GLOBAL FINITE ENERGY WEAK SOLUTIONS TO THE COMPRESSIBLE NEMATIC LIQUID CRYSTAL FLOW IN DIMENSION THREE

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ABSTRACT. In this paper, we consider the initial and boundary value problem of a simplified compressible nematic liquid crystal flow in $\Omega \subset \mathbb{R}^3$. We establish the existence of global weak solutions, provided the initial orientational director field d_0 lies in the hemisphere \mathbb{S}^2_{\pm} .

1. INTRODUCTION

The continuum theory of liquid crystals was developed by Ericksen [3] and Leslie [7] during the period of 1958 through 1968, see also the book by De Gennes [2]. Since then there have been remarkable research developments in liquid crystals from both theoretical and applied aspects. When the fluid containing nematic liquid crystal materials is at rest, we have the well-known Oseen-Frank theory for static nematic liquid crystals, see Hardt-Lin-Kinderlehrer [8] on the analysis of energy minimal configurations of nematic liquid crystals. In general, the motion of fluid always takes place. The so-called Ericksen-Leslie system is a macroscopic continuum description of the time evolution of the material under influence of both the flow velocity field u and the macroscopic description of the microscopic orientation configurations d of rod-like liquid crystals.

When the fluid is an incompressible, viscous fluid, Lin [10] first derived a simplified Ericksen-Leslie system (i.e. $\rho = 1$ and divu = 0 in the equation (1.1) below) modeling liquid crystal flows in 1989. Subsequently, Lin and Liu [11, 12] have made some important analytic studies, such as the global existence of weak and strong solutions and the partial regularity of suitable weak solutions, of the simplified Ericksen-Leslie system, under the assumption that the liquid crystal director field is of varying length by Leslie's terminology or variable degree of orientation by Ericksen's terminology. When dealing with the system (1.1) with $\rho = 1$ and divu = 0, in dimension two Lin-Lin-Wang [13] and Lin-Wang [14] have established the existence of a unique global weak solution, that has at most finitely many possible singular time, for the initial-boundary value problem in bounded domains (see also Hong [9], Xu-Zhang [36], and Lei-Li-Zhang [15] for some related works); and in dimension three Lin-Wang [18] have obtained the existence of global weak solutions very recently when the initial director field d_0 maps to the hemisphere \mathbb{S}^2_+ .

When the fluid is compressible, the simplified Ericksen-Leslie system (1.1) becomes more complicate, which is a strongly coupling system between the compressible Navier-Stokes equation and the transported harmonic map heat flow to \mathbb{S}^2 . It seems worthwhile to be explored for the mathematical analysis of (1.1). We would like to mention that there have been both modeling study, see Morro [24], and numerical study, see Zakharov-Vakulenko [25], on the hydrodynamics of compressible nematic liquid crystals under the influence of temperature gradient or electromagnetic forces.

Now let's introduce the simplified Ericksen-Leslie system for compressible nematic liquid crystal flow. Let $\Omega \subset \mathbb{R}^3$ be a bounded, smooth domain, $\mathbb{S}^2 \subset \mathbb{R}^3$ be the unit sphere, and $0 < T \leq +\infty$. We will consider a simplified version of the three dimensional hydrodynamic flow of the compressible

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nematic liquid crystal flow in $\Omega \times (0,T)$, i.e., $(\rho, u, d) : \Omega \times (0,T) \to \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{S}^2$ solves

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + a \nabla \rho^{\gamma} = \mathcal{L} u - \nabla \cdot \left(\nabla d \odot \nabla d - \frac{1}{2} |\nabla d|^2 \mathbb{I}_3 \right), \\ \partial_t d + u \cdot \nabla d = \Delta d + |\nabla d|^2 d, \end{cases}$$
(1.1)

under the initial and boundary condition:

$$\begin{cases} \rho(x,0) = \rho_0(x), \ \rho u(x,0) = m_0(x), \ d(x,0) = d_0(x), \ x \in \Omega, \\ u(x,t) = 0, \ d(x,t) = d_0(x), \ x \in \partial\Omega, \ t > 0, \end{cases}$$
(1.2)

where $\rho: \Omega \times [0,T) \to \mathbb{R}_+$ denotes density function of the fluid, $u: \Omega \times [0,T) \to \mathbb{R}^3$ denotes velocity field of the fluid, $d: \Omega \times [0,T) \to \mathbb{S}^2$ denotes direction field of the averaged macroscopic molecular orientations, $\nabla \cdot$ denotes the divergence operator in \mathbb{R}^3 , \mathbb{I}_3 is the 3×3 identity matrix, $P(\rho) = a\rho^{\gamma}$, with a > 0 and $\gamma > 1$, denotes the pressure function associated with an isentropic fluid, \mathcal{L} is the Lamé operator defined by

$$\mathcal{L}u = \mu \triangle u + (\mu + \lambda) \nabla (\nabla \cdot u),$$

where μ and λ represent the shear viscosity and the bulk viscosity coefficients of the fluid respectively, which satisfy the natural physical condition:

$$\mu > 0, \quad \widetilde{\mu} := \mu + \lambda \ge 0, \tag{1.3}$$

 $\nabla d \odot \nabla d$ denotes the 3 × 3 matrix valued function whose (i, j)-entry is $\langle \partial_{x_i} d, \partial_{x_j} d \rangle$ for $1 \le i, j \le 3$, and $u \otimes u = (u^i u^j)_{1 \le i, j \le 3}$.

Throughout this paper, we denote $\mathbb{S}^2_+ = \{y = (y^1, y^2, y^3) \in \mathbb{S}^2 : y^3 \ge 0\}$ as the upper hemisphere, χ_E denote the characteristic function of a set $E \subset \mathbb{R}^3$,

$$H^1(\Omega, \mathbb{S}^2) = \Big\{ d \in H^1(\Omega, \mathbb{R}^3) : \ d(x) \in \mathbb{S}^2 \text{ a.e. } x \in \Omega \Big\},\$$

and $A: B = \sum_{i,j=1}^{3} A_{ij}B_{ij}$ denotes the scalar product of two 3×3 matrices. For $0 < T \le +\infty$, denote

$$Q_T = \Omega \times (0,T), \ \partial_p Q_T = (\Omega \times \{0\}) \cup (\partial \Omega \times (0,T)), \ \mathcal{D}'(Q_T) = (C_0^{\infty}(Q_T))'.$$

We say $(\rho, u, d) : \Omega \times [0, T) \to \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{S}^2$ is a *finite energy* weak solution of the initial-boundary value problem (1.1)-(1.2) if the following properties hold:

- (i) $\rho \ge 0, \ \rho \in L^{\infty}((0,T), L^{\gamma}(\Omega)), \ u \in L^{2}((0,T), H^{1}(\Omega, \mathbb{R}^{3})), \text{ and } d \in L^{2}((0,T), H^{1}(\Omega, \mathbb{S}^{2})).$
- (ii) the system (1.1) holds in $\mathcal{D}'(Q_T)$, (1.1)₁ also holds in $\mathcal{D}'(\mathbb{R}^3 \times (0,T))$ provided (ρ, u) is prolonged by zero in $\mathbb{R}^3 \setminus \Omega$, $(\rho, \rho u, d)(x, 0) = (\rho_0(x), m_0(x), d_0(x))$ for a.e. $x \in \Omega$, and $(u, d)(x, t) = (0, d_0(x))$ on $\partial\Omega \times (0, T)$ in the sense of traces.
- (iii) (ρ, u) satisfies $(1.1)_1$ in the sense of the renormalized solutions introduced by DiPerna-Lions [26], that is, (ρ, u) satisfies

$$\partial_t (b(\rho)) + \nabla \cdot (b(\rho)u) + (b'(\rho)\rho - b(\rho))\nabla \cdot u = 0, \qquad (1.4)$$

in the sense of distributions in $\mathbb{R}^3 \times (0, +\infty)$ for any $b \in C^1((0, +\infty)) \cap C([0, +\infty))$ such that

$$b'(z) = 0$$
 for all $z \in (0, +\infty)$ large enough, say $z \ge M$, (1.5)

where the constant M > 0 may vary for different functions b's. Here (ρ, u) is prolonged by zero outside Ω .

(iv) (ρ, u, d) satisfies the following energy inequality

$$\mathbf{E}(t) + \int_0^t \int_\Omega \left(\mu |\nabla u|^2 + \widetilde{\mu} |\nabla \cdot u|^2 + |\Delta d + |\nabla d|^2 d|^2 \right) \le \mathbf{E}(0), \tag{1.6}$$

for almost all 0 < t < T. Here

$$\mathbf{E}(t) := \int_{\Omega} \left(\frac{1}{2} \rho |u|^2 + \frac{a \rho^{\gamma}}{\gamma - 1} + \frac{1}{2} |\nabla d|^2 \right)(t)$$
(1.7)

is the total energy of (ρ, u, d) at time t > 0, and

$$\mathbf{E}(0) = \int_{\Omega} \left(\frac{|m_0|^2}{2\rho_0} \chi_{\{\rho_0 \ge 0\}} + \frac{a\rho_0^{\gamma}}{\gamma - 1} + \frac{1}{2} |\nabla d_0|^2 \right)$$
(1.8)

is the initial energy.

There have been some earlier results on (1.1). In dimension one, the existence of global strong solutions and weak solutions to (1.1) has been obtained by [28] and [29] respectively. In dimension two, the existence of global weak solution of (1.1), under the condition that the image of d_0 is contained in \mathbb{S}^2_+ , was obtained by [33]. In dimension three, the local existence of strong solutions of (1.1) has been studied by [30] and [31]. The compressible limit of compressible nematic liquid crystal flow (1.1) has been studied by [32]. We also mention a related work [34]. When considering the compressible nematic liquid crystal flow (1.1) under the assumption that the director d has variable degree of orientations, the global existence of weak solutions in dimension three has been obtained by [27] and [35] respectively.

In this paper, we are mainly interested in the existence of finite energy weak solutions of (1.1)-(1.2) in dimension three. Our main states as follows.

Theorem 1.1. Assume $\gamma > \frac{3}{2}$ and the condition (1.3) holds. If the initial data (ρ_0, m_0, d_0) satisfies the following condition:

$$0 \le \rho_0 \in L^{\gamma}(\Omega), \tag{1.9}$$

$$m_0 \in L^{\frac{2\gamma}{\gamma+1}}(\Omega), \quad m_0\chi_{\{\rho_0=0\}} = 0, \quad \frac{|m_0|^2}{\rho_0}\chi_{\{\rho_0>0\}} \in L^1(\Omega),$$
 (1.10)

and

$$d_0 \in H^1(\Omega, \mathbb{S}^2), \text{ with } d_0(x) \in \mathbb{S}^2_+ \text{ a.e. } x \in \Omega.$$

$$(1.11)$$

Then there exists a global finite energy weak solution $(\rho, u, d) : \Omega \times [0, +\infty) \to \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{S}^2$ to the initial and boundary value problem (1.1)-(1.2) such that

(i) $d = (d^1, d^2, d^3) \in L^{\infty}((0, +\infty), H^1(\Omega, \mathbb{S}^2))$ and $d^3(x, t) \ge 0$ a.e. $(x, t) \in \Omega \times (0, +\infty)$. (ii) *it holds*

$$\int_{0}^{\infty} \eta(t) \int_{\Omega} \left(\nabla d \odot \nabla d - \frac{1}{2} |\nabla d|^{2} \mathbb{I}_{3} \right) : \nabla X + \int_{0}^{\infty} \eta(t) \int_{\Omega} \left\langle \partial_{t} d + u \cdot \nabla d, X \cdot \nabla d \right\rangle = 0, \quad (1.12)$$

for any
$$X \in C_0^1(\Omega, \mathbb{R}^3)$$
 and $\eta \in C_0^1((0, +\infty))$.

The main ideas of proof of Theorem 1.1 rely on (i) the precompactness results, due to Lin-Wang [18], on approximated Ginzburg-Landau equations $\{d_{\epsilon}\}$ with bounded energies, bounded L^2 -tension fields, and the condition $|d_{\epsilon}| \leq 1$ and $d_{\epsilon}^3 \geq -1+\delta$ for $\delta > 0$, and (ii) suitable adaption of compactness properties of renormalized solutions of compressible Navier-Stokes equations established by Lions [26] and Feireisl and his collaborators [4], [5], and [6].

For any global finite energy weak solutions of (1.1) and (1.2) that satisfies the properties stated in Theorem 1.1, we are able to establish the following preliminary result on its large time asymptotic behavior.

Corollary 1.2. Under the same assumptions of Theorem 1.1, let $(\rho, u, d) : \Omega \times [0, +\infty) \to \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{S}^2$ be any global finite energy weak solution of (1.1) and (1.2) that satisfies the properties of Theorem 1.1. Then there exist $t_n \to \infty$ and a harmonic map $d_{\infty} \in H^1 \cap C^{\infty}(\Omega, \mathbb{S}^2_+)$, with $d_{\infty} = d_0$ on $\partial\Omega$, such that

$$\left(\rho(\cdot, t_n), u(\cdot, t_n), d(\cdot, t_n)\right) \to \left(\rho_{0,\infty}, 0, d_\infty\right) \text{ in } L^{\gamma}(\Omega) \times L^p(\Omega) \times H^1_{\text{loc}}(\Omega), \tag{1.13}$$

for any
$$1 , where $\rho_{0,\infty} := \frac{1}{|\Omega|} \int_{\Omega} \rho_0 > 0$ is the average of the initial mass.$$

Remark 1.3. It is a very interesting question to ask whether the convergence in (1.13) holds for $t \to +\infty$. We plan to address it in a future work. We would like to point out that such a property has been established by [5] for the compressible Navier-Stokes equation. For the compressible flow of nematic liquid crystals with variable degree of orientations, see Wang-Yu [35] for the large time asymptotic behavior of global weak solutions.

The paper is written as follows. In section 2, we provide some preliminary estimates of (1.1). In section 3, we briefly review a compactness theorem due to Lin and Wang [18]. In section 4, we review the main results by Wang-Yu [35] on nematic liquid crystal flows with variable lengths of directors. In section 5, we prove Theorem 1.1. In section 6, we prove Corollary 1.2.

2. Global energy inequality and estimates based on the maximum principle

In this section, we will provide several basic properties of the hydrodynamic flow of compressible nematic liquid crystals (1.1) and (1.2). First, we will derive an energy equality for sufficiently smooth solutions of (1.1) and (1.2).

Lemma 2.1. Assume the conditions (1.3), (1.9), (1.10), and (1.11) hold. For $0 < T \le +\infty$, if $(\rho, u, d) \in C^1(Q_T, \mathbb{R}_+) \times C^2(Q_T, \mathbb{R}^3) \times C^2(Q_T, \mathbb{S}^2)$ is a solution of (1.1) and (1.2), then the following energy equality

$$\mathbf{E}(t) + \int_0^t \int_\Omega \left(\mu |\nabla u|^2 + \widetilde{\mu} |\nabla \cdot u|^2 + |\Delta d + |\nabla d|^2 d|^2 \right) = \mathbf{E}(0),$$
(2.1)

holds for any $0 \le t < T$, where $\mathbf{E}(t)$ and $\mathbf{E}(0)$ are given by (1.7) and (1.8) respectively.

Proof. Multiplying $(1.1)_2$ by u, integrating the resulting equation over Ω , applying integration by parts, and using $(1.1)_1$, we obtain

$$\frac{d}{dt} \int_{\Omega} \left(\frac{1}{2} \rho |u|^2 + \frac{a \rho^{\gamma}}{\gamma - 1} \right) + \int_{\Omega} \left(\mu |\nabla u|^2 + \widetilde{\mu} |\nabla \cdot u|^2 \right)
= -\int_{\Omega} \nabla \cdot \left(\nabla d \odot \nabla d - \frac{1}{2} |\nabla d|^2 \mathbb{I}_3 \right) u,$$
(2.2)

where we have used the fact

$$\int_{\Omega} \rho^{\gamma} \nabla \cdot u = \int_{\Omega} \rho^{\gamma-1} \rho \nabla \cdot u = -\int_{\Omega} (\partial_t \rho + u \cdot \nabla \rho) \rho^{\gamma-1} = -\frac{d}{dt} \int_{\Omega} \frac{\rho^{\gamma}}{\gamma} + \int_{\Omega} \frac{\rho^{\gamma}}{\gamma} \nabla \cdot u,$$

so that

$$-a\int_{\Omega}\rho^{\gamma}\nabla\cdot u = \frac{d}{dt}\int_{\Omega}\frac{a\rho^{\gamma}}{\gamma-1}.$$

Direct calculations show

$$abla \cdot (
abla d \odot
abla d - \frac{1}{2} |
abla d|^2 \mathbb{I}_3) = \langle riangle d,
abla d
angle.$$

Note also, since |d| = 1, that we have

$$\langle \partial_t d, d \rangle = \langle \nabla d, d \rangle = 0$$

and hence

$$-\int_{\Omega} \nabla \cdot (\nabla d \cdot \nabla d - \frac{1}{2} |\nabla d|^2 \mathbb{I}_3) u = -\int_{\Omega} u \cdot \langle \Delta d + |\nabla d|^2 d, \nabla d \rangle.$$
(2.3)

Multiplying $(1.1)_3$ by $-(\triangle d + |\nabla d|^2 d)$ and integrating over Ω yields that

$$\frac{d}{dt} \int_{\Omega} \frac{1}{2} |\nabla d|^2 + \int_{\Omega} |\Delta d + |\nabla d|^2 d|^2 = \int_{\Omega} u \cdot \langle \Delta d + |\nabla d|^2 d, \nabla d \rangle.$$
(2.4)

Putting (2.2), (2.3), and (2.4) together implies

$$\frac{d}{dt}\mathbf{E}(t) + \int_{\Omega} \left(\mu|\nabla u|^2 + \widetilde{\mu}|\nabla \cdot u|^2 + |\Delta d + |\nabla d|^2 d|^2\right) = 0.$$
(2.5)

This, after integrating over t, implies (2.1).

In order to construct global finite energy weak solutions to (1.1)-(1.2), we need some important estimates of transported Ginzburg-Landau equations based on the maximum principle.

Lemma 2.2. For $\epsilon > 0$, T > 0, and $u_{\epsilon} \in L^{2}([0,T], L^{\infty}(\Omega, \mathbb{R}^{3}))$, assume $d_{\epsilon} \in L^{2}([0,T], H^{1}(\Omega, \mathbb{R}^{3}))$, with $(1 - |d_{\epsilon}|^{2}) \in L^{2}(Q_{T})$, solves the transported Ginzburg-Landau equation:

$$\begin{cases} \partial_t d_\epsilon + u_\epsilon \cdot \nabla d_\epsilon = \triangle d_\epsilon + \frac{1}{\epsilon^2} (1 - |d_\epsilon|^2) d_\epsilon, & \text{in } Q_T, \\ d_\epsilon = g_\epsilon, & \text{on } \partial_p Q_T. \end{cases}$$
(2.6)

If $g_{\epsilon} \in H^1(\Omega, \mathbb{R}^3)$ satisfies $|g_{\epsilon}(x)| \leq 1$ for a.e. $x \in \Omega$, then

$$|d_{\epsilon}(x,t)| \leq 1$$
 for a.e. $(x,t) \in Q_T$.

Proof. We will follow the proof of Lemma 2.1 of Lin-Wang [18] with some modifications. For any k > 1, define $f_{\epsilon}^k : Q_T \to \mathbb{R}_+$ by

$$f_{\epsilon}^{k} = \begin{cases} k^{2} - 1, & \text{if } |d_{\epsilon}(x,t)| > k, \\ |d_{\epsilon}(x,t)|^{2} - 1, & \text{if } 1 < |d_{\epsilon}(x,t)| \le k, \\ 0, & \text{if } |d_{\epsilon}(x,t)| \le 1. \end{cases}$$

By direct calculations, we have that f^k_ϵ satisfies, in the sense of distributions,

$$\begin{cases} \partial_t f_{\epsilon}^k + u_{\epsilon} \cdot \nabla f_{\epsilon}^k = \triangle f_{\epsilon}^k - 2\chi_{\{1 < |d_{\epsilon}| \le k\}} \left(|\nabla d_{\epsilon}|^2 + \frac{1}{\epsilon^2} (|d_{\epsilon}|^2 - 1)|d_{\epsilon}|^2 \right) \le \triangle f_{\epsilon}^k \text{ in } Q_T, \\ f_{\epsilon}^k = 0 & \text{ on } \partial_p Q_T. \end{cases}$$
(2.7)

Multiplying (2.7) by f_{ϵ}^k and integrating over Ω , we obtain

$$\frac{d}{dt} \int_{\Omega} |f_{\epsilon}^{k}|^{2} + 2 \int_{\Omega} |\nabla f_{\epsilon}^{k}|^{2} \leq 2 \int_{\Omega} u_{\epsilon} \cdot \nabla f_{\epsilon}^{k} f_{\epsilon}^{k} \\
\leq \int_{\Omega} |\nabla f_{\epsilon}^{k}|^{2} + \|u_{\epsilon}(t)\|_{L^{\infty}(\Omega)}^{2} \int_{\Omega} |f_{\epsilon}^{k}|^{2}.$$

Hence we have

$$\frac{d}{dt} \int_{\Omega} |f_{\epsilon}^{k}|^{2} dx \leq 2 \|u_{\epsilon}(\cdot)\|_{L^{\infty}(\Omega)}^{2} \int_{\Omega} |f_{\epsilon}^{k}|^{2} dx.$$

$$(2.8)$$

Since $u_{\epsilon} \in L^2([0,T], L^{\infty}(\Omega))$ and $f_{\epsilon}^k(x,0) = 0$ for a.e. $x \in \Omega$, applying Gronwall's inequality to (2.8) yields that $f_{\epsilon}^k = 0$ a.e. in Q_T . By the definition of f_{ϵ}^k , this implies that $d_{\epsilon} \leq 1$ a.e. in Q_T . \Box

We also have the following lemma.

Lemma 2.3. For $\epsilon > 0$, T > 0, and $u_{\epsilon} \in L^{2}([0,T], L^{\infty}(\Omega, \mathbb{R}^{3}))$, assume $d_{\epsilon} \in L^{2}([0,T], H^{1}(\Omega, \mathbb{R}^{3}))$, with $(1 - |d_{\epsilon}|^{2}) \in L^{2}(Q_{T})$, solves the transported Ginzburg-Landau equation (2.6). If $g_{\epsilon} \in H^{1}(\Omega, \mathbb{R}^{3})$ satisfies

$$|g_{\epsilon}(x)| \leq 1 \text{ and } g_{\epsilon}^{3}(x) \geq 0 \text{ for a.e. } x \in \Omega,$$

then

$$|d_{\epsilon}(x,t)| \leq 1$$
 and $d_{\epsilon}^{3}(x,t) \geq 0$ for a.e. $(x,t) \in Q_{T}$

Proof. We will modify the proof of Lemma 2.2 by Lin-Wang [18]. First it follows from Lemma 2.2 that

$$0 \le \frac{1}{\epsilon^2} (1 - |d_\epsilon|^2) \le \frac{1}{\epsilon^2}.$$

Set $\widetilde{d}_{\epsilon}^3 := e^{-\frac{t}{\epsilon^2}} d_{\epsilon}^3$. Then we have

$$\partial_t \widetilde{d_\epsilon^3} + u_\epsilon \cdot \nabla \widetilde{d_\epsilon^3} - \bigtriangleup \widetilde{d_\epsilon^3} = h_\epsilon \widetilde{d_\epsilon^3},$$

where

$$h_{\epsilon}(x,t) = \left(\frac{1}{\epsilon^2}(1-|d_{\epsilon}|^2) - \frac{1}{\epsilon^2}\right) \le 0 \quad \text{a.e.} \ (x,t) \in Q_T.$$

Since $\tilde{d}_{\epsilon}^3 \ge 0$ on $\partial_p Q_T$, we have that $(\tilde{d}_{\epsilon}^3)^- := -\min\left\{\tilde{d}_{\epsilon}^3, 0\right\}$ satisfies

$$\begin{cases} \partial_t (\widetilde{d}^3_{\epsilon})^- + u_{\epsilon} \cdot \nabla (\widetilde{d}^3_{\epsilon})^- - \triangle (\widetilde{d}^3_{\epsilon})^- = h_{\epsilon} (\widetilde{d}^3_{\epsilon})^-, \text{ in } Q_T, \\ (\widetilde{d}^3_{\epsilon})^- = 0, & \text{ on } \partial_p Q_T. \end{cases}$$
(2.9)

Multiplying (2.9)₁ by $(\tilde{d}_{\epsilon}^3)^-$ and integrating the resulting equation over Ω , we have

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} |(\tilde{d}_{\epsilon}^{3})^{-}|^{2} + 2 \int_{\Omega} |\nabla(\tilde{d}_{\epsilon}^{3})^{-}|^{2} \\ &= -2 \int_{\Omega} u_{\epsilon} \cdot \nabla(\tilde{d}_{\epsilon}^{3})^{-} (\tilde{d}_{\epsilon}^{3})^{-} + 2 \int_{\Omega} h_{\epsilon} |(\tilde{d}_{\epsilon}^{3})^{-}|^{2} \\ &\leq -2 \int_{\Omega} u_{\epsilon} \cdot \nabla(\tilde{d}_{\epsilon}^{3})^{-} (\tilde{d}_{\epsilon}^{3})^{-} \\ &\leq \int_{\Omega} |\nabla(\tilde{d}_{\epsilon}^{3})^{-}|^{2} + ||u_{\epsilon}(t)||_{L^{\infty}(\Omega)}^{2} \int_{\Omega} |(\tilde{d}_{\epsilon}^{3})^{-}|^{2}, \end{aligned}$$

where we have used the fact that $h_{\epsilon}(x,t) \leq 0$ a.e. $(x,t) \in Q_T$. Thus we have

$$\frac{d}{dt} \int_{\Omega} |(\widetilde{d}_{\epsilon}^{3})^{-}|^{2} \leq ||u_{\epsilon}(t)||_{L^{\infty}(\Omega)}^{2} \int_{\Omega} |(\widetilde{d}_{\epsilon}^{3})^{-}|^{2}.$$

Applying Gronwall's inequality and using the initial condition $(d_{\epsilon}^3)^-(x,0) = 0$ a.e. $x \in \Omega$, we obtain that $(\tilde{d}_{\epsilon}^3)^- = 0$ a.e. in Q_T . Therefore $d_{\epsilon}^3 \ge 0$ a.e. Q_T . This completes the proof of Lemma 2.3. \Box

3. Review of Lin-Wang's compactness results

In order to show that a family of global finite weak solutions $(\rho_{\epsilon}, u_{\epsilon}, d_{\epsilon})$ to the Ginzburg-Landau approximation of compressible nematic liquid crystal flow converges to a global finite weak solution (ρ, u, d) of the compressible nematic liquid crystal flow (1.1) and (1.2), we need to establish the compactness of d_{ϵ} in $L^2_{loc}([0, T], H^1_{loc}(\Omega, \mathbb{R}^3))$. Under suitable conditions, this has recently been achieved by Lin-Wang [18] in their studies of the existence of global weak solutions to the incompressible nematic liquid crystal flow.

Since such a compactness property also plays a crucial role in this paper, we will state it and refer the interested readers to the paper [18] for more detail. For $a \in (0, 2]$, denote

$$\mathbb{S}^{2}_{-1+a} = \Big\{ y = (y^{1}, y^{2}, y^{3}) \in \mathbb{S}^{2} | y^{3} \ge -1 + a \Big\}.$$

For any $a \in (0,2]$, $L_1 > 0$ and $L_2 > 0$, let $\mathbf{X}(L_1, L_2, a; \Omega)$ denote the set consisting of all maps $d_{\epsilon} \in H^1(\Omega, \mathbb{R}^3)$, with $\epsilon \in (0, 1]$, that are solutions of

$$\Delta d_{\epsilon} + \frac{1}{\epsilon^2} (1 - |d_{\epsilon}|^2) d_{\epsilon} = \tau_{\epsilon} \quad \text{in } \ \Omega, \text{ with } \tau_{\epsilon} \in L^2(\Omega, \mathbb{R}^3),$$
(3.1)

such that for all $0 < \epsilon \leq 1$, the following properties hold:

(i) $|d_{\epsilon}| \leq 1$ and $d_{\epsilon}^{3} \geq -1 + a$ for a.e. $x \in \Omega$. (ii) $\mathbf{E}_{\epsilon}(d_{\epsilon}) := \int_{\Omega} \left(\frac{1}{2}|\nabla d_{\epsilon}|^{2} + \frac{3}{4\epsilon^{2}}(1 - |d_{\epsilon}|^{2})^{2}\right) \leq L_{1}$.

(iii)
$$\left\| \tau_{\epsilon} \right\|_{L^{2}(\Omega)} \leq L_{2}.$$

We have

Theorem 3.1. ([18]) For any $a \in (0, 2]$, $L_1 > 0$, and $L_2 > 0$, the set $\mathbf{X}(L_1, L_2, a; \Omega)$ is precompact in $H^1_{loc}(\Omega, \mathbb{R}^3)$. In particular, if for $\epsilon \to 0$, $\{d_{\epsilon}\} \subset H^1(\Omega, \mathbb{R}^3)$ is a sequence of maps in $\mathbf{X}(L_1, L_2, a; \Omega)$, then there exists a map $d \in H^1(\Omega, \mathbb{S}^2_{-1+a}) \cap \mathbf{Y}(L_1, L_2, a; \Omega)$ such that after passing to possible subsequences, $d_{\epsilon} \to d$ in $H^1_{loc}(\Omega, \mathbb{R}^3)$ and

$$e_{\epsilon}(d_{\epsilon}) dx := \left(\frac{1}{2} |\nabla d_{\epsilon}|^2 + \frac{(1 - |d_{\epsilon}|^2)^2}{4\epsilon^2}\right) dx \rightharpoonup \frac{1}{2} |\nabla d|^2 dx$$

as convergence of Radon measures.

The idea of proof of Theorem 3.1 is based on: (1) almost energy monotonicity inequality of $d_{\epsilon} \in \mathbf{X}(L_1, L_2, a; \Omega);$ (2) an δ_0 -regularity and compactness property of $d_{\epsilon} \in \mathbf{X}(L_1, L_2, a; \Omega);$ (3) the blowing-up analysis of $d_{\epsilon} \in \mathbf{X}(L_1, L_2, a; \Omega)$ as $\epsilon \to 0$ in terms of both the concentration set Σ and the defect measure ν , motivated by that of harmonic maps by Lin [16] and approximated harmonic maps [20, 21, 22]; and (4) the ruling out of possible harmonic \mathbb{S}^2 's generated at Σ .

In order to study the large time behavior of global finite energy weak solutions to the compressible nematic liquid crystal flow (1.1) and (1.2), we also need the following compactness result on approximated harmonic maps to \mathbb{S}^2_{-1+a} for $0 < a \leq 2$.

For $0 < a \leq 2, L_1 > 0$, and $L_2 > 0$, let $\mathbf{Y}(L_1, L_2, a; \Omega)$ be the set consisting of maps $d \in H^1(\Omega, \mathbb{S}^2)$ that are approximated harmonic maps, i.e.,

$$\Delta d + |\nabla d|^2 d = \tau \text{ in } \Omega, \text{ with } \tau \in L^2(\Omega, \mathbb{R}^3), \tag{3.2}$$

that satisfy the following properties:

- (i) $d^3(x) \ge -1 + a$ for a.e. $x \in \Omega$. (ii) $\mathbf{F}(d) := \frac{1}{2} \int_{\Omega} |\nabla d|^2 \le L_1$.
- (iii) $\|\tau\|_{L^2(\Omega)} \le L_2.$
- (iv) (almost energy monotonicity inequality) for any $x_0 \in \Omega$ and $0 < r \le R < d(x_0, \partial \Omega)$,

$$\Psi_R(d, x_0) \ge \Psi_r(d, x_0) + \frac{1}{2} \int_{B_R(x_0) \setminus B_r(x_0)} |x - x_0|^{-1} \left| \frac{\partial d}{\partial |x - x_0|} \right|^2,$$
(3.3)

where

$$\Psi_r(d, x_0) := \frac{1}{r} \int_{B_r(x_0)} \left(\frac{1}{2} |\nabla d|^2 - \langle (x - x_0) \cdot \nabla d, \tau \rangle \right) + \frac{1}{2} \int_{B_r(x_0)} |x - x_0| |\tau|^2.$$

Theorem 3.2. ([18]) For any $a \in (0, 2]$, $L_1 > 0$, and $L_2 > 0$, the set $\mathbf{Y}(L_1, L_2, a; \Omega)$ is precompact in $H^1_{loc}(\Omega, \mathbb{S}^2)$. In particular, if $\{d_i\} \subset H^1(\Omega, \mathbb{R}^3)$ is a sequence of approximated harmonic maps in $\mathbf{Y}(L_1, L_2, a; \Omega)$ with tension fields $\{\tau_i\}$, then there exist $\tau_0 \in L^2(\Omega, \mathbb{R}^3)$ and an approximated harmonic map $d_0 \in \mathbf{Y}(L_1, L_2, a; \Omega)$ with tension field τ_0 such that after passing to possible subsequences, $d_i \to d_0$ in $H^1_{\text{loc}}(\Omega, \mathbb{S}^2)$ and $\tau_i \to \tau_0$ in $L^2(\Omega, \mathbb{R}^3)$. In fact, $\{d_i\}$ is bounded in $H^2_{\text{loc}}(\Omega, \mathbb{S}^2)$. In particular, $d_0 \in H^2_{loc}(\Omega, \mathbb{S}^2)$.

4. GINZBURG-LANDAU APPROXIMATION OF COMPRESSIBLE NEMATIC LIQUID CRYSTAL FLOW

In this section, we will consider the Ginzburg-Landau approximation of compressible nematic liquid crystal flow and state the existence of global weak solutions, which is an improved version of an earlier result obtained by Wang-Yu [35] (see also [27]).

For $\epsilon > 0$ and $0 < T \leq +\infty$, the Ginzburg-Landau approximation equation of (1.1) and (1.2) seeks $(\rho_{\epsilon}, u_{\epsilon}, d_{\epsilon}) : Q_T \to \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$ that satisfies:

$$\begin{cases} \partial_t \rho_{\epsilon} + \nabla \cdot (\rho u_{\epsilon}) = 0, \\ \partial_t (\rho_{\epsilon} u_{\epsilon}) + \nabla \cdot (\rho_{\epsilon} u_{\epsilon} \otimes u_{\epsilon}) + a \nabla \rho_{\epsilon}^{\gamma} \\ = \mathcal{L} u_{\epsilon} - \nabla \cdot (\nabla d_{\epsilon} \odot \nabla d_{\epsilon} - (\frac{1}{2} |\nabla d_{\epsilon}|^2 + \frac{1}{4\epsilon^2} (1 - |d_{\epsilon}|^2)^2) \mathbb{I}_3), \\ \partial_t d_{\epsilon} + u_{\epsilon} \cdot \nabla d_{\epsilon} = \Delta d_{\epsilon} + \frac{1}{\epsilon^2} (1 - |d_{\epsilon}|^2) d_{\epsilon}, \end{cases}$$

$$\tag{4.1}$$

along with the initial and boundary condition (1.2). We would like to point out that the notion of finite energy weak solutions of (4.1) and (1.2) can be defined in the same way as that of (1.1) and (1.2) given in §1.

Theorem 4.1. Assume $\gamma > \frac{3}{2}$ and the condition (1.3), and (ρ_0, m_0, d_0) satisfies (1.9), (1.10), (1.11). Then there exists a global finite energy weak solution $(\rho_{\epsilon}, u_{\epsilon}, d_{\epsilon}) : \Omega \times [0, +\infty) \to \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$ to the system (4.1), under the initial and boundary condition (1.2), such that

- (i) $d_{\epsilon} = (d_{\epsilon}^1, d_{\epsilon}^2, d_{\epsilon}^3) \in L^{\infty}((0, \infty), H^1(\Omega, \mathbb{R}^3))$, with $|d_{\epsilon}| \leq 1$ and $d_{\epsilon}^3 \geq 0$ for a.e. $(x, t) \in \Omega \times (0, \infty)$.
- (ii) $(\rho_{\epsilon}, u_{\epsilon}, d_{\epsilon})$ satisfies the global energy inequality

$$\frac{d}{dt}\mathbf{F}_{\epsilon}(t) + \int_{\Omega} \left(\mu |\nabla u_{\epsilon}|^{2} + \widetilde{\mu} |\nabla \cdot u_{\epsilon}|^{2} + |\Delta d_{\epsilon} + \frac{1}{\epsilon^{2}} (1 - |d_{\epsilon}|^{2}) d_{\epsilon}|^{2} \right)(t) \leq 0$$
(4.2)

in $\mathcal{D}'((0,+\infty))$, where

$$\mathbf{F}_{\epsilon}(t) := \int_{\Omega} \left(\frac{1}{2} \rho_{\epsilon} |u_{\epsilon}|^2 + \frac{a \rho_{\epsilon}^{\gamma}}{\gamma - 1} + \left(\frac{1}{2} |\nabla d_{\epsilon}|^2 + \frac{1}{4\epsilon^2} (1 - |d_{\epsilon}|^2)^2 \right) \right) (t)$$

Proof. The existence of finite energy weak solutions has been established by Wang-Yu [35], which uses a three level approximation scheme similar to that of compressible Navier-Stokes equation by [4] and [6]. It consists of Faedo-Galerkin approximation, artificial viscosity, and artificial pressure. The reader can consult the proof of [35] Theorem 2.1 for the detail.

Here we only indicate the proof of (i). Let $\epsilon > 0$ be fixed. Recall that the first level of Faedo-Galerkin's approximation involves to solve the initial and boundary value problems of (4.1) as follows. For any $\alpha > 0$, $\delta > 0$, and $0 < T < +\infty$, we first approximate the initial data (ρ_0, m_0, d_0) by $(\rho_{0,\delta}, m_{0,\delta}, d_{0,\delta}) \in C^2(\overline{\Omega}, \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3_+)$ such that the following conditions hold:

$$\begin{cases} \delta \leq \rho_{0,\delta} \leq \delta^{-1} \text{ in } \Omega, \frac{\partial \rho_{0,\delta}}{\partial \nu} \Big|_{\partial\Omega} = 0, \text{ and } \rho_{0,\delta} \to \rho_0 \text{ in } L^{\gamma}(\Omega), \\ m_{0,\delta} \to m_0 \text{ in } L^{\frac{2\gamma}{\gamma+1}}(\Omega), \frac{|m_{0,\delta}|^2}{\rho_{0,\delta}} \to \frac{|m_0|^2}{\rho_0} \chi_{\{\rho_0 > 0\}} \text{ in } L^1(\Omega), \\ |d_{0,\delta}(x)| \leq 1, \ d^3_{0,\delta}(x) \geq 0 \text{ a.e. } x \in \Omega, \ d_{0,\delta} \to d_0 \text{ in } H^1(\Omega, \mathbb{R}^3), \end{cases}$$
(4.3)

as $\delta \to 0$.

For
$$u \in C^1([0,T], C_0^2(\overline{\Omega}, \mathbb{R}^3))$$
, with $u\big|_{t=0} = u_{0,\delta} \equiv \frac{m_{0,\delta}}{\rho_{0,\delta}}$, let $d_{\delta} = d_{\delta}([u]) \in C^1([0,T], C^2(\overline{\Omega}, \mathbb{R}^3))$

be the unique solution of (see [35] Lemma 3.1 and Lemma 3.2):

$$\begin{cases} \partial_t d + u \cdot \nabla d = \Delta d + \frac{1}{\epsilon^2} (1 - |d|^2) d & \text{in } Q_T, \\ d = d_{0,\delta} & \text{on } \partial_p Q_T. \end{cases}$$
(4.4)

Since $|d_{0,\delta}(x)| \leq 1$ and $d_{0,\delta}^3(x) \geq 0$ for $x \in \Omega$, it follows from Lemma 2.2 and Lemma 2.3 that d_{δ} satisfies

$$d_{\delta}(x,t)| \leq 1 \text{ and } d^3_{\delta}(x,t) \geq 0, \ \forall \ (x,t) \in Q_T.$$

$$(4.5)$$

Now let $\rho_{\alpha,\delta} = \rho_{\alpha,\delta}([u]) \in C^1([0,T], C^2(\overline{\Omega}))$ be the unique solution of the problem:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = \alpha \Delta \rho & \text{in } Q_T, \\ \rho(x, 0) = \rho_{0,\delta}(x) & \text{in } \Omega, \\ \frac{\partial \rho}{\partial \nu} = 0 & \text{on } \partial \Omega \times (0, T). \end{cases}$$
(4.6)

While for u, it involves to employ first the Galerkin method and then the fixed point theorem to solve $u = u_{\alpha,\delta}([u])$ to the problem: for some $\beta > \max\{4, \gamma\}$,

$$\begin{cases} \partial_t(\rho_{\alpha,\delta}u) + \nabla \cdot (\rho_{\alpha,\delta}u \otimes u) + a\nabla(\rho_{\alpha,\delta}^{\gamma}) + \delta\nabla(\rho_{\alpha,\delta}^{\beta}) + \alpha\nabla u \cdot \nabla\rho_{\alpha,\delta} \\ = \mathcal{L}u - \nabla \cdot \left[\nabla d_\delta \odot \nabla d_\delta - \left(\frac{1}{2}|\nabla d_\delta|^2 + \frac{1}{4\epsilon^2}(1 - |d_\delta|^2)^2\right)\mathbb{I}_3\right], \text{ in } Q_T, \\ u = u_{0,\delta} \quad \text{on } \partial_p Q_T. \end{cases}$$
(4.7)

Since the global weak solution $(\rho_{\epsilon}, u_{\epsilon}, d_{\epsilon})$ to the system (4.1), under the initial and boundary condition (1.2), constructed in [35], was obtained as a strong limit of $(\rho_{\alpha,\delta}, u_{\alpha,\delta}, d_{\delta})$ in $L^{\gamma}(Q_T) \times L^2(Q_T) \times L^2([0, T], H^1(\Omega, \mathbb{R}^3))$ for any $0 < T < +\infty$, as viscosity coefficients $\alpha \to 0$ first and then artificial pressure coefficients $\delta \to 0$. It is readily seen that d_{ϵ} satisfies the property (ii).

5. EXISTENCE OF GLOBAL WEAK SOLUTIONS

In this section, we will prove Theorem 1.1 by studying in depth the convergence of sequences of solutions $(\rho_{\epsilon}, u_{\epsilon}, d_{\epsilon})$, constructed by Theorem 4.1, as $\epsilon \to 0^+$.

Proof of Theorem 1.1.

To prove the existence of global finite energy weak solutions to (1.1), let $(\rho_{\epsilon}, u_{\epsilon}, d_{\epsilon}) : \Omega \times [0, +\infty) \rightarrow \mathbb{R}_{+} \times \mathbb{R}^{3} \times \mathbb{R}^{3}$, $0 < \epsilon \leq 1$, be a family of finite energy weak solutions to the system (4.1), under the initial and boundary condition (1.2), constructed by Theorem 4.1. Since $|d_{0}| = 1$ and $d_{0}^{3} \geq 0$ a.e. in Ω , $(\rho_{\epsilon}, u_{\epsilon}, d_{\epsilon})$ satisfies all these properties in Theorem 4.1. In particular, it follows from (4.2) that

$$\sup_{\epsilon>0} \left[\sup_{0< t<\infty} \int_{\Omega} \left(\frac{1}{2} \rho_{\epsilon} |u_{\epsilon}|^{2} + \frac{a}{\gamma - 1} \rho_{\epsilon}^{\gamma} + \left(\frac{1}{2} |\nabla d_{\epsilon}|^{2} + \frac{1}{4\epsilon^{2}} (1 - |d_{\epsilon}|^{2})^{2} \right) \right)(t) \\
+ \int_{0}^{\infty} \int_{\Omega} \left(\mu |\nabla u_{\epsilon}|^{2} + \widetilde{\mu} |\nabla \cdot u_{\epsilon}|^{2} + |\triangle d_{\epsilon} + \frac{1}{\epsilon^{2}} (1 - |d_{\epsilon}|^{2}) d_{\epsilon}|^{2} \right) \right] \\
\leq \int_{\Omega} \left(\frac{|m_{0}|^{2}}{2\rho_{0}} \chi_{\{\rho_{0}>0\}} + \frac{a}{\gamma - 1} \rho_{0}^{\gamma} + \frac{1}{2} |\nabla d_{0}|^{2} \right) := \mathbf{E}(0).$$
(5.1)

By (5.1), we may assume that there exists $(\rho, u, d) : \Omega \times [0, +\infty) \to \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{S}^2$ such that after passing to a subsequence,

$$\begin{cases} \rho_{\epsilon} \rightharpoonup \rho \text{ weak}^* \text{ in } L^{\infty}([0,T], L^{\gamma}(\Omega)), \\ u_{\epsilon} \rightharpoonup u \text{ in } L^2([0,T], H_0^1(\Omega)), \\ d_{\epsilon} \rightharpoonup d \text{ weak}^* \text{ in } L^{\infty}([0,T], H^1(\Omega)), \end{cases}$$
(5.2)

as $\epsilon \to 0$, for any $0 < T < +\infty$.

We will prove that (ρ, u, d) is a global finite energy weak solution to (1.1) and (1.2). The proof will be divided into several subsections.

5.1. $d_{\epsilon} \to d$ strongly in $L^2([0,T], H^1_{\text{loc}}(\Omega))$. This will be achieved by applying Theorem 3.1, similar to that of [18]. First it follows from the equation (4.1)₃ and the inequality (5.1) that $\partial_t d_{\epsilon} \in L^2([0,T], L^{\frac{3}{2}}(\Omega)) + L^2([0,T], L^2(\Omega))$ so that $\partial_t d_{\epsilon} \in L^2([0,T], H^{-1}(\Omega))$ and

$$\sup_{0<\epsilon\leq 1} \left\|\partial_t d_\epsilon\right\|_{L^2(0,T;H^{-1}(\Omega))} < +\infty.$$
(5.3)

By Aubin-Lions' lemma, we conclude that

$$d_{\epsilon} \to d \text{ in } L^2(Q_T) \text{ and } \nabla d_{\epsilon} \rightharpoonup \nabla d \text{ in } L^2([0,T], L^2(\Omega)).$$
 (5.4)

By Fatou's lemma, (5.1) implies that

$$\int_{0}^{T} \liminf_{\epsilon \to 0} \int_{\Omega} \left| \Delta d_{\epsilon} + \frac{1}{\epsilon^{2}} (1 - |d_{\epsilon}|^{2}) d_{\epsilon} \right|^{2} \leq \mathbf{E}(0).$$
(5.5)

For sufficiently large $\Lambda > 1$, define the set of good time slice, G_{Λ}^{T} , by

$$G_{\Lambda}^{T} := \Big\{ t \in [0,T] \ \Big| \ \liminf_{\epsilon \to 0} \int_{\Omega} \Big| \triangle d_{\epsilon} + \frac{1}{\epsilon^{2}} (1 - |d_{\epsilon}|^{2}) d_{\epsilon} \Big|^{2}(t) \le \Lambda \Big\},$$

and the set of bad time slices, B_{Λ}^T , by

$$B_{\Lambda}^{T} := [0,T] \setminus G_{\Lambda}^{T} = \Big\{ t \in [0,T] \ \Big| \ \liminf_{\epsilon \to 0} \int_{\Omega} \big| \triangle d_{\epsilon} + \frac{1}{\epsilon^{2}} (1 - |d_{\epsilon}|^{2}) d_{\epsilon} \big|^{2}(t) > \Lambda \Big\}.$$

It is easy to see from (5.5) that

$$\left|B_{\Lambda}^{T}\right| \leq \frac{\mathbf{E}(0)}{\Lambda}.$$
(5.6)

By (5.1) and (5.6), we obtain

$$\int_{B_{\Lambda}^{T}} \int_{\Omega} \left[|\nabla d_{\epsilon} - \nabla d|^{2} + \frac{1}{\epsilon^{2}} (1 - |d_{\epsilon}|^{2})^{2} (t) \right] \leq C \left| B_{\Lambda}^{T} \right| \sup_{0 < t < T} \mathbf{F}_{\epsilon}(t) \leq \frac{C \mathbf{E}(0)}{\Lambda}.$$
(5.7)

For any $t \in G_{\Lambda}^{T}$, set $\tau_{\epsilon}(t) = \left(\triangle d_{\epsilon} + \frac{1}{\epsilon^{2}} (1 - |d_{\epsilon}|^{2}) d_{\epsilon} \right)(t)$. Then it follows from the definition of G_{Λ}^{T} that there exists $\tau(t) \in L^{2}(\Omega, \mathbb{R}^{3})$ such that, after passing to a subsequence, $\tau_{\epsilon}(t) \rightharpoonup \tau(t)$ in $L^{2}(\Omega)$. Since $\{d_{\epsilon}(t)\} \subset \mathbf{X}(\mathbf{E}(0), \Lambda, 1; \Omega)$, Theorem 3.1 implies that there exists $d(t) \in \mathbf{Y}(\mathbf{E}(0), \Lambda, 1; \Omega)$ such that after passing to a subsequence, $d_{\epsilon}(t) \rightarrow d(t)$ strongly in $H^{1}_{\text{loc}}(\Omega)$ and $\frac{1}{\epsilon^{2}}(1 - |d_{\epsilon}(t)|^{2})^{2} \rightarrow 0$ in $L^{1}_{\text{loc}}(\Omega)$.

Now we want to show that, after passing to a subsequence,

$$\nabla d_{\epsilon} \to \nabla d \text{ in } L^2_{\text{loc}}(\Omega \times G^T_{\Lambda}).$$
 (5.8)

This can be done similarly to Claim 8.2 of [18]. Here we provide it. Suppose (5.8) were false. Then there exist a subdomain $\widetilde{\Omega} \subset \subset \Omega$, $\delta_0 > 0$, and $\epsilon_i \to 0$ such that

$$\int_{\widetilde{\Omega} \times G_{\Lambda}^{T}} |\nabla (d_{\epsilon_{i}} - d)|^{2} \ge \delta_{0}.$$
(5.9)

Note that from (5.4) we have

$$\lim_{\epsilon_i \to 0} \int_{\Omega \times G_\Lambda^T} |d_{\epsilon_i} - d|^2 = 0.$$
(5.10)

By Fubini's theorem, (5.9), and (5.10), we have that there exists $t_i \in G_{\Lambda}^T$ such that

$$\lim_{\epsilon_i \to 0} \int_{\Omega} |d_{\epsilon_i}(t_i) - d(t_i)|^2 = 0, \qquad (5.11)$$

and

$$\int_{\widetilde{\Omega}} \left| \nabla (d_{\epsilon_i}(t_i) - d(t_i)) \right|^2 \ge \frac{2\delta_0}{T}.$$
(5.12)

It is easy to see that $\{d_{\epsilon_i}(t_i)\} \subset \mathbf{X}(\mathbf{E}(0), \Lambda, 1; \Omega)$ and $\{d(t_i)\} \subset \mathbf{Y}(\mathbf{E}(0), \Lambda, 1; \Omega)$. It follows from Theorem 3.1 and Theorem 3.2 that there exist $d_1, d_2 \in \mathbf{Y}(\mathbf{E}(0), \Lambda, 1; \Omega)$ such that

$$d_{\epsilon_i}(t_i) \to d_1 \text{ and } d(t_i) \to d_2 \text{ in } L^2(\Omega) \cap H^1(\overline{\Omega}).$$

This and (5.12) imply that

$$\int_{\widetilde{\Omega}} \left| \nabla (d_1 - d_2) \right|^2 \ge \frac{2\delta_0}{T}.$$
(5.13)

On the other hand, from (5.11), we have that

$$\int_{\Omega} |d_1 - d_2|^2 = 0.$$
(5.14)

It is clear that (5.13) contradicts (5.14). Hence (5.8) is proven. Similar to Lemma 4.1, Claim 4.4 in [18], we also have

$$\int_{\widetilde{\Omega}\times G_{\Lambda}^{T}} \frac{1}{\epsilon^{2}} (1-|d_{\epsilon}|^{2})^{2} \to 0 \text{ as } \epsilon \to 0.$$
(5.15)

Combining (5.7), (5.8), with (5.15), we obtain

$$\lim_{\epsilon \to 0} \left[\|d_{\epsilon} - d\|_{L^{2}([0,T], H^{1}(\widetilde{\Omega}))} + \int_{\widetilde{\Omega} \times [0,T]} \frac{(1 - |d_{\epsilon}|^{2})^{2}}{\epsilon^{2}} \right] \le C\Lambda^{-1}.$$
(5.16)

Since $\Lambda > 1$ can be chosen arbitrarily large, we conclude that

$$d_{\epsilon} \to d \text{ in } L^{2}([0,T], H^{1}_{\text{loc}}(\Omega)) \text{ and } \frac{(1-|d_{\epsilon}|^{2})^{2}}{\epsilon^{2}} \to 0 \text{ in } L^{1}([0,T], L^{1}_{\text{loc}}(\Omega)).$$
 (5.17)

5.2. $\rho_{\epsilon}u_{\epsilon} \to \rho u$ in the sense of distributions. By (5.1), $\sqrt{\rho_{\epsilon}}$ is bounded in $L^{\infty}([0,T], L^{2\gamma}(\Omega))$ and $\sqrt{\rho_{\epsilon}}u_{\epsilon}$ is bounded in $L^{\infty}([0,T], L^{2}(\Omega))$. Thus $\rho_{\epsilon}u_{\epsilon}$ is bounded in $L^{\infty}([0,T], L^{\frac{2\gamma}{\gamma+1}}(\Omega))$ and $\partial_{t}\rho_{\epsilon} = -\nabla \cdot (\rho_{\epsilon}u_{\epsilon})$ is bounded in $L^{\infty}([0,T], W^{-1,\frac{2\gamma}{\gamma+1}}(\Omega))$. Applying [26] Lemma C.1, we have

$$\rho_{\epsilon} \to \rho \text{ in } C([0,T], L^{\gamma}_{\text{weak}}(\Omega)).$$
(5.18)

Since $L^{\gamma}(\Omega) \subset H^{-1}(\Omega)$ is compact, we conclude that

$$\rho_{\epsilon} \to \rho \text{ in } C([0,T], H^{-1}(\Omega)).$$
(5.19)

Thus we show that

$$\rho_{\epsilon} u_{\epsilon} \to \rho u \text{ in } \mathcal{D}'(Q_T).$$
(5.20)

5.3. Higher integrability estimates of ρ_{ϵ} . There exist $\theta > 0$ and C > 0 depending only on γ and T such that for any $0 < \epsilon \leq 1$, it holds

$$\int_0^T \int_\Omega \rho_\epsilon^{\gamma+\theta} \le C. \tag{5.21}$$

By Theorem 4.1, $(\rho_{\epsilon}, u_{\epsilon})$ is a renormalized solution of $(4.1)_1$. Let $(\rho_{\epsilon}, u_{\epsilon}) : \mathbb{R}^3 \times (0, T) \to \mathbb{R}_+ \times \mathbb{R}^3$ be the extension of $(\rho_{\epsilon}, u_{\epsilon})$ from Ω such that $(\rho_{\epsilon}, u_{\epsilon}) = (0, 0)$ in $\mathbb{R}^3 \setminus \Omega$. Then $(\rho_{\epsilon}, u_{\epsilon})$ satisfies, in the sense of distributions, that

$$\partial_t(b(\rho_\epsilon)) + \nabla \cdot (b(\rho_\epsilon)u_\epsilon) + (b'(\rho_\epsilon)\rho_\epsilon - b(\rho_\epsilon))\nabla \cdot u_\epsilon = 0 \text{ in } \mathbb{R}^3 \times (0,T),$$
(5.22)

for any bounded function $b \in C^1((0, +\infty)) \cap C([0, +\infty))$ (see, e.g., [6]).

As in [4], [6] and [35], we can employ suitable approximations so that (5.22) also holds for $b(\rho_{\epsilon}) = \rho_{\epsilon}^{\theta}$ for $0 < \theta < 1$. Note that $\rho_{\epsilon}^{\theta} \in L^{\frac{\gamma}{\theta}}(Q_T)$. For $m \ge 1$, let $S_m(f) = \eta_{\frac{1}{m}} * f$ denote the standard mollification of $f \in L^1(\mathbb{R}^3)$. Then we have

$$\partial_t \left(S_m(\rho_\epsilon^\theta) \right) + \nabla \cdot \left(S_m(\rho_\epsilon^\theta) u_\epsilon \right) - (1 - \theta) S_m(\rho_\epsilon^\theta \nabla \cdot u_\epsilon) = q_m \quad \text{in } \mathbb{R}^3 \times (0, T), \tag{5.23}$$

where

$$q_m = \nabla \cdot (S_m(\rho_{\epsilon}^{\theta})u_{\epsilon}) - S_m(\nabla \cdot (\rho_{\epsilon}^{\theta}u_{\epsilon})).$$

By virtue of [26] Lemma 2.3, $\rho_{\epsilon}^{\theta} \in L^{\infty}([0,T], L^{\frac{\gamma}{\theta}}(\Omega))$, and $u_{\epsilon} \in L^{2}([0,T], H_{0}^{1}(\Omega))$, we have that

$$\lim_{m \to \infty} \|q_m\|_{L^2([0,T], L^{\lambda}(\mathbb{R}^3))} = 0, \text{ with } \frac{1}{\lambda} = \frac{\theta}{\gamma} + \frac{1}{2},$$
(5.24)

provided $\theta < \frac{\gamma}{2}$.

As in [4] and [26], define the (inverse of divergence) operator

$$\mathcal{B}: \left\{ f \in L^p(\Omega) \mid \int_{\Omega} f = 0 \right\} \mapsto W_0^{1,p}(\Omega, \mathbb{R}^3)$$

such that for any 1 ,

$$\begin{cases} \nabla \cdot \mathcal{B}(f) = f \text{ in } \Omega, \quad \mathcal{B}(f) = 0 \text{ on } \partial\Omega, \\ \left\| \mathcal{B}(f) \right\|_{W_0^{1,p}(\Omega)} \le C(p) \left\| f \right\|_{L^p(\Omega)}. \end{cases}$$
(5.25)

Set $\oint_{\Omega} f = \frac{1}{|\Omega|} \int_{\Omega} f$. For $\varphi \in C_0^{\infty}((0,T))$, with $0 \le \varphi \le 1$, let $\phi(x,t) = \phi(t) \mathcal{B} \left[S_{-}(a^{\theta}) - \oint_{0} S_{m}(a^{\theta}) \right]$

$$\phi(x,t) = \varphi(t)\mathcal{B}\Big[S_m(\rho_{\epsilon}^{\theta}) - \oint_{\Omega} S_m(\rho_{\epsilon}^{\theta})\Big](x,t)$$

By (5.1), we see that for sufficiently small θ ,

$$\left[S_m(\rho_{\epsilon}^{\theta}) - \oint_{\Omega} S_m(\rho_{\epsilon}^{\theta})\right] \in C([0,T], L^p(\Omega)), \ \forall \ p \in (1, +\infty).$$

By (5.25) and the Sobolev embedding theorem, we have that $\phi \in C(Q_T)$. Thus we can test the equation $(4.1)_2$ by ϕ and obtain

$$\begin{split} & a \int_{0}^{T} \int_{\Omega} \varphi(t) \rho_{\epsilon}^{\gamma} S_{m}(\rho_{\epsilon}^{\theta}) \\ &= a \int_{0}^{T} \varphi(t) \Big(\int_{\Omega} \rho_{\epsilon}^{\gamma} \Big) \Big(\oint_{\Omega} S_{m}(\rho_{\epsilon}^{\theta}) \Big) \\ &- \int_{0}^{T} \int_{\Omega} \varphi'(t) \rho_{\epsilon} u_{\epsilon} \mathcal{B} \big[S_{m}(\rho_{\epsilon}^{\theta}) - \oint_{\Omega} S_{m}(\rho_{\epsilon}^{\theta}) \big] \\ &+ \int_{0}^{T} \int_{\Omega} \varphi(t) (\mu \nabla u_{\epsilon} - \rho_{\epsilon} u_{\epsilon} \otimes u_{\epsilon}) \nabla \mathcal{B} \big[S_{m}(\rho_{\epsilon}^{\theta}) - \oint_{\Omega} S_{m}(\rho_{\epsilon}^{\theta}) \big] \\ &+ \int_{0}^{T} \int_{\Omega} \varphi(t) \tilde{\mu} \nabla \cdot u_{\epsilon} \nabla \cdot \mathcal{B} \big[S_{m}(\rho_{\epsilon}^{\theta}) - \oint_{\Omega} S_{m}(\rho_{\epsilon}^{\theta}) \big] \\ &+ (1 - \theta) \int_{0}^{T} \int_{\Omega} \varphi(t) \rho_{\epsilon} u_{\epsilon} \mathcal{B} \big[S_{m}(\rho_{\epsilon}^{\theta} \nabla \cdot u_{\epsilon}) - \oint_{\Omega} S_{m}(\rho_{\epsilon}^{\theta} \nabla \cdot u_{\epsilon}) \big] \\ &+ \int_{0}^{T} \int_{\Omega} \varphi(t) (\Delta d_{\epsilon} + \frac{1}{\epsilon^{2}} (1 - |d_{\epsilon}|^{2}) d_{\epsilon}) \cdot \nabla d_{\epsilon} \mathcal{B} \big[S_{m}(\rho_{\epsilon}^{\theta}) - \oint_{\Omega} S_{m}(\rho_{\epsilon}^{\theta}) \big] \\ &- \int_{0}^{T} \int_{\Omega} \varphi(t) \rho_{\epsilon} u_{\epsilon} \mathcal{B} \big[\nabla \cdot (S_{m}(\rho_{\epsilon}^{\theta}) u_{\epsilon}) \big] \\ &+ \int_{0}^{T} \int_{\Omega} \varphi(t) \rho_{\epsilon} u_{\epsilon} \mathcal{B} \big[q_{m} - \oint_{\Omega} q_{m} \big] \\ &= \sum_{i=1}^{7} L_{i}^{m} + \int_{0}^{T} \int_{\Omega} \varphi(t) \rho_{\epsilon} u_{\epsilon} \mathcal{B} \big[q_{m} - \oint_{\Omega} q_{m} \big]. \end{split}$$

Since $\rho_{\epsilon}u_{\epsilon}$ is bounded in $L^{\infty}([0,T], L^{\frac{2\gamma}{\gamma+1}}(\Omega))$ and q_m satisfies (5.24), it follows from (5.25), the Sobolev embedding theorem, and the Hölder inequality that

$$\lim_{m \to \infty} \int_0^T \int_\Omega \varphi(t) \rho_\epsilon u_\epsilon \mathcal{B} \big[q_m - \oint_\Omega q_m \big] = 0.$$

Hence, after taking $m \to \infty$, we have

$$\int_{(0,T)\times\Omega} \varphi \rho_{\epsilon}^{\gamma+\theta} \le \limsup_{m\to\infty} \sum_{i=1}^{7} L_i^m.$$
(5.26)

Now we estimate L_1^m, \cdots, L_7^m as follows.

(1) For L_1^m , we have that

$$\left|\lim_{m\to\infty}L_1^m\right| = \left|a\int_0^T\int_\Omega\varphi\rho_\epsilon^\gamma(\oint_\Omega\rho_\epsilon^\theta)\right| \le C\Big(\int_0^T\int_\Omega\rho_\epsilon^\gamma\Big)\Big\|\rho\epsilon\Big\|_{L^\infty([0,T],L^\gamma(\Omega))}^\theta$$

is uniformly bounded.

(2) For L_2^m , we have that

$$\begin{aligned} |L_{2}^{m}| &= \left| \int_{0}^{T} \int_{\Omega} \varphi'(t) \rho_{\epsilon} u_{\epsilon} \mathcal{B} \left[q_{m} - \oint_{\Omega} q_{m} \right] \right| \\ &\leq C \int_{0}^{T} \left\| \rho_{\epsilon} u_{\epsilon} \right\|_{L^{\frac{2\gamma}{\gamma+1}}(\Omega)} \left\| \mathcal{B} \left[q_{m} - \oint_{\Omega} q_{m} \right] \right\|_{L^{\frac{2\gamma}{\gamma-1}}(\Omega)} dt \\ &\leq C \int_{0}^{T} \left\| \mathcal{B} \left[q_{m} - \oint_{\Omega} q_{m} \right] \right\|_{W^{1,\lambda}(\Omega)} dt \\ &\leq C \int_{0}^{T} \left\| q_{m} - \oint_{\Omega} q_{m} \right\|_{L^{\lambda}(\Omega)} dt \\ &\leq C \left\| q_{m} \right\|_{L^{2}([0,T], L^{\lambda}(\Omega))} \to 0 \text{ as } m \to +\infty, \end{aligned}$$

provided $\theta < \frac{\gamma}{3} - \frac{1}{2}$. (3) For L_3^m , we have

$$\begin{aligned} |L_{3}^{m}| &\leq C \int_{0}^{T} \left\{ \left\| u_{\epsilon} \right\|_{H^{1}(\Omega)} \left\| \mathcal{B} \left[\rho_{\epsilon}^{\theta} - \oint \rho_{\epsilon}^{\theta} \right] \right\|_{H^{1}(\Omega)} \\ &+ \left\| \rho_{\epsilon} \right\|_{L^{\gamma}(\Omega)} \left\| u_{\epsilon} \right\|_{L^{6}(\Omega)}^{2} \left\| \mathcal{B} \left[\rho_{\epsilon}^{\theta} - \oint \rho_{\epsilon}^{\theta} \right] \right\|_{W^{1,\frac{3\gamma}{2\gamma-3}}(\Omega)} \right\} dt \\ &\leq C \int_{0}^{T} \left(\left\| u_{\epsilon} \right\|_{H^{1}(\Omega)} \left\| \rho_{\epsilon} \right\|_{L^{2\theta}(\Omega)}^{\theta} + \left\| \rho_{\epsilon} \right\|_{L^{\gamma}(\Omega)} \left\| u_{\epsilon} \right\|_{H^{1}(\Omega)}^{2} \left\| \rho_{\epsilon} \right\|_{L^{\frac{3\gamma\theta}{2\gamma-3}}(\Omega)}^{\theta} \right) dt \end{aligned}$$

is uniformly bounded, provided $\theta < \min\left\{\frac{\gamma}{2}, \frac{2\gamma}{3} - 1\right\}$. (4) For L_4^m , we have

$$\begin{aligned} |L_4^m| &= \left| \int_0^T \int_{\Omega} \widetilde{\mu} \varphi \nabla \cdot u_{\epsilon} \nabla \cdot \mathcal{B} \left[\rho_{\epsilon}^{\theta} - \oint_{\Omega} \rho_{\epsilon}^{\theta} \right] \right| \\ &= \left| \int_0^T \int_{\Omega} \widetilde{\mu} \varphi \nabla \cdot u_{\epsilon} \left(\rho_{\epsilon}^{\theta} - \oint_{\Omega} \rho_{\epsilon}^{\theta} \right) \right| \\ &\leq C \left\| u_{\epsilon} \right\|_{L^2([0,T], H^1(\Omega))} \left\| \rho_{\epsilon} \right\|_{L^{\infty}([0,T], L^{2\theta}(\Omega))}^{\theta} \end{aligned}$$

is uniformly bounded, provided $\theta \leq \frac{\gamma}{2}$. (5) For L_5^m , we have

$$\begin{aligned} |L_{5}^{m}| &= (1-\theta) \Big| \int_{0}^{T} \int_{\Omega} \varphi \rho_{\epsilon} u_{\epsilon} \mathcal{B} \Big[\rho_{\epsilon}^{\theta} \nabla \cdot u_{\epsilon} - \oint_{\Omega} \rho_{\epsilon}^{\theta} \nabla \cdot u_{\epsilon} \Big] \Big| \\ &\leq C \int_{0}^{T} \| \rho_{\epsilon} \|_{L^{\gamma}(\Omega)} \| u_{\epsilon} \|_{L^{6}(\Omega)} \| \mathcal{B} \big[\rho_{\epsilon}^{\theta} \nabla \cdot u_{\epsilon} - \oint_{\Omega} \rho_{\epsilon}^{\theta} \nabla \cdot u_{\epsilon} \big] \|_{L^{\frac{6\gamma}{5\gamma-6}}(\Omega)} dt \\ &\leq C \int_{0}^{T} \| \rho_{\epsilon} \|_{L^{\gamma}(\Omega)} \| u_{\epsilon} \|_{L^{6}(\Omega)} \| \rho_{\epsilon}^{\theta} \nabla \cdot u_{\epsilon} \|_{L^{\frac{6\gamma}{7\gamma-6}}(\Omega)} dt \\ &\leq C \int_{0}^{T} \| \rho_{\epsilon} \|_{L^{\gamma}(\Omega)} \| u_{\epsilon} \|_{H^{1}(\Omega)} \| \rho_{\epsilon} \|_{L^{\frac{3\gamma\theta}{2\gamma-3}}(\Omega)}^{\theta} \| \nabla \cdot u_{\epsilon} \|_{L^{2}(\Omega)} dt \end{aligned}$$

is uniformly bounded, provided $\theta < \frac{2\gamma}{3} - 1$.

(6) For L_6^m , we have

$$\begin{split} L_{6}^{m} &| = \left\| \int_{0}^{T} \int_{\Omega} \varphi \left(\bigtriangleup d_{\epsilon} + \frac{1}{\epsilon^{2}} (1 - |d_{\epsilon}|^{2}) d_{\epsilon} \right) \cdot \nabla d_{\epsilon} \mathcal{B} \left[\rho_{\epsilon}^{\theta} - \oint_{\Omega} \rho_{\epsilon}^{\theta} \right] \\ &\leq C \left\| \bigtriangleup d_{\epsilon} + \frac{1}{\epsilon^{2}} (1 - |d_{\epsilon}|^{2}) d_{\epsilon} \right\|_{L^{2}(Q_{T})} \cdot \\ & \left\| \nabla d_{\epsilon} \right\|_{L^{\infty}([0,T],L^{2}(\Omega))} \left\| \mathcal{B} \left[\rho_{\epsilon}^{\theta} - \oint_{\Omega} \rho_{\epsilon}^{\theta} \right] \right\|_{L^{2}([0,T],L^{\infty}(\Omega))} \\ &\leq C \left\| \left\| \mathcal{B} \left[\rho_{\epsilon}^{\theta} - \oint_{\Omega} \rho_{\epsilon}^{\theta} \right] \right\|_{W^{1,\frac{\gamma}{\theta}}(\Omega)} \right\|_{L^{2}([0,T])} \\ &\leq C \sup_{0 < t \leq T} \left\| \rho_{\epsilon}^{\theta} \right\|_{L^{\frac{\gamma}{\theta}}(\Omega)} \leq C \left\| \rho_{\epsilon} \right\|_{L^{\infty}([0,T],L^{\gamma}(\Omega))}^{\theta}, \end{split}$$

provided $\theta < \frac{\gamma}{3}$, where we have used the energy inequality (5.1) and the Sobolev embedding theorem $W^{1,\frac{\gamma}{\theta}}(\Omega) \subset L^{\infty}(\Omega).$

(7) For L_7^m , we have

$$\begin{aligned} |L_{7}^{m}| &= \left| \int_{0}^{T} \int_{\Omega} |\varphi \rho_{\epsilon} u_{\epsilon} \mathcal{B} \left[\nabla \cdot (\rho_{\epsilon}^{\theta} u_{\epsilon}) \right] \right| \\ &\leq C \int_{0}^{T} \left\| \rho_{\epsilon} \right\|_{L^{\gamma}(\Omega)} \left\| u_{\epsilon} \right\|_{L^{6}(\Omega)} \left\| \mathcal{B} \left[\nabla \cdot (\rho_{\epsilon}^{\theta} u_{\epsilon}) \right] \right\|_{L^{\frac{6\gamma}{5\gamma-6}}(\Omega)} dt \\ &\leq C \int_{0}^{T} \left\| \rho_{\epsilon} \right\|_{L^{\gamma}(\Omega)} \left\| u_{\epsilon} \right\|_{L^{6}(\Omega)} \left\| \rho_{\epsilon}^{\theta} u_{\epsilon} \right\|_{L^{\frac{6\gamma}{5\gamma-6}}(\Omega)} dt \\ &\leq C \int_{0}^{T} \left\| \rho_{\epsilon} \right\|_{L^{\gamma}(\Omega)} \left\| u_{\epsilon} \right\|_{L^{6}(\Omega)}^{2} \left\| \rho_{\epsilon} \right\|_{L^{\frac{3\gamma\theta}{2\gamma-3}}(\Omega)} dt \end{aligned}$$

is uniformly bounded, provided $\theta < \frac{2\gamma}{3} - 1$. It is clear that we can choose sufficiently small $\theta > 0$ depending only on γ such that all these estimates on L_i^m , $i = 1, \dots, 7$, hold. Therefore, by putting together (1), \dots , (7), we obtain the estimate (5.21).

5.4. $\rho_{\epsilon}u_{\epsilon} \otimes u_{\epsilon} \to \rho u \otimes u$ in the sense of distributions. As $\rho_{\epsilon}u_{\epsilon}$ is bounded in $L^{\infty}([0,T], L^{\frac{2\gamma}{\gamma+1}}(\Omega))$, it follows from the section 5.3 that

$$\rho_{\epsilon} u_{\epsilon} \rightharpoonup \rho u \text{ weak}^* \text{ in } L^{\infty}([0,T], L^{\frac{2\gamma}{\gamma+1}}(\Omega)).$$
(5.27)

Meanwhile, since

$$\begin{aligned} \partial_t(\rho_\epsilon u_\epsilon) &= -\nabla \cdot \left(\rho_\epsilon u_\epsilon \otimes u_\epsilon\right) + a\nabla \rho_\epsilon^\gamma \\ &+ \mathcal{L}u_\epsilon - \nabla \cdot \left[\nabla d_\epsilon \odot \nabla d_\epsilon - \left(\frac{1}{2}|\nabla d_\epsilon|^2 + \frac{(1-|d_\epsilon|^2)^2}{4\epsilon^2}\right)\mathbb{I}_3\right] \\ &\in L^2([0,T], W^{-1,\frac{6\gamma}{4\gamma+3}}(\Omega)) + L^{\frac{\gamma+\theta}{\gamma}}([0,T], W^{-1,\frac{\gamma+\theta}{\gamma}}(\Omega)) \\ &+ L^2([0,T], H^{-1}(\Omega)) + L^\infty([0,T], W^{-2,\frac{5}{4}}(\Omega)), \end{aligned}$$

we have that

$$\rho_{\epsilon} u_{\epsilon} \to \rho u \text{ in } C([0,T], L_{\text{weak}}^{\frac{2\gamma}{\gamma+1}}(\Omega)) \text{ and } \rho_{\epsilon} u_{\epsilon} \to \rho u \text{ in } C([0,T], H^{-1}(\Omega)).$$
(5.28)

Hence we obtain

$$\rho_{\epsilon} u_{\epsilon} \otimes u_{\epsilon} \to \rho u \otimes u \quad \text{in} \quad \mathcal{D}'(Q_T).$$
(5.29)

It follows from §5.1, §5.2, §5.3, and §5.4 that, after sending $\epsilon \to 0^+$ in the equation (4.1), (ρ, u, d) satisfies the following system:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + a \nabla \overline{\rho^{\gamma}} = \mathcal{L} u - \nabla \cdot \left[\nabla d \odot \nabla d - \frac{1}{2} |\nabla d|^2 \mathbb{I}_3 \right], \\ \partial_t d + u \cdot \nabla d = \triangle d + |\nabla d|^2 d, \end{cases}$$
(5.30)

in the sense of distributions, where $\overline{\rho^{\gamma}}$ is a weak limit of ρ_{ϵ}^{γ} in $L^{\frac{\gamma+\theta}{\gamma}}(Q_T)$.

It is straightforward that (ρ, u, d) satisfies the first two equations of (5.30). To see (u, d) solves the third equation of (5.30), we employ the standard technique, due to Chen [1], as follows. Let \times denote the cross product in \mathbb{R}^3 . Then the equation (4.1)₃ for $(u_{\epsilon}, d_{\epsilon})$ can be rewritten as

$$(\partial_t d_\epsilon + u_\epsilon \cdot \nabla d_\epsilon) \times d_\epsilon = \Delta d_\epsilon \times d_\epsilon, \text{ in } \mathcal{D}'(Q_T).$$

After taking $\epsilon \to 0$, we have that (u, d) satisfies

$$(\partial_t d + u \cdot \nabla d) \times d = \Delta d \times d, \text{ in } \mathcal{D}'(Q_T).$$
(5.31)

Since |d| = 1, the equation (5.31) is equivalent to $(5.30)_3$.

In order to identify $\overline{\rho^{\gamma}}$, we need to establish the strong convergence of ρ_{ϵ} to ρ in $L^{\gamma}(Q_T)$. To do it, we need to have fine estimates of the effective viscous flux, which has played important rules in the study of compressible Navier-Stokes equations (see [4] and [26]).

For $k \geq 1$, define $T_k(z) = kT(\frac{z}{k}) : \mathbb{R} \to \mathbb{R}$, where $T(z) \in C^{\infty}(\mathbb{R})$ is a concave function such that

$$T(z) = \begin{cases} z, & \text{if } z \le 1, \\ 2, & \text{if } z \ge 3. \end{cases}$$

5.5. Fine estimates of effective viscous flux $H_{\epsilon} := a\rho_{\epsilon}^{\gamma} - \tilde{\mu}\nabla \cdot u_{\epsilon}$. For any fixed $k \ge 1$, there holds

$$\lim_{\epsilon \to 0} \int_0^T \int_\Omega \psi \phi \big(a \rho_\epsilon^\gamma - \widetilde{\mu} \nabla \cdot u_\epsilon \big) T_k(\rho_\epsilon) = \int_0^T \int_\Omega \psi \phi \big(a(\overline{\rho^\gamma}) - \widetilde{\mu} \nabla \cdot u \big) \overline{T_k(\rho)}, \tag{5.32}$$

for any $\psi \in C_0^{\infty}((0,T))$ and $\phi \in C_0^{\infty}(\Omega)$. By density arguments, similar to [4], it is not hard to see that (5.32) remains to be true for $\phi = \psi = 1$.

Since $(\rho_{\epsilon}, u_{\epsilon})$ is a renormalized solution to $(4.1)_1$ in Q_T , it is clear that if we extend $(\rho_{\epsilon}, u_{\epsilon})$ to \mathbb{R}^3 by letting it to be zero in $\mathbb{R}^3 \setminus \Omega$, then $(\rho_{\epsilon}, u_{\epsilon})$ is also a renormalized solution of $(4.1)_1$ in \mathbb{R}^3 . Replacing b(z) by $T_k(z)$ in $(4.1)_1$ yields

$$\partial_t \big(T_k(\rho_\epsilon) \big) + \nabla \cdot (T_k(\rho_\epsilon) u_\epsilon) + \big(T'(\rho_\epsilon) \rho_\epsilon - T_k(\rho_\epsilon) \big) \nabla \cdot u_\epsilon = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3 \times (0, T)).$$
(5.33)

Since $T_k(\rho_{\epsilon})$ is bounded in $L^{\infty}(Q_T)$, we have

$$T_k(\rho_{\epsilon}) \rightharpoonup \overline{T_k(\rho)}$$
 weak*in $L^{\infty}(Q_T)$.

This, combined with the equation (5.33), implies that for any $p \in (1, +\infty)$,

$$T_k(\rho_\epsilon) \to \overline{T_k(\rho)}$$
 in $C([0,T], L^p_{\text{weak}}(\Omega))$ and in $C([0,T], H^{-1}(\Omega)).$ (5.34)

Hence, after sending $\epsilon \to 0$ in the equation (5.33), we have

$$\partial_t \overline{T_k(\rho)} + \nabla \cdot (\overline{T_k(\rho)}u) + \overline{(T'_k(\rho)\rho - T_k(\rho))}\nabla \cdot u = 0 \quad \text{in } \mathcal{D}'(Q_T),$$
(5.35)

where $(T'_k(\rho)\rho - T_k(\rho))\nabla \cdot u$ is a weak limit of $(T'_k(\rho_\epsilon)\rho_\epsilon - T_k(\rho_\epsilon))\nabla \cdot u_\epsilon$ in $L^2(Q_T)$.

Now we need to estimate the effective viscous flux $(a\rho_{\epsilon}^{\gamma} - \tilde{\mu}\nabla \cdot u_{\epsilon})$. Define the operator $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ by letting

$$\mathcal{A}_i = \partial_{x_i} \triangle^{-1}$$

for i = 1, 2, 3, where \triangle^{-1} denote the inverse of the Laplace operator on \mathbb{R}^3 (see [6]). By the L^p regularity theory of the Laplace equation, we have

$$\begin{cases} \|\mathcal{A}v\|_{W^{1,s}(\Omega)} \leq C \|v\|_{L^{s}(\mathbb{R}^{3})}, & 1 < s < +\infty, \\ \|\mathcal{A}v\|_{L^{q}(\Omega)} \leq C \|v\|_{L^{s}(\mathbb{R}^{3})}, & \frac{1}{q} \geq \frac{1}{s} - \frac{1}{3}, \\ \|\mathcal{A}v\|_{L^{\infty}(\Omega)} \leq C \|v\|_{L^{s}(\mathbb{R}^{3})}, & s > 3, \end{cases}$$
(5.36)

where C > 0 depends only on s and Ω .

Testing the equation (5.33) by $\mathcal{A}_i[\varphi]$ for $\varphi \in C_0^{\infty}(Q_T)$ yields

$$\partial_t \left(\mathcal{A}_i[T_k(\rho_\epsilon)] \right) + \nabla \cdot \left(\mathcal{A}_i[(T_k(\rho_\epsilon)u_\epsilon)] \right) + \mathcal{A}_i \left[(T'_k(\rho_\epsilon)\rho_\epsilon - T_k(\rho_\epsilon))\nabla \cdot u_\epsilon \right] = 0, \tag{5.37}$$

in $\mathcal{D}'(\mathbb{R}^3 \times (0,T)) \cap L^2(\mathbb{R}^3 \times (0,T))$. This implies $\partial_t (\mathcal{A}_i[T_k(\rho_\epsilon)]) \in L^2(\mathbb{R}^3 \times (0,T))$. Hence we can test the equation $(4.1)_2$ by $\psi \phi \mathcal{A}[T_k(\rho_\epsilon)]$, for $\phi \in C_0^{\infty}(\Omega)$ and $\psi \in C_0^{\infty}((0,T))$, and obtain

$$\int_{0}^{T} \int_{\Omega} \psi \phi(a\rho_{\epsilon}^{\gamma} - \tilde{\mu}\nabla \cdot u_{\epsilon})T_{k}(\rho_{\epsilon})$$

$$= \int_{0}^{T} \int_{\Omega} \psi(\tilde{\mu}\nabla \cdot u_{\epsilon} - a\rho_{\epsilon}^{\gamma})\nabla \phi \mathcal{A}[T_{k}(\rho_{\epsilon})]$$

$$+ \mu \int_{0}^{T} \int_{\Omega} \psi\{\nabla \phi \nabla u_{\epsilon}^{i}\mathcal{A}_{i}[T_{k}(\rho_{\epsilon})] - u_{\epsilon}^{i}\nabla \phi \nabla \mathcal{A}_{i}[T_{k}(\rho_{\epsilon})] + u_{\epsilon}\nabla \phi T_{k}(\rho_{\epsilon})\}$$

$$- \int_{0}^{T} \int_{\Omega} \phi \rho_{\epsilon} u_{\epsilon}\{\partial_{t}\psi\mathcal{A}[T_{k}(\rho_{\epsilon})] + \psi\mathcal{A}[(T_{k}(\rho_{\epsilon}) - T_{k}'(\rho_{\epsilon})\rho_{\epsilon})\nabla \cdot u_{\epsilon}]\}$$

$$- \int_{0}^{t} \int_{\Omega} \psi \rho_{\epsilon} u_{\epsilon}^{i}u_{\epsilon}^{j}\partial_{j}\phi\mathcal{A}_{i}[T_{k}(\rho_{\epsilon})]$$

$$+ \int_{0}^{T} \int_{\Omega} \psi u_{\epsilon}^{i}\{T_{k}(\rho_{\epsilon})\mathcal{R}_{ij}[\phi\rho_{\epsilon}u_{\epsilon}^{i}] - \phi\rho_{\epsilon}u_{\epsilon}^{j}\mathcal{R}_{ij}[T_{k}(\rho_{\epsilon})]\}$$

$$- \int_{0}^{T} \int_{\Omega} \psi\nabla \cdot \{\nabla d_{\epsilon} \odot \nabla d_{\epsilon} - [\frac{1}{2}|\nabla d_{\epsilon}|^{2} + \frac{1}{4\epsilon^{2}}(1 - |d_{\epsilon}|^{2})^{2}]\mathbb{I}_{3}\} \cdot \phi\mathcal{A}[T_{k}(\rho_{\epsilon})], \quad (5.38)$$

where $\mathcal{R}_{ij} = \partial_{x_j} \mathcal{A}_i$ is the Riesz transform. Similarly, we can test the equation (5.30)₂ by $\psi \phi \mathcal{A}_i[T_k(\rho)]$ and obtain

$$\int_{0}^{T} \int_{\Omega} \psi \phi(a\rho^{\gamma} - \tilde{\mu}\nabla \cdot u) T_{k}(\rho)$$

$$= \int_{0}^{T} \int_{\Omega} \psi(\tilde{\mu}\nabla \cdot u - a\overline{\rho^{\gamma}}) \nabla \phi \mathcal{A}[T_{k}(\rho)]$$

$$+ \mu \int_{0}^{T} \int_{\Omega} \psi \{\nabla \phi \nabla u^{i} \mathcal{A}_{i}[T_{k}(\rho)] - u^{i} \nabla \phi \nabla \mathcal{A}_{i}[T_{k}(\rho)] + u \nabla \phi T_{k}(\rho)\}$$

$$- \int_{0}^{T} \int_{\Omega} \phi \rho u \{\partial_{t} \psi \mathcal{A}[T_{k}(\rho)] + \psi \mathcal{A}[(T_{k}(\rho) - T_{k}'(\rho)\rho)\nabla \cdot u]\}$$

$$- \int_{0}^{t} \int_{\Omega} \psi \rho u^{i} u^{j} \partial_{j} \phi \mathcal{A}_{i}[T_{k}(\rho)]$$

$$+ \int_{0}^{T} \int_{\Omega} \psi u^{i} \{T_{k}(\rho) \mathcal{R}_{ij}[\phi \rho u^{i}] - \phi \rho u^{j} \mathcal{R}_{ij}[T_{k}(\rho)]\}$$

$$- \int_{0}^{T} \int_{\Omega} \psi \nabla \cdot \left[\nabla d \odot \nabla d - \frac{1}{2} |\nabla d|^{2} \mathbb{I}_{3}\right] \cdot \phi \mathcal{A}[T_{k}(\rho)].$$
(5.39)

To prove (5.32), it suffices to show that each term in the right hand side of (5.38) converges to the corresponding term in the right hand side of (5.39). Since the convergence of the first five terms in the right hand side of (5.38) can be done in the exact same way as in [6] (see also [35]), we only indicate how to show the convergence of the last term in the right hand side of (5.38), namely,

$$\int_{0}^{T} \int_{\Omega} \psi \nabla \cdot \left[\nabla d_{\epsilon} \odot \nabla d_{\epsilon} - \left(\frac{1}{2} |\nabla d_{\epsilon}|^{2} + \frac{1}{4\epsilon^{2}} (1 - |d_{\epsilon}|^{2})^{2} \right) \mathbb{I}_{3} \right] \cdot \phi \mathcal{A}[T_{k}(\rho_{\epsilon})]$$

$$\rightarrow \int_{0}^{T} \int_{\Omega} \psi \nabla \cdot \left[\nabla d \odot \nabla d - \frac{1}{2} |\nabla d|^{2} \mathbb{I}_{3} \right] \cdot \phi \mathcal{A}[T_{k}(\rho)] \text{ as } \epsilon \to 0.$$
(5.40)

To see this, first observe that $T_k(\rho_{\epsilon})$ is bounded in $L^{\infty}(Q_T)$ and hence we have, by (5.37), that (see also [6])

$$\mathcal{A}[T_k(\rho_{\epsilon})] \to \mathcal{A}[T_k(\rho)] \text{ in } C(\overline{\Omega} \times [0,T]).$$
(5.41)

Secondly, observe that a.e. in Q_T , there holds

$$\nabla \cdot \left[\nabla d_{\epsilon} \odot \nabla d_{\epsilon} - \left(\frac{1}{2} |\nabla d_{\epsilon}|^{2} + \frac{1}{4\epsilon^{2}} (1 - |d_{\epsilon}|^{2})^{2} \right) \mathbb{I}_{3} \right] = \left(\Delta d_{\epsilon} + \frac{1}{\epsilon^{2}} (1 - |d_{\epsilon}|^{2}) d_{\epsilon} \right) \cdot \nabla d_{\epsilon}$$
$$= \left(\partial_{t} d_{\epsilon} + u_{\epsilon} \cdot \nabla d_{\epsilon} \right) \cdot \nabla d_{\epsilon}.$$

By the energy inequality (5.1), we see that $(\partial_t d_\epsilon + u_\epsilon \cdot \nabla d_\epsilon)$ is bounded in $L^2(Q_T)$ and hence there exists $v \in L^2(Q_T)$ such that

$$(\partial_t d_\epsilon + u_\epsilon \cdot \nabla d_\epsilon) \rightharpoonup v \text{ in } L^2(Q_T).$$
 (5.42)

On the other hand, since $d_{\epsilon} \to d$ in $L^2([0,T], H^1_{\text{loc}}(\Omega))$ and $u_{\epsilon} \rightharpoonup u$ in $L^2([0,T], H^1_0(\Omega))$, we have that

$$(\partial_t d_\epsilon + u_\epsilon \cdot \nabla d_\epsilon) \to (\partial_t d + u \cdot \nabla d)$$
 in $\mathcal{D}'(Q_T)$.

Hence we have

$$v = \partial_t d + u \cdot \nabla d \quad \text{in} \quad Q_T. \tag{5.43}$$

By (5.41) and the local L^2 -convergence of ∇d_{ϵ} to ∇d in Q_T , we know that

$$\nabla d_{\epsilon} \phi \mathcal{A}[T_k(\rho_{\epsilon})] \to \nabla d\phi \mathcal{A}[T_k(\rho)] \quad \text{in} \quad L^2(Q_T).$$

Hence we obtain

$$\int_{0}^{T} \int_{\Omega} \psi \nabla \cdot \left[\nabla d_{\epsilon} \odot \nabla d_{\epsilon} - \left(\frac{1}{2} |\nabla d_{\epsilon}|^{2} + \frac{1}{4\epsilon^{2}} (1 - |d_{\epsilon}|^{2})^{2} \right) \mathbb{I}_{3} \right] \cdot \phi \mathcal{A}[T_{k}(\rho_{\epsilon})]$$

$$\rightarrow \int_{0}^{T} \int_{\Omega} \psi(\partial_{t} d + u \cdot \nabla d) \nabla d \cdot \phi \mathcal{A}[T_{k}(\rho)] \text{ as } \epsilon \to 0.$$
(5.44)

Applying the equation $(5.30)_3$ and the fact that $\langle |\nabla d|^2 d, \nabla d \rangle = 0$ a.e. in Q_T , we obtain

$$\int_{0}^{T} \int_{\Omega} \psi(\partial_{t}d + u \cdot \nabla d) \nabla d \cdot \phi \mathcal{A}[T_{k}(\rho)]$$

$$= \int_{0}^{T} \int_{\Omega} \psi(\Delta d + |\nabla d|^{2}d) \nabla d \cdot \phi \mathcal{A}[T_{k}(\rho)]$$

$$= \int_{0}^{T} \int_{\Omega} \psi \Delta d \nabla d \cdot \phi \mathcal{A}[T_{k}(\rho)]$$

$$= \int_{0}^{T} \int_{\Omega} \psi \nabla \cdot \left[\nabla d \odot \nabla d - \frac{1}{2} |\nabla d|^{2} \mathbb{I}_{3} \right] \cdot \phi \mathcal{A}[T_{k}(\rho)].$$
(5.45)

It is easy to see that (5.40) follows from (5.44) and (5.45).

In order to show the strong convergence of ρ_{ϵ} , we also need to estimate on the oscillation defect measure of $(\rho_{\epsilon} - \rho)$ in $L^{\gamma}(Q_T)$.

5.6. Estimate of oscillation of defect measures. There exists C > 0 such that for any $k \ge 1$, there holds

$$\limsup_{\epsilon \to 0} \left\| T_k(\rho_{\epsilon}) - T_k(\rho) \right\|_{L^{\gamma+1}(Q_T)}^{\gamma+1} \le \lim_{\epsilon \to 0} \int_0^T \int_\Omega \left[\rho_{\epsilon}^{\gamma} T_k(\rho_{\epsilon}) - \overline{\rho^{\gamma}} \overline{T_k(\rho)} \right] \le C,$$
(5.46)

where $\overline{T_k(\rho)}$ is a weak^{*} limit of $T_k(\rho_{\epsilon})$ in $L^{\infty}(Q_T)$.

Following the lines of argument presented in [6] and using (5.32), we obtain

$$\begin{split} & \limsup_{\epsilon \to 0} \left\| T_k(\rho_{\epsilon}) - T_k(\rho) \right\|_{L^{\gamma+1}(Q_T)}^{\gamma+1} \\ & \leq \lim_{\epsilon \to 0} \int_0^T \int_\Omega \left[\rho_{\epsilon}^{\gamma} T_k(\rho_{\epsilon}) - \overline{\rho^{\gamma}} \overline{T_k(\rho)} \right] \\ & = \frac{\widetilde{\mu}}{a} \lim_{\epsilon \to 0} \int_0^T \int_\Omega \left[(\nabla \cdot u_{\epsilon}) \left(T_k(\rho_{\epsilon}) - \overline{T_k(\rho)} \right) \right] \\ & \leq C \Big(\sup_{\epsilon > 0} \left\| \nabla u_{\epsilon} \right\|_{L^2(Q_T)} \Big) \limsup_{\epsilon \to 0} \left[\left\| T_k(\rho_{\epsilon}) - T_k(\rho) \right\|_{L^2(Q_T)} \\ & + \left\| T_k(\rho) - \overline{T_k(\rho)} \right\|_{L^2(Q_T)} \right] \\ & \leq C \limsup_{\epsilon \to 0} \left\| T_k(\rho_{\epsilon}) - T_k(\rho) \right\|_{L^2(Q_T)}, \end{split}$$

this implies (5.46) by applying Young's inequality and using $\gamma + 1 > 2$.

We now want to show

Claim 1. (ρ, u) is a renormalized solution to the equation $(5.30)_1$.

Observe that (5.35) also holds for $(\rho_{\epsilon}, u_{\epsilon})$ in \mathbb{R}^3 provided it is set to be zero in $\mathbb{R}^3 \setminus \Omega$. Hence we have

$$\partial_t \overline{T_k(\rho)} + \nabla \cdot (\overline{T_k(\rho)}u) + \overline{(T'_k(\rho)\rho - T_k(\rho))\nabla \cdot u} = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^3 \times (0,T)).$$
(5.47)

As in the step 3, we can mollify (5.47) and obtain

$$\partial_t \left(S_m \left[\overline{T_k(\rho)} \right] \right) + \nabla \cdot \left(S_m \left[\overline{T_k(\rho)} \right] u \right) + S_m \left[\overline{[T'_k(\rho)\rho - T_k(\rho)]\nabla \cdot u} \right] = q_m, \tag{5.48}$$

where

$$q_m := \nabla \cdot \left(S_m \Big[\overline{T_k(\rho)} \Big] u \right) - S_m \Big[\nabla \cdot (\overline{T_k(\rho)} u) \Big] \to 0 \text{ in } L^2([0,T], L^s(\Omega)), \text{ as } m \to \infty,$$

for any $s \in [1, 2)$, by virtue of Lemma 2.3 in [26].

Let b be a test function in the definition of renormalized solutions of $(5.30)_1$. Multiplying (5.48) by $b'\left(S_m\left[\overline{T_k(\rho)}\right]\right)$ yields

$$\partial_t \left(b(S_m \left[\overline{T_k(\rho)} \right]) \right) + \nabla \cdot \left(b\left(S_m \left[\overline{T_k(\rho)} \right] \right) u \right) + \left(b'(S_m \left[\overline{T_k(\rho)} \right]) S_m \left[\overline{T_k(\rho)} \right] - b(S_m \left[\overline{T_k(\rho)} \right]) \right) \nabla \cdot u \\ = -b'(S_m \left[\overline{T_k(\rho)} \right]) S_m \left(\overline{[T'(\rho)\rho - T_k(\rho)]} \nabla \cdot u \right) + b'(S_m \left[\overline{T_k(\rho)} \right]) q_m.$$

Sending $m \to +\infty$ in the above equation yields that

$$\partial_t \left(b\left(\overline{T_k(\rho)}\right) \right) + \nabla \cdot \left(b\left(\overline{T_k(\rho)}\right) u \right) + \left(b'\left(\overline{T_k(\rho)}\right) \overline{T_k(\rho)} - b\left(\overline{T_k(\rho)}\right) \right) \nabla \cdot u \\ = -b'\left(\overline{T_k(\rho)}\right) \overline{\left[T'(\rho)\rho - T_k(\rho)\right]} \nabla \cdot u \quad \text{in} \quad \mathcal{D}'\left(\mathbb{R}^3 \times (0,T)\right).$$
(5.49)

On the other hand, for $p \in [1, \gamma)$, we have

$$\left\|\overline{T_k(\rho)} - \rho\right\|_{L^p(Q_T)}^p \le \liminf_{\epsilon \to 0} \left\|T_k(\rho_\epsilon) - \rho_\epsilon\right\|_{L^p(Q_T)}^p.$$
(5.50)

On the other hand, we have

$$\begin{aligned} \|T_k(\rho_{\epsilon}) - \rho_{\epsilon}\|_{L^p(Q_T)}^p &\leq \int_{\{\rho_{\epsilon} \geq k\}} \left|kT(\frac{\rho_{\epsilon}}{k}) - \rho_{\epsilon}\right|^p \\ &\leq 2^p \int_{\{\rho_{\epsilon} \geq k\}} \left|\rho_{\epsilon}\right|^p \left(\text{since } kT(\frac{\rho_{\epsilon}}{k}) \leq \rho_{\epsilon}\right) \\ &\leq 2^p k^{-\gamma+p} \int_{\{\rho_{\epsilon} \geq k\}} \rho_{\epsilon}^{\gamma} \\ &\leq Ck^{-\gamma+p} \to 0, \text{ as } k \to +\infty, \text{uniformly in } \epsilon. \end{aligned}$$
(5.51)

It follows from (5.50) and (5.51) that

$$\lim_{k \to +\infty} \left\| \overline{T_k(\rho)} - \rho \right\|_{L^p(Q_T)} = 0, \text{ for } p \in [1, \gamma).$$
(5.52)

For any M > 0 so large that b'(z) = 0 for $z \ge M$, we set

$$Q_{k,M} := \left\{ (x,t) \in Q_T \mid \overline{T_k(\rho)} \le M \right\}.$$

Then

$$\int_{0}^{T} \int_{\Omega} \left| b'\left(\overline{T_{k}(\rho)}\right) \overline{\left[T_{k}'(\rho)\rho - T_{k}(\rho)\right]\nabla \cdot u} \right| \\
= \int_{Q_{k,M}} \left| b'\left(\overline{T_{k}(\rho)}\right) \overline{\left[T_{k}'(\rho)\rho - T_{k}(\rho)\right]\nabla \cdot u} \right| \\
\leq \sup_{Q_{k,M}} \left| b'(\overline{T_{k}(\rho)}) \right| \int_{Q_{k,M}} \left| \overline{\left[T_{k}'(\rho)\rho - T_{k}(\rho)\right]\nabla \cdot u} \right| \\
\leq \sup_{0 \le z \le M} \left| b'(z) \right| \liminf_{\epsilon \to 0} \int_{Q_{k,M}} \left| \left[T_{k}'(\rho_{\epsilon})\rho_{\epsilon} - T_{k}(\rho_{\epsilon})\right]\nabla \cdot u_{\epsilon} \right| \\
\leq C \liminf_{\epsilon \to 0} \left\| \nabla u_{\epsilon} \right\|_{L^{2}(Q_{T})} \left\| T_{k}'(\rho_{\epsilon})\rho_{\epsilon} - T_{k}(\rho_{\epsilon}) \right\|_{L^{2}(Q_{k,M})} \\
\leq C \liminf_{\epsilon \to 0} \left\| T_{k}'(\rho_{\epsilon})\rho_{\epsilon} - T_{k}(\rho_{\epsilon}) \right\|_{L^{2}(Q_{T})}^{\frac{1}{2} - \frac{1}{2\gamma}} \left\| T_{k}'(\rho_{\epsilon})\rho_{\epsilon} - T_{k}(\rho_{\epsilon}) \right\|_{L^{\gamma+1}(Q_{k,M})}^{\frac{1}{2} + \frac{1}{2\gamma}}.$$
(5.53)

Now we can estimate

$$\left\|T_{k}'(\rho_{\epsilon})\rho_{\epsilon} - T_{k}(\rho_{\epsilon})\right\|_{L^{1}(Q_{T})} \leq 2k^{1-\gamma} \sup_{\epsilon} \|\rho_{\epsilon}\|_{L^{\gamma}}^{\gamma} \leq Ck^{1-\gamma} \to 0, \text{ as } k \to \infty.$$
(5.54)

On the other hand, since $T_k(z)$ is a concave function and $T_k''(z) \leq 0$, we have, by Taylor's expansion, that

$$0 = T_k(z) - T'_k(z)z + \frac{1}{2}T''(\xi z)z^2 \text{ for some } \xi \in (0,1).$$

In particular, we have $T_k'(z)z \leq T_k(z)$ and hence

$$\begin{split} & \left\| T_{k}'(\rho_{\epsilon})\rho_{\epsilon} - T_{k}(\rho_{\epsilon}) \right\|_{L^{\gamma+1}(Q_{k,M})} \leq 2 \left\| T_{k}(\rho_{\epsilon}) \right\|_{L^{\gamma+1}(Q_{k,M})} \\ & \leq 2 \Big(\left\| T_{k}(\rho_{\epsilon}) - T_{k}(\rho) \right\|_{L^{\gamma+1}(Q_{k,M})} + \left\| T_{k}(\rho) - \overline{T_{k}(\rho)} \right\|_{L^{\gamma+1}(Q_{k,M})} + \left\| \overline{T_{k}(\rho)} \right\|_{L^{\gamma+1}(Q_{k,M})} \Big) \\ & \leq 2 \Big(\left\| T_{k}(\rho_{\epsilon}) - T_{k}(\rho) \right\|_{L^{\gamma+1}(Q_{T})} + \left\| T_{k}(\rho) - \overline{T_{k}(\rho)} \right\|_{L^{\gamma+1}(Q_{T})} + M |Q_{k,M}|^{\frac{1}{\gamma+1}} \Big). \end{split}$$

Applying (5.46), we then obtain that there exists C > 0 independent of k such that

$$\limsup_{\epsilon \to 0} \left\| T'_k(\rho_\epsilon)\rho_\epsilon - T_k(\rho_\epsilon) \right\|_{L^{\gamma+1}(Q_{k,M})} \le C \left(1 + M |Q_{k,M}|^{\frac{1}{\gamma+1}} \right) \le C.$$
(5.55)

Substituting (5.54) and (5.55) into (5.53) yields

$$\lim_{k \to \infty} \int_0^T \int_\Omega \left| b' \left(\overline{T_k(\rho)} \right) \overline{\left[T'_k(\rho) \rho - T_k(\rho) \right] \nabla \cdot u} \right| = 0.$$
(5.56)

Sending $k \to \infty$ into the equation (5.49) and applying (5.52), (5.56), we conclude that (ρ, u) is a renormalized solution of the equation (5.30). This proves Claim 1.

5.7. $\rho_{\epsilon} \to \rho$ strongly in $L^{p}(Q_{T})$ for any $1 \le p < \gamma + \theta$. Hence $\overline{\rho^{\gamma}} = \rho^{\gamma}$ a.e. in Q_{T} . It suffices to show that $\rho_{\epsilon} \to \rho$ in $L^{1}(Q_{T})$. This can be done in the exactly same lines as in [6]. Here we sketch it for the readers' convenience. Let $L_{k}(z) \in C^{1}(0, +\infty) \cap C([0, +\infty))$ be defined by

$$L_k(z) = \begin{cases} z \ln z, & 0 \le z \le k \\ z \ln k + z \int_k^z \frac{T_k(s)}{s^2} ds, & z > k. \end{cases}$$

Note that for z large enough, $L_k(z)$ is a linear function, i.e., for $z \ge 3k$,

$$L_k(z) = \beta_k z - 2k$$
, with $\beta_k = \ln k + \int_k^{3k} \frac{T_k(s)}{s^2} ds + \frac{2}{3}$.

Therefore $b_k(z) := L_k(z) - \beta_k z \in C^1(0, +\infty) \cap C([0, +\infty))$ satisfies $b'_k(z) = 0$ for z large enough. Moreover, it is easy to see

$$b'_k(z)z - b_k(z) = T_k(z).$$

Since $(\rho_{\epsilon}, u_{\epsilon})$ is a renormalized solution of the equation $(4.1)_1$ and (ρ, u) is a renormalized solution of the equation $(5.30)_2$, we can take $b(z) = b_k(z)$ in the definition of the renormalized solutions to get that

$$\partial_t L_k(\rho_\epsilon) + \nabla \cdot \left(L_k(\rho_\epsilon) u_\epsilon \right) + T_k(\rho_\epsilon) \nabla \cdot u_\epsilon = 0, \text{ in } \mathcal{D}'(Q_T), \tag{5.57}$$

and

$$\partial_t L_k(\rho) + \nabla \cdot (L_k(\rho)u) + T_k(\rho)\nabla \cdot u = 0, \text{ in } \mathcal{D}'(Q_T).$$
(5.58)

Subtracting (5.57) from (5.58) gives

$$\partial_t \big(L_k(\rho_\epsilon) - L_k(\rho) \big) + \nabla \cdot \big(L_k(\rho_\epsilon) u_\epsilon - L_k(\rho) u \big) + \big(T_k(\rho_\epsilon) \nabla \cdot u_\epsilon - T_k(\rho) \nabla \cdot u \big) = 0, \tag{5.59}$$

in $\mathcal{D}'(Q_T)$.

Since $L_k(z)$ is a linear function for z sufficiently large, we have that $L_k(\rho_{\epsilon})$ is bounded in $L^{\infty}([0,T], L^{\gamma}(\Omega))$, uniformly in ϵ . Thus we have

$$L_k(\rho_{\epsilon}) \rightharpoonup \overline{L_k(\rho)}$$
 weak^{*} in $L^{\infty}([0,T], L^{\gamma}(\Omega))$, as $\epsilon \to 0$.

This, combined with the equation (5.58), implies

$$L_k(\rho_{\epsilon}) \rightharpoonup \overline{L_k(\rho)}$$
 in $C([0,T], L^{\gamma}_{\text{weak}}(\Omega)) \cap C([0,T], H^{-1}(\Omega))$, as $\epsilon \to 0.$ (5.60)

In particular, we have

$$L_k(\rho_\epsilon), \ L_k(\rho) \in C([0,T], L^{\gamma}_{\text{weak}}(\Omega)).$$
 (5.61)

Hence we can multiply the equation (5.59) by $\phi \in C_0^{\infty}(\Omega)$ and integrate the resulting equation over $Q_t, 0 < t \leq T$, to obtain

$$\int_{\Omega} [L_k(\rho_{\epsilon}) - L_k(\rho)](t)\phi$$

=
$$\int_0^t \int_{\Omega} \left\{ [L_k(\rho_{\epsilon})u_{\epsilon} - L_k(\rho)u] \cdot \nabla\phi + [T_k(\rho)\nabla \cdot u - T_k(\rho_{\epsilon})\nabla \cdot u_{\epsilon}]\phi \right\},$$

where we have used the fact that $[L_k(\rho_{\epsilon}) - L_k(\rho)]|_{t=0} = 0$. Taking $\epsilon \to 0$ in the above equation yields

$$\int_{\Omega} \left[\overline{L_k(\rho)} - L_k(\rho) \right](t) \phi = \int_0^t \int_{\Omega} \left[\overline{L_k(\rho)} - L_k(\rho) \right] u \cdot \nabla \phi + \lim_{\epsilon \to 0} \int_0^t \int_{\Omega} \left[T_k(\rho) \nabla \cdot u - T_k(\rho_\epsilon) \nabla \cdot u_\epsilon \right] \phi.$$
(5.62)

As in [6], we can choose $\phi = \phi_m \in C_0^{\infty}(\Omega)$ in (5.62), which approximates the characteristic function of Ω , i.e.,

$$\begin{cases} 0 \le \phi_m \le 1, \quad \phi_m(x) = 1 \text{ for } x \in \Omega \text{ with } \operatorname{dist}(x, \partial \Omega) \ge \frac{1}{m}, \\ \phi_m \to 1 \text{ in } \Omega \text{ as } m \to \infty \text{ and } |\nabla \phi_m(x)| \le 2m \text{ for all } x \in \Omega. \end{cases}$$
(5.63)

We then obtain that for $0 < t \leq T$, it holds

$$\int_{\Omega} \left[\overline{L_k(\rho)} - L_k(\rho) \right](t) = \lim_{\epsilon \to 0} \int_0^t \int_{\Omega} \left[T_k(\rho) \nabla \cdot u - T_k(\rho_\epsilon) \nabla \cdot u_\epsilon \right]$$

Hence we have

$$\begin{split} &\int_{\Omega} \left[\overline{L_k(\rho)} - L_k(\rho) \right](t) \\ &= \int_0^t \int_{\Omega} T_k(\rho) \nabla \cdot u - \lim_{\epsilon \to 0} \int_0^t \int_{\Omega} T_k(\rho_{\epsilon}) \nabla \cdot u_{\epsilon} \\ &= \int_0^t \int_{\Omega} T_k(\rho) \nabla \cdot u + \frac{1}{\tilde{\mu}} \lim_{\epsilon \to 0} \int_0^t \int_{\Omega} (a\rho_{\epsilon}^{\gamma} - \tilde{\mu} \nabla \cdot u_{\epsilon}) T_k(\rho_{\epsilon}) - \frac{a}{\tilde{\mu}} \lim_{\epsilon \to 0} \int_0^t \int_{\Omega} \rho_{\epsilon}^{\gamma} T_k(\rho_{\epsilon}) \\ &= \int_0^t \int_{\Omega} T_k(\rho) \nabla \cdot u + \frac{1}{\tilde{\mu}} \int_0^t \int_{\Omega} \left(a\overline{\rho^{\gamma}} - \tilde{\mu} \nabla \cdot u \right) \overline{T_k(\rho)} - \frac{a}{\tilde{\mu}} \lim_{\epsilon \to 0} \int_0^t \int_{\Omega} \rho_{\epsilon}^{\gamma} T_k(\rho_{\epsilon}) (by (5.32)) \\ &= \int_0^t \int_{\Omega} \left[T_k(\rho) - \overline{T_k(\rho)} \right] \nabla \cdot u - \frac{a}{\tilde{\mu}} \lim_{\epsilon \to 0} \int_0^t \int_{\Omega} \left[\rho_{\epsilon}^{\gamma} T_k(\rho_{\epsilon}) - \overline{\rho^{\gamma}} \overline{T_k(\rho)} \right] \\ &\leq \int_0^t \int_{\Omega} \left[T_k(\rho) - \overline{T_k(\rho)} \right] \nabla \cdot u (by (5.46)) \\ &\leq \left\| T_k(\rho) - \overline{T_k(\rho)} \right\|_{L^2(\{\rho \ge k\})} \left\| \nabla \cdot u \right\|_{L^2(\{\rho \ge k\})} + \left\| T_k(\rho) - \overline{T_k(\rho)} \right\|_{L^2(\{\rho \le k\})} \left\| \nabla \cdot u \right\|_{L^2(\{\rho \le k\})} \right) \\ &\leq C \left(\left\| \nabla \cdot u \right\|_{L^2(\{\rho \ge k\})} + \left\| T_k(\rho) - \overline{T_k(\rho)} \right\|_{L^1(\{\rho \le k\})} \right) \left\| T_k(\rho) - \overline{T_k(\rho)} \right\|_{L^2(\{\rho \le k\})} \right) \\ &\leq C \left(\left\| \nabla \cdot u \right\|_{L^2(\{\rho \ge k\})} + \left\| T_k(\rho) - \overline{T_k(\rho)} \right\|_{L^1(\{\rho \le k\})} \right) \right) \end{split}$$

where we have used (5.46) that guarantees

$$\left\| T_k(\rho) - \overline{T_k(\rho)} \right\|_{L^{\gamma+1}(Q_T)} \le \liminf_{\epsilon \to 0} \left\| T_k(\rho_\epsilon) - T_k(\rho) \right\|_{L^{\gamma+1}(Q_T)} \le C,$$

uniformly in k.

Since T_k is concave, it follows $\overline{T_k(\rho)} \leq T_k(\rho)$. By the definition of T_k , we also have $T_k(\rho) \leq \rho$. Hence we have

$$\begin{aligned} \left\| T_k(\rho) - \overline{T_k(\rho)} \right\|_{L^1(\{\rho \le k\})} &\le \left\| \rho - \overline{T_k(\rho)} \right\|_{L^1(\{\rho \le k\})} \\ &\le \left\| \rho - \overline{T_k(\rho)} \right\|_{L^1(Q_T)} \to 0 \text{ as } k \to \infty \text{ (by (5.52))} \end{aligned}$$

Since $\nabla \cdot u \in L^2(Q_T)$, it follows that

$$\lim_{k \to \infty} \left\| \nabla \cdot u \right\|_{L^2(\{\rho \ge k\})} = 0$$

Therefore we obtain

$$\lim_{k \to \infty} \int_{\Omega} \left[\overline{L_k(\rho)} - L_k(\rho) \right](t) \le 0, \ t \in (0, T).$$
(5.64)

It follows from the definition of L_k that

$$\int_{0}^{T} \int_{\Omega} \left| L_{k}(\rho) - \rho \ln \rho \right| \leq \left\| L_{k}(\rho) - \rho \ln \rho \right\|_{L^{1}(\{\rho \ge k\})} \\
\leq \int \int_{\{\rho \ge k\}} \left| \rho \ln \rho \right| \to 0, \text{ as } k \to +\infty,$$
(5.65)

and

$$\begin{aligned} \left\| L_{k}(\rho_{\epsilon}) - \rho_{\epsilon} \ln \rho_{\epsilon} \right\|_{L^{1}(Q_{T})} &\leq \int \int_{\{\rho_{\epsilon} \geq k\}} \left| L_{k}(\rho_{\epsilon}) - \rho_{\epsilon} \ln \rho_{\epsilon} \right| \\ &\leq \int \int_{\{\rho_{\epsilon} \geq k\}} \frac{\left| L_{k}(\rho_{\epsilon}) \right| + \left| \rho_{\epsilon} \ln \rho_{\epsilon} \right|}{\rho_{\epsilon}^{\gamma}} \rho_{\epsilon}^{\gamma} \\ &\leq C(\delta) \int \int_{\{\rho_{\epsilon} \geq k\}} \frac{\rho_{\epsilon}^{\gamma}}{\rho_{\epsilon}^{\gamma-1-\delta}} \quad (\delta > 0 \text{ is sufficiently small}) \\ &\leq Ck^{-\gamma+1+\delta} \to 0, \text{ as } k \to +\infty, \text{ uniformly in } \epsilon, \end{aligned}$$
(5.66)

so that by the lower semicontinuity we have

$$\lim_{k \to \infty} \left\| \overline{L_k(\rho)} - \overline{\rho \ln \rho} \right\|_{L^1(Q_T)} \le \lim_{k \to \infty} \liminf_{\epsilon \to 0} \left\| L_k(\rho_\epsilon) - \rho_\epsilon \ln \rho_\epsilon \right\|_{L^1(Q_T)} = 0.$$
(5.67)

Combining (5.64), (5.66), (5.66), with (5.67) implies that

$$\int_{\Omega} \left[\overline{\rho \ln \rho} - \rho \ln \rho \right](t) \le 0, \quad t \in (0, T)$$

Since $\overline{\rho \ln \rho} \ge \rho \ln \rho$ a.e. in Q_T , this implies that

$$\overline{\rho \ln \rho} = \rho \ln \rho$$
 a.e. in Q_T .

By the convexity of the function $\omega(z) = z \ln z : (0, +\infty) \to \mathbb{R}$, this implies that

$$\rho_{\epsilon} \to \rho \text{ in } L^1(Q_T).$$

Since ρ_{ϵ} is bounded in $L^{\gamma+\theta}(Q_T)$, it follow from a simple interpolation that

$$\rho_{\epsilon} \to \rho \text{ in } L^p(Q_T) \text{ for any } 1 \le p < \gamma + \theta.$$

Thus $\overline{\rho^{\gamma}} = \rho^{\gamma}$ a.e. in Q_T .

The energy inequality (1.6) for (ρ, u, d) follows from the energy inequality (4.2) for $(\rho_{\epsilon}, u_{\epsilon}, d_{\epsilon})$. In fact, (4.2) implies that for almost all $0 < t < \infty$, it holds

$$\mathbf{F}_{\epsilon}(t) + \int_{0}^{t} \int_{\Omega} \left(\mu |\nabla u_{\epsilon}|^{2} + \widetilde{\mu} |\nabla \cdot u_{\epsilon}|^{2} + |\Delta d_{\epsilon} + \frac{1}{\epsilon^{2}} (1 - |d_{\epsilon}|^{2}) d_{\epsilon}|^{2} \right) \leq \mathbf{F}_{\epsilon}(0) = \mathbf{E}(0).$$
(5.68)

On the other hand, by the lower semicontinuity, we have that for almost all $t \in (0, +\infty)$

$$\mathbf{E}(t) + \int_{0}^{t} \int_{\Omega} \left(\mu |\nabla u|^{2} + \widetilde{\mu} |\nabla \cdot u|^{2} + |\Delta d + |\nabla d|^{2} d|^{2} \right)$$

$$\leq \liminf_{\epsilon \to 0} \left\{ \mathbf{F}_{\epsilon}(t) + \int_{0}^{t} \int_{\Omega} \left(\mu |\nabla u_{\epsilon}|^{2} + \widetilde{\mu} |\nabla \cdot u_{\epsilon}|^{2} + |\Delta d_{\epsilon} + \frac{1}{\epsilon^{2}} (1 - |d_{\epsilon}|^{2}) d_{\epsilon} |^{2} \right) \right\}, \quad (5.69)$$

where we have used the observation that

$$\Delta d_{\epsilon} + \frac{1}{\epsilon^2} (1 - |d_{\epsilon}|^2) d_{\epsilon} = \partial_t d_{\epsilon} + u_{\epsilon} \cdot \nabla d_{\epsilon} \rightharpoonup \partial_t d + u \cdot \nabla d = \Delta d + |\nabla d|^2 d \text{ in } L^2(Q_t).$$

It is clear that (5.68) and (5.69) imply (1.6).

After these steps, we conclude that (ρ, u, d) is a global finite energy weak solution of the system (1.1), under the initial and boundary condition (1.2), that satisfies the properties (i) of Theorem 1.1.

The property (ii) for (u, d) follows from the strong convergence of d_{ϵ} to d in $L^2_{\rm loc}((0, +\infty), H^1_{\rm loc}(\Omega))$. In fact, it is easy to see that $d_{\epsilon} \in L^2_{\rm loc}((0, +\infty), H^2_{\rm loc}(\Omega))$. For any $X \in C^1_0(\Omega, \mathbb{R}^3)$ and $\eta \in C^1_0((0, +\infty))$, we can multiply the equation $(4.1)_3$ by $\eta(t)X(x) \cdot \nabla d_{\epsilon}(x)$ and integrate the resulting equation over $\Omega \times (0, +\infty)$ and apply the integration by parts a few times to obtain

$$\int_{0}^{\infty} \eta(t) \int_{\Omega} \left(e_{\epsilon}(d_{\epsilon}) \nabla \cdot X - \nabla d_{\epsilon} \odot \nabla d_{\epsilon} : \nabla X \right) = \int_{0}^{\infty} \eta(t) \int_{\Omega} \left\langle \partial_{t} d_{\epsilon} + u_{\epsilon} \cdot \nabla d_{\epsilon}, X \cdot \nabla d_{\epsilon} \right\rangle, \quad (5.70)$$

where $e_{\epsilon}(d_{\epsilon}) := \frac{1}{2} |\nabla d_{\epsilon}|^2 + \frac{1}{4\epsilon^2} (1 - |d_{\epsilon}|^2)^2$. Since

$$\partial_t d_\epsilon + u_\epsilon \nabla d_\epsilon \rightharpoonup \partial_t d + u \cdot \nabla d$$
 in $L^2(\Omega \times (0, +\infty))$, as $\epsilon \to 0$.

we obtain, by sending $\epsilon \to 0$ in (5.70) and applying both Theorem 3.1 and 5.1, that

$$\int_{0}^{T} \eta(t) \int_{\Omega} \left(\frac{1}{2} |\nabla d|^{2} \nabla \cdot X - \nabla d \odot \nabla d : \nabla X\right) = \int_{0}^{T} \eta(t) \int_{\Omega} \left\langle \partial_{t} d + u \cdot \nabla d, X \cdot \nabla d \right\rangle.$$
(5.71)
roof of Theorem 1.1 is now complete.

The proof of Theorem 1.1 is now complete.

6. Large time behavior of finite energy solutions and proof of corollary 1.2

In this section, we will study the large time asymptotic behavior of the global finite energy weak solutions obtained in Theorem 1.1 and give a proof of Corollary 1.2.

Proof of Corollary 1.2:

First it follows from (1.6) that

$$\operatorname{esssup}_{t>0} \mathbf{E}(t) + \int_0^\infty \int_\Omega \left(\mu |\nabla u|^2 + |\Delta d + |\nabla d|^2 d|^2 \right) \le \mathbf{E}(0).$$
(6.1)

For any positive integer m, define $(\rho_m, u_m, d_m) : Q_1 \to \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{S}^2$ by

$$\begin{cases} \rho_m(x,t) = \rho(x,t+m), \\ u_m(x,t) = u(x,t+m), \\ d_m(x,t) = d(x,t+m). \end{cases}$$

Then (ρ_m, u_m, d_m) is a sequence of finite energy weak solutions of (1.1) in Q_1 . It follows from (6.1) that

$$\begin{aligned} \|\rho_m\|_{L^{\infty}([0,1],L^{\gamma}(\Omega))} + \|\rho_m^{\frac{1}{2}}u_m\|_{L^{\infty}([0,1],L^{2}(\Omega))} + \|\rho_m u_m\|_{L^{\infty}([0,1],L^{\frac{2\gamma}{2\gamma+1}}(\Omega))} \\ + \|d_m\|_{L^{\infty}([0,1],H^{1}(\Omega))} \le C(\mathbf{E}(0)), \end{aligned}$$
(6.2)

and

$$\lim_{m \to \infty} \int_0^1 \left(\left\| \nabla u_m \right\|_{L^2(\Omega)}^2 + \left\| \Delta d_m + |\nabla d_m|^2 d_m \right\|_{L^2(\Omega)}^2 \right) = 0.$$
(6.3)

After passing to a subsequence, we may assume that as $m \to \infty$,

$$\rho_m \rightharpoonup \rho_\infty \text{ in } L^{\gamma}(Q_1), \ u_m \rightharpoonup u_\infty \text{ in } L^2([0,1], H^1_0(\Omega)), \ d_m \rightharpoonup d_\infty \text{ in } L^2([0,1], H^1(\Omega)).$$

Applying (6.3) and the Poincaré inequality, we have

$$\lim_{m \to \infty} \int_0^1 \|u_m\|_{L^2(\Omega)}^2 = 0,$$

and hence $u_{\infty} = 0$ a.e. in Q_1 .

Sending $m \to \infty$ in $(1.1)_3$, we see that d_∞ solves

$$\partial_t d_\infty = \Delta d_\infty + |\nabla d_\infty|^2 d_\infty$$
 in Q_1 .

On the other hand, by the lower semicontinuity and (6.3) we have

$$\int_0^1 \int_\Omega \left| \Delta d_\infty + |\nabla d_\infty|^2 d_\infty \right|^2 = 0.$$

Hence $\partial_t d_{\infty} = 0$ in Q_1 and $d_{\infty}(x,t) = d_{\infty}(x) \in H^1(\Omega, \mathbb{S}^2_+)$ is a harmonic map, with $d_{\infty} = d_0$ on $\partial\Omega$. By Hölder's inequality, (6.2), and (6.3), we have

$$\lim_{m \to \infty} \int_{0}^{1} \left(\left\| \rho_{m} u_{m} \right\|_{L^{\frac{6\gamma}{\gamma+6}}(\Omega)}^{2} + \left\| \rho_{m} |u_{m}|^{2} \right\|_{L^{\frac{3\gamma}{\gamma+3}}(\Omega)}^{2} \right) = 0.$$
(6.4)

Since (ρ_m, u_m, d_m) solves $(1.1)_1$ in Q_1 , we have

 $\partial_t(\rho_m u_m) + \nabla \cdot (\rho_m u_m \otimes u_m) + a \nabla \rho_m^{\gamma} = \mu \Delta u_m + \widetilde{\mu} \nabla (\nabla \cdot u_m) - (\Delta d_m + |\nabla d_m|^2 d_m) \cdot \nabla d_m \text{ in } Q_1,$ which, after sending $m \to \infty$ and applying (6.2), (6.3), (6.4), and Claim 3 below, implies

$$\nabla \rho_{\infty}^{\gamma} = 0$$
 in Q_1 .

Hence ρ_{∞} is x-independent in Q_1 . On the other hand, since ρ_{∞} is a weak solution of

$$\partial_t \rho_\infty + \nabla \cdot (\rho_\infty u_\infty) = 0 \quad \text{in} \quad Q_{1,2}$$

so that $\partial_t \rho_{\infty} = 0$ and ρ_{∞} is t-independent in Q_1 . Thus ρ_{∞} is a constant.

It remains to show $(\rho_m, d_m) \to (\rho_\infty, d_\infty)$ in $L^{\gamma}(Q_1) \times L^2([0, 1], H^1_{\text{loc}}(\Omega))$. This is divided into two separate claims.

Claim 2. $d_m \to d_\infty$ in $L^2([0,1], H^1_{loc}(\Omega))$. The idea is based on the compactness Theorem 3.2, and the argument is similar to that given in §5.1 and [18] Theorem 1.3. For the convenience of readers, we sketch it here. As in §5.1, for $\Lambda > 1$ define

$$G_{\Lambda} = \Big\{ t \in [0,1] \ \Big| \ \liminf_{m \to \infty} \int_{\Omega} \Big| \Delta d_m + |\nabla d_m|^2 d_m \Big|^2 \le \Lambda \Big\},$$

and

$$B_{\Lambda} = [0,1] \setminus G_{\Lambda}$$

From (6.3), we have

$$\left|B_{\Lambda}\right| \leq \Lambda^{-1} \liminf_{m \to \infty} \int_{0}^{1} \int_{\Omega} \left|\Delta d_{m} + |\nabla d_{m}|^{2} d_{m}\right|^{2} = 0.$$

$$(6.5)$$

Since d_m satisfies (1.12) for any $X \in C_0^1(\Omega)$ and $\eta \in C_0^1((0,1))$, it is not hard to check that there exists a subset $Z \subset G_\Lambda$, with |Z| = 0, such that for any $t \in G_\Lambda \setminus Z$, it holds

$$\int_{\Omega} \left(\nabla d_m \odot \nabla d_m - \frac{1}{2} |\nabla d_m|^2 \mathbb{I}_3 \right)(t) : \nabla X = -\int_{\Omega} \left\langle (\partial_t d_m + u_m \cdot \nabla d_m)(t), X \cdot \nabla d_m(t) \right\rangle$$
$$= -\int_{\Omega} \left\langle (\Delta d_m + |\nabla d_m|^2 d_m)(t), X \cdot \nabla d_m(t) \right\rangle \quad (by \ (1.1)_3)$$
(6.6)

It is standard (see [23]) that (6.6) implies that $d_m(t)$, $t \in G_{\Lambda} \setminus Z$, satisfies the almost energy monotonicity inequality (3.3), i.e., $x_0 \in \Omega$ and $0 < r \le R < d(x_0, \partial\Omega)$,

$$\Psi_R(d_m(t), x_0) \ge \Psi_r(d_m(t), x_0) + \frac{1}{2} \int_{B_R(x_0) \setminus B_r(x_0)} |x - x_0|^{-1} \left| \frac{\partial d_m(t)}{\partial |x - x_0|} \right|^2, \tag{6.7}$$

where

$$\Psi_r(d_m(t), x_0) = \frac{1}{r} \int_{B_r(x_0)} \left(\frac{1}{2} |\nabla d_m|^2(t) - \langle (x - x_0) \cdot \nabla d_m(t), \tau_m(t) \rangle \right) + \frac{1}{2} \int_{B_r(x_0)} |x - x_0| |\tau_m(t)|^2,$$

and

$$\overline{T}_m(t) = (\Delta d_m + |\nabla d_m|^2 d_m)(t).$$

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From the definition of G_{Λ} , we have

$$\left\|\tau_m(t)\right\|_{L^2(\Omega)} \le \Lambda, \ \forall t \in G_\Lambda \setminus Z.$$
(6.8)

From (6.2), we see

$$E(d_m(t)) = \frac{1}{2} \int_{\Omega} |\nabla d_m(t)|^2 \le C(\mathbf{E}(0)).$$
(6.9)

Note also that

$$d_m^3(x,t) \ge 0 \text{ a.e. } x \in \Omega, \ \forall \ t \in G_\Lambda \setminus Z.$$
(6.10)

From (6.7), (6.8), (6.9), and (6.10), we conclude that $\{d_m(t)\}_{m\geq 1} \subset \mathbf{Y}(C(\mathbf{E}(0)), \Lambda, 0; \Omega)$ for any $t \in G_{\Lambda} \setminus Z$. Hence, by Theorem 3.2, we have that $\{d_m\}_{m\geq 1}$ is bounded in $H^2_{\text{loc}}(\Omega, \mathbb{S}^2)$ and precompact in $H^1(\Omega, \mathbb{S}^2)$.

Since

$$\partial_t d_m = -u_m \cdot \nabla d_m + (\Delta d_m + |\nabla d_m|^2 d_m) \in L^2([0,1], L^{\frac{3}{2}}(\Omega)) + L^2([0,1], L^2(\Omega)),$$

and

$$\sup_{m \ge 1} \left\| \partial_t d_m \right\|_{L^2([0,1], L^{\frac{3}{2}}(\Omega)) + L^2([0,1], L^2(\Omega))} \le C.$$

We can apply Aubin-Lions' lemma, similar to §5.1, to conclude that for any open set $\Omega \subset \Omega$, after taking a subsequence, there holds

$$\lim_{m \to \infty} \left\| \nabla (d_m - d_\infty) \right\|_{L^2(\widetilde{\Omega} \times (G_\Lambda \setminus Z))} = 0.$$
(6.11)

On the other hand, by (6.2), we have

$$\sup_{m\geq 1} \left\| \nabla (d_m - d_\infty) \right\|_{L^2(\tilde{\Omega} \times (B_\Lambda \cup Z))} \leq C(\mathbf{E}(0)) \left| B_\Lambda \cup Z \right| = 0.$$
(6.12)

Putting (6.11) and (6.12) together yields

$$\lim_{m \to \infty} \left\| \nabla (d_m - d_\infty) \right\|_{L^2(\tilde{\Omega} \times (0,1))} = 0.$$
(6.13)

Claim 2 follows from (6.13).

Claim 3. $\rho_m \to \rho_{0,\infty}$ in $L^{\gamma}(Q_1)$. To show this claim, first observe that by the same lines of argument in §5.3 with $(\rho_{\epsilon}, u_{\epsilon}, d_{\epsilon})$ replaced by (ρ_m, u_m, d_m) , we can obtain that there exist $\theta > 0$ and C > 0independent of m such that

$$\int_0^1 \int_\Omega \rho_m^{\gamma+\theta} \le C, \ \forall m \ge 1.$$
(6.14)

From (6.14), we may assume that

$$\rho_m^{\gamma} \rightharpoonup \overline{\rho_{\infty}^{\gamma}} \text{ in } L^{p_1}(Q_1), \ 1 < p_1 \le \frac{\gamma + \theta}{\gamma}(Q_1).$$
(6.15)

There are two methods to prove that $\overline{\rho_{\infty}^{\gamma}} = \rho_{\infty}^{\gamma}$ a.e. in Q_1 and $\rho_m \to \rho_{\infty}$ in $L^{\gamma}(Q_1)$: the first is to repeat the same lines of arguments given by §5.5, §5.6, and §5.7 with $(\rho_{\epsilon}, u_{\epsilon}, d_{\epsilon})$ replaced by (ρ_m, u_m, d_m) ; and the second is to apply the div-curl lemma, similar to [5] Proposition 4.1. Here we sketch it. For simplicity, assume the pressure coefficient a = 1. Let Div and Curl denote the divergence and curl operators in Q_1 . As pointed out by [6] Remark 1.1, (1.4) also holds for $b(\rho_m) = G(\rho_m^{\gamma})$ when $G(z) = z^{\alpha}$, with

$$0 < \alpha < \min\left\{\frac{1}{2\gamma}, \frac{\theta}{\theta + \gamma}\right\}.$$

Using the equation (1.4), one can check that

 $\operatorname{Div}[0,0,0,G(\rho_m^{\gamma})]$ is precompact in $W^{-1,q_1}(Q_1)$

for some $q_1 > 1$.

While, using the equation $(1.1)_2$ and (6.2), one can check

 $\operatorname{Curl}[0,0,0,\rho_m^{\gamma}]$ is precompact in $W^{-1,q_2}(Q_1)$

for some $q_2 > 1$.

Assume

$$G(\rho_m^{\gamma}) \rightharpoonup \overline{G(\rho_{\infty}^{\gamma})}$$
 in $L^{p_2}(Q_1)$,

and

$$G(\rho_m^{\gamma})\rho_m^{\gamma} \rightharpoonup \overline{G(\rho_\infty^{\gamma})\rho_\infty^{\gamma}} \text{ in } L^r(Q_1),$$

with

$$p_2 = \frac{1}{\alpha}, \ \frac{1}{r} = \frac{1}{p_2} + \frac{1}{p_1}.$$

Then by the div-curl lemma we conclude that

$$\overline{G(\rho_{\infty}^{\gamma})\rho_{\infty}^{\gamma}} = \overline{G(\rho_{\infty}^{\gamma})} \ \overline{\rho_{\infty}^{\gamma}}$$

As G is strictly monotone, this implies $\overline{G(\rho_{\infty}^{\gamma})} = G(\overline{\rho_{\infty}^{\gamma}})$. Since L^{p_2} is uniformly convex, this implies that the convergence in (6.15) is strong in $L^1(Q_1)$. Hence we have that

 $\rho_m \to \rho_\infty \text{ in } L^{\gamma}(Q_1).$

Since $\int_{\Omega} \rho_m(t) = \int_{\Omega} \rho_0$ for 0 < t < 1 and ρ_∞ is constant, it follows that $\rho_\infty \equiv \frac{1}{|\Omega|} \int_{\Omega} \rho_0$ (:= $\rho_{0,\infty}$).

From claim 2, claim 3, and (6.2), we can apply Fubini's theorem to conclude that there exists $t_m \in (m, m+1)$ such that as $m \to \infty$,

$$(\rho(t_m), d(t_m)) \to (\rho_{0,\infty}, d_\infty)$$
 in $L^{\gamma}(\Omega) \times H^1_{\text{loc}}(\Omega, \mathbb{S}^2)$,

and

$$\left\| u(t_m) \right\|_{H^1(\Omega)} \to 0.$$

Hence by Sobolev's embedding theorem we have that $u(t_m) \to 0$ in $L^p(\Omega)$ for any 1 . The proof is now complete.

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