A characterization of $K_{2,4}$-minor-free graphs

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Abstract

We provide a complete structural characterization of $K_{2,4}$-minor-free graphs. The 3-connected $K_{2,4}$-minor-free graphs consist of nine small graphs on at most eight vertices, together with a family of planar graphs that contains $2n-8$ nonisomorphic graphs of order n for each $n \geq 5$ as well as K_4. To describe the 2-connected $K_{2,4}$-minor-free graphs we use *xy-outerplanar* graphs, graphs embeddable in the plane with a Hamilton xy-path so that all other edges lie on one side of this path. We show that, subject to an appropriate connectivity condition, xy-outerplanar graphs are precisely the graphs that have no rooted $K_{2,2}$ minor where x and y correspond to the two vertices on one side of the bipartition of $K_{2,2}$. Each 2-connected $K_{2,4}$-minor-free graph is then (i) outerplanar, (ii) the union of three xy-outerplanar graphs and possibly the edge xy, or (iii) obtained from a 3-connected $K_{2,4}$-minor-free graph by replacing each edge x_iy_i in a set $\{x_1y_1, x_2y_2, \ldots, x_ky_k\}$ satisfying a certain condition by an x_iy_i-outerplanar graph.

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From our characterization it follows that a $K_{2,4}$-minor-free graph has a hamilton cycle if it is 3-connected and a hamilton path if it is 2-connected. Also, every 2-connected $K_{2,4}$-minor-free graph is either planar, or else toroidal and projective-planar.

1 Introduction

The Robertson-Seymour Graph Minors project has shown that minor-closed classes of graphs can be described by finitely many forbidden minors. Excluding a small number of minors can give graph classes with interesting properties. The first such result was Wagner's demonstration [17] that planar graphs are precisely the graphs that are K_5- and $K_{3,3}$-minor-free.

Excluding certain special classes of graphs as minors seems to give close connections to other graph properties. One of the most important open problems at present is Hadwiger's Conjecture, which relates excluded complete graph minors to chromatic number. Our interest is in excluding complete bipartite graphs as minors. Together with connectivity conditions, and possibly other assumptions, graphs with no $K_{s,t}$ as a minor can be shown to have interesting properties relating to toughness, hamiltonicity, and other traversability properties. The simplest result of this kind follows from a well-known consequence of Wagner's characterization of planar graphs. This consequence says that 2-connected $K_{2,3}$-minor-free graphs are outerplanar or K_4; hence, they are hamiltonian. For some recent examples of this type of result, involving toughness, circumference, and spanning trees of bounded degree, see [1, 2, 13].

Our work was originally motivated by trying to find forbidden minor conditions to make 3-connected planar graphs, or 3-connected graphs more generally, hamiltonian. In examining the hamiltonicity of 3-connected $K_{2,4}$-minor-free graphs we were led to a complete picture of their structure, which we then extended to $K_{2,4}$-minor-free graphs in general. Using this, we show in Section 4 that 3-connected $K_{2,4}$-minor-free graphs are hamiltonian, and that 2-connected $K_{2,4}$-minor-free graphs have hamilton paths.

For $K_{2,4}$-minor-free graphs, or $K_{2,t}$-minor-free graphs in general, there are a number of previous results. Dieng and Gavoille (see Dieng's thesis [5]) showed that every 2-connected $K_{2,4}$-minor-free graph contains two vertices whose removal leaves the graph outerplanar. Streib and Young [16] used Dieng and Gavoille's result to show that the dimension of the minor poset of a connected graph G with no $K_{2,4}$ minor is polynomial in $|E(G)|$. Chen et al. [2] proved that 2-connected $K_{2,t}$-minor-free graphs have a cycle of length at least n/t^{t-1}. Myers [11] proved that a $K_{2,t}$-minor-free graph G with $t \geq 10^{29}$ satisfies $|E(G)| \leq (1/2)(t+1)(n-1)$; more recently Chudnovsky, Reed and Seymour [3] showed that this is valid for all $t \geq 2$, and provided stronger bounds for 2-, 3- and 5-connected graphs. Our results improve their bound for 3-connected graphs when $t = 4$. An unpublished paper of Ding [7] proposes that $K_{2,t}$-minor-free graphs can be built from slight variations of outerplanar graphs and graphs of bounded order by adding 'strips' and 'fans' using an operation that is a variant of a 2-sum (and which corresponds to the idea of replacing subdividable sets of edges that is used later in this paper). Ding's result involves subgraphs that have $K_{2,4}$ minors, and so not all aspects of his structure can be present in the case of $K_{2,4}$-minor-free graphs; our results illuminate the extent to which Ding's structure still holds.

As part of our work we use rooted minors, where particular vertices of G must correspond to certain vertices of H when we find H as a minor in G. For example, Robertson and Seymour [14] characterized all 3-connected graphs that have no $K_{2,3}$ minor rooted at the three vertices on one side of the bipartition. Fabila-Monroy and Wood [8] characterized graphs with no K_4 minor rooted at all four vertices. Demasi [4] characterized all 3-connected planar graphs with no $K_{2,4}$ minor rooted at the four vertices on one side of the bipartition. In this paper we characterize all graphs with no $K_{2,2}$ minor rooted at two vertices on one side of the bipartition. This result is useful not only here, but also in the authors' proof that 3-connected $K_{2,5}$-minor-free planar graphs are hamiltonian (see [9]).

We begin with some definitions and notation. All graphs are simple. We use '$-$' to denote set difference and deletion of vertices from a graph, '$\backslash$' to denote deletion of edges, '/' to denote contraction of edges, and '+' to denote both addition of edges and join of graphs. Since we work with simple graphs, when we contract an edge any parallel edges formed are reduced to a single edge.

A graph H is a *minor* of a graph G if H is isomorphic to a graph formed from G by contracting and deleting edges of G and deleting isolated vertices of G. We delete multiple edges and loops, so all minors are simple. Another way to think of a k-vertex minor H of G is as a collection of disjoint subsets of the vertices of G, $(V_1, V_2, \ldots, V_k)$ where each V_i corresponds to a vertex v_i of H, where $G[V_i]$ (the subgraph of G induced by the vertex set V_i) is connected for $1 \le i \le k$, and for each edge $v_iv_j \in E(H)$ there is at least one edge between V_i and V_j in G. We call this a *model* of H in G. We will often identify minors in graphs by describing the sets $(V_1, V_2, \ldots, V_k)$. The set V_i is known as the *branch set* of v_i, and may be thought of as the set of vertices in G that contracts to v_i in H.

Suppose we are given $S \subseteq V(G)$, $T \subseteq V(H)$, and a bijection $f : S \to T$. We say that a model of H in G is a *minor rooted at S in G and at T in H by f* if each $v \in S$ belongs to the branch set of $f(v) \in T$. If the symmetric group on T is a subgroup of the automorphism group of H (as it will be in our case) then the exact bijection f between S and T does not matter.

A graph is *H-minor-free* if it does not contain H as a minor. A *k-separation* in a graph G is a pair (H, J) of edge-disjoint subgraphs of G with $G = H \cup J$, $|V(H) \cap V(J)| = k$, $V(H) - V(J) \neq \emptyset$, and $V(J) - V(H) \neq \emptyset$.

Suppose $K_{2,t}$ has bipartition $(\{a_1, a_2\}, \{b_1, b_2, \ldots, b_t\})$. Let R_1 and R_2 be the branch sets of a_1 and a_2 in a model of $K_{2,t}$ in a graph G. Suppose B is the branch set of b_i for some i. Then there is a path $v_1v_2\ldots v_k$, $k \ge 3$, with $v_1 \in R_1$, $v_k \in R_2$, and $v_i \in B$ for $2 \le i \le k-1$. Let $B' = \{v_2\}$ and let $R_2' = R_2 \cup \{v_3, \ldots, v_{k-1}\}$. We can replace B with B' and R_2 with R_2' and still have a model of $K_{2,t}$ (possibly using fewer vertices of G than before). Hence without loss of generality we may assume that the branch set of each vertex b_i, $1 \le i \le t$, contains a single vertex s_i. Let $S = \{s_1, s_2, \ldots, s_t\}$. We say $(R_1, R_2; S)$ represents a *standard $K_{2,t}$ minor*. Observe that G contains a $K_{2,t}$ minor if and only if G contains a standard $K_{2,t}$ minor. Note that the standard model also applies to $K_{2,t}$ minors rooted at two vertices corresponding to a_1 and a_2.

A *wheel* is a graph $W_n = K_1 + C_{n-1}$ with $n \ge 4$. A vertex of degree $n-1$ in W_n is a *hub* and its incident edges are *spokes* while the remaining edges form a cycle called the *rim*. In $W_4 = K_4$ every vertex is a hub and every edge is both a spoke and a rim edge, but in W_n for $n \ge 5$ there is a unique hub and the edges are partitioned into spokes and rim edges. Note that we identify wheels by their number of vertices, rather than their number of spokes.

A graph is *outerplanar* if it has an *outerplane embedding*, an embedding in the plane with every vertex on the outer face.

In the next section, we define a class of graphs and describe several small examples which together make up all 3-connected $K_{2,4}$-minor-free graphs. We begin with 3-connected graphs because all 4-connected graphs on at least six vertices have a $K_{2,4}$ minor. This is obvious for complete graphs. Otherwise, a pair of nonadjacent vertices and the four internally disjoint paths between them guaranteed by Menger's Theorem yield a $K_{2,4}$ minor. In Section 3 we extend the characterization to 2-connected graphs. The generalization to all graphs follows because a graph that is not 2-connected is $K_{2,4}$-minor-free if and only if each of its blocks is $K_{2,4}$-minor-free. Section 4 presents applications of our characterization to hamiltonicity, topological properties, counting, and edge bounds.

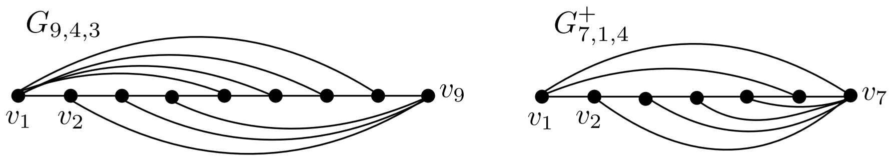


Figure 1

2 The 3-connected case

All graphs G with $|V(G)| < 6$ are trivially $K_{2,4}$-minor-free; the 3-connected ones are K_5, $K_5\backslash e$, W_5, and $K_4 = W_4$. For $|V(G)| \geq 6$, first we define a class of graphs and identify those that are 3-connected and $K_{2,4}$-minor-free. We then look at some small graphs that do not fit into this class. Finally, we show that every 3-connected $K_{2,4}$-minor-free graph is one of these we have described.

2.1 A class of graphs $G^{(+)}_{n,r,s}$

For $n \geq 3$ and $r, s \in \{0, 1, \ldots, n-3\}$, let $G_{n,r,s}$ consist of a spanning path $v_1v_2 \ldots v_n$, which we call the *spine*, and edges v_1v_{n-i} for $1 \leq i \leq r$ and v_nv_{1+j} for $1 \leq j \leq s$. The graph $G^+_{n,r,s}$ is $G_{n,r,s} + v_1v_n$; we call v_1v_n the *plus edge*. All graphs $G^{(+)}_{n,r,s}$ are planar. The graph $G^+_{n,1,n-3}$ is a wheel W_n with hub v_n. Examples are shown in Figure 1. Since $G^{(+)}_{n,r,s} \cong G^{(+)}_{n,s,r}$ we often assume $r \leq s$.

In the following three lemmas we first determine when a graph $G^{(+)}_{n,r,s}$ is 3-connected, and then when it is $K_{2,4}$-minor-free.

Lemma 2.1. *For $n \geq 4$, $G = G^{(+)}_{n,r,s}$ is 3-connected if and only if (i) $r = 1$, $s = n-3$, and the plus edge is present (or symmetrically $s = 1$, $r = n-3$, and the plus edge is present) or (ii) $r, s \geq 2$ and $r + s \geq n-2$.*

Proof. Assume that $r \leq s$. To prove the forward direction, assume G is 3-connected and first suppose $r = 1$. If the plus edge is not present, then v_1 has degree 2 and $\{v_2, v_{n-1}\}$ is a 2-cut. Similarly if $s \leq n-4$, then v_{n-2} has degree 2 and $\{v_{n-3}, v_{n-1}\}$ is a 2-cut. Next suppose $r, s \geq 2$. If $r + s \leq n-3$, then there is necessarily a degree 2 vertex v_i with $4 \leq i \leq n-3$ and hence a 2-cut in G.

To prove the reverse direction, assume (i) or (ii). If (i) holds, G is a wheel, which is 3-connected, so we may assume that (ii) holds. To show 3-connectedness we find three internally disjoint paths between each possible pair of vertices. For v_1 and v_n we have paths $v_1v_2v_n$, $v_1v_{n-1}v_n$, and $v_1v_{n-2}v_{n-3}...v_{n-r}v_{1+s}v_n$ (where possibly $v_{n-r} = v_{1+s}$). Next suppose that only one of v_1 and v_n is in the considered pair, say v_1 without loss of generality. First consider v_1 and v_i where $n-r \leq i \leq n-1$. When $v_1v_{i+1} \in E(G)$, then the three disjoint paths are $v_1v_2...v_i$, v_1v_i, and $v_1v_{i+1}v_i$. When $v_1v_{i+1} \notin E(G)$, then $v_1v_{i-1} \in E(G)$ and $v_{i-1} \neq v_2$ and the three disjoint paths are $v_1v_2v_nv_{n-1}...v_i$, $v_1v_{i-1}v_i$, and v_1v_i. Now consider v_1 and v_i where $2 \leq i \leq n-r-1$. Then the three disjoint paths are $v_1v_2...v_i$, $v_1v_{n-r}v_{n-r-1}...v_i$, and $v_1v_{n-r+1}v_{n-r+2}...v_nv_i$. Finally consider v_i and v_j where $i < j$ and $i, j \neq 1, n$. If v_i and v_j are both adjacent to the same end vertex, say v_1, where $i, j \neq 2$, then the three disjoint paths are $v_iv_{i+1}...v_j$, $v_iv_1v_j$, and $v_iv_{i-1}...v_2v_nv_{n-1}...v_j$. Otherwise the three disjoint paths are $v_iv_{i+1}...v_j$, $v_iv_{i-1}...v_1v_j$, and $v_jv_{j+1}...v_nv_i$. □

Lemma 2.2. *For $n \geq 6$, $G = G^{(+)}_{n,r,s}$ is $K_{2,4}$-minor-free if and only if $r + s \leq n-1$.*

Proof. To prove the forward direction, suppose $r + s \geq n$. Then there are vertices v_i and v_{i+1} such that both v_1 and v_n are adjacent to both v_i and v_{i+1} and $3 \leq i \leq n-3$. Then there is a standard $K_{2,4}$ minor $(R_1, R_2; S)$ in G: let $S = \{v_2, v_i, v_{i+1}, v_{n-1}\}$, $R_1 = \{v_1\}$, and $R_2 = \{v_n\}$.

Now suppose that $r + s \leq n-1$. We claim that if G has a standard $K_{2,4}$ minor $(R_1, R_2; S)$, then $v_1 \in R_1$ and $v_n \in R_2$ (or vice versa). The graph $G - v_1$ is outerplanar and thus has no $K_{2,3}$ minor. Therefore, if G

has a $K_{2,4}$ minor, then it must include v_1. We cannot have $v_1 \in S$ because then the outerplanar graph $G - v_1$ would have a $K_{2,3}$ minor. By symmetry, v_n must also be included in the minor and $v_n \notin S$. If $v_1, v_n \in R_i$, then $G - \{v_1, v_n\}$ has a $K_{1,4}$ minor, but $G - \{v_1, v_n\}$ is a path and there is no $K_{1,4}$ minor in a path. The only remaining possibility is $v_1 \in R_1$ and $v_n \in R_2$ (or vice versa).

Let $N(v)$ denote the set of neighbors of v. Let $A = N(v_1) - \{v_2\} = \{v_{n-r}, v_{n-r+1}, \ldots, v_{n-1}\}$ and $B = N(v_n) - \{v_{n-1}\} = \{v_2, v_3, \ldots, v_{s+1}\}$, which intersect only if $v_{n-r} = v_{s+1}$. Suppose G has a standard $K_{2,4}$ minor $(R_1, R_2; S)$. Then by the claim proved in the previous paragraph, $v_1 \in R_1$ and $v_n \in R_2$. We consider the makeup of S. Suppose $\{s_1, s_2, s_3\} \subseteq S \cap A$, in that order along the spine. Since $\{v_1, s_1, s_3\} \subseteq R_1 \cup \{s_1, s_3\}$ separates s_2 and v_n, and $v_n \in R_2$, we cannot have R_2 adjacent to s_2, which is a contradiction. Thus $|S \cap A| \leq 2$. Symmetrically, $|S \cap B| \leq 2$. We must have $s_1, s_2 \in S \cap A$ and $s_3, s_4 \in S \cap B$ in the order s_4, s_3, s_2, s_1 along the spine. Since $v_n \in R_2$, there must be a $v_n s_2$-path in $G - \{v_1, s_1, s_3, s_4\}$, and hence $s_3 \neq v_{s+1}$. Then v_{s+1} is a cutvertex separating v_n and s_2 in $G - \{v_1, s_1, s_3, s_4\}$, so $v_{s+1} \in R_2$. Now there must also be a $v_1 s_3$-path in $G - \{v_n, v_{s+1}, s_4\}$ but no such path exists. Thus there is no $K_{2,4}$ minor. $\square$

Define $\mathcal{G}$ to be the set of (labeled) graphs of the form $G^{(+)}_{n,r,s}$ that are both 3-connected and $K_{2,4}$-minor-free. Of the four 3-connected graphs on fewer than six vertices, three are planar, and all three belong to $\mathcal{G}$: $K_5 \backslash e \cong G^+_{5,2,2}$, $W_5 \cong G^+_{5,1,2} \cong G_{5,2,2}$, and $K_4 = W_4 \cong G^+_{4,1,1}$. From this and Lemmas 2.1 and 2.2 we get

$$\mathcal{G} = \{G^+_{n,1,n-3}, G^+_{n,n-3,1} : n \geq 4\} \cup \{G^{(+)}_{n,r,s} : n \geq 5,\ r, s \in \{2, 3, \ldots, n-3\},\ r + s = n-1 \text{ or } n-2\}.$$

Let $\widetilde{\mathcal{G}}$ denote the class of all graphs isomorphic to a graph in $\mathcal{G}$. Note that graphs in $\mathcal{G}$ are 3-sums of two wheels, a fact we will see in more detail later on.

There are some isomorphisms between graphs in $\mathcal{G}$ and also symmetries within certain graphs of the class. Let $\rho = \rho_n$ be the involution with $\rho(v_i) = v_{n+1-i}$ for $1 \leq i \leq n$. Then ρ provides the isomorphism (in both directions) between $G^{(+)}_{n,r,s}$ and $G^{(+)}_{n,s,r}$ that we have already noted; if $r = s$ it is an automorphism. The graph $G^+_{n,1,n-3}$ is isomorphic to W_n, with v_n as a hub. It has the obvious symmetries.

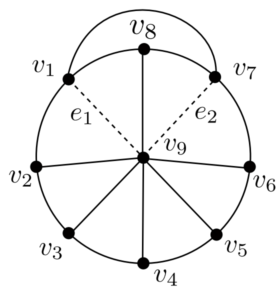


Figure 2

Define $\sigma = \sigma_n$ to be the involution fixing v_{n-1} and v_n and with $\sigma(v_i) = v_{n-1-i}$ for $1 \leq i \leq n-2$. Then σ is an automorphism of $G_{n,2,n-4}$, an isomorphism (in both directions) between $G^+_{n,2,n-4}$ and $G_{n,2,n-3}$, and an automorphism of $G^+_{n,2,n-3}$. The case $n = 9$ is illustrated in Figure 2, where $\sigma = \sigma_9$ corresponds to reflection about a vertical axis. The graph without the dashed edges e_1 and e_2 is $G_{9,2,5}$. With the edge e_1, the graph is $G^+_{9,2,5}$ and with e_2, the graph is $G_{9,2,6}$. With both edges e_1 and e_2, the graph is $G^+_{9,2,6}$. In general σ maps the spine $P = v_1 v_2 \ldots v_n$ to the path $\sigma(P) = v_{n-2} v_{n-3} \ldots v_2 v_1 v_{n-1} v_n$. For $G^{(+)}_{n,r,s}$ with $r = 2$ we call $\sigma(P)$ the *second spine*. When $s = 2$ we have a similar involution σ', and the path $\sigma'(P) = v_1 v_2 v_n v_{n-1} \ldots v_4 v_3$ can be regarded as an extra spine. When $r = s = 2$, $\sigma'(P)$ is the image of $\sigma(P)$ under the automorphism ρ.

Finally, besides some obvious special symmetries when $n = 4$ or 5, $G_{6,2,2}$ is vertex-transitive and is isomorphic to the triangular prism.

These symmetries and isomorphisms will be important later, particularly in Section 3 when we discuss which edges of $G \in \mathcal{G}$ can be subdivided without creating a $K_{2,4}$ minor. Up to isomorphism the class $\mathcal{G}$ contains one 4-vertex graph and $2n-8$ n-vertex graphs for each $n \geq 5$.

We now examine the effect of deleting or contracting a single edge of a graph in $\mathcal{G}$.

Lemma 2.3. *Suppose $G = G^{(+)}_{n,r,s} \in \mathcal{G}$ and $e \in E(G)$. The following are equivalent.*
(i) $G\backslash e \in \mathcal{G}$.
(ii) $G\backslash e$ is 3-connected.
(iii) G is not a wheel and either e is a plus edge, or $r+s=n-1$ and $e \in \{v_1v_{n-r}, v_nv_{1+s}\}$.

Proof. Clearly (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii). If (iii) does not hold then $G\backslash e$ has at least one vertex of degree 2, so (ii) does not hold; thus (ii) $\Rightarrow$ (iii). □

Table 1: Contracting an edge e in $G = G^{(+)}_{n,r,s}$ with $r, s \geq 2$

e	G/e isomorphic to	G/e is 3-conn.?	$G/e \in \widetilde{\mathcal{G}}$?
spine edges			
$*\ v_{1+s}v_{n-r},\quad r+s=n-2$	$G^{(+)}_{n-1,r,s}$	yes	yes
$v_{n-i}v_{n-i+1},\quad 2 \leq i \leq r,\ r \geq 3$	$G^{(+)}_{n-1,r-1,s}$	yes	yes
$v_{n-2}v_{n-1},\quad r=2$	$G^{(+)}_{n-1,1,n-4}$	if plus edge	if plus edge
$*\ v_{n-1}v_n,\quad r \geq 3$	$G^{+}_{n-1,r-1,s}$	yes	yes
$*\ v_{n-1}v_n,\quad r=2$	$G^{+}_{n-1,1,n-4} \cong W_{n-1}$	yes	yes
non-spine edges			
v_1v_n (plus edge)	K_1+P_{n-2}	no	no
$v_1v_{n-i},\quad 2 \leq i \leq r-1$	$G^{(+)}_{n-1,r-1,s}\backslash v_{n-i-1}v_{n-i}$	no	no
$v_1v_{n-1},\quad r \geq 3$ or $r=2$ and $s=n-4$	$G^{+}_{n-1,r-1,s}\backslash v_{n-2}v_{n-1}$	no	no
$v_1v_{n-1},\quad r=2$ and $s=n-3$	$G^{+}_{n-1,1,n-4} \cong W_{n-1}$	yes	yes
$v_1v_{n-r},\quad r+s=n-2$	$G^{(+)}_{n-1,r,s}\backslash v_{n-r-1}v_{n-r}$	no	no
$v_1v_{n-r},\quad r+s=n-1,\ s \geq 3$	$G^{+}_{n-1,r,s-1}\backslash v_{n-r-1}v_{n-r}$	no	no
$v_1v_{n-r},\quad r+s=n-1,\ s=2$	$G^{+}_{n-1,n-4,1}\backslash v_2v_3$	no	no

Now consider contracting an edge e of $G = G^{(+)}_{n,r,s}$. If $n=4$ then $G = K_4$ and $G/e \cong K_3$ for any edge e, so assume that $n \geq 5$. If G is a wheel W_n then we obtain W_{n-1} if we contract a rim edge, and K_1+P_{n-2} if we contract a spoke. Therefore assume G is not a wheel, so $r, s \geq 2$. The effects of contracting edges in this case are shown in Table 1. Here the superscript '$^{(+)}$' means that the plus edge is present in G/e precisely if it is present in G. Edges not included in the table are covered by the symmetry ρ that swaps r and s, v_i and v_{n+1-i}. We may summarize the results as follows.

Lemma 2.4. *Suppose $G = G^{(+)}_{n,r,s} \in \mathcal{G}$ and $e \in E(G)$.*
(i) If $n \geq 5$ then G/e is isomorphic to a graph in $\mathcal{G}$ with at most one edge deleted.
(ii) If $n \geq 4$ then $G/e \in \widetilde{\mathcal{G}}$ if and only if G/e is 3-connected.
(iii) If G is a wheel with $n \geq 5$ then some G/e is isomorphic to W_{n-1}, and every $G/e \in \widetilde{\mathcal{G}}$ is isomorphic to W_{n-1}. If G is not a wheel then (from the starred entries in Table 1) some G/e is isomorphic to each of $G^{+}_{n-1,r-1,\min(s,n-4)}$, $G^{+}_{n-1,\min(r,n-4),s-1}$ and, if $r+s=n-2$, also $G^{(+)}_{n-1,r,s}$; and any $G/e \in \widetilde{\mathcal{G}}$ is isomorphic to a spanning subgraph of one of these.

Now we apply these results to the structure of minors of graphs in $\mathcal{G}$ or $\widetilde{\mathcal{G}}$.

Corollary 2.5. *Every minor of a graph in $\widetilde{\mathcal{G}}$ is a subgraph of some graph in $\widetilde{\mathcal{G}}$.*

Proof. Apply Lemma 2.4(i) repeatedly to replace contractions by deletions (details are left to the reader). □

Lemma 2.6. *If a 3-connected graph H is a minor of a 3-connected graph G, then there is a sequence of 3-connected graphs $G_0, G_1, \ldots, G_k$ where $G_0 \cong G$, $G_k \cong H$, and each G_{i+1} is obtained from G_i by contraction or deletion of a single edge.*

Proof. Seymour's Splitter Theorem [15] as applied to graphs, or a similar result of Negami [12], says that our result is true if H is not a wheel, or if H is the largest wheel minor of G. Seymour's operations and connectivity are defined for graphs with loops and multiple edges, not simple graphs, which is why a sequence of minors $G_0, G_1, \ldots, G_i$ cannot be continued to reduce a large wheel minor G_i to a smaller one. In particular, contracting a rim edge of a wheel in his definition yields a pair of parallel edges and so by his definition the graph is not 3-connected. With our definition, where we reduce parallel edges to a single edge after contraction, we can contract a rim edge of a wheel W_ℓ, $\ell \geq 5$, to obtain the smaller wheel $W_{\ell-1}$, which is still 3-connected. Therefore, we can continue the sequence of operations to also reach wheel minors H that are not the largest wheel minor. □

Corollary 2.7. *If H is a 3-connected minor of $G \in \widetilde{\mathcal{G}}$ then $H \in \widetilde{\mathcal{G}}$.*

Proof. Take the 3-connected sequence $G \cong G_0, G_1, \ldots, G_k \cong H$ given by Lemma 2.6. From Lemmas 2.3 and 2.4(ii), if $G_i \in \widetilde{\mathcal{G}}$ then $G_{i+1} \in \widetilde{\mathcal{G}}$ also, and the result follows by induction. □

2.2 Small cases

Figure 3 shows nine small graphs that are 3-connected (easily checked), not in $\mathcal{G}$ (also easily checked; all but D have a $K_{3,3}$ minor and so are nonplanar) and $K_{2,4}$-minor-free. The first graph, K_5, is the only 3-connected graph on fewer than six vertices that is not in $\mathcal{G}$. To prove that the other eight graphs are $K_{2,4}$-minor-free we examine the two maximal graphs C^+ and D, and show that the rest are minors of C^+.

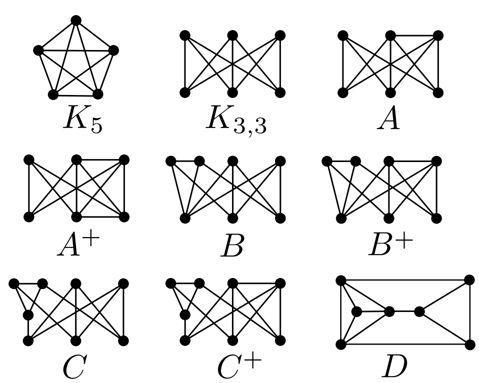


Figure 3

Lemma 2.8. *The graph C^+ is $K_{2,4}$-minor-free.*

Proof. Consider C^+ with vertices labeled as on the left in Figure 4. Suppose there is a standard $K_{2,4}$ minor $(R_1, R_2; S)$ in C^+ and suppose $|R_1| = 1$. Then R_1 must be either v_4 or v_5 since these are the only vertices of degree 4. Say, without loss of generality, $R_1 = \{v_4\}$. Then $S = \{v_5, v_6, v_7, v_8\}$, and R_2 must be a subset of $\{v_1, v_2, v_3\}$. None of these three vertices are adjacent to v_5, however, so we cannot have R_2 adjacent to v_5 and thus we cannot have $|R_1| = 1$, or symmetrically $|R_2| = 1$. Thus $|R_1| \geq 2$ and $|R_2| \geq 2$ and since $|V(C^+)| = 8$, $|R_1| = |R_2| = 2$.

Let T be a triangle with a set N of neighbors with $|N| = 3$. Suppose $R_1 \subseteq V(T)$. Then we would have $N \subseteq S$ along with the third vertex t of T, but N separates t from the rest of the graph so R_2 cannot be adjacent to t. Thus R_1 (or symmetrically R_2) cannot consist of two vertices in a triangle with only three neighbors. In C^+, we have the following triples of vertices which form such triangles: $\{v_1, v_2, v_3\}$, $\{v_4, v_5, v_6\}$, $\{v_4, v_5, v_7\}$, and $\{v_4, v_5, v_8\}$. The only remaining pairs of adjacent vertices that could make up R_1 or R_2 are $\{v_3, v_6\}$, $\{v_2, v_8\}$, and $\{v_1, v_7\}$ where all three cases are symmetric. If $R_1 = \{v_3, v_6\}$, then R_2 must be $\{v_7, v_8\}$ but this set is not an option for R_2. $\square$

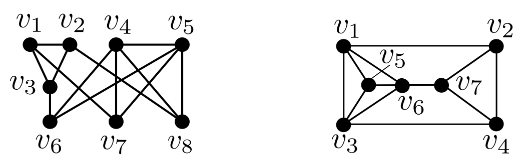


Figure 4

Lemma 2.9. *The graph D is $K_{2,4}$-minor-free.*

Proof. It is easy to check that D has no subgraph isomorphic to $K_{2,4}$, nor does D/e for any $e \in E(D)$. Hence D is $K_{2,4}$-minor-free since $|V(D)| = 7$. $\square$

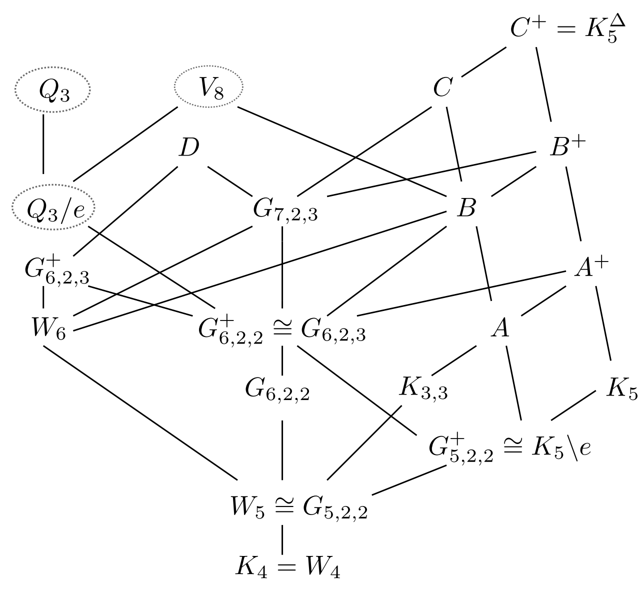


Figure 5: Q_3, V_8, $C^+ = K_5^\Delta$, D, and their 3-connected minors

Figure 5 shows what we will prove is the Hasse diagram for the minor ordering of all 3-connected minors of C^+ (also labeled K_5^Δ, following Ding and Liu [6]) and D. For future reference the figure also includes three additional, circled graphs Q_3 (the cube), Q_3/e (contract any edge of Q_3) and V_8 (the 8-vertex twisted cube or Möbius ladder). Unlike the other graphs, these three have $K_{2,4}$ minors, as shown by the minor in Q_3/e on the right in Figure 6. Here, and later, a $K_{2,4}$ minor is indicated by two groups of vertices circled by

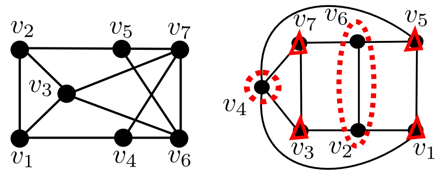


Figure 6

dotted curves representing the two vertices in one part of the bipartition of $K_{2,4}$, and four triangular vertices representing the other part.

Lemma 2.10. *Figure 5 is the Hasse diagram for all* 3*-connected minors (up to isomorphism) of* C^+*,* D*,* Q_3 *and* V_8*.*

Proof. By Lemma 2.6, we can proceed by single edge deletions and contractions, and we do not need to consider further minors once we reach a graph that is not 3-connected. The figure is clearly correct for the 3-connected graphs on four or five vertices, so we consider only graphs with at least six vertices. Also, the 3-connected minors for graphs in $\mathcal{G}$ follow from Lemmas 2.3 and 2.4(iii), so we consider only graphs not in $\mathcal{G}$.

In what follows results of all deletions or contractions are identified only up to isomorphism. When we lose 3-connectivity, in all but one case there will be at least one vertex of degree 2. We work upwards in the figure.

For the graphs $K_{3,3}$, A and A^+ label the vertices consecutively along the top row then the bottom row in Figure 3. For $K_{3,3}$, deleting any edge loses 3-connectivity; contracting any edge results in W_5. For A, deleting v_2v_3 yields $K_{3,3}$, and deleting any other edge loses 3-connectivity. Contracting an edge incident with v_1 yields $K_5\backslash e$, contracting v_2v_3 loses 3-connectivity, and contracting any other edge incident with v_2 or v_3 yields W_5. For A^+, all edges are equivalent up to symmetry to one of v_1v_4, v_1v_5, v_2v_3 or v_2v_5. Deleting v_1v_4 or v_1v_5 loses 3-connectivity, deleting v_2v_3 gives A, and deleting v_2v_5 gives $G_{6,2,3}$. Contracting v_1v_4 gives K_5, contracting v_1v_5 gives $K_5\backslash e$, contracting v_2v_3 loses 3-connectivity, and contracting v_2v_5 yields W_5.

For B and B^+ we redraw B^+ as on the left in Figure 6 and take $B = B^+\backslash v_6v_7$. For B, deleting any edge loses 3-connectivity. Up to symmetry, there are five edge contractions to consider: v_1v_2, v_1v_3, v_1v_4, v_3v_6 and v_4v_6. Contracting v_1v_2 yields $K_{3,3}$, contracting v_1v_3 loses 3-connectivity, contracting v_1v_4 yields A, contracting v_3v_6 results in W_6, and contracting v_4v_6 gives $G_{6,2,3}$. For B^+, all edges are equivalent up to symmetry to six possibilities: v_1v_2, v_1v_3, v_1v_4, v_3v_6, v_4v_6 and v_6v_7. Deleting v_1v_2, v_1v_3, v_1v_4 or v_4v_6 loses 3-connectivity, deleting v_3v_6 yields $G_{7,2,3}$, and deleting v_6v_7 results in B. Contracting v_1v_3 or v_6v_7 loses 3-connectivity, contracting v_1v_2 results in A, contracting v_1v_4 yields A^+, contracting v_3v_6 gives W_6, and contracting v_4v_6 gives $G_{6,2,3}$.

For C^+ we label the vertices as on the left in Figure 4 and take $C = C^+\backslash v_4v_5$. Deleting any edge of C loses 3-connectivity. Up to symmetry, there are three edge contractions to consider: v_1v_2, v_1v_7 and v_4v_6. Contracting v_1v_2 loses 3-connectivity, contracting v_1v_7 results in B, and contracting v_4v_6 yields $G_{7,2,3}$. For C^+, deleting v_4v_5 yields C and deleting any other edge loses 3-connectivity. Up to symmetry, there are four edge contractions of C^+ to consider: v_1v_2, v_1v_7, v_4v_5 and v_4v_6. Contracting v_1v_2 or v_4v_5 loses 3-connectivity, contracting v_1v_7 results in B^+, and contracting v_4v_6 gives $G_{7,2,3}$.

We label D as on the right in Figure 4. Up to symmetry all edges are equivalent to one of four edges: v_1v_3, v_2v_4, v_5v_6 and v_6v_7. Deleting v_1v_3 results in $G_{7,2,3}$, and deleting any of the other three edges loses 3-connectivity. Contracting v_1v_3 or v_2v_4 loses 3-connectivity, contracting v_5v_6 yields the triangular prism $G_{6,2,2}$, and contracting v_6v_7 results in $G^+_{6,2,3}$.

Finally, consider Q_3/e, Q_3 and V_8. Label Q_3/e as shown on the right in Figure 6. Every edge in Q_3/e is adjacent to a degree 3 vertex so deleting any edge loses 3-connectivity. Up to symmetry, there are four edge contractions to consider: v_1v_2, v_3v_4, v_2v_6 and v_3v_7. Contracting v_3v_4 loses 3-connectivity, and contracting v_2v_6 also loses 3-connectivity (without creating a vertex of degree 2). Contracting v_1v_2 results in $G_{6,2,3}$, and contracting v_3v_7 yields $G_{6,2,2}$. In the cube Q_3 all edges are symmetric; deleting any edge loses 3-connectivity, and contracting any edge yields Q_3/e. We may take V_8 to be $C_8 = (v_1v_2 \dots v_8)$ with added diagonals v_iv_{i+4} for $1 \leq i \leq 4$. Deleting any edge loses 3-connectivity, contracting a C_8 edge results in B, and contracting a diagonal yields Q_3/e. □

Considering the minors of C^+, we obtain the following.

Corollary 2.11. *The graphs C, B^+, B, A^+, A, and $K_{3,3}$ are $K_{2,4}$-minor-free.*

2.3 Characterization of 3-connected graphs

Theorem 2.12. *Let G be a 3-connected graph. Then G is $K_{2,4}$-minor-free if and only if $G \in \widetilde{\mathcal{G}}$ or G is isomorphic to one of the nine small exceptions shown in Figure 3.*

Our original proof of this theorem examined the structure of a 3-connected $K_{2,4}$-minor-free graph relative to a longest non-hamilton cycle in the graph. We analyzed cases and either derived a contradiction with a longer non-hamilton cycle or a $K_{2,4}$ minor, or found a desired graph. However, we then discovered the recent systematic investigation by Ding and Liu [6], characterizing H-minor-free graphs for all 3-connected graphs H on at most eleven edges. These allow us to give a shorter proof, which we present here.

First we give some definitions. Denote by Oct$\backslash e$ the graph obtained from the octahedron by removing one edge. A 3-*sum* of two 3-connected graphs G_1 and G_2 is a graph G obtained by identifying a triangle of G_1 with a triangle of G_2 and possibly deleting some of the edges of the common triangle as long as no degree 2 vertices are created. Any 2-cut in G would lead to a 2-cut in either G_1 or G_2 so G is 3-connected. An example is the graph C^+ which is a 3-sum of K_5 and a triangular prism. A *common* 3-*sum* of three or more graphs is formed by specifying one triangle in each graph and identifying all as a single triangle called the *common triangle*; again edges of the common triangle may be deleted as long as no degree 2 vertices are created. Let $\mathcal{S}$ be the set of all graphs formed by taking common 3-sums of wheels and triangular prisms. All graphs in $\mathcal{S}$ are 3-connected. We use the following result due to Ding and Liu.

Theorem 2.13 (Ding and Liu [6])**.** *Up to isomorphism the family of 3-connected Oct$\backslash e$-minor-free graphs consists of graphs in $\mathcal{S}$ and 3-connected minors of V_8, Q_3, and C^+.*

Proof of Theorem 2.12. The results of subsections 2.1 and 2.2 give the reverse direction of the proof.

For the forward direction, Oct$\backslash e$ contains $K_{2,4}$ as a subgraph, so all 3-connected $K_{2,4}$-minor-free graphs must be Oct$\backslash e$-minor-free graphs as described in Theorem 2.13. We must decide which of those graphs are actually $K_{2,4}$-minor-free. By Lemma 2.10, Figure 5 gives all 3-connected minors of V_8, Q_3, and C^+ up to isomorphism. The $K_{2,4}$-minor-free ones are uncircled; all are in $\mathcal{G}$ or one of the nine small exceptions.

So we must determine which members of $\mathcal{S}$ are $K_{2,4}$-minor-free. Any common 3-sum of four or more graphs has a $K_{3,4}$ minor (the three vertices of the common triangle form the part of size three) and hence a $K_{2,4}$ minor. Thus, we consider common 3-sums of at most three graphs, analyzed according to the numbers of wheels and prisms.

First consider a common 3-sum of three wheels, W_k, W_ℓ, and W_m. For $k = \ell = 5$ and $m = 4$, since all vertices of $W_4 = K_4$ are equivalent, there are two ways up to symmetry to form a common 3-sum (disregarding the possible existence of the edges of the common triangle): the hubs of the two wheels are either identified or not. Both result in a $K_{2,4}$ minor, as shown in the left and middle pictures of Figure 7. The dashed edges are the edges of the common triangle which may or may not be present in the common 3-sum. Since graphs with $k, \ell \geq 5$ and $m \geq 4$ have one of these two graphs as a minor, these graphs also

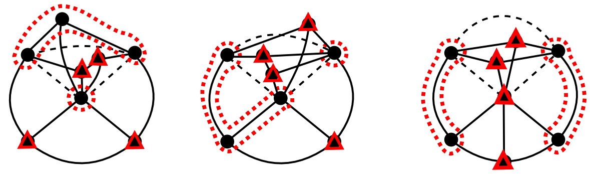


Figure 7

have $K_{2,4}$ minors. Hence at most one of k, ℓ, m can be greater than 4. When $k = 6$ and $\ell = m = 4$, there is again a $K_{2,4}$ minor, shown on the right in Figure 7. Graphs with $k > 6$ and $\ell = m = 4$ have this graph as a minor and hence also have a $K_{2,4}$ minor. For $k = 5$ and $\ell = m = 4$, we have the graph shown on the left and middle in Figure 8. With no dashed edges of the common triangle, this graph is isomorphic to B. With at least one dashed edge there is a $K_{2,4}$ minor as shown on the left of the figure for e_1 (e_2 is symmetric) or in the middle for e_3. Hence $k = \ell = m = 4$, and we have the graph shown on the right in Figure 8. With any two dashed edges, the graph has a $K_{2,4}$ minor, shown in the figure for e_1 and e_2. With no or one dashed edge, the graph is isomorphic to $K_{3,3}$ or A, respectively.

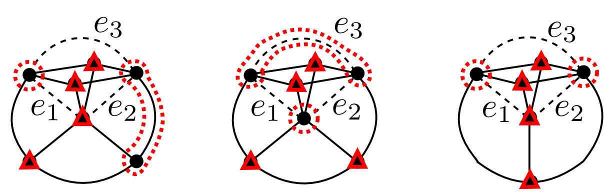


Figure 8

Next consider a common 3-sum of two wheels and a prism. If the wheels are W_5 and W_4, then all common 3-sums have the $K_{2,4}$ minor shown on the left in Figure 9. Any other combination of wheels gives this, and hence $K_{2,4}$, as a minor, unless both wheels are W_4. Then we have the graph shown on the right in Figure 9. With any dashed edge we have a $K_{2,4}$ minor, shown in the figure for e_1. With no dashed edges, the graph is isomorphic to C.

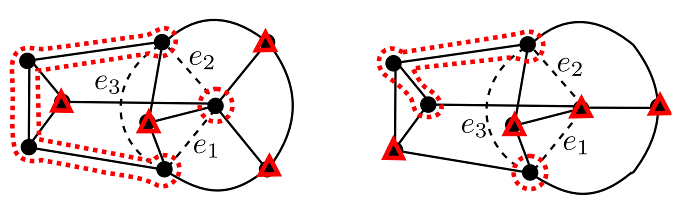


Figure 9

Now consider a common 3-sum of two wheels W_k and W_ℓ. Suppose the hubs of the wheels are not identified, or $k = 4$ or $\ell = 4$. We have the graph shown on the left in Figure 10. At least one of the edges labeled e_1 and e_2 must be present in the common 3-sum to ensure there are no degree 2 vertices. Let $n = k + \ell - 3$. With e_1 and e_2, the graph is isomorphic to $G_{n,k-2,\ell-2}$. With e_1 (or symmetrically e_2), the graph is isomorphic to either $G_{n,k-3,\ell-2}$ or $G_{n,k-2,\ell-3}$. In all cases e_3 is the optional plus edge. The spine is shown in the figure as the thick, highlighted path. Hence we obtain graphs in $\widetilde{\mathcal{G}}$.

Now suppose that $k, \ell \geq 5$ and the hubs of W_k and W_ℓ are identified in the common 3-sum. The graph with $k = \ell = 5$ appears on the right in Figure 10. With the edge labeled e_1, we have the $K_{2,4}$ minor shown, and if $k, \ell \geq 5$ we get a similar minor. Without e_1, both e_2 and e_3 must be present to ensure there are no vertices of degree 2, and the graph is isomorphic in the general case to $W_{k+\ell-3} \in \widetilde{\mathcal{G}}$.

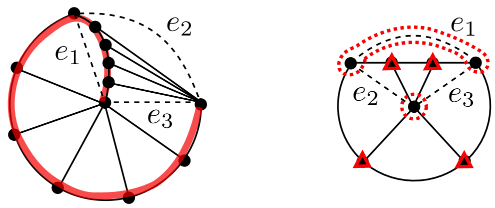


Figure 10

Now consider a common 3-sum of two prisms and one wheel. For W_4 we have the graph on the left in Figure 11 with the $K_{2,4}$ minor shown; for any larger wheel we get this graph, and hence $K_{2,4}$, as a minor.

Next consider a common 3-sum of two or three prisms. For two prisms we have the graph on the right in Figure 11. At least two dashed edges are needed to prevent a degree 2 vertex and so we have the $K_{2,4}$ minor shown. In a common 3-sum of three prisms, the dashed edges need not be present to ensure 3-connectivity. However, instead of using one of the dashed edges in the $K_{2,4}$ minor as on the right in Figure 11, we can use a path between these two vertices through the third prism. Hence a similar $K_{2,4}$ minor exists.

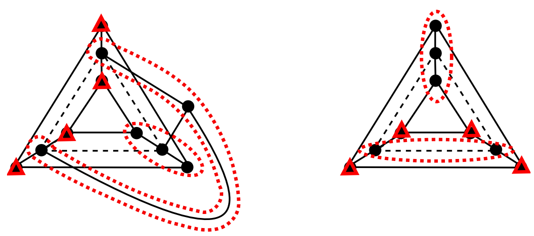


Figure 11

Consider a common 3-sum of one wheel W_k and one prism; this is unique up to isomorphism. Figure 12 shows the graph for $k = 5$ on the left. To prevent vertices of degree 2, either e_1 is present, in which case we have the $K_{2,4}$ minor shown, or the other two dashed edges must exist, and the graph is isomorphic to $G_{8,2,4}$. For $k \geq 6$ there is a similar minor or the graph is isomorphic to $G_{k+3,2,k-1}$. The graph for $k = 4$ is shown on the right in Figure 12. At least two dashed edges must be present to prevent degree 2 vertices. With two or three dashed edges the graph is isomorphic to $G_{7,2,3}$ or D, respectively.

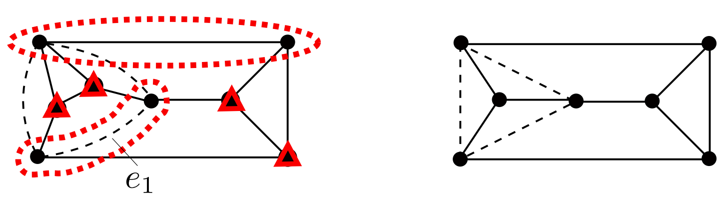


Figure 12

Finally, a common 3-sum of a single graph is $W_k \cong G^+_{k,1,k-3} \in \mathcal{G}$ or the triangular prism, isomorphic to $G_{6,2,2} \in \mathcal{G}$. $\square$

In [6] Ding and Liu also prove the following result, where $K^{\ddagger}_{3,3}$ is the graph $K_{3,3}$ with two additional edges added on the same side of the bipartition.

Theorem 2.14 (Ding and Liu [6])**.** *The family of all* 3*-connected* $K^{\ddagger}_{3,3}$*-minor-free graphs consists of* 3*-connected planar graphs and* 3*-connected minors of three small graphs on at most ten vertices.*

Because $K_{2,4}$ is a subgraph of $K_{3,3}^{\ddagger}$, $K_{2,4}$-minor-free graphs must be a subset of the graphs described in Theorem 2.14. Combining this with Theorem 2.13, we conclude that all large enough $K_{2,4}$-minor-free graphs G must be planar (and so $K_{3,3}$-minor-free) members of $\mathcal{S}$, hence common 3-sums of at most two graphs, which reduces the work needed to conclude that $G \in \widetilde{\mathcal{G}}$. The analysis required for small graphs is not simplified by using Theorem 2.14, however, so we provide the full analysis using only Theorem 2.13.

3 The 2-connected case

We begin this section by looking at how $K_{2,t}$ minors interact with separations in a graph. We will mostly be concerned with 2-separations.

Lemma 3.1. *Suppose (H, J) is a 2-separation in a graph G with $V(H) \cap V(J) = \{x, y\}$. If G contains a standard $K_{2,t}$ minor $(R_1, R_2; S)$ with $t \geq 3$, then one of the following hold:*
(i) there exists a $K_{2,t}$ minor in $H + xy$,
(ii) there exists a $K_{2,t}$ minor in $J + xy$, or
(iii) $x \in R_1$ and $y \in R_2$ (or vice versa).

Proof. Let $H' = H - \{x, y\}$ and $J' = J - \{x, y\}$. Assume (iii) does not hold, then $\{x, y\} \cap R_i = \emptyset$ for at least one i; we may suppose that $\{x, y\} \cap R_2 = \emptyset$. Since R_2 induces a connected subgraph, this means that $R_2 \subseteq V(H')$ or $R_2 \subseteq V(J')$; without loss of generality we assume that $R_2 \subseteq V(H')$. Then necessarily $S \subseteq V(H)$, and so $(R_1 \cap V(H), R_2; S)$ is a standard $K_{2,t}$ minor in $H + xy$ and (i) holds. □

By a $K_{2,t}$ minor $(R_1, R_2; S)$ *rooted at x and y*, we mean $x \in R_1$ and $y \in R_2$. If part (iii) of Lemma 3.1 holds, then the $K_{2,t}$ minor splits into two minors, K_{2,t_1} and K_{2,t_2} with $t_1 + t_2 = t$, both rooted at x and y. For $K_{2,4}$ minors this means that we will be concerned with rooted $K_{2,2}$ minors; we will describe the structure of graphs without rooted $K_{2,2}$ minors. Note that Demasi [4, Lemma 2.2.2] has characterized graphs without $K_{2,2}$ minors rooted at all four vertices, in terms of disjoint paths.

An *xy-outerplane embedding* of a connected graph G with $x, y \in V(G)$ is an embedding of G in a closed disk D such that a hamilton xy-path P of G is contained in the boundary of D. This is equivalent to embedding G in the plane so that the outer facial walk contains P as an uninterrupted subwalk, or so that all edges not in P lie 'on the same side' of P; we use this as our practical definition. The path P is called the *outer path*. A graph is *xy-outerplanar*, or generically *path-outerplanar*, if it has an xy-outerplane embedding.

A *block* is a connected graph without a cutvertex: an isolated vertex, an edge, or a 2-connected graph. The *blocks of a graph G* are the maximal blocks that are subgraphs of G. The *block-cutvertex tree* of a connected graph G is a tree whose vertices are the blocks and cutvertices of G; a block B and cutvertex v are adjacent if $v \in V(B)$.

The following useful properties are obvious, so we omit their proofs.

Lemma 3.2. *(i) If G is xy-outerplanar, H is yz-outerplanar, and $V(G) \cap V(H) = \{y\}$ then $G \cup H$ is xz-outerplanar.*

(ii) Suppose $x \neq y$. Then G is xy-outerplanar if and only if $G + xy$ is a block with an outerplane embedding in which xy is on the outer face. Such an embedding of $G + xy$ is also xy-outerplane.

We now characterize rooted $K_{2,2}$-minor-free graphs.

Lemma 3.3. *Suppose x and y are distinct vertices of G and $G' = G + xy$ is a block. Then G has no $K_{2,2}$ minor rooted at x and y if and only if G is xy-outerplanar.*

Proof. ($\Leftarrow$) Assume an xy-outerplane embedding of G. Add a vertex z and edges xz, yz to G in the outer face; the resulting graph G'' is outerplanar. If G has a $K_{2,2}$ minor rooted at x and y, then G'' has a $K_{2,3}$ minor, which is a contradiction since outerplanar graphs are $K_{2,3}$-minor-free.

($\Rightarrow$) Proceed by induction on $|E(G)|$. The base case for G is K_2 which has no $K_{2,2}$ minor rooted at x and y and is clearly xy-outerplanar. Now assume the claim holds for all graphs on at most $m \geq 1$ edges and suppose $|E(G)| = m + 1$. Then G' is 2-connected.

First assume there is a cutvertex v in G. Since G' is 2-connected, the block-cutvertex tree of G must be a path $B_1 v_1 B_2 v_2 \ldots v_{k-1} B_k$ where $k \geq 2$, $x \in V(B_1) - \{v_1\}$ and $y \in V(B_k) - \{v_{k-1}\}$. Define $v_0 = x$ and $v_k = y$. Because G has no $K_{2,2}$ minor rooted at x and y, each block B_i has no $K_{2,2}$ minor rooted at v_{i-1} and v_i for $1 \leq i \leq k$. Thus, by induction each block B_i is $v_{i-1}v_i$-outerplanar. By Lemma 3.2(i), the outerplane embeddings of the blocks can then be combined to create an xy-outerplane embedding of G, as in Figure 13.

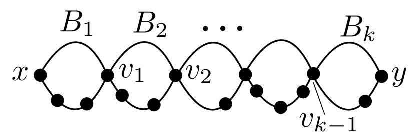


Figure 13

Now suppose G has no cutvertex (G is 2-connected). Assume that G contains the edge xy. Then by induction, $G \backslash xy$ has an xy-outerplane embedding. By Lemma 3.2(ii), G also has an xy-outerplane embedding. Therefore, we may assume that G does not contain the edge xy. Since G has no cutvertex, there exist two internally disjoint xy-paths. Since $xy \notin E(G)$, each path has an internal vertex, and hence they yield a $K_{2,2}$ minor rooted at x and y, a contradiction. □

In order to describe the structure of 2-connected $K_{2,4}$-minor-free graphs, we need the following lemma:

Lemma 3.4. *Suppose $t \geq 3$. Let z be a degree 2 vertex in a graph G with neighbors x and y. Let G' be the graph formed from G by replacing the path xzy with an xy-outerplanar graph J on at least three vertices. Then G is $K_{2,t}$-minor-free if and only if G' is $K_{2,t}$-minor-free.*

Proof. ($\Leftarrow$) G is a minor of G' so if G' is $K_{2,t}$-minor-free then so is G.

($\Rightarrow$) Let $H = G - z$. Then (H, J) is a 2-separation in G' with $V(H) \cap V(J) = \{x, y\}$. Because G is $K_{2,t}$-minor-free, we know that $H + xy$ is $K_{2,t}$-minor-free and also there is no $K_{2,t-1}$ minor in H rooted at x and y. Because $J + xy$ is outerplanar, $J + xy$ is $K_{2,t}$-minor-free. Thus by Lemma 3.1, if G' has a $K_{2,t}$ minor, then $x \in R_1$ and $y \in R_2$. If $|S \cap V(J)| \geq 2$, then J has a $K_{2,2}$ minor rooted at x and y which contradicts Lemma 3.3. Thus $|S \cap V(H)| \geq t - 1$ but now we have a $K_{2,t-1}$ minor rooted at x and y in H which is a contradiction. Hence G' is $K_{2,t}$-minor-free. □

We can now describe the structure of 2-connected $K_{2,4}$-minor-free graphs using one new concept. If G is $K_{2,4}$-minor-free then $F \subseteq E(G)$ is *subdividable* if the graph formed from G by subdividing all edges of F (replacing each edge by a path of length 2) is $K_{2,4}$-minor-free. The edge e is *subdividable* if $\{e\}$ is subdividable. If F is a subdividable set then every edge of F is subdividable, but the converse is not true.

Theorem 3.5. *Let G be a block. Then G is $K_{2,4}$-minor-free if and only if one of the following holds.*

(i) G is outerplanar.

(ii) G is the union of three xy-outerplanar graphs H_1, H_2, H_3 and possibly the edge xy, where $|V(H_i)| \geq 3$ for each i and $V(H_i) \cap V(H_j) = \{x, y\}$ for $i \neq j$.

(iii) G is obtained from a 3-connected $K_{2,4}$-minor-free graph G_0 by replacing each edge x_iy_i in a (possibly empty) subdividable set of edges $\{x_1y_1, x_2y_2, \ldots, x_ky_k\}$ by an x_iy_i-outerplanar graph H_i, where $V(H_i) \cap V(G_0) = \{x_i, y_i\}$ for each i, and $V(H_i) \cap V(H_j) \subseteq V(G_0)$ for $i \neq j$.

Proof. ($\Leftarrow$) For (i), all outerplanar graphs are $K_{2,4}$-minor-free since they are $K_{2,3}$-minor-free. To show that a graph G in (ii) is $K_{2,4}$-minor-free, we use Lemma 3.4. G is $K_{2,4}$-minor-free if the graph formed from G by replacing each of the three outerplanar pieces with a single vertex is $K_{2,4}$-minor-free. This graph is either $K_{2,3}$ or $K_{1,1,3}$ and is thus $K_{2,4}$-minor-free. We use Lemma 3.4 again to show that graphs in (iii) are $K_{2,4}$-minor-free. Let G' be formed from a 3-connected $K_{2,4}$-minor-free graph by subdividing a subdividable set of edges. G' is still $K_{2,4}$-minor-free by the definition of subdividable set. Now replace each subdivided edge $x_iz_iy_i$ with an x_iy_i-outerplanar graph; by Lemma 3.4, the resulting graph is still $K_{2,4}$-minor-free.

($\Rightarrow$) Suppose G is a $K_{2,4}$-minor-free block. We proceed by induction on $n = |V(G)|$. As the basis, if $n \leq 4$ then G is one of K_1, K_2, K_3, $K_{1,1,2}$ or C_4, which are outerplanar and covered by (i), or K_4, which is 3-connected and covered by (iii). If G is 3-connected then (iii) holds.

So we may assume that $n \geq 5$ and G has a 2-cut $\{x, y\}$. Let $H'_1, H'_2, \ldots, H'_\ell$, where $\ell \geq 2$, be the components of $G - \{x, y\}$, and for each i let H_i be the subgraph induced by $V(H'_i) \cup \{x, y\}$.

If $\ell \geq 4$, then G has a $K_{2,4}$ minor with $x \in R_1$, $y \in R_2$, and S consisting of a vertex from each of H'_1, H'_2, H'_3, H'_4. This is a contradiction.

Suppose $\ell = 3$. If some H_i is not xy-outerplanar, then we have a $K_{2,4}$ minor: by Lemma 3.3, there is a $K_{2,2}$ minor rooted at x and y in H_i, to which we may add one vertex from each of the two other components of $G - \{x, y\}$. Thus, H_1, H_2, H_3 are all xy-outerplanar and (ii) holds.

Now suppose $\ell = 2$. If neither H_1 nor H_2 is xy-outerplanar, then G contains a $K_{2,4}$ minor. If both are xy-outerplanar then G is outerplanar as in (i). Hence one, say H_1, is not xy-outerplanar and the other, H_2, is xy-outerplanar. Let $H_1^+ = H_1 + xy$. Since $|V(H_1^+)| < |V(G)|$, by induction H_1^+ is in (i), (ii), or (iii). By Lemma 3.2(ii), because H_1 is not xy-outerplanar, H_1^+ is not outerplanar and hence not in (i). If H_1^+ is in (ii), then the 2-cut $\{u, v\}$ in H_1^+ giving three components is also a 2-cut in G giving three components, and so, applying the argument for $\ell = 3$ to $\{u, v\}$, (ii) holds for G.

Now assume H_1^+ is in (iii): H_1^+ is a 3-connected $K_{2,4}$-minor-free graph G_0 with each edge $f = uv$ of a subdividable set F replaced by a uv-outerplanar graph $J(f)$. Let H_2^* be $H_2 + xy$ if $xy \in E(G)$, and H_2 otherwise. In either case H_2^* is xy-outerplanar and G is obtained from H_1^+ by replacing xy by H_2^*.

Suppose first that $xy \notin \bigcup_{f \in F} E(J(f))$; then $xy \in E(G_0) - F$. If we let $J(xy) = H_2^*$, then G is obtained from G_0 by replacing each $f \in F \cup \{xy\}$ by $J(f)$. The graph obtained from G by replacing every $J(f)$, $f \in F \cup \{xy\}$, by a path of length two with the same ends as f is the same as the graph obtained from G_0 by subdividing every edge of $F \cup \{xy\}$. Since G is $K_{2,4}$-minor-free, this graph is also $K_{2,4}$-minor-free by repeated application of Lemma 3.4, so $F \cup \{xy\}$ is subdividable in G_0. Hence (iii) holds for G.

Next suppose xy is an edge of some $J(f)$, $f = uv \in F$, with outer path P. Suppose $xy \notin E(P)$. Then there is at least one vertex in the subpath Q of P between, but not including, x and y. No vertex of Q is adjacent to a vertex of $V(H_1^+) - V(J(f))$ or, because $xy \in E(J(f))$, to a vertex of $V(J(f)) - (V(Q) \cup \{x, y\})$. Now there exists $w \in V(H_1^+) - V(J(f))$, and Q and w are in different components of $H_1^+ - \{x, y\} = H'_1$, contradicting the fact that H'_1 is connected. So $xy \in E(P)$. Then the graph $J'(f)$ obtained by replacing xy in $J(f)$ with the xy-outerplanar graph H_2^* is still uv-outerplanar. Thus G is again in (iii). □

To complete the 2-connected case, it remains to find all subdividable sets of edges in part (iii) of Theorem 3.5 for each 3-connected $K_{2,4}$-minor-free graph. If a set of edges is subdividable, then all subsets of that set are also subdividable, so it suffices to state the maximal (under inclusion) subdividable sets of edges in each graph. We start with graphs in $\mathcal{G}$ with $n \geq 6$. The graphs $G_{6,2,2}$, $G_{6,2,2}^+ \cong G_{6,2,3}$, and $G_{7,2,3}$ need special treatment and are dealt with later.

Theorem 3.6. *Consider $G^{(+)}_{n,r,s} \in \mathcal{G}$ with $r \le s$ and $n \ge 6$. (Results for $r > s$ may be obtained using the isomorphism between $G^{(+)}_{n,r,s}$ and $G^{(+)}_{n,s,r}$.)*

(i) When $r = 1$, the wheel $G^+_{n,1,n-3}$ has $n-1$ maximal subdividable sets of edges. Each one includes all edges of the rim as well as one of the spokes.

(ii) When $r = 2$, $G_{n,2,s}$ with $s \ge 4$ or $G^+_{n,2,s}$ with $s \ge 3$ has two maximal subdividable sets of edges: the edge sets of the spine, $v_1v_2\dots v_n$, and second spine, $v_{n-2}v_{n-3}\dots v_1v_{n-1}v_n$.

(iii) When $r \ge 3$ the only maximal subdividable set of edges is the edge set of the spine, $v_1v_2\dots v_n$.

Proof. We first show that each claimed subdividable set is subdividable. For the wheel $G^+_{n,1,n-3}$, subdividing all edges of the rim and one spoke gives a graph isomorphic to a subgraph of $G_{2n,2,2n-4}$, and hence $K_{2,4}$-minor-free. For $r \ge 2$ the graph formed by subdividing all edges of the spine in $G^{(+)}_{n,r,s}$ is isomorphic to a subgraph of another graph in $\mathcal{G}$ with $2n-1$ vertices, and thus $K_{2,4}$-minor-free. So the edge set of the spine is subdividable. When $r = 2$ the second spine is the image under an isomorphism of the spine in another (or possibly the same) member of $\mathcal{G}$, and hence the edge set of the second spine is also subdividable.

Now we show that the sets of edges listed are maximal and are the only subdividable sets. Begin with the wheel $G^+_{n,1,n-3}$. All edges of the rim are in each set so we consider the spokes. If we subdivide two adjacent spokes, we have the $K_{2,4}$ minor shown on the left in Figure 14. A similar minor exists if we subdivide nonadjacent spokes as long as $n \ge 6$. Hence we cannot divide two spokes and the sets listed are maximal and are the only subdividable sets of edges.

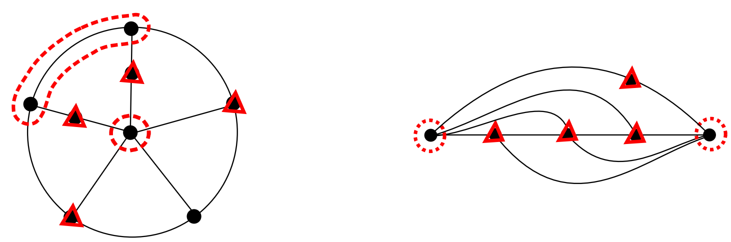


Figure 14

Now assume $r, s \ge 2$. For this portion of the proof, we remove the assumption that $r \le s$ (which is just for brevity in stating our results). Denote by $G \circ e$ the graph formed from G by subdividing the edge e. We consider subdivision of non-spine edges v_1v_{n-i} for $0 \le i \le r$; edges v_nv_{1+j} for $0 \le j \le s$ are handled by symmetry. The situations $i = 0$ and $j = 0$ correspond to a plus edge.

We describe two cases in which we can find a $K_{2,4}$ minor. The first, Case A, is the $K_{2,4}$ minor in $G^+_{5,2,2} \circ v_1v_5$ shown on the right in Figure 14. If $s \ge 2$ and $0 \le i \le r-2$, then we form $G^+_{5,2,2} \circ v_1v_5$, and hence $K_{2,4}$, as a minor from $G^{(+)}_{n,r,s} \circ v_1v_{n-i}$ by contracting all edges of the paths $v_3v_4\dots v_{n-i-2}$ and $v_{n-i}v_{n-i+1}\dots v_n$ and deleting multiple edges.

The second case, Case B, is the $K_{2,4}$ minor in $G^+_{5,2,2} \circ v_1v_3$ shown on the left in Figure 15. Note that the minor does not use the edge v_2v_5. As with Case A, this minor is inherited by the following larger graphs that have $G^+_{5,2,2} \circ v_1v_3$ as a minor:

(B1) $G^+_{n,r,s} \circ v_1v_{n-i}$ with $s \ge 2$ and $2 \le i \le r$;

(B2) $G_{n,r,s} \circ v_1v_{n-i}$ with $s \ge 2$ and $3 \le i \le r$; and

(B3) $G^{(+)}_{n,r,s} \circ v_1v_{n-2}$ with $s \ge 3$.

For graphs in (B1), form $G^+_{5,2,2} \circ v_1v_3$ as a minor from $G^+_{n,r,s} \circ v_1v_{n-i}$ by contracting all edges of the paths $v_3v_4\dots v_{n-i}$ and $v_{n-i+1}v_{n-i+2}\dots v_{n-1}$ and deleting multiple edges as well as the edge v_1v_3 if it is present after contraction. Similarly for graphs in (B2), contract all edges of the paths $v_3v_4\dots v_{n-i}$ and $v_{n-i+2}v_{n-i+3}\dots v_n$

and delete multiple edges and v_1v_3. For graphs in (B3), contract v_1v_2 and all edges of the path $v_4v_5 \dots v_{n-2}$ and delete multiple edges and v_1v_3.

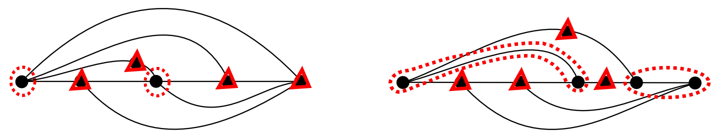


Figure 15

For $G^{(+)}_{n,r,s}$ with $r, s \geq 3$, Case A shows that $v_1v_{n-r-2}, v_1v_{n-r-1}, \dots, v_1v_{n-1}$ and (if present) v_1v_n are not subdividable. Case B shows that $v_1v_{n-r}, v_1v_{n-r+1}, \dots, v_1v_{n-2}$ are not subdividable. By symmetry all non-spine edges incident to v_n are not subdividable, and hence the spine is the only maximal subdividable set of edges.

Now either $r = 2$ or $s = 2$. For our stated result we only need the case $r = 2$ with s as in (ii). Consider $G^{(+)}_{n,2,s}$. Case A forbids subdivision of v_1v_n (if present) and (B3) does the same for v_1v_{n-2}. Applying symmetry, Case A forbids subdivision of v_nv_{1+j} for $1 \leq j \leq s-2$, and Case B covers v_nv_{1+j} for $2 \leq j \leq s$ if there is a plus edge and $3 \leq j \leq s$ otherwise. The conditions in (ii) mean that these cover v_nv_{1+j} for all j, $1 \leq j \leq s$. So the only possible subdividable non-spine edge is v_1v_{n-1}, which we already know is subdividable along with all edges of the spine other than $v_{n-2}v_{n-1}$, as this is the edge set of the second spine. So consider v_1v_{n-1} and $v_{n-2}v_{n-1}$ together. We use the $K_{2,4}$ minor in $G_{6,2,2} \circ v_1v_5 \circ v_4v_5$ shown on the right in Figure 15. When $n \geq 6$, $G^{(+)}_{n,2,s} \circ v_1v_{n-1} \circ v_{n-2}v_{n-1}$ has $G_{6,2,2} \circ v_1v_5 \circ v_4v_5$, and hence $K_{2,4}$, as a minor: delete v_nv_{1+j} with $j = 0$ (if present) and $3 \leq j \leq s$, then contract all edges of $v_4v_5 \dots v_{n-2}$. Therefore $\{v_1v_{n-1}, v_{n-2}v_{n-1}\}$ is not subdividable, and the only maximal subdividable sets are the edge sets of the spine and second spine. $\square$

All remaining small graphs are covered by Table 2. Verifying these results is straightforward; complete proofs may be found in [9, Section 5.2]. These results were also confirmed by computer (the program may be obtained from the first author). The dashed edges in the table indicate edges present in one graph but not the other. For example, in the row for C and C^+, the dashed edge is present in C^+ but not C.

Lemma 3.7. *The maximal subdividable sets of edges for the nine small cases not in $\mathcal{G}$ as well as $K_4 = W_4$, $W_5 \cong G_{5,2,2}$, $K_5 \backslash e \cong G^+_{5,2,2}$, $G_{6,2,2}$, $G^+_{6,2,2} \cong G_{6,2,3}$ and $G_{7,2,3}$ are listed in Table 2.*

As mentioned earlier, a graph G is $K_{2,4}$-minor-free if and only if each of its blocks is $K_{2,4}$-minor-free, so our overall result can now be stated as follows.

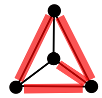
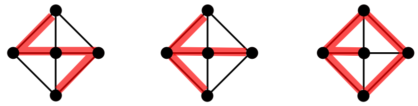
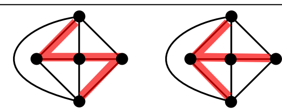
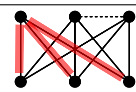
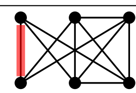
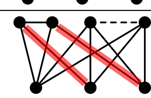
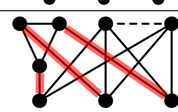
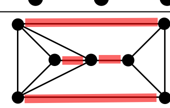
Theorem 3.8 (Characterization of $K_{2,4}$-minor-free graphs)**.** *A graph is $K_{2,4}$-minor-free if and only if each of its blocks is described by Theorem 3.5, where for Theorem 3.5 (iii), the 3-connected graphs are given in Theorem 2.12 and the subdividable sets are described in Theorem 3.6 and Lemma 3.7.*

4 Consequences

Our characterization has a number of consequences. First, as mentioned in the introduction, we are interested in hamiltonian properties of $K_{2,4}$-minor-free graphs.

Corollary 4.1. *(i) Every 3-connected $K_{2,4}$-minor-free graph has a hamilton cycle.*
(ii) There are 2-connected $K_{2,4}$-minor-free planar graphs that have no spanning closed trail and hence no hamilton cycle.
(iii) However, every 2-connected $K_{2,4}$-minor-free graph has a hamilton path.

Table 2

Graph	Maximal Subdividable Sets of Edges	Number of Symmetric Copies
$K_4 = W_4$		12
W_5 ($\cong G_{5,2,2}$)		4 of each
$G^+_{5,2,2}$ ($\cong K_5 \backslash e$)		6 of each
$G_{6,2,2}$	edge set of spine	6
$G^+_{6,2,2}$ ($\cong G_{6,2,3}$)	edge set of spine in $G^+_{6,2,2}$ edge set of second spine in $G^+_{6,2,2}$	spine: 1 second spine: 2
$G_{7,2,3}$	edge set of spine, edge set of second spine, $\{v_1v_2, v_4v_5, v_6v_7, v_3v_7\}$	1 of each
K_5	$\emptyset$	1
$A, K_{3,3}$		A: 1 $K_{3,3}$: 6
A^+		1
B, B^+		1
C, C^+		1
D		3

Proof. (i) The graph $G^{(+)}_{n,r,s} \in \mathcal{G}$ has a hamilton cycle $(v_1v_2 \ldots v_{s+1}v_nv_{n-1} \ldots v_{s+2})$. The graphs in Figure 3 are also all hamiltonian.

(ii) A 2-connected graph described by Theorem 3.5(ii) is planar and has no closed spanning trail; the simplest example is $K_{2,3}$. (It is also possible to construct examples using Theorem 3.5(iii).)

(iii) Define a *hamilton base* in a graph to be a hamilton path extended by a new edge at one or both ends, i.e., a trail of the form $x_0x_1x_2 \ldots x_nx_{n+1}$, $x_1x_2 \ldots x_nx_{n+1}$, or $x_1x_2 \ldots x_n$, where $x_1x_2 \ldots x_n$ is a hamilton path. If B is a hamilton base in a graph G_0 and G_1 is obtained from G_0 by subdividing the elements of a subset of $E(B)$ arbitrarily many times, we observe that G_1 has a hamilton path.

Now consider a 2-connected $K_{2,4}$-minor-free graph G. If G is described by Theorem 3.5(i) then G is hamiltonian. Suppose G is described by Theorem 3.5(iii), as constructed from a 3-connected $K_{2,4}$-minor-free graph G_0 by replacing each edge of a subdividable set $S = \{x_1y_1, x_2y_2, \ldots, x_ky_k\}$ by an x_iy_i-outerplanar graph. Then G has a spanning subgraph G_1 which is obtained from G_0 by subdividing each edge of S some number of times. If G_0 has a hamilton base containing S, then G_1, and hence G, has a hamilton path. So we just need to verify that each maximal subdividable set of edges in G_0 is contained in a hamilton base.

Each subdividable set from Theorem 3.6 itself forms a hamilton base, and it is not difficult to show that the subdividable sets from Lemma 3.7 (Table 2) are contained in hamilton bases; we omit the details. Finally, if G is described by Theorem 3.5(ii) then G has a spanning subgraph G_1 obtained by subdividing edges of $G_0 = K_{2,3}$, and $K_{2,3}$ has a hamilton base containing all its edges, so a similar argument applies. □

Second, a theorem of Dieng and Gavoille mentioned earlier can be derived from our results. We state it and just outline a proof.

Corollary 4.2 (Dieng and Gavoille, see [5, Théorème 3.2])**.** *For every 2-connected $K_{2,4}$-minor-free graph G there is $U \subseteq V(G)$ with $|U| \leq 2$ ($|U| \leq 1$ if G is planar) such that $G - U$ is outerplanar.*

Sketch of proof. Consider the structure of G as described in Theorem 3.5. If (i) holds no vertices need to be deleted, and if (ii) holds then one of x or y can be deleted. To verify the result when (iii) holds, it suffices to show that for every 3-connected $K_{2,4}$-minor-free G_0 and every maximal subdividable set of edges F in G_0, there is $U \subseteq V(G_0)$ with $|U| \leq 2$ ($|U| \leq 1$ if G_0 is planar) so that $G_0 - U$ has an outerplane embedding with all remaining edges of F (those not incident with U) on the outer face. If $G_0 = G^{(+)}_{n,r,s} \in \mathcal{G}$ is covered by Theorem 3.6 then $G_0 - v_n$ always works. The result must be checked for the small graphs in Table 2. □

Dieng and Gavoille in fact showed that there is an $O(n)$ time algorithm to find either a $K_{2,4}$ minor or a set U as in Corollary 4.2 in any n-vertex graph.

Third, our result also gives bounds on genus.

Corollary 4.3. *Every 2-connected $K_{2,4}$-minor-free graph is either planar or else toroidal and projective-planar. Thus, its orientable and nonorientable genus are at most 1.*

Proof. The 3-connected graphs described in Theorem 2.12 are planar or minors of C^+, and it is not difficult to find toroidal and projective-planar embeddings of C^+. For connectivity 2 the graphs G constructed in Theorem 3.5 are either planar or have the same genus as some 3-connected $K_{2,4}$-minor-free graph G_0. □

Note that Corollary 4.3 does not follow from Dieng and Gavoille's result, Corollary 4.2, since a result of Mohar [10] implies that graphs which become outerplanar after deleting two vertices can have arbitrarily high (orientable) genus.

Fourth, our result shows that the number of 3-connected $K_{2,4}$-minor-free graphs grows only linearly. For $n \geq 9$ the only such n-vertex graphs are those in $\widetilde{\mathcal{G}}$, and there are only $2n - 8$ nonisomorphic such graphs. Although we have not done so, it should also be possible to deduce counting results for 2-connected $K_{2,4}$-minor-free graphs from our characterization.

Finally, Chudnovsky, Reed and Seymour [3] showed that the number of edges in a 3-connected $K_{2,t}$-minor-free graph is at most $5n/2 + c(t)$. They provide examples to show that this is in a sense best possible for $t \geq 5$. Theorem 2.12 shows that this can be improved when $t = 4$. Using Theorem 3.5 we can also obtain a result for 2-connected $K_{2,4}$-minor-free graphs. We omit the straightforward proofs, which use the fact that an n-vertex outerplanar graph has at most $2n - 3$ edges.

Corollary 4.4. *(i) Every 3-connected $K_{2,4}$-minor-free n-vertex graph with $n \geq 7$ has at most $2n - 2$ edges, and such graphs with $2n - 2$ edges exist for all $n \geq 7$. (K_5 has $2n$ edges, and A^+ has $2n - 1$ edges.)*
(ii) Every 2-connected $K_{2,4}$-minor-free n-vertex graph with $n \geq 6$ has at most $2n - 1$ edges, and such graphs with $2n - 1$ edges exist for all $n \geq 6$. (K_5 has $2n$ edges.)

References

[1] Guantao Chen, Yoshimi Egawa, Ken-ichi Kawarabayashi, Bojan Mohar and Katsuhiro Ota, Toughness of $K_{a,t}$-minor-free graphs, *Electron. J. Combin.* 18 no. 1 (2011) #P148 (6 pages).

[2] Guantao Chen, Laura Sheppardson, Xingxing Yu and Wenan Zang, The circumference of a graph with no $K_{3,t}$-minor, *J. Combin. Theory Ser. B* 96 (2006) 822–845.

[3] Maria Chudnovsky, Bruce Reed and Paul Seymour, The edge-density for $K_{2,t}$ minors, *J. Combin. Theory Ser. B* 101 (2011) 18-46.

[4] Lino Demasi, Rooted minors and delta-wye transformations. Ph.D. thesis, Simon Fraser University, October 2012. `http://summit.sfu.ca/system/files/iritems1/12552/etd7556_LDemasi.pdf`

[5] Youssou Dieng, Décomposition arborescente des graphes planaires et routage compact, Ph.D. thesis, Université Bordeaux I, October 2009. `http://ori-oai.u-bordeaux1.fr/pdf/2009/DIENG_YOUSSOU_2009.pdf`

[6] Guoli Ding and Cheng Liu, Excluding a small minor, *Discrete Appl. Math.* 161 (2013) 355–368.

[7] Guoli Ding, Graphs without large $K_{2,n}$-minors. `https://www.math.lsu.edu/~ding/k2n.ps` (downloaded 8 May 2014)

[8] Ruy Fabila-Monroy and David R. Wood, Rooted K_4-minors, *Electron. J. Combin.* 20 no. 2 (2013) #P64 (19 pages).

[9] Emily A. Marshall, Hamiltonicity and structure of classes of minor-free graphs, Ph.D. dissertation, Vanderbilt University, May 2014. `http://etd.library.vanderbilt.edu/available/etd-03212014-152116/unrestricted/Emily_Marshall_dissertation.pdf`

[10] Bojan Mohar, Face covers and the genus problem for apex graphs, *J. Combin. Theory Ser. B* 82 (2001) 102–117.

[11] Joseph Samuel Myers, The extremal function for unbalanced bipartite minors, *Discrete Math.* 271 (2003) 209–222.

[12] Seiya Negami. A characterization of 3-connected graphs containing a given graph, *J. Combin. Theory Ser. B* 32 (1982) 69–74.

[13] Katsuhiro Ota and Kenta Ozeki, Spanning trees in 3-connected $K_{3,t}$-minor-free graphs, *J. Combin. Theory Ser. B* 102 (2012) 1179–1188.

[14] Neil Robertson and P. D. Seymour, Graph minors. IX. Disjoint crossed paths, *J. Combin. Theory Ser. B* 49 (1990) 40–77.

[15] P. D. Seymour. Decomposition of regular matroids, *J. Combin. Theory Ser. B* 28 (1980) 305–359.

[16] Noah Streib and Stephen J. Young, Dimension and structure for a poset of graph minors. `http://www.math.louisville.edu/~syoung/research/papers/DimensionMinorPoset.pdf` (downloaded 22 May 2014)

[17] K. Wagner, Über eine Eigenschaft der ebenen Komplexe, *Math. Ann.* 114 (1937) 570–590.