POINTWISE SECOND-ORDER NECESSARY CONDITIONS FOR STOCHASTIC OPTIMAL CONTROLS, PART I: THE CASE OF CONVEX CONTROL CONSTRAINT[*]

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Abstract. This paper is the first part of our series work to establish pointwise second-order necessary conditions for stochastic optimal controls. In this part, both drift and diffusion terms may contain the control variable but the control region is assumed to be convex. Under some assumptions in terms of Malliavin calculus, we establish the desired necessary condition for stochastic singular optimal controls in the classical sense.

Key words. Stochastic optimal control, Malliavin calculus, pointwise second-order necessary condition, variational equation, adjoint equation.

AMS subject classifications. Primary 93E20; Secondary 60H07, 60H10.

1. Introduction. Let $T > 0$ and $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a complete filtered probability space (satisfying the usual conditions), on which a 1-dimensional standard Wiener process $W(\cdot)$ is defined such that $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ is the natural filtration generated by $W(\cdot)$ (augmented by all of the P-null sets).

In this paper, we shall consider the following controlled stochastic differential equation

$$(1.1) \qquad \begin{cases} dx(t) = b(t, x(t), u(t))dt + \sigma(t, x(t), u(t))dW(t), & t \in [0, T], \\ x(0) = x_0, \end{cases}$$

with a cost functional

$$(1.2) \qquad J(u(\cdot)) = \mathbb{E}\Big[\int_0^T f(t, x(t), u(t))dt + h(x(T)) \Big].$$

Here $u(\cdot)$ is the control variable valued in a set $U \subset \mathbb{R}^m$ (for some $m \in \mathbb{N}$), $x(\cdot)$ is the state variable valued in $\mathbb{R}^n$ (for some $n \in \mathbb{N}$), and $b, \sigma : [0, T] \times \mathbb{R}^n \times U \times \Omega \to \mathbb{R}^n$, $f : [0, T] \times \mathbb{R}^n \times U \times \Omega \to \mathbb{R}$ and $h : \mathbb{R}^n \times \Omega \to \mathbb{R}$ are given functions (satisfying some conditions to be given later). As usual, when the context is clear, we omit the $\omega(\in \Omega)$ argument in the defined functions.

Denote by $\mathcal{B}(\mathcal{X})$ the Borel σ-field of a metric space $\mathcal{X}$, and by $\mathcal{U}_{ad}$ the set of $\mathcal{B}([0, T]) \otimes \mathcal{F}$-measurable and $\mathbb{F}$-adapted stochastic processes valued in U. Any $u(\cdot) \in \mathcal{U}_{ad}$ is called an admissible control. The stochastic optimal control problem considered in this paper is to find a control $\bar{u}(\cdot) \in \mathcal{U}_{ad}$ such that

$$(1.3) \qquad J(\bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}_{ad}} J(u(\cdot)).$$

Any $\bar{u}(\cdot) \in \mathcal{U}_{ad}$ satisfying (1.3) is called an optimal control. The corresponding state $\bar{x}(\cdot)$ (to (1.1)) is called an optimal state, and $(\bar{x}(\cdot), \bar{u}(\cdot))$ is called an optimal pair.

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In optimal control theory, one of the central topics is to establish the first-order necessary condition for optimal controls. We refer to [15] for an early study on the first-order necessary condition for stochastic optimal controls. After that, many authors contributed on this topic, see [2, 3, 12] and references cited therein. Compared to the deterministic setting, new phenomenon and difficulties appear when the diffusion term of the stochastic control system contains the control variable and the control region is nonconvex. The corresponding first-order necessary condition for this general case was established in [18].

For some optimal controls, it may happen that the first-order necessary conditions turn out to be trivial. For deterministic control systems, there are two types of such optimal controls. One of them, called the singular optimal control in the classical sense, is the optimal control for which the gradient and the Hessian of the corresponding Hamiltonian with respect to the control variable vanish/degenerate. The other one, called the singular optimal control in the sense of Pontryagin-type maximum principle, is the optimal control for which the corresponding Hamiltonian is equal to a constant in the control region. When an optimal control is singular, the first-order necessary condition cannot provide enough information for the theoretical analysis and numerical computing, and therefore one needs to study the second-order necessary conditions. In the deterministic setting, one can find many references in this direction (See [1, 7, 9, 10, 11, 13, 14, 16] and references cited therein).

Compared to the deterministic control systems, there are only two papers ([4, 19]) addressed to the second-order necessary condition for stochastic optimal controls. In [19], a pointwise second-order maximum principle for stochastic singular optimal controls in the sense of Pontryagin-type maximum principle was established for the case that the diffusion term $\sigma(t, x, u)$ is independent of the control u; while in [4], an integral-type second-order necessary condition for stochastic optimal controls was derived under the assumption that the control region U is convex.

The main purpose of this paper is to establish a *pointwise* second-order necessary condition for stochastic optimal controls. In this work, both drift and diffusion terms, i.e., $b(t, x, u)$ and $\sigma(t, x, u)$, may contain the control variable u, and we assume that the control region U is convex. The key difference between [4] and our work is that we consider here the *pointwise* second-order necessary condition, which is easier to be verified in practical applications. We remark that, quite different from the deterministic setting, there exist some essential difficulties to derive the pointwise second-order necessary condition from an integral-type one when the diffusion term of the control system contains the control variable, *even for the case of convex control constraint* (See the first 4 paragraphs of Subsection 3.2 for a detailed explanation). We overcome these difficulties by means of some technique from the Malliavin calculus. The method developed in this work can be adopted to establish a pointwise second-order necessary condition for stochastic optimal controls for the general case when the control region is nonconvex but the analysis is much more complicated, and therefore we shall give the details in another paper [21].

The rest of this paper is organized as follows. In Section 2, we list some notations, spaces and preliminary results from Malliavin calculus. In Section 3, we introduce the main results of this paper and give some examples. Finally, in Section 4 we give the proofs of the main results.

2. Some preliminaries. In this section, we present some preliminaries.

2.1. Some notations and spaces. We introduce some notations and spaces which will be used in the sequel.

Denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ respectively the inner product and norm in $\mathbb{R}^n$ or $\mathbb{R}^m$, which can be identified from the contexts. Let $\mathbb{R}^{n \times m}$ be the space of all $n \times m$-matrices. For any $A \in \mathbb{R}^{n \times m}$, denote by $A^\top$ the transpose of A and by $|A| = \sqrt{tr\{AA^\top\}}$ the norm of A. Also, write $\mathbf{S}^n := \{ A \in \mathbb{R}^{n \times n} \mid A^\top = A \}$.

Let $\varphi : [0, T] \times \mathbb{R}^n \times U \times \Omega \to \mathbb{R}^d$ be a given function. For a.e. $(t, \omega) \in [0, T] \times \Omega$, we denote by $\varphi_x(t, x, u)$, $\varphi_u(t, x, u)$ the first order partial derivatives of φ with respect to x and u at (t, x, u, ω), by $\varphi_{(x,u)^2}(t, x, u)$ the Hessian of φ with respect to (x, u) at (t, x, u, ω), and by $\varphi_{xx}(t, x, u)$, $\varphi_{xu}(t, x, u)$, $\varphi_{uu}(t, x, u)$ the second order partial derivatives of φ with respect to x and u at (t, x, u, ω).

For any $\alpha, \beta \in [1, +\infty)$ and $t \in [0, T]$, we denote by $L_{\mathcal{F}_t}^\beta(\Omega; \mathbb{R}^n)$ the space of $\mathbb{R}^n$-valued, $\mathcal{F}_t$ measurable random variables ξ such that $\mathbb{E}\, |\xi|^\beta < +\infty$; by $L^\beta([0, T] \times \Omega; \mathbb{R}^n)$ the space of $\mathbb{R}^n$-valued, $\mathcal{B}([0, T]) \otimes \mathcal{F}$-measurable processes φ such that $\|\varphi\|_\beta := \left[\mathbb{E} \int_0^T |\varphi(t)|^\beta dt \right]^{\frac{1}{\beta}} < +\infty$; by $L_\mathbb{F}^\beta(\Omega; L^\alpha(0, T; \mathbb{R}^n))$ the space of $\mathbb{R}^n$-valued, $\mathcal{B}([0, T]) \otimes \mathcal{F}$-measurable, $\mathbb{F}$-adapted processes φ such that $\|\varphi\|_{\alpha,\beta} := \left[\mathbb{E} \left(\int_0^T |\varphi(t)|^\alpha dt \right)^{\frac{\beta}{\alpha}} \right]^{\frac{1}{\beta}} < +\infty$; by $L_\mathbb{F}^\beta(\Omega; C([0, T]; \mathbb{R}^n))$ the space of $\mathbb{R}^n$-valued, $\mathcal{B}([0, T]) \otimes \mathcal{F}$-measurable, and $\mathbb{F}$-adapted continuous processes φ such that $\|\varphi\|_{\infty,\beta} := \left[\mathbb{E} \left(\sup_{t \in [0, T]} |\varphi(t)|^\beta \right) \right]^{\frac{1}{\beta}} < +\infty$, by $L^\infty([0, T] \times \Omega; \mathbb{R}^n)$ the space of $\mathbb{R}^n$-valued, $\mathcal{B}([0, T]) \otimes \mathcal{F}$-measurable processes φ such that $\|\varphi\|_\infty := \mathrm{ess} \sup_{(t,\omega) \in [0, T] \times \Omega} |\varphi(t, \omega)| < +\infty$; and by $L^\beta(0, T; L_\mathbb{F}^\beta([0, T] \times \Omega; \mathbb{R}^n))$ the $\mathbb{R}^n$-valued, $\mathcal{B}([0, T]) \otimes \mathcal{B}([0, T]) \otimes \mathcal{F}$ measurable functions φ such that for any $t \in [0, T]$, $\varphi(\cdot, t)$ is $\mathbb{F}$-adapted and $\|\varphi\|_\beta := \left[\mathbb{E} \int_0^T \int_0^T |\varphi(s, t)|^\beta ds dt \right]^{\frac{1}{\beta}} < +\infty$.

2.2. Some concepts and results from Malliavin calculus. In this subsection, we recall some concepts and results from Malliavin calculus (See [17] for a detailed discussion on this topic).

Denote by $C_b^\infty(\mathbb{R}^d; \mathbb{R}^n)$ the set of C^∞-smooth functions with bounded partial derivatives. For any $h \in L^2(0, T)$, write $W(h) = \int_0^T h(t) dW(t)$. Define

$$
(2.1) \qquad \mathcal{S} := \left\{ \zeta = \varphi(W(h_1),\ W(h_2),\ \cdots,\ W(h_d)) \ \middle|\ \varphi \in C_b^\infty(\mathbb{R}^d; \mathbb{R}^n),\ d \in \mathbb{N}, \right.
$$
$$
\left. h_1, h_2, \cdots, h_d \in L^2(0, T) \right\}.
$$

Clearly, $\mathcal{S}$ is a linear subspace of $L_{\mathcal{F}_T}^2(\Omega; \mathbb{R}^n)$. For any $\zeta \in \mathcal{S}$ (in the form of that in (2.1)), its Malliavin derivative is defined as follows:

$$
\mathcal{D}_s \zeta := \sum_{i=1}^d h_i(s) \frac{\partial \varphi}{\partial x_i}(W(h_1),\ W(h_2),\ \cdots,\ W(h_d)), \qquad s \in [0, T].
$$

Write

$$
|||\zeta|||_2 := \left[\mathbb{E}\, |\zeta|^2 + \mathbb{E} \int_0^T |\mathcal{D}_s \zeta|^2 ds \right]^{\frac{1}{2}}.
$$

Obviously, $||| \cdot |||_2$ is a norm on $\mathcal{S}$. It is shown in [17] that the operator $\mathcal{D}$ has a closed extension to the space $\mathbb{D}^{1,2}(\mathbb{R}^n)$, the completion of $\mathcal{S}$ with respect to the norm $||| \cdot |||_2$. When $\zeta \in \mathbb{D}^{1,2}(\mathbb{R}^n)$, the following Clark-Ocone representation formula holds:

$$
(2.2) \qquad \zeta = \mathbb{E}\, \zeta + \int_0^T \mathbb{E}\, (\mathcal{D}_s \zeta \mid \mathcal{F}_s) dW(s).
$$

Furthermore, if ζ is $\mathcal{F}_t$-measurable, then $\mathcal{D}_s \zeta = 0$ for any $s \in (t, T]$.

Define $\mathbb{L}^{1,2}(\mathbb{R}^n)$ to be the space of processes $\varphi \in L^2([0, T] \times \Omega; \mathbb{R}^n)$ such that

(i) For *a.e.* $t \in [0, T]$, $\varphi(t, \cdot) \in \mathbb{D}^{1,2}(\mathbb{R}^n)$;

(ii) The function $\mathcal{D}_s\varphi(t, \omega) : [0, T] \times [0, T] \times \Omega \to \mathbb{R}^n$ admits a measurable version; and

(iii) $\||\varphi\||_{1,2} := \left[\mathbb{E} \int_0^T |\varphi(t)|^2 dt + \mathbb{E} \int_0^T \int_0^T |\mathcal{D}_s\varphi(t)|^2 ds dt \right]^{\frac{1}{2}} < +\infty.$

Denote by $\mathbb{L}_{\mathbb{F}}^{1,2}(\mathbb{R}^n)$ the set of all adapted processes in $\mathbb{L}^{1,2}(\mathbb{R}^n)$.

In addition, write

$$\mathbb{L}_{2+}^{1,2}(\mathbb{R}^n) := \left\{ \varphi(\cdot) \in \mathbb{L}^{1,2}(\mathbb{R}^n) \mid \exists \, \mathcal{D}^+\varphi(\cdot) \in L^2([0,T] \times \Omega; \mathbb{R}^n) \text{ such that} \right.$$

$$f_\varepsilon(s) := \sup_{s < t < (s+\varepsilon) \wedge T} \mathbb{E} \left| \mathcal{D}_s\varphi(t) - \mathcal{D}^+\varphi(s) \right|^2 < \infty, \quad a.e. \ s \in [0, T],$$

$$\left. f_\varepsilon(\cdot) \text{ is measurable on } [0, T] \text{ for any } \varepsilon > 0, \text{ and } \lim_{\varepsilon \to 0^+} \int_0^T f_\varepsilon(s) ds = 0 \right\};$$

$$\mathbb{L}_{2-}^{1,2}(\mathbb{R}^n) := \left\{ \varphi(\cdot) \in \mathbb{L}^{1,2}(\mathbb{R}^n) \mid \exists \, \mathcal{D}^-\varphi(\cdot) \in L^2([0,T] \times \Omega; \mathbb{R}^n) \text{ such that} \right.$$

$$g_\varepsilon(s) := \sup_{(s-\varepsilon) \vee 0 < t < s} \mathbb{E} \left| \mathcal{D}_s\varphi(t) - \mathcal{D}^-\varphi(s) \right|^2 < \infty, \quad a.e. \ s \in [0, T],$$

$$\left. g_\varepsilon(\cdot) \text{ is measurable on } [0, T] \text{ for any } \varepsilon > 0, \text{ and } \lim_{\varepsilon \to 0^+} \int_0^T g_\varepsilon(s) ds = 0 \right\}.$$

Denote

$$\mathbb{L}_2^{1,2}(\mathbb{R}^n) = \mathbb{L}_{2+}^{1,2}(\mathbb{R}^n) \cap \mathbb{L}_{2-}^{1,2}(\mathbb{R}^n).$$

For any $\varphi(\cdot) \in \mathbb{L}_2^{1,2}(\mathbb{R}^n)$, denote $\nabla \varphi(\cdot) = \mathcal{D}^+\varphi(\cdot) + \mathcal{D}^-\varphi(\cdot)$.

When φ is adapted, $\mathcal{D}_s\varphi(t) = 0$ for any $t < s$. In this case, $\mathcal{D}^-\varphi(\cdot) = 0$, and $\nabla\varphi(\cdot) = \mathcal{D}^+\varphi(\cdot)$. Denote by $\mathbb{L}_{2,\mathbb{F}}^{1,2}(\mathbb{R}^n)$ the set of all adapted processes in $\mathbb{L}_2^{1,2}(\mathbb{R}^n)$.

Roughly speaking, an element $\varphi \in \mathbb{L}_2^{1,2}(\mathbb{R}^n)$ is a stochastic process whose Malliavin derivative has suitable continuity on some neighbourhood of $\{(t, t) \mid t \in [0, T]\}$. Examples of such process can be found in [17]. Especially, if $(s, t) \mapsto \mathcal{D}_s\varphi(t)$ is continuous from $V_\delta := \{(s, t) \mid |s - t| < \delta, \ s, t \in [0, T]\}$ (for some $\delta > 0$) to $L_{\mathcal{F}_T}^2(\Omega; \mathbb{R}^n)$, then $\varphi \in \mathbb{L}_2^{1,2}(\mathbb{R}^n)$ and, $\mathcal{D}^+\varphi(t) = \mathcal{D}^-\varphi(t) = \mathcal{D}_t\varphi(t)$.

To end this section, we show the following technical result which will be use in the sequel.

LEMMA 2.1. *Let* $\varphi(\cdot) \in \mathbb{L}_{2,\mathbb{F}}^{1,2}(\mathbb{R}^n)$. *Then, there exists a sequence* $\{\theta_n\}_{n=1}^\infty$ *of positive numbers such that* $\theta_n \to 0^+$ *as* $n \to \infty$ *and*

$$(2.3) \qquad \lim_{n \to \infty} \frac{1}{\theta_n^2} \int_\tau^{\tau+\theta_n} \int_\tau^t \mathbb{E} \left| D_s\varphi(t) - \nabla\varphi(s) \right|^2 ds dt = 0, \quad a.e. \ \tau \in [0, T].$$

Proof. For any $\tau, \theta \in [0, \infty)$, we take the convention that

$$\sup_{t \in [\tau, \tau+\theta] \cap [0,T]} \mathbb{E} \left| D_\tau\varphi(t) - \nabla\varphi(\tau) \right|^2 = 0$$

whenever $[\tau, \tau + \theta] \cap [0, T] = \emptyset$. From the definition of $\mathbb{L}_{2,\mathbb{F}}^{1,2}(\mathbb{R}^m)$, it follows that

$$\lim_{\theta \to 0^+} \frac{1}{\theta^2} \int_0^T \int_\tau^{\tau+\theta} \int_\tau^t \mathbb{E} \left| D_s\varphi(t) - \nabla\varphi(s) \right|^2 ds dt d\tau$$

$$= \lim_{\theta \to 0^+} \frac{1}{\theta^2} \int_0^T \int_\tau^{\tau+\theta} \int_s^{\tau+\theta} \mathbb{E}\big|D_s\varphi(t) - \nabla\varphi(s)\big|^2 dt ds d\tau$$

$$\leq \lim_{\theta \to 0^+} \frac{1}{\theta} \int_0^T \int_\tau^{\tau+\theta} \Big[\sup_{t \in [s,s+\theta] \cap [0,T]} \mathbb{E}\big|D_s\varphi(t) - \nabla\varphi(s)\big|^2 \Big] ds d\tau$$

$$\leq \lim_{\theta \to 0^+} \frac{1}{\theta} \int_0^T \int_0^\theta \Big[\sup_{t \in [s+\tau,s+\tau+\theta] \cap [0,T]} \mathbb{E}\big|D_{s+\tau}\varphi(t) - \nabla\varphi(s+\tau)\big|^2 \Big] ds d\tau$$

$$\leq \lim_{\theta \to 0^+} \frac{1}{\theta} \int_0^\theta \int_0^T \Big[\sup_{t \in [s+\tau,s+\tau+\theta] \cap [0,T]} \mathbb{E}\big|D_{s+\tau}\varphi(t) - \nabla\varphi(s+\tau)\big|^2 \Big] d\tau ds$$

$$\leq \lim_{\theta \to 0^+} \frac{1}{\theta} \int_0^\theta \int_s^T \Big[\sup_{t \in [\tau,\tau+\theta] \cap [0,T]} \mathbb{E}\big|D_\tau\varphi(t) - \nabla\varphi(\tau)\big|^2 \Big] d\tau ds$$

$$\leq \lim_{\theta \to 0^+} \frac{1}{\theta} \int_0^\theta \int_0^T \Big[\sup_{t \in [\tau,\tau+\theta] \cap [0,T]} \mathbb{E}\big|D_\tau\varphi(t) - \nabla\varphi(\tau)\big|^2 \Big] d\tau ds$$

$$\leq \lim_{\theta \to 0^+} \int_0^T \Big[\sup_{t \in [\tau,\tau+\theta] \cap [0,T]} \mathbb{E}\big|D_\tau\varphi(t) - \nabla\varphi(\tau)\big|^2 \Big] d\tau$$

$$= 0,$$

which implies (2.3). $\square$

3. Second-order necessary conditions. In this section, we shall present several second-order necessary conditions for stochastic optimal controls.

To begin with, we assume that

(C1) *The control region U is nonempty, bounded, and convex.*

(C2) *The functions b, σ, f, and h satisfy the following:*

(i) *For any $(x,u) \in \mathbb{R}^n \times U$, the stochastic processes $b(\cdot,x,u) : [0,T] \times \Omega \to \mathbb{R}^n$ and $\sigma(\cdot,x,u) : [0,T] \times \Omega \to \mathbb{R}^n$ are $\mathcal{B}([0,T]) \otimes \mathcal{F}$-measurable and $\mathbb{F}$-adapted. For a.e. $(t,\omega) \in [0,T] \times \Omega$, the functions $b(t,\cdot,\cdot) : \mathbb{R}^n \times U \to \mathbb{R}^n$ and $\sigma(t,\cdot,\cdot) : \mathbb{R}^n \times U \to \mathbb{R}^n$ are continuously differentiable up to order 2, and all of their partial derivatives are uniformly bounded (with respect to $(t,\omega) \in [0,T] \times \Omega$). There exists a constant $L > 0$ such that for a.e. $(t,\omega) \in [0,T] \times \Omega$ and for any x, $\tilde{x} \in \mathbb{R}^n$ and u, $\tilde{u} \in U$,*

$$\begin{cases} |b(t,0,u)| + |\sigma(t,0,u)| \leq L, \\ |b_{(x,u)^2}(t,x,u) - b_{(x,u)^2}(t,\tilde{x},\tilde{u})| \leq L(|x-\tilde{x}| + |u-\tilde{u}|), \\ |\sigma_{(x,u)^2}(t,x,u) - \sigma_{(x,u)^2}(t,\tilde{x},\tilde{u})| \leq L(|x-\tilde{x}| + |u-\tilde{u}|). \end{cases}$$

(ii) *For any $(x,u) \in \mathbb{R}^n \times U$, the stochastic process $f(\cdot,x,u) : [0,T] \times \Omega \to \mathbb{R}$ is $\mathcal{B}([0,T]) \otimes \mathcal{F}$-measurable and $\mathbb{F}$-adapted, and the random variable $h(x)$ is $\mathcal{F}_T$-measurable. For a.e. $(t,\omega) \in [0,T] \times \Omega$, the functions $f(t,\cdot,\cdot) : \mathbb{R}^n \times U \to \mathbb{R}$ and $h(\cdot) : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable up to order 2, and for any x, $\tilde{x} \in \mathbb{R}^n$ and u, $\tilde{u} \in U$,*

$$\begin{cases} |f(t,x,u)| \leq L(1 + |x|^2 + |u|^2), \\ |f_x(t,x,u)| + |f_u(t,x,u)| \leq L(1 + |x| + |u|), \\ |f_{xx}(t,x,u)| + |f_{xu}(t,x,u)| + |f_{uu}(t,x,u)| \leq L, \\ |f_{(x,u)^2}(t,x,u) - f_{(x,u)^2}(t,\tilde{x},\tilde{u})| \leq L(|x-\tilde{x}| + |u-\tilde{u}|), \\ |h(x)| \leq L(1 + |x|^2), \ |h_x(x)| \leq L(1 + |x|), \\ |h_{xx}(x)| \leq L, \ |h_{xx}(x) - h_{xx}(\tilde{x})| \leq L|x-\tilde{x}|. \end{cases}$$

When the condition $(C2)$ is satisfied, the state $x(\cdot)$ (of (1.1)) is uniquely defined by any given initial datum $x_0 \in \mathbb{R}^n$ and admissible control $u(\cdot) \in \mathcal{U}_{ad}$, and the cost functional (1.2) is well-defined on $\mathcal{U}_{ad}$. In what follows, C represents a generic constant, depending on T and L, but independent of any other parameter, which can be different from line to line.

3.1. Integral-type second-order conditions. Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be an optimal pair, and $u(\cdot) \in \mathcal{U}_{ad}$ be any given admissible control. Let $\varepsilon \in (0, 1)$, and write

$$(3.1) \qquad\qquad v(\cdot) = u(\cdot) - \bar{u}(\cdot), \qquad u^\varepsilon(\cdot) = \bar{u}(\cdot) + \varepsilon v(\cdot).$$

Since U is convex, $u^\varepsilon(\cdot) \in \mathcal{U}_{ad}$. Denote by $x^\varepsilon(\cdot)$ the state of (1.1) with respect to the control $u^\varepsilon(\cdot)$, and put $\delta x(\cdot) = x^\varepsilon(\cdot) - \bar{x}(\cdot)$. For $\varphi = b, \sigma, f$, denote

$$\begin{aligned}
\varphi_x(t) &= \varphi_x(t, \bar{x}(t), \bar{u}(t)), & \varphi_u(t) &= \varphi_u(t, \bar{x}(t), \bar{u}(t)), \\
\varphi_{xx}(t) &= \varphi_{xx}(t, \bar{x}(t), \bar{u}(t)), & \varphi_{xu}(t) &= \varphi_{xu}(t, \bar{x}(t), \bar{u}(t)), \\
\varphi_{uu}(t) &= \varphi_{uu}(t, \bar{x}(t), \bar{u}(t)).
\end{aligned}$$

First, similar to [4], we introduce the following two variational equations:

$$(3.2) \quad \begin{cases} dy_1(t) = \Big[b_x(t) y_1(t) + b_u(t) v(t) \Big] dt \\ \qquad\qquad + \Big[\sigma_x(t) y_1(t) + \sigma_u(t) v(t) \Big] dW(t), \qquad t \in [0, T], \\ y_1(0) = 0 \end{cases}$$

and

$$(3.3) \quad \begin{cases} dy_2(t) = \Big[b_x(t) y_2(t) + y_1(t)^\top b_{xx}(t) y_1(t) + 2v(t)^\top b_{xu}(t) y_1(t) \\ \qquad\qquad + v(t)^\top b_{uu}(t) v(t) \Big] dt + \Big[\sigma_x(t) y_2(t) + y_1(t)^\top \sigma_{xx}(t) y_1(t) \\ \qquad\qquad + 2v(t)^\top \sigma_{xu}(t) y_1(t) + v(t)^\top \sigma_{uu}(t) v(t) \Big] dW(t), \qquad t \in [0, T], \\ y_2(0) = 0. \end{cases}$$

By (3.2)–(3.3) and similar to [4, Lemmas 3.5 and 3.11], one has the following estimates.

PROPOSITION 3.1. *Let (C2) hold. Then, for any $\kappa \geq 2$,*

$$\|y_1\|_{\infty, \kappa}^\kappa \leq C, \quad \|y_2\|_{\infty, \kappa}^\kappa \leq C, \quad \|\delta x\|_{\infty, \kappa}^\kappa \leq C\varepsilon^\kappa,$$

$$\|\delta x - \varepsilon y_1\|_{\infty, \kappa}^\kappa \leq C\varepsilon^{2\kappa}, \quad \Big\| \delta x - \varepsilon y_1 - \frac{\varepsilon^2}{2} y_2 \Big\|_{\infty, \kappa}^\kappa \leq C\varepsilon^{3\kappa}.$$

Proof. The proof is very close to that of [4, Lemmas 3.5 and 3.11], and therefore, we omit the details. $\square$

Next, define the Hamiltonian

$$(3.4) \qquad H(t, x, u, y_1, z_1) := \langle y_1, b(t, x, u) \rangle + \langle z_1, \sigma(t, x, u) \rangle - f(t, x, u),$$

$(t, x, u, y_1, z_1) \in [0, T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R}^n$. We introduce respectively the following two adjoint equations for (3.2)–(3.3):

$$(3.5) \quad \begin{cases} dP_1(t) = -\Big[b_x(t)^\top P_1(t) \\ \qquad\qquad + \sigma_x(t)^\top Q_1(t) - f_x(t) \Big] dt + Q_1(t) dW(t), \ t \in [0, T], \\ P_1(T) = -h_x(\bar{x}(T)) \end{cases}$$

and

$$(3.6) \quad \begin{cases} dP_2(t) = -\Big[b_x(t)^\top P_2(t) + P_2(t)b_x(t) + \sigma_x(t)^\top P_2(t)\sigma_x(t) + \sigma_x(t)^\top Q_2(t) \\ \qquad\qquad +Q_2(t)\sigma_x(t) + H_{xx}(t)\Big]dt + Q_2(t)dW(t), \ t \in [0,T], \\ P_2(T) = -h_{xx}(\bar{x}(T)), \end{cases}$$

where $H_{xx}(t) = H_{xx}(t, \bar{x}(t), \bar{u}(t), P_1(t), Q_1(t))$.

From [8], it is easy to check that, for any $\beta \geq 1$, the equation (3.5) admits a unique strong solution $(P_1(\cdot), Q_1(\cdot)) \in L^\beta_\mathbb{F}(\Omega; C([0,T]; \mathbb{R}^n)) \times L^\beta_\mathbb{F}(\Omega; L^2(0, T; \mathbb{R}^n))$, and (3.6) admits a unique strong solution $(P_2(\cdot), Q_2(\cdot)) \in L^\beta_\mathbb{F}(\Omega; C([0,T]; \mathbf{S}^n)) \times L^\beta_\mathbb{F}(\Omega; L^2(0, T; \mathbf{S}^n))$.

Also, we define

$$(3.7) \qquad \mathbb{S}(t, x, u, y_1, z_1, y_2, z_2) := H_{xu}(t, x, u, y_1, z_1) + b_u(t, x, u)^\top y_2$$
$$+\sigma_u(t, x, u)^\top z_2 + \sigma_u(t, x, u)^\top y_2 \sigma_x(t, x, u),$$

$(t, x, u, y_1, z_1, y_2, z_2) \in [0, T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbf{S}^n \times \mathbf{S}^n$, and denote

$$(3.8) \qquad \mathbb{S}(t) = \mathbb{S}(t, \bar{x}(t), \bar{u}(t), P_1(t), Q_1(t), P_2(t), Q_2(t)), \quad t \in [0, T].$$

We have the following result.

PROPOSITION 3.2. *Let (C1)–(C2) hold. Then, the following variational equality holds for any $u(\cdot) \in \mathcal{U}_{ad}$:*

$$(3.9) \quad J(u^\varepsilon(\cdot)) - J(\bar{u}(\cdot))$$
$$= -\mathbb{E}\int_0^T \Big[\varepsilon\,\langle H_u(t), v(t)\rangle + \frac{\varepsilon^2}{2}\,\langle H_{uu}(t)v(t), v(t)\rangle$$
$$+\frac{\varepsilon^2}{2}\,\langle P_2(t)\sigma_u(t)v(t), \sigma_u(t)v(t)\rangle + \varepsilon^2\,\langle\mathbb{S}(t)y_1(t), v(t)\rangle\Big]dt + o(\varepsilon^2), \ (\varepsilon \to 0^+),$$

where $H_u(t) = H_u(t, \bar{x}(t), \bar{u}(t), P_1(t), Q_1(t))$, $H_{uu}(t) = H_{uu}(t, \bar{x}(t), \bar{u}(t), P_1(t), Q_1(t))$.

Proof. By (3.1), using Taylor's formula and Proposition 3.1, similar to [4, Subsection 3.2], we have

$$(3.10) \quad J(u^\varepsilon) - J(\bar{u})$$
$$= \mathbb{E}\int_0^T \Big[\langle f_x(t), \delta x(t)\rangle + \varepsilon\,\langle f_u(t), v(t)\rangle + \frac{1}{2}\,\langle f_{xx}(t)\delta x(t), \delta x(t)\rangle$$
$$+\varepsilon\,\langle f_{xu}(t)\delta x(t), v(t)\rangle + \frac{\varepsilon^2}{2}\,\langle f_{uu}(t)v(t), v(t)\rangle\Big]dt$$
$$+\mathbb{E}\Big[\langle h_x(\bar{x}(T)), \delta x(T)\rangle + \frac{1}{2}\,\langle h_{xx}(\bar{x}(T))\delta x(T), \delta x(T)\rangle\Big] + o(\varepsilon^2) \quad (\varepsilon \to 0^+)$$
$$= \mathbb{E}\int_0^T \Big[\varepsilon\,\langle f_x(t), y_1(t)\rangle + \frac{\varepsilon^2}{2}\,\langle f_x(t), y_2(t)\rangle + \varepsilon\,\langle f_u(t), v(t)\rangle$$
$$+\frac{\varepsilon^2}{2}\Big(\langle f_{xx}(t)y_1(t), y_1(t)\rangle + 2\,\langle f_{xu}(t)y_1(t), v(t)\rangle + \langle f_{uu}(t)v(t), v(t)\rangle\Big)\Big]dt$$
$$+\mathbb{E}\Big[\varepsilon\,\langle h_x(\bar{x}(T)), y_1(T)\rangle + \frac{\varepsilon^2}{2}\,\langle h_x(\bar{x}(T)), y_2(T)\rangle$$
$$+\frac{\varepsilon^2}{2}\,\langle h_{xx}(\bar{x}(T))y_1(T), y_1(T)\rangle\Big] + o(\varepsilon^2), \quad (\varepsilon \to 0^+).$$

By Itô's formula, we have

(3.11) $\quad \mathbb{E}\ \langle h_x(\bar{x}(T)), y_1(T)\rangle = -\mathbb{E}\ \langle P_1(T), y_1(T)\rangle$

$$= -\mathbb{E}\int_0^T \Big[\langle P_1(t), b_u(t)v(t)\rangle + \langle Q_1(t), \sigma_u(t)v(t)\rangle + \langle f_x(t), y_1(t)\rangle \Big]dt,$$

(3.12) $\quad \mathbb{E}\ \langle h_x(\bar{x}(T)), y_2(T)\rangle = -\mathbb{E}\ \langle P_1(T), y_2(T)\rangle$

$$= -\mathbb{E}\int_0^T \Big[\langle P_1(t), y_1(t)^\top b_{xx}(t)y_1(t)\rangle + 2\langle P_1(t), v(t)^\top b_{xu}(t)y_1(t)\rangle$$
$$+ \langle P_1(t), v(t)^\top b_{uu}(t)v(t)\rangle + \langle Q_1(t), y_1(t)^\top \sigma_{xx}(t)y_1(t)\rangle$$
$$+2\langle Q_1(t), v(t)^\top \sigma_{xu}(t)y_1(t)\rangle + \langle Q_1(t), v(t)^\top \sigma_{uu}(t)v(t)\rangle$$
$$+ \langle f_x(t), y_2(t)\rangle \Big]dt,$$

and (noting that $P_2(t)^\top = P_2(t)$ and $Q_2(t)^\top = Q_2(t)$)

(3.13) $\quad \mathbb{E}\ \langle h_{xx}(\bar{x}(T))y_1(T), y_1(T)\rangle = -\mathbb{E}\ \langle P_2(T)y_1(T), y_1(T)\rangle$

$$= -\mathbb{E}\int_0^T \Big[\langle P_2(t)y_1(t), b_u(t)v(t)\rangle + \langle P_2(t)b_u(t)v(t), y_1(t)\rangle$$
$$+ \langle P_2(t)\sigma_x(t)y_1(t), \sigma_u(t)v(t)\rangle + \langle P_2(t)\sigma_u(t)v(t), \sigma_x(t)y_1(t)\rangle$$
$$+ \langle P_2(t)\sigma_u(t)v(t), \sigma_u(t)v(t)\rangle + \langle Q_2(t)\sigma_u(t)v(s), y_1(t)\rangle$$
$$+ \langle Q_2(t)y_1(t), \sigma_u(t)v(t)\rangle - \langle H_{xx}(t)y_1(t), y_1(t)\rangle \Big]dt$$

$$= -\mathbb{E}\int_0^T \Big[2\langle P_2(t)y_1(t), b_u(t)v(t)\rangle + 2\langle P_2(t)\sigma_x(t)y_1(t), \sigma_u(t)v(t)\rangle$$
$$+ \langle P_2(t)\sigma_u(t)v(t), \sigma_u(t)v(t)\rangle + 2\langle Q_2(t)\sigma_u(t)v(s), y_1(t)\rangle$$
$$- \langle H_{xx}(t)y_1(t), y_1(t)\rangle \Big]dt.$$

Substituting (3.11), (3.12) and (3.13) into (3.10), we obtain that

$$J(u^\varepsilon) - J(\bar{u})$$
$$= -\mathbb{E}\int_0^T \Big[\varepsilon\Big(\langle P_1(t), b_u(t)v(t)\rangle + \langle Q_1(t), \sigma_u(t)v(t)\rangle - \langle f_u(t), v(t)\rangle\Big)$$
$$+ \frac{\varepsilon^2}{2}\Big(\langle P_1(t), v(t)^\top b_{uu}(t)v(t)\rangle + \langle Q_1(t), v(t)^\top \sigma_{uu}(t)v(t)\rangle$$
$$- \langle f_{uu}(t)v(t), v(t)\rangle\Big) + \frac{\varepsilon^2}{2}\langle P_2(t)\sigma_u(t)v(t), \sigma_u(t)v(t)\rangle$$
$$+\varepsilon^2\Big(\langle P_1(t), v(t)^\top b_{xu}(t)y_1(t)\rangle + \langle Q_1(t), v(t)^\top \sigma_{xu}(t)y_1(t)\rangle$$
$$- \langle f_{xu}(t)y_1(t), v(t)\rangle + \langle b_u(t)^\top P_2(t)y_1(t), v(t)\rangle$$
$$+ \langle \sigma_u(t)^\top P_2(t)\sigma_x(t)y_1(t), v(t)\rangle$$
$$+ \langle \sigma_u(t)^\top Q_2(t)y_1(t), v(t)\rangle\Big)\Big]dt + o(\varepsilon^2) \qquad (\varepsilon \to 0^+)$$
$$= -\mathbb{E}\int_0^T \Big[\varepsilon\langle H_u(t), v(t)\rangle + \frac{\varepsilon^2}{2}\langle H_{uu}(t)v(t), v(t)\rangle$$

$$+\frac{\varepsilon^2}{2}\left\langle P_2(t)\sigma_u(t)v(t), \sigma_u(t)v(t)\right\rangle$$

$$+\varepsilon^2\left\langle \mathbb{S}(t)y_1(t), v(t)\right\rangle\Big]dt + o(\varepsilon^2), \quad (\varepsilon \to 0^+).$$

This completes the proof of Proposition 3.2. □

Now, we establish an integral-type second-order necessary condition for stochastic optimal controls. Stimulated by [10], we introduce the following notion.

DEFINITION 3.3. *We call a control $\tilde{u}(\cdot) \in \mathcal{U}_{ad}$ a singular control in the classical sense if $\tilde{u}(\cdot)$ satisfies*

$$\begin{cases} H_u(t, \tilde{x}(t), \tilde{u}(t), \tilde{P}_1(t), \tilde{Q}_1(t)) = 0, \quad a.s., \ a.e. \ t \in [0,T], \\ H_{uu}(t, \tilde{x}(t), \tilde{u}(t), \tilde{P}_1(t), \tilde{Q}_1(t)) + \sigma_u(t, \tilde{x}(t), \tilde{u}(t))^\top \tilde{P}_2(t)\sigma_u(t, \tilde{x}(t), \tilde{u}(t)) = 0, \\ \qquad a.s., \ a.e. \ t \in [0,T], \end{cases}$$

(3.14)

where $\tilde{x}(\cdot)$ is the state with respect to $\tilde{u}(\cdot)$, and $(\tilde{P}_1(\cdot), \tilde{Q}_1(\cdot))$ and $(\tilde{P}_2(\cdot), \tilde{Q}_2(\cdot))$ be the adjoint processes given respectively by (3.5) and (3.6) with $(\bar{x}(\cdot), \bar{u}(\cdot))$ replaced by $(\tilde{x}(\cdot), \tilde{u}(\cdot))$.

Remark 3.1. Since the diffusion term $\sigma(t, x, u)$ contains the control variable u, in order to represent the stochastic maximum principle, one needs to introduce the following $\mathcal{H}$-function:

$$\mathcal{H}(t, x, u) := H(t, x, u, \tilde{P}_1(t), \tilde{Q}_1(t)) - \frac{1}{2}\left\langle \tilde{P}_2(t)\sigma(t, \tilde{x}(t), \tilde{u}(t)), \sigma(t, \tilde{x}(t), \tilde{u}(t))\right\rangle$$

$$+\frac{1}{2}\left\langle \tilde{P}_2(t)\big(\sigma(t, x, u) - \sigma(t, \tilde{x}(t), \tilde{u}(t))\big), \sigma(t, x, u) - \sigma(t, \tilde{x}(t), \tilde{u}(t))\right\rangle.$$

The stochastic maximum principle (see [18]) says, if $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ is an optimal pair, then

(3.15) $$\mathcal{H}(t, \tilde{x}(t), \tilde{u}(t)) = \max_{v \in U} \mathcal{H}(t, \tilde{x}(t), v), \quad a.s., \ a.e. \ t \in [0,T].$$

A singular control in the classical sense is the one that satisfies trivially the first- and second-order necessary conditions in optimization theory for the maximization problem (3.15), i.e.,

(3.16) $$\begin{cases} \mathcal{H}_u(t, \tilde{x}(t), \tilde{u}(t)) = 0, \quad a.s., \ a.e. \ t \in [0,T], \\ \mathcal{H}_{uu}(t, \tilde{x}(t), \tilde{u}(t) = 0, \quad a.s., \ a.e. \ t \in [0,T]. \end{cases}$$

It is easy to see that (3.16) is equivalent to (3.14). On the other hand, one can consider (stochastic) singular optimal controls in other senses, say in the sense of Pontryagin-type maximum principle. Due to the space limitation, we shall present our results in this respect elsewhere.

By Proposition 3.2, we obtain the following integral-type second-order necessary condition.

THEOREM 3.4. *Let (C1)–(C2) hold. If $\bar{u}(\cdot)$ is a singular optimal control in the classical sense, then*

(3.17) $$\mathbb{E}\int_0^T \left\langle \mathbb{S}(t)y_1(t), v(t)\right\rangle dt \le 0,$$

for any $v(\cdot) = u(\cdot) - \bar{u}(\cdot), \ u(\cdot) \in \mathcal{U}_{ad}$.

Proof. By (3.9) and Definition 3.3, we have

$$(3.18) \qquad 0 \leq \lim_{\varepsilon \to 0^+} \frac{J(u^\varepsilon) - J(\bar{u})}{\varepsilon^2} = -\mathbb{E} \int_0^T \langle \mathbb{S}(t) y_1(t), v(t) \rangle \, dt,$$

as stated. □

In [4], the authors obtained the following integral-type first- and second-order necessary conditions for stochastic optimal controls:

THEOREM 3.5. *Let (C1)–(C2) hold. If $\bar{u}(\cdot)$ is an optimal control, then*

$$\int_0^T \langle H_u(t), w(t) \rangle \, dt \leq 0, \qquad \forall \ w(\cdot) \in cl_{2,2}\big(\mathcal{R}_{\mathcal{U}_{ad}}(\bar{u}) \cap L_{\mathbb{F}}^4(\Omega; L^4(0, T; \mathbb{R}^m)))\big).$$

Furthermore, for any $w(\cdot) \in cl_{4,4}\big(\mathcal{R}_{\mathcal{U}_{ad}}(\bar{u}) \cap L^\infty([0,T] \times \Omega; \mathbb{R}^m) \cap \Upsilon(\bar{u})\big)$ the following second-order necessary condition holds:

$$\mathbb{E} \int_0^T \Big[\langle H_{xx}(t) y_1(t), y_1(t) \rangle + 2 \langle H_{xu}(t) y_1(t), w(t) \rangle$$

$$(3.19) \qquad + \langle H_{uu}(t) w(t), w(t) \rangle \Big] dt + \mathbb{E} \langle h_{xx}(\bar{x}(T)) y_1(T), y_1(T) \rangle \leq 0.$$

Here,

$$\mathcal{R}_{\mathcal{U}_{ad}}(\bar{u}) := \Big\{ \alpha u(\cdot) - \alpha \bar{u}(\cdot) \ \Big| \ u(\cdot) \in \mathcal{U}_{ad}, \alpha > 0 \Big\},$$

$$\Upsilon(\bar{u}) := \Big\{ w(\cdot) \in L_{\mathbb{F}}^2(\Omega; L^2(0, T; \mathbb{R}^m)) \ \Big| \ \int_0^T \langle H_u(t), w(t) \rangle \, dt = 0 \Big\}$$

and, $cl_{2,2}(A)$ and $cl_{4,4}(A)$ are the closure of a set A under the norms $\| \cdot \|_{2,2}$ and $\| \cdot \|_{4,4}$, respectively.

There are some second-order terms with respect to $y_1(\cdot)$ in (3.19). These terms are eliminated in (3.17) by introducing the second-order adjoint process $(P_2(\cdot), Q_2(\cdot))$. Note also that, the second-order necessary condition we consider in this paper is for the singular optimal controls in the classical sense, hence the second order terms $\langle H_{uu}(t) v(t), v(t) \rangle$ and $\langle P_2(t) \sigma_u(t) v(t), \sigma_u(t) v(t) \rangle$ appearing in the variational formulation (3.9) do not enter into (3.17).

3.2. Second-order necessary condition in term of martingale representation. Let us recall that, in order to derive pointwise necessary conditions for optimal controls, one needs to establish first some suitable integral-type necessary conditions. It is well-known that there is no difficulty to establish the pointwise first-order necessary condition for optimal controls whenever an integral-type one is obtained. However, the classical method of deriving the pointwise condition from the integral-type one cannot be used directly to establish the pointwise second-order condition in the general stochastic setting.

Note that the solution $y_1(\cdot)$ to the first variational equation (3.2) appears in the integral-type second-order condition (3.17). By [20, Theorem 1.6.14, p.47], $y_1(\cdot)$ enjoys an explicit representation:

$$y_1(t) = \Phi(t) \int_0^t \Phi(s)^{-1} \big(b_u(s) - \sigma_x(s) \sigma_u(s) \big) v(s) ds$$

$$(3.20) \qquad + \Phi(t) \int_0^t \Phi(s)^{-1} \sigma_u(s) v(s) dW(s),$$

where $\Phi(\cdot)$ is the solution to the following matrix-valued stochastic differential equation

$$(3.21) \qquad \begin{cases} d\Phi(t) = b_x(t)\Phi(t)dt + \sigma_x(t)\Phi(t)dW(t), & t \in [0,T], \\ \Phi(0) = I, \end{cases}$$

and I stands for the identity matrix in $\mathbb{R}^{n\times n}$. Substituting the explicit representation (3.20) of $y_1(\cdot)$ into (3.17), we see that there will appear a "bad" term of the following form:

$$(3.22) \qquad \mathbb{E}\int_0^T \left\langle \mathbb{S}(t)\Phi(t)\int_0^t \Phi(s)^{-1}\sigma_u(s)v(s)dW(s), v(t)\right\rangle dt.$$

To see (3.22) is "bad", let us choose $\tau \in [0,T)$, $v \in U$, $E_\theta = [\tau, \tau+\theta)$ such that $\theta > 0$ and $\tau + \theta \leq T$. Denote by $\chi_{E_\theta}(\cdot)$ the characteristic function of the set E_θ. As usual, though the control region U is convex, in order to derive a pointwise second-order necessary condition from the integral one (3.17), people need to choose the following needle variation for the optimal control $\bar{u}(\cdot)$:

$$(3.23) \qquad u(t) = \begin{cases} v, & t \in E_\theta, \\ \bar{u}(t), & t \in [0,T] \setminus E_\theta. \end{cases}$$

For this $u(\cdot)$, it is clear that $v(\cdot) = u(\cdot) - \bar{u}(\cdot) = (v - \bar{u}(\cdot))\chi_{E_\theta}(\cdot)$, and (3.22) is reduced to

$$(3.24) \qquad \mathbb{E}\int_\tau^{\tau+\theta} \left\langle \mathbb{S}(t)\Phi(t)\int_\tau^t \Phi(s)^{-1}\sigma_u(s)(v - \bar{u}(s))dW(s), v - \bar{u}(t)\right\rangle dt.$$

Since an Itô integral appears in (3.24), we have

$$\mathbb{E}\int_\tau^{\tau+\theta} \left\langle \mathbb{S}(t)\Phi(t)\int_\tau^t \Phi(s)^{-1}\sigma_u(s)(v - \bar{u}(s))dW(s), v - \bar{u}(t)\right\rangle dt$$

$$\leq \left[\mathbb{E}\int_\tau^{\tau+\theta} \left|(\mathbb{S}(t)\Phi(t))^\top (v - \bar{u}(t))\right|^2 dt\right]^{\frac{1}{2}} \left[\mathbb{E}\int_\tau^{\tau+\theta}\int_\tau^t \left|\Phi(s)^{-1}\sigma_u(s)(v - \bar{u}(s))\right|^2 dsdt\right]^{\frac{1}{2}}$$

$$= O(\theta^{\frac{3}{2}}), \quad (\theta \to 0^+).$$

Because of this, it seems that (3.24) is not an infinitesimal of order 2 but only that of order $\frac{3}{2}$ with respect to θ (as $\theta \to 0^+$).

However, by the properties of Itô's integral, we find that

$$\lim_{\theta\to 0^+} \left|\frac{1}{\theta^{\frac{3}{2}}}\mathbb{E}\int_\tau^{\tau+\theta} \left\langle \mathbb{S}(t)\Phi(t)\int_\tau^t \Phi(s)^{-1}\sigma_u(s)(v - \bar{u}(s))dW(s), v - \bar{u}(t)\right\rangle dt\right|$$

$$\leq \lim_{\theta\to 0^+} \left|\frac{1}{\theta^{\frac{3}{2}}}\mathbb{E}\int_\tau^{\tau+\theta} \left\langle (\mathbb{S}(t)\Phi(t))^\top (v - \bar{u}(t)) - (\mathbb{S}(\tau)\Phi(\tau))^\top (v - \bar{u}(\tau)),\right.\right.$$

$$\left.\left.\int_\tau^t \Phi(s)^{-1}\sigma_u(s)(v - \bar{u}(s))dW(s)\right\rangle dt\right|$$

$$+ \lim_{\theta\to 0^+} \left|\frac{1}{\theta^{\frac{3}{2}}}\mathbb{E}\int_\tau^{\tau+\theta} \left\langle (\mathbb{S}(\tau)\Phi(\tau))^\top (v - \bar{u}(\tau)), \int_\tau^t \Phi(s)^{-1}\sigma_u(s)(v - \bar{u}(s))dW(s)\right\rangle dt\right|$$

$$\leq \lim_{\theta\to 0^+} \frac{1}{\theta^{\frac{3}{2}}}\left[\mathbb{E}\int_\tau^{\tau+\theta} \left|(\mathbb{S}(t)\Phi(t))^\top (v - \bar{u}(t)) - (\mathbb{S}(\tau)\Phi(\tau))^\top (v - \bar{u}(\tau))\right|^2 dt\right]^{\frac{1}{2}} \cdot$$

$$\left[\mathbb{E}\int_\tau^{\tau+\theta}\int_\tau^t \left|\Phi(s)^{-1}\sigma_u(s)(v - \bar{u}(s))\right|^2 dsdt\right]^{\frac{1}{2}}$$

$$= 0, \qquad a.e. \ \tau \in [0,T).$$

This indicates that, (3.22) is actually a higher order infinitesimal of $\theta^{\frac{3}{2}}$ (as $\theta \to 0^+$).

Essentially, the above problem is caused by the Itô integral. Indeed, one cannot use the Lebesgue differentiation theorem directly to treat the Itô integral appeared in (3.24). In this subsection, we shall reduce the Itô-Lebesgue integral term (3.24) to a double Lebesgue integral term by means of the property of Itô's integrals and the martingale representation theorem, and obtain a second-order necessary condition for singular optimal controls.

We need the following technical result (which should be known but we do not find an exact reference).

LEMMA 3.6. *Let* $\varphi(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^2(0, T; \mathbb{R}^n))$. *Then, there exists a* $\phi(\cdot, \cdot) \in L^2(0, T; L^2_{\mathbb{F}}([0, T] \times \Omega; \mathbb{R}^n))$ *such that*

$$(3.25) \qquad \varphi(t) = \mathbb{E}\, \varphi(t) + \int_0^t \phi(s, t) dW(s), \quad a.s., \ a.e. \ t \in [0, T].$$

Proof. Let $\{\varphi_j(\cdot)\}_{j=1}^\infty$ be a sequence in $L^2_{\mathbb{F}}(\Omega; L^2(0, T; \mathbb{R}^n))$ such that

$$\mathbb{E} \int_0^T \left|\varphi_j(t) - \varphi(t)\right|^2 dt \to 0, \qquad \text{as } j \to \infty,$$

where $\varphi_j(\cdot) = \sum_{k=0}^{K_j} \xi_j^k \chi_{[t_k, t_{k+1})}(t)$, $K_j \in \mathbb{N}$, $0 = t_0 < t_1 < \cdots < t_{K_j+1} = T$ is a partition of $[0, T]$, and $\xi_j^k \in L^2_{\mathcal{F}_{t_k}}(\Omega; \mathbb{R}^n)$.

For any fixed j and k, since $\xi_j^k \in L^2_{\mathcal{F}_{t_k}}(\Omega; \mathbb{R}^n)$, by the martingale representation theorem, there exists a stochastic process $\phi_j^k(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^2(0, T; \mathbb{R}^n))$ such that

$$\xi_j^k = \mathbb{E}\, \xi_j^k + \int_0^{t_k} \phi_j^k(s) dW(s), \quad a.s.$$

Define

$$\phi_j(s, t) = \sum_{k=0}^{K_j} \phi_j^k(s) \chi_{[0, t_k]}(s) \chi_{[t_k, t_{k+1})}(t), \quad (s, t) \in [0, T] \times [0, T].$$

Clearly, $\varphi_j(\cdot)$ can be represented as

$$\varphi_j(t) = \mathbb{E}\, \varphi_j(t) + \int_0^t \phi_j(s, t) dW(s), \quad a.s., \ a.e. \ t \in [0, T].$$

Consequently, we have

$$\mathbb{E} \int_0^T \int_0^T \left|\phi_j(s, t) - \phi_m(s, t)\right|^2 dsdt = \mathbb{E} \int_0^T \int_0^t \left|\phi_j(s, t) - \phi_m(s, t)\right|^2 dsdt$$

$$= \int_0^T \mathbb{E} \left| \int_0^t \left[\phi_j(s, t) - \phi_m(s, t)\right] dW(s)\right|^2 dt$$

$$= \int_0^T \mathbb{E} \left|\varphi_j(t) - \varphi_m(t) - \mathbb{E}\left[\varphi_j(t) - \varphi_m(t)\right]\right|^2 dt \leq 4\mathbb{E} \int_0^T \left|\varphi_j(t) - \varphi_m(t)\right|^2 dt.$$

Since $\varphi_j(\cdot)$ converges strongly to $\varphi(\cdot)$, $\{\phi_j(\cdot, \cdot)\}_{j=1}^\infty$ is a Cauchy sequence in $L^2(0, T; L^2_{\mathbb{F}}([0, T] \times \Omega; \mathbb{R}^n))$. Hence, there exists a $\phi(\cdot, \cdot) \in L^2(0, T; L^2_{\mathbb{F}}([0, T] \times \Omega; \mathbb{R}^n))$ such that

$$\mathbb{E} \int_0^T \int_0^T \left|\phi_j(s, t) - \phi(s, t)\right|^2 dsdt \to 0, \qquad \text{as } j \to \infty,$$

and

$$\mathbb{E} \int_0^T \left| \varphi(t) - \mathbb{E}\, \varphi(t) - \int_0^t \phi(s,t) dW(s) \right|^2 dt$$

$$= \mathbb{E} \int_0^T \Big| \varphi(t) - \varphi_j(t) + \varphi_j(t) - \mathbb{E}\, \varphi(t) + \mathbb{E}\, \varphi_j(t) - \mathbb{E}\, \varphi_j(t)$$

$$- \int_0^t \phi(s,t) dW(s) + \int_0^t \phi_j(s,t) dW(s) - \int_0^t \phi_j(s,t) dW(s) \Big|^2 dt$$

$$\le C\mathbb{E} \int_0^T \left| \varphi(t) - \varphi_j(t) \right|^2 dt + C \int_0^T \left| \mathbb{E}\, \varphi(t) - \mathbb{E}\, \varphi_j(t) \right|^2 dt$$

$$+ C\mathbb{E} \int_0^T \left| \int_0^t \phi(s,t) dW(s) - \int_0^t \phi_j(s,t) dW(s) \right|^2 dt$$

$$+ C\mathbb{E} \int_0^T \left| \varphi_j(t) - \mathbb{E}\, \varphi_j(t) - \int_0^t \phi_j(s,t) dW(s) \right|^2 dt$$

$$\le C\mathbb{E} \int_0^T \left| \varphi(t) - \varphi_j(t) \right|^2 dt + C\mathbb{E} \int_0^T \int_0^T \left| \phi(s,t) - \phi_j(s,t) \right|^2 ds dt$$

$$\to 0, \qquad \text{as } j \to \infty.$$

Therefore, (3.25) holds. □

Also, we need the following simple result.

LEMMA 3.7. *Let (C1)–(C2) hold. Then* $\mathbb{S}(\cdot) \in L_{\mathbb{F}}^4(\Omega; L^2(0,T;\mathbb{R}^{m \times n}))$.

Proof. We only need to prove that

$$\mathbb{E} \left[\int_0^T \left| \mathbb{S}(t) \right|^2 dt \right]^2 < \infty.$$

By *(C1)–(C2)*,

$$|f_{xu}(t)| \le C, \quad a.s., \ a.e. \ t \in [0,T],$$

and, for $\varphi = b, \ \sigma$,

$$|\varphi_x(t)| + |\varphi_u(t)| + |\varphi_{xu}(t)| \le C, \quad a.s., \ a.e. \ t \in [0,T].$$

Therefore,

$$\mathbb{E} \left[\int_0^T \left| \mathbb{S}(t) \right|^2 dt \right]^2$$

$$= \mathbb{E} \left[\int_0^T \Big| H_{xu}(t,\bar{x}(t),\bar{u}(t),P_1(t),Q_1(t)) + b_u(t,\bar{x}(t),\bar{u}(t))^\top P_2(t) \right.$$

$$\left. + \sigma_u(t,\bar{x}(t),\bar{u}(t))^\top Q_2(t) + \sigma_u(t,\bar{x}(t),\bar{u}(t))^\top P_2(t)\sigma_x(t,\bar{x}(t),\bar{u}(t)) \Big|^2 dt \right]^2$$

$$\le C + C\mathbb{E} \left[\int_0^T \left(|P_1(t)|^2 + |Q_1(t)|^2 + |P_2(t)|^2 + |Q_2(t)|^2 \right) dt \right]^2$$

$$\le C + C \left(\|P_1\|_{\infty,4}^4 + \|Q_1\|_{2,4}^4 + \|P_2\|_{\infty,4}^4 + \|Q_2\|_{2,4}^4 \right)$$

$$< \infty,$$

which completes the proof of Lemma 3.7. □

By Lemma 3.7, $\mathbb{S}(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^2(0,T; \mathbb{R}^{m \times n}))$. Then, by our assumption (C1) and Lemma 3.6, for any $v \in U$, there exists a $\phi_v(\cdot, \cdot) \in L^2(0,T; L^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R}^n))$ such that for a.e. $t \in [0,T]$,

$$(3.26) \qquad \mathbb{S}(t)^\top (v - \bar{u}(t)) = \mathbb{E}\left[\mathbb{S}(t)^\top (v - \bar{u}(t))\right] + \int_0^t \phi_v(s,t) dW(s), \quad a.s.$$

Using (3.26), we obtain the following second-order necessary condition, which is pointwise with respect to the time variable (but it is still in the integral form with respect to the sample point ω).

THEOREM 3.8. *Let (C1)–(C2) hold. If $\bar{u}(\cdot)$ is a singular optimal control in the classical sense, then for any $v \in U$, it holds that*

$$\begin{aligned}
& \mathbb{E} \ \langle \mathbb{S}(\tau) b_u(\tau)(v - \bar{u}(\tau)), v - \bar{u}(\tau)\rangle \\
(3.27) \qquad & + \partial_\tau^+ \left(\mathbb{S}(\tau)^\top (v - \bar{u}(\tau)), \sigma_u(\tau)(v - \bar{u}(\tau))\right) \le 0, \ \ a.e. \ \tau \in [0,T],
\end{aligned}$$

where,

$$\partial_\tau^+ \left(\mathbb{S}(\tau)^\top (v - \bar{u}(\tau)), \sigma_u(\tau)(v - \bar{u}(\tau))\right)$$

$$(3.28)$$
$$:= 2 \limsup_{\theta \to 0^+} \frac{1}{\theta^2} \mathbb{E} \int_\tau^{\tau+\theta} \int_\tau^t \left\langle \phi_v(s,t), \Phi(\tau)\Phi(s)^{-1} \sigma_u(s)(v - \bar{u}(s))\right\rangle ds dt,$$

$\phi_v(\cdot, \cdot)$ is determined by (3.26), and $\Phi(\cdot)$ is the solution to the stochastic differential equation (3.21).

The proof of Theorem 3.8 will be given in Subsection 4.1.

3.3. Second-order necessary condition in term of Malliavin derivative. In Theorem 3.8 we obtain a second-order necessary condition in term of martingale representation. From the martingale representation theorem, we only know that, for any $v \in U$, $\phi_v(\cdot, \cdot) \in L^2(0,T; L^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R}^n))$, and hence, for each $\tau \in [0,T]$, the function

$$\varphi(s,t) := \mathbb{E} \ \langle \phi_v(s,t), \Phi(\tau)\Phi(s)^{-1} \sigma_u(s)(v - \bar{u}(s))\rangle, \quad (s,t) \in [0,T] \times [0,T]$$

is in $L^1([0,T] \times [0,T])$. However, the condition $\varphi(\cdot, \cdot) \in L^1([0,T] \times [0,T])$ is not sufficient to ensure that, for a.e. $\tau \in [0,T]$, the limit

$$(3.29) \qquad \lim_{\theta \to 0^+} \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \int_\tau^t \varphi(s,t) ds dt$$

exists.

Example 3.1. Let $a_n = \frac{2}{3^n}$, $n = 0, 1, 2, \cdots$. Then, $\sum_{n=1}^\infty a_n = 1$ and $\sum_{k=n+1}^\infty a_k = \frac{a_n}{2}$. Let $T = \sqrt{2}$ and define $\varphi(\cdot, \cdot) \in L^1([0,\sqrt{2}] \times [0,\sqrt{2}])$ as follows:

$$\varphi(s,t) = \begin{cases} 1, & (s,t) \in \left([0,\sqrt{2}] \times [0,\sqrt{2}]\right) \bigcap \left\{\frac{a_n}{2} \le \frac{t-s}{\sqrt{2}} < a_n, \ n = 1, \ 2, \ \cdots\right\}, \\ -1, & (s,t) \in \left([0,\sqrt{2}] \times [0,\sqrt{2}]\right) \bigcap \left\{a_n \le \frac{t-s}{\sqrt{2}} < \frac{a_{n-1}}{2}, \ n = 1, \ 2, \ \cdots\right\}, \\ 0, & otherwise. \end{cases}$$

Fixed a $\tau \in [0,\sqrt{2})$ arbitrarily. If $\theta_n = \frac{\sqrt{2} a_{n-1}}{2}$, $\tau + \theta_n \le \sqrt{2}$, then

$$\lim_{n \to \infty} \frac{1}{\theta_n^2} \int_\tau^{\tau+\theta_n} \int_\tau^t \varphi(s,t) ds dt = \lim_{n \to \infty} \frac{\sum_{k=n}^\infty (\frac{\sqrt{2} a_k}{2})^2}{(\frac{\sqrt{2} a_{n-1}}{2})^2} = \lim_{n \to \infty} \frac{\frac{1}{4 \cdot 9^{n-1}}}{\frac{2}{9^{n-1}}} = \frac{1}{8}.$$

On the other hand, if $\theta_n = \sqrt{2}a_n$, $\tau + \theta_n \leq \sqrt{2}$, then

$$\lim_{n\to\infty} \frac{1}{\theta_n^2} \int_\tau^{\tau+\theta_n} \int_\tau^t \varphi(s,t)dsdt = \lim_{n\to\infty} \frac{\frac{1}{2}(\frac{\sqrt{2}a_n}{2})^2 + \sum_{k=n}^{\infty}(\frac{\sqrt{2}a_{k+1}}{2})^2}{(\sqrt{2}a_n)^2}$$

$$= \lim_{n\to\infty} \frac{\frac{5}{4\cdot9^n}}{\frac{8}{9^n}} = \frac{5}{32}.$$

Example 3.2. Let $T = 1$. Define

$$\varphi(s,t) = \begin{cases} 0, & t \leq s, \ s,t \in [0,1], \\ -\frac{1}{(t-s)^{\frac{1}{2}}}, & t > s, \ s,t \in [0,1]. \end{cases}$$

Obviously, $\varphi \in L^1([0,1]\times[0,1])$. But, for any $\tau \in [0,1)$ and $\theta > 0$ satisfying $\tau+\theta \leq 1$,

$$\frac{\int_\tau^{\tau+\theta}\int_\tau^t \varphi(s,t)dsdt}{\theta^2} = \frac{-\int_\tau^{\tau+\theta} 2(t-\tau)^{\frac{1}{2}}dt}{\theta^2} = \frac{-\frac{4}{3}\theta^{\frac{3}{2}}}{\theta^2} \to -\infty, \quad (\theta \to 0^+).$$

The above two examples show that, in general, the superior limit

$$\limsup_{\theta\to0^+} \frac{1}{\theta^2} \mathbb{E} \int_\tau^{\tau+\theta} \int_\tau^t \Big\langle \phi_v(s,t), \Phi(\tau)\Phi(s)^{-1}\sigma_u(s)(v-\bar{u}(s)) \Big\rangle dsdt$$

(in (3.28)) cannot be refined to be the limit, and even worse, this superior limit may be equal to $-\infty$. If the superior limit in (3.28) is equal to $-\infty$ for a.e. $\tau \in [0,T]$, the second-order necessary condition (3.27) turns out to be trivial. On the other hand, even this superior limit is finite for a.e. $\tau \in [0,T]$, it is still difficult to obtain the continuity of the function

$$v \mapsto \partial_\tau^+ \big(\mathbb{S}(\tau)^\top (v-\bar{u}(\tau)), \sigma_u(\tau)(v-\bar{u}(\tau)) \big).$$

All the problems mentioned in the above are caused by the lack of further information for $\phi_v(\cdot,\cdot)$. If both $\mathbb{S}(\cdot)$ and $\bar{u}(\cdot)$ are regular enough, the function $\phi_v(\cdot,\cdot)$ has an explicit representation and then we can improve the result obtained in Theorem 3.8. To this end, we assume that

(C3)

$$\bar{u}(\cdot) \in \mathbb{L}_{2,\mathbb{F}}^{1,2}(\mathbb{R}^m), \ \mathbb{S}(\cdot) \in \mathbb{L}_{2,\mathbb{F}}^{1,2}(\mathbb{R}^{m\times n}) \cap L^\infty([0,T]\times\Omega;\mathbb{R}^{m\times n}).$$

We have the following pointwise second-order necessary condition for singular optimal controls.

THEOREM 3.9. *Let (C1)–(C3) hold. If $\bar{u}(\cdot)$ is a singular optimal control in the classical sense, then for a.e. $\tau \in [0,T]$, it holds that*

(3.30)
$$\langle \mathbb{S}(\tau)b_u(\tau)(v-\bar{u}(\tau)), v-\bar{u}(\tau)\rangle$$
$$+ \langle \nabla\mathbb{S}(\tau)\sigma_u(\tau)(v-\bar{u}(\tau)), v-\bar{u}(\tau)\rangle$$
$$- \langle \mathbb{S}(\tau)\sigma_u(\tau)(v-\bar{u}(\tau)), \nabla\bar{u}(\tau)\rangle \leq 0, \quad \forall\, v \in U, \ a.s.$$

The proof of Theorem 3.9 will be given in Subsection 4.2.

Remark 3.2. In some special cases, the regularity assumption on $\mathbb{S}(\cdot)$ holds automatically. One of them is the linear quadratic optimal control problem with convex

control constraints. In this case, the functions b, σ, f and h in (1.1)-(1.2) are given as follows:

$$b(t,x,u) = A(t)x + B(t)u, \quad \sigma(t,x,u) = C(t)x + D(t)u, \quad h(x) = \frac{1}{2}\langle Gx, x \rangle,$$

$$f(t,x,u) = \frac{1}{2}\big[\langle R(t)x, x \rangle + 2\langle M(t)x, u \rangle + \langle N(t)u, u \rangle\big], \quad (t,x,u) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^m,$$

where $A(\cdot), C(\cdot) \in C([0,T]; \mathbb{R}^{n \times n})$, $B(\cdot), D(\cdot) \in C([0,T]; \mathbb{R}^{n \times m})$, $R(\cdot) \in C([0,T]; \mathbf{S}^n)$, $M(\cdot) \in C([0,T]; \mathbb{R}^{m \times n})$ and $N(\cdot) \in C([0,T]; \mathbf{S}^m)$ are deterministic matrix-valued functions, and $G \in \mathbf{S}^n$ is a (deterministic) matrix.

Indeed, for this problem, the second-order adjoint equation is

$$(3.31) \quad \begin{cases} dP_2(t) = -\Big[A(t)^\top P_2(t) + P_2(t)A(t) + C(t)^\top P_2(t)C(t) + C(t)^\top Q_2(t) \\ \qquad\qquad + Q_2(t)C(t) - R(t)\Big]dt + Q_2(t)dW(t), \quad t \in [0,T], \\ P_2(T) = -G. \end{cases}$$

Since $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$, $R(\cdot)$, $M(\cdot)$, $N(\cdot)$ and G are deterministic, the equa-tio(3.31) admits a unique deterministic solution $(P_2(\cdot), 0)$, where $P_2(\cdot)$ is the solution to the following differential equation

$$(3.32) \quad \begin{cases} \dot{P}_2(t) = -A(t)^\top P_2(t) - P_2(t)A(t) - C(t)^\top P_2(t)C(t) + R(t), \quad t \in [0,T], \\ P_2(T) = -G. \end{cases}$$

Hence, for this case,

$$\mathbb{S}(\cdot) = -M(\cdot) + B(\cdot)^\top P_2(\cdot) + D(\cdot)^\top P_2(\cdot)C(\cdot)$$

is a deterministic continuous matrix-valued function, hence it belongs to the space $\mathbb{L}^{1,2}_{2,\mathbb{F}}(\mathbb{R}^{m \times n}) \cap L^\infty([0,T] \times \Omega; \mathbb{R}^{m \times n})$.

In general, to obtain the regularity of $\mathbb{S}(\cdot)$, we need the regularity of $(\bar{u}(\cdot), \bar{x}(\cdot))$, $(P_1(\cdot), Q_1(\cdot))$ and $(P_2(\cdot), Q_2(\cdot))$. From the regularity results for solutions to stochastic differential equations (see [8] and [17]), the optimal control $\bar{u}(\cdot)$ needs to be regular enough. In the deterministic setting, the regularity of optimal controls has been studied by many authors (see [5, 6] and references cited therein). However, to the best of our knowledge, there exists no reference addressing the regularity of stochastic optimal controls. We will discuss this topic in our forthcoming paper.

    To end this section, we give two examples to explain how to distinguish singular optimal controls from others by using the pointwise second-order necessary conditions established in Theorem 3.9.

    Example 3.3. Let $n = m = 1$, $T = 1$, $U = [-1,1]$. Consider the following one-dimensional control system

$$(3.33) \quad \begin{cases} dx(t) = u(t)dt + u(t)dW(t), \quad t \in [0,1], \\ x(0) = 0 \end{cases}$$

and the cost functional

$$J(u(\cdot)) = \frac{1}{2}\mathbb{E}\int_0^1 |u(t)|^2 dt - \frac{1}{2}\mathbb{E}|x(1)|^2.$$

For this optimal control problem, the Hamiltonian is given by

$$H(t,x,u,y_1,z_1) = y_1 u + z_1 u - \frac{1}{2}u^2, \quad (t,x,u,y_1,z_1) \in [0,1] \times \mathbb{R} \times U \times \mathbb{R} \times \mathbb{R}.$$

Let $(\bar{x}(t), \bar{u}(t)) \equiv (0,0)$. The corresponding two adjoint equations are

(3.34)
$$\begin{cases} dP_1(t) = Q_1(t)dW(t), \ t \in [0,1], \\ P_1(1) = 0, \end{cases}$$

and

(3.35)
$$\begin{cases} dP_2(t) = Q_2(t)dW(t), \ t \in [0,1], \\ P_2(1) = 1. \end{cases}$$

Obviously,

$$(P_1(t), Q_1(t)) \equiv (0,0), \qquad (P_2(t), Q_2(t)) \equiv (1,0).$$

Then, we have for all $(t, \omega) \in [0,1] \times \Omega$,

$$H_u(t, \bar{x}(t), \bar{u}(t), P_1(t), Q_1(t)) = 0,$$

and

$$H_{uu}(t, \bar{x}(t), \bar{u}(t), P_1(t), Q_1(t)) + \sigma_u(t, \bar{x}(t), \bar{u}(t))^\top P_2(t)\sigma_u(t, \bar{x}(t), \bar{u}(t)) = 0.$$

That is, $\bar{u}(t) \equiv 0$ is a singular control in the classical sense. Let $\hat{u}(t) \equiv -1$, we have

$$-\frac{1}{2} = J(\hat{u}(\cdot)) < J(\bar{u}(\cdot)) = 0.$$

Therefore, $\bar{u}(t) \equiv 0$ is not an optimal control.

Now, we show that $\bar{u}(t) \equiv 0$ does not satisfy the second-order necessary condition (3.30). Actually,

$$\mathbb{S}(t) \equiv 1, \quad \nabla\mathbb{S}(t) \equiv 0, \quad \nabla\bar{u}(t) \equiv 0.$$

Let $v = 1$, we find that

$$\langle \mathbb{S}(\tau)b_u(\tau)(v - \bar{u}(\tau)), v - \bar{u}(\tau) \rangle$$
$$+ \langle \nabla\mathbb{S}(\tau)\sigma_u(\tau)(v - \bar{u}(\tau)), v - \bar{u}(\tau) \rangle$$
$$- \langle \mathbb{S}(\tau)\sigma_u(\tau)(v - \bar{u}(\tau)), \nabla\bar{u}(\tau) \rangle$$
$$= 1 > 0, \quad \forall\, (\tau, \omega) \in [0,1] \times \Omega.$$

Hence, the condition (3.30) fails at $v = 1$.

Example 3.4. Let $n = m = 1$, $U = [-1,1] \times [-1,1]$. Consider the control system

(3.36)
$$\begin{cases} dx(t) = Bu(t)dt + Du(t)dW(t), \qquad t \in [0,T], \\ x(0) = 0 \end{cases}$$

with the following cost functional

$$J(u(\cdot)) = \frac{1}{2}\mathbb{E}\langle Gx(T), x(T) \rangle,$$

where

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad G = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

For this optimal control problem, the Hamiltonian is given by

$$H(t, x, u, y_1, z_1) = \langle y_1, Bu \rangle + \langle z_1, Du \rangle, \quad (t, x, u, y_1, z_1) \in [0, T] \times \mathbb{R}^2 \times U \times \mathbb{R}^2 \times \mathbb{R}^2.$$

Clearly, $(\bar{x}(t), \bar{u}(t)) \equiv (0, 0)$ is an optimal pair, and the corresponding adjoint equations are respecitvely

$$(3.37) \qquad \begin{cases} dP_1(t) = Q_1(t)dW(t), & t \in [0, T], \\ P_1(T) = 0 \end{cases}$$

and

$$(3.38) \qquad \begin{cases} dP_2(t) = Q_2(t)dW(t), & t \in [0, T], \\ P_2(T) = -G. \end{cases}$$

Obviously, $(P_1(t), Q_1(t)) \equiv (0, 0)$, $(P_2(t), Q_2(t)) \equiv (-G, 0)$, and

$$H_u(t, \bar{x}(t), \bar{u}(t), P_1(t), Q_1(t)) \equiv 0,$$

$$H_{uu}(t, \bar{x}(t), \bar{u}(t), P_1(t), Q_1(t)) + D^\top P_2(t)D \equiv 0.$$

Therefore, $\bar{x}(t) \equiv 0$ is a singular optimal control in the classical sense.
Since for this case,

$$\mathbb{S}(t) \equiv -B^\top G, \quad \nabla\mathbb{S}(t) \equiv 0, \quad \nabla\bar{u}(t) \equiv 0,$$

we have

$$\begin{aligned}
&\langle \mathbb{S}(\tau)b_u(\tau)(v - \bar{u}(\tau)), v - \bar{u}(\tau) \rangle \\
&+ \langle \nabla\mathbb{S}(\tau)\sigma_u(\tau)(v - \bar{u}(\tau)), v - \bar{u}(\tau) \rangle \\
&- \langle \mathbb{S}(\tau)\sigma_u(\tau)(v - \bar{u}(\tau)), \nabla\bar{u}(\tau) \rangle \\
&= - \langle B^\top GBv, v \rangle \leq 0, \quad \forall\, v \in U, \quad \forall\, (t, \omega) \in [0, T] \times \Omega.
\end{aligned}$$

That is, the necessary condition (3.30) holds.

4. Proofs of the main results. This section is devoted to proving Theorems 3.8 and 3.9. Firstly, we show a technical result.

LEMMA 4.1. *Let $\Phi(\cdot), \Psi(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^2(0, T; \mathbb{R}^n))$. Then, for a.e. $\tau \in [0, T)$,*

$$(4.1) \qquad \lim_{\theta \to 0^+} \frac{1}{\theta^2} \mathbb{E} \int_\tau^{\tau+\theta} \left\langle \Phi(\tau), \int_\tau^t \Psi(s)ds \right\rangle dt = \frac{1}{2} \mathbb{E} \left\langle \Phi(\tau), \Psi(\tau) \right\rangle,$$

$$(4.2) \qquad \lim_{\theta \to 0^+} \frac{1}{\theta^2} \mathbb{E} \int_\tau^{\tau+\theta} \left\langle \Phi(t), \int_\tau^t \Psi(s)ds \right\rangle dt = \frac{1}{2} \mathbb{E} \left\langle \Phi(\tau), \Psi(\tau) \right\rangle.$$

Proof. The equality (4.1) is a corollary of the Lebesgue differentiation theorem. Now, we prove (4.2). For any $\tau \in [0, T)$, let $\theta > 0$ and $\tau + \theta < T$. By the Lebesgue differentiation theorem, we have

$$\lim_{\theta \to 0^+} \frac{1}{\theta} \int_\tau^{\tau+\theta} \mathbb{E} \left| \Phi(t) - \Phi(\tau) \right|^2 dt = 0, \quad a.e. \ \tau \in [0, T),$$

and

$$\lim_{\theta \to 0^+} \frac{1}{\theta^2} \mathbb{E} \int_\tau^{\tau+\theta} \int_\tau^t |\Psi(s)|^2 ds dt = \frac{1}{2} \mathbb{E} |\Psi(\tau)|^2, \quad a.e.\ \tau \in [0,T).$$

Therefore,

$$(4.3) \quad \lim_{\theta \to 0^+} \left| \frac{1}{\theta^2} \mathbb{E} \int_\tau^{\tau+\theta} \left\langle \Phi(t) - \Phi(\tau), \int_\tau^t \Psi(s) ds \right\rangle dt \right|$$

$$\leq \lim_{\theta \to 0^+} \frac{1}{\theta^2} \left[\int_\tau^{\tau+\theta} \mathbb{E} \left| \Phi(t) - \Phi(\tau) \right|^2 dt \right]^{\frac{1}{2}} \left[\int_\tau^{\tau+\theta} (t-\tau) \mathbb{E} \int_\tau^t |\Psi(s)|^2 ds dt \right]^{\frac{1}{2}}$$

$$\leq \lim_{\theta \to 0^+} \frac{1}{\theta^{\frac{3}{2}}} \left[\int_\tau^{\tau+\theta} \mathbb{E} \left| \Phi(t) - \Phi(\tau) \right|^2 dt \right]^{\frac{1}{2}} \left[\int_\tau^{\tau+\theta} \mathbb{E} \int_\tau^t |\Psi(s)|^2 ds dt \right]^{\frac{1}{2}}$$

$$= 0, \quad a.e.\ \tau \in [0,T).$$

Combining (4.3) and (4.1), we obtain (4.2). This completes the proof of Lemma 4.1.
□

4.1. Proof of Theorem 3.8. For any $v \in U$, $\tau \in [0,T)$ and $\theta \in (0, T - \tau)$, let $E_\theta = [\tau, \tau+\theta)$ and $u(\cdot)$ be defined by (3.23). Then, $v(\cdot) = u(\cdot) - \bar{u}(\cdot) = (v - \bar{u}(\cdot)) \chi_{E_\theta}(\cdot)$ and the corresponding solution $y_1(\cdot)$ to the equation (3.2) is given by

$$(4.4) \qquad y_1(t) = \Phi(t) \int_0^t \Phi(s)^{-1} \left(b_u(s) - \sigma_x(s) \sigma_u(s) \right) \left(v - \bar{u}(s) \right) \chi_{E_\theta}(s) ds$$

$$+ \Phi(t) \int_0^t \Phi(s)^{-1} \sigma_u(s) (v - \bar{u}(s)) \chi_{E_\theta}(s) dW(s).$$

Substituting $v(\cdot) = (v - \bar{u}(\cdot)) \chi_{E_\theta}(\cdot)$ and (4.4) into (3.17), we have

$$0 \geq \frac{1}{\theta^2} \mathbb{E} \int_\tau^{\tau+\theta} \langle \mathbb{S}(t) y_1(t), v - \bar{u}(t) \rangle dt$$

$$= \frac{1}{\theta^2} \mathbb{E} \int_\tau^{\tau+\theta} \left\langle \mathbb{S}(t) \Phi(t) \int_\tau^t \Phi(s)^{-1} \left(b_u(s) - \sigma_x(s) \sigma_u(s) \right) \cdot \right.$$

$$(4.5) \qquad\qquad\qquad\qquad\qquad (v - \bar{u}(s)) ds, v - \bar{u}(t) \Big\rangle dt$$

$$+ \frac{1}{\theta^2} \mathbb{E} \int_\tau^{\tau+\theta} \left\langle \mathbb{S}(t) \Phi(t) \int_\tau^t \Phi(s)^{-1} \sigma_u(s) \cdot \right.$$

$$(v - \bar{u}(s)) dW(s), v - \bar{u}(t) \Big\rangle dt.$$

By Lemma 4.1, we have, for a.e. $\tau \in [0,T)$,

$$\lim_{\theta \to 0^+} \frac{1}{\theta^2} \mathbb{E} \int_\tau^{\tau+\theta} \left\langle \mathbb{S}(t) \Phi(t) \int_\tau^t \Phi(s)^{-1} \left(b_u(s) - \sigma_x(s) \sigma_u(s) \right) \cdot \right.$$

$$(v - \bar{u}(s)) ds, v - \bar{u}(t) \Big\rangle dt$$

$$(4.6) \quad = \frac{1}{2} \mathbb{E} \left\langle \mathbb{S}(\tau) \left(b_u(\tau) - \sigma_x(\tau) \sigma_u(\tau) \right) (v - \bar{u}(\tau)), v - \bar{u}(\tau) \right\rangle.$$

On the other hand, by (3.21), we have

$$(4.7) \qquad \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \mathbb{E} \left\langle \mathbb{S}(t) \Phi(t) \int_\tau^t \Phi(s)^{-1} \sigma_u(s) (v - \bar{u}(s)) dW(s), v - \bar{u}(t) \right\rangle dt$$

$$= \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \mathbb{E} \left\langle \mathbb{S}(t)\Phi(\tau) \int_\tau^t \Phi(s)^{-1}\sigma_u(s)(v-\bar{u}(s))dW(s), v-\bar{u}(t) \right\rangle dt$$

$$+ \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \mathbb{E} \left\langle \mathbb{S}(t) \int_\tau^t b_x(s)\Phi(s)ds \cdot \right.$$
$$\left. \int_\tau^t \Phi(s)^{-1}\sigma_u(s)(v-\bar{u}(s))dW(s), v-\bar{u}(t) \right\rangle dt$$

$$+ \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \mathbb{E} \left\langle \mathbb{S}(t) \int_\tau^t \sigma_x(s)\Phi(s)dW(s) \cdot \right.$$
$$\left. \int_\tau^t \Phi(s)^{-1}\sigma_u(s)(v-\bar{u}(s))dW(s), v-\bar{u}(t) \right\rangle dt.$$

Substituting (3.26) into the fist term of the right hand of (4.7), we get that

$$(4.8) \quad \limsup_{\theta\to 0^+} \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \mathbb{E} \left\langle \mathbb{S}(t)\Phi(\tau) \int_\tau^t \Phi(s)^{-1}\sigma_u(s)(v-\bar{u}(s))dW(s), v-\bar{u}(t) \right\rangle dt$$

$$= \limsup_{\theta\to 0^+} \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \mathbb{E} \left\langle \int_\tau^t \Phi(\tau)\Phi(s)^{-1}\sigma_u(s)(v-\bar{u}(s))dW(s), \right.$$
$$\left. \mathbb{E}\left[\mathbb{S}(t)^\top(v-\bar{u}(t))\right] \right\rangle dt$$

$$+ \limsup_{\theta\to 0^+} \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \mathbb{E} \left\langle \int_\tau^t \Phi(\tau)\Phi(s)^{-1}\sigma_u(s)(v-\bar{u}(s))dW(s), \right.$$
$$\left. \int_0^t \phi_v(s,t)dW(s) \right\rangle dt$$

$$= \limsup_{\theta\to 0^+} \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \int_\tau^t \mathbb{E} \left\langle \Phi(\tau)\Phi(s)^{-1}\sigma_u(s)(v-\bar{u}(s)), \phi_v(s,t) \right\rangle ds dt$$

$$= \frac{1}{2}\partial_\tau^+ \left(\mathbb{S}(\tau)^\top(v-\bar{u}(\tau)), \sigma_u(\tau)(v-\bar{u}(\tau))\right), \qquad \forall\, \tau \in [0,T).$$

Next, by Lemma 3.7, $\mathbb{S}(\cdot) \in L^4(\Omega; L^2(0,T;\mathbb{R}^{m\times n})) \subset L^2(\Omega; L^2(0,T;\mathbb{R}^{m\times n}))$. Then, by Condition (C1), we have

$$\lim_{\theta\to 0^+} \left| \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \mathbb{E} \left\langle \mathbb{S}(t) \int_\tau^t b_x(s)\Phi(s)ds \cdot \right. \right.$$
$$\left. \left. \int_\tau^t \Phi(s)^{-1}\sigma_u(s)(v-\bar{u}(s))dW(s), v-\bar{u}(t) \right\rangle dt \right|$$

$$\leq \lim_{\theta\to 0^+} \frac{C}{\theta^2} \int_\tau^{\tau+\theta} \mathbb{E} \left| \mathbb{S}(t) \int_\tau^t b_x(s)\Phi(s)ds \int_\tau^t \Phi(s)^{-1}\sigma_u(s)(v-\bar{u}(s))dW(s) \right| dt$$

$$\leq \lim_{\theta\to 0^+} \frac{C}{\theta^2} \int_\tau^{\tau+\theta} \left\{ \left[\mathbb{E}\,|\mathbb{S}(t)|^2\right]^{\frac{1}{2}} \cdot \left[\mathbb{E}\left|\int_\tau^t b_x(s)\Phi(s)ds\right|^4\right]^{\frac{1}{4}} \cdot \right.$$
$$\left. \left[\mathbb{E}\left|\int_\tau^t \left[\Phi(s)^{-1}\sigma_u(s)(v-\bar{u}(s))\right]dW(s)\right|^4\right]^{\frac{1}{4}} \right\} dt$$

$$\leq \lim_{\theta\to 0^+} \frac{C}{\theta^2} \int_\tau^{\tau+\theta} \left\{ \left[\mathbb{E}\,|\mathbb{S}(t)|^2\right]^{\frac{1}{2}} \cdot \left[\mathbb{E}\left|\int_\tau^t b_x(s)\Phi(s)ds\right|^4\right]^{\frac{1}{4}} \cdot \right.$$
$$\left. \left[\mathbb{E}\left(\int_\tau^t \left|\Phi(s)^{-1}\sigma_u(s)(v-\bar{u}(s))\right|^2 ds\right)^2\right]^{\frac{1}{4}} \right\} dt$$

$$\leq \lim_{\theta \to 0^+} \frac{C}{\theta^2} \int_\tau^{\tau+\theta} (t-\tau)^{\frac{3}{2}} \left[\mathbb{E} \left| \mathbb{S}(t) \right|^2 \right]^{\frac{1}{2}} dt$$
$$= 0, \qquad a.e. \ \ \tau \in [0, T).$$

This implies that

$$(4.9) \qquad \lim_{\theta \to 0^+} \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \mathbb{E} \left\langle \mathbb{S}(t) \int_\tau^t b_x(s)\Phi(s)ds \cdot \right.$$
$$\left. \int_\tau^t \Phi(s)^{-1}\sigma_u(s)(v - \bar{u}(s))dW(s), v - \bar{u}(t) \right\rangle dt$$
$$= 0 \qquad a.e. \ \ \tau \in [0, T).$$

Furthermore, since

$$\lim_{\theta \to 0^+} \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \mathbb{E} \left\langle \mathbb{S}(t) \int_\tau^t \sigma_x(s)\Phi(s)dW(s) \cdot \right.$$
$$\left. \int_\tau^t \Phi(s)^{-1}\sigma_u(s)(v - \bar{u}(s))dW(s), v - \bar{u}(t) \right\rangle dt$$
$$= \lim_{\theta \to 0^+} \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \mathbb{E} \left\langle \int_\tau^t \sigma_x(s)\Phi(s)dW(s) \int_\tau^t \Phi(s)^{-1}\sigma_u(s)(v - \bar{u}(s))dW(s), \right.$$
$$\left. \mathbb{S}(t)^\top (v - \bar{u}(t)) - \mathbb{S}(\tau)^\top (v - \bar{u}(\tau)) \right\rangle dt$$
$$+ \lim_{\theta \to 0^+} \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \mathbb{E} \left\langle \mathbb{S}(\tau) \int_\tau^t \sigma_x(s)\Phi(s)dW(s) \cdot \right.$$
$$\left. \int_\tau^t \Phi(s)^{-1}\sigma_u(s)(v - \bar{u}(s))dW(s), v - \bar{u}(\tau) \right\rangle dt,$$

and

$$\lim_{\theta \to 0^+} \frac{1}{\theta^2} \left| \int_\tau^{\tau+\theta} \mathbb{E} \left\langle \int_\tau^t \sigma_x(s)\Phi(s)dW(s) \int_\tau^t \Phi(s)^{-1}\sigma_u(s)(v - \bar{u}(s))dW(s), \right.\right.$$
$$\left.\left. \mathbb{S}(t)^\top (v - \bar{u}(t)) - \mathbb{S}(\tau)^\top (v - \bar{u}(\tau)) \right\rangle dt \right|$$
$$\leq \lim_{\theta \to 0^+} \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \left[\mathbb{E} \left| \int_\tau^t \left| \sigma_x(s)\Phi(s) \right|^2 ds \right|^2 \right]^{\frac{1}{4}} \cdot$$
$$\left[\mathbb{E} \left| \int_\tau^t \left| \Phi(s)^{-1}\sigma_u(s)(v - \bar{u}(s)) \right|^2 ds \right|^2 \right]^{\frac{1}{4}} \cdot$$
$$\left[\mathbb{E} \left| \mathbb{S}(t)^\top (v - \bar{u}(t)) - \mathbb{S}(\tau)^\top (v - \bar{u}(\tau)) \right|^2 \right]^{\frac{1}{2}} dt$$
$$\leq \lim_{\theta \to 0^+} \frac{C}{\theta^{\frac{1}{2}}} \left[\int_\tau^{\tau+\theta} \mathbb{E} \left| \mathbb{S}(t)^\top (v - \bar{u}(t)) - \mathbb{S}(\tau)^\top (v - \bar{u}(\tau)) \right|^2 dt \right]^{\frac{1}{2}}$$
$$= 0, \qquad a.e. \ \ \tau \in [0, T),$$

then, by Lemma 4.1, we have,

$$(4.10) \qquad \lim_{\theta \to 0^+} \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \mathbb{E} \left\langle \mathbb{S}(t) \int_\tau^t \sigma_x(s)\Phi(s)dW(s) \cdot \right.$$

$$\int_\tau^t \Phi(s)^{-1}\sigma_u(s)(v - \bar{u}(s))dW(s), v - \bar{u}(t)\Big\rangle dt$$

$$= \lim_{\theta \to 0^+} \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \mathbb{E}\Big\langle \mathbb{S}(\tau)\int_\tau^t \sigma_x(s)\sigma_u(s)(v - \bar{u}(s))ds, v - \bar{u}(\tau)\Big\rangle dt$$

$$= \frac{1}{2}\mathbb{E}\langle \mathbb{S}(\tau)\sigma_x(\tau)\sigma_u(\tau)(v - \bar{u}(\tau)), v - \bar{u}(\tau)\rangle, \quad a.e. \ \tau \in [0, T).$$

Therefore, by (4.7)–(4.10), we have, for a.e. $\tau \in [0, T)$,

(4.11) $\displaystyle \limsup_{\theta \to 0^+} \frac{1}{\theta^2}\mathbb{E}\int_\tau^{\tau+\theta}\Big\langle \mathbb{S}(t)\Phi(t)\int_\tau^t \Phi(s)^{-1}\sigma_u(s)(v - \bar{u}(s))dW(s), v - \bar{u}(t)\Big\rangle dt$

$$= \frac{1}{2}\partial_\tau^+\big(\mathbb{S}(\tau)^\top(v - \bar{u}(\tau)), \sigma_u(\tau)(v - \bar{u}(\tau))\big)$$

$$+ \frac{1}{2}\mathbb{E}\langle \mathbb{S}(\tau)\sigma_x(\tau)\sigma_u(\tau)(v - \bar{u}(\tau)), v - \bar{u}(\tau)\rangle.$$

Finally, by (4.5), (4.6) and (4.11), we obtain that

$$0 \geq \limsup_{\theta \to 0^+}\frac{1}{\theta^2}\mathbb{E}\int_\tau^{\tau+\theta}\langle \mathbb{S}(t)y_1(t), v - \bar{u}(t)\rangle\, dt$$

$$= \frac{1}{2}\mathbb{E}\Big\langle \mathbb{S}(\tau)\big(b_u(\tau) - \sigma_x(\tau)\sigma_u(\tau)\big)(v - \bar{u}(\tau)), v - \bar{u}(\tau)\Big\rangle$$

$$+ \frac{1}{2}\partial_\tau^+\big(\mathbb{S}(\tau)^\top(v - \bar{u}(\tau)), \sigma_u(\tau)(v - \bar{u}(\tau))\big)$$

$$+ \frac{1}{2}\mathbb{E}\langle \mathbb{S}(\tau)\sigma_x(\tau)\sigma_u(\tau)(v - \bar{u}(\tau)), v - \bar{u}(\tau)\rangle$$

$$= \frac{1}{2}\mathbb{E}\langle \mathbb{S}(\tau)b_u(\tau)(v - \bar{u}(\tau)), v - \bar{u}(\tau)\rangle$$

$$+ \frac{1}{2}\partial_\tau^+\big(\mathbb{S}(\tau)^\top(v - \bar{u}(\tau)), \sigma_u(\tau)(v - \bar{u}(\tau))\big), \quad a.e. \ \tau \in [0, T),$$

which gives (3.27). This completes the proof of Theorem 3.8.

4.2. Proof of Theorem 3.9. Since $W(\cdot)$ is a continuous stochastic process, $\mathcal{F}_t$ is countably generated for any $t \in [0, T]$. Hence, one can fined a sequence $\{A_l\}_{l=1}^\infty \subset \mathcal{F}_t$ such that for any $A \in \mathcal{F}_t$, there exists a subsequence $\{A_{l_n}\}_{n=1}^\infty \subset \{A_l\}_{l=1}^\infty$ such that $\lim_{n \to \infty} P(A\Delta A_{l_n}) = 0$, where $A\Delta A_{l_n} = (A \setminus A_{l_n})\bigcup(A_{l_n} \setminus A)$. $\mathcal{F}_t$ is also said to be generated by the sequence $\{A_l\}_{l=1}^\infty$.

Denote by $\{t_i\}_{i=1}^\infty$ the sequence of rational numbers in $[0, T)$, by $\{v^k\}_{k=1}^\infty$ a dense subset of U. As in [12, 19], we choose $\{A_{ij}\}_{j=1}^\infty(\subset \mathcal{F}_{t_i})$ to be a sequence generating $\mathcal{F}_{t_i}$ (for each $i \in \mathbb{N}$). Fix $i, j, k \in \mathbb{N}$ arbitrarily. For any $\tau \in [t_i, T)$ and $\theta \in (0, T - \tau)$, write $E_\theta^i = [\tau, \tau + \theta)$, and define

$$u_{ij}^k(t, \omega) = \begin{cases} v^k, & (t, \omega) \in E_\theta^i \times A_{ij}, \\ \bar{u}(t, \omega), & (t, \omega) \in \big([0, T] \times \Omega\big) \setminus \big(E_\theta^i \times A_{ij}\big). \end{cases}$$

Clearly, $u_{ij}^k(\cdot) \in \mathcal{U}_{ad}$. Choosing a "test" function $v(\cdot)$ in (3.17) as

$$v_{ij}^k(t, \omega) = u_{ij}^k(t, \omega) - \bar{u}(t, \omega) = \big(v^k - \bar{u}(t, \omega)\big)\chi_{A_{ij}}(\omega)\chi_{E_\theta^i}(t), \quad (t, \omega) \in [0, T] \times \Omega,$$

we obtain that

(4.12) $\displaystyle \mathbb{E}\int_\tau^{\tau+\theta}\langle \mathbb{S}(t)y_{ij}^k(t), v^k - \bar{u}(t)\rangle\,\chi_{A_{ij}}(\omega)dt \leq 0,$

where $y_{ij}^k(\cdot)$ is the solution to the variational equation (3.2) with $v(\cdot)$ replaced by $v_{ij}^k(\cdot)$. By (3.20),

$$(4.13) \quad y_{ij}^k(t) = \Phi(t) \int_0^t \Phi(s)^{-1} \big(b_u(s) - \sigma_x(s)\sigma_u(s) \big) \big(v^k - \bar{u}(s) \big) \chi_{E_\theta^i}(s) \chi_{A_{ij}}(\omega) ds$$

$$+ \Phi(t) \int_0^t \Phi(s)^{-1} \sigma_u(s) \big(v^k - \bar{u}(s) \big) \chi_{E_\theta^i}(s) \chi_{A_{ij}}(\omega) dW(s).$$

Substituting (4.13) into (4.12), we have

$$(4.14) \quad 0 \geq \frac{1}{\theta^2} \mathbb{E} \int_\tau^{\tau+\theta} \Big\langle \mathbb{S}(t)\Phi(t) \int_\tau^t \Big[\Phi(s)^{-1} \big(b_u(s) - \sigma_x(s)\sigma_u(s) \big) \cdot$$

$$(v^k - \bar{u}(s)) \chi_{A_{ij}}(\omega) \Big] ds, v^k - \bar{u}(t) \Big\rangle \chi_{A_{ij}}(\omega) dt$$

$$+ \frac{1}{\theta^2} \mathbb{E} \int_\tau^{\tau+\theta} \Big\langle \mathbb{S}(t)\Phi(t) \int_\tau^t \Big[\Phi(s)^{-1} \sigma_u(s) \cdot$$

$$(v^k - \bar{u}(s)) \chi_{A_{ij}}(\omega) \Big] dW(s), v^k - \bar{u}(t) \Big\rangle \chi_{A_{ij}}(\omega) dt.$$

By Lemma 4.1, it is immediate that for a.e. $\tau \in [t_i, T)$,

$$(4.15) \quad \lim_{\theta \to 0^+} \frac{1}{\theta^2} \mathbb{E} \int_\tau^{\tau+\theta} \Big\langle \mathbb{S}(t)\Phi(t) \int_\tau^t \Big[\Phi(s)^{-1} \big(b_u(s) - \sigma_x(s)\sigma_u(s) \big) \cdot$$

$$(v^k - \bar{u}(s)) \chi_{A_{ij}}(\omega) \Big] ds, v^k - \bar{u}(t) \Big\rangle \chi_{A_{ij}}(\omega) dt$$

$$= \frac{1}{2} \mathbb{E} \Big[\big\langle \mathbb{S}(\tau)\big(b_u(\tau) - \sigma_x(\tau)\sigma_u(\tau) \big)(v^k - \bar{u}(\tau)), v^k - \bar{u}(\tau) \big\rangle \chi_{A_{ij}}(\omega) \Big].$$

Next, we prove that there exists a sequence $\{\theta_n\}_{n=1}^\infty$ such that $\theta_n \to 0^+$ as $n \to \infty$ and

$$(4.16) \quad \lim_{n \to \infty} \frac{1}{\theta_n^2} \mathbb{E} \int_\tau^{\tau+\theta_n} \Big\langle \mathbb{S}(t)\Phi(t) \int_\tau^t \Big[\Phi(s)^{-1} \sigma_u(s) \cdot$$

$$(v^k - \bar{u}(s)) \chi_{A_{ij}}(\omega) \Big] dW(s), v^k - \bar{u}(t) \Big\rangle \chi_{A_{ij}}(\omega) dt$$

$$= \frac{1}{2} \mathbb{E} \Big[\big\langle \nabla\mathbb{S}(\tau)\sigma_u(\tau)(v^k - \bar{u}(\tau)), v^k - \bar{u}(\tau) \big\rangle \chi_{A_{ij}}(\omega) \Big]$$

$$- \frac{1}{2} \mathbb{E} \Big[\big\langle \mathbb{S}(\tau)\sigma_u(\tau)(v^k - \bar{u}(\tau)), \nabla\bar{u}(\tau) \big\rangle \chi_{A_{ij}}(\omega) \Big]$$

$$+ \frac{1}{2} \mathbb{E} \Big[\big\langle \mathbb{S}(\tau)\sigma_x(\tau)\sigma_u(\tau)(v^k - \bar{u}(\tau)), v^k - \bar{u}(\tau) \big\rangle \chi_{A_{ij}}(\omega) \Big], \quad a.e. \ \tau \in [t_i, T).$$

By (3.21),

$$(4.17) \quad \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \mathbb{E} \Big\{ \Big\langle \mathbb{S}(t)\Phi(t) \int_\tau^t \Big[\Phi(s)^{-1} \sigma_u(s) \cdot$$

$$(v^k - \bar{u}(s)) \chi_{A_{ij}}(\omega) \Big] dW(s), v^k - \bar{u}(t) \Big\rangle \chi_{A_{ij}}(\omega) \Big\} dt$$

$$= \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \mathbb{E} \Big\{ \Big\langle \mathbb{S}(t)\Phi(\tau) \int_\tau^t \Big[\Phi(s)^{-1} \sigma_u(s) \cdot$$

$$(v^k - \bar{u}(s))\chi_{A_{ij}}(\omega)\Big]dW(s), v^k - \bar{u}(t)\Big\rangle\chi_{A_{ij}}(\omega)\Big\}dt$$

$$+\frac{1}{\theta^2}\int_\tau^{\tau+\theta}\mathbb{E}\left\{\Big\langle\mathbb{S}(t)\int_\tau^t b_x(s)\Phi(s)ds\int_\tau^t\Big[\Phi(s)^{-1}\sigma_u(s)\cdot\right.$$

$$(v^k - \bar{u}(s))\chi_{A_{ij}}(\omega)\Big]dW(s), v^k - \bar{u}(t)\Big\rangle\chi_{A_{ij}}(\omega)\Big\}dt$$

$$+\frac{1}{\theta^2}\int_\tau^{\tau+\theta}\mathbb{E}\left\{\Big\langle\mathbb{S}(t)\int_\tau^t\sigma_x(s)\Phi(s)dW(s)\int_\tau^t\Big[\Phi(s)^{-1}\sigma_u(s)\cdot\right.$$

$$(v^k - \bar{u}(s))\chi_{A_{ij}}(\omega)\Big]dW(s), v^k - \bar{u}(t)\Big\rangle\chi_{A_{ij}}(\omega)\Big\}dt.$$

Therefore, we can divide the computation for the left hand side of (4.16) into three parts.

Similar to respectively (4.9) and (4.10) (in the proof of Theorem 3.8), we get that

(4.18) $\quad\displaystyle\lim_{\theta\to0^+}\frac{1}{\theta^2}\int_\tau^{\tau+\theta}\mathbb{E}\left\{\Big\langle\mathbb{S}(t)\int_\tau^t b_x(s)\Phi(s)ds\int_\tau^t\Big[\Phi(s)^{-1}\sigma_u(s)\cdot\right.$

$$(v^k - \bar{u}(s))\chi_{A_{ij}}(\omega)\Big]dW(s), v^k - \bar{u}(t)\Big\rangle\chi_{A_{ij}}(\omega)\Big\}dt$$

$$= 0, \quad a.e. \ \tau\in[t_i, T),$$

and

(4.19) $\quad\displaystyle\lim_{\theta\to0^+}\frac{1}{\theta^2}\int_\tau^{\tau+\theta}\mathbb{E}\left\{\Big\langle\mathbb{S}(t)\int_\tau^t\sigma_x(s)\Phi(s)dW(s)\int_\tau^t\Big[\Phi(s)^{-1}\sigma_u(s)\cdot\right.$

$$(v^k - \bar{u}(s))\chi_{A_{ij}}(\omega)\Big]dW(s), v^k - \bar{u}(t)\Big\rangle\chi_{A_{ij}}(\omega)\Big\}dt$$

$$= \frac{1}{2}\mathbb{E}\left[\Big\langle\mathbb{S}(\tau)\sigma_x(\tau)\sigma_u(\tau)(v^k - \bar{u}(\tau), v^k - \bar{u}(\tau)\Big\rangle\chi_{A_{ij}}(\omega)\right], \quad a.e. \ \tau\in[t_i, T).$$

It remains to prove that there exists a sequence $\{\theta_n\}_{n=1}^\infty$ such that $\theta_n\to0^+$ as $n\to\infty$ and

(4.20) $\quad\displaystyle\lim_{n\to\infty}\frac{1}{\theta_n^2}\int_\tau^{\tau+\theta_n}\mathbb{E}\left\{\Big\langle\mathbb{S}(t)\Phi(\tau)\int_\tau^t\Big[\Phi(s)^{-1}\sigma_u(s)\cdot\right.$

$$(v^k - \bar{u}(s))\chi_{A_{ij}}(\omega)\Big]dW(s), v^k - \bar{u}(t)\Big\rangle\chi_{A_{ij}}(\omega)\Big\}dt$$

$$= \frac{1}{2}\mathbb{E}\left[\Big\langle\nabla\mathbb{S}(\tau)\sigma_u(\tau)(v^k - \bar{u}(\tau)), v^k - \bar{u}(\tau)\Big\rangle\chi_{A_{ij}}(\omega)\right]$$

$$-\frac{1}{2}\mathbb{E}\left[\Big\langle\mathbb{S}(\tau)\sigma_u(\tau)(v^k - \bar{u}(\tau)), \nabla\bar{u}(\tau)\Big\rangle\chi_{A_{ij}}(\omega)\right], \quad a.e. \ \tau\in[t_i, T).$$

By the boundness of U and the regularity assumption *(C3)*, it holds that

$$\mathbb{S}(\cdot)^\top(v^k - \bar{u}(\cdot))\in\mathbb{L}_\mathbb{F}^{1,2}(\mathbb{R}^n)\cap L^\infty([0, T]\times\Omega; \mathbb{R}^n),$$

Then, by the Clark-Ocone formula, for a.e. $t\in[0, T]$,

$$\mathbb{S}(t)^\top(v^k - \bar{u}(t)) = \mathbb{E}\left[\mathbb{S}(t)^\top(v^k - \bar{u}(t))\right]$$

(4.21)

$$+\int_0^t\mathbb{E}\left[\mathcal{D}_s\big(\mathbb{S}(t)^\top(v^k - \bar{u}(t))\big)\ \Big|\ \mathcal{F}_s\right]dW(s).$$

Substituting (4.21) into the first term of the right hand of (4.17), we obtain that

$$(4.22) \quad \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \mathbb{E} \left\{ \left\langle \mathbb{S}(t)\Phi(\tau) \int_\tau^t \left[\Phi(s)^{-1}\sigma_u(s) \right. \right. \right.$$

$$\left. \left. \left. (v^k - \bar{u}(s))\chi_{A_{ij}}(\omega) \right] dW(s), v^k - \bar{u}(t) \right\rangle \chi_{A_{ij}}(\omega) \right\} dt$$

$$= \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \mathbb{E} \left\{ \left\langle \int_\tau^t \Phi(\tau)\Phi(s)^{-1}\sigma_u(s)(v^k - \bar{u}(s))\chi_{A_{ij}}(\omega)dW(s), \right. \right.$$

$$\left. \left. \mathbb{E}\left[\mathbb{S}(t)^\top (v^k - \bar{u}(t)) \right] \right\rangle \chi_{A_{ij}}(\omega) \right\} dt$$

$$+ \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \mathbb{E} \left\{ \left\langle \int_\tau^t \Phi(\tau)\Phi(s)^{-1}\sigma_u(s)(v^k - \bar{u}(s))\chi_{A_{ij}}(\omega)dW(s), \right. \right.$$

$$\left. \left. \int_0^t \mathbb{E}\left[\mathcal{D}_s\big(\mathbb{S}(t)^\top (v^k - \bar{u}(t))\big) \mid \mathcal{F}_s \right] dW(s) \right\rangle \chi_{A_{ij}}(\omega) \right\} dt$$

$$= \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \int_\tau^t \mathbb{E}\left[\left\langle \Phi(\tau)\Phi(s)^{-1}\sigma_u(s)(v^k - \bar{u}(s)), \right. \right.$$

$$\left. \left. \mathcal{D}_s\big(\mathbb{S}(t)^\top (v^k - \bar{u}(t))\big) \right\rangle \chi_{A_{ij}}(\omega) \right] ds dt.$$

The last equality in (4.22) follows from the fact that $A_{ij} \in \mathcal{F}_{t_i} \subset \mathcal{F}_\tau$ and

$$\mathbb{E}\left\{ \left\langle \int_\tau^t \Phi(\tau)\Phi(s)^{-1}\sigma_u(s)(v^k - \bar{u}(s))\chi_{A_{ij}}(\omega)dW(s), \right. \right.$$

$$\left. \left. \int_0^t \mathbb{E}\left[\mathcal{D}_s\big(\mathbb{S}(t)^\top (v^k - \bar{u}(t))\big) \mid \mathcal{F}_s \right] dW(s) \right\rangle \chi_{A_{ij}}(\omega) \right\}$$

$$= \mathbb{E}\left\{ \chi_{A_{ij}}(\omega)\mathbb{E}\left(\left\langle \int_\tau^t \Phi(\tau)\Phi(s)^{-1}\sigma_u(s)(v^k - \bar{u}(s))\chi_{A_{ij}}(\omega)dW(s), \right. \right. \right.$$

$$\left. \left. \left. \int_\tau^t \mathbb{E}\left[\mathcal{D}_s\big(\mathbb{S}(t)^\top (v^k - \bar{u}(t))\big) \mid \mathcal{F}_s \right] dW(s) \right\rangle \mid \mathcal{F}_\tau \right) \right\}$$

$$= \mathbb{E}\left\{ \chi_{A_{ij}}(\omega)\mathbb{E}\left(\int_\tau^t \left\langle \Phi(\tau)\Phi(s)^{-1}\sigma_u(s)(v^k - \bar{u}(s))\chi_{A_{ij}}(\omega), \right. \right. \right.$$

$$\left. \left. \left. \mathbb{E}\left[\mathcal{D}_s\big(\mathbb{S}(t)^\top (v^k - \bar{u}(t))\big) \mid \mathcal{F}_s \right] \right\rangle ds \mid \mathcal{F}_\tau \right) \right\}$$

$$= \mathbb{E}\int_\tau^t \left\langle \Phi(\tau)\Phi(s)^{-1}\sigma_u(s)(v^k - \bar{u}(s))\chi_{A_{ij}}(\omega), \right.$$

$$\left. \mathbb{E}\left[\mathcal{D}_s\big(\mathbb{S}(t)^\top (v^k - \bar{u}(t))\big) \mid \mathcal{F}_s \right] \right\rangle \chi_{A_{ij}}(\omega) ds$$

$$= \int_\tau^t \mathbb{E}\left[\left\langle \Phi(\tau)\Phi(s)^{-1}\sigma_u(s)(v^k - \bar{u}(s)), \mathcal{D}_s\big(\mathbb{S}(t)^\top (v^k - \bar{u}(t))\big) \right\rangle \chi_{A_{ij}}(\omega) \right] ds.$$

Note that

$$\mathcal{D}_s\big(\mathbb{S}(t)^\top (v^k - \bar{u}(t))\big) = \mathcal{D}_s\mathbb{S}(t)^\top (v^k - \bar{u}(t)) - \mathbb{S}(t)^\top \mathcal{D}_s\bar{u}(t).$$

We have,

$$(4.23) \quad \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \int_\tau^t \mathbb{E}\left[\left\langle \Phi(\tau)\Phi(s)^{-1}\sigma_u(s)(v^k - \bar{u}(s)), \right. \right.$$

$$\mathcal{D}_s\big(\mathbb{S}(t)^\top(v^k-\bar{u}(t)))\big)\Big\rangle\chi_{A_{ij}}(\omega)\Big]dsdt$$

$$=\frac{1}{\theta^2}\int_\tau^{\tau+\theta}\int_\tau^t\mathbb{E}\ \Big[\Big\langle\Phi(\tau)\Phi(s)^{-1}\sigma_u(s)(v^k-\bar{u}(s)),$$

$$\mathcal{D}_s\mathbb{S}(t)^\top(v^k-\bar{u}(t))\Big\rangle\chi_{A_{ij}}(\omega)\Big]dsdt$$

$$-\frac{1}{\theta^2}\int_\tau^{\tau+\theta}\int_\tau^t\mathbb{E}\ \Big[\ \big\langle\Phi(\tau)\Phi(s)^{-1}\sigma_u(s)(v^k-\bar{u}(s)),\mathbb{S}(t)^\top\mathcal{D}_s\bar{u}(t)\big\rangle\ \chi_{A_{ij}}(\omega)\Big]dsdt.$$

For the first part in the right hand side of (4.23),

$$(4.24)\qquad\frac{1}{\theta^2}\int_\tau^{\tau+\theta}\int_\tau^t\mathbb{E}\ \Big[\Big\langle\Phi(\tau)\Phi(s)^{-1}\sigma_u(s)(v^k-\bar{u}(s)),$$

$$\mathcal{D}_s\mathbb{S}(t)^\top(v^k-\bar{u}(t)))\Big\rangle\chi_{A_{ij}}(\omega)\Big]dsdt$$

$$=\frac{1}{\theta^2}\int_\tau^{\tau+\theta}\int_\tau^t\mathbb{E}\ \Big[\Big\langle\Phi(\tau)\Phi(s)^{-1}\sigma_u(s)(v^k-\bar{u}(s)),$$

$$\big(\mathcal{D}_s\mathbb{S}(t)-\nabla\mathbb{S}(s)\big)^\top(v^k-\bar{u}(t))\big)\Big\rangle\chi_{A_{ij}}(\omega)\Big]dsdt$$

$$+\frac{1}{\theta^2}\int_\tau^{\tau+\theta}\int_\tau^t\mathbb{E}\ \Big[\Big\langle\Phi(\tau)\Phi(s)^{-1}\sigma_u(s)(v^k-\bar{u}(s)),$$

$$\nabla\mathbb{S}(s)^\top(v^k-\bar{u}(t)))\Big\rangle\chi_{A_{ij}}(\omega)\Big]dsdt.$$

Since

$$\Big|\frac{1}{\theta^2}\int_\tau^{\tau+\theta}\int_\tau^t\mathbb{E}\Big[\Big\langle\Phi(\tau)\Phi(s)^{-1}\sigma_u(s)(v^k-\bar{u}(s)),$$

$$\big(\mathcal{D}_s\mathbb{S}(t)-\nabla\mathbb{S}(s)\big)^\top(v^k-\bar{u}(t))\big)\Big\rangle\chi_{A_{ij}}(\omega)\Big]dsdt\Big|$$

$$\leq\frac{C}{\theta^2}\int_\tau^{\tau+\theta}\int_\tau^t\mathbb{E}\ \Big[\big|\Phi(\tau)\Phi(s)^{-1}\sigma_u(s)(v^k-\bar{u}(s))\big|\cdot\big|\mathcal{D}_s\mathbb{S}(t)-\nabla\mathbb{S}(s)\big|\Big]dsdt$$

$$\leq\frac{C}{\theta}\Big[\mathbb{E}\Big(\sup_{s\in[\tau,T]}|\Phi(\tau)\Phi(s)^{-1}|^2\Big)\Big]^{\frac{1}{2}}\cdot\Big[\mathbb{E}\int_\tau^{\tau+\theta}\int_\tau^t\big|\mathcal{D}_s\mathbb{S}(t)-\nabla\mathbb{S}(s)\big|^2dsdt\Big]^{\frac{1}{2}}$$

$$\leq\frac{C}{\theta}\Big[\mathbb{E}\int_\tau^{\tau+\theta}\int_\tau^t\big|\mathcal{D}_s\mathbb{S}(t)-\nabla\mathbb{S}(s)\big|^2dsdt\Big]^{\frac{1}{2}},$$

by Lemma 2.1, there exists a sequence $\{\theta_n\}_{n=1}^\infty$ such that $\theta_n\to0^+$ as $n\to\infty$ and

$$(4.25)\qquad\lim_{n\to\infty}\frac{1}{\theta_n^2}\int_\tau^{\tau+\theta_n}\int_\tau^t\mathbb{E}\Big[\Big\langle\Phi(\tau)\Phi(s)^{-1}\sigma_u(s)(v^k-\bar{u}(s)),$$

$$\big(\mathcal{D}_s\mathbb{S}(t)-\nabla\mathbb{S}(s)\big)^\top(v^k-\bar{u}(t))\Big\rangle\chi_{A_{ij}}(\omega)\Big]dsdt$$

$$=0\qquad a.e.\tau\in[0,T).$$

For the second part in the right hand side of (4.24), by Lemma 4.1 it follows that

$$(4.26)\qquad\lim_{\theta\to0^+}\frac{1}{\theta^2}\int_\tau^{\tau+\theta}\int_\tau^t\mathbb{E}\ \Big[\Big\langle\Phi(\tau)\Phi(s)^{-1}\sigma_u(s)(v^k-\bar{u}(s)),$$

$$\nabla\mathbb{S}(s)^\top(v^k - \bar{u}(t)))\Big\rangle\chi_{A_{ij}}(\omega)\Big]dsdt$$

$$= \frac{1}{2}\mathbb{E}\left[\Big\langle\nabla\mathbb{S}(\tau)\sigma_u(\tau)(v^k - \bar{u}(\tau)), v^k - \bar{u}(\tau)\Big\rangle\chi_{A_{ij}}(\omega)\right], \quad a.e. \ \tau \in [t_i, T).$$

Therefore, by (4.24)–(4.26), we conclude that

$$\text{(4.27)} \qquad \lim_{n\to\infty}\frac{1}{\theta_n^2}\int_\tau^{\tau+\theta_n}\int_\tau^t\mathbb{E}\left[\Big\langle\Phi(\tau)\Phi(s)^{-1}\sigma_u(s)(v^k - \bar{u}(s)),\right.$$

$$\mathcal{D}_s\mathbb{S}(t)^\top(v^k - \bar{u}(t))\Big\rangle\chi_{A_{ij}}(\omega)\Big]dsdt$$

$$= \frac{1}{2}\mathbb{E}\left[\Big\langle\nabla\mathbb{S}(\tau)\sigma_u(\tau)(v^k - \bar{u}(\tau)), v^k - \bar{u}(\tau)\Big\rangle\chi_{A_{ij}}\right], \quad a.e. \ \tau \in [t_i, T).$$

In a similar way, we can prove that there exists a subsequence $\{\theta_{n_l}\}_{l=1}^\infty$ of $\{\theta_n\}_{n=1}^\infty$ such that

$$\lim_{l\to\infty}\frac{1}{\theta_{n_l}^2}\int_\tau^{\tau+\theta_{n_l}}\int_\tau^t\mathbb{E}\left[\Big\langle\Phi(\tau)\Phi(s)^{-1}\sigma_u(s)(v^k - \bar{u}(s)), \mathbb{S}(t)^\top\mathcal{D}_s\bar{u}(t)\Big\rangle\chi_{A_{ij}}(\omega)\right]dsdt$$

$$= \frac{1}{2}\mathbb{E}\left[\Big\langle\mathbb{S}(\tau)\sigma_u(\tau)(v^k - \bar{u}(\tau)), \nabla\bar{u}(\tau)\Big\rangle\chi_{A_{ij}}(\omega)\right], \quad a.e. \ \tau \in [t_i, T).$$

To simplify the notation, we assume that the above $\{\theta_{n_l}\}_{l=1}^\infty$ is $\{\theta_n\}_{n=1}^\infty$ itself, that is

$$\lim_{n\to\infty}\frac{1}{\theta_n^2}\int_\tau^{\tau+\theta_n}\int_\tau^t\mathbb{E}\left[\Big\langle\Phi(\tau)\Phi(s)^{-1}\sigma_u(s)(v^k - \bar{u}(s)), \mathbb{S}(t)^\top\mathcal{D}_s\bar{u}(t)\Big\rangle\chi_{A_{ij}}(\omega)\right]dsdt$$

$$= \frac{1}{2}\mathbb{E}\left[\Big\langle\mathbb{S}(\tau)\sigma_u(\tau)(v^k - \bar{u}(\tau)), \nabla\bar{u}(\tau)\Big\rangle\chi_{A_{ij}}(\omega)\right], \quad a.e. \ \tau \in [t_i, T).$$
(4.28)

Combining (4.22), (4.23), (4.27) and (4.28), we obtain (4.20). Then, by (4.17)–(4.20), we obtain (4.16).

Finally, by (4.14), (4.15) and (4.16) we conclude that, for any $i, j, k \in \mathbb{N}$, there exists a Lebesgue measurable set $E_{i,j}^k \subset [t_i, T)$ with $|E_{i,j}^k| = 0$ such that

$$\text{(4.29)} \quad 0 \geq \frac{1}{2}\mathbb{E}\left[\Big\langle\mathbb{S}(\tau)b_u(\tau)(v^k - \bar{u}(\tau)), v^k - \bar{u}(\tau)\Big\rangle\chi_{A_{ij}}(\omega)\right]$$

$$+ \frac{1}{2}\mathbb{E}\left[\Big\langle\nabla\mathbb{S}(\tau)\sigma_u(\tau)(v^k - \bar{u}(\tau)), v^k - \bar{u}(\tau)\Big\rangle\chi_{A_{ij}}(\omega)\right]$$

$$- \frac{1}{2}\mathbb{E}\left[\Big\langle\mathbb{S}(\tau)\sigma_u(\tau)(v^k - \bar{u}(\tau)), \nabla\bar{u}(\tau)\Big\rangle\chi_{A_{ij}}(\omega)\right], \quad \forall \ \tau \in [t_i, T) \setminus E_{i,j}^k.$$

Let $E_0 = \bigcup_{i,j,k\in\mathbb{N}}E_{i,j}^k$, then $|E_0| = 0$, and for any $i, j, k \in \mathbb{N}$,

$$\mathbb{E}\left[\Big\langle\mathbb{S}(\tau)b_u(\tau)(v^k - \bar{u}(\tau)), v^k - \bar{u}(\tau)\Big\rangle\chi_{A_{ij}}(\omega)\right]$$

$$+ \mathbb{E}\left[\Big\langle\nabla\mathbb{S}(\tau)\sigma_u(\tau)(v^k - \bar{u}(\tau)), v^k - \bar{u}(\tau)\Big\rangle\chi_{A_{ij}}(\omega)\right]$$

$$- \mathbb{E}\left[\Big\langle\mathbb{S}(\tau)\sigma_u(\tau)(v^k - \bar{u}(\tau)), \nabla\bar{u}(\tau)\Big\rangle\chi_{A_{ij}}(\omega)\right]$$

$$\leq 0, \quad \forall \ \tau \in [t_i, T) \setminus E_0.$$

By the construction of $\{A_{ij}\}_{i=1}^{\infty}$, the continuity of the filter $\mathbb{F}$ and the density of $\{v^k\}_{k=1}^{\infty}$, we conclude that

$$
\begin{aligned}
&\langle \mathbb{S}(\tau)b_u(\tau)(v - \bar{u}(\tau)), v - \bar{u}(\tau)\rangle \\
&\quad + \langle \nabla\mathbb{S}(\tau)\sigma_u(\tau)(v - \bar{u}(\tau)), v - \bar{u}(\tau)\rangle \\
&\quad - \langle \mathbb{S}(\tau)\sigma_u(\tau)(v - \bar{u}(\tau)), \nabla\bar{u}(\tau)\rangle \\
&\leq 0, \quad a.s., \qquad \forall\, (\tau, v) \in ([0, T] \setminus E_0) \times U.
\end{aligned}
$$

This completes the proof of Theorem 3.9.

REFERENCES

[1] D. J. Bell and D. H. Jacobson, *Singular optimal control problems*, Mathematics in Science and Engineering, Vol. 117. Academic Press, London-New York, 1975.

[2] A. Bensoussan, *Lectures on stochastic control, in Nonlinear Filtering and Stochastic Control, pp. 1–62*, Lecture Notes in Math., 972, Springer-Verlag, Berlin, 1981.

[3] J. M. Bismut, *An introductory approach to duality in optimal stochastic control*, SIAM Rev., 20 (1978), pp. 62–78.

[4] J. F. Bonnans and F. J. Silva, *First and second order necessary conditions for stochastic optimal control problems*, Appl. Math. Optim., 65 (2012), pp. 403–439.

[5] P. Cannarsa, H. Frankowska and E. M. Marchini, *Existence and Lipschitz regularity of solutions to Bolza problems in optimal control*, Trans. Amer. Math. Soc., 361 (2009), pp. 4491–4517.

[6] F. H. Clarke and R. B. Vinter, *Regularity properties of optimal controls*, SIAM J. Control Optim., 28 (1990), pp. 980–997.

[7] D. J. Clements and B. D. O. Anderson, *Singular optimal control: the linear-quadratic problem*, Lecture Notes in Control and Information Sciences, Vol. 5. Springer-Verlag, Berlin-New York, 1978.

[8] N. El Karoui, S. Peng, and M. C. Quenez, *Backward stochastic differential equations in fanance*, Math. Finance, 7 (1997), pp. 1–71.

[9] H. Frankowska and D. Tonon, *Pointwise second-order necessary optimality conditions for the Mayer problem with control constraints*, SIAM, J. Control Optim., 51 (2013), pp. 3814–3843.

[10] R. Gabasov and F. M. Kirillova, *High order necessary conditions for optimality*, SIAM J. Control, 10 (1972), pp. 127–168.

[11] B. S. Goh, *Necessary conditions for singular extremals involving multiple control variables*, SIAM J. Control, 4 (1966), pp. 716–731.

[12] U. G. Haussmann, *General necessary conditions for optimal control of stochastic systems*, Math. Program. Study, 6 (1976), pp. 30–48.

[13] H.-W. Knobloch, *Higher order necessary conditions in optimal control theory*, Lecture Notes in Computer Science, 34. Springer-Verlag, Berlin-New York, 1981.

[14] A. J. Krener, *The high order maximal principle and its application to singular extremals*, SIAM J. Control Optim., 15 (1977), pp. 256–293.

[15] H. J. Kushner, *Necessary conditions for continuous parameter stochastic optimization problems*, SIAM J. Control Optim., 10 (1972), pp. 550–565.

[16] H. Lou, *Second-order necessary/sufficient conditions for optimal control problems in the absence of linear structure*, Discrete Contin. Dyn. Syst. Ser. B, 14 (2010), pp. 1445–1464.

[17] D. Nualart, *The Malliavin Calculus and Related Topics*, Second edition, Springer-Verlag, Berlin, 2006.

[18] S. Peng, *A general stochastic maximum principle for optimal control problems*, SIAM J. Control Optim., 28 (1990), pp. 966–979.

[19] S. Tang, *A second-order maximum principle for singular optimal stochastic controls*, Discrete Contin. Dyn. Syst. Ser. B, 14 (2010), pp. 1581–1599.

[20] J. Yong and X.Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Springer-Verlag, New York, Berlin, 2000.

[21] H. Zhang and X. Zhang, *Pointwise second-order necessary conditions for stochastic optimal controls, Part II: The general case*, preprint.