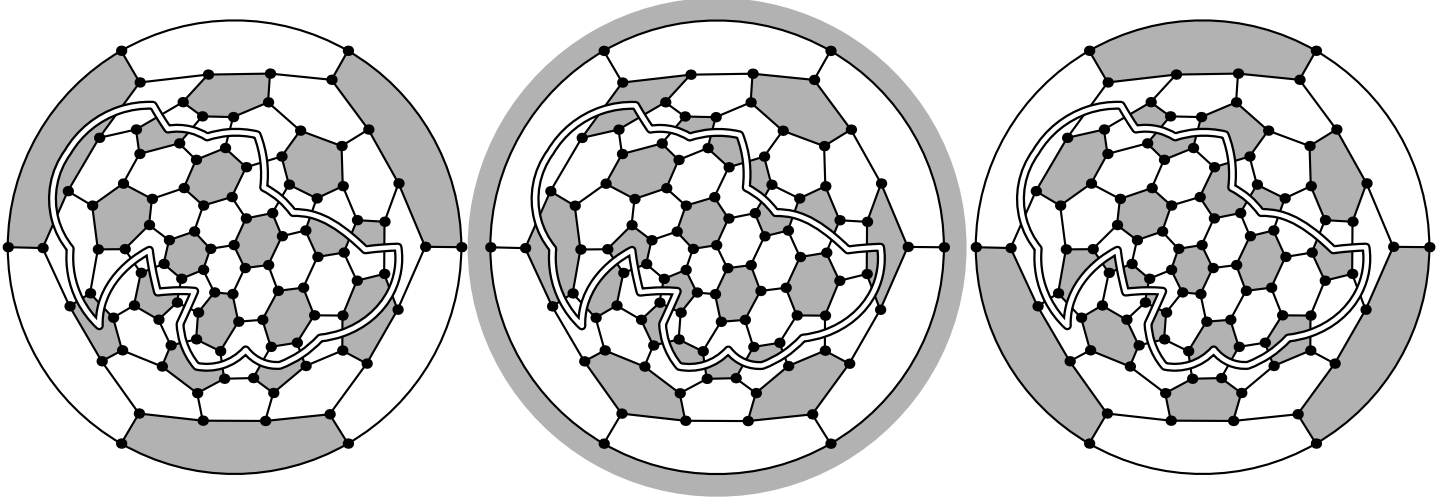


Figure 12: Three of the nine black-and-white colorings of the graph in Figure 10, (combinations of three different colorings of G_1 and three different colorings of G_2) corresponding to the given order of pentagons.

in precisely six of them. For example, the segments p_6p_7 and $p_{12}p_1$ are inactive in all the three colorings depicted in Figure 12, the segment p_4p_5 is active in all the three colorings, whereas the segment p_9p_{10} is inactive in the first coloring and active in the other two.

When switching from P_i^* to P_{i+1}^* , if the i -th small face is a quadrangle, we have $\varphi_i = \varphi_{i+1}$. If the i -th small face is a pentagon, the difference $\varphi_{i+1} \circ \varphi_i^{-1}$ is a permutation of the colors $\{A, B, C\}$ such that the color of the pentagon is stable and the two other colors are switched – a transposition. See Figure 13 for illustration.

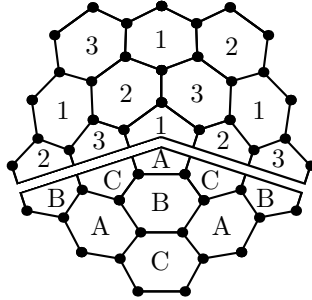


Figure 13: A pentagon always causes a single switch of colors – the two colors different from its color are switched.

Let p_i be a pentagonal face of G such that the segments P_{i-1}^* and P_i^* meet at p_i . Then exactly one of the following happens:

- (i) if $\varphi_{i-1}(1) = \varphi_i(1) = A$, then both P_{i-1}^* and P_i^* are inactive, p_i generates a switch of

B and C , thus it is colored A and it is black in both subgraphs;

- (ii.a) if $\varphi_{i-1}(1) = A$ and $\varphi_i(1) \neq A$, then P_{i-1}^* is inactive and P_i^* is active, p_i generates a switch of A and $\varphi_i(1)$, thus it is colored neither A nor 1, so it is white in both subgraphs;
- (ii.b) if $\varphi_{i-1}(1) \neq A$ and $\varphi_i(1) = A$, then P_{i-1}^* is active and P_i^* is inactive, p_i generates a switch of A and $\varphi_{i-1}(1)$, thus it is colored neither A nor 1, so it is white in both subgraphs;
- (iii.a) if $\varphi_{i-1}(1) = \varphi_i(1) \neq A$, then both P_{i-1}^* and P_i^* are active, p_i generates a switch of A and the third color, thus it is colored $\varphi_{i-1}(1)$, so it is black in G_1 and white in G_2 ;
- (iii.b) if $\{\varphi_{i-1}(1), \varphi_i(1)\} = \{B, C\}$, then both P_{i-1}^* and P_i^* are active, p_i generates a switch of B and C , thus it is colored A , so it is white in G_1 and black in G_2 .

In order to transform $\phi^{(i)}$ into a black-and-white face-coloring of G corresponding to a good 2-factor of G , we reroute slightly the cut C^* in a way described in the following subsection.

3.2 Phase 2: Approximate the cut by Γ -paths

Let P_i^* be an active segment, let $\varphi_i(1) = B$. Suppose without loss of generality that $\varphi_i^{-1}(A) = 2$. Then all the faces of P_i colored A (and 2) or 1 (and B) are partially black and partially white; both parts of each face of P_i^* colored C and 3 are white.

We approximate the dual path P_i^* by a sequence Q_i of faces colored C and/or 3, each consecutive pair of faces in a mutual position $(1, 1)$.

Let f be a white (C - and 3-colored) hexagonal face of Q_i . Then among its neighbors, there is a cyclic sub-sequence of A - and B -faces colored alternatively black and white, and another cyclic sub-sequence of 1- and 2-faces colored alternatively black and white, with the coloring being the opposite of the first one. Therefore, there are exactly two pairs (not necessarily disjoint) of adjacent faces of the same color: each pair is either a black A -face adjacent to a black 1-face, or a white B -face adjacent to a white 2-face. Therefore, f is a white non-resonant hexagon, corresponding to a vertex of degree 2 in the future auxiliary graph Γ being constructed – we will call it a Γ -face.

Let f and f' be two consecutive Γ -faces. If the two faces adjacent both to f and f' are black, then the two cycles of the 2-factors in G_1 and G_2 are merged. If the two faces adjacent both to f and f' are white, then a new 2-cycle of the 2*-factor is created. In the first case, the Γ -edge ff' is black, in the second case it is a white one.

Two consecutive Γ -edges of Q_i of the same color always form a 180° angle, otherwise it could be possible to simplify Q_i by removing a face from Q_i . Similarly, two consecutive edges of Q_i of different colors always form an angle of $\pm 120^\circ$.

The resulting structure of $\phi^{(i)}$ along P_i^* is the following: All vertices are covered by cycles of length 6 (single faces), 10 (two adjacent black hexagons, both incident to a black Γ -edge), or 2 (white Γ -edges). A Γ -path Q_i separates the two subgraphs of regular coloring. See Figure 14 for illustration.

The first (the last) Γ -face of Q_i is the pentagon p_i (p_{i+1}) if and only if the segment P_{i-1}^* (P_{i+1}^*) is inactive; otherwise the first (the last) Γ -face of Q_i is a hexagon adjacent to p_i (p_{i+1}) and it is the last (the first) Γ -face of Q_{i-1} (Q_{i+1} , respectively).

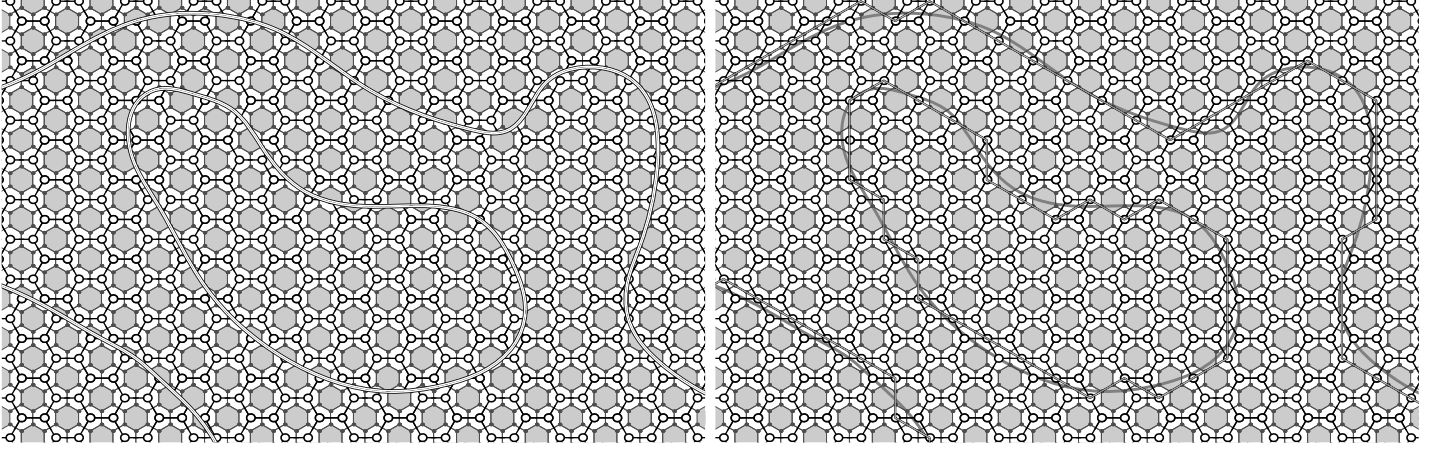


Figure 14: Several hexagonal patterns meeting along some cut curves (left). As the cutting lines are approximated by Γ -paths, a good 2^* -factor is created (right).

White non-resonant hexagons where two consecutive sequences Q_{i-1} and Q_i meet are the only occasion where two Γ -edges of the same color might form a 60° angle – if only they are both incident to the same pentagon.

Let us explicit the structure of $H^{(i)} = H_1 \cup H_2$ and of Γ now: Vertices of $H^{(i)}$ are all the vertices corresponding to faces of G white in G_1 or in G_2 ; each vertex of Γ where two black edges meet corresponds to a 2-vertex in $H^{(i)}$ (the corresponding face of G is a non-resonant white hexagon adjacent to four black faces belonging to two different components of the 2^* -factor); each vertex of Γ where two white edges meet corresponds to a 4-vertex in $H^{(i)}$ (the corresponding face of G is incident to four different compents of the 2^* -factor, including two 2-cycles); each vertex of Γ where a black and a white edge meet at a 120° angle corresponds to a 3-vertex in $H^{(i)}$ (the corresponding face of G being incident to three different components of the 2^* -factor: a 2-cycle, a 6-cycle and a 10-cycle).

If there are q_5 white pentagons, then Γ is composed of $q_5/2$ paths. A white quadrangle is either an isolated vertex of Γ (if both incident segments are inactive) or it is an internal vertex of a path (otherwise).

3.3 Phase 3: Change the parity of the 2^* -factor

It follows from Lemma 7 that whenever we want to transform an even 2^* -factor into an odd one, it suffices either to increase or decrease the number of black quadrangles by 1, or to increase or decrease the number of black pentagons by 2. In other words, it suffices either to change the number of isolated vertices in Γ by 1 or change the number of Γ -paths by 1.

3.3.1 Changing the parity using a quadrangle

Let q be a quadrangular face of G . For three of the nine colorings of G_1 and G_2 , both segments incident to q are inactive; moreover, for two out of the three q is a white face. In

Phase 1, we choose one of these two.

If the good 2^* -factor obtained in Phase 1 is even, it can be transformed into an odd one by recoloring q black. This way an isolated vertex of Γ is transformed into a cycle of length 2, see Figure 15 for illustration.

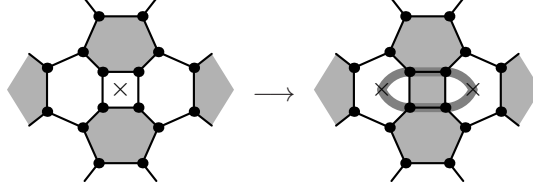


Figure 15: We can use a white quadrangle to change the parity of a 2^* -factor. The times sign marks non-resonant faces – vertices of Γ ; edges of Γ are drawn using a thick grey line.

3.3.2 Changing the parity using two pentagons

From this point on we may assume that G has no quadrangular faces – it is a (fullerene) graph having 12 pentagonal faces.

Suppose first that some pair of consecutive pentagons p_i and p_{i+1} (consecutive along the cut C) are in the mutual position (c_1, c_2) , $c_1 \geq c_2 \geq 0$, with $3 \mid (c_1 - c_2)$. Then in the coloring of G_1 with colors 1, 2, 3 (and of G_2 with A, B, C) the partial faces corresponding to the pentagons p_i and p_{i+1} have the same color. Therefore, for two of the nine colorings the segment P_i joining p_i and p_{i+1} is active whereas the neighboring segments P_{i-1} and P_{i+1} are inactive.

For both such colorings, after Phase 2 there is a Γ -path with endvertices at p_i and p_{i+1} , and the vertex set of this path can be chosen to be the same in both colorings. If this is the case, then each Γ -edge white in one coloring is black in the other and vice versa. Among the two colorings, we may fix the one where the number of white Γ -edges is maximised.

We transform the Γ -path into a Γ -cycle, increasing the number of black pentagons by 2, in the following way: For each black Γ -edge, we recolor both black hexagons forming a black 10-cycle white; then we recolor all faces corresponding to the vertices of the Γ -path black, including the first and the last one (p_i and p_{i+1}). We will denote this operation O_1 . See Figure 16 for illustration.

From this point on we may assume that there is no pair of consecutive pentagons with the same color in G_1 (or in G_2). Then for every pair of consecutive pentagons the nine colorings look like depicted in Figure 17.

Let ϕ_i^j be the angle between the two segments meeting at pentagon p_i in G_j , $j = 1, 2$. Clearly, $\phi_i^1 + \phi_i^2 = 300^\circ$. When following the segments composing the cut in an ascending order, say G_1 is to the left and G_2 to the right. If $\phi_i^1 > 150^\circ > \phi_i^2$, then there is a right turn at p_i when switching from P_{i-1} to P_i . If $\phi_i^1 < 150^\circ < \phi_i^2$, then there is a left turn at p_i when switching from P_{i-1} to P_i . The value $\phi_i^1 = \phi_i^2 = 150^\circ$ means that the segment P_i continues in the same direction as P_{i-1} .

Let $\phi_i = \phi_i^1 - \phi_i^2$ for $i = 1, \dots, 12$. It is easy to see that $\sum_{i=1}^{12} \phi_i = 0$, since $\sum_{i=1}^{12} \phi_i^1 = \sum_{i=1}^{12} \phi_i^2 = 1800^\circ$. Therefore, there exist i such that $\phi_i \cdot \phi_{i+1} \leq 0$ (indices modulo 12). We

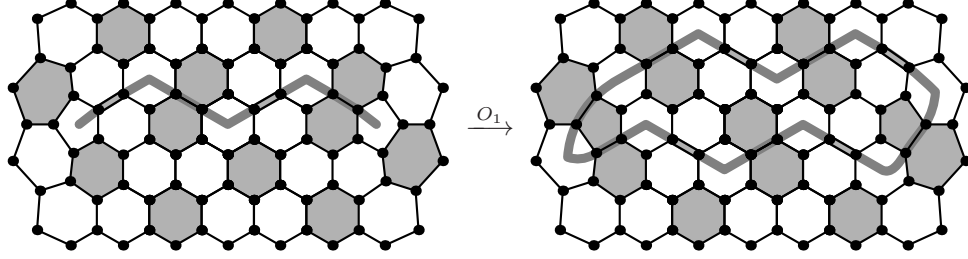


Figure 16: Operation O_1 : The parity of a 2^* -factor can be changed by modifying a Γ -path joining two consecutive pentagons into a Γ -cycle.

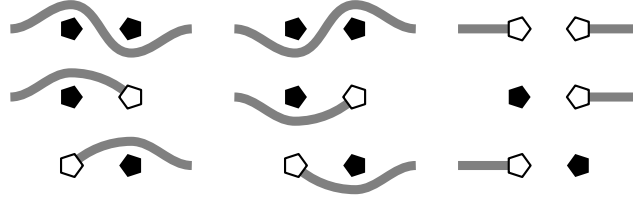


Figure 17: A schematic drawing of the position of the Γ -paths in the neighborhood of two consecutive pentagons of different colors.

fix i such that $\phi_i \cdot \phi_{i+1} \leq 0$ and the difference $|\phi_i - \phi_{i+1}|$ is as big as possible.

Without loss of generality we may assume that $\phi_i \geq 0$ and $\phi_{i+1} \leq 0$. In other words, there is a right turn at p_i followed by a left turn at p_{i+1} . There are two colorings in which the segments P_{i-1} , P_i , and P_{i+1} are active; among them we choose the one where p_i is black in G_2 and p_{i+1} is black in G_1 .

We can now change the parity of the 2^* -factor (if needed) by decreasing the number of black pentagons in the following way: For each black Γ -edge of P_i , we recolor both black hexagons forming a black 10-cycle white; then we recolor all faces corresponding to the vertices of the Γ -subpath Q_i black, including the first and the last one (those adjacent to p_i and p_{i+1} , respectively); we recolor p_i and p_{i+1} white. As the last step, we simplify unnecessary 60° turns. We will denote this operation O_2 . See Figures 18, 19 and 20 for illustration.

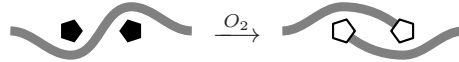


Figure 18: A schematic drawing of the operation O_2 .

3.4 Phase 4: Transform a good odd 2^* -factor into a simple 2-factor

It suffices now, as the last phase, to transform a good odd 2^* -factor into a simple (odd) 2-factor. We do it in the following way:

In a good 2^* -factor, each 2-cycle corresponds to a white Γ -edge ff' , incident to two white resonant hexagons h_1 and h_2 (one in each of G_1 and G_2). We can choose either h_1 or h_2 ,

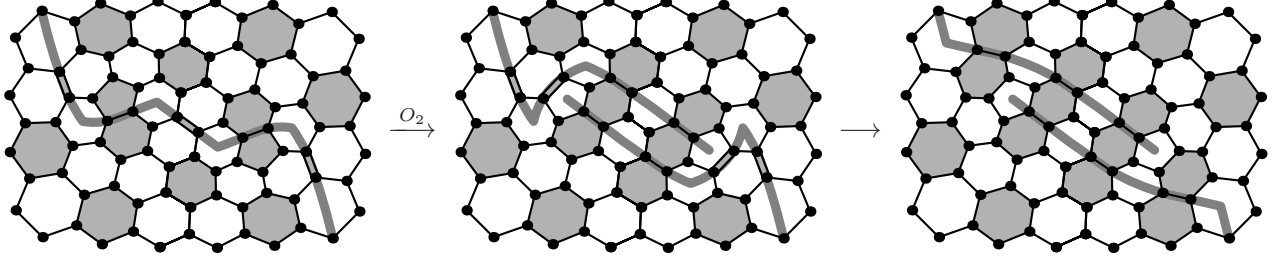


Figure 19: Operation O_2 : The parity of a 2^* -factor can be changed by transforming a Γ -path passing by two consecutive pentagons into two different Γ -paths.

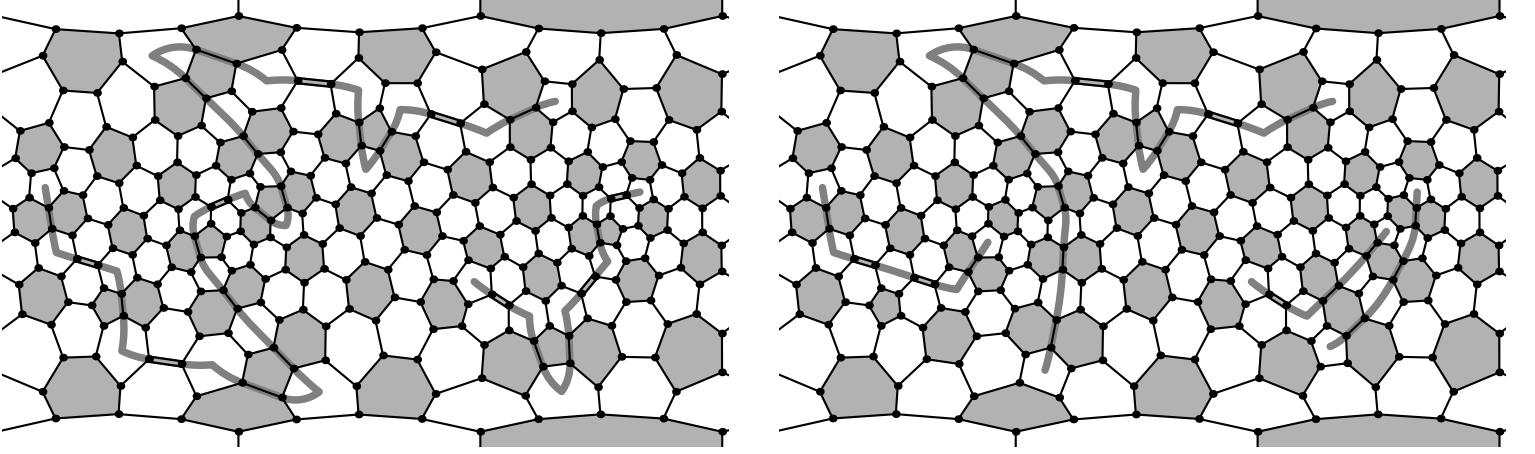


Figure 20: The good 2^* -factor induced by one of the nine possible black-and-white face-colorings of the graph in Figure 11 (left). It is already an odd 2^* -factor; there are several pairs of pentagons for which the operation O_2 is admissible. Another good 2^* -factor of the same graph obtained by two applications of O_2 (right).

say h_i , and recolor it black: By doing this, the 2-cycle is merged with two other cycles in G_i ; the other face f_0 incident to both cycles being merged loses its resonantness, it becomes another Γ -face inserted to the Γ -path between f and f' , joint now to f and f' by two black Γ -edges forming a 60° angle and replacing the original white Γ -edge. In $H^{(i)}$, a vertex of degree 3 is removed, and thus the degree of three other vertices is decreased by 1: one of them corresponds to f_0 , the other two correspond to f and f' .

Observe that this operation decreases the number of components of the factor by 2, therefore, starting with an odd factor we can only obtain odd factors.

We make a decision for all white Γ -edges sequentially according to their order along Q_i , according to the following rules: If a white Γ -edge e_j forms a 180° angle with e_{j-1} (which has to have been white in this case) and that we have decided to recolor black a hexagon in G_i , $i = 1, 2$, incident to e_{j-1} , then we decide to recolor black a hexagon in G_{3-i} incident to e_j . If a white Γ -edge e_j forms a 120° angle with a black e_{j-1} , we decide to recolor black a

hexagon incident to e_j in such a way that one of the new black Γ -edges forms a 180° angle with e_{j-1} .

The resulting structure in G is the following: All the Γ -paths and Γ -cycles are formed of black Γ -edges only. Each vertex of Γ of degree 1 or 2 corresponds to a 2-vertex in $H^{(i)}$.

Finally, to obtain H , we suppress all the 2-vertices in $H^{(i)}$; for each Γ -edge we merge the incident partial faces of $H^{(i)}$.

To describe the structure of H , we introduce the following notation: A vertex of Γ is called *direct* if it corresponds to a pentagon or if the two incident (black) Γ -edges form a 180° degree; otherwise it is called *sharp*.

We claim that there cannot be three consecutive sharp Γ -vertices along any Q_i : Suppose some Q_i contains a subpath $f_0f_1f_2f_3f_4$ with all of f_1 , f_2 , and f_3 sharp and f_0 direct. If f_1f_3 had been a white Γ -edge after the Phase 2, we would not have decided to choose f_2 . Therefore, f_2 was a Γ -vertex already after Phase 2, which means that f_0f_2 was a white Γ -edge after Phase 2. If f_2f_4 was also a white Γ -edge after Phase 2, we would have decided one of them in the other way. Therefore, f_3 was a Γ -vertex already after Phase 2, but not f_4 , which means that f_4 is sharp. As f_1 must have been chosen because of the other Γ -edge incident to f_0 , f_4 should never have been chosen, a contradiction.

A (black) Γ -edge joining two direct Γ -vertices f and f' completes the boundary of two partial faces in H_1 and H_2 , each having three incident 3-vertices. After the suppression of 2-vertices in $H^{(i)}$, in H these two partial faces are merged into a hexagon.

The 60° angle at a sharp Γ -vertex f contains a partial face of $H^{(i)}$ having one 3-vertex, which is to be merged with (at least) two other partial faces.

If both Γ -vertices adjacent to f in Γ are direct, then a face of size 7 is created in H by merging two partial faces each having three incident 3-vertices in H_i with a partial face having one incident 3-vertex in H_{3-i} . On the other hand, opposite to this one, there is a face of $H^{(i)}$ whose size is decreased by 1 by the suppression of the 2-vertex f – a pentagonal face is created in H .

If one of the vertices adjacent to a sharp vertex in Γ is a sharp one, they are transformed into a face of size 8 and two pentagons in H . See Figures 21, 22, and 23 for illustration.

4 Checking the correctness of the algorithm in the neighborhood of small faces close to each other: the computer-assisted part

Let G be a Barnette graph. Let $S(G)$ be the set of the *small* faces (faces of size 4 or 5) of G . It is straightforward to derive from the Euler's formula that $2f_4 + f_5 = 12$, where f_4 and f_5 are the numbers of quadrangles and pentagons in G , respectively.

4.1 Patches

A *patch* is a 2-connected subcubic plane graph P , having at most one face of size different from 4, 5 and 6, and such that all vertices of P of degree 2 are incident to this special face, often referred to as the outer face of the patch; moreover, P contains no pair of adjacent

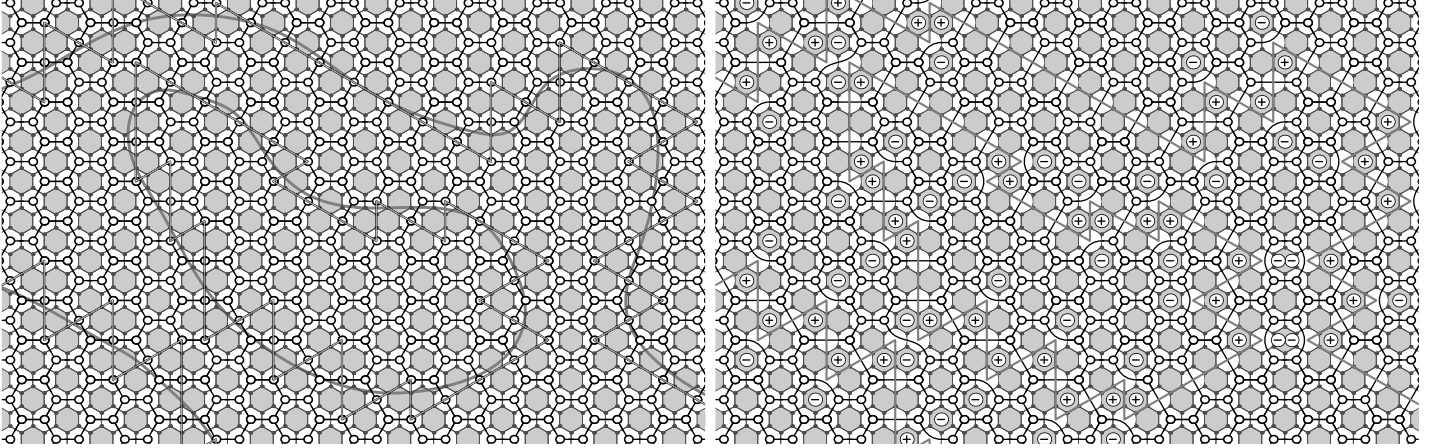


Figure 21: The intermediate structure after eliminating the white Γ -edges (left) and the final simple 2-factor, with the face size changes in H (with respect to the initial size of 6) marked with plus and minus signs.

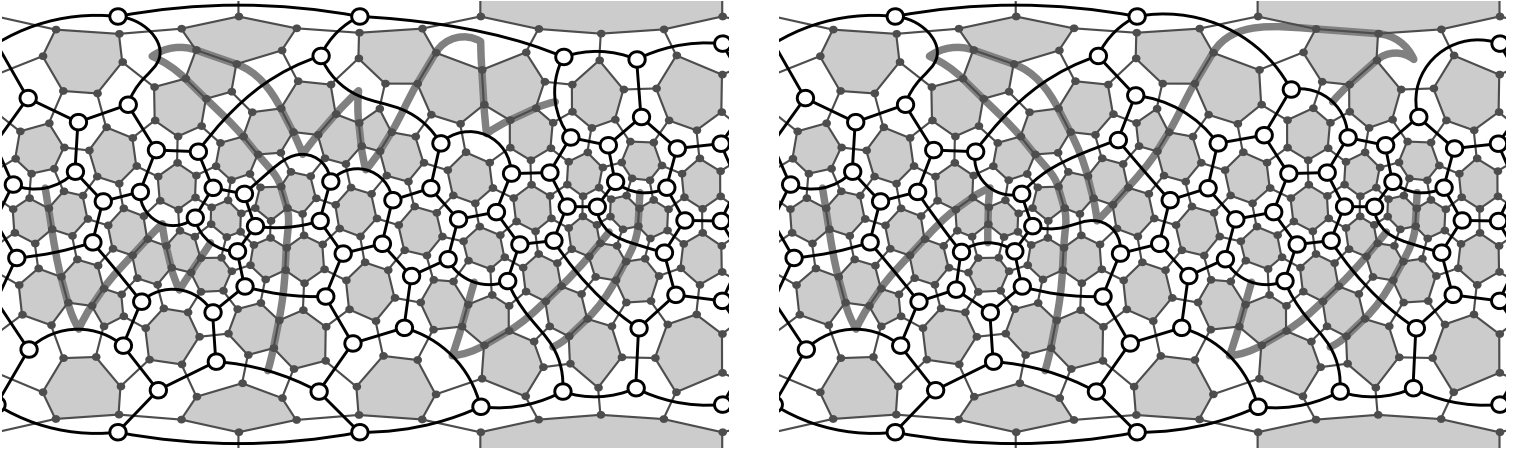


Figure 22: The simple 2-factor obtained from the good 2^* -factor in Figure 20, depicted together with the residual graph H and the auxiliary graph Γ (left). Another simple 2-factor obtained from the previous one by "flipping out" unnecessary zig-zags of sharp Γ -vertices (right). Observe that for the latter, the residual graph H has faces of size 5, 6, and 7 only and it is cyclically 5-edge-connected.

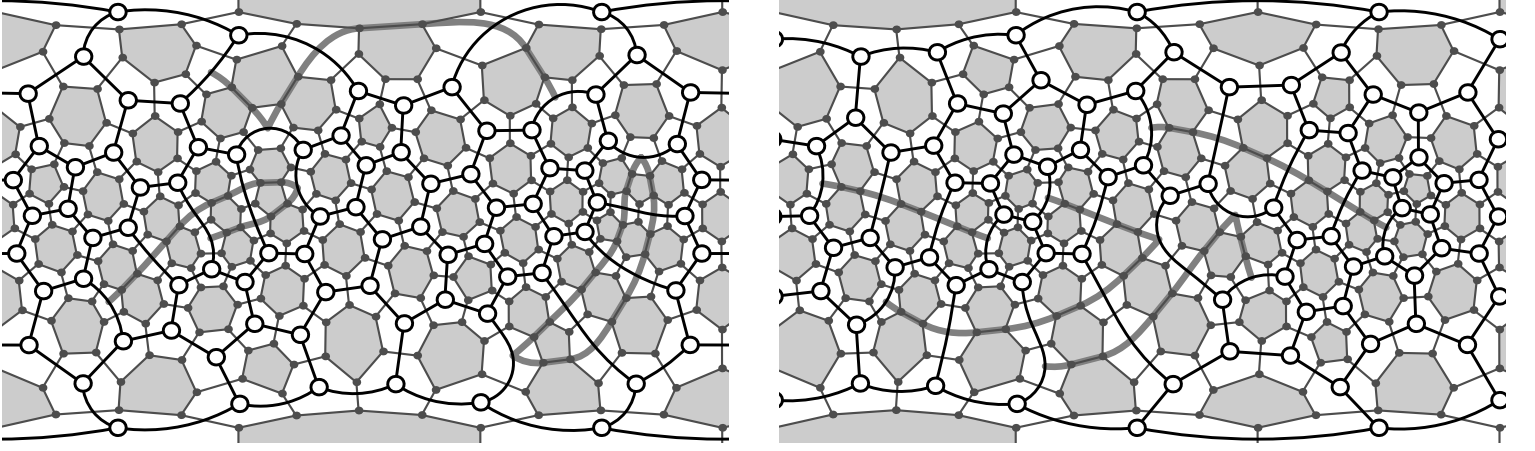


Figure 23: Two different odd simple 2-factors of the graph in Figure 11 with the largest number of vertices of the residual graph H (82) we were able to find.

4-faces. When a patch is depicted, there are additional pending half-edges at vertices of degree 2 towards the outer face.

The *curvature* of a patch P , denoted by $\mu(P)$, is equal to $2f_4(P) + f_5(P)$, where $f_4(P)$ and $f_5(P)$ are the numbers of quadrangles and pentagons in P (distinct from the outer face of P), respectively.

We denote $\partial(P)$ the *boundary* of a patch P – the facial cycle of the outer face of P ; we denote $\delta(P)$ the *perimeter* of a patch P , the number of 2-vertices in P .

The *boundary vector* $\sigma(P)$ of a patch P is a cyclic sequence of distances between consecutive 2-vertices on the boundary cycle of P . The length of $\sigma(P)$ is equal to $\delta(P)$ and its sum is equal to the length of $\partial(P)$. When expliciting elements of a cyclic sequence σ , we write x^k as a shortcut for k consecutive occurrences of a value x in σ .

Each vertex of $\partial(P)$ is either a 2-vertex or a 3-vertex in P . The proportion of 2-vertices along $\partial(P)$ is determined by the curvature of P , as is stated explicitly in the following lemma, which is a generalisation of an observation from [10] and can be derived directly from Euler's formula by the same double-counting arguments.

Lemma 10 *Let P be a patch of curvature μ . Then*

$$2\delta(P) - |\partial(P)| = 6 - \mu.$$

Observe that for patches of curvature (greater than, less than) six, the average value of $\sigma(P)$ is (greater than, less than, respectively) two.

A patch P of curvature $\mu \leq 4$ ($\mu \geq 8$) is called *convex* if its boundary vector $\sigma(P)$ only contains '1's and '2's ('2's and '3's, respectively). A patch P with $\mu = 5$ ($\mu = 7$) is called *convex* if $\sigma(P)$ does not contain 32^j3 (does not contain 12^j1 , respectively). A patch P of curvature 6 is called *convex* if $\sigma(P)$ contains at most one subsequence 32^j3 ; if this is the case,

$j > \delta(P)/2$. For instance, all the caps of nanotubes in Figures 5 and 7 are convex patches of curvature 6.

Note that, according to Lemma 10, the boundary vector of a convex patch P of curvature $\mu \leq 4$ has the form $(12^{k_1}12^{k_2} \dots 12^{k_t})$ where $t = 6 - \mu$, $k_1, k_2, \dots, k_t \in \mathbb{N}_0$, and $k_1 + k_2 + \dots + k_t = p - t$.

We denote $P^{i \leftarrow j}$ a patch obtained from P by adding a face of size j to P along the path corresponding to the i -th element of $\sigma(P)$, if such a patch exists, see Figure 24 for illustration.

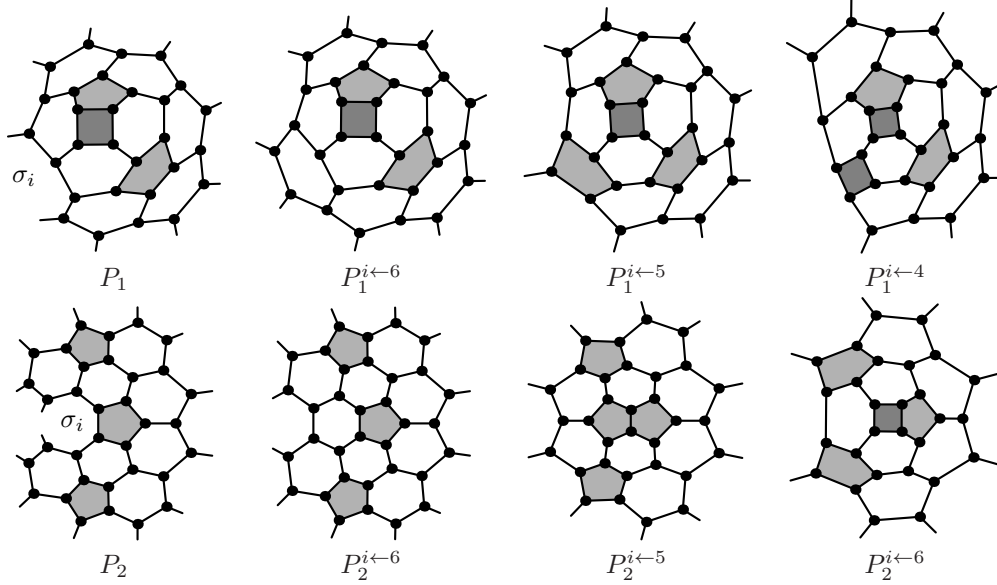


Figure 24: Two different examples of three different patches obtained from a given patch (on the left) by inserting a new face at the element σ_i of its boundary vector.

It may happen that while adding a new face to a patch, we have to identify some elements (vertices/edges/faces) of the patch, as in the second row of Figure 24. It may even happen that adding a new face of some desired size to a specific place of a patch is not possible, since the faces to be identified are not of the same size.

4.1.1 Patches in Barnette graphs

Let G be a Barnette graph. We say that a patch P is contained in G if there is a graph homomorphism $\varphi : P \rightarrow G$ such that all faces of the patch (except for the outer face) are also faces of G . We say that a patch P is *realizable* if it is contained in some Barnette graph.

Observe that a patch P of perimeter 0 is contained in a Barnette graph G if and only if $P = G$ and the outer face of P is a face of G . Similarly, a patch P of perimeter 2 is contained in a Barnette graph G if and only if $P = G \setminus e$ for some edge e of G and the outer face of P is the union of the two faces incident to e in G . Finally, since Barnette graphs are cyclically 4-edge-connected, a patch P of perimeter 3 is contained in a Barnette graph G if and only if $P = G \setminus v$ for some vertex v of G and the outer face of P is the union of the three faces

incident to v in G . On the other hand, no patch of perimeter 1 can be realizable, since it would correspond to a cut-edge in a Barnette graph.

Some (but not all) realizable patches can be obtained in the following way: For any induced cycle C of a Barnette graph G , there are two distinct (but not disjoint) patches P and \bar{P} contained in G such that $\partial(P) = \partial(\bar{P}) = C$. It is easy to see that we have $\mu(P) + \mu(\bar{P}) = 12$ and that $\delta(P)$ is equal to the number of edges of the cut separating P from $G \setminus P$.

Moreover, as each vertex of C is either a 2-vertex in P or a 2-vertex in \bar{P} , $\delta(P) + \delta(\bar{P})$ is equal to the length of $\partial(P)$.

As a direct consequence of Lemma 10 we obtain the following observation.

Lemma 11 *Let C be an induced cycle in a Barnette graph and let P and \bar{P} be the two corresponding patches. Then*

$$\delta(P) - \delta(\bar{P}) = 6 - \mu(P) = 6 + \mu(\bar{P}).$$

However, there are patches contained in Barnette graphs which cannot be obtained this way: it is not always true that the facial cycle of the outer face of a patch corresponds to an induced cycle of the host Barnette graph – a patch can even be self-overlapping.

Lemma 12 *Let P be a realizable patch of perimeter at least 2. For every element σ_i of its boundary vector there exists $j \in \{4, 5, 6\}$ such that $P^{i \leftarrow j}$ is also a realizable patch, moreover, it is contained (at least) in the same Barnette graph as P .*

Proof. Let P be contained in a Barnette graph G . Each element of σ is a path contained in a facial cycle of some face of G of a certain size $j \in \{4, 5, 6\}$. Therefore, the face added to the patch corresponds to a face of G . \square

4.1.2 Primitive patches

A convex patch of curvature $\mu \leq 5$ is *primitive* if the arrangement of its small faces is the same as in one of the patches depicted in Figure 25 or it has no small faces at all (for $\mu = 0$).

Observe that each convex patch with at most one small face is primitive.

Lemma 13 *Let P be a convex patch of curvature $\mu \leq 5$ which is not primitive. Then there exists another patch P' with the same curvature and the same boundary vector as P on a bigger number of vertices.*

Proof. Suppose that there exists a convex patch of curvature $\mu \leq 5$ which is not primitive, and all the convex patches of given curvature and boundary vector have at most as many vertices as P .

If P has at most one small face, then it is primitive by definition, a contradiction. Therefore, we may assume that P has at least two small faces.

If all the small faces of P are pairwise adjacent to each other, then P has at most three small faces, moreover, if it has three small faces, at most one of them is a quadrangle. In all the cases the patch is primitive, a contradiction.

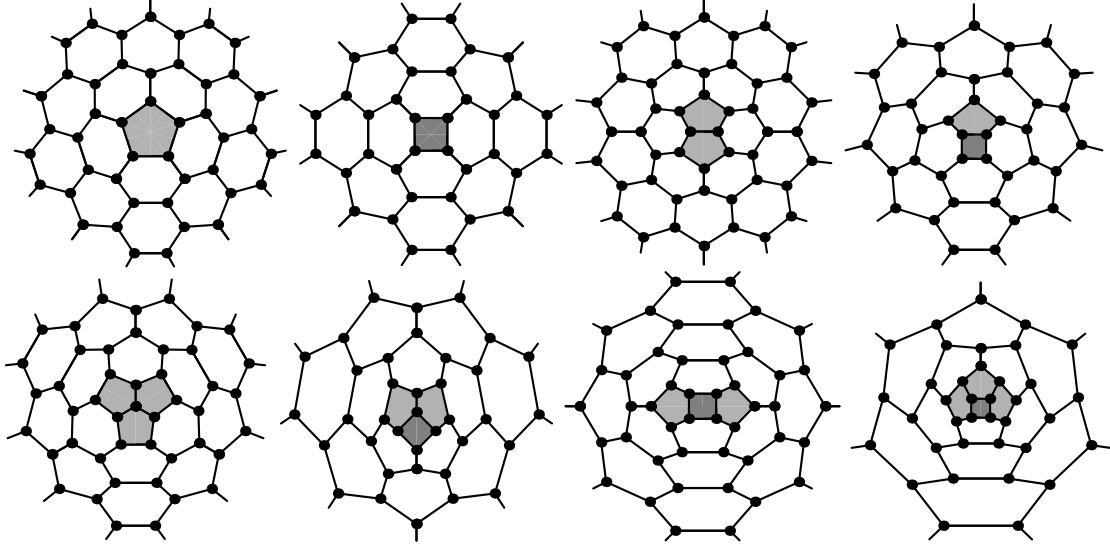


Figure 25: Arrangements of small faces in primitive patches for different values of curvature $1 \leq \mu \leq 5$.

We may suppose that P has two small faces f_1 and f_2 which are not adjacent to each other. We claim that f_1 and f_2 are at mutual position $(1, 1)$ and the edge connecting them is incident to a quadrangle:

Suppose f_1 and f_2 are two small faces in mutual position (c_1, c_2) such that $c_1 \geq c_2 \geq 1$ and $c_1 \geq 2$. Then there exists a new patch P' with the same boundary vector and the same curvature, but with a bigger number of vertices: P' can be found by inserting two pentagons and $c_1 + c_2 - 3$ hexagons along a shortest path joining f_1 and f_2 . (The path is in P due to convexity of P .) By applying this operation, the size of f_1 and f_2 is increased by one; the mutual position of the two new pentagons is $(c_1 - 1, c_2 - 1)$, see Figure 26 for illustration. The patch P' is indeed a patch of a Barnette graph, since no pair of adjacent quadrangles can be created this way.

Similarly, if $c_1 \geq 3$ and $c_2 = 0$, then there is a sequence h_1, \dots, h_{c_1-1} of hexagons forming a dual path joining f_1 and f_2 . We subdivide the edge between f_1 and h_1 and the edge between h_{c_1-1} and f_2 once; we subdivide each edge between h_i and h_{i+1} ($1 \leq i \leq c_1 - 2$) twice; we join the new vertices in such a way that h_1 and h_{c_1-1} are split into a pentagon and a hexagon and that all other hexagons in the sequence are split into two new hexagons. Again, the size of f_1 and f_2 is increased by one and a new pair of pentagons at mutual position $(c_1 - 2, 1)$ is created, see Figure 26 for illustration.

Analogously, if $(c_1, c_2) = (2, 0)$, then there is a hexagon h adjacent to both f_1 and f_2 . To obtain P' , it suffices to subdivide the two edges h shares with f_1 and f_2 , respectively, and join the two new vertices by a new edge. This way h is split into two pentagons and the size of f_1 and f_2 is increased by one, see Figure 26 for illustration.

Finally, let $(c_1, c_2) = (1, 1)$. Then f_1 and f_2 are connected by an edge e . If the edge e is not incident to any quadrangle, then new patch P' can be obtained by replacing e by a

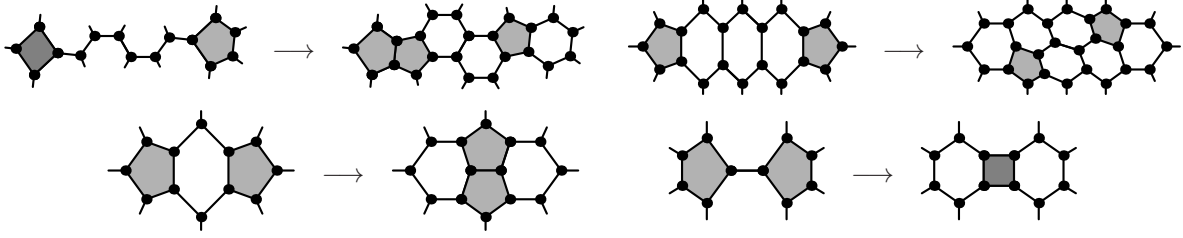


Figure 26: If a patch contains at least two non-adjacent small faces, it can be transformed to another one with more vertices, unless the two small faces are in position (1,1) and the edge connecting them is incident to a quadrangle: Two generic cases (top) and two special cases (bottom). The size of the two small faces is always increased by one (so if they were pentagons, they are no more small); two new pentagons or one new quadrangle are created.

quadrangle, see Figure 26 for illustration. Since the size of f_1 and f_2 is increased by one, there can not be two adjacent quadrangles in the patch P' .

To conclude, for every pair of non-adjacent small faces of P , there is a quadrangle adjacent to both of them, so both of them are pentagons, and so $\mu \geq 4$ and P contains a quadrangle adjacent to two pentagons (which are not adjacent to each other). If $\mu = 4$, then P has no other small faces, so it is primitive, a contradiction. If $\mu = 5$, then P contains an additional pentagon, which, due to the previous observations, has to be adjacent to (the only) quadrangle – again we obtain a primitive patch, a contradiction. \square

Corollary 1 *For a given curvature and given boundary vector, a convex patch with maximal number of vertices has to be a primitive one.*

Lemma 14 *Let P be a convex patch of curvature $\mu \leq 5$ and boundary vector σ . Then there exists a unique primitive patch $\bar{P}(\mu, \sigma)$ with the same curvature and the same boundary vector.*

Proof. The existence is given by the previous lemma. The uniqueness can be proven by induction, by adding/removing rows of hexagons from a patch, or, alternatively, by considering embeddings of patches onto infinite hexagonal cones. We omit the details. \square

Lemma 15 *Let P be a convex patch of perimeter p and curvature $\mu \leq 5$. Then P has at most $\frac{p^2}{6-\mu}$ vertices.*

Proof. It suffices to count the numbers of vertices of primitive convex patches. We omit the details. \square

It is worth mentioning that the bound from Lemma 15 is tight only if $\mu(P) \leq 2$ and the patch contains at most one small face.

Corollary 2 *Let P be a convex patch of curvature $\mu \geq 7$. Then P can be realized only in finitely many Barnette graphs.*

The largest Barnette graph containing a given realizable convex patch of curvature $\mu \geq 7$ can be found by adding the corresponding (unique) primitive patch of curvature $12 - \mu$.

4.1.3 Patch closure and essential patches

A k -disc centered at a face f of a plane graph G , denoted by $B_k(f)$, is a subgraph of G composed of facial cycles of faces at (dual) distance at most k from the face f . Note that if k is large enough, then $B_k(f) = G$ for any f .

A patch P' is called a *closure* of a patch P , if

1. P is contained in P' ,
2. every small face of P' corresponds to a small face of P ,
3. P' contains the 2-discs centered at the small faces of P , and
4. P' is convex.

A patch P is called *closed* if it is a closure of itself.

Clearly, if P' is a closure of P , then P' can be obtained from P by adding a finite number of hexagons.

Let P be a patch with boundary vector $\sigma(P) = \sigma_1\sigma_2\ldots\sigma_k$. The *small face distance* of a value σ_i is equal to the minimum of the distances $d(f^*, g^*)$, where f is the new face of the patch $P^{i \leftarrow j}$ (for some j sufficiently big), g run the set of small faces of P , and the distances are taken in the inner dual (dual without the vertex representing the outer face) of $P^{i \leftarrow j}$.

Let P be a patch which is not convex. Then we set all the values of its boundary vector as *admissible*.

Let P be a convex patch with boundary vector $\sigma(P) = \sigma_1\sigma_2\ldots\sigma_k$. A value σ_i is called *admissible*, if the small face distance of σ_i is at most 2.

Observe that boundary vectors of closed patches have no admissible values.

Let P be a patch with boundary vector $\sigma(P) = \sigma_1\sigma_2\ldots\sigma_k$ which is not closed. A *critical element* of the boundary vector of P is an admissible value σ_i such that

- σ_i is maximal, and then
- the sum $\sigma_{i-1} + \sigma_{i+1}$ (incides taken modulo k) is maximal, unless $\mu(P) \geq 5$ and $\max_{i=1}^k \sigma_i = 3$; in which case we choose $\sigma_i = 3$ contained in a subsequence 32^j3 of minimum length, and then
- the small face distance of σ_i is minimal.

Lemma 16 *Let G be a Barnette graph and let f be a small face of G . Then there exists a finite sequence of patches $\{P_k\}_{k=1}^t$ contained in G such that*

- P_1 is a cycle of length equal to the size of f ;
- $P_{k+1} = P_k^{i \leftarrow j}$, where $j \in \{4, 5, 6\}$ and σ_i is a critical element of $\sigma(P_k)$;
- for each k , in the embedding of P_k into G , the face f corresponds to a face of P_k ;
- either P_t is the first closed patch of the sequence or $P_t = G$.

Proof. The existence of the sequence is guaranteed by Lemma 12. Either adding faces one by one yields a closed patch, or all the faces of G are eventually added. In both cases the sequence is finite. \square

Observe that the sequence $\{P_k\}_{k=1}^t$ of patches contained in a Barnette graph G starting with a fixed small face f of G given by Lemma 12 is not unique – it may depend on the choice of a critical element.

Let f be a small face of a Barnette graph G and let P be a patch. If $P = P_t$ for some sequence described in Lemma 12 starting with f , then we call P an *essential patch* for f in G .

4.2 Patches and the general procedure

Let P be a patch essential from some small face of a Barnette graph G . Then the Hamiltonian cycle C_T of the triangulation T capturing the mutual position of small faces of G enters and leaves P at least once.

We will modify the general procedure in order to ensure that we can choose a cycle C_T entering and leaving P exactly once: For essential patches of curvature at least 6 this is automatically true due to convexity of the patch and minimality of the cycle. For each essential patch P of curvature at most 5 we can temporarily replace P by the corresponding primitive patch \bar{P} ; in the resulting graph we find the cycle C^* visiting each small face exactly once. Since in \bar{P} the small faces are adjacent to each other, they are consecutive along C^* by minimality of C^* . When replacing back the primitive patches by the actual patches, we keep the order in which the (primitive) patches were covered by C^* and we keep the position of the segments joining different patches. We disregard the way how C^* visits the small faces inside each essential patch, since we will inspect that in details later.

From this point on we may assume that for each essential patch P there are exactly two segments leaving P , say P_i^* and P_j^* . For any position of the segments $P_{i+1}^*, \dots, P_{j-1}^*$ inside P , the difference $\varphi_j \circ \varphi_i^{-1}$ is a permutation of three elements which is even if and only if $\mu(P)$ is even (each pentagon of P contributes with a single transposition).

If $\mu(P)$ is odd, then the difference $\varphi_j \circ \varphi_i^{-1}$ is an odd permutation – a transposition. Therefore, among the nine choices of colorings of G_1 and G_2 , for one choice both segments leaving P are inactive, for four choices one of them is active and the other one is inactive, and for the remaining four both segments are active – the patch behaves like a pentagon. We will call these patches type 1.

If $\mu(P)$ is even and the difference $\varphi_j \circ \varphi_i^{-1}$ is the identity, then among the nine choices of colorings of G_1 and G_2 , for three of them both segments are inactive and for the remaining six both segments are active – the patch behaves like a quadrangle. We will call these patches type 0.

If $\mu(P)$ is even and the difference $\varphi_j \circ \varphi_i^{-1}$ is an even permutation different from the identity, then it has to be a cycle of length three. Therefore, among the nine choices of colorings of G_1 and G_2 , for three of them both segments are active and for the remaining six there is one active and one inactive segment – the patch behaves like a pair of pentagons of different colors. We will call these patches type 2.

Let P be a patch essential from some small face of a Barnette graph G . Let the position of two segments leaving P and all the segments inside P be fixed. Let one of the nine colorings of G_1 and G_2 be chosen. Let the procedure described in Section 3 be applied. We first obtain

a 2*-factor, which is then transformed into at most two 2-factors (depending on the order of decisions at 2-cycles of the 2*-factor).

Let H_P^0 be the subgraph of the residual graph H induced by the vertices corresponding to the faces of P and faces adjacent to faces of P in G . There can be vertices of degree 1 or 2 in H_P^0 . We add $3 - d$ new vertices adjacent to each vertex of degree d in H_P^0 inside the outer face; we then connect all these new vertices by a new cycle. This way we obtain a plane cubic graph H_P , we call it *partial residual graph*.

Let f^* be a vertex of H_P corresponding to a face f of P . The face f is a white resonant hexagon. If we recolor f black, then three different components of the underlying 2-factor are merged into a single cycle; the vertex f^* is deleted from H_P and the three resulting 2-vertices are suppressed. We call this operation *elimination* of f^* .

We say that a plane cubic graph is *strongly essentially 4-edge-connected*, if it is cyclically 3-edge-connected, and every cyclic 3-edge-cut separates a triangle adjacent to the outer face from the rest of the graph.

We say that a plane cubic graph is *essentially 4-edge-connected* if it can be transformed into a strongly essentially 4-edge-connected plane graph by a vertex elimination.

We say that a patch P is *regular*, if for every possible position of a pair of segments leaving P and for every choice of the colors of G_1 and G_2 , there is a permutation of small faces of P such that for each of the (at most) two 2-factors obtained by the general procedure the corresponding partial residual graph is essentially 4-edge-connected. See Figures 27 and 28 for illustration.

We say that a patch P is *weakly regular*, if for every possible position of a pair of segments leaving P there exists a choice of the colors of G_1 and G_2 such that there is a permutation of small faces of P such that for at least one 2-factor obtained by the general procedure the corresponding partial residual graph is essentially 4-edge-connected.

We say that a patch P is *parity-switching* if for every possible position of a pair of segments leaving P there exists a choice of the colors of G_1 and G_2 such that there exists a permutation of small faces of P such that one of the operations O_1 and O_2 can be applied inside P ; for both 2*-factors (before and after the operation), for at least one 2-factor the corresponding partial residual graph is essentially 4-edge-connected.

4.3 Generation of patches

Theorem 5 *There exists a finite set \mathcal{P} of patches such that for every Barnette graph G on at least 318 vertices and every small face f of G , there exists a patch $P \in \mathcal{P}$ essential for f in G .*

Proof. We prove the claim by construction. We used Algorithm 1 to generate all the patches in \mathcal{P} , by two calls of the procedure GENERATE(), passing as a parameter first a 4-cycle and then a 5-cycle, with the database of patches containing initially the closures of the two initial patches. The procedure uses Algorithm 2 as a subroutine to calculate a closure of a given patch.

If the insertion at lines 6, 11, or 16 of Algorithm 1 fails, it means that there is no Barnette graph containing the current patch P such that the element σ_i corresponds to a j -face for $j = 4, 5$, or 6 , respectively. If this is the case, the following lines are ignored until the next insertion. Similarly for the insertion at line 6 of Algorithm 2. \square

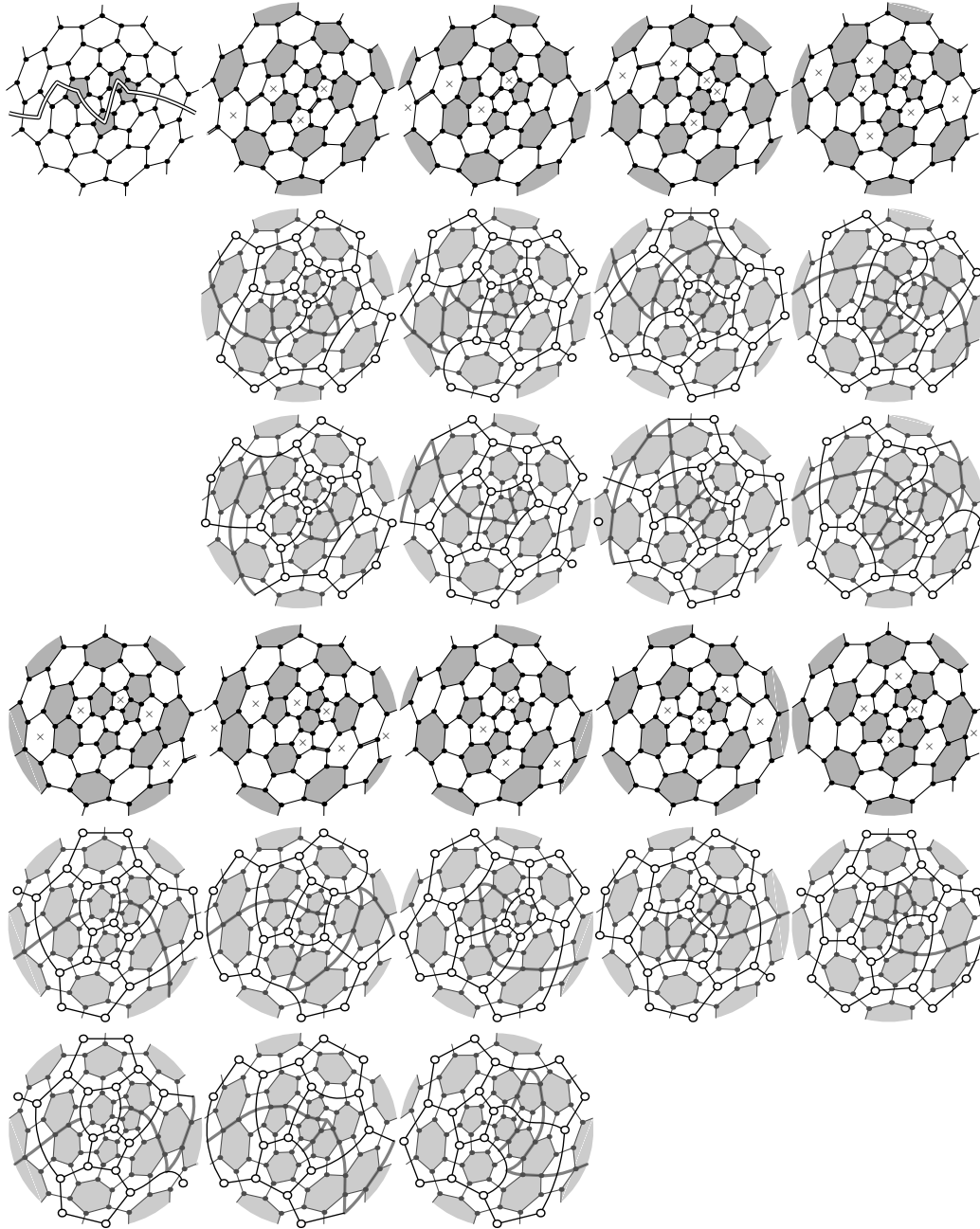


Figure 27: For a patch P with four pentagons and a fixed position of two segments leaving P , for each of the nine colorings of G_1 and G_2 the 2*-factor and (at most) two simple 2-factors obtained by the general procedure are depicted. The third drawing in the third column of the second row proves that for this position of the segments leaving P the patch P is parity-switching.

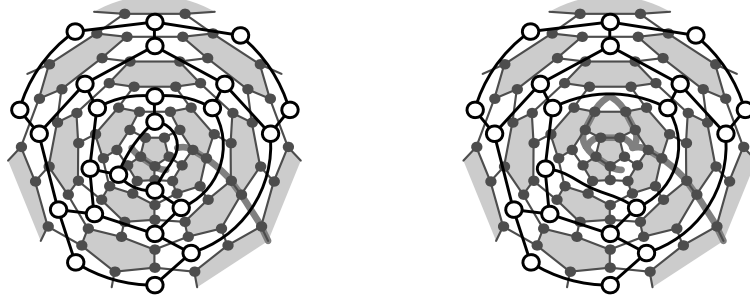


Figure 28: For a few patches with many small faces adjacent to each other, the first outcome of the general procedure is a 2-factor such that the corresponding partial residual graph is not strongly essentially 4-edge-connected (left). However, to obtain a strongly essentially 4-edge-connected graph, it suffices to eliminate a vertex incident to a short cycle (right).

Algorithm 1 Generation of all closed patches containing a given patch

```

1: procedure GENERATE(patch  $P$ )
2:   if  $\mu(P) \geq 7$  and the largest graph containing  $P$  has at most 316 vertices then return
3:   else
4:     let  $\sigma_i$  be a critical element of the boundary of  $P$ 
5:     if the path along  $\sigma_i$  is not adjacent to a 4-face then
6:        $P' \leftarrow P^{i \leftarrow 4}$ 
7:        $P'' \leftarrow \text{CLOSURE}(P')$ 
8:       if  $P''$  is not in the database of patches then
9:         Add  $P''$  to the database of patches
10:      GENERATE( $P'$ )
11:      $P' \leftarrow P^{i \leftarrow 5}$ 
12:      $P'' \leftarrow \text{CLOSURE}(P')$ 
13:     if  $P''$  is not in the database of patches then
14:       Add  $P''$  to the database of patches
15:     GENERATE( $P'$ )
16:      $P' \leftarrow P^{i \leftarrow 6}$ 
17:      $P'' \leftarrow \text{CLOSURE}(P')$ 
18:     if  $P'' \neq P'$  then
19:       GENERATE( $P'$ )

```

Algorithm 2 Computation of a closure of a given patch

```

1: procedure CLOSURE(patch  $P$ )
2:   if  $P$  is closed then
3:     return  $P$ 
4:   else
5:     let  $\sigma_i$  be a critical element of the boundary of  $P$ 
6:      $P' \leftarrow P^{i \leftarrow 6}$ 
7:     return CLOSURE( $P'$ )

```

The counts of patches generated in the proof of Theorem 5 are depicted in Table 1.

$f_4 \setminus f_5$	1	2	3	4	5	6	7
0	1	3	12	92	1202	8821	679
1	3	24	354	3279			
2	37	383					

Table 1: Numbers of essential patches in \mathcal{P} , given number of pentagons and quadrangles. Amongst the patches of curvature greater than 6, only patches contained in at least one graph on at least 318 vertices are counted.

4.4 Analyse of patches

The following statements were checked by computer:

Theorem 6 *There is no patch $P \in \mathcal{P}$ with $\mu(P) \geq 8$.*

This means that for every patch P of curvature at least 8 considered by the generating algorithm, the largest graph containing P has less than 318 vertices. See line 2 of Algorithm 1 and the remark after Corollary 2.

Theorem 7 *Every patch $P \in \mathcal{P}$ with $\mu(P) \leq 5$ is regular. Every patch $P \in \mathcal{P}$ with $\mu(P) \in \{6, 7\}$ is weakly regular, unless P contains a cap of a nanotube of type (p_1, p_2) with $(p_1, p_2) \in \{(4, 0), (5, 0), (4, 1), (5, 1), (3, 2), (4, 2), (3, 3), (4, 3)\}$.*

Theorem 7 guarantees the existence of a simple 2-factor such that the residual graph is cyclically 4-edge-connected. The only missing part is that we cannot be sure that this 2-factor is odd.

We do not need to check for regularity of patches of curvature 6 and 7, weak regularity suffices instead: If a Barnette graph contains an essential patch of curvature $\mu = 7$, then it only contains one. Therefore, we can chose the coloring of G_1 and G_2 such that no segment leaving P is active.

If a Barnette graph G contains an essential patch P of curvature $\mu = 6$, then P contains a cap of a nanotube, and we can choose the coloring of G_1 and G_2 such that the tubical part of G is traversed by at most one active segment (one if P is type 2, none if P is type 0).

If G is a nanotube of type (c_1, c_2) , $c_1 \geq c_2 \geq 0$, and the caps are (contained in) patches of type 0, then $3 \mid (c_1 - c_2)$, so we can write $(c_1, c_2) = (3a + b, b)$ for some integers $a, b \geq 0$. If we choose any of the three colorings of G_1 and G_2 such that no active segment traverses the tubical part of G , then the residual graph H is a nanotube of type $(a + b, a)$.

If G is a nanotube of type (c_1, c_2) , $c_1 \geq c_2 \geq 0$, and the caps are (contained in) patches of type 2, then $3 \nmid (c_1 - c_2)$, so we can write $(c_1, c_2) = (3a + b, b + 1)$ or $(c_1, c_2) = (3a + b + 1, b)$ for some integers $a, b \geq 0$, or $(c_1, c_2) = (3a + 2, 0)$ for some integer $a \geq 1$. If we choose a coloring of G_1 and G_2 such that one active segment traverses the tubical part of G , then the residual graph H is a nanotube of type $(a + b, a)$ (in the first two cases) or $(a + 1, 1)$ (in the third case).

This is the reason for excluding the aforementioned 8 types of nanotubes.

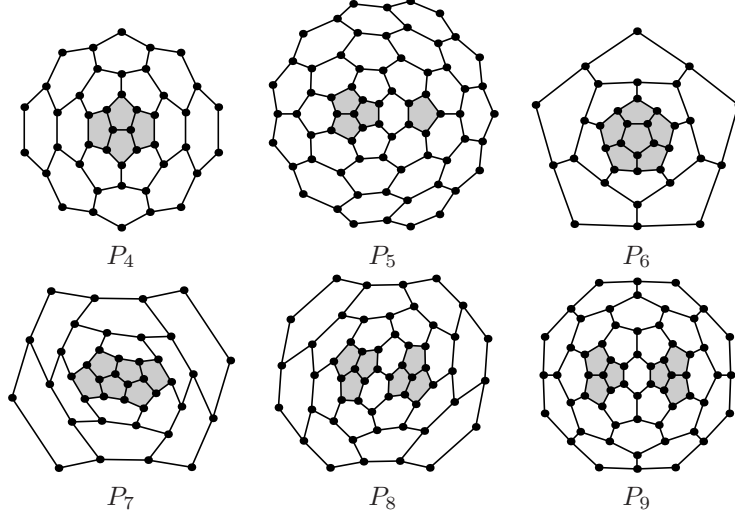


Figure 29: Patches with 4 and 6 pentagons for which it is not possible to increase or decrease the number of black pentagons by 2.

Theorem 8 *Let $P \in \mathcal{P}$ with $\mu(P) \leq 7$. Then P is parity-switching, unless P is one of the following exceptional patches:*

- *the patch P_1 of curvature 1 having one pentagon,*
- *the patch P_2 of curvature 2 containing two adjacent pentagons,*
- *the patch P_3 with three pentagons sharing a common vertex,*
- *two patches P_4 and P_5 with four pentagons (the type 0 patches obtained from P_3 by adding a pentagon at distance at most two),*
- *four patches P_6, P_7, P_8, P_9 with six pentagons, depicted in Figure 29.*

There is a combinatorial reason for the patches P_3 - P_9 not to be parity-switching: if three pentagons share a vertex, either one or two of them have to be black, so we do not have the freedom to change their colors independently.

As a consequence of Theorem 8, if a Barnette graph contains at least one parity-switching essential patch, we choose the coloring of G_1 and G_2 that allows to change the parity of the 2-factor, and, by regularity, we are done.

It remains to consider Barnette graphs (in fact, fullerene graphs) only containing patches P_1 - P_9 and verify that we can use parity-switching operations using pentagons from different patches.

If a fullerene graph contains P_6 , it is a nanotube of type $(5,0)$, and it is known to be Hamiltonian [12]. If a fullerene graph contains P_7 (P_8, P_9 , respectively), then it is a nanotube of type $(4,2)$ (of type $(6,2), (8,0)$) – the patch itself already contains a corresponding ring. Out of all the possible patches (caps) to close the other end of the tube, P_7 (P_8, P_9) is the

only one that is not parity-switching, as it was verified by a computer. However, if both caps of a nanotube are P_7 (P_8, P_9), then it has an even number of hexagons and exactly 6 black and 6 white pentagons, so by Lemma 7 the number of cycles in the 2-factor is odd.

It remains to consider fullerene graphs only having patches P_1, P_2, P_3, P_4 , and P_5 .

It was verified by computer that for each of the five patches, for each active segment leaving the patch, the Γ -path can be transformed into a pair of Γ -paths (interconnected inside the patch or not) – it is nothing else than applying a half of one of the operations O_1 and O_2 (or its inverse) inside the patch and the other half inside another.

In each of the patches this modification corresponds to increasing or decreasing the number of black pentagons by one. In most of the cases both are possible. More precisely, for each segment leaving P_1, P_2, P_4 or P_5 , out of the nine possible colorings, for three colorings the segment is inactive, for at least two colorings it is possible to increase the number of black pentagons by one, and for at least four colorings it is possible to decrease the number of black pentagons by one. It means that if two of these patches are consecutive along C^* , then there exists a coloring such that we can decrease the number of black pentagons in each of them by one.

On the other hand, for P_3 , it is possible to increase the number of black pentagons by one for four colorings and decrease it for two of them. Again, if there are two such patches consecutive along C^* , there exists a coloring such that we can increase the number of black pentagons in each of them by one.

It remains to consider fullerene graphs such that along C^* , the patches P_3 alternate with other types of patches among $\{P_1, P_2, P_4, P_5\}$. Since each P_3 contains three pentagons and there are twelve pentagons altogether, it is easy to see that the number of P_3 patches is either 2 or 3.

If there are two P_3 patches, the other two patches have six pentagons, and hence one of them is P_2 and the other one is either P_4 or P_5 . The patch with four pentagons has to be far from each of the P_3 patches, otherwise the graph would be too small (see Lemma 15). The patches P_4 and P_5 are both type 0. That is why we may omit the four-pentagon patch and search only for a cycle passing through the eight pentagons of the other three patches; we consider P_4 or P_5 as if no segment leaving it was active. As a consequence, we find two P_3 patches consecutive along C^* .

If there are three P_3 patches, the other three patches can only have one pentagon each. Moreover, the condition that for each segment joining a P_3 to a P_1 the two colorings allowing to decrease the number of black pentagons in P_3 correspond to the two colorings allowing to increase the number of black pentagons in the other patch implies that out of the nine colorings, there is one with no active segment joining a P_3 to a P_1 , there are four colorings with three active segments and three inactive segments alternating, and there are four colorings with all the six segments active. In all the cases there are three Γ -paths in G . (In the case of no active segments joining different patches, there is still a Γ -path joining different pentagons inside each P_3 .)

If we replace a vertex incident to three pentagons inside each P_3 by a triangle temporarily, then the graph will contain three pentagons and three triangles (and all the other faces will be hexagons). Moreover, in the coloring of G_1 and G_2 all the six small faces have the same color.

By the structural theorem of Alexandrov, such a graph can be isometrically embedded onto a surface of a (possibly degenerate) convex polyhedron, say P . The polyhedron P has

six vertices, and the cycle C^* is a Hamiltonian cycle in some triangulation of P .

The cycle C^* cuts the polyhedron P into two hexagons. In the two hexagons the angles at a fixed P_3 -vertex (center of a triangle) sum up to 180° , and hence they are both always convex (smaller than 180°). For the angles at the P_1 -vertices (centers of isolated pentagons), in at least one hexagon the angle is convex. Therefore, it is always possible to permute a P_1 patch with a P_3 patch to obtain a new cycle with two consecutive patches of the same type, which gives us a possibility to change the parity of the number of cycles. See Figure 30 for illustration.

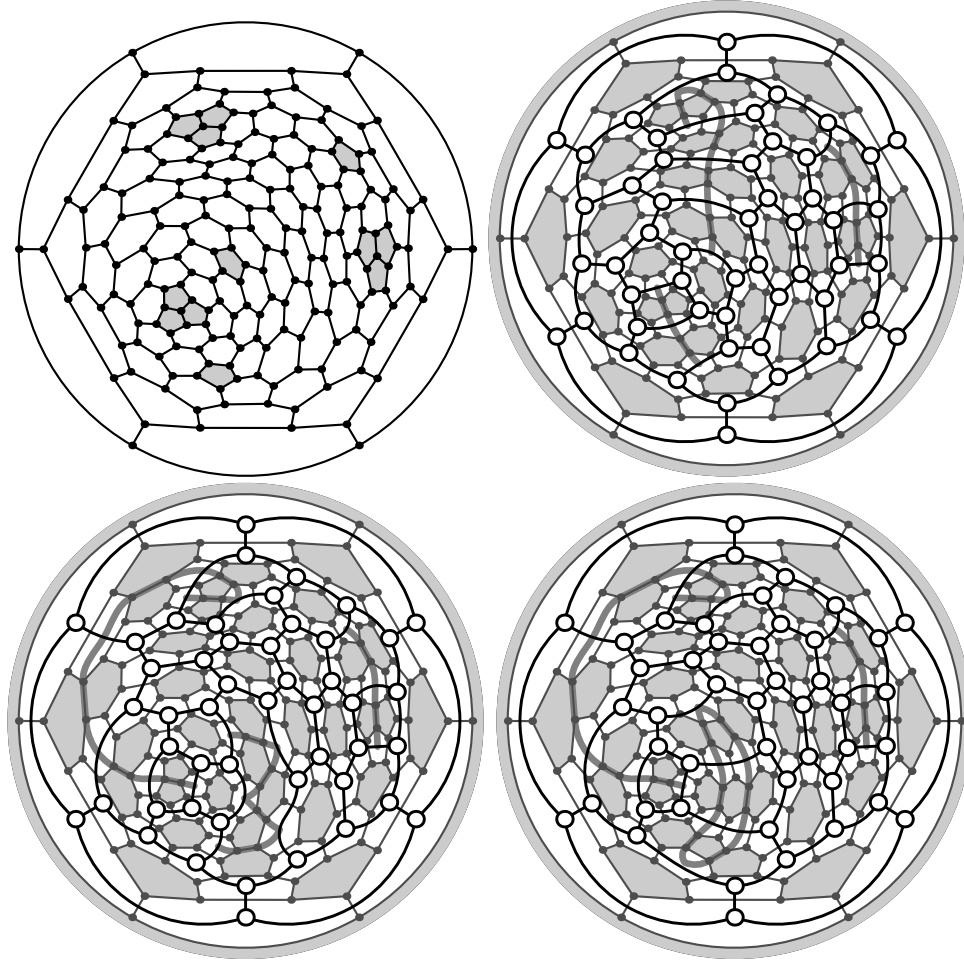


Figure 30: Top to bottom, left to right: An example of a fullerene graph on 198 vertices containing three patches P_3 and three patches P_1 . An even 2-factor with three Γ -paths without a possibility to apply O_1 or O_2 . Another even 2-factor obtained by switching the order of the patches. An odd 2-factor after applying O_1 .

5 Concluding remarks

Similar technique could be used to prove Hamiltonicity of related graph classes: planar cubic graphs with only a few faces of size larger than six; projective-planar graphs with faces of size at most six (except, of course, for the Petersen graph), etc.

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