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ON TAMED EULER APPROXIMATIONS OF SDES DRIVEN BY LÉVY NOISE WITH APPLICATIONS TO DELAY EQUATIONS*

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Abstract. We extend the taming techniques for explicit Euler approximations of stochastic differential equations driven by Lévy noise with superlinearly growing drift coefficients. Strong convergence results are presented for the case of locally Lipschitz coefficients. Moreover, rate of convergence results are obtained in agreement with classical literature when the local Lipschitz continuity assumptions are replaced by global assumptions and, in addition, the drift coefficients satisfy polynomial Lipschitz continuity. Finally, we further extend these techniques to the case of delay equations.

Key words. explicit Euler approximations, rate of convergence, local Lipschitz condition, superlinear growth, SDEs driven by Lévy noise, delay equations

AMS subject classifications. Primary, 60H35; Secondary, 65C30

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1. Introduction.

In economics, finance, medical sciences, ecology, engineering, and many other branches of science, one often encounters problems which are influenced by event-driven uncertainties. For example, in finance, the unpredictable nature of important events such market crashes, announcements made by central banks, changes in credit ratings, defaults, etc., might have sudden and significant impacts on the stock price movements. Stochastic differential equations (SDEs) with jumps, or more precisely SDEs driven by Lévy noise, have been widely used to model such event-driven phenomena. The interested reader may refer, for example, to [3, 18, 23] and references therein.

Many such SDEs do not have explicit solutions, and therefore one requires numerical schemes so as to approximate their solutions. Over the past few years, several explicit and implicit schemes of SDEs driven by Lévy noise have been studied and results on their strong and weak convergence have been proved. For a comprehensive discussion on these schemes, one could refer to [2, 8, 9, 14, 20] and references therein.

It is also known, however, that the computationally efficient explicit Euler schemes of SDEs (even without jumps) may not convergence in strong ($\mathcal{L}^q$) sense when the drift coefficients are allowed to grow superlinearly; see, for example, [11]. The development of tamed Euler schemes was a recent breakthrough in order to address this problem; one may consult [12, 21] as well as [10, 22, 24] and references therein for a thorough investigation of the subject.

In this article, we propose explicit tamed Euler schemes to numerically solve SDEs with random coefficients driven by Lévy noise. The taming techniques developed here allow one to approximate these SDEs with drift coefficients that grow superlinearly. By adopting the approach of [21], we prove strong convergence in (uniform) $\mathcal{L}^q$ sense of these tamed schemes by assuming one-sided local Lipschitz condition on drift and

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local Lipschitz conditions on both diffusion and jump coefficients. Moreover, our technical calculations are more refined than those of [12, 21] in that we develop new techniques to overcome the challenges arising due to jumps. In addition, explicit formulations of the tamed Euler schemes are presented at the end of section 3 for the case of SDEs driven by Lévy noise which have nonrandom coefficients.

To the best of the authors' knowledge, the results obtained in this article are the first for the case of superlinear coefficients in this area. Moreover, the techniques developed here allow for further investigation of convergence properties of higher order explicit numerical schemes for SDEs driven by Lévy noise with superlinear coefficients.

As an application of our approach which considers random coefficients, we also present in this article uniform $\mathcal{L}^q$ convergence results of explicit tamed Euler schemes for the case of stochastic delay differential equations (SDDEs) driven by Lévy noise. The link between delay equations and random coefficients utilizes ideas from [7]. The aforementioned results are derived under the assumptions of one-sided local Lipschitz condition on drift and local Lipschitz conditions on both diffusion and jump coefficients with respect to nondelay variables, whereas these coefficients are only asked to be continuous with respect to arguments corresponding to delay variables. It is worth mentioning here that our approach allows one to use our schemes to approximate SDDEs with jumps when drift coefficients can have superlinear growth in both delay and nondelay arguments. Thus, the proposed tamed Euler schemes provide significant improvements over the existing results available on numerical techniques of SDDEs, for example, [1, 15]. It should also be noted that, by adopting the approach of [7], we prove the existence of a unique solution to the SDDEs driven by Lévy noise under more relaxed conditions than those existing in the literature, for example, [13], whereby we ask for the local Lipschitz continuity only with respect to the nondelay variables.

Finally, rate of convergence results are obtained (which are in agreement with classical literature) when the local Lipschitz continuity assumptions are replaced by global assumptions and, in addition, the drift coefficients satisfy polynomial Lipschitz continuity. Similar results are also obtained for delay equations when the following assumptions hold—(a) drift coefficients satisfy one-sided Lipschitz and polynomial Lipschitz conditions in nondelay variables whereas polynomial Lipschitz conditions in delay variables and (b) diffusion and jump coefficients satisfy Lipschitz conditions in nondelay variables whereas polynomial Lipschitz conditions in delay variables. This finding is itself a significant improvement over recent results in the area; see, for example, [1] and references therein.

We conclude this section by introducing some basic notation. For a vector $x \in \mathbb{R}^d$, we write $|x|$ for its Euclidean norm, and for a $d \times m$ matrix σ, we write $|\sigma|$ for its Hilbert–Schmidt norm and σ^* for its transpose. Also for $x, y \in \mathbb{R}^d$, xy denotes the inner product of these two vectors. Further, the indicator function of a set A is denoted by I_A, whereas $[x]$ stands for the integer part of a real number x. Let $\mathscr{P}$ be the predictable sigma-algebra on $\Omega \times \mathbb{R}_+$ and $\mathscr{B}(V)$, the sigma-algebra of Borel sets of a topological space V. Also, let $T > 0$ be fixed and $\mathbb{L}^p$ denote the set of nonnegative measurable functions g on $[0, T]$, such that $\int_0^T |g_t|^p dt < \infty$. Finally, for a random variable X, the notation $X \in \mathcal{L}^p$ means $E|X|^p < \infty$.

2. SDE with random coefficients driven by Lévy noise.

Let us assume that $(\Omega, \{\mathscr{F}_t\}_{t\geq 0}, \mathscr{F}, P)$ denotes a probability space equipped with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ which is assumed to satisfy the usual conditions, i.e., $\mathscr{F}_0$ contains all P-null sets and the filtration is right continuous. Let w be an $\mathbb{R}^m$-valued standard Wiener process.

Further assume that $(Z, \mathscr{Z}, \nu)$ is a σ-finite measure space and $N(dt, dz)$ is a Poisson random measure defined on $(Z, \mathscr{Z}, \nu)$ with intensity $\nu \not\equiv 0$. (In the case $\nu \equiv 0$, one could consult [21].) Also let the compensated poisson random measure be denoted by $\tilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt$.

Let $b_t(x)$ and $\sigma_t(x)$ be $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}^d)$-measurable functions which, respectively, take values in $\mathbb{R}^d$ and $\mathbb{R}^{d\times m}$. Further assume that $\gamma_t(x, z)$ is $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}^d) \otimes \mathscr{Z}$-measurable function which takes values in $\mathbb{R}^d$. Also assume that t_0 and t_1 are fixed constants satisfying $0 \le t_0 < t_1 \le T$.

We consider the SDE

$$dx_t = b_t(x_t)dt + \sigma_t(x_t)dw_t + \int_Z \gamma_t(x_t, z)\tilde{N}(dt, dz) \tag{2.1}$$

almost surely for any $t \in [t_0, t_1]$ with initial value x_{t_0} which is an $\mathscr{F}_{t_0}$-measurable random variable in $\mathbb{R}^d$.

Remark 2.1. For notational convenience, we write x_t instead of x_{t-} on the right-hand side of the above equation. This does not cause any problem since the compensators of the martingales driving the equation are continuous. This notational convention shall be adopted throughout this article.

Remark 2.2. In this article, we use $K > 0$ to denote a generic constant which varies at different occurrences.

The proof for the following lemma can be found in [16].

LEMMA 2.1. *Let $r \ge 2$. There exists a constant K, depending only on r, such that for every real-valued, $\mathscr{P} \otimes \mathscr{Z}$-measurable function g satisfying*

$$\int_0^T \int_Z |g_t(z)|^2 \nu(dz)dt < \infty$$

almost surely, the following estimate holds:

$$\begin{aligned} E \sup_{0\le t\le T} \left| \int_0^t \int_Z g_s(z)\tilde{N}(ds, dz)\right|^r &\le KE\left(\int_0^T \int_Z |g_t(z)|^2 \nu(dz)dt\right)^{r/2} \\ &\quad + KE\int_0^T \int_Z |g_t(z)|^r \nu(dz)dt. \end{aligned} \tag{2.2}$$

It is known that if $1 \le r \le 2$, then the second term in (2.2) *can be dropped.*

2.1. Existence and uniqueness. Let $\mathcal{A}$ denote the class of nonnegative predictable processes $L := (L_t)_{t\in[0,T]}$ such that

$$\int_0^T L_t dt < \infty$$

for almost every $\omega \in \Omega$.

For the purpose of this section, the set of assumptions are listed below.

A-1. *There exists an $\mathcal{M} \in \mathcal{A}$ such that*

$$xb_t(x) + |\sigma_t(x)|^2 + \int_Z |\gamma_t(x, z)|^2 \nu(dz) \le \mathcal{M}_t(1 + |x|^2)$$

almost surely for any $t \in [t_0, t_1]$ and $x \in \mathbb{R}^d$.

A-2. *For every $R > 0$, there exists an $\mathcal{M}(R) \in \mathcal{A}$ such that*

$$(x - \bar{x})(b_t(x) - b_t(\bar{x})) + |\sigma_t(x) - \sigma_t(\bar{x})|^2 + \int_Z |\gamma_t(x,z) - \gamma_t(\bar{x},z)|^2 \nu(dz) \\ \leq \mathcal{M}_t(R)|x - \bar{x}|^2$$

almost surely for any $t \in [t_0, t_1]$ whenever $|x|, |\bar{x}| \leq R$.

A-3. *For any $t \in [t_0, t_1]$ and $\omega \in \Omega$, the function $b_t(x)$ is continuous in $x \in \mathbb{R}^d$.*

The proof for the following theorem can be found in [6].

Theorem 2.2. *Let assumptions* A-1 *to* A-3 *be satisfied. Then, there exists a unique solution to SDE* (2.1).

2.2. Moment bounds. We make the following assumptions on the coefficients of SDE (2.1).

A-4. *For a fixed $p \geq 2$, $E|x_{t_0}|^p < \infty$.*

A-5. *There exist a constant $L > 0$ and a nonnegative random variable M satisfying $EM^{\frac{p}{2}} < \infty$ such that*

$$x b_t(x) \vee |\sigma_t(x)|^2 \vee \int_Z |\gamma_t(x,z)|^2 \nu(dz) \leq L(M + |x|^2)$$

almost surely for any $t \in [t_0, t_1]$ and $x \in \mathbb{R}^d$.

A-6. *There exist a constant $L > 0$ and a nonnegative random variable M' satisfying $EM' < \infty$ such that*

$$\int_Z |\gamma_t(x,z)|^p \nu(dz) \leq L(M' + |x|^p)$$

almost surely for any $t \in [t_0, t_1]$ and $x \in \mathbb{R}^d$.

The following is probably well known. However, the proof is provided for the sake of completeness and for the justification of finiteness of the right-hand side when applying Gronwall's lemma, something that is missing from the existing literature.

Lemma 2.3. *Let assumptions* A-2 *to* A-6 *be satisfied. Then there exists a unique solution $(x_t)_{t \in [t_0, t_1]}$ of SDE* (2.1) *and the following estimate holds:*

$$E \sup_{t_0 \leq t \leq t_1} |x_t|^p \leq K$$

with $K := K(t_0, t_1, L, p, E|x_{t_0}|^p, EM^{\frac{p}{2}}, EM')$.

Proof. The existence and uniqueness of solution to SDE (2.1) follows immediately from Theorem 2.2 by noting that due to assumption A-5, assumption A-1 is satisfied.

Let us first define the stopping time $\pi_R := \inf\{t \geq t_0 : |x_t| > R\} \wedge t_1$, and notice that $|x_{t-}| \leq R$ for $t_0 \leq t \leq \pi_R$. By Itô's formula,

$$\begin{aligned} |x_t|^p = {}& |x_{t_0}|^p + p\int_{t_0}^t |x_s|^{p-2} x_s b_s(x_s) ds + p\int_{t_0}^t |x_s|^{p-2} x_s \sigma_s(x_s) dw_s \\ & + \frac{p(p-2)}{2}\int_{t_0}^t |x_s|^{p-4} |\sigma_s^*(x_s)x_s|^2 ds + \frac{p}{2}\int_{t_0}^t |x_s|^{p-2}|\sigma_s(x_s)|^2 ds \\ & + p\int_{t_0}^t\int_Z |x_s|^{p-2} x_s \gamma_s(x_s,z)\tilde{N}(ds,dz) \\ & + \int_{t_0}^t\int_Z \{|x_s + \gamma_s(x_s,z)|^p - |x_s|^p - p|x_s|^{p-2}x_s\gamma_s(x_s,z)\}N(ds,dz) \end{aligned} \tag{2.3}$$

almost surely for any $t \in [t_0, t_1]$. By virtue of assumption A-5 and Young's inequality, one can estimate the second, fourth, and fifth terms of (2.3) by

$$KM^{\frac{p}{2}} + K\int_{t_0}^t |x_s|^p ds. \tag{2.4}$$

Further, since the map $y \to |y|^p$ is of class C^2, by the formula for the remainder, for any $y_1, y_2 \in \mathbb{R}^d$, one gets

$$\begin{aligned} |y_1 + y_2|^p - |y_1|^p - p|y_1|^{p-2}y_1y_2 &\leq K\int_0^1 |y_1 + qy_2|^{p-2}|y_2|^2 dq \\ &\leq K(|y_1|^{p-2}|y_2|^2 + |y_2|^p). \end{aligned} \tag{2.5}$$

Hence the last term of (2.3) can be estimated by

$$K\int_{t_0}^t\int_Z \{|x_s|^{p-2}|\gamma_s(x_s,z)|^2 + |\gamma_s(x_s,z)|^p\}\ N(ds,dz). \tag{2.6}$$

One substitutes the estimates from (2.4) and (2.6) in (2.3), which by taking suprema over $[t_0, u \wedge \pi_R]$ for $u \in [t_0, t_1]$ and expectations gives

$$\begin{aligned} E\sup_{t_0 \leq t \leq u\wedge\pi_R} |x_t|^p \leq {}& E|x_{t_0}|^p + KEM^{\frac{p}{2}} + KE\int_{t_0}^{u\wedge\pi_R} |x_s|^p ds \\ & + pE\sup_{t_0 \leq t \leq u\wedge\pi_R}\left|\int_{t_0}^t |x_s|^{p-2}x_s\sigma_s(x_s)dw_s\right| \\ & + pE\sup_{t_0 \leq t \leq u\wedge\pi_R}\left|\int_{t_0}^t\int_Z |x_s|^{p-2}x_s\gamma_s(x_s,z)\tilde{N}(ds,dz)\right| \\ & + KE\int_{t_0}^{u\wedge\pi_R}\int_Z \{|x_s|^{p-2}|\gamma_s(x_s,z)|^2 + |\gamma_s(x_s,z)|^p\}\ N(ds,dz) \\ =:{}& C_1 + C_2 + C_3 + C_4 + C_5. \end{aligned} \tag{2.7}$$

Here $C_1 := E|x_{t_0}|^p + KEM^{\frac{p}{2}}$. By the Burkholder–Davis–Gundy inequality, C_3 can be estimated as

$$\begin{aligned} C_3 &= pE\sup_{t_0 \leq t < u\wedge\pi_R}\left|\int_{t_0}^t |x_{s-}|^{p-2}x_{s-}\sigma_s(x_{s-})dw_s\right| \\ &\leq KE\sup_{t_0 \leq t \leq u\wedge\pi_R} |x_{t-}|^{p-1}\left(\int_{t_0}^{u\wedge\pi_R} |\sigma_s(x_{s-})|^2 ds\right)^{1/2}, \end{aligned}$$

which on the application of Young's inequality gives

$$C_3 \le \frac{1}{4} E \sup_{t_0 \le t \le u\wedge\pi_R} |x_{t-}|^p + KE\left(\int_{t_0}^{u\wedge\pi_R} |\sigma_s(x_s)|^2 ds\right)^{p/2},$$

and then due to Hölder's inequality and assumption A-5, one has

$$C_3 \le \frac{1}{4} E \sup_{t_0 \le t \le u\wedge\pi_R} |x_{t-}|^p + KEM^{\frac{p}{2}} + KE\int_{t_0}^{u\wedge\pi_R} |x_r|^p ds < \infty. \tag{2.8}$$

To estimate C_4, one uses Lemma 2.1 to write

$$\begin{aligned} C_4 &:= pE \sup_{t_0 \le t \le u\wedge\pi_R} \left|\int_{t_0}^{t}\int_Z |x_{s-}|^{p-2} x_{s-}\gamma_s(x_{s-},z)\tilde{N}(ds,dz)\right| \\ &\le KE\left(\int_{t_0}^{u\wedge\pi_R}\int_Z |x_{s-}|^{2p-2}|\gamma_s(x_{s-},z)|^2\nu(dz)ds\right)^{\frac{1}{2}} \\ &\le KE \sup_{t_0 \le t \le u\wedge\pi_R} |x_{t-}|^{p-1}\left(\int_{t_0}^{u\wedge\pi_R}\int_Z |\gamma_s(x_s,z)|^2\nu(dz)ds\right)^{\frac{1}{2}}, \end{aligned}$$

which due to Young's inequality, assumption A-5, and Hölder's inequality implies

$$C_4 \le \frac{1}{4} E \sup_{t_0 \le t \le u\wedge\pi_R} |x_{t-}|^p + KEM^{\frac{p}{2}} + KE\int_{t_0}^{u\wedge\pi_R} |x_r|^p ds < \infty. \tag{2.9}$$

For C_5, by assumptions A-5 and A-6 and Young's inequality,

$$\begin{aligned} C_5 &:= KE\int_{t_0}^{u\wedge\pi_R}\int_Z \left(|x_s|^{p-2}|\gamma_s(x_s,z)|^2 + |\gamma_s(x_s,z)|^p\right)\ \nu(dz)ds \\ &\le KE\int_{t_0}^{u\wedge\pi_R}\left\{|x_s|^{p-2}(M+|x_s|^2) + M' + |x_s|^p\right\}ds \\ &\le KEM^{\frac{p}{2}} + KEM' + EK\int_{t_0}^{u\wedge\pi_R} |x_r|^p ds < \infty. \end{aligned} \tag{2.10}$$

By substituting the estimates from (2.8)–(2.10) in (2.7), one has

$$E \sup_{t_0 \le t \le u\wedge\pi_R} |x_t|^p \le K + \frac{1}{2} E \sup_{t_0 \le t \le u\wedge\pi_R} |x_{t-}|^p + KE\int_{t_0}^{u\wedge\pi_R} |x_r|^p ds < \infty \tag{2.11}$$

for any $u \in [t_0, t_1]$. In particular we obtain

$$E \sup_{t_0 \le t \le t_1\wedge\pi_R} |x_t|^p < \infty.$$

Since it holds that

$$E \sup_{t_0 \le t \le u\wedge\pi_R} |x_{t-}|^p \le E \sup_{t_0 \le t \le u\wedge\pi_R} |x_t|^p,$$

by rearranging in (2.11), we obtain

$$\begin{aligned} E \sup_{t_0 \le t \le u\wedge\pi_R} |x_t|^p &\le K + KE\int_{t_0}^{u\wedge\pi_R} |x_r|^p ds \\ &\le K + E\int_{t_0}^{u} \sup_{t_0 \le t \le s\wedge\pi_R} |x_t|^p ds < \infty. \end{aligned} \tag{2.12}$$

From here we can finish the proof by Gronwall's and Fatou's lemmas. ∎

3. Tamed Euler scheme. For every $n \in \mathbb{N}$, let $b_t^n(x)$ and $\sigma_t^n(x)$ are $\mathscr{P}\otimes\mathscr{B}(\mathbb{R}^d)$-measurable functions which, respectively, take values in $\mathbb{R}^d$ and $\mathbb{R}^{d\times m}$. Also, for every $n \in \mathbb{N}$, let $\gamma_t^n(x,z)$ be a $\mathscr{P}\otimes\mathscr{B}(\mathbb{R}^d)\otimes\mathscr{Z}$-measurable function which takes values in $\mathbb{R}^d$. For every $n \in \mathbb{N}$, we consider a scheme of SDE (2.1) as defined below,

$$dx_t^n = b_t^n(x_{\kappa(n,t)}^n)dt + \sigma_t^n(x_{\kappa(n,t)}^n)dw_t + \int_Z \gamma_t^n(x_{\kappa(n,t)}^n, z)\tilde{N}(dt,dz), \tag{3.1}$$

almost surely for any $t \in [t_0,t_1]$, where the initial value $x_{t_0}^n$ is an $\mathscr{F}_{t_0}$-measurable random variable which takes values in $\mathbb{R}^d$ and function κ is defined by

$$\kappa(n,t) := \frac{[n(t-t_0)]}{n} + t_0 \tag{3.2}$$

for any $t \in [t_0,t_1]$.

3.1. Moment bounds. We make the following assumptions on the coefficients of the scheme (3.1).

B-1. *We have* $\sup_{n\in\mathbb{N}} E|x_{t_0}^n|^p < \infty$.

B-2. *There exist a constant* $L>0$ *and a sequence* $(M_n)_{n\in\mathbb{N}}$ *of nonnegative random variables satisfying* $\sup_{n\in\mathbb{N}} EM_n^{\frac{p}{2}} < \infty$ *such that*

$$xb_t^n(x) \vee |\sigma_t^n(x)|^2 \vee \int_Z |\gamma_t^n(x,z)|^2\nu(dz) \le L(M_n + |x|^2)$$

almost surely for any $t \in [t_0,t_1]$, $n \in \mathbb{N}$, *and* $x \in \mathbb{R}^d$.

B-3. *There exist a constant* $L>0$ *and a sequence* $(M_n')_{n\in\mathbb{N}}$ *of nonnegative random variables satisfying* $\sup_{n\in\mathbb{N}} EM_n' < \infty$ *such that*

$$\int_Z |\gamma_t^n(x,z)|^p\nu(dz) \le L(M_n' + |x|^p)$$

almost surely for any $t \in [t_0,t_1]$, $n \in \mathbb{N}$, *and* $x \in \mathbb{R}^d$.

Below is our taming assumption on drift coefficient of scheme (3.1) following the approach of [21].

B-4. *For any* $t \in [t_0,t_1]$ *and* $x \in \mathbb{R}^d$,

$$|b_t^n(x)| \le n^{\theta}$$

almost surely with $\theta \in (0,\frac{1}{2}]$ *for every* $n \in \mathbb{N}$.

Remark 3.1. Note that due to B-4, for each $n \ge 1$, the norm of b^n is a bounded function of t and x which along with B-1 and B-2 guarantee the existence of a unique solution to (3.1). Moreover, they also guarantee that for each $n \ge 1$,

$$E\sup_{0\le t\le T}|x_t^n|^p < \infty.$$

Clearly, one cannot claim at this point that this bound is independent of n. Nevertheless, as a result of this observation, one need not apply stopping time arguments, similar to the one used in the proof of Lemma 2.3, in the proofs of Lemmas 3.1 and 3.2 mentioned below.

LEMMA 3.1. *Let assumptions* B-2 *to* B-4 *hold. Then*

$$\int_{t_0}^{u} E|x_t^n - x_{\kappa(n,t)}^n|^p dt \leq Kn^{-1} + Kn^{-1}\int_{t_0}^{u} E|x_{\kappa(n,t)}^n|^p dt$$

for any $u \in [t_0, t_1]$ *with* $K := K(t_0, t_1, L, p, \sup_{n\in\mathbb{N}} EM_n^{\frac{p}{2}}, \sup_{n\in\mathbb{N}} EM_n')$, *which does not depend on* n.

Proof. From the definition of scheme (3.1), one writes

$$E|x_t^n - x_{\kappa(n,t)}^n|^p \leq KE\left|\int_{\kappa(n,t)}^{t} b_s^n(x_{\kappa(n,s)}^n)ds\right|^p + KE\left|\int_{\kappa(n,t)}^{t} \sigma_s^n(x_{\kappa(n,s)}^n)dw_s\right|^p$$
$$+ KE\left|\int_{\kappa(n,t)}^{t}\int_Z \gamma_s^n(x_{\kappa(n,s)}^n, z)\tilde{N}(ds,dz)\right|^p,$$

which on the application of Hölder's inequality and an elementary stochastic inequalities gives

$$E|x_t^n - x_{\kappa(n,t)}^n|^p \leq Kn^{-(p-1)}E\int_{\kappa(n,t)}^{t} |b_s^n(x_{\kappa(n,s)}^n)|^p ds$$
$$+ KE\left(\int_{\kappa(n,t)}^{t} |\sigma_s^n(x_{\kappa(n,s)}^n)|^2 ds\right)^{\frac{p}{2}}$$
$$+ KE\left(\int_{\kappa(n,t)}^{t}\int_Z |\gamma_s^n(x_{\kappa(n,s)}^n, z)|^2 \nu(dz)ds\right)^{\frac{p}{2}}$$
$$+ KE\int_{\kappa(n,t)}^{t}\int_Z |\gamma_s^n(x_{\kappa(n,s)}^n, z)|^p \nu(dz)ds.$$

On using assumptions B-2, B-3, and B-4, one obtains

$$E|x_t^n - x_{\kappa(n,t)}^n|^p \leq K\Big(n^{-p(1-\theta)} + n^{-\frac{p}{2}}E(M_n + |x_{\kappa(n,t)}^n|^2)^{\frac{p}{2}} + n^{-1}E(M_n' + |x_{\kappa(n,t)}^n|^p)\Big),$$

which completes the proof by noticing that $\theta \in (0, \frac{1}{2}]$ and $p \geq 2$. ∎

LEMMA 3.2. *Let assumptions* B-1 *to* B-4 *be satisfied. Then,*

$$\sup_{n\in\mathbb{N}} E \sup_{t_0\leq t\leq t_1} |x_t^n|^p \leq K$$

with $K := K(t_0, t_1, L, p, \sup_{n\in\mathbb{N}} E|x_{t_0}^n|^p, \sup_{n\in\mathbb{N}} EM_n^{\frac{p}{2}}, \sup_{n\in\mathbb{N}} EM_n')$, *which is independent of* n.

Proof. By the application of Itô formula, one gets

$$\begin{aligned}
|x^n_t|^p = {}& |x^n_{t_0}|^p + p\int_{t_0}^t |x^n_s|^{p-2} x^n_s b^n_s(x^n_{\kappa(n,s)})ds + p\int_{t_0}^t |x^n_s|^{p-2} x^n_s \sigma^n_s(x^n_{\kappa(n,s)})dw_s \\
&+ \frac{p(p-2)}{2}\int_{t_0}^t |x^n_s|^{p-4} |\sigma^{n*}_s(x^n_{\kappa(n,s)})x^n_s|^2 ds + \frac{p}{2}\int_{t_0}^t |x^n_s|^{p-2}|\sigma^n_s(x^n_{\kappa(n,s)})|^2 ds \\
&+ p\int_{t_0}^t\int_Z |x^n_s|^{p-2} x^n_s \gamma^n_s(x^n_{\kappa(n,s)},z)\tilde{N}(ds,dz) \\
&+ \int_{t_0}^t\int_Z \{|x^n_s + \gamma^n_s(x^n_{\kappa(n,s)},z)|^p - |x^n_s|^p - p|x^n_s|^{p-2}x^n_s\gamma^n_s(x^n_{\kappa(n,s)},z)\}N(ds,dz)
\end{aligned} \tag{3.3}$$

almost surely for any $t \in [t_0, t_1]$. In order to estimate second term of (3.3), one writes

$$x^n_s b^n_s(x^n_{\kappa(n,s)}) = (x^n_s - x^n_{\kappa(n,s)}) b^n_s(x^n_{\kappa(n,s)}) + x^n_{\kappa(n,s)} b^n_s(x^n_{\kappa(n,s)}),$$

which due to assumption B-2 and (3.1) gives

$$\begin{aligned}
x^n_s b^n_s(x^n_{\kappa(n,s)}) \le |b^n_s(x^n_{\kappa(n,s)})| \Bigg\{ & \left|\int_{\kappa(n,s)}^s b^n_r(x^n_{\kappa(n,r)})dr\right| + \left|\int_{\kappa(n,s)}^s \sigma^n_r(x^n_{\kappa(n,r)})dw_r\right| \\
&+ \left|\int_{\kappa(n,s)}^s \int_Z \gamma^n_r(x^n_{\kappa(n,r)},z)\tilde{N}(dr,dz)\right| \Bigg\}, \\
&+ K(M_n + |x^n_{\kappa(n,s)}|^2)
\end{aligned}$$

and then assumption B-4 implies

$$\begin{aligned}
x^n_s b^n_s(x^n_{\kappa(n,s)}) \le {}& n^{2\theta-1} + n^\theta \left|\int_{\kappa(n,s)}^s \sigma^n_r(x^n_{\kappa(n,r)})dw_r\right| \\
&+ n^\theta \left|\int_{\kappa(n,s)}^s \int_Z \gamma^n_r(x^n_{\kappa(n,r)},z)\tilde{N}(dr,dz)\right| + K(M_n + |x^n_{\kappa(n,s)}|^2)
\end{aligned}$$

almost surely for any $s \in [t_0, t_1]$. By using the fact that $\theta \in (0, 1/2]$ implies $2\theta - 1 \le 0$, one obtains

$$\begin{aligned}
|x^n_s|^{p-2} x^n_s b^n_s(x^n_{\kappa(n,s)}) \le {}& |x^n_s|^{p-2} + n^\theta |x^n_s|^{p-2} \left|\int_{\kappa(n,s)}^s \sigma^n_r(x^n_{\kappa(n,r)})dw_r\right| \\
&+ n^\theta |x^n_s|^{p-2} \left|\int_{\kappa(n,s)}^s \int_Z \gamma^n_r(x^n_{\kappa(n,r)},z)\tilde{N}(dr,dz)\right| \\
&+ K|x^n_s|^{p-2}(M_n + |x^n_{\kappa(n,s)}|^2),
\end{aligned}$$

which on using Young's inequality along with the inequality $|x^n_s|^{p-2} \le 2^{p-3}|x^n_s -$

$x^n_{\kappa(n,s)}|^{p-2} + 2^{p-3}|x^n_{\kappa(n,s)}|^{p-2}$ gives

$$
\begin{aligned}
|x^n_s|^{p-2} x^n_s b^n_s(x^n_{\kappa(n,s)}) \le{}& 1 + K|x^n_s|^p + Kn^{\theta\frac{p}{2}} \left| \int_{\kappa(n,s)}^s \sigma^n_r(x^n_{\kappa(n,r)}) dw_r \right|^{\frac{p}{2}} \\
&+ Kn^{\theta} |x^n_{\kappa(n,s)}|^{p-2} \left| \int_{\kappa(n,s)}^s \int_Z \gamma^n_r(x^n_{\kappa(n,r)}, z) \tilde{N}(dr, dz) \right| \\
&+ Kn^{\theta} |x^n_s - x^n_{\kappa(n,s)}|^{p-2} \left| \int_{\kappa(n,s)}^s \int_Z \gamma^n_r(x^n_{\kappa(n,r)}, z) \tilde{N}(dr, dz) \right| \\
&+ K(M_n^{\frac{p}{2}} + |x^n_{\kappa(n,s)}|^p)
\end{aligned} \tag{3.4}
$$

almost surely for any $s \in [t_0, t_1]$. Therefore, by substituting estimates from (2.5) and (3.4) into (3.3), one obtains for $u \in [t_0, t_1]$,

$$
\begin{aligned}
&E \sup_{t_0 \le t \le u} |x^n_t|^p \\
&\le E|x^n_{t_0}|^p + K + KE \int_{t_0}^u |x^n_s|^p ds \\
&\quad + Kn^{\theta\frac{p}{2}} E \int_{t_0}^u \left| \int_{\kappa(n,s)}^s \sigma^n_r(x^n_{\kappa(n,r)}) dw_r \right|^{\frac{p}{2}} ds \\
&\quad + Kn^{\theta} E \int_{t_0}^u \left| \int_{\kappa(n,s)}^s \int_Z |x^n_{\kappa(n,s)}|^{p-2} \gamma^n_r(x^n_{\kappa(n,r)}, z) \tilde{N}(dr, dz) \right| ds \\
&\quad + Kn^{\theta} E \int_{t_0}^u |x^n_s - x^n_{\kappa(n,s)}|^{p-2} \left| \int_{\kappa(n,s)}^s \int_Z \gamma^n_r(x^n_{\kappa(n,r)}, z) \tilde{N}(dr, dz) \right| ds \\
&\quad + KE \int_{t_0}^u (M_n^{\frac{p}{2}} + |x^n_{\kappa(n,s)}|^p) ds \\
&\quad + pE \sup_{t_0 \le t \le u} \left| \int_{t_0}^t |x^n_s|^{p-2} x^n_s \sigma^n_s(x^n_{\kappa(n,s)}) dw_s \right| \\
&\quad + KE \int_{t_0}^u |x^n_s|^{p-2} |\sigma^n_s(x^n_{\kappa(n,s)})|^2 ds \\
&\quad + pE \sup_{t_0 \le t \le u} \left| \int_{t_0}^t \int_Z |x^n_s|^{p-2} x^n_s \gamma^n_s(x^n_{\kappa(n,s)}, z) \tilde{N}(ds, dz) \right| \\
&\quad + E \int_{t_0}^u \int_Z \{ |x^n_s|^{p-2} |\gamma^n_s(x^n_{\kappa(n,s)}, z)|^2 + |\gamma^n_s(x^n_{\kappa(n,s)}, z)|^p \} N(ds, dz) \\
&=: E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7 + E_8 + E_9 + E_{10}.
\end{aligned} \tag{3.5}
$$

Here $E_1 := E|x^n_{t_0}|^p + K$. One estimates E_2 by

$$
E_2 := KE \int_{t_0}^u |x^n_s|^p ds \le K \int_{t_0}^u E \sup_{t_0 \le r \le s} |x^n_r|^p ds. \tag{3.6}
$$

In order to estimate E_3, one applies an elementary stochastic inequality to obtain

$$E_3 := Kn^{\theta\frac{p}{2}} E \int_{t_0}^{u} \left| \int_{\kappa(n,s)}^{s} \sigma_r^n(x_{\kappa(n,r)}^n) dw_r \right|^{\frac{p}{2}} ds$$
$$\leq Kn^{\theta\frac{p}{2}} \int_{t_0}^{u} E\left(\int_{\kappa(n,s)}^{s} |\sigma_r^n(x_{\kappa(n,r)}^n)|^2 dr \right)^{\frac{p}{4}} ds,$$

and then on the application of assumption B-2, one obtains

$$E_3 \leq Kn^{\frac{p}{2}(\theta-\frac{1}{2})} \int_{t_0}^{u} E(M_n + |x_{\kappa(n,s)}^n|^2)^{\frac{p}{4}} ds,$$

which, by noting that $\frac{p}{2}(\theta - \frac{1}{2}) \in (-\frac{p}{4}, 0]$, implies

$$E_3 \leq K \int_{t_0}^{u} E\{1 + (M_n + |x_{\kappa(n,s)}^n|^2)^{\frac{p}{2}}\} ds \leq K + \int_{t_0}^{u} E \sup_{t_0 \leq r \leq s} |x_r^n|^p ds. \tag{3.7}$$

By Lemma 2.1, one estimates E_4 as

$$E_4 := Kn^{\theta} E \int_{t_0}^{u} \left| \int_{\kappa(n,s)}^{s} \int_Z |x_{\kappa(n,s)}^n|^{p-2} \gamma_r^n(x_{\kappa(n,r)}^n, z) \tilde{N}(dr, dz) \right| ds$$
$$\leq Kn^{\theta} \int_{t_0}^{u} E\left(\int_{\kappa(n,s)}^{s} \int_Z |x_{\kappa(n,s)}^n|^{2p-4} |\gamma_r^n(x_{\kappa(n,r)}^n, z)|^2 \nu(dz) dr \right)^{\frac{1}{2}} ds,$$

which due to assumption B-2 gives

$$E_4 \leq KE \sup_{t_0 \leq s \leq u} |x_s^n|^{p-2} n^{\theta - \frac{1}{2}} \int_{t_0}^{u} (M_n + |x_{\kappa(n,s)}^n|^2)^{\frac{1}{2}} ds,$$

and then on using Young's inequality and Hölder's inequality, one obtains

$$E_4 \leq \frac{1}{8} E \sup_{t_0 \leq s \leq u} |x_s^n|^p + Kn^{\frac{p}{2}(\theta - \frac{1}{2})} E \int_{t_0}^{u} (M_n + |x_{\kappa(n,s)}^n|^2)^{\frac{p}{4}} ds.$$

By noticing that $\theta \in (0, \frac{1}{2}]$, one has

$$E_4 \leq \frac{1}{8} E \sup_{t_0 \leq s \leq u} |x_s^n|^p + KE \int_{t_0}^{u} \{1 + (M_n + |x_{\kappa(n,s)}^n|^2)^{\frac{p}{2}}\} ds.$$
$$\leq \frac{1}{8} E \sup_{t_0 \leq s \leq u} |x_s^n|^p + K + K \int_{t_0}^{u} E \sup_{t_0 \leq r \leq s} |x_r^n|^p ds. \tag{3.8}$$

Further, to estimate E_5, one uses Young's inequality and Hölder's inequality to write

$$E_5 := Kn^{\theta} E \int_{t_0}^{u} |x_s^n - x_{\kappa(n,s)}^n|^{p-2} \left| \int_{\kappa(n,s)}^{s} \int_Z \gamma_r^n(x_{\kappa(n,r)}^n, z) \tilde{N}(dr, dz) \right| ds$$
$$\leq Kn^{\theta} \int_{t_0}^{u} E|x_s^n - x_{\kappa(n,s)}^n|^p ds + Kn^{\theta} \int_{t_0}^{u} E \left| \int_{\kappa(n,s)}^{s} \int_Z \gamma_r^n(x_{\kappa(n,r)}^n, z) \tilde{N}(dr, dz) \right|^{\frac{p}{2}} ds$$
$$\leq Kn^{\theta} \int_{t_0}^{u} E|x_s^n - x_{\kappa(n,s)}^n|^p ds + 1$$
$$+ Kn^{2\theta} \int_{t_0}^{u} E \left| \int_{\kappa(n,s)}^{s} \int_Z \gamma_r^n(x_{\kappa(n,r)}^n, z) \tilde{N}(dr, dz) \right|^{p} ds,$$

which on the application of Lemmas 2.1 and 3.1 implies

$$\begin{aligned} E_5 &\leq 1 + Kn^{\theta-1} + Kn^{\theta-1}\int_{t_0}^{u} E|x^n_{\kappa(n,s)}|^p ds \\ &\quad + Kn^{2\theta}\int_{t_0}^{u} E\left(\int_{\kappa(n,s)}^{s}\int_Z |\gamma^n_r(x^n_{\kappa(n,r)},z)|^2\nu(dz)dr\right)^{\frac{p}{2}} ds \\ &\quad + Kn^{2\theta}\int_{t_0}^{u} E\int_{\kappa(n,s)}^{s}\int_Z |\gamma^n_r(x^n_{\kappa(n,r)},z)|^p\nu(dz)drds. \end{aligned}$$

By using assumptions B-2 and B-3, one obtains

$$\begin{aligned} E_5 &\leq 1 + Kn^{\theta-1} + Kn^{\theta-1}\int_{t_0}^{u} E|x^n_{\kappa(n,s)}|^p ds + Kn^{2\theta-\frac{p}{2}}\int_{t_0}^{u} E\big(M_n + |x^n_{\kappa(n,s)}|^2\big)^{\frac{p}{2}} ds \\ &\quad + Kn^{2\theta-1}\int_{t_0}^{u} E(M'_n + |x^n_{\kappa(n,s)}|^p)ds. \end{aligned}$$

Notice that $2\theta-1 \in (-1,0]$ and $p \geq 2$. Hence one has

$$E_5 \leq K + KEM_n^{\frac{p}{2}} + EM'_n + K\int_{t_0}^{u} E|x^n_{\kappa(n,s)}|^p ds \leq K + K\int_{t_0}^{u} E\sup_{t_0\leq r\leq s}|x^n_r|^p ds. \tag{3.9}$$

It is easy to observe that E_6 can be estimated by

$$E_6 := KE\int_{t_0}^{u}(M_n^{\frac{p}{2}} + |x^n_{\kappa(n,s)}|^p)ds \leq K + K\int_{t_0}^{u} E\sup_{t_0\leq r\leq s}|x^n_r|^p ds. \tag{3.10}$$

By using the Burkholder–Davis–Gundy inequality and assumption B-2, one obtains the following estimates of E_7:

$$\begin{aligned} E_7 &:= pE\sup_{t_0\leq t\leq u}\left|\int_{t_0}^{t}|x^n_s|^{p-2}x^n_s\sigma^n_s(x^n_{\kappa(n,s)})dw_s\right| \\ &\leq KE\left(\int_{t_0}^{u}|x^n_s|^{2p-2}|\sigma^n_s(x^n_{\kappa(n,s)})|^2 ds\right)^{\frac{1}{2}} \\ &\leq KE\left(\int_{t_0}^{u}|x^n_s|^{2p-2}(M_n + |x^n_{\kappa(n,s)}|^2)ds\right)^{\frac{1}{2}} \\ &\leq KE\sup_{t_0\leq s\leq u}|x^n_s|^{p-1}\left(\int_{t_0}^{u}(M_n + |x^n_{\kappa(n,s)}|^2)ds\right)^{\frac{1}{2}}, \end{aligned}$$

which due to Young's inequality and Hölder's inequality gives

$$E_7 \leq \frac{1}{8}E\sup_{t_0\leq s\leq u}|x^n_s|^p + K + KE\int_{t_0}^{u} E\sup_{t_0\leq r\leq s}|x^n_r|^p ds. \tag{3.11}$$

Similarly, by using assumption B-2 and Young's inequality, E_8 can be estimated by

$$E_8 := KE\int_{t_0}^{u}|x^n_s|^{p-2}|\sigma^n_s(x^n_{\kappa(n,s)})|^2 ds \leq K + K\int_{t_0}^{u} E\sup_{t_0\leq r\leq s}|x^n_r|^p ds. \tag{3.12}$$

Further one uses Lemma 2.1 and assumption B-2 to estimate E_9 by

$$\begin{aligned}E_9 &:= pE \sup_{t_0\le t\le u}\left|\int_{t_0}^t \int_Z |x_s^n|^{p-2} x_s^n \gamma_s^n(x_{\kappa(n,s)}^n, z)\tilde{N}(ds,dz)\right| \\ &\le KE\left(\int_{t_0}^u \int_Z |x_s^n|^{2p-2}|\gamma_s^n(x_{\kappa(n,s)}^n,z)|^2\nu(dz)ds\right)^{\frac{1}{2}} \\ &\le KE\left(\int_{t_0}^u |x_s^n|^{2p-2}(M_n+|x_{\kappa(n,s)}^n|^2)ds\right)^{\frac{1}{2}},\end{aligned}$$

which due to Young's inequality and Hölder's inequality gives

$$E_9 \le \frac{1}{8}E\sup_{t_0\le s\le u}|x_s^n|^p + K + KE\int_{t_0}^u E\sup_{t_0\le r\le s}|x_r^n|^p ds. \tag{3.13}$$

Finally, due to assumptions B-2 and B-3, E_{10} can be estimated as follows,

$$\begin{aligned}E_{10} &:= E\int_{t_0}^u \int_Z \{|x_s^n|^{p-2}|\gamma_s^n(x_{\kappa(n,s)}^n,z)|^2 + |\gamma_s^n(x_{\kappa(n,s)}^n,z)|^p\}N(ds,dz) \\ &= E\int_{t_0}^u \int_Z \{|x_s^n|^{p-2}|\gamma_s^n(x_{\kappa(n,s)}^n,z)|^2 + |\gamma_s^n(x_{\kappa(n,s)}^n,z)|^p\}\nu(dz)ds \\ &= E\int_{t_0}^u |x_s^n|^{p-2}(M_n+|x_{\kappa(n,s)}^n|^2)ds + E\int_{t_0}^u (M_n'+|x_{\kappa(n,s)}^n|^p)ds,\end{aligned}$$

and then Young's inequality implies

$$E_{10} \le K + \int_{t_0}^u E\sup_{t_0\le r\le s}|x_r^n|^p ds. \tag{3.14}$$

By substituting estimates from (3.6)–(3.14) in (3.5), one obtains

$$E\sup_{t_0\le t\le u}|x_t^n|^p \le \frac{1}{2}E\sup_{t_0\le t\le u}|x_t^n|^p + K + KE\int_{t_0}^u E\sup_{t_0\le r\le s}|x_r^n|^p ds.$$

The application of Gronwall's lemma completes the proof. ∎

Remark 3.2. Due to assumptions B-2 and B-3, there exist a constant $L>0$ and a sequence $(M_n')_{n\in\mathbb{N}}$ of nonnegative random variables satisfying $\sup_{n\in\mathbb{N}} EM_n' < \infty$ such that

$$\int_Z |\gamma_t^n(x,z)|^r \nu(dz) \le L(M_n' + |x|^r)$$

almost surely for any $2\le r\le p$, $t\in[t_0,t_1]$, $n\in\mathbb{N}$, and $x\in\mathbb{R}^d$.

LEMMA 3.3. *Let assumptions* B-1 *to* B-4 *be satisfied. Then*

$$\sup_{t_0\le t\le t_1} E|x_t^n - x_{\kappa(n,t)}^n|^r \le Kn^{-1}$$

for any $2\le r\le p$ *with* $K := K(t_0,t_1,L,p,\sup_{n\in\mathbb{N}}E|x_{t_0}^n|^p, \sup_{n\in\mathbb{N}}EM_n^{\frac{p}{2}}, \sup_{n\in\mathbb{N}}EM_n')$ *which does not depend on* n.

Proof. The lemma follows immediately from Lemmas 3.1 and 3.2. ∎

3.2. Convergence in $\mathcal{L}^q$. For every $R > 0$, we consider $\mathscr{F}_{t_0}$-measurable random variables C_R which satisfy

$$\lim_{R\to\infty} P(C_R > f(R)) = 0 \tag{3.15}$$

for a nondecreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$. This notation for the family of random variables with the above property will be used throughout this article.

A-7. *For every $R > 0$ and $t \in [t_0, t_1]$,*

$$(x-\bar{x})(b_t(x)-b_t(\bar{x})) \vee |\sigma_t(x)-\sigma_t(\bar{x})|^2 \vee \int_Z |\gamma_t(x,z)-\gamma_t(\bar{x},z)|^2\nu(dz) \le C_R|x-\bar{x}|^2$$

almost surely whenever $|x|, |\bar{x}| \le R$.

A-8. *For every $R > 0$ and $t \in [t_0, t_1]$,*

$$\sup_{|x|\le R} |b_t(x)| \le C_R$$

almost surely.

B-5. *For every $R > 0$ and $B(R) := \{\omega \in \Omega : C_R \le f(R)\}$,*

$$\lim_{n\to\infty} E\int_{t_0}^{t_1} I_{B(R)} \sup_{|x|\le R}\{|b_t^n(x)-b_t(x)|^2 + |\sigma_t^n(x)-\sigma_t(x)|^2\}dt = 0$$
$$\lim_{n\to\infty} E\int_{t_0}^{t_1} I_{B(R)} \sup_{|x|\le R}\int_Z |\gamma_t^n(x,z)-\gamma_t(x,z)|^2\nu(dz)dt = 0.$$

B-6. *For every $n \in \mathbb{N}$, the initial values of SDE* (2.1) *and scheme* (3.1) *satisfy $|x_{t_0} - x_{t_0}^n| \xrightarrow{P} 0$ as $n \to \infty$.*

We introduce families of stopping times that shall be used frequently in this report. For every $R > 0$ and $n \in \mathbb{N}$, let

$$\pi_R := \inf\{t \ge t_0 : |x_t| \ge R\},\ \pi_{nR} := \inf\{t \ge t_0 : |x_t^n| \ge R\},\ \tau_{nR} := \pi_R \wedge \pi_{nR} \tag{3.16}$$

almost surely.

THEOREM 3.4. *Let assumptions* A-3 *to* A-8 *be satisfied. Also assume that* B-1 *to* B-6 *hold. Then,*

$$\lim_{n\to\infty} E \sup_{t_0\le t\le t_1} |x_t - x_t^n|^q = 0$$

for all $q < p$.

Proof. Let $e_t^n := x_t - x_t^n$ and define

$$\bar{b}_t^n := b_t(x_t) - b_t^n(x_{\kappa(n,t)}^n), \bar{\sigma}_t^n := \sigma_t(x_t) - \sigma_t^n(x_{\kappa(n,t)}^n), \bar{\gamma}_t^n(z) := \gamma_t(x_t,z) - \gamma_t^n(x_{\kappa(n,t)}^n, z) \tag{3.17}$$

almost surely for any $t \in [t_0, t_1]$. In this simplified notation, e_t^n can be written as

$$e_t^n = e_{t_0}^n + \int_{t_0}^t \bar{b}_s^n ds + \int_{t_0}^t \bar{\sigma}_s^n dw_s + \int_{t_0}^t\int_Z \bar{\gamma}_s^n(z)\tilde{N}(ds,dz) \tag{3.18}$$

almost surely for any $t \in [t_0, t_1]$. Further, by using the stopping times defined in (3.16) and random variables defined in (3.15), let us partition the sample space Ω into two parts Ω_1 and Ω_2 where

$$\begin{aligned}\Omega_1 &= \{\omega \in \Omega : \pi_R \le t_1 \,\text{or}\, \pi_{nR} \le t_1 \,\text{or}\, C_R > f(R)\ \}\\ &= \{\omega \in \Omega : \pi_R \le t_1\} \cup \{\omega \in \Omega : \pi_{nR} \le t_1\} \cup \{\omega \in \Omega : C_R > f(R)\}\\ \Omega_2 &= \Omega \backslash \Omega_1 = \{\omega \in \Omega : \pi_R > t_1\} \cap \{\omega \in \Omega : \pi_{nR} > t_1\} \cap B(R),\end{aligned}$$

where $B(R) := \{\omega \in \Omega : C_R \le f(R)\}$ as defined in assumption B-5. Also note that $I_\Omega = I_{\Omega_1 \cup \Omega_2} \le I_{\Omega_1} + I_{\Omega_2}$. By using this fact, for any $q < p$, one could write the following:

$$E \sup_{t_0 \le t \le t_1} |e_t^n|^q = E \sup_{t_0 \le t \le t_1} |e_t^n|^q I_{\Omega_1} + E \sup_{t_0 \le t \le t_1} |e_t^n|^q I_{\Omega_2} =: D_1 + D_2. \tag{3.19}$$

By the application of Hölder's inequality and Lemmas 2.3 and 3.2 one could write

$$\begin{aligned}D_1 := E \sup_{t_0 \le t \le t_1} |e_t^n|^q I_{\Omega_1} &\le \left(E \sup_{t_0 \le t \le t_1} |e_t^n|^{q\frac{p}{q}}\right)^{\frac{q}{p}} \left(EI_{\Omega_1}\right)^{\frac{p-q}{p}}\\ &\le K \left(\frac{E|x_{\pi_R}|^p}{R^p} + \frac{E|x^n_{\pi_{nR}}|^p}{R^p} + P(\{\omega \in \Omega : C_R > f(R)\})\right)^{\frac{p-q}{p}}\\ &\le K \left(\frac{1}{R^p} + P(\{\omega \in \Omega : C_R > f(R)\})\right)^{\frac{p-q}{p}}, \qquad (3.20)\end{aligned}$$

where the constant $K > 0$ does not depend on n. Having obtained estimates for D_1, we now proceed to obtain the estimates for D_2. For this, we recall (3.18) and use Itô formula to obtain the following:

$$\begin{aligned}|e_t^n|^2 = |e_{t_0}^n|^2 &+ 2\int_{t_0}^t e_s^n \bar{b}_s^n ds + 2\int_{t_0}^t e_s^n \bar{\sigma}_s^n dw_s + \int_{t_0}^t |\bar{\sigma}_s^n|^2 ds\\ &+ 2\int_{t_0}^t \int_Z e_s^n \bar{\gamma}_s^n(z) \tilde{N}(ds,dz) + \int_{t_0}^t \int_Z |\bar{\gamma}_s^n(z)|^2 N(ds,dz), \qquad (3.21)\end{aligned}$$

almost surely for any $t \in [t_0, t_1]$. Also, to estimate the second term of (3.21), one uses the following splitting:

$$\begin{aligned}e_s^n \bar{b}_s^n = &(x_s - x^n_{\kappa(n,s)})(b_s(x_s) - b_s(x^n_{\kappa(n,s)}))\\ &+ (x_s - x^n_{\kappa(n,s)})(b_s(x^n_{\kappa(n,s)}) - b^n_s(x^n_{\kappa(n,s)}))\\ &+ (x^n_{\kappa(n,s)} - x^n_s)(b_s(x_s) - b_s(x^n_{\kappa(n,s)}))\\ &+ (x^n_{\kappa(n,s)} - x^n_s)(b_s(x^n_{\kappa(n,s)}) - b^n_s(x^n_{\kappa(n,s)})) \qquad (3.22)\end{aligned}$$

almost surely for any $s \in [t_0, t_1]$. Notice that D_2 is nonzero only on Ω_2, and thus one can henceforth restrict all the calculations in the estimation of D_2 on the interval $[t_0, t_1 \wedge \tau_{nR})$, which also means that $|x_t| \vee |x^n_t| < R$ for any $t \in [t_0, t_1 \wedge \tau_{nR})$. As a consequence, on the application of assumption A-7 and the Cauchy–Schwarz inequality, one obtains

$$\begin{aligned}e_s^n \bar{b}_s^n \le C_R |x_s - x^n_{\kappa(n,s)}|^2 &+ |x_s - x^n_{\kappa(n,s)}||b_s(x^n_{\kappa(n,s)}) - b^n_s(x^n_{\kappa(n,s)})|\\ &+ |x^n_{\kappa(n,s)} - x^n_s||b_s(x_s) - b_s(x^n_{\kappa(n,s)})| + |x^n_{\kappa(n,s)} - x^n_s||b_s(x^n_{\kappa(n,s)}) - b^n_s(x^n_{\kappa(n,s)})|\end{aligned}$$

almost surely for any $s \in [t_0, t_1 \wedge \tau_{nR})$. By using assumption A-8, this can further be estimated as

$$\begin{aligned} e_s^n \bar{b}_s^n \leq (2C_R+1)|x_s - x_s^n|^2 &+ \left(2C_R + \frac{3}{2}\right)|x_s^n - x_{\kappa(n,s)}^n|^2 \\ &+ |b_s(x_{\kappa(n,s)}^n) - b_s^n(x_{\kappa(n,s)}^n)|^2 \\ &+ 2C_R|x_s^n - x_{\kappa(n,s)}^n| \end{aligned} \tag{3.23}$$

almost surely for any $s \in [t_0, t_1 \wedge \tau_{nR})$. Now, by using the definition of Ω_2 and of τ_{nR} in (3.16), one has

$$D_2 := E \sup_{t_0 \leq t \leq t_1} |e_t^n|^q I_{\Omega_2} \leq E \sup_{t_0 \leq t \leq t_1} |e_{t\wedge\tau_{nR}}^n|^q I_{B(R)}. \tag{3.24}$$

Thus using the estimate obtained in (3.23), one obtains

$$\begin{aligned} &E \sup_{t_0 \leq t \leq u} |e_{t\wedge\tau_{nR}}^n|^2 I_{B(R)} \\ &\quad \leq E|e_{t_0}^n|^2 + E(2C_R+1)\int_{t_0}^{u\wedge\tau_{nR}} |e_s^n|^2 I_{B(R)} ds \\ &\qquad + E\left(2C_R + \frac{3}{2}\right)\int_{t_0}^{u\wedge\tau_{nR}} |x_s^n - x_{\kappa(n,s)}^n|^2 I_{B(R)} ds \\ &\qquad + 2EC_R \int_{t_0}^{u\wedge\tau_{nR}} |x_s^n - x_{\kappa(n,s)}^n| I_{B(R)} ds \\ &\qquad + E\int_{t_0}^{u\wedge\tau_{nR}} |b_s(x_{\kappa(n,s)}^n) - b_s^n(x_{\kappa(n,s)}^n)|^2 I_{B(R)} ds \\ &\qquad + 2E \sup_{t_0 \leq t \leq u} \left|\int_{t_0}^{t\wedge\tau_{nR}} I_{B(R)} e_s^n \bar{\sigma}_s^n dw_s\right| + E\int_{t_0}^{u\wedge\tau_{nR}} |\bar{\sigma}_s^n|^2 I_{B(R)} ds \\ &\qquad + 2E \sup_{t_0 \leq t \leq u} \left|\int_{t_0}^{t\wedge\tau_{nR}} \int_Z I_{B(R)} e_s^n \bar{\gamma}_s^n(z) \tilde{N}(ds,dz)\right| \\ &\qquad + E \sup_{t_0 \leq t \leq u} \int_{t_0}^{t\wedge\tau_{nR}} \int_Z I_{B(R)} |\bar{\gamma}_s^n(z)|^2 N(ds,dz) \\ &\quad =: F_1 + F_2 + F_3 + F_4 + F_5 + F_6 + F_7 + F_8 + F_9 \end{aligned} \tag{3.25}$$

for every $R > 0$ and $u \in [t_0, t_1 \wedge \tau_{nR})$. Here $F_1 := E|e_{t_0}^n|^2$. F_2 is estimated easily by

$$\begin{aligned} F_2 &:= E(2C_R+1)\int_{t_0}^{u\wedge\tau_{nR}} |e_s^n|^2 I_{B(R)} ds \\ &\leq (2f(R)+1)\int_{t_0}^{u} E \sup_{t_0 \leq r \leq s} |e_{r\wedge\tau_{nR}}^n|^2 I_{B(R)} ds \end{aligned} \tag{3.26}$$

for every $R > 0$ and $u \in [t_0, t_1 \wedge \tau_{nR})$. Further,

$$\begin{aligned} F_3 &:= E\left(2C_R + \frac{3}{2}\right)\int_{t_0}^{u\wedge\tau_{nR}} |x_s^n - x_{\kappa(n,s)}^n|^2 I_{B(R)} ds \\ &\leq (f(R)+1)K \sup_{t_0 \leq t \leq t_1} E|x_t^n - x_{\kappa(n,t)}^n|^2 \end{aligned} \tag{3.27}$$

and similarly, term F_4 can be estimated by

$$F_4 := 2EC_R\int_{t_0}^{u\wedge\tau_{nR}} |x_s^n - x_{\kappa(n,s)}^n| I_{B(R)} ds \le f(R)K \sup_{t_0\le t\le t_1} E|x_t^n - x_{\kappa(n,t)}^n| \tag{3.28}$$

for every $R>0$. Again, term F_5 has the following estimate:

$$\begin{aligned} F_5 &:= E\int_{t_0}^{u\wedge\tau_{nR}} |b_s(x_{\kappa(n,s)}^n) - b_s^n(x_{\kappa(n,s)}^n)|^2 I_{B(R)} ds \\ &\le E\int_{t_0}^{t_1} I_{\{t_0\le s<\tau_{nR}\}} I_{B(R)} |b_s(x_{\kappa(n,s)}^n) - b_s^n(x_{\kappa(n,s)}^n)|^2 ds. \end{aligned}\tag{3.29}$$

To estimate the term F_6, one uses the Burkholder–Davis–Gundy inequality to write

$$\begin{aligned} F_6 := 2E\sup_{t_0\le t\le u}\left|\int_{t_0}^{t\wedge\tau_{nR}} I_{B(R)} e_s^n \bar{\sigma}_s^n dw_s\right| &\le KE\left(\int_{t_0}^{u\wedge\tau_{nR}} I_{B(R)}|e_s^n|^2|\bar{\sigma}_s^n|^2 ds\right)^{\frac12} \\ \le KE\sup_{t_0\le s\le u}|e_{s\wedge\tau_{nR}}^n| I_{B(R)}\left(\int_{t_0}^{u\wedge\tau_{nR}} I_{B(R)}|\bar{\sigma}_s^n|^2 ds\right)^{\frac12}, \end{aligned}$$

which on the application of Young's inequality gives

$$F_6 + F_7 \le \frac18 E\sup_{t_0\le s\le u}|e_{s\wedge\tau_{nR}}^n|^2 I_{B(R)} + KE\int_{t_0}^{u\wedge\tau_{nR}} I_{B(R)}|\bar{\sigma}_s^n|^2 ds \tag{3.30}$$

for any $R>0$ and $u\in[t_0,t_1]$, where constant $K>0$ does not depend on R and n. In order to estimate the second term of the above inequality, one uses the following splitting of $\bar{\sigma}_s^n$:

$$\bar{\sigma}_s^n = (\sigma_s(x_s) - \sigma_s(x_s^n)) + (\sigma_s(x_s^n) - \sigma_s(x_{\kappa(n,s)}^n)) + (\sigma_s(x_{\kappa(n,s)}^n) - \sigma_s^n(x_{\kappa(n,s)}^n)) \tag{3.31}$$

almost surely for any $s\in[t_0,t_2]$. As before, one again notices that $|x_s|\le R$ and $|x_s^n|\le R$ whenever $s\in[t_0,t_1\wedge\tau_{nR})$. Thus on the application of assumption A-7, one obtains

$$|\bar{\sigma}_s^n|^2 \le 3C_R|e_s^n|^2 + 3C_R|x_s^n - x_{\kappa(n,s)}^n|^2 + 3|\sigma_s(x_{\kappa(n,s)}^n) - \sigma_s^n(x_{\kappa(n,s)}^n)|^2$$

almost surely $s\in[t_0,t_1\wedge\tau_{nR})$. Hence substituting this estimate in inequality (3.30) gives

$$\begin{aligned} F_6 + F_7 \le \frac18 E\sup_{t_0\le s\le u}|e_{s\wedge\tau_{nR}}^n|^2 I_{B(R)} &+ Kf(R)\int_{t_0}^{u} E\sup_{t_0\le r\le s}|e_{r\wedge\tau_{nR}}^n|^2 I_{B(R)} ds \\ &+ Kf(R)\sup_{t_0\le s\le t_1} E|x_s^n - x_{\kappa(n,s)}^n|^2 \\ &+ KE\int_{t_0}^{t_1} I_{\{t_0\le s<\tau_{nR}\}} I_{B(R)}|\sigma_s(x_{\kappa(n,s)}^n) - \sigma_s^n(x_{\kappa(n,s)}^n)|^2 ds \end{aligned}\tag{3.32}$$

for any $u\in[t_0,t_1]$. Further, one proceeds as above in the similar way to the derivation of (3.30) and uses Lemma 2.1 to obtain

$$\begin{aligned} F_8 + F_9 \le \frac18 E&\sup_{t_0\le s\le u}|e_{s\wedge\tau_{nR}}^n|^2 I_{B(R)} \\ &+ KE\int_{t_0}^{u\wedge\tau_{nR}}\int_Z I_{B(R)}|\bar{\gamma}_s^n(z)|^2\nu(dz)ds \end{aligned}\tag{3.33}$$

for any $u \in [t_0, t_1]$. In order to estimate the second term of the above inequality, one uses the following splitting:

$$\begin{aligned}\bar{\gamma}_s^n(z) = {} & (\gamma(x_s, z) - \gamma_s(x_s^n, z)) + (\gamma_s(x_s^n, z) - \gamma_s(x_{\kappa(n,s)}^n, z) \\ & + (\gamma_s(x_{\kappa(n,s)}^n, z) - \gamma_s^n(x_{\kappa(n,s)}^n, z))\end{aligned} \tag{3.34}$$

almost surely for any $s \in [t_0, t_1]$. Thus, by using the assumption A-7, one has

$$\begin{aligned}F_8 + F_9 \leq {} & \frac{1}{8} E \sup_{t_0 \leq s \leq u} |e_{s\wedge\tau_{nR}}^n|^2 I_{B(R)} + Kf(R) E \int_{t_0}^{u} E \sup_{t_0 \leq r \leq s} |e_{r\wedge\tau_{nR}}^n|^2 I_{B(R)} ds \\ & + Kf(R) \sup_{t_0 \leq s \leq t_1} E|x_s^n - x_{\kappa(n,s)}^n|^2 \\ & + KE \int_{t_0}^{t_1} \int_Z I_{\{t_0 \leq s < \tau_{nR}\}} I_{B(R)} |\gamma_s(x_{\kappa(n,s)}^n, z) - \gamma_s^n(x_{\kappa(n,s)}^n, z)|^2 \nu(dz) ds\end{aligned} \tag{3.35}$$

for any $u \in [t_0, t_1]$. On combining estimates obtained in (3.26), (3.27), (3.28), (3.32), and (3.35) in (3.25) and then applying Gronwall's inequality, one obtains

$$\begin{aligned}& E \sup_{t_0 \leq t \leq t_1} |e_{t\wedge\tau_{nR}}^n|^2 I_{B(R)} \\ & \quad \leq \exp(Kf(R)) \Bigg\{ E|e_{t_0}^n|^2 + Kf(R) \sup_{t_0 \leq s \leq t_1} E|x_s^n - x_{\kappa(n,s)}^n|^2 \\ & \qquad + Kf(R) \left(\sup_{t_0 \leq s \leq t_1} E|x_s^n - x_{\kappa(n,s)}^n|^2 \right)^{\frac{1}{2}} \\ & \qquad + KE \int_{t_0}^{t_1} I_{\{t_0 \leq s < \tau_{nR}\}} I_{B(R)} |b_s(x_{\kappa(n,s)}^n) - b_s^n(x_{\kappa(n,s)}^n)|^2 ds \\ & \qquad + KE \int_{t_0}^{t_1} I_{\{t_0 \leq s < \tau_{nR}\}} I_{B(R)} |\sigma_s(x_{\kappa(n,s)}^n) - \sigma_s^n(x_{\kappa(n,s)}^n)|^2 ds \\ & \qquad + KE \int_{t_0}^{t_1} \int_Z I_{\{t_0 \leq s < \tau_{nR}\}} I_{B(R)} |\gamma_s(x_{\kappa(n,s)}^n, z) \\ & \qquad\qquad - \gamma_s^n(x_{\kappa(n,s)}^n, z)|^2 \nu(dz) ds \Bigg\}.\end{aligned}$$

Hence, by the application of Lemma 3.3 and assumptions B-5 and B-6, one obtains

$$E \sup_{t_0 \leq t \leq t_1} |e_{t\wedge\tau_{nR}}^n|^2 I_{B(R)} \to 0 \text{ as } n \to \infty$$

for every $R > 0$. Consequently $\sup_{t_0 \leq t \leq t_1} |e_{t\wedge\tau_{nR}}^n| I_{B(R)} \to 0$ in probability, as $n \to \infty$. By Lemmas 2.3 and 3.2, we have that the sequence of random variables $(\sup_{t_0 \leq t \leq t_1} |e_{t\wedge\tau_{nR}}^n|^q I_{B(R)})_{n\in\mathbb{N}}$ is uniformly integrable for any $q < p$. Hence, for each $R > 0$ we have

$$E \sup_{t_0 \leq t \leq t_1} |e_{t\wedge\tau_{nR}}^n|^q I_{B(R)} \to 0, \text{as } n \to \infty,$$

which implies from inequality (3.24) that $D_2 \to 0$ as $n \to \infty$ for every $R > 0$. Also by choosing sufficiently large $R > 0$ in inequality (3.20) along with (3.15), one obtains $D_1 \to 0$. This complete the proof. ∎

3.3. Rate of convergence. In order to obtain the rate of convergence of the scheme (3.1), one replaces assumption A-7 by the following assumptions.

A-9. *There exist constants* $C > 0$, $q \geq 2$, *and* $\chi > 0$ *such that*

$$(x-\bar{x})(b_t(x)-b_t(\bar{x})) \vee |\sigma_t(x)-\sigma_t(\bar{x})|^2 \vee \int_Z |\gamma_t(x,z)-\gamma_t(\bar{x},z)|^2\nu(dz) \leq C|x-\bar{x}|^2$$

$$\int_Z |\gamma_t(x,z)-\gamma_t(\bar{x},z)|^q\nu(dz) \leq C|x-\bar{x}|^q$$

$$|b_t(x)-b_t(\bar{x})|^2 \leq C(1+|x|^\chi+|\bar{x}|^\chi)|x-\bar{x}|^2 \tag{3.36}$$

almost surely for any $t \in [t_0, t_1]$, $x, \bar{x} \in \mathbb{R}^d$, *and a* $\delta \in (0,1)$ *such that* $\max\{(\chi+2)q, \frac{q\chi}{2}\frac{q+\delta}{\delta}\} \leq p$.

Remark 3.3. Due to (3.36) and assumption A-8, one immediately obtains

$$|b_t(x)|^2 \leq K(1+|x|^{\chi+2})$$

almost surely for any $t \in [t_0, t_1]$ and $x \in \mathbb{R}^d$.

Furthermore, one replaces assumption B-5 by the following assumption.

B-7. *There exists a constant* $C > 0$ *such that*

$$E\int_{t_0}^{t_1} \{|b_t^n(x_{\kappa(n,t)}^n)-b_t(x_{\kappa(n,t)}^n)|^q+|\sigma_t^n(x_{\kappa(n,t)}^n)-\sigma_t(x_{\kappa(n,t)}^n)|^q\}dt \leq Cn^{-\frac{q}{q+\delta}}$$

$$E\int_{t_0}^{t_1}\left(\int_Z |\gamma_t^n(x_{\kappa(n,t)}^n,z)-\gamma_t(x_{\kappa(n,t)}^n,z)|^\zeta\nu(dz)\right)^{\frac{q}{\zeta}}dt \leq Cn^{-\frac{q}{q+\delta}}$$

for $\zeta = 2, q$.

Finally, assumption B-6 is replaced by the following assumption.

B-8. *There exists a constant* $C > 0$ *such that*

$$E|x_{t_0}-x_{t_0}^n|^q \leq Cn^{-\frac{q}{q+\delta}}.$$

THEOREM 3.5. *Let assumptions* A-3 *to* A-6, A-8, *and* A-9 *be satisfied. Also suppose that assumptions* B-1 *to* B-4, B-7, *and* B-8 *hold. Then*

$$E\sup_{t_0\leq t\leq t_1}|x_t-x_t^n|^q \leq Kn^{-\frac{q}{q+\delta}},$$

where constant $K > 0$ *does not depend on* n.

Proof. First, let us recall the notation used in the proof of Theorem 3.4. By the application of Itô formula, one obtains

$$\begin{aligned}|e_t^n|^q = {}& |e_{t_0}^n|^q + q\int_{t_0}^t |e_s^n|^{q-2}e_s^n\bar{b}_s^n ds + q\int_{t_0}^t |e_s^n|^{q-2}e_s^n\bar{\sigma}_s^n dw_s \\ &+ \frac{q(q-2)}{2}\int_{t_0}^t |e_s^n|^{q-4}|\bar{\sigma}_s^{n*}e_s^n|^2 ds + \frac{q}{2}\int_{t_0}^t |e_s^n|^{q-2}|\bar{\sigma}_s^n|^2 ds \\ &+ q\int_{t_0}^t\int_Z |e_s^n|^{q-2}e_s^n\bar{\gamma}_s^n(z)\tilde{N}(ds,dz) \\ &+ \int_{t_0}^t\int_Z\{|e_s^n+\bar{\gamma}_s^n(z)|^q-|e_s^n|^q-q|e_s^n|^{q-2}e_s^n\bar{\gamma}_s^n(z)\}N(ds,dz)\end{aligned} \tag{3.37}$$

almost surely for any $t \in [t_0, t_1]$. In Theorem 3.4, the splitting given in (3.22) is used to prove the $\mathcal{L}^q$ convergence of the scheme (3.1). In order to obtain a rate of convergence of scheme (3.1), one uses the splitting

$$\begin{aligned} e_s^n \bar{b}_s^n = {} & (x_s - x_s^n)(b_s(x_s) - b_s(x_s^n)) + (x_s - x_s^n)(b_s(x_s^n) - b_s(x^n_{\kappa(n,s)})) \\ & + (x_s - x_s^n)(b_s(x^n_{\kappa(n,s)}) - b_s^n(x^n_{\kappa(n,s)})), \end{aligned} \tag{3.38}$$

which on the application of assumption A-9, the Cauchy–Schwarz inequality, and Young's inequality gives

$$\begin{aligned} |e_s^n|^{q-2} e_s^n \bar{b}_s^n \leq {} & K|e_s^n|^q + K|b_s(x_s^n) - b_s(x^n_{\kappa(n,s)})|^q \\ & + K|b_s(x^n_{\kappa(n,s)}) - b_s^n(x^n_{\kappa(n,s)})|^q \end{aligned} \tag{3.39}$$

almost surely for any $s \in [t_0, t_1]$. Therefore by taking suprema over $[t_0, u]$ for any $u \in [t_0, t_1]$ and expectations, one has

$$\begin{aligned} E \sup_{t_0 \leq t \leq u} |e_t^n|^q \leq {} & E|e_{t_0}^n|^q + KE\int_{t_0}^u |e_s^n|^q ds + KE\int_{t_0}^u |b_s(x_s^n) - b_s(x^n_{\kappa(n,s)})|^q ds \\ & + KE\int_{t_0}^u |b_s(x^n_{\kappa(n,s)}) - b_s^n(x^n_{\kappa(n,s)})|^q ds \\ & + qE \sup_{t_0 \leq t \leq u} \left| \int_{t_0}^t |e_s^n|^{q-2} e_s^n \bar{\sigma}_s^n dw_s \right| \\ & + \frac{q(q-2)}{2} E\int_{t_0}^u |e_s^n|^{q-4} |\bar{\sigma}_s^{n*} e_s^n|^2 ds + \frac{q}{2} E\int_{t_0}^u |e_s^n|^{q-2} |\bar{\sigma}_s^n|^2 ds \\ & + qE \sup_{t_0 \leq t \leq u} \left| \int_{t_0}^t \int_Z |e_s^n|^{q-2} e_s^n \bar{\gamma}_s^n(z) \tilde{N}(ds, dz) \right| \\ & + E \sup_{t_0 \leq t \leq u} \int_{t_0}^t \int_Z \{|e_s^n|^{q-2} |\bar{\gamma}_s^n(z)|^2 + |\bar{\gamma}_s^n(z)|^q\} N(ds, dz) \\ = {} & G_1 + G_2 + G_3 + G_4 + G_5 + G_6 + G_7 + G_8 + G_9 \end{aligned} \tag{3.40}$$

for any $u \in [t_0, t_1]$. Here $G_1 := E|e_{t_0}^n|^q$ and G_2 can be estimated by

$$G_2 := KE\int_{t_0}^u |e_s^n|^q ds \leq K\int_{t_0}^u E \sup_{t_0 \leq r \leq s} |e_r^n|^q ds \tag{3.41}$$

for any $u \in [t_0, t_1]$. By the application of assumption A-9, Hölder's inequality, and Lemma 3.2, G_3 can be estimated by

$$\begin{aligned} G_3 := {} & KE\int_{t_0}^u |b_s(x_s^n) - b_s(x^n_{\kappa(n,s)})|^q ds \\ \leq {} & K\int_{t_0}^u \left(1 + E|x_s^n|^{\chi \frac{q}{2} \frac{q+\delta}{\delta}} + E|x^n_{\kappa(n,s)}|^{\chi \frac{q}{2} \frac{q+\delta}{\delta}}\right)^{\frac{\delta}{q+\delta}} \left(E|x_s^n - x^n_{\kappa(n,s)}|^{q+\delta}\right)^{\frac{q}{q+\delta}} ds \\ \leq {} & K\int_{t_0}^{t_1} \left(E|x_s^n - x^n_{\kappa(n,s)}|^{q+\delta}\right)^{\frac{q}{q+\delta}} ds. \end{aligned} \tag{3.42}$$

Further, G_4 can be estimated by

$$G_4 := KE\int_{t_0}^{u} |b_s(x^n_{\kappa(n,s)}) - b^n_s(x^n_{\kappa(n,s)})|^q ds$$

$$\le KE\int_{t_0}^{t_1} |b_s(x^n_{\kappa(n,s)}) - b^n_s(x^n_{\kappa(n,s)})|^q ds. \tag{3.43}$$

By the application of the Burkholder–Davis–Gundy inequality, one obtains

$$G_5 := qE\sup_{t_0\le t\le u}\left|\int_{t_0}^{t} |e^n_s|^{q-2} e^n_s \bar\sigma^n_s dw_s\right| \le KE\left(\int_{t_0}^{u} |e^n_s|^{2q-2}|\bar\sigma^n_s|^2 ds\right)^{\frac{1}{2}}$$

$$\le KE\sup_{t_0\le s\le u}|e^n_s|^{q-1}\left(\int_{t_0}^{u}|\bar\sigma^n_s|^2 ds\right)^{\frac{1}{2}},$$

which due to Young's inequality and Hölder's inequality gives

$$G_5 \le \frac{1}{8}E\sup_{t_0\le s\le u}|e^n_s|^q + KE\int_{t_0}^{u}|\bar\sigma^n_s|^q ds \tag{3.44}$$

for any $u \in [t_0, t_1]$. Further, due to the Cauchy–Schwarz inequality and Young's inequality, G_6 and G_7 can be estimated together by

$$G_6 + G_7 := \frac{q(q-2)}{2}E\int_{t_0}^{u} |e^n_s|^{q-4}|\bar\sigma^{n*}_s e^n_s|^2 ds + \frac{q}{2}E\int_{t_0}^{u}|e^n_s|^{q-2}|\bar\sigma^n_s|^2 ds$$

$$\le KE\int_{t_0}^{u}|e^n_s|^{q-2}|\bar\sigma^n_s|^2 ds \le K\int_{t_0}^{u}E\sup_{t_0\le r\le s}|e^n_r|^q ds + KE\int_{t_0}^{u}|\bar\sigma^n_s|^q ds \tag{3.45}$$

for any $u \in [t_0, t_1]$. On combining the estimates from (3.44) and (3.45), one has

$$G_5 + G_6 + G_7 \le \frac{1}{8}E\sup_{t_0\le s\le u}|e^n_s|^q + K\int_{t_0}^{u}E\sup_{t_0\le r\le s}|e^n_r|^q ds + KE\int_{t_0}^{u}|\bar\sigma^n_s|^q ds \tag{3.46}$$

for any $u \in [t_0, t_1]$. Now, one uses the splitting of $\bar\sigma^n_s$ given in (3.31) along with assumption A-9 to write

$$G_5 + G_6 + G_7 \le \frac{1}{8}E\sup_{t_0\le s\le u}|e^n_s|^q + K\int_{t_0}^{u}E\sup_{t_0\le r\le s}|e^n_r|^q ds + K\int_{t_0}^{t_1}E|x^n_s - x^n_{\kappa(n,s)}|^q ds$$

$$+ KE\int_{t_0}^{t_1}|\sigma_s(x^n_{\kappa(n,s)}) - \sigma^n_s(x^n_{\kappa(n,s)})|^q ds \tag{3.47}$$

for any $u \in [t_0, t_1]$. Further, for estimating G_8, one uses the splitting of $\bar\gamma^n_s(z)$ given in (3.34) to write

$$G_8 \le E\sup_{t_0\le t\le u}\left|\int_{t_0}^{t}\int_Z |e^n_s|^{q-2}e^n_s\{\gamma_s(x_s,z) - \gamma_s(x^n_s,z)\}\tilde N(ds,dz)\right|$$

$$+ E\sup_{t_0\le t\le u}\left|\int_{t_0}^{t}\int_Z |e^n_s|^{q-2}e^n_s\{\gamma_s(x^n_s,z) - \gamma_s(x^n_{\kappa(n,s)},z)\}\tilde N(ds,dz)\right|$$

$$+ E\sup_{t_0\le t\le u}\left|\int_{t_0}^{t}\int_Z |e^n_s|^{q-2}e^n_s\{\gamma_s(x^n_{\kappa(n,s)},z) - \gamma^n_s(x^n_{\kappa(n,s)},z)\}\tilde N(ds,dz)\right|,$$

which due to Lemma 2.1 gives

$$G_8 \leq E\left(\int_{t_0}^{u}\int_Z |e_s^n|^{2q-2}|\gamma_s(x_s,z)-\gamma_s(x_s^n,z)|^2\nu(dz)ds\right)^{\frac{1}{2}}$$
$$+E\left(\int_{t_0}^{u}\int_Z |e_s^n|^{2q-2}|\gamma_s(x_s^n,z)-\gamma_s(x_{\kappa(n,s)}^n,z)|^2\nu(dz)ds\right)^{\frac{1}{2}}$$
$$+E\left(\int_{t_0}^{u}\int_Z |e_s^n|^{2q-2}|\gamma_s(x_{\kappa(n,s)}^n,z)-\gamma_s^n(x_{\kappa(n,s)}^n,z)|^2\nu(dz)ds\right)^{\frac{1}{2}}$$

for any $u \in [t_0,t_1]$. Then on the application of Young's inequality and Hölder's inequality, one obtains

$$G_8 \leq \frac{1}{8}E\sup_{t_0\leq s\leq u}|e_s^n|^q + E\int_{t_0}^{u}\left(\int_Z|\gamma_s(x_s,z)-\gamma_s(x_s^n,z)|^2\nu(dz)\right)^{\frac{q}{2}}ds$$
$$+E\int_{t_0}^{u}\left(\int_Z|\gamma_s(x_s^n,z)-\gamma_s(x_{\kappa(n,s)}^n,z)|^2\nu(dz)\right)^{\frac{q}{2}}ds$$
$$+E\int_{t_0}^{u}\left(\int_Z|\gamma_s(x_{\kappa(n,s)}^n,z)-\gamma_s^n(x_{\kappa(n,s)}^n,z)|^2\nu(dz)\right)^{\frac{q}{2}}ds.$$

Thus by using assumption A-9, one has

$$G_8 \leq \frac{1}{8}E\sup_{t_0\leq s\leq u}|e_s^n|^q + \int_{t_0}^{u}E\sup_{t_0\leq r\leq s}|e_s^n|^q ds + \int_{t_0}^{t_1}E|x_s^n - x_{\kappa(n,s)}^n|^q ds$$
$$+E\int_{t_0}^{t_1}\left(\int_Z|\gamma_s(x_{\kappa(n,s)}^n,z)-\gamma_s^n(x_{\kappa(n,s)}^n,z)|^2\nu(dz)\right)^{\frac{q}{2}}ds \tag{3.48}$$

for any $u \in [t_0,t_1]$. Finally, one could write G_9 as

$$G_9 := E\sup_{t_0\leq t\leq u}\int_{t_0}^{t}\int_Z\{|e_s^n|^{q-2}|\bar{\gamma}_s^n(z)|^2+|\bar{\gamma}_s^n(z)|^q\}N(ds,dz)$$
$$= E\int_{t_0}^{u}\int_Z|e_s^n|^{q-2}|\bar{\gamma}_s^n(z)|^2\nu(dz)ds + E\int_{t_0}^{u}\int_Z|\bar{\gamma}_s^n(z)|^q\nu(dz)ds =: H_1+H_2 \tag{3.49}$$

for any $u \in [t_0,t_1]$. In order to estimate the first term H_1 on the right-hand side of the inequality (3.49) along with assumption A-9, one recalls the splitting of $\gamma_s^n(z)$ given in (3.34) to get the following estimate:

$$H_1 \leq KE\int_{t_0}^{u}|e_s^n|^q ds + KE\int_{t_0}^{u}|e_s^n|^{q-2}|x_s^n - x_{\kappa(n,s)}^n|^2 ds$$
$$+E\int_{t_0}^{u}\int_Z|e_s^n|^{q-2}|\gamma_s(x_{\kappa(n,s)}^n,z)-\gamma_s^n(x_{\kappa(n,s)}^n,z)|^2\nu(dz)ds$$

for any $u \in [t_0,t_1]$. By the application of Young's inequality, one obtains

$$H_1 \leq K\int_{t_0}^{u}E\sup_{t_0\leq r\leq s}|e_r^n|^q ds + K\int_{t_0}^{t_1}E|x_s^n - x_{\kappa(n,s)}^n|^q ds$$
$$+KE\int_{t_0}^{t_1}\left(\int_Z|\gamma_s(x_{\kappa(n,s)}^n,z)-\gamma_s^n(x_{\kappa(n,s)}^n,z)|^2\nu(dz)\right)^{\frac{q}{2}}ds \tag{3.50}$$

for any $u \in [t_0, t_1]$. For the second term H_2 on the right-hand side of the inequality (3.49) along with assumption A-9, one again uses the splitting of $\bar{\gamma}_s^n(z)$ given in (3.34) to get the following estimate:

$$
\begin{aligned}
H_2 \leq{}& K \int_{t_0}^{u} E \sup_{t_0 \leq r \leq s} |e_r^n|^q ds + K \int_{t_0}^{t_1} E|x_s^n - x_{\kappa(n,s)}^n|^q ds \\
&+ KE \int_{t_0}^{t_1} \int_Z |\gamma_s(x_{\kappa(n,s)}^n, z) - \gamma_s^n(x_{\kappa(n,s)}^n, z)|^q \nu(dz) ds
\end{aligned} \tag{3.51}
$$

for any $u \in [t_0, t_1]$. Hence on combining the estimates obtained in (3.50) and (3.51) in (3.49), one obtains

$$
\begin{aligned}
G_9 \leq{}& K \int_{t_0}^{u} E \sup_{t_0 \leq r \leq s} |e_r^n|^q ds + K \int_{t_0}^{t_1} E|x_s^n - x_{\kappa(n,s)}^n|^q ds \\
&+ KE \int_{t_0}^{t_1} \left(\int_Z |\gamma_s(x_{\kappa(n,s)}^n, z) - \gamma_s^n(x_{\kappa(n,s)}^n, z)|^2 \nu(dz) \right)^{\frac{q}{2}} ds \\
&+ KE \int_{t_0}^{t_1} \int_Z |\gamma_s(x_{\kappa(n,s)}^n, z) - \gamma_s^n(x_{\kappa(n,s)}^n, z)|^q \nu(dz) ds
\end{aligned} \tag{3.52}
$$

for any $u \in [t_0, t_1]$.

Thus one can substitute estimates from (3.41), (3.42), (3.43), (3.47), (3.48), and (3.52) in (3.40) and then apply Gronwall's inequality to obtain

$$
\begin{aligned}
E \sup_{t_0 \leq t \leq t_1} |e_t^n|^q \leq{}& E|e_{t_0}^n|^q + K \int_{t_0}^{t_1} \left(E|x_s^n - x_{\kappa(n,s)}^n|^{q+\delta} \right)^{\frac{q}{q+\delta}} ds \\
&+ K \int_{t_0}^{t_1} E|x_s^n - x_{\kappa(n,s)}^n|^q ds + KE \int_{t_0}^{t_1} |b_s(x_{\kappa(n,s)}^n) - b_s^n(x_{\kappa(n,s)}^n)|^q ds \\
&+ KE \int_{t_0}^{t_1} |\sigma_s(x_{\kappa(n,s)}^n) - \sigma_s^n(x_{\kappa(n,s)}^n)|^q ds \\
&+ KE \int_{t_0}^{t_1} \left(\int_Z |\gamma_s(x_{\kappa(n,s)}^n, z) - \gamma_s^n(x_{\kappa(n,s)}^n, z)|^2 \nu(dz) \right)^{\frac{q}{2}} ds \\
&+ KE \int_{t_0}^{t_1} \int_Z |\gamma_s(x_{\kappa(n,s)}^n, z) - \gamma_s^n(x_{\kappa(n,s)}^n, z)|^q \nu(dz) ds.
\end{aligned}
$$

By the application of assumptions B-7 and B-8 and Lemma 3.3, one obtains

$$
E \sup_{t_0 \leq t \leq t_1} |e_t^n|^q \leq K n^{-\frac{q}{q+\delta}},
$$

which completes the proof. ∎

3.4. A simple example. We now introduce a tamed Euler scheme of SDEs driven by Lévy noise which have coefficients that are not random. For this purpose, we only highlight the modifications needed in the settings of our previous discussion. In SDE (2.1), $b_t(x)$ and $\sigma_t(x)$ are $\mathscr{B}([0,T]) \otimes \mathscr{B}(\mathbb{R}^d)$-measurable functions with values in $\mathbb{R}^d$ and $\mathbb{R}^{d \times m}$, respectively. Also $\gamma_t(x,z)$ is a $\mathscr{B}([0,T]) \otimes \mathscr{B}(\mathbb{R}^d) \otimes \mathscr{Z}$-measurable function with values in $\mathbb{R}^d$. Moreover, one modifies assumptions A-5 and A-6 by assigning $M = M' = 1$. Further, for every $n \in \mathbb{N}$, the scheme (3.1) is given by defining

$$b_t^n(x) = \frac{b_t(x)}{1+n^{-\theta}|b_t(x)|}, \sigma_t^n(x) = \sigma_t(x) \text{ and } \gamma_t^n(x,z) = \gamma_t(x,z) \tag{3.53}$$

with $\theta \in (0, \frac{1}{2}]$ for any $t \in [t_0, t_1]$, $x \in \mathbb{R}^d$, and $z \in Z$. Then, it is easy to observe that assumptions B-2 to B-4 hold since $M_n = M_n' = 1$ and $\theta \in (0, \frac{1}{2}]$. Hence Lemmas 2.3, 3.1, 3.2, 3.3 follow immediately. Finally, the $\mathscr{F}_{t_0}$-measurable random variable C_R in assumptions A-7 and A-8 is a constant for every R. In this new settings, one obtains the following corollaries for SDE (2.1) and scheme (3.1) with coefficients given by (3.53).

Corollary 3.6. *Let assumptions* A-3 *to* A-8 *be satisfied by the coefficients of SDE given immediately above. Also assume that* B-1 *and* B-6 *hold. Then, the numerical scheme* (3.1) *with coefficients given by* (3.53) *converges to the solution of SDE* (2.1) *in* $\mathcal{L}^q$ *sense, i.e.,*

$$\lim_{n\to\infty} E \sup_{t_0 \le t \le t_1} |x_t - x_t^n|^q = 0$$

for all $q < p$.

Proof. Assumptions A-7 and A-8 are satisfied on taking $f(R) = C_R$ in (3.15). For assumption B-5, one observes due to (3.53) and assumption A-8,

$$\begin{aligned} E\int_{t_0}^{t_1} I_{B(R)} \sup_{|x|\le R} |b_t^n(x) - b_t(x)|^2 dt &\le n^{-2\theta} E\int_{t_0}^{t_1} I_{B(R)} \sup_{|x|\le R} |b_t(x)|^4 dt \\ &\le K f(R)^4 n^{-2\theta} \to 0 \end{aligned}$$

as $n \to \infty$ for every R. Also for diffusion and jump coefficients, assumption B-5 holds trivially. Thus, Theorem 3.4 completes the proof. ∎

For the rate of convergence of scheme (3.1), one takes $\theta = \frac{1}{2}$ in (3.53).

Corollary 3.7. *Let assumptions* A-3 *to* A-6, A-8, *and* A-9 *be satisfied by the coefficients of SDE given immediately above. Also suppose that assumptions* B-1 *and* B-8 *hold. Then, the numerical scheme* (3.1) *with coefficients given by* (3.53) *achieves the classical rate (of Euler scheme) in* $\mathcal{L}^q$ *sense, i.e.,*

$$E \sup_{t_0 \le t \le t_1} |x_t - x_t^n|^q \le K n^{-\frac{q}{q+\delta}}, \tag{3.54}$$

where constant $K > 0$ *does not depend on* n.

Proof. By using (3.53) and Remark 3.3, one obtains

$$\begin{aligned} E\int_{t_0}^{t_1} |b_t^n(x_{\kappa(n,t)}^n) - b_t(x_{\kappa(n,t)}^n)|^q dt &\le n^{-2\theta} E\int_{t_0}^{t_1} |b_t(x_{\kappa(n,t)}^n)|^{2q} dt \\ &\le K n^{-1} \left(1 + E \sup_{t_0 \le t \le t_1} |x_t^n|^{q(\chi+2)}\right) \end{aligned}$$

since $\theta = \frac{1}{2}$. Hence assumption B-7 for drift coefficients follows due to Lemma 3.2. For diffusion and jump coefficients, assumption B-7 holds trivially. The proof is completed by Theorem 3.5. ∎

4. Application to delay equations. Let us assume that $\beta_t(y_1,\dots,y_k,x)$ and $\alpha_t(y_1,\dots,y_k,x)$ are $\mathscr{B}([0,T])\otimes\mathscr{B}(\mathbb{R}^{d\times k})\otimes\mathscr{B}(\mathbb{R}^d)$-measurable functions and take values in $\mathbb{R}^d$ and $\mathbb{R}^{d\times m}$, respectively. Also let $\lambda_t(y_1,\dots,y_k,x,z)$ be a $\mathscr{B}([0,T])\otimes\mathscr{B}(\mathbb{R}^{d\times k})\otimes\mathscr{B}(\mathbb{R}^d)\otimes\mathscr{Z}$-measurable function and take values in $\mathbb{R}^d$. For fixed $H>0$, we consider a d-dimensional SDDE on $(\Omega,\{\mathscr{F}_t\}_{t\geq 0},\mathscr{F},P)$ defined by

$$
\begin{aligned}
&dx_t=\beta_t(y_t,x_t)dt+\alpha_t(y_t,x_t)dw_t+\int_Z\lambda_t(y_t,x_t,z)\tilde N(dt,dz),\ \ t\in[0,T],\\
&x_t=\xi_t,\ \ t\in[-H,0], \qquad (4.1)
\end{aligned}
$$

where $\xi:[-H,0]\times\Omega\to\mathbb{R}^d$ and $y_t:=(x_{\delta_1(t)},\dots,x_{\delta_k(t)})$. The delay parameters $\delta_1(t),\dots,\delta_k(t)$ are increasing functions of t and satisfy $-H\leq\delta_j(t)\leq[t/h]h$ for some $h>0$ and $j=1,\dots,k$.

Remark 4.1. In the following, we assume, without loss of generality, that T is a multiple of h. If not, then SDDE (4.1) can be defined for $T'>T$ so that $T'=N'h$, where N' is a positive integer. The results proved in this article are then recovered for the original SDDE (4.1) by choosing parameters as $\beta I_{t\leq T}$, $\alpha I_{t\leq T}$ and $\lambda I_{t\leq T}$.

Remark 4.2. We remark that two popular cases of delay, viz., $\delta_i(t)=t-h$ and $\delta_i(t)=[t/h]h$, can be addressed by our findings which have been widely used in the literature, for example, [4, 5, 17, 19] and references therein.

4.1. Existence and uniqueness. To prove the existence and uniqueness of the solution of SDDE (4.1), we make the following assumptions.

C-1. *For every $R>0$, there exists an $M(R)\in\mathbb{L}^1$ such that*

$$
x\beta_t(y,x)+|\alpha_t(y,x)|^2+\int_Z|\lambda_t(y,x,z)|^2\nu(dz)\leq M_t(R)(1+|x|^2)
$$

for any $t\in[0,T]$ whenever $|y|\leq R$ and $x\in\mathbb{R}^d$.

C-2. *For every $R>0$, there exists an $M(R)\in\mathbb{L}^1$ such that*

$$
\begin{aligned}
(x-\bar x)(\beta_t(y,x)-\beta_t(y,\bar x))+|\alpha_t(y,x)-\alpha_t(y,\bar x)|^2+\int_Z|\lambda_t(y,x,z)-\lambda_t(y,\bar x,z)|^2\nu(dz)\\
\leq M_t(R)|x-\bar x|^2
\end{aligned}
$$

for any $t\in[0,T]$ whenever $|x|,|\bar x|,|y|\leq R$.

C-3. *The function $\beta_t(y,x)$ is continuous in x for any t and y.*

THEOREM 4.1. *Let assumptions* C-1 *to* C-3 *be satisfied. Then there exists a unique solution to SDDE* (4.1).

Proof. We adopt the approach of [7] and consider SDDE (4.1) as a special case of SDE (2.1) by assigning the following values to the coefficients:

$$
b_t(x)=\beta_t(y_t,x),\sigma_t(x)=\alpha_t(y_t,x),\gamma_t(x,z)=\lambda_t(y_t,x,z) \qquad (4.2)
$$

almost surely for any $t\in[0,T]$. Then the proof is a straightforward generalization of Theorem 2.1 of [7] and follows due to Theorem 2.2. ∎

4.2. Tamed Euler scheme. For every $n \in \mathbb{N}$, define the following tamed Euler scheme:

$$\begin{aligned} dx_t^n &= \beta_t^n(y_t^n, x_{\kappa(n,t)}^n)dt + \alpha_t(y_t^n, x_{\kappa(n,t)}^n)dw_t \\ &\quad + \int_Z \lambda_t(y_t^n, x_{\kappa(n,t)}^n, z)\tilde{N}(dt, dz), \quad t \in [0,T], \\ x_t^n &= \xi_t, \quad t \in [-H, 0], \end{aligned} \tag{4.3}$$

where $y_t^n := (x_{\delta_1(t)}^n, \ldots, x_{\delta_k(t)}^n)$ and κ is defined by (3.2) with $t_0 = 0$. Furthermore, for every $n \in \mathbb{N}$, the drift coefficient is given by

$$\beta_t^n(y,x) := \frac{\beta_t(y,x)}{1 + n^{-\theta}|\beta_t(y,x)|},$$

which satisfies

$$|\beta_t^n(y,x)| \le \min(n^\theta, |\beta_t(y,x)|) \tag{4.4}$$

for any $t \in [0,T]$, $x \in \mathbb{R}^d$, and $y \in \mathbb{R}^{d\times k}$.

C-4. *For a fixed $p \ge 2$, $E\sup_{-H\le t\le 0}|\xi_t|^p < \infty$.*

C-5. *There exist constants $G > 0$ and $\chi \ge 2$ such that*

$$x\beta_t(y,x) \vee |\alpha_t(y,x)|^2 \vee \int_Z |\lambda_t(y,x,z)|^2\nu(dz) \le G(1 + |y|^\chi + |x|^2)$$

for any $t \in [0,T]$, $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^{d\times k}$.

C-6. *There exist constants $G > 0$ and $\chi \ge 2$ such that*

$$\int_Z |\lambda_t(y,x,z)|^p\nu(dz) \le G(1 + |y|^{\chi\frac{p}{2}} + |x|^p)$$

for any $t \in [0,T]$, $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^{d\times k}$.

C-7. *For every $R > 0$, there exists a constant $K_R > 0$ such that*

$$\begin{aligned} (x-\bar{x})(\beta_t(y,x) - \beta_t(y,\bar{x})) \vee |\alpha_t(y,x) - \alpha_t(y,\bar{x})|^2 \vee \int_Z |\lambda_t(y,x,z) - \lambda_t(y,\bar{x},z)|^2\nu(dz) \\ \le K_R|x-\bar{x}|^2 \end{aligned}$$

for any $t \in [0,T]$ whenever $|x|, |y|, |\bar{x}| < R$.

C-8. *For every $R > 0$, there exists a constant $K_R > 0$ such that*

$$\sup_{|x|\le R}\sup_{|y|\le R}|\beta_t(y,x)|^2 \le K_R$$

for any $t \in [0,T]$.

C-9. *For every $R > 0$ and $t \in [0,T]$,*

$$\sup_{|x|\le R}\left\{|\beta_t(y,x) - \beta_t(y',x)|^2 + |\alpha_t(y,x) - \alpha_t(y',x)|^2 + \int_Z |\lambda_t(y,x,z) - \lambda_t(y',x,z)|^2\nu(dz)\right\} \to 0$$

when $y' \to y$.

Let us also define

$$p_i = \left(\frac{2}{\chi}\right)^i p \tag{4.5}$$

for $i = 1, \ldots, N'$, where χ and p satisfy $p/2 \geq (\chi/2)^{N'}$. Also

$$p^* = \min_i p_i = \left(\frac{2}{\chi}\right)^{N'} p. \tag{4.6}$$

The following corollary is a consequence of Theorem 3.4.

COROLLARY 4.2. *Let assumptions* C-3 *to* C-9 *hold, and then*

$$\lim_{n\to\infty} E \sup_{0\leq t\leq T} |x_t - x_t^n|^q = 0$$

for any $q < p^*$.

Proof. First, as before, one observes that SDDE (4.1) can be regarded as a special case of SDE (2.1) with coefficients given by (4.2). Moreover, tamed Euler scheme (4.3) is a special of (3.1) with coefficients given by

$$b_t^n(x) = \frac{\beta_t(y_t^n, x)}{1 + n^{-\theta}|\beta_t(y_t^n, x)|}, \sigma_t^n(x) = \alpha_t(y_t^n, x), \gamma_t^n(x, z) = \lambda_t(y_t^n, x, z) \tag{4.7}$$

almost surely for any $t \in [0, T]$ and $x \in \mathbb{R}^d$. We shall use inductive arguments to show

$$\lim_{n\to\infty} E \sup_{(i-1)h\leq t\leq ih} |x_t - x_t^n|^q = 0 \tag{4.8}$$

for any $q < p_i$ and for every $i \in \{1, \ldots, N'\}$.

Case: $\mathbf{t} \in [\mathbf{0}, \mathbf{h}]$. For $t \in [0, h]$, one could consider SDDE (4.1) and their tamed Euler scheme (4.3) as SDE (2.1) and scheme (3.1), respectively, with $t_0 = 0$, $t_1 = h$, $x_0 = x_0^n = \xi_0$ and with coefficients given in (4.2) and (4.7). Further, one observes that assumptions A-3 to A-8 and B-1 to B-6 hold due to assumptions C-3 to C-9. In particular, assumption A-3 holds due to assumption C-3 while assumptions A-4 and B-1 due to assumption C-4. Further assumptions A-5, A-6, B-2, and B-3 hold due to assumptions C-5 and C-6 with $L = G$, $M = M_n = 1 + \Psi^\chi \in \mathcal{L}^{\frac{p_1}{2}}$, and $M' = M_n' = 1 + \Psi^{\chi\frac{p_1}{2}} \in \mathcal{L}^1$, where $\Psi := \sup_{t\in[0,h]} |(\xi_{\delta_1(t)}, \ldots, \xi_{\delta_k(t)})| \in \mathcal{L}^p$. Also assumption A-7 holds due to assumption C-7 with

$$C_R := K_R I_{\Omega_R} + \sum_{j=R}^{\infty} K_{j+1} I_{\Omega_{j+1}\setminus\Omega_j},$$

where $\Omega_j := \{\omega \in \Omega : \Psi \leq j\}$. Further one takes $f(R) := K_R$ and then

$$P(C_R > f(R)) \leq P(\Psi > R) \leq \frac{E\Psi}{R} \to 0$$

as $R \to \infty$. This also implies that assumption A-8 holds due to assumption C-8. To verify assumption B-5, one observes that

$$b_t^n(x) = \frac{\beta_t(\xi_{\delta_1(t)}, \ldots, \xi_{\delta_k(t)}, x)}{1 + n^{-\theta}|\beta_t(\xi_{\delta_1(t)}, \ldots, \xi_{\delta_k(t)}, x)|} \to \beta_t(\xi_{\delta_1(t)}, \ldots, \xi_{\delta_k(t)}, x) = b_t(x)$$

as $n \to \infty$ and sequence

$$\left\{ I_{B(R)} \sup_{|x| \le R} |b_t^n(x) - b_t(x)|^2 \right\}_{\{n \in \mathbb{N}\}}$$

is uniformly integrable, which implies

$$\lim_{n \to \infty} E \int_{t_0}^{t_1} I_{B(R)} \sup_{|x| \le R} |b_t^n(x) - b_t(x)|^2 dt = 0$$

and similarly for diffusion and jump coefficients. Finally, assumption B-6 holds trivially.

Therefore (4.8) holds due to Theorem 3.4 and Lemmas 2.3 and 3.2 when $i = 1$. We note that the convergence here is achieved for all $q < p_1$ and as we proceed to the next interval $[h, 2h]$, the convergence is achieved in the lower space, i.e., $q < p_2$ due to assumptions C-5 and C-6. Therefore for the inductive arguments, we assume that the convergence in the interval $[(r-1)h, rh]$ is achieved for all $q < p_r$, i.e., we assume that Theorem 3.4 and Lemmas 2.3 and 3.2 hold for any $q < p_r$ when $i = r$.

Case: $\mathbf{t} \in [\mathbf{rh}, (\mathbf{r}+\mathbf{1})\mathbf{h}]$. When $t \in [rh, (r+1)h]$, SDDE (4.1) and scheme (4.3) become SDE (2.1) and scheme (3.1), respectively, with $t_0 = rh$, $t_1 = (r+1)h$, $x_{t_0} = x_{rh}$, $x_{t_0}^n = x_{rh}^n$, and coefficients given by (4.2) and (4.7).

Verify A-3. Assumption A-3 holds due to assumption C-3 trivially.

Verify A-4 *and* B-1. Assumptions A-4 and B-1 hold due to Lemmas 2.3 and 3.2 and inductive assumptions.

Verify A-5, A-6, B-2, *and* B-3. Assumptions A-5 and B-2 hold due to assumption C-5 with $M := 1 + \sup_{rh \le t \le (r+1)h} |y_t|^\chi$ and $M_n := 1 + \sup_{rh \le t \le (r+1)h} |y_t^n|^\chi$, which are bounded in $\mathcal{L}^{\frac{p_{r+1}}{2}}$ due to Lemmas 2.3 and 3.2 and inductive assumptions. Furthermore assumptions A-6 and B-3 hold with $M' := 1 + \sup_{rh \le t \le (r+1)h} |y_t|^{\chi \frac{p_{r+1}}{2}}$ and $M_n' := 1 + \sup_{rh \le t \le (r+1)h} |y_t^n|^{\chi \frac{p_{r+1}}{2}}$, which are bounded in $\mathcal{L}^1$ due to Lemmas 2.3 and 3.2 and inductive assumptions.

Verify A-7. For every $R > 0$, $|x|, |\bar{x}| \le R$, and $t \in [rh, (r+1)h]$, assumption A-7 holds due to assumption C-7 with $\mathcal{F}_{rh}$-measurable random variable C_R given by

$$C_R := K_R I_{\Omega_R} + \sum_{j=R}^{\infty} K_{j+1} I_{\Omega_{j+1} \setminus \Omega_j}, \tag{4.9}$$

where $\Omega_j := \{\omega \in \Omega : \sup_{t \in [rh,(r+1)h]} |y_t| \le j\}$. Further one takes $f(R) := K_R$ and then

$$P(C_R > f(R)) \le P\left(\sup_{rh \le t < (r+1)h} |y_t| > R \right) \to 0 \,\text{as}\, R \to \infty. \tag{4.10}$$

Verify A-8. For every $R > 0$ and any $t \in [rh, (r+1)h]$, we take C_R as defined in (4.9), $f(R) = K_R$. Then one uses (4.10) to establish A-8.

Verify B-5. The inductive assumption implies $|y_t^n - y_t| \to 0$ in probability and thus due to assumption C-9, $\sup_{|x| \le R} |\beta_t^n(y_t^n, x) - \beta_t(y_t, x)| \to 0$ in probability. Furthermore the sequence

$$I_{B(R)} \left\{ \sup_{|x| \le R} |\beta_t^n(y_t^n, x) - \beta_t(y_t, x)|^2 \right\}_{\{n \in \mathbb{N}\}}$$

is uniformly integrable due to assumption C-8 and an inductive assumption, which implies

$$\lim_{n\to\infty} E\int_{rh}^{(r+1)h} \sup_{|x|\le R} |b_t^n(x)-b_t(x)|^2 = 0.$$

For the diffusion coefficient, due to the inductive assumption,

$$\left\{\sup_{rh\le t\le (r+1)h} |y_t^n - y_t|^\chi\right\}_{\{n\in\mathbb{N}\}} \text{ and hence } \left\{\sup_{rh\le t\le (r+1)h} |y_t^n|^\chi\right\}_{\{n\in\mathbb{N}\}}$$

are uniformly integrable, which on using assumptions C-5 to C-6 imply that

$$\left\{\sup_{|x|\le R} |\alpha_t(y_t^n,x)-\alpha_t(y_t,x)|^2\right\}_{\{n\in\mathbb{N}\}}$$

is uniformly integrable. Moreover due to assumption C-9,

$$\sup_{|x|\le R} |\alpha_t(y_t^n,x)-\alpha_t(y_t,x)|^2 \to 0$$

in probability as $n\to\infty$ and therefore assumption B-5 holds for the diffusion coefficients. One adopts similar arguments for jump coefficients.

Verify B-6. This follows due to the inductive assumptions.
This completes the proof. ◻

We now proceed to obtain the rate of convergence of the scheme (4.3). For this purpose, we replace assumptions C-7 and C-9 by the following assumptions.

C-10. *There exist constants $C>0$, $q\ge 2$, and $\chi>0$ such that*

$$\begin{aligned}(x-\bar{x})(\beta_t(y,x)-\beta_t(y,\bar{x})) \vee |\alpha_t(y,x)-\alpha_t(y,\bar{x})|^2 \vee\\ \int_Z |\lambda_t(y,x,z)-\lambda_t(y,\bar{x},z)|^2\nu(dz) \le C|x-\bar{x}|^2\\ \int_Z |\lambda_t(y,x,z)-\lambda_t(y,\bar{x},z)|^q\nu(dz) \le C|x-\bar{x}|^q\\ |\beta_t(y,x)-\beta_t(y,\bar{x})|^2 \le C(1+|x|^\chi+|\bar{x}|^\chi)|x-\bar{x}|^2\end{aligned}$$

for any $t\in[0,T]$, $x,\bar{x}\in\mathbb{R}^d$, $y\in\mathbb{R}^{d\times k}$, and a $\delta\in(0,1)$ such that $\max\{(\chi+2)q, \frac{q\chi}{2}\frac{q+\delta}{\delta}\}\le p^*$.

C-11. *Assume that*

$$\begin{aligned}|\beta_t(y,x)-\beta_t(\bar{y},x)|^2 \vee |\alpha_t(y,x)-\alpha_t(\bar{y},x)|^2 \vee \left(\int_Z |\lambda_t(y,x)-\lambda_t(\bar{y},x)|^\zeta\nu(dz)\right)^{\frac{q}{\zeta}}\\ \le C(1+|y|^\chi+|\bar{y}|^\chi)|y-\bar{y}|^2,\end{aligned}$$

where $\zeta=2,q$ for any $t\in[0,T]$, $x\in\mathbb{R}^d$, and $y,\bar{y}\in\mathbb{R}^{d\times k}$.

Remark 4.3. Due to assumptions C-8, C-10, and C-11, there exists a constant $C>0$ such that

$$|\beta_t(y,x)|^2 \le C(1+|y|^{\chi+2}+|x|^{\chi+2})$$

for any $t\in[0,T]$, $x\in\mathbb{R}^d$, and $y\in\mathbb{R}^{d\times k}$.

In the following corollary, we obtain a convergence rate for the tamed Euler scheme (4.3), which is equal to the classical convergence rate of Euler scheme. For this purpose, one can take $\theta = \frac{1}{2}$.

Corollary 4.3. *Let assumptions* C-3 *through* C-6, C-8, C-10, *and* C-11 *be satisfied. Then*

$$E \sup_{0 \le t \le T} |x_t - x_t^n|^q \le K n^{-\frac{q}{q+N'\delta}} \tag{4.11}$$

for any $q < p^*$ where constant $K > 0$ does not depend on n.

Proof. The corollary can be proved by adopting similar arguments as used in the proof of Corollary 4.2. For this purpose, one can use Theorem 3.5 inductively to show that for every $i = 1, \ldots, N'$,

$$E \sup_{(i-1)h \le t \le ih} |x_t - x_t^n|^q \le K n^{-\frac{q}{q+i\delta}}$$

for any $q < p_i$ where constant $K > 0$ does not depend on n. Now, notice that assumptions A-3 through A-6, A-8, and B-1 through B-3 have already been verified in the proof of Corollary 4.2. Hence, one only needs to verify assumptions A-9, B-7, and B-8.

Case $\mathbf{t} \in [\mathbf{0}, \mathbf{h}]$. As before, one considers SDDE (4.1) as a special case of SDE (2.1) with $t_0 = 0$, $t_1 = h$, $x_{t_0} = \xi_0$, and coefficients given by (4.2). Also, scheme (4.3) can be considered as a special case of scheme (3.1) with $t_0 = 0$, $t_1 = h$, $x_{t_0} = \xi_0$, and coefficients given by (4.7).

Verify A-9. Assumption A-9 follows from assumption C-10 trivially.

Verify B-7. Notice that $y_t = y_t^n =: \Phi_t$ for $t \in [0, h]$, which implies

$$E \int_0^h |b_t^n(x_{\kappa(n,t)}^n) - b_t(x_{\kappa(n,t)}^n)|^q dt \le n^{-q\theta} E \int_0^h \left|\beta_t(\Phi_t, x_{\kappa(n,t)}^n)\right|^{2q} dt,$$

which on using Remark 4.3, assumption C-4, and Lemma 3.2 gives

$$\begin{aligned} E \int_0^h |b_t^n(x_{\kappa(n,t)}^n) - b_t(x_{\kappa(n,t)}^n)|^q dt &\le n^{-q\theta} K \left(1 + E\Psi^{(\chi+2)q} + E \sup_{0 \le t \le h} |x_{\kappa(n,t)}^n|^{(\chi+2)q}\right) \\ &\le K n^{-\frac{q}{2}} \end{aligned}$$

for any $q < p_1$ because $\theta = \frac{1}{2}$.

Verify B-8. This holds trivially.

Thus, by Theorem 3.5, one obtains that (4.11) holds for $i = 1$. For inductive arguments, one assumes that (4.11) holds for $i = r$ and then verifies it for $i = 1 + r$.

Case $\mathbf{t} \in [\mathbf{rh}, (\mathbf{r} + \mathbf{1})\mathbf{h}]$. Again, consider SDDE (4.1) as a special case of SDE (2.1) with $t_0 = rh$, $t_1 = (r+1)h$, $x_{t_0} = x_{rh}$ and coefficients given by (4.2). Similarly, consider scheme (4.3) as a special case of scheme (3.1) with $t_0 = rh$, $t_1 = (r+1)h$, $x_{t_0} = x_{rh}$, and coefficients given by (4.7).

Verify A-9. Assumption A-9 follows from assumption C-10 trivially.

Verify B-7. One observes that

$$
\begin{aligned}
E\int_0^h &|b_t^n(x^n_{\kappa(n,t)}) - b_t(x^n_{\kappa(n,t)})|^q dt \\
&\le KE\int_0^h \left|\frac{\beta_t(y_t^n, x^n_{\kappa(n,t)})}{1+n^{-\theta}|\beta_t(y_t^n, x^n_{\kappa(n,t)})|} - \beta_t(y_t^n, x^n_{\kappa(n,t)})\right|^q dt \\
&\quad + KE\int_0^h \left|\beta_t(y_t^n, x^n_{\kappa(n,t)}) - \beta_t(y_t, x^n_{\kappa(n,t)})\right|^q dt \\
&\le Kn^{-q\theta}E\int_0^h |\beta_t(y_t^n, x^n_{\kappa(n,t)})|^{2q} dt + KE\int_0^h \left|\beta_t(y_t^n, x^n_{\kappa(n,t)}) - \beta_t(y_t, x^n_{\kappa(n,t)})\right|^q dt,
\end{aligned}
$$

which on the application of Remark 4.3 and assumption C-11 gives

$$
\begin{aligned}
E\int_0^h |b_t^n(x^n_{\kappa(n,t)}) - b_t(x^n_{\kappa(n,t)})|^q dt \le Kn^{-q\theta}E\int_0^h (1+|y_t^n|^{(\chi+2)q} + |x^n_{\kappa(n,t)}|^{(\chi+2)q}) dt \\
+ KE\int_0^h (1+|y_t|^{\frac{q\chi}{2}} + |y_t^n|^{\frac{q\chi}{2}})|y_t - y_t^n|^q dt
\end{aligned}
$$

and then on the application of Hölder's inequality and Lemmas 2.3 and 3.2 along with inductive assumptions gives

$$
E\int_0^h |b_t^n(x^n_{\kappa(n,t)}) - b_t(x^n_{\kappa(n,t)})|^q dt \le Kn^{-q\theta} + KE\int_0^h (E|y_t - y_t^n|^{q+\delta})^{\frac{q}{q+\delta}}.
$$

Finally, on using the inductive assumption and $\theta = \frac{1}{2}$, one obtains

$$
E\int_0^h |b_t^n(x^n_{\kappa(n,t)}) - b_t(x^n_{\kappa(n,t)})|^q dt \le Kn^{-\frac{q}{2}} + Kn^{-\frac{q}{q+(r+1)\delta}}
$$

and hence (4.11) holds for $i = r+1$.

Verify B-8. This holds due to inductive assumptions.

Thus, by Theorem 3.5, one obtains that (4.11) holds for $i = r+1$. This completes the proof. ∎

5. Numerical illustrations. We demonstrate our results numerically with the help of following examples.

Example 1. Consider the following SDE:

$$
dx_t = -\, x_t^5 dt + x_t dw_t + \int_{\mathbb{R}} x_t z\tilde{N}(dt, dz) \tag{5.1}
$$

for any $t \in [0,1]$ with initial value $x_0 = 1$. The jump size follows standard normal distribution, and the jump intensity is 3. The tamed Euler scheme with step-size 2^{-21} is taken as true solution. Table 1 and Figure 1(a) are based on 1000 simulations.

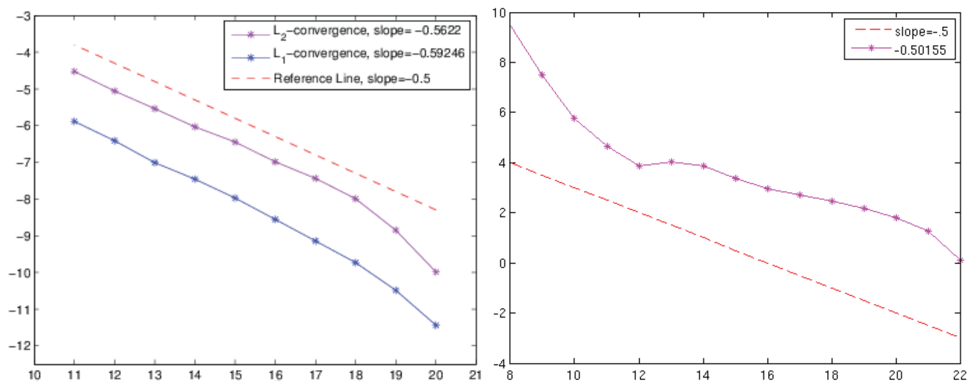
Example 2. Consider the following SDDE:

$$
dx_t = (x_t - x_t^3 + y_t^2)dt + (x_t + y_t^3)dw_t + \int_{\mathbb{R}} (x_t + y_t)z\tilde{N}(dt, dz), \tag{5.2}
$$

where $y_t = x_{t-1}$ for $t \in [0,2]$ with initial data $\xi_t = t+1$ for $t \in [-1,0]$. The jump size follows standard normal distribution, and the jump intensity is 3. The tamed scheme with step-size 2^{-23} is taken as the true solution. Figure 1(b) is based on 300 sample paths.

Table 1
SDE: Errors in the tamed Euler scheme.

Step-size	$\sqrt{E\|x_t - x_t^n\|^2}$	$E\|x_t - x_t^n\|$
2^{-20}	0.000983465083412957	0.000359729516674718
2^{-19}	0.00216716723504906	0.000696592563650715
2^{-18}	0.00392575778408420	0.00117629823362591
2^{-17}	0.00577090918102760	0.00176826651345228
2^{-16}	0.00788070333470230	0.00265746428431957
2^{-15}	0.0114588451477506	0.00398287796962204
2^{-14}	0.0152592153162732	0.00568182096844841
2^{-13}	0.0214987425830999	0.00775473960140893
2^{-12}	0.0300412202466655	0.0117456051149168
2^{-11}	0.0434809466351964	0.0168998838844189



(a) SDE: $\mathcal{L}^1$ and $\mathcal{L}^2$ convergence with rate (b) SDDE: $\mathcal{L}^2$ convergence with rate

Fig. 1. *Tamed Euler schemes of SDE* (5.1) *and SDDE* (5.2).

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