# On partial regularity for the $3 D$ non-stationary Hall magnetohydrodynamics equations on the plane 

Dongho Chae* and Jörg Wolf ${ }^{\dagger}$<br>*Department of Mathematics<br>Chung-Ang University<br>Seoul 156-756, Republic of Korea<br>e-mail: dchae@cau.ac.kr<br>and<br>$\dagger$ Department of Financial Engineering<br>Ajou University<br>Suwon 443-749, Republic of Korea<br>e-mail: jwolf@math.hu-berlin.de


#### Abstract

We study partial regularity of weak solutions of the 3D valued non-stationary Hall magnetohydrodynamics equations on $\mathbb{R}^{2}$. In particular we prove the existence of a weak solution whose set of possible singularities has the space-time Hausdorff dimension at most two.


AMS Subject Classification Number: 35Q35, 35Q85,76W05
keywords: non-stationary Hall-MHD equations, partial regularity

## 1 Introduction and the main theorem

We consider the homogeneous incompressible Hall magnetohydrodynamics(Hall-MHD) equations:

$$
\left\{\begin{array}{l}
\frac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{u} \cdot \nabla \boldsymbol{u}+\nabla p=(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}+\nu \Delta \boldsymbol{u}+\boldsymbol{f} \\
\frac{\partial \boldsymbol{B}}{\partial t}-\nabla \times(\boldsymbol{u} \times \boldsymbol{B})+\nabla \times((\nabla \times \boldsymbol{B}) \times \boldsymbol{B})=\mu \Delta B+\nabla \times \boldsymbol{g} \\
\nabla \cdot \boldsymbol{u}=0, \nabla \cdot \boldsymbol{B}=0
\end{array}\right.
$$

where the three dimensional vector fields $\boldsymbol{u}=\boldsymbol{u}(x, t)$ and $\boldsymbol{B}=\boldsymbol{B}(x, t)$ are the fluid velocity and the magnetic field respectively. The scalar field $p=p(x, t)$ is the pressure, while the positive constants $\nu$ and $\mu$ represent the viscosity and the magnetic resistivity
respectively. The given vector fields $\boldsymbol{f}$ and $\nabla \times \boldsymbol{g}$ are external forces on the magnetically charged fluid flows. The system has been studied first by Lighthill [13] in 1960. We notice that comparing with the usual MHD system, the Hall-MHD system contains the extra term $\nabla \times((\nabla \times \boldsymbol{B}) \times \boldsymbol{B})$, called the Hall term. The inclusion of this term is essential in understanding the phenomena of magnetic reconnection, meaning the change of the topology of magnetic field lines. This is observed in real physical situations such as space plasma [9, 11], star formation [21] and neutron star [19]. There are also many other physical phenomena that requires the Hall-MHD system to describe them (see e.g. [15, 20, 16] and the references therein). The Hall term is quadratically nonlinear, containing the second order derivative, and it causes major difficulties in the mathematical study of the Hall-MHD system. Thanks to the orthogonality in $L^{2}$ of the Hall term with $\boldsymbol{B}$, however, the energy inequality similar to the usual MHD case holds true. Using this fact the construction of the global in time weak solution can be achieved without any difficulties, as has been shown in [1]. Observing similar cancellation properties of the Hall term, the local in time well-posedness as well as the global in time well-posedness for small initial data was also established in [3], and has been refined in (4]. Regarding the question of energy conservation for weak solutions in the inviscid case we refer to [7]. For a special form of axially symmetric initial data the authors of [8] proved the global in time existence of classical solutions to the system. On the other hand, the optimal temporal decay estimates are obtained in [5].

Concerning the regularity of weak solutions, one can expect that the problem is more difficult than the Navier-Stokes equations and the usual MHD system. Even the problem of regularity of stationary weak solution has essential difficulty with current methods of the regularity theory, which is contrary to the case of the stationary Navier-Stokes equations. The partial regularity of stationary weak solutions has been obtained recently by the current authors (cf. [6]). In the present paper we investigate the partial regularity of weak solutions of the non-stationary system. For the Navier-Stokes equations there are many publications on this direction of study (see e.g. [18, 2, ,12, 14, 23]). In the case of the 3D Hall-MHD system in $\mathbb{R}^{3}$, however, we encounter again essential difficulties in constructing suitable weak solutions, satisfying desired form of localized energy inequality.

In the current paper we focus on the case of 3D valued Hall-MHD system on the plane, which is sometimes called the $2 \frac{1}{2}$ dimensional system. Physically the situation corresponds to the full 3D system having the translational symmetry in the $x_{3}$ direction. In this case, as will be shown in detail below, although we cannot construct suitable weak solution, satisfying the localized energy inequality, instead, we could construct an approximate system, for which we can deduce Caccioppoli-type inequalities to obtain "approximate singular set", and then by passing to a limit in an appropriate sense, we can show that there exists a possible singular set for the limiting weak solution, whose Hausdorff dimension is at most two. When we try to apply the similar idea to the full 3 D non-stationary system defined on $\mathbb{R}^{3}$, we have difficulty in constructing a sequence of the approximate weak solutions, the compactness of which is strong enough to pass to the limit. Therefore, we leave the proof of partial regularity of the full 3D non-stationary system as an open problem. We now formulate our problem more precisely, and state our main result.

We concentrate on the following $3 D$ valued Hall-MHD system in $Q=\mathbb{R}^{2} \times(0, T)$.

$$
\begin{align*}
\partial_{t} \boldsymbol{u}+(\boldsymbol{u} \cdot \nabla) \boldsymbol{u}-\Delta \boldsymbol{u} & =-\nabla p+(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}+\boldsymbol{f},  \tag{1.1}\\
\partial_{t} \boldsymbol{B}+\nabla \times(\boldsymbol{B} \times \boldsymbol{u})-\Delta \boldsymbol{B} & =-\nabla \times((\nabla \times \boldsymbol{B}) \times \boldsymbol{B})+\nabla \times \boldsymbol{g},  \tag{1.2}\\
\nabla \cdot \boldsymbol{u} & =0, \quad \nabla \cdot \boldsymbol{B}=0 \tag{1.3}
\end{align*}
$$

together with the initial condition

$$
\begin{equation*}
\boldsymbol{u}=\boldsymbol{u}_{0}, \quad \boldsymbol{B}=\boldsymbol{B}_{0} \quad \text { on } \quad \mathbb{R}^{2} \times\{0\} \tag{1.4}
\end{equation*}
$$

which satisfy

$$
\begin{equation*}
\nabla \cdot \boldsymbol{u}_{0}=\nabla \cdot \boldsymbol{B}_{0}=0 \quad \text { on } \quad \mathbb{R}^{2} \tag{1.5}
\end{equation*}
$$

Here, $\boldsymbol{u}=\left(u_{1}, u_{2}, u_{3}\right), \boldsymbol{B}=\left(B_{1}, B_{2}, B_{3}\right)$, where $u_{j}=u_{j}\left(x_{1}, x_{2}, t\right), B_{j}=B_{j}\left(x_{1}, x_{2}, t\right), j=$ $1,2,3$, and $p=p\left(x_{1}, x_{2}, t\right),(x, t)=\left(x_{1}, x_{2}, t\right) \in Q$. Note that we set $\nu=\mu=1$ for convenience. For the definition of weak solution see Definition 1.1 below. The aim of the present paper is to prove the existence of a weak solution to the Hall-MHD system (1.1)-(1.3), which is Hölder continuous outside of a possible singular set together with the estimation of its Hausdorff dimension. We set $L_{\text {div }}^{2}=\left\{\boldsymbol{u} \in L^{2} \mid \nabla \cdot \boldsymbol{u}=0\right\}$, where the derivative is defined in the sense of distribution. We also define $V^{2}(Q)=L^{\infty}\left(0, T ; L^{2}\right) \cap$ $L^{2}\left(0, T ; W^{1,2}\right)$. By $V_{\text {div }}^{2}(Q)$ we denote the space of all $\boldsymbol{u} \in V^{2}(Q)$ such that $\nabla \cdot \boldsymbol{u}=0$ in the sense of distribution in $Q$.

Notice that using the formula $(\boldsymbol{u} \cdot \nabla) \boldsymbol{u}=(\nabla \times \boldsymbol{u}) \times \boldsymbol{u}+\frac{1}{2}|\boldsymbol{u}|^{2}$, one can rewrite (1.1) into

$$
\begin{equation*}
\partial_{t} \boldsymbol{u}+(\nabla \times \boldsymbol{u}) \times \boldsymbol{u}-\Delta \boldsymbol{u}=-\nabla\left(p+\frac{|\boldsymbol{u}|^{2}}{2}\right)+(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}+\boldsymbol{f} \quad \text { in } \quad Q \tag{1.6}
\end{equation*}
$$

Applying $\nabla \times$ to the both sides of the above, we get

$$
\begin{equation*}
\partial_{t} \boldsymbol{\omega}+\nabla \times(\boldsymbol{\omega} \times \boldsymbol{u})-\Delta \boldsymbol{\omega}=\nabla \times((\nabla \times \boldsymbol{B}) \times \boldsymbol{B})+\nabla \times \boldsymbol{f} \quad \text { in } \quad Q, \tag{1.7}
\end{equation*}
$$

where $\boldsymbol{\omega}$ stands for the vorticity $\nabla \times \boldsymbol{u}$. Taking the sum of (1.2) and (1.7), we are led to

$$
\begin{equation*}
\partial_{t} \boldsymbol{V}+\nabla \times(\boldsymbol{V} \times \boldsymbol{u})-\Delta \boldsymbol{V}=\nabla \times(\boldsymbol{f}+\boldsymbol{g}) \quad \text { in } \quad Q, \tag{1.8}
\end{equation*}
$$

where

$$
\begin{equation*}
\boldsymbol{V}=\boldsymbol{B}+\boldsymbol{\omega} \tag{1.9}
\end{equation*}
$$

Since $\nabla \cdot \boldsymbol{V}=0$, there exists a solenoidal potential $\boldsymbol{v}$ such that $\nabla \times \boldsymbol{v}=\boldsymbol{V}$. From (1.8) we deduce that $\boldsymbol{v}$ solves the following system in $Q$,

$$
\begin{align*}
\nabla \cdot \boldsymbol{v} & =0  \tag{1.10}\\
\partial_{t} \boldsymbol{v}+(\boldsymbol{v} \cdot \nabla) \boldsymbol{v}-\Delta \boldsymbol{v} & =-\nabla \pi+(\nabla \times \boldsymbol{v}) \times \boldsymbol{b}+\boldsymbol{f}+\boldsymbol{g}, \tag{1.11}
\end{align*}
$$

where $\boldsymbol{b}=\boldsymbol{v}-\boldsymbol{u}$. Clearly, $\nabla \times \boldsymbol{b}=\boldsymbol{B}$.
We now introduce the notion of a weak solution to the system (1.1)-(1.5).

Definition 1.1. Let $\boldsymbol{f}, \boldsymbol{g} \in L^{2}(Q)$. We say $(\boldsymbol{u}, p, \boldsymbol{B}) \in V_{\text {div }}^{2}(Q) \times L^{2}\left(0, T ; L_{\mathrm{loc}}^{2}\right) \times V_{\text {div }}^{2}(Q)$ is a weak solution to (1.3)-(1.4) if

$$
\begin{align*}
& \int_{Q}\left(-\boldsymbol{u} \cdot \partial_{t} \boldsymbol{\varphi}+\nabla \boldsymbol{u}: \nabla \boldsymbol{\varphi}-\boldsymbol{u} \otimes \boldsymbol{u}: \nabla \boldsymbol{\varphi}\right) d x d t \\
& \quad=\int_{Q} p \nabla \cdot \boldsymbol{\varphi} d x d t+\int_{Q}((\nabla \times \boldsymbol{B}) \times \boldsymbol{B}+\boldsymbol{f}) \cdot \boldsymbol{\varphi} d x d t+\int_{\mathbb{R}^{2}} \boldsymbol{u}_{0} \cdot \boldsymbol{\varphi}(0) d x  \tag{1.12}\\
& \int_{Q}\left(\boldsymbol{B} \cdot \partial_{t} \boldsymbol{\varphi}+\nabla \boldsymbol{B}: \nabla \boldsymbol{\varphi}+\boldsymbol{B} \times \boldsymbol{u}: \nabla \times \boldsymbol{\varphi}\right) d x d t \\
& \quad=\int_{Q}((\nabla \times \boldsymbol{B}) \times \boldsymbol{B}+\boldsymbol{g}) \cdot \nabla \times \boldsymbol{\varphi} d x d t+\int_{\mathbb{R}^{2}} \boldsymbol{B}_{0} \cdot \boldsymbol{\varphi}(0) d x \tag{1.13}
\end{align*}
$$

for all $\boldsymbol{\varphi} \in C_{\mathrm{c}}^{\infty}\left(\mathbb{R}^{2} \times[0, T)\right)$. Here we used the notation $\boldsymbol{A}: \boldsymbol{B}=\sum_{i, j=1}^{3} A_{i j} B_{i j}$ for matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{3 \times 3}$.

By $\mathcal{M}_{\text {loc }}^{2, \lambda}(Q)$ we denote the local Morrey space, which is defined in Section 3 below. Our main result is the following theorem.

Theorem 1.2. Let $\boldsymbol{u}_{0} \in L_{\mathrm{div}}^{2}, \boldsymbol{B}_{0} \in L^{2}$ and $\boldsymbol{f}, \boldsymbol{g} \in L^{2}(Q)$. Moreover, we suppose that $\boldsymbol{g} \in \mathcal{M}_{\mathrm{loc}}^{2, \lambda}(Q)$ for some $2<\lambda<4$. Then, there exists a weak solution $(\boldsymbol{u}, p, \boldsymbol{B}) \in$ $V_{\text {div }}^{2}(Q) \times L^{2}\left(0, T ; L_{\text {loc }}^{2}\right) \times V_{\text {div }}^{2}(Q)$ of (1.1) $-(1.5)$ being $\alpha$-Hölder continuous outside of a closed subset set $\Sigma(\boldsymbol{B}) \subset Q$ of Hausdorff dimension less than or equal to two, where $0<\alpha<\frac{\lambda-2}{2}$.

The paper is organized as follows. In Section 2 we discuss local estimates of weak solutions to the approximate system related to (1.2) involving the magnetic field $\boldsymbol{B}$. Thanks to the validity of the local energy equality (see (2.4) below) we are able to establish a Caccioppoli-type inequality, which plays a central role in the proof of the fundamental estimate in Section 3 (cf. Lemma3.2). To achieve this result we make use of an indirect argument together with the fundamental estimate which holds true for the corresponding linear limit system (cf. Lemma3.1). The aim of section Section 4 is the construction of an approximate solution to system (1.1)-(1.5) along with the required a priori estimates. Furthermore, passing to the limit in the approximate system we get a weak solution to (1.1)-(1.5). In Section 5 we prove that the weak solution constructed in Section 4 fulfills the required partial regularity property stated in Theorem1.2, the main result of the paper. We wish to remark that even for the weak solution to the system under consideration constructed in a suitable way, a corresponding local energy inequality similar to the case of the Navier-Stokes equation may not be available. For this reason in the proof of the main theorem we are only able to work on the approximate solutions using Lemma 3.2. The estimation of the parabolic Hausdorff dimension of the singular set is obtained by Theorem[5.1, the proof of which can be found at the end of Section 5. For readers convenience we added an appendix which contains the definition of the parabolic Hölder space $C^{\alpha, \alpha / 2}(Q)$, the parabolic version of the Poincaré inequality and an algebraic lemma which will be used in the proof of Theorem5.1.

## 2 Caccioppoli-type inequality for the approximate $B$ system

Let $\boldsymbol{g}, \boldsymbol{u} \in L^{2}(Q)$ be given. For fixed $0<\delta<1$ we consider the following system for $\boldsymbol{B}$ approximating (1.2)

$$
\begin{align*}
& \partial_{t} \boldsymbol{B}-\Delta \boldsymbol{B} \\
& \quad=-\nabla \times\left(\nabla \times \boldsymbol{B} \times \frac{\boldsymbol{B}}{1+\delta|\boldsymbol{B}|}\right)+\nabla \times\left(\boldsymbol{u} \times \frac{\boldsymbol{B}}{1+\delta|\boldsymbol{B}|}\right)+\nabla \times \boldsymbol{g} \quad \text { in } \quad Q . \tag{2.1}
\end{align*}
$$

We start our discussion with the following notions of a weak solution to (2.1).
Definition 2.1. A vector field $\boldsymbol{B} \in V^{2}(Q)$ is said to be a weak solution to (2.1) if

$$
\begin{align*}
\int_{Q} & \left(-\boldsymbol{B} \cdot \partial_{t} \boldsymbol{\varphi}+\nabla \boldsymbol{B}: \nabla \boldsymbol{\varphi}\right) d x d t \\
& =-\int_{Q}(\nabla \times \boldsymbol{B}-\boldsymbol{u}) \times \frac{\boldsymbol{B}}{1+\delta|\boldsymbol{B}|} \cdot \nabla \times \boldsymbol{\varphi} d x d t+\int_{Q} \boldsymbol{g} \cdot \nabla \times \boldsymbol{\varphi} d x d t \tag{2.2}
\end{align*}
$$

for all $\varphi \in C_{\mathrm{c}}^{\infty}(\Omega)$.
Remark 2.2. Let $\boldsymbol{B}$ be a weak solution to (2.1). Then, (2.2) yields the existence of the distributional time derivative $\boldsymbol{B}^{\prime} \in L^{2}\left(0, T ; W^{-1,2}\right)$, determined by the identity

$$
\begin{align*}
& \int_{\mathbb{R}^{2}}\left\langle\boldsymbol{B}^{\prime}(s), \boldsymbol{\psi}\right\rangle d x+\int_{\mathbb{R}^{2}} \nabla \boldsymbol{B}(s): \nabla \boldsymbol{\psi} d x \\
& =-\int_{\mathbb{R}^{2}}(\nabla \times \boldsymbol{B}(s)-\boldsymbol{u}(s)) \times \frac{\boldsymbol{B}(s)}{1+\delta|\boldsymbol{B}(s)|} \cdot \nabla \times \boldsymbol{\psi} d x+\int_{\mathbb{R}^{2}} \boldsymbol{g}(s) \cdot \nabla \times \boldsymbol{\psi} d x \tag{2.3}
\end{align*}
$$

for all $\boldsymbol{\psi} \in W^{1,2}\left(\mathbb{R}^{2}\right)$ and for a.e. $s \in(0, T)$. Inserting $\boldsymbol{\psi}(x, s)=\phi(x, s)(\boldsymbol{B}(x, s)-\boldsymbol{\Lambda})$ into (2.3) with $\phi \in C_{\mathrm{c}}^{\infty}(Q)$ and a constant vector $\boldsymbol{\Lambda} \in \mathbb{R}^{3}$, integrating the result over ( $0, t$ )
$(t \in(0, T))$ and using integrating by parts, we obtain the following local energy equality

$$
\begin{align*}
& \frac{1}{2} \int_{\mathbb{R}^{2}} \phi(t)|\boldsymbol{B}(t)-\boldsymbol{\Lambda}|^{2} d x+\int_{0}^{t} \int_{Q} \phi|\nabla \boldsymbol{B}|^{2} d x d s \\
& =\frac{1}{2} \int_{0}^{t} \int_{\mathbb{R}^{2}}\left(\partial_{t} \phi+\Delta \phi\right)|\boldsymbol{B}-\boldsymbol{\Lambda}|^{2} d x d s \\
& \quad+\int_{0}^{t} \int_{\mathbb{R}^{2}}(\nabla \times \boldsymbol{B}-\boldsymbol{u}) \times \frac{\boldsymbol{B}}{1+\delta|\boldsymbol{B}|} \cdot((\boldsymbol{B}-\boldsymbol{\Lambda}) \times \nabla \phi) d x d s \\
& \quad+\int_{0}^{t} \int_{\mathbb{R}^{2}} \phi \boldsymbol{u} \times \frac{\boldsymbol{B}}{1+\delta|\boldsymbol{B}|} \cdot \nabla \times \boldsymbol{B} d x d s \\
& \quad+\int_{0}^{t} \int_{\mathbb{R}^{2}}(\phi \boldsymbol{g} \cdot \nabla \times \boldsymbol{B}-\boldsymbol{g} \cdot(\boldsymbol{B}-\boldsymbol{\Lambda}) \times \nabla \phi) d x d s \tag{2.4}
\end{align*}
$$

First, let us fix some notations which is used throughout the present and subsequent sections. Let $X_{0}=\left(x_{0}, t_{0}\right) \in \mathbb{R}^{3}$ and $0<r<+\infty$ by $Q_{r}=Q_{r}\left(X_{0}\right)$ we denote the parabolic cylinder $B_{r}\left(x_{0}\right) \times\left(t_{0}-r^{2}, t_{0}\right)$. Furthermore, for a function $f \in L^{1}\left(Q_{r}\right)$ we define

$$
f_{r, X_{0}}:=f_{Q_{r}}=\int_{Q_{r}} f d x d t=\frac{1}{\operatorname{mes} Q_{r}} \int_{Q_{r}} f d x d t
$$

where mes $Q_{r}$ stands for the three dimensional Lebesgue measure of $Q_{r}$.
Let $0<\rho<r$. We call $\theta \in C^{\infty}\left(\mathbb{R}^{3}\right)$ a cut-off function suitable for $Q_{r}$ and $Q_{\rho}$ if $0 \leq \theta \leq 1$ in $\mathbb{R}^{3}, \theta \equiv 1$ on $Q_{\rho}, \theta \equiv 0$ in $\left(\mathbb{R}^{3} \backslash B_{r}\right) \times\left(t_{0}-r^{2}, t_{0}\right) \cup \mathbb{R}^{2} \times\left(-\infty, t_{0}-r^{2}\right)$ and $\left|\partial_{t} \theta\right|+|\nabla \theta|^{2}+\left|\nabla^{2} \theta\right| \leq c(r-\rho)^{-2}$ in $\mathbb{R}^{3}$.

Now, we state the following Caccioppoli-type inequality.
Lemma 2.3. Let $\boldsymbol{g} \in L^{2}(Q), \boldsymbol{u} \in L^{4}(Q)$ be given, and let $\boldsymbol{B} \in V^{2}(Q)$ be a weak solution to (2.1). Then, for every cylinder $\bar{Q}_{r}=\overline{Q_{r}\left(X_{0}\right)} \subset Q$ and $0<\rho<r$ there holds

$$
\begin{align*}
& \operatorname{ess~sup}_{t \in\left(t_{0}-r^{2}, t_{0}\right)} \int_{B_{r}} \theta^{4}\left|\boldsymbol{B}-\boldsymbol{B}_{r, X_{0}}\right|^{2} d x+\int_{Q_{r}} \theta^{4}|\nabla \boldsymbol{B}|^{2} d x d t \\
& \leq \frac{c}{(r-\rho)^{2}}\left(1+\left|\boldsymbol{B}_{r, X_{0}}\right|^{2}\right) \int_{Q_{r}}\left|\boldsymbol{B}-\boldsymbol{B}_{r, X_{0}}\right|^{2} d x d t \\
& \quad+\frac{c}{r-\rho}\left(\int_{Q_{r}} \theta^{3+\gamma}\left|\boldsymbol{B}-\boldsymbol{B}_{r, X_{0}}\right|^{4} d x d t\right)^{1 / 2}\left(\int_{Q_{r}} \theta^{3-\gamma}|\nabla \boldsymbol{B}|^{2} d x d t\right)^{1 / 2} \\
& \quad+c \int_{Q_{r}}\left(|\boldsymbol{g}|^{2}+\theta^{4}|\boldsymbol{B}|^{2}|\boldsymbol{u}|^{2}\right) d x d t \tag{2.5}
\end{align*}
$$

for all cut-off function $\theta$ suitable for $Q_{r}$ and $Q_{\rho}(\gamma \in[-3,3])$, and

$$
\begin{align*}
E(\rho)^{4} \leq & \frac{c r^{4}}{(r-\rho)^{4}}\left(1+\left|\boldsymbol{B}_{r, X_{0}}\right|^{2}\right)\left(G(r)^{4}+F(r)^{4}\right) \\
& +\frac{c r^{6}}{(r-\rho)^{6}}\left(G(r)^{6}+F(r)^{6}\right) \\
& +\frac{c}{(r-\rho)^{2}}\left\{\int_{Q_{r}}|\boldsymbol{g}|^{2} d x d t+\left|\boldsymbol{B}_{r, X_{0}}\right|^{2} \int_{Q_{r}}|\boldsymbol{u}|^{2} d x d t\right\}\left(G(r)^{2}+F(r)^{2}\right) \\
& +\frac{c r^{4}}{(r-\rho)^{4}} \int_{Q_{r}}|\boldsymbol{u}|^{4} d x d t\left(G(r)^{4}+F(r)^{4}\right) \tag{2.6}
\end{align*}
$$

where $c=$ const $>0$ denotes a universal constant, and

$$
\begin{aligned}
& E(r)=E\left(r, X_{0}\right)=\left(f_{Q_{r}\left(X_{0}\right)}\left|\boldsymbol{B}-\boldsymbol{B}_{r, X_{0}}\right|^{4} d x d t\right)^{1 / 4}, \\
& F(r)=F\left(r, X_{0}\right)=\left(r^{-2} \int_{Q_{r}\left(X_{0}\right)}|\nabla \boldsymbol{B}|^{2} d x d t\right)^{1 / 2}, \\
& G(r)=G\left(r, X_{0}\right)=\left(f_{Q_{r\left(X_{0}\right)}}\left|\boldsymbol{B}-\boldsymbol{B}_{r, X_{0}}\right|^{2} d x d t\right)^{1 / 2}, 0<r<\sqrt{t_{0}} .
\end{aligned}
$$

Proof Let $\bar{Q}_{r}=\overline{Q_{r}\left(X_{0}\right)} \subset Q$ be a fixed cylinder. For $0<\rho<r$ we take a cut-off function $\theta \in C^{\infty}\left(\mathbb{R}^{3}\right)$ suitable for $Q_{r}$ and $Q_{\rho}$.

From (2.4) with $\phi=\theta^{4}$ and $\boldsymbol{\Lambda}=\boldsymbol{B}_{r, X_{0}}$ we obtain the following Caccioppoli-type inequality

$$
\begin{align*}
& \operatorname{esssup}_{t \in\left(t_{0}-r^{2}, t_{0}\right)} \int_{B_{r}} \theta^{4}\left|\boldsymbol{B}-\boldsymbol{B}_{r, X_{0}}\right|^{2} d x+\int_{Q_{r}} \theta^{4}|\nabla \boldsymbol{B}|^{2} d x d t \\
& \leq \frac{c}{(r-\rho)^{2}} \int_{Q_{r}}\left|\boldsymbol{B}-\boldsymbol{B}_{r, X_{0}}\right|^{2} d x d t+c \int_{Q_{r}}|\boldsymbol{g}|^{2}+\theta^{4}|\boldsymbol{B}|^{2}|\boldsymbol{u}|^{2} d x d t \\
& \quad+\frac{c}{r-\rho} \int_{Q_{r}} \theta^{3}|\nabla \boldsymbol{B}||\boldsymbol{B}|\left|\boldsymbol{B}-\boldsymbol{B}_{r, X_{0}}\right| d x d t \\
& \quad=\frac{c}{(r-\rho)^{2}} \int_{Q_{r}}\left|\boldsymbol{B}-\boldsymbol{B}_{r, X_{0}}\right|^{2} d x d t+c \int_{Q_{r}}|\boldsymbol{g}|^{2}+\theta^{4}|\boldsymbol{B}|^{2}|\boldsymbol{u}|^{2} d x d t+J . \tag{2.7}
\end{align*}
$$

Let $\gamma \in[-3,3]$. Applying Hölder's and Young's inequality, we estimate

$$
\begin{aligned}
& J \leq \frac{c}{(r-\rho)^{2}}\left|\boldsymbol{B}_{r, X_{0}}\right|^{2} \int_{Q_{r}}\left|\boldsymbol{B}-\boldsymbol{B}_{r, X_{0}}\right|^{2} d x d t \\
&+\frac{c}{r-\rho}\left(\int_{Q_{r}} \theta^{3+\gamma}\left|\boldsymbol{B}-\boldsymbol{B}_{r, X_{0}}\right|^{4} d x d t\right)^{1 / 2}\left(\int_{Q_{r}} \theta^{3-\gamma}|\nabla \boldsymbol{B}|^{2} d x d t\right)^{1 / 2} \\
&+\frac{1}{2} \int_{Q_{r}} \theta^{4}|\nabla \boldsymbol{B}|^{2} d x d t
\end{aligned}
$$

Inserting the estimate of $J$ into (2.7), we are led to

$$
\begin{align*}
& \underset{t \in\left(t_{0}-r^{2}, t_{0}\right)}{\operatorname{ess} \sup _{B_{r}}} \int_{\theta^{4}} \theta^{4} \boldsymbol{B}-\left.\boldsymbol{B}_{r, X_{0}}\right|^{2} d x+\int_{Q_{r}} \theta^{4}|\nabla \boldsymbol{B}|^{2} d x d t \\
& \leq \frac{c}{(r-\rho)^{2}}\left(1+\left|\boldsymbol{B}_{r, X_{0}}\right|^{2}\right) \int_{Q_{r}}\left|\boldsymbol{B}-\boldsymbol{B}_{r, X_{0}}\right|^{2} d x d t \\
& \quad+\frac{c}{r-\rho}\left(\int_{Q_{r}} \theta^{3+\gamma}\left|\boldsymbol{B}-\boldsymbol{B}_{r, X_{0}}\right|^{4} d x d t\right)^{1 / 2}\left(\int_{Q_{r}} \theta^{3-\gamma}|\nabla \boldsymbol{B}|^{2} d x d t\right)^{1 / 2} \\
& \quad+c \int_{Q_{r}}\left(|\boldsymbol{g}|^{2}+\theta^{4}|\boldsymbol{B}|^{2}|\boldsymbol{u}|^{2}\right) d x d t . \tag{2.8}
\end{align*}
$$

This proves (2.5). On the other hand, by means of Sobolev's embedding theorem we get

$$
\begin{align*}
& f_{Q_{r}} \theta^{4}\left|\boldsymbol{B}-\boldsymbol{B}_{r, X_{0}}\right|^{4} d x d t \\
& \left.\leq c r^{-4}\left\|\theta^{2}\left(\boldsymbol{B}-\boldsymbol{B}_{r, X_{0}}\right)\right\|_{L^{\infty}\left(t_{0}-r^{2}, t_{0} ; L^{2}\left(B_{r}\right)\right)}^{2}\right)\|\nabla \boldsymbol{B}\|_{L^{2}\left(Q_{r}\right)}^{2} \\
& \quad+c r^{-4}(r-\rho)^{-2} \| \theta^{2}\left(\boldsymbol{B}-\boldsymbol{B}_{\left.r, X_{0}\right)}^{2}\left\|_{L^{\infty}\left(t_{0}-r^{2}, t_{0} ; L^{2}\left(B_{r}\right)\right)}^{2}\right\| \boldsymbol{B}-\boldsymbol{B}_{r, X_{0}} \|_{L^{2}\left(Q_{r}\right)}^{2}\right. \\
& \leq \frac{c}{(r-\rho)^{2}}\left\|\theta^{2}\left(\boldsymbol{B}-\boldsymbol{B}_{r, X_{0}}\right)\right\|_{L^{\infty}\left(t_{0}-r^{2}, t_{0} ; L^{2}\left(B_{r}\right)\right)}^{2}\left(F(r)^{2}+G(r)^{2}\right) . \tag{2.9}
\end{align*}
$$

Combining (2.8) with $\gamma=1$ and (2.9) with help of Young's inequality, we get

$$
\begin{align*}
f_{Q_{r}} & \theta^{4}\left|\boldsymbol{B}-\boldsymbol{B}_{r, X_{0}}\right|^{4} d x d t \\
\leq & \frac{c r^{4}}{(r-\rho)^{4}}\left(1+\left|\boldsymbol{B}_{r, X_{0}}\right|^{2}\right) G(r)^{2}\left(F(r)^{2}+G(r)^{2}\right) \\
& +\frac{c r^{6}}{(r-\rho)^{6}}\left(F(r)^{6}+G(r)^{6}\right) \\
& \quad+\frac{c}{(r-\rho)^{2}} \int_{Q_{r}}\left(|\boldsymbol{g}|^{2}+\theta^{4}|\boldsymbol{B}|^{2}|\boldsymbol{u}|^{2}\right) d x d t\left(F(r)^{2}+G(r)^{2}\right) \tag{2.10}
\end{align*}
$$

Estimating $|\boldsymbol{B}|^{2} \leq 2\left|\boldsymbol{B}-\boldsymbol{B}_{r, X_{0}}\right|^{2}+2\left|\boldsymbol{B}_{r, X_{0}}\right|^{2}$ and applying Young's inequality, we obtain (2.6). Thus, the proof of the Lemma is complete.

Remark 2.4. From (2.5) with $\gamma=-1$ along with Young's inequality we get

$$
\begin{align*}
& \left(\frac{1}{r^{2}} \underset{t \in\left(t_{0}-\rho^{2}, t_{0}\right)}{\operatorname{ess} \sup _{B_{\rho}}}\left|\boldsymbol{B}(t)-\boldsymbol{B}_{r, X_{0}}\right|^{2} d x\right)^{1 / 2}+F(\rho) \\
& \leq \frac{c r}{r-\rho}\left\{\left(1+\left|\boldsymbol{B}_{r, X_{0}}\right|\right) E(r)+E(r)^{2}\right\} \\
& \quad+\frac{c}{\rho}\left\{\|\boldsymbol{u}\|_{2, Q_{r}}\left(E(r)+\left|\boldsymbol{B}_{r, X_{0}}\right|\right)+\|\boldsymbol{g}\|_{2, Q_{r}}\right\} . \tag{2.11}
\end{align*}
$$

Furthermore, using the parabolic Poincaré-type inequality (cf. Lemma A.1, appendix below), we find

$$
\begin{align*}
& f_{Q_{r}}\left|\boldsymbol{B}-\boldsymbol{B}_{r, X_{0}}\right|^{2} d x d t \\
& \qquad \begin{array}{l}
\leq c\left(1+\left|\boldsymbol{B}_{r, X_{0}}\right|^{2}\right) r^{-2} \int_{Q_{r}}|\nabla \boldsymbol{B}|^{2} d x d t \\
\quad+c\left(1+\left|\boldsymbol{B}_{r, X_{0}}\right|^{2}\right) r^{-2} \int_{Q_{r}}\left(|\boldsymbol{g}|^{2}+|\boldsymbol{u}|^{2}\right) d x d t \\
\quad+C_{1} r^{-2} \int_{Q_{r}}\left(|\nabla \boldsymbol{B}|^{2}+|\boldsymbol{u}|^{2}\right) d x d t \int_{Q_{r}}\left|\boldsymbol{B}-\boldsymbol{B}_{r, X_{0}}\right|^{2} d x d t
\end{array}, \quad .
\end{align*}
$$

with an absolute constant $C_{1}>0$. Thus, assuming that

$$
\begin{equation*}
C_{1}\left\{r^{-2} \int_{Q_{r}}|\nabla \boldsymbol{B}|^{2} d x d t+4\left(\int_{Q_{r}}|\boldsymbol{u}|^{4} d x d t\right)^{1 / 2}\right\} \leq \frac{1}{2} \tag{2.13}
\end{equation*}
$$

(2.12) leads to

$$
\begin{equation*}
G(r) \leq c\left(1+\left|\boldsymbol{B}_{r, X_{0}}\right|\right)(F(r)+H(r)) \tag{2.14}
\end{equation*}
$$

where

$$
H(r)=H\left(r, X_{0}\right)=r^{-1}\|\boldsymbol{g}\|_{2, Q_{r}}+\|\boldsymbol{u}\|_{4, Q_{r}}, \quad 0<r<\sqrt{t_{0}} .
$$

Substituting $G(r)$ on the right of (2.6) by (2.14), setting $\rho=\frac{r}{2}$ therein, we arrive at

$$
\begin{equation*}
E(r / 2) \leq C_{2}\left(1+\left|\boldsymbol{B}_{r, X_{0}}\right|^{2}\right)\left\{F(r)+F(r)^{2}+H(r)+H(r)^{2}\right\} \tag{2.15}
\end{equation*}
$$

with an absolute constant $C_{2}>0$, provided (2.13) is fulfilled.
From (2.11) with $\rho=\frac{r}{2}$ we deduce that

$$
\begin{equation*}
F(r / 2) \leq C_{3}\left(1+\left|\boldsymbol{B}_{r, X_{0}}\right|\right)\left\{E(r)+E(r)^{2}+H(r)+H(r)^{2}\right\} \tag{2.16}
\end{equation*}
$$

with an absolute constant $C_{3}>0$.

## 3 Blow-up lemma

In what follows we define the space

$$
V^{2}\left(Q_{r}\right)=L^{\infty}\left(t_{0}-r^{2}, t_{0} ; W^{1,2}\left(B_{r}\left(x_{0}\right)\right)\right) \cap L^{\infty}\left(t_{0}-r^{2}, t_{0} ; L^{2}\left(B_{r}\left(x_{0}\right)\right)\right)
$$

for $X_{0}=\left(x_{0}, t_{0}\right)$ and $0<r<+\infty$.
We begin our discussion with the following fundamental estimate for solutions to the model problem in $Q_{1}=Q_{1}(0,0)$, which will be used in the blow-up lemma below.

Lemma 3.1. Let $\boldsymbol{\Lambda} \in \mathbb{R}^{3}$. Let $\boldsymbol{W} \in L^{4}\left(Q_{1}\right)$ such that $\left.\boldsymbol{W}\right|_{Q_{\sigma}} \in V^{2}\left(Q_{\sigma}\right)$ for all $0<\sigma<1$ solves

$$
\begin{equation*}
\partial_{t} \boldsymbol{W}-\Delta \boldsymbol{W}=-\nabla \times((\nabla \times \boldsymbol{W}) \times \boldsymbol{\Lambda}) \quad \text { in } \quad Q_{1} \tag{3.1}
\end{equation*}
$$

in sense of distributions, i.e.

$$
\begin{align*}
& \int_{B_{1}} \boldsymbol{W}(t) \cdot \boldsymbol{\Phi}(t) d x+\int_{-1}^{t} \int_{B_{1}}\left(-\boldsymbol{W} \cdot \partial_{t} \boldsymbol{\Phi}+\nabla \boldsymbol{W}: \nabla \boldsymbol{\Phi}\right) d x d s \\
& \quad=-\int_{-1}^{t} \int_{B_{1}}((\nabla \times \boldsymbol{W}) \times \boldsymbol{\Lambda}) \cdot \nabla \times \boldsymbol{\Phi} d x d s \tag{3.2}
\end{align*}
$$

for all $\mathbf{\Phi} \in W^{1,2}\left(Q_{1}\right)$ compactly supported in $Q_{1}$, for a.e. $t \in(-1,0)$. Then,

$$
\begin{equation*}
\left(f_{Q_{\tau}}\left|\boldsymbol{W}-\boldsymbol{W}_{Q_{\tau}}\right|^{4} d x d t\right)^{1 / 4} \leq C_{0} \tau\left(1+|\boldsymbol{\Lambda}|^{5}\right)\left(f_{Q_{1}}\left|\boldsymbol{W}-\boldsymbol{W}_{Q_{1}}\right|^{4} d x d t\right)^{1 / 4} \tag{3.3}
\end{equation*}
$$

for all $0<\tau<1$, where $C_{0}>0$ denotes a universal constant.
Proof Since the assertion is trivial for $\frac{1}{4}<\tau<1$, we may assume that $0<\tau \leq \frac{1}{4}$. Let $\zeta \in C_{\mathrm{c}}^{\infty}\left(\mathbb{R}^{3}\right)$ be a suitable cut-off function for $Q_{\tau}$ and $Q_{1 / 2}$. Inserting the admissible test function $\boldsymbol{\Phi}=\zeta^{2 m}\left(\boldsymbol{W}-\boldsymbol{W}_{B_{1}}\right)(m \in \mathbb{N})$ into (3.2), by using Cauchy-Schwarz's inequality along with Young's inequality, we are led to

$$
\begin{align*}
& \underset{t \in(-1,0)}{\operatorname{ess} \sup } \int_{B_{1}} \zeta^{2 m}|\boldsymbol{W}(t)|^{2} d x+\int_{Q_{1}} \zeta^{2 m}|\nabla \boldsymbol{W}|^{2} d x d t \\
& \quad \leq c\left(1+|\boldsymbol{\Lambda}|^{2}\right) \int_{Q_{1}} \zeta^{2 m-2}\left|\boldsymbol{W}-\boldsymbol{W}_{Q_{1}}\right|^{2} d x d t . \tag{3.4}
\end{align*}
$$

If $\boldsymbol{W}$ is smooth in $Q_{1}$, since (3.1) is a linear system, the same inequality holds true for $D^{\alpha} \boldsymbol{W}$ in place of $\boldsymbol{W}$ for any multi-index $\alpha$. By a standard mollifying argument together with Sobolev's embedding theorem we see that $\boldsymbol{W}$ is smooth in $Q_{1}$. By an iterative application of (3.4) with $m=4,3,2,1$ we obtain

$$
\begin{equation*}
\underset{t \in(-1,0)}{\operatorname{ess} \sup } \int_{B_{1}} \zeta^{8}\left|D^{\alpha} \boldsymbol{W}\right|^{2} d x \leq c\left(1+|\boldsymbol{\Lambda}|^{8}\right) \int_{Q_{1}}\left|\boldsymbol{W}-\boldsymbol{W}_{Q_{1}}\right|^{2} d x d t \quad \forall|\alpha| \leq 3 \tag{3.5}
\end{equation*}
$$

By means of Sobolev's embedding theorem and Jensen's inequality we get

$$
\begin{equation*}
\|\nabla \boldsymbol{W}\|_{\infty, Q_{1 / 2}}^{4} \leq c\left(1+|\boldsymbol{\Lambda}|^{16}\right) \int_{Q_{1}}\left|\boldsymbol{W}-\boldsymbol{W}_{Q_{1}}\right|^{4} d x d t \tag{3.6}
\end{equation*}
$$

Applying Poincaré's inequality, we arrive at

$$
\begin{equation*}
f_{Q_{\tau}}\left|\boldsymbol{W}-\boldsymbol{W}_{Q_{\tau}}\right|^{4} d x d t \leq c \tau^{4}\left(1+|\Lambda|^{4}\right)\|\nabla \boldsymbol{W}\|_{\infty, Q_{1 / 2}}^{4} \tag{3.7}
\end{equation*}
$$

Combination of (3.6) and (3.7) gives the desired estimate.
In our discussion below we make use of the notion of the Morrey space. Let $K \subset Q$ be a compact set. Define, $d_{K}=\min \{t \in(0, T) \mid t \in K\}$. We say $f$ belongs to the Morrey space $\mathcal{M}^{p, \lambda}(K)$ if

$$
[f]_{\mathcal{M}^{p, \lambda}, K}:=\sup \left\{r^{-\lambda} \int_{Q_{r}\left(X_{0}\right)}|f|^{p} d x d t \mid X_{0} \in K, 0<r \leq d_{K}\right\}<+\infty .
$$

Furthermore, by $f \in \mathcal{M}_{\mathrm{loc}}^{p, \lambda}(Q)$ we mean $f \in \mathcal{M}^{p, \lambda}(K)$ for all compact set $K \subset Q$.
Now we are ready to state the following key lemma.
Lemma 3.2. Let $\boldsymbol{g} \in \mathcal{M}_{\mathrm{loc}}^{2, \lambda}(Q)$ for some $2<\lambda<4$. For every $0<\tau<\frac{1}{2}, 0<$ $M, L<+\infty$, compact set $K \subset Q$ and $0<\alpha<\frac{\lambda-2}{2}$, there exist positive numbers $\varepsilon_{0}=$ $\varepsilon_{0}(\tau, M, L, K, \alpha), R_{0}=R_{0}(\tau, M, L, K, \alpha)<d_{K}$ and $\delta_{0}=\delta_{0}(\tau, M, L, K, \alpha) \leq 1$ such that, if $\boldsymbol{B} \in V^{2}(Q)$ is a weak solution to (2.1) with $0<\delta \leq \delta_{0}$ and $\boldsymbol{u} \in L^{8 /(4-\lambda)}(Q)$ such that

$$
\begin{equation*}
\|\boldsymbol{u}\|_{8 /(4-\lambda), Q} \leq L \tag{3.8}
\end{equation*}
$$

and if for $X_{0} \in K$ and $0<R \leq R_{0}$ the following condition is fulfilled

$$
\begin{equation*}
\left|\boldsymbol{B}_{R, X_{0}}\right| \leq M, \quad E\left(R, X_{0}\right)+R^{\alpha} \leq \varepsilon_{0} \tag{3.9}
\end{equation*}
$$

then there holds

$$
\begin{equation*}
E\left(\tau R, X_{0}\right) \leq 2 \tau C_{0}\left(1+M^{5}\right)\left(E\left(R, X_{0}\right)+R^{\alpha}\right) \tag{3.10}
\end{equation*}
$$

where $C_{0}>0$ stands for the constant appearing on the right hand side of (3.3).
Proof Assume the assertion of the Lemma is not true. Then there exist $0<\tau<$ $\frac{1}{2}, 0<M, L<+\infty$, a compact set $K \subset \Omega$ and $0<\alpha<\frac{\lambda-2}{2}$ as well as sequences $\left\{\varepsilon_{k}\right\},\left\{\delta_{k}\right\} \subset(0,1)$ with $\varepsilon_{k}, \delta_{k} \rightarrow 0$ as $k \rightarrow+\infty,\left\{R_{k}\right\} \subset\left(0, d_{K}\right),\left\{X_{k}\right\}=\left\{\left(x_{k}, t_{k}\right)\right\} \subset K$, $\left\{\boldsymbol{u}^{(k)}\right\} \subset L^{8 /(4-\lambda)}(Q)$ fulfilling

$$
\begin{equation*}
\left\|\boldsymbol{u}^{(k)}\right\|_{8 /(4-\lambda)} \leq L \quad \forall k \in \mathbb{N} \tag{3.11}
\end{equation*}
$$

and a sequence $\left\{\boldsymbol{B}^{(k)}\right\} \subset V^{2}(Q)$, being a weak solutions to (2.1) replacing $\boldsymbol{u}$ by $\boldsymbol{u}^{(k)}$ and $\delta$ by $\delta_{k}$ respectively, such that

$$
\begin{equation*}
\left|\boldsymbol{B}_{R_{k}, X_{k}}^{(k)}\right| \leq M, \quad E_{k}\left(R_{k}, X_{k}\right)+R_{k}^{\alpha}=\varepsilon_{k} \tag{3.12}
\end{equation*}
$$

and

$$
\begin{equation*}
E_{k}\left(\tau R_{k}, X_{k}\right)>2 \tau C_{0}\left(1+M^{5}\right)\left(E_{k}\left(R_{k}, X_{k}\right)+R_{k}^{\alpha}\right) \tag{3.13}
\end{equation*}
$$

Here we have used the notation

$$
E_{k}\left(r, X_{k}\right)=\left(f_{Q_{r}\left(X_{k}\right)}\left|\boldsymbol{B}^{(k)}-\boldsymbol{B}_{r, X_{k}}^{(k)}\right|^{4} d x d t\right)^{1 / 4}, \quad X_{k} \in K, 0<r \leq d_{K}
$$

$(k \in \mathbb{N})$. Note that (3.12) yields $R_{k} \rightarrow 0$ as $k \rightarrow+\infty$.
Next, for $Y:=(y, s) \in Q_{1}(0)$ we define

$$
\begin{aligned}
\boldsymbol{W}_{k}(Y) & =\frac{1}{\varepsilon_{k}}\left(\boldsymbol{B}^{(k)}\left(x_{k}+R_{k} y, t_{k}+R_{k}^{2} s\right)-\boldsymbol{B}_{R_{k}, X_{k}}^{(k)}\right), \\
\boldsymbol{v}_{k}(Y) & =\boldsymbol{u}^{(k)}\left(x_{k}+R_{k} y, t_{k}+R_{k}^{2} s\right) \\
\boldsymbol{g}_{k}(Y) & =\boldsymbol{g}\left(x_{k}+R_{k} y, t_{k}+R_{k}^{2} s\right)
\end{aligned}
$$

$(k \in \mathbb{N})$. Furthermore, we set

$$
\mathscr{E}_{k}(\sigma)=\left(f_{Q_{\sigma}}\left|\boldsymbol{W}_{k}-\left(\boldsymbol{W}_{k}\right)_{Q_{\sigma}}\right|^{4} d y d s\right)^{1 / 4}, \quad 0<\sigma \leq 1
$$

Then (3.12) and (3.13) turn into

$$
\begin{equation*}
\left|\boldsymbol{B}_{R_{k}, X_{k}}^{(k)}\right| \leq M, \quad \mathscr{E}_{k}(1)+\frac{R_{k}^{\alpha}}{\varepsilon_{k}}=1 \tag{3.14}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathscr{E}_{k}(\tau)>2 \tau C_{0}\left(1+M^{5}\right)\left(\mathscr{E}_{k}(1)+\frac{R_{k}^{\alpha}}{\varepsilon_{k}}\right)=2 \tau C_{0}\left(1+M^{5}\right) \tag{3.15}
\end{equation*}
$$

respectively.
Using the chain rule, restriction of system (2.1) to $Q_{R_{k}}\left(X_{k}\right)$ takes the form

$$
\begin{align*}
\partial_{t} \boldsymbol{W}_{k} & -\Delta \boldsymbol{W}_{k} \\
= & -\nabla \times\left(\left(\nabla \times \boldsymbol{W}_{k}\right) \times \frac{\varepsilon_{k} \boldsymbol{W}_{k}+\boldsymbol{B}_{R_{k}, X_{k}}^{(k)}}{1+\delta_{k}\left|\varepsilon_{k} \boldsymbol{W}_{k}+\boldsymbol{B}_{R_{k}, X_{k}}^{(k)}\right|}\right) \\
& +\frac{R_{k}}{\varepsilon_{k}} \nabla \times\left(\boldsymbol{v}_{k} \times \frac{\varepsilon_{k} \boldsymbol{W}_{k}+\boldsymbol{B}_{R_{k}, X_{k}}^{(k)}}{1+\delta_{k}\left|\varepsilon_{k} \boldsymbol{W}_{k}+\boldsymbol{B}_{R_{k}, X_{k}}^{k(k)}\right|}\right)+\frac{R_{k}}{\varepsilon_{k}} \nabla \times \boldsymbol{g}_{k} \tag{3.16}
\end{align*}
$$

in $Q_{1}$. Thus, $\boldsymbol{W}_{k} \in V^{2}\left(Q_{1}\right)$ is a weak solution to (3.16).
Let $0<\sigma<1$. Using the transformation formula, noticing that $\left|\boldsymbol{B}_{R_{k}, X_{k}}^{(k)}\right| \leq M$, the Caccioppoli-type inequality (2.11) with $r=R_{k}$ and $\rho=\sigma R_{k}$ turns into

$$
\begin{align*}
& \left\|\boldsymbol{W}_{k}\right\|_{L^{\infty}\left(-\sigma^{2}, 0 ; L^{2}\left(B_{\sigma}\right)\right)}+\left\|\nabla \boldsymbol{W}_{k}\right\|_{2, B_{\sigma}} \\
& \quad \leq \quad c(1-\sigma)^{-1}\left((1+M) \mathscr{E}_{k}(1)+\varepsilon_{k} \mathscr{E}_{k}(1)^{2}\right) \\
& \quad+\frac{c R_{k}^{-1}}{\varepsilon_{k}}\left(\left\|\boldsymbol{u}^{(k)}\right\|_{2, Q_{R_{k}}\left(X_{k}\right)}\left(\varepsilon_{k} \mathscr{E}_{k}(1)+M\right)+\|\boldsymbol{g}\|_{2, Q_{R_{k}}\left(X_{k}\right)}\right) . \tag{3.17}
\end{align*}
$$

As $\boldsymbol{g}_{k} \in \mathcal{M}^{2, \lambda}(K)$ observing (3.14), we see that

$$
\begin{equation*}
\frac{R_{k}^{-1}}{\varepsilon_{k}}\|\boldsymbol{g}\|_{2, Q_{R_{k}}\left(x_{k}\right)} \leq \frac{R_{k}^{(\lambda-2) / 2}}{\varepsilon_{k}}[\boldsymbol{g}]_{\mathcal{M}^{2, \lambda}(K)} \leq R_{k}^{(\lambda-2) / 2-\alpha}[\boldsymbol{g}]_{\mathcal{M}^{2, \lambda}(K)} \tag{3.18}
\end{equation*}
$$

Similarly, by (3.11) and (3.14) we get

$$
\begin{equation*}
\frac{R_{k}^{-1}}{\varepsilon_{k}}\left\|\boldsymbol{u}^{(k)}\right\|_{2, Q_{R_{k}}\left(x_{k}\right)} \leq c \frac{R_{k}^{(\lambda-2) / 2}}{\varepsilon_{k}}\left\|\boldsymbol{u}^{(k)}\right\|_{8 / 4-\lambda, Q} \leq c R_{k}^{(\lambda-2) / 2-\alpha} L \tag{3.19}
\end{equation*}
$$

Thus, from (3.17) with help of (3.18), (3.19) and (3.14) we obtain

$$
\begin{equation*}
\left\|\boldsymbol{W}_{k}\right\|_{L^{\infty}\left(-\sigma^{2}, 0 ; L^{2}\left(B_{\sigma}\right)\right)}+\left\|\nabla \boldsymbol{W}_{k}\right\|_{2, Q_{\sigma}} \leq c(1-\sigma)^{-1}(M+1)+c\left([\boldsymbol{g}]_{\mathcal{M}^{2, \lambda}(K)}+L\right) . \tag{3.20}
\end{equation*}
$$

In addition, in view of (3.14) we estimate

$$
\begin{equation*}
\left\|\boldsymbol{W}_{k}\right\|_{4, Q_{1}}=\left(\operatorname{mes} B_{1}\right)^{1 / 4} \mathscr{E}_{k}(1) \leq\left(\operatorname{mes} B_{1}\right)^{1 / 4} \tag{3.21}
\end{equation*}
$$

From (3.20) and (3.21) it follows that $\left\{\boldsymbol{W}_{k}\right\}$ is bounded in $V^{2}\left(Q_{\sigma}\right)$ for all $0<\sigma<1$ and bounded in $L^{4}\left(Q_{1}\right)$. Thus, by means of reflexivity, eventually passing to subsequences, we get $\boldsymbol{W} \in L^{4}\left(Q_{1}\right)$ with $\boldsymbol{W} \in V^{2}\left(Q_{\sigma}\right)$ for all $0<\sigma<1$ and $\boldsymbol{\Lambda} \in \mathbb{R}^{3}$ such that

$$
\begin{align*}
\boldsymbol{W}_{k} & \rightarrow \boldsymbol{W} \quad \text { weakly in } L^{4}\left(Q_{1}\right) \quad \text { as } \quad k \rightarrow+\infty,  \tag{3.22}\\
\nabla \boldsymbol{W}_{k} & \rightarrow \nabla \boldsymbol{W} \quad \text { weakly in } L^{2}\left(Q_{\sigma}\right) \text { as } \quad k \rightarrow+\infty \quad \forall 0<\sigma<1,  \tag{3.23}\\
\boldsymbol{W}_{k} & \rightarrow \boldsymbol{W} \quad \text { weakly* in } L^{\infty}\left(-\sigma^{2}, 0 ; L^{2}\left(B_{\sigma}\right)\right) \text { as } k \rightarrow+\infty \quad \forall 0<\sigma<1,  \tag{3.24}\\
\boldsymbol{\Lambda}_{k} & \rightarrow \boldsymbol{\Lambda} \text { in } \mathbb{R}^{3} \text { as } k \rightarrow+\infty \tag{3.25}
\end{align*}
$$

On the other hand, from (3.16) we deduce that the sequence of distributive time derivative $\left\{\boldsymbol{W}_{k}^{\prime}\right\}$ is bounded in $L^{4 / 3}\left(-\sigma^{2}, 0 ; W^{-1,4 / 3}\left(B_{\sigma}\right)\right)$. From this fact together with (3.22) it follows that

$$
\begin{equation*}
\boldsymbol{W}_{k} \rightarrow \boldsymbol{W} \quad \text { strongly in } \quad L^{2}\left(Q_{\sigma}\right) \quad \text { as } \quad k \rightarrow+\infty \quad \forall 0<\sigma<1 . \tag{3.26}
\end{equation*}
$$

Thus, we are in a position to carry out the passage to the limit $k \rightarrow+\infty$ in the weak formulation of (3.16) to deduce that $\boldsymbol{W}$ is a weak solution to the linear system (3.1).

Our next aim is to prove the strong convergence of $\boldsymbol{W}_{k} \rightarrow \boldsymbol{W}$ in $L^{4}\left(Q_{\sigma}\right) \quad(0<\sigma<1)$. We first state the following energy equality,

$$
\begin{aligned}
& \frac{1}{2} \int_{B_{1}} \phi^{2}(t)\left|\boldsymbol{W}_{k}(t)\right|^{2} d y+\int_{-1}^{t} \int_{B_{1}} \phi^{2}\left|\nabla \boldsymbol{W}_{k}\right|^{2} d y d s \\
& =\frac{1}{2} \int_{-1}^{t} \int_{B_{1}}\left(\partial_{t} \phi^{2}+\Delta \phi^{2}\right)\left|\boldsymbol{W}_{k}\right|^{2} d y d s \\
& \quad+\int_{-1}^{t} \int_{B_{1}}\left(\nabla \times \boldsymbol{W}_{k}\right) \times \frac{\varepsilon_{k} \boldsymbol{W}_{k}+\boldsymbol{B}_{R_{k}, X_{k}}^{(k)}}{1+\delta_{k}\left|\varepsilon_{k} \boldsymbol{W}_{k}+\boldsymbol{B}_{R_{k}, X_{k}}^{(k)}\right|} \cdot\left(\boldsymbol{W}_{k} \times \nabla \phi^{2}\right) d y d s \\
& \quad+\frac{R_{k}}{\varepsilon_{k}} \int_{-1}^{t} \int_{B_{1}}\left\{\boldsymbol{v}_{k} \times \frac{\varepsilon_{k} \boldsymbol{W}_{k}+\boldsymbol{B}_{R_{k}, X_{k}}^{(k)}}{1+\delta_{k}\left|\varepsilon_{k} \boldsymbol{W}_{k}+\boldsymbol{B}_{R_{k}, X_{k}}^{(k)}\right|}+\boldsymbol{g}_{k}\right\} \nabla \times\left(\phi^{2} \boldsymbol{W}_{k}\right) d y d s
\end{aligned}
$$

for all $t \in[-1,0]$. In view of (3.22), (3.23), (3.25) and (3.26) on both sides of (3.27) with $t=0$ letting $k \rightarrow+\infty$, we infer

$$
\begin{align*}
& \lim _{k \rightarrow \infty}\left(\frac{1}{2} \int_{B_{1}} \phi^{2}(0)\left|\boldsymbol{W}_{k}(0)\right|^{2} d y+\int_{Q_{1}} \phi^{2}\left|\nabla \boldsymbol{W}_{k}\right|^{2} d y d s\right) \\
& =\frac{1}{2} \int_{Q_{1}}\left(\partial_{t} \phi^{2}+\Delta \phi^{2}\right)|\boldsymbol{W}|^{2} d y d s-\int_{Q_{1}}(\nabla \times \boldsymbol{W}) \times \boldsymbol{\Lambda} \cdot\left(\boldsymbol{W} \times \nabla \phi^{2}\right) d y d s . \tag{3.28}
\end{align*}
$$

Since $\boldsymbol{W}$ is a weak solution to (3.1), there holds

$$
\begin{align*}
& \frac{1}{2} \int_{B_{1}} \phi^{2}(0)|\boldsymbol{W}(0)|^{2} d y+\int_{Q_{1}} \phi^{2}|\nabla \boldsymbol{W}|^{2} d y d s \\
& =\frac{1}{2} \int_{Q_{1}}\left(\partial_{t} \phi^{2}+\Delta \phi^{2}\right)|\boldsymbol{W}|^{2} d y d s-\int_{Q_{1}}(\nabla \times \boldsymbol{W}) \times \boldsymbol{\Lambda} \cdot\left(\boldsymbol{W} \times \nabla \phi^{2}\right) d y d s . \tag{3.29}
\end{align*}
$$

Noticing that

$$
\left\{\begin{array}{l}
\left(\phi(0) \boldsymbol{W}_{k}(0), \phi \nabla \boldsymbol{W}_{k}\right) \rightarrow(\phi(0) \boldsymbol{W}(0), \phi \nabla \boldsymbol{W}) \\
\text { weakly in } \quad L^{2}\left(B_{1}\right) \times L^{2}\left(Q_{1}\right) \quad \text { as } \quad k \rightarrow+\infty
\end{array}\right.
$$

from (3.28) and (3.29), we deduce that

$$
\nabla \boldsymbol{W}_{k} \rightarrow \nabla \boldsymbol{W} \quad \text { strongly in } \quad L^{2}\left(Q_{\sigma}\right) \quad \text { as } \quad k \rightarrow+\infty \quad \forall 0<\sigma<1 .
$$

Accordingly,

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \mathscr{E}_{k}(\sigma)=\mathscr{E}(\sigma) \quad \forall 0<\sigma<1 \tag{3.30}
\end{equation*}
$$

where $\mathscr{E}(\sigma)=\left(f_{B_{\sigma}}\left|\boldsymbol{W}-\boldsymbol{W}_{B_{\sigma}}\right|^{4} d y\right)^{1 / 2}$. In particular, thanks to (3.30) (with $\sigma=\tau$ ) from (3.15) we get

$$
\begin{equation*}
\mathscr{E}(\tau) \geq 2 \tau C_{0}\left(1+M^{5}\right) \tag{3.31}
\end{equation*}
$$

Since $\boldsymbol{W}$ is a weak solution to (3.1) and $|\boldsymbol{\Lambda}| \leq M$, appealing to Lemma3.1, we find

$$
\begin{equation*}
\mathscr{E}(\tau) \leq \tau C_{0}\left(1+M^{5}\right) \mathscr{E}(1) . \tag{3.32}
\end{equation*}
$$

On the other hand, by virtue of the lower semi continuity of the norm together with (3.15) and (3.30) we get

$$
\begin{aligned}
\mathscr{E}(1) & \leq \liminf _{k \rightarrow \infty}\left(\mathscr{E}_{k}(1)+\frac{R_{k}^{\alpha}}{\varepsilon_{k}}\right) \leq \frac{1}{2 \tau C_{0}\left(1+M^{5}\right)} \lim _{k \rightarrow \infty} \mathscr{E}_{k}(\tau) \\
& =\frac{1}{2 \tau C_{0}\left(1+M^{5}\right)} \mathscr{E}(\tau)
\end{aligned}
$$

Estimating the right of (3.32) by the inequality, we have just obtained we are led to $\mathscr{E}(\tau) \leq \frac{1}{2} \mathscr{E}(\tau)$ and hence $\mathscr{E}(\tau)=0$, which contradicts to (3.31). Whence, the assumption cannot be true, which completes the proof of the Lemma.

## 4 Construction of approximate solutions

The aim of the present section is to construct a weak solution of the Hall-MHD system (1.1)-(1.5) as a limit of a sequence of solutions to the a corresponding approximate system. As we will see in the following section, such solution will satisfy the desired partial regularity as stated in the main result of the present paper.

Let $\left\{\delta_{m}\right\} \subset(0,1) \quad(m \in \mathbb{N})$ be a sequence, such that $\delta_{m} \rightarrow 0$ as $m \rightarrow+\infty$. Now, we consider the following approximate system

$$
\begin{align*}
& \partial_{t} \boldsymbol{u}_{m}+\frac{\boldsymbol{\omega}_{m}}{1+\delta_{m}\left|\boldsymbol{B}_{m}\right|} \times \boldsymbol{u}_{m}-\Delta \boldsymbol{u}_{m} \\
& \quad=-\nabla p_{m}+\left(\nabla \times \boldsymbol{B}_{m}\right) \times \frac{\boldsymbol{B}_{m}}{1+\delta_{m}\left|\boldsymbol{B}_{m}\right|}+\boldsymbol{f}  \tag{4.1}\\
& \begin{aligned}
& \partial_{t} \boldsymbol{B}_{m}+\nabla \times\left(\frac{\boldsymbol{B}_{m}}{1+\delta_{m}\left|\boldsymbol{B}_{m}\right|} \times \boldsymbol{u}_{m}\right)-\Delta \boldsymbol{B}_{m} \\
&=-\nabla \times\left(\nabla \times \boldsymbol{B}_{m} \times \frac{\boldsymbol{B}_{m}}{1+\delta_{m}\left|\boldsymbol{B}_{m}\right|}\right)+\nabla \times \boldsymbol{g} \\
& \nabla \cdot \boldsymbol{u}_{m}=0, \quad \nabla \cdot \boldsymbol{B}_{m}=0
\end{aligned}
\end{align*}
$$

in $Q=\mathbb{R}^{2} \times(0, T)$, together with the initial condition

$$
\begin{equation*}
\boldsymbol{u}_{m}=\boldsymbol{u}_{0}, \quad \boldsymbol{B}_{m}=\boldsymbol{B}_{0}, \quad \text { in } \quad \mathbb{R}^{2} \times\{0\} . \tag{4.4}
\end{equation*}
$$

Here $\left(\boldsymbol{u}_{m}, p_{m}, \boldsymbol{B}_{m}\right) \in V_{\text {div }}^{2}(Q) \times L^{2}(Q) \times V_{\text {div }}^{2}(Q)$ is called a weak solution to (4.1)-(4.3) if

$$
\begin{align*}
& \int_{Q}\left(-\boldsymbol{u}_{m} \cdot \partial_{t} \boldsymbol{\varphi}+\nabla \boldsymbol{u}_{m}: \nabla \boldsymbol{\varphi}-\boldsymbol{u}_{m} \otimes \boldsymbol{u}_{m}: \nabla \boldsymbol{\varphi}\right) d x d t \\
& \quad=\int_{Q} p_{m} \nabla \cdot \boldsymbol{\varphi} d x d t+\int_{Q}\left(\left(\nabla \times \boldsymbol{B}_{m}\right) \times \frac{\boldsymbol{B}_{m}}{1+\delta_{m}\left|\boldsymbol{B}_{m}\right|}\right) \cdot \boldsymbol{\varphi} d x d t+\int_{Q} \boldsymbol{f} \cdot \boldsymbol{\varphi} d x d t  \tag{4.5}\\
& \int_{Q}\left(-\boldsymbol{B}_{m} \cdot \partial_{t} \boldsymbol{\varphi}+\nabla \boldsymbol{B}_{m}: \nabla \boldsymbol{\varphi}\right) d x d t \\
& \quad=-\int_{Q}\left(\left(\nabla \times \boldsymbol{B}_{m}-\boldsymbol{u}_{m}\right) \times \frac{\boldsymbol{B}_{m}}{1+\delta_{m}\left|\boldsymbol{B}_{m}\right|}\right) \cdot \nabla \times \boldsymbol{\varphi} d x+\int_{Q} \boldsymbol{g} \cdot \nabla \times \boldsymbol{\varphi} d x d t \tag{4.6}
\end{align*}
$$

for all $\varphi \in C_{\mathrm{c}}^{\infty}(Q)$.
The existence of weak solutions to (4.1)-(4.4) is given by the following
Lemma 4.1. Let $\boldsymbol{u}_{0} \in L_{\text {div }}^{2}, \boldsymbol{B}_{0} \in L^{2}$ and $\boldsymbol{f}, \boldsymbol{g} \in L^{2}(Q)$. Then for every $m \in \mathbb{N}$ there exists a weak solution $\left(\boldsymbol{u}_{m}, p_{m}, \boldsymbol{B}_{m}\right) \in V_{\mathrm{div}}^{2}(Q) \times L^{2}\left(0, T ; L_{\mathrm{loc}}^{2}\right) \times V_{\mathrm{div}}^{2}(Q)$ to (4.1) -(4.4), such that

$$
\begin{equation*}
\nabla \boldsymbol{u}_{m} \in V^{2}\left(Q_{r}\right), \quad \forall \bar{Q}_{r} \subset Q \tag{4.7}
\end{equation*}
$$

Furthermore, this solution fulfills the energy equality

$$
\begin{align*}
& \frac{1}{2}\left\|\boldsymbol{u}_{m}(t)\right\|_{2}^{2}+\frac{1}{2}\left\|\boldsymbol{B}_{m}(t)\right\|_{2}^{2}+\int_{0}^{t}\left(\left\|\nabla \boldsymbol{u}_{m}(s)\right\|_{2}^{2}+\left\|\nabla \boldsymbol{B}_{m}(s)\right\|_{2}^{2}\right) d s \\
& \quad=\frac{1}{2}\left\|\boldsymbol{u}_{0}\right\|_{2}^{2}+\frac{1}{2}\left\|\boldsymbol{B}_{0}\right\|_{2}^{2}+\int_{0}^{t} \int_{\mathbb{R}^{2}}\left(\boldsymbol{f} \cdot \boldsymbol{u}_{m}+\boldsymbol{g} \cdot \nabla \times \boldsymbol{B}_{m}\right) d x d s \tag{4.8}
\end{align*}
$$

for a.e. $t \in(0, T)$.
Proof Let $m \in \mathbb{N}$ be fixed. Let $\beta_{l} \rightarrow 0^{+}$as $l \rightarrow+\infty$. By using the well-known monotone operator theory we get a weak solution $\left(\boldsymbol{u}_{m, l}, p_{m, l}, \boldsymbol{B}_{m, l}\right) \in V_{\text {div }}^{2}(Q) \times L^{2}\left(0, T ; L_{\text {loc }}^{2}\right) \times V_{\text {div }}^{2}(Q)$ of the following approximate system

$$
\begin{gather*}
\partial_{t} \boldsymbol{u}_{m, l}+\frac{\boldsymbol{\omega}_{m, l}}{1+\delta_{m}\left|\boldsymbol{B}_{m, l}\right|+\beta_{l}\left|\boldsymbol{V}_{m, l}\right|} \times \boldsymbol{u}_{m, l}-\Delta \boldsymbol{u}_{m, l} \\
=-\nabla p_{m, l}+\left(\nabla \times \boldsymbol{B}_{m, l}\right) \times \frac{\boldsymbol{B}_{m, l}}{1+\delta_{m}\left|\boldsymbol{B}_{m, l}\right|+\beta_{l}\left|\boldsymbol{V}_{m, l}\right|}+\boldsymbol{f},  \tag{4.9}\\
\begin{aligned}
& \partial_{t} \boldsymbol{B}_{m, l}+\nabla \times \frac{\boldsymbol{B}_{m, l}}{1+\delta_{m}\left|\boldsymbol{B}_{m, l}\right|+\beta_{l}\left|\boldsymbol{V}_{m, l}\right|} \times \boldsymbol{u}_{m, l}-\Delta \boldsymbol{B}_{m, l} \\
&=-\nabla \times\left(\nabla \times \boldsymbol{B}_{m, l} \times \frac{\boldsymbol{B}_{m, l}}{1+\delta_{m}\left|\boldsymbol{B}_{m, l}\right|+\beta_{l}\left|\boldsymbol{V}_{m, l}\right|}\right)+\nabla \times \boldsymbol{g} \\
& \nabla \cdot \boldsymbol{u}_{m, l}=0, \quad \nabla \cdot \boldsymbol{B}_{m, l}=0
\end{aligned}
\end{gather*}
$$

in $Q=\mathbb{R}^{2} \times(0, T)$ together with the initial condition

$$
\begin{equation*}
\boldsymbol{u}_{m, l}=\boldsymbol{u}_{0}, \quad \boldsymbol{B}_{m, l}=\boldsymbol{B}_{0}, \quad \text { in } \quad \mathbb{R}^{2} \times\{0\} \tag{4.12}
\end{equation*}
$$

where

$$
\boldsymbol{V}_{m, l}=\boldsymbol{\omega}_{m, l}+\boldsymbol{B}_{m, l} .
$$

Clearly, the energy equality (4.8) holds true with $\boldsymbol{u}_{m, l}$ in place of $\boldsymbol{u}_{m}$ and $\boldsymbol{B}_{m, l}$ in place of $\boldsymbol{B}_{m}$ respectively. In particular, both $\left\{\boldsymbol{u}_{m, l}\right\}$ and $\left\{\boldsymbol{B}_{m, l}\right\}$ are bounded in $V^{2}(Q)$. Thus, by a standard reflexivity argument along with Banach-Alaoglu's compactness lemma, eventually passing to a subsequence, we may assume there exist $\boldsymbol{u}_{m} \in V_{\text {div }}^{2}(Q)$ and $\boldsymbol{B}_{m} \in$ $V_{\text {div }}^{2}(Q)$ such that

$$
\begin{align*}
\nabla \boldsymbol{u}_{m, l} & \rightarrow \nabla \boldsymbol{u}_{m}, \quad \nabla \boldsymbol{B}_{m, l} \rightarrow \nabla \boldsymbol{B}_{m} \quad \text { weakly in } \quad L^{2}(Q),  \tag{4.13}\\
\boldsymbol{u}_{m, l} & \rightarrow \boldsymbol{u}_{m}, \quad \boldsymbol{B}_{m, l} \rightarrow \boldsymbol{B}_{m} \quad \text { weakly* in } \quad L^{\infty}\left(0, T ; L^{2}\right) \quad \text { as } \quad l \rightarrow+\infty . \tag{4.14}
\end{align*}
$$

Furthermore, by Lions-Aubin's compactness lemma we see that

$$
\begin{equation*}
\boldsymbol{u}_{m, l} \rightarrow \boldsymbol{u}_{m}, \quad \boldsymbol{B}_{m, l} \rightarrow \boldsymbol{B}_{m} \quad \text { strongly in } \quad L^{2}(Q) \quad \text { as } \quad l \rightarrow+\infty \tag{4.15}
\end{equation*}
$$

Hence, thanks to (4.13), (4.14) and (4.15) we are in a position to carry out the passage to the limit $l \rightarrow+\infty$ in the weak formulation of (4.9)-(4.11). Accordingly, there exists
$p_{m} \in L^{2}\left(0, T ; L_{\text {loc }}^{2}\right)$ such that $\left(\boldsymbol{u}_{m}, p_{m}, \boldsymbol{B}_{m}\right)$ is a weak solution to (4.1)-(4.4). Verifying that $\boldsymbol{u}_{m}$ and $\boldsymbol{B}_{m}$ satisfying the energy equality (4.8), it follows that

$$
\begin{equation*}
\nabla \boldsymbol{u}_{m, l} \rightarrow \nabla \boldsymbol{u}_{m}, \quad \nabla \boldsymbol{B}_{m, l} \rightarrow \nabla \boldsymbol{B}_{m} \quad \text { strongly in } \quad L^{2}(Q) \quad \text { as } \quad l \rightarrow+\infty . \tag{4.16}
\end{equation*}
$$

As $V^{2}(Q) \hookrightarrow L^{4}(Q)$ from (4.16) we infer

$$
\begin{equation*}
\boldsymbol{u}_{m, l} \rightarrow \boldsymbol{u}_{m}, \quad \boldsymbol{B}_{m, l} \rightarrow \boldsymbol{B}_{m} \quad \text { strongly in } \quad L^{4}(Q) \quad \text { as } \quad l \rightarrow+\infty . \tag{4.17}
\end{equation*}
$$

Next, applying $\nabla \times$ to both sides of (4.9) and combining the result with (4.10), we are led to

$$
\begin{equation*}
\partial_{t} \boldsymbol{V}_{m, l}-\Delta \boldsymbol{V}_{m, l}=-\nabla \times\left(\frac{\boldsymbol{V}_{m, l}}{1+\delta_{m}\left|\boldsymbol{B}_{m, l}\right|+\beta_{l}\left|\boldsymbol{V}_{m, l}\right|} \times \boldsymbol{u}_{m, l}\right)+\nabla \times \boldsymbol{h} \quad \text { in } \quad Q, \tag{4.18}
\end{equation*}
$$

where $\boldsymbol{h}=\boldsymbol{g}+\boldsymbol{f}$. By using a routine smoothing argument one gets $\boldsymbol{V}_{m, l} \in V^{2}\left(Q_{r}\right)$ for all $\bar{Q}_{r} \subset Q$.

Now, let $\bar{Q}_{r}=\overline{Q_{r}\left(X_{0}\right)} \subset Q$ be arbitrarily chosen. Let $\theta \in C_{\mathrm{c}}^{\infty}\left(B_{r} \times\left(t_{0}-r^{2}, t_{0}\right]\right)$ be a test function suitable for $Q_{r / 2}$. Testing (4.18) by $\theta^{2} \boldsymbol{V}_{m, l}$, we get

$$
\begin{align*}
& \frac{1}{2} \int_{B_{r}} \theta^{2}(t)\left|\boldsymbol{V}_{m, l}(t)\right|^{2} d x+\int_{t_{0}-r^{2}}^{t} \int_{B_{r}} \theta^{2}\left|\nabla \boldsymbol{V}_{m, l}\right|^{2} d x d s \\
& \quad=\frac{1}{2} \int_{t_{0}-r^{2}}^{t} \int_{B_{r}}\left(\partial_{t} \theta^{2}+\Delta \theta^{2}\right)\left|\boldsymbol{V}_{m, l}\right|^{2} d x d s \\
& \quad-\int_{t_{0}-r^{2}}^{t} \int_{B_{r}}\left(\frac{\boldsymbol{V}_{m, l}}{1+\delta_{m}\left|\boldsymbol{B}_{m, l}\right|+\beta_{l}\left|\boldsymbol{V}_{m, l}\right|} \times \boldsymbol{u}_{m, l}-\boldsymbol{h}\right) \cdot \nabla \times\left(\theta^{2} \boldsymbol{V}_{m, l}\right) d x d s \tag{4.19}
\end{align*}
$$

for a.e. $t \in\left(t_{0}-r^{2}, t_{0}\right)$. From the above identity using the embedding $V^{2}\left(Q_{r}\right) \hookrightarrow L^{4}\left(Q_{r}\right)$, it is readily seen that

$$
\begin{align*}
& \left(\int_{Q_{r}} \theta^{4}\left|\boldsymbol{V}_{m, l}\right|^{4} d x d t\right)^{1 / 2} \\
& \leq c \underset{t \in\left(t_{0}-r^{2}, t_{0}\right)}{\operatorname{esssup}} \int_{B_{r}} \theta^{2}(t)\left|\boldsymbol{V}_{m, l}(t)\right|^{2} d x+c \int_{Q_{r}} \theta^{2}\left|\nabla \boldsymbol{V}_{m, l}\right|^{2}+r^{-2}\left|\boldsymbol{V}_{m, l}\right|^{2}+|\boldsymbol{h}|^{2} d x d t \\
& \leq c r^{-2}\left(1+\left\|\boldsymbol{u}_{m, l}\right\| \|_{4}^{2}\right) \int_{Q_{r}}\left|\boldsymbol{V}_{m, l}\right|^{2} d x d t+c\|\boldsymbol{h}\|_{2}^{2} \\
& \quad+\hat{C}\left\|\boldsymbol{u}_{m, l}\right\|_{4, Q_{r}}\left(\int_{Q_{r}} \theta^{4}\left|\boldsymbol{V}_{m, l}\right|^{4} d x d t\right)^{1 / 2}, \tag{4.20}
\end{align*}
$$

with an absolute constant $\hat{C}>0$. As $\boldsymbol{u}_{m} \in L^{4}(Q)$, we may choose $0<r<\sqrt{t_{0}}$ such that $\hat{C}\left\|\boldsymbol{u}_{m}\right\|_{4, Q_{r}} \leq \frac{1}{4}$. Observing (4.17), there exists $l_{0} \in \mathbb{N}$ such that $\hat{C}\left\|\boldsymbol{u}_{m, l}\right\|_{4, Q_{r}} \leq \frac{1}{2}$ for all $l \geq l_{0}$. Accordingly, (4.20) implies

$$
\begin{equation*}
\left(\int_{Q_{r}} \theta^{4}\left|\boldsymbol{V}_{m, l}\right|^{4} d x d t\right)^{1 / 2} \leq c r^{-2}\left(1+\left\|\boldsymbol{u}_{m, l}\right\|_{4}^{2}\right) \int_{Q_{r}}\left|\boldsymbol{\omega}_{m, l}+\boldsymbol{V}_{m, l}\right|^{2} d x d t+c\|\boldsymbol{h}\|_{2}^{2} \tag{4.21}
\end{equation*}
$$

for $l \geq l_{0}$. Since the right of (4.21) is bounded independently of $l \in \mathbb{N}$, by a constant $C\left(\boldsymbol{u}_{0}, \boldsymbol{B}_{0}, \boldsymbol{f}, \boldsymbol{g}\right)$ by virtue of the lower semi continuity of the norm from (4.21) together with (4.19) and (4.20) we get

$$
\begin{equation*}
\left\|\nabla \boldsymbol{V}_{m}\right\|_{2, Q_{r / 2}}+\left\|\boldsymbol{V}_{m}\right\|_{L^{\infty}\left(t_{0}-r^{2} / 4, t_{0} ; L^{2}\left(B_{r / 2}\right)\right)}+\left\|\boldsymbol{V}_{m}\right\|_{4, Q_{r / 2}} \leq C\left(\boldsymbol{u}_{0}, \boldsymbol{B}_{0}, \boldsymbol{f}, \boldsymbol{g}\right) \tag{4.22}
\end{equation*}
$$

where $\boldsymbol{V}_{m}=\boldsymbol{B}_{m}+\nabla \times \boldsymbol{u}_{m}$. By applying a standard covering argument, since $\boldsymbol{B}_{m} \in L^{2}$ we see that $\nabla \times \boldsymbol{u}_{m} \in V^{2}\left(Q_{r}\right)$ for all $\bar{Q}_{r} \subset Q$. Whence, the assertion follows from the inequality

$$
\left\|\nabla \boldsymbol{u}_{m}\right\|_{2, Q_{r / 2}} \leq c r^{-1}\left(\left\|\boldsymbol{u}_{m}\right\|_{2, Q_{r}}+\left\|\nabla \times \boldsymbol{u}_{m}\right\|_{2, Q_{r}}\right)
$$

which completes the proof of the lemma.
Next, we are going to carry out the passage to the limit $m \rightarrow+\infty$, which can be done by an analogous argument used in the proof of Lemma4.1. Observing the energy equality (4.8), we find that both $\left\{\boldsymbol{u}_{m}\right\}$ and $\left\{\boldsymbol{B}_{m}\right\}$ are bounded in $V^{2}(Q)$. Eventually passing to a subsequence, we get the existence of $\boldsymbol{u}, \boldsymbol{B} \in V_{\text {div }}^{2}(Q)$ such that

$$
\begin{align*}
\nabla \boldsymbol{u}_{m} & \rightarrow \nabla \boldsymbol{u}, \quad \nabla \boldsymbol{B}_{m} \rightarrow \nabla \boldsymbol{B} \quad \text { weakly in } \quad L^{2}(Q),  \tag{4.23}\\
\boldsymbol{u}_{m} & \rightarrow \boldsymbol{u}, \quad \boldsymbol{B}_{m, l} \rightarrow \boldsymbol{B}_{m} \quad \text { weakly }{ }^{*} \text { in } \quad L^{\infty}\left(0, T ; L^{2}\right) \quad \text { as } \quad m \rightarrow+\infty . \tag{4.24}
\end{align*}
$$

Furthermore, by Lions-Aubin's compactness lemma we see that

$$
\begin{equation*}
\boldsymbol{u}_{m} \rightarrow \boldsymbol{u}, \quad \boldsymbol{B}_{m} \rightarrow \boldsymbol{B} \quad \text { strongly in } \quad L^{2}(Q) \quad \text { as } \quad m \rightarrow+\infty . \tag{4.25}
\end{equation*}
$$

With the aid of (4.23), (4.24) and (4.25) we are in a position to carry out the passage to the limit $m \rightarrow+\infty$ in the weak formulation of (4.1)-(4.4), which yields a weak solution $(\boldsymbol{u}, p, \boldsymbol{B})$ to (1.1)-(1.4).

Our next aim is to get a strong $L^{4}$ convergence of $\boldsymbol{u}_{m}$.
Lemma 4.2. Let $\left\{\left(\boldsymbol{u}_{m}, p_{m}, \boldsymbol{B}_{m}\right)\right\}$ be a sequence of weak solutions to (4.1) -(4.4) obtained by Lemma 4.1. Furthermore, suppose (4.23)-(4.25). Then, for every $\bar{Q}_{r} \subset Q$ there holds

$$
\begin{equation*}
\boldsymbol{u}_{m} \rightarrow \boldsymbol{u} \quad \text { strongly in } \quad \boldsymbol{L}^{4}\left(Q_{r}\right) \quad \text { as } \quad m \rightarrow+\infty . \tag{4.26}
\end{equation*}
$$

In addition, for every $X_{0} \in Q$ there exists $0<r=r\left(X_{0}\right)<\sqrt{t_{0}}$ such that

$$
\begin{align*}
& \left\|\nabla \boldsymbol{\omega}_{m}\right\|_{2, Q_{r}}+\left\|\boldsymbol{\omega}_{m}\right\|_{L^{\infty}\left(t_{0}-r^{2}, t_{0} ; L^{2}\left(B_{r}\right)\right)}+\left\|\boldsymbol{\omega}_{m}\right\|_{4, Q_{r}} \\
& \quad \leq C\left(\boldsymbol{u}_{0}, \boldsymbol{B}_{0}, \boldsymbol{f}, \boldsymbol{g}\right) \quad \forall m \in \mathbb{N} . \tag{4.27}
\end{align*}
$$

Proof Let $m \in \mathbb{N}$. In view of Lemma4.1, taking the sum of (4.1) and (4.2), we see that $\boldsymbol{V}_{m}=\boldsymbol{\omega}_{m}+\boldsymbol{B}_{m} \in V_{\mathrm{loc}}^{2}(Q)$ is a weak solution to the following system

$$
\begin{equation*}
\partial_{t} \boldsymbol{V}_{m}-\Delta \boldsymbol{V}_{m}=-\nabla \times\left(\frac{\boldsymbol{V}_{m}}{1+\delta_{m}\left|\boldsymbol{B}_{m}\right|} \times \boldsymbol{u}_{m}\right)+\nabla \times \boldsymbol{h} \quad \text { in } \quad Q \tag{4.28}
\end{equation*}
$$

Here $V_{\text {loc }}^{2}(Q)$ contains all $\varphi \in L^{2}(Q)$ such that $\left.\varphi\right|_{Q_{r}} \in V^{2}\left(Q_{r}\right)$ for all $\bar{Q}_{r} \subset Q$.
Clearly, there exists $\boldsymbol{v}_{m} \in V_{\text {loc }}^{2}(Q)$ such that $\nabla \times \boldsymbol{v}_{m}=\boldsymbol{V}_{m}$. Thus, from (4.28) we infer that

$$
\begin{equation*}
\partial_{t} \boldsymbol{v}_{m}-\Delta \boldsymbol{v}_{m}=-\nabla \pi_{m}-\frac{\boldsymbol{V}_{m}}{1+\delta_{m}\left|\boldsymbol{B}_{m}\right|} \times\left(\boldsymbol{v}_{m}-\boldsymbol{b}_{m}\right)+\boldsymbol{h} \quad \text { in } \quad Q \tag{4.29}
\end{equation*}
$$

where $\boldsymbol{b}_{m}=\boldsymbol{v}_{m}-\boldsymbol{u}_{m}$. By the definition of $\boldsymbol{v}_{m}$ we have $\nabla \times \boldsymbol{b}_{m}=\boldsymbol{B}_{m}$.
Let $\bar{Q}_{r} \subset Q$ be fixed. Eventually, replacing $\boldsymbol{v}_{m}$ by $\boldsymbol{v}_{m}(t)-\left(\boldsymbol{v}_{m}(t)\right)_{x_{0}, r}\left(t \in t_{0}-r^{2}, t_{0}\right)$, observing (4.23), (4.24) and (4.25), by virtue of Sobolev's embedding theorem we easily verify that

$$
\begin{align*}
\boldsymbol{V}_{m} & \rightarrow \boldsymbol{V} \quad \text { weakly in } L^{2}\left(Q_{r}\right),  \tag{4.30}\\
\boldsymbol{b}_{m} & \rightarrow \boldsymbol{b} \quad \text { strongly in } \quad L^{6}\left(Q_{r}\right) \quad \text { as } \quad m \rightarrow+\infty . \tag{4.31}
\end{align*}
$$

Indeed, we note that $\left|\boldsymbol{b}_{m}(t)_{x_{0}, B_{r}}\right|=\left|\boldsymbol{u}_{m}(t)_{x_{0}, B_{r}}\right| \leq\left\|\boldsymbol{u}_{m}\right\|_{L^{\infty}\left(0, T ; L^{2}\right)}$. Consequently, by Sobolev-Poincaré's inequality we see that $\left\|\boldsymbol{b}_{m}\right\|_{q, Q_{r}} \leq c\left\|\boldsymbol{u}_{m}\right\|_{L^{\infty}\left(0, T ; L^{2}\right)}+c\left\|\boldsymbol{B}_{m}\right\|_{L^{\infty}\left(0, T ; L^{2}\right)}$ for every $1 \leq q<+\infty$. Once more appealing to (4.25), eventually passing to a subsequence we may assume that

$$
\begin{equation*}
\boldsymbol{B}_{m} \rightarrow \boldsymbol{B} \quad \text { a.e. in } \quad Q \quad \text { as } \quad m \rightarrow+\infty \tag{4.32}
\end{equation*}
$$

By means of Vitali's convergence theorem, making use of (4.31) and (4.32), we get

$$
\begin{equation*}
\frac{\boldsymbol{b}_{m}}{1+\delta_{m}\left|\boldsymbol{B}_{m}\right|} \rightarrow \boldsymbol{b} \quad \text { strongly in } \quad L^{6}\left(Q_{r}\right) \quad \text { as } \quad m \rightarrow+\infty \tag{4.33}
\end{equation*}
$$

Next, we define the local pressure

$$
\begin{aligned}
\nabla \pi_{m, 1} & =\boldsymbol{E}_{B_{r}}\left(\Delta \boldsymbol{v}_{m}\right) \\
\nabla \pi_{m, 2} & =\boldsymbol{E}_{B_{r}}\left(-\frac{\boldsymbol{V}_{m}}{1+\delta_{m}\left|\boldsymbol{B}_{m}\right|} \times\left(\boldsymbol{v}_{m}-\boldsymbol{b}_{m}\right)+\boldsymbol{h}\right), \\
\nabla \pi_{m, \mathrm{hm}} & =-\boldsymbol{E}_{B_{r}}\left(\boldsymbol{v}_{m}\right)
\end{aligned}
$$

where $\boldsymbol{E}_{B_{r}}: W^{-1, q}\left(B_{r}\right) \rightarrow W^{-1, q}\left(B_{r}\right)$ stands for the projection defined by the Stokes equation. Note that the restriction of $\boldsymbol{E}_{B_{r}}$ to $L^{q}\left(Q_{r}\right)(1<q<+\infty)$ defines a projection in $L^{q}\left(Q_{r}\right)$ (cf. [23, 24] for details). We also note that $\pi_{m, \mathrm{hm}}(t)$ is harmonic in $B_{r}$ for a.e. all $t \in\left(t_{0}-r^{2}, t_{0}\right)$. As it has been proved in [23],(4.29) implies that the function $\boldsymbol{z}_{m}=\boldsymbol{v}_{m}+\nabla \pi_{m, \mathrm{hm}} \in V^{2}\left(Q_{r}\right)$ solves the following system in sense of distributions

$$
\begin{equation*}
\partial_{t} \boldsymbol{z}_{m}-\Delta \boldsymbol{z}_{m}=-\nabla\left(\pi_{m, 1}+\pi_{m, 2}\right)-\frac{\boldsymbol{V}_{m}}{1+\delta_{m}\left|\boldsymbol{B}_{m}\right|} \times \boldsymbol{u}_{m}+\boldsymbol{h} \quad \text { in } \quad Q_{r} \tag{4.34}
\end{equation*}
$$

Let $\phi \in C_{\mathrm{c}}^{\infty}\left(Q_{r}\right)$ be a non-negative cut-off function. Testing (4.34) by $\phi \boldsymbol{z}_{m}$, we obtain the following energy equality

$$
\begin{align*}
& \int_{Q_{r}} \phi\left|\nabla \boldsymbol{z}_{m}\right|^{2} d x d t \\
& =\frac{1}{2} \int_{Q_{r}}\left(\partial_{t} \phi+\Delta \phi\right)\left|\boldsymbol{z}_{m}\right|^{2} d x d t+\int_{Q_{r}}\left(\frac{\boldsymbol{V}_{m}}{1+\delta_{m}\left|\boldsymbol{B}_{m}\right|} \times \boldsymbol{b}_{m}+\boldsymbol{h}\right) \cdot \phi \boldsymbol{z}_{m} d x d t \\
& \quad \quad+\int_{Q_{r}}\left(\pi_{m, 1}+\pi_{m, 2}\right) \nabla \phi \cdot \boldsymbol{z}_{m} d x d t \tag{4.35}
\end{align*}
$$

Verifying

$$
\left\|\frac{\boldsymbol{V}_{m}}{1+\delta_{m}\left|\boldsymbol{B}_{m}\right|} \times \boldsymbol{u}_{m}\right\|_{L^{3 / 2}\left(0, T ; L^{6 / 5}\right)} \leq\left\|\boldsymbol{V}_{m}\right\|_{2}\left\|\boldsymbol{u}_{m}\right\|_{L^{6}\left(0, T ; L^{3}\right)} \leq C\left(\boldsymbol{u}_{0}, \ldots\right)
$$

we may estimate the pressure $\pi_{m, 2}$ in $L^{3 / 2}\left(Q_{r}\right)$ by using the Sobolev-Poincaré inequality as follows

$$
\begin{align*}
\left\|\pi_{m, 2}\right\|_{3 / 2, Q_{r}} & \leq c\left\|\nabla \pi_{m, 2}\right\|_{L^{3 / 2}\left(t_{0}-r^{2}, t_{0} ; L^{6 / 5}\left(B_{r}\right)\right)} \\
& \leq c\left\|\frac{\boldsymbol{V}_{m}}{1+\delta_{m}\left|\boldsymbol{B}_{m}\right|} \times \boldsymbol{u}_{m}\right\|_{L^{3 / 2}\left(0, T ; L^{6 / 5}\right)}+c\|\boldsymbol{h}\|_{2} \leq C\left(\boldsymbol{u}_{0}, \ldots\right) \tag{4.36}
\end{align*}
$$

Furthermore, we immediately get

$$
\begin{equation*}
\left\|\pi_{m, 1}\right\|_{2, Q_{r}} \leq c\left\|\nabla \boldsymbol{v}_{m}\right\|_{2} \leq c\left\|\nabla \boldsymbol{u}_{m}\right\|_{2}+c\left\|\boldsymbol{B}_{m}\right\|_{2} \leq C\left(\boldsymbol{u}_{0}, \ldots\right) \tag{4.37}
\end{equation*}
$$

Observing (4.25) along with (4.31), we find

$$
\begin{equation*}
\boldsymbol{v}_{m} \rightarrow \boldsymbol{v} \quad \text { strongly in } \quad L^{3}\left(Q_{r}\right) \quad \text { as } \quad m \rightarrow+\infty, \tag{4.38}
\end{equation*}
$$

where $\boldsymbol{v}=\boldsymbol{u}+\boldsymbol{b}$. Thus, having

$$
\begin{equation*}
\nabla \pi_{m, \mathrm{hm}} \rightarrow \nabla \pi_{\mathrm{hm}} \quad \text { strongly in } \quad L^{3}\left(Q_{r}\right) \quad \text { as } \quad m \rightarrow \infty \tag{4.39}
\end{equation*}
$$

where $\nabla \pi_{\mathrm{hm}}=-\boldsymbol{E}_{B_{r}}(\boldsymbol{v})$, it follows that

$$
\begin{equation*}
\boldsymbol{z}_{m} \rightarrow \boldsymbol{z} \quad \text { strongly in } \quad L^{3}\left(Q_{r}\right) \quad \text { as } \quad m \rightarrow+\infty \tag{4.40}
\end{equation*}
$$

Now, with help of (4.36), (4.37) and (4.40) we get

$$
\lim _{m \rightarrow \infty} \int_{Q_{r}}\left(\pi_{m, 1}+\pi_{m, 2}\right) \nabla \phi \cdot \boldsymbol{z}_{m} d x d t=\int_{Q_{r}}\left(\pi_{1}+\pi_{2}\right) \nabla \phi \cdot \boldsymbol{z} d x d t
$$

where

$$
\begin{aligned}
\nabla \pi_{1} & =\boldsymbol{E}_{B_{r}}(\Delta \boldsymbol{v}), \\
\nabla \pi_{2} & =\boldsymbol{E}_{B_{r}}(-\boldsymbol{V} \times \boldsymbol{u}+\boldsymbol{h})
\end{aligned}
$$

On the other hand, making use of (4.33), together with (4.25) and (4.40) we see that

$$
\lim _{m \rightarrow \infty} \int_{Q_{r}}\left(\frac{\boldsymbol{V}_{m}}{1+\delta_{m}\left|\boldsymbol{B}_{m}\right|} \times \boldsymbol{b}_{m}+\boldsymbol{h}\right) \cdot \phi \boldsymbol{z}_{m} d x d t=\int_{Q_{r}}(\boldsymbol{V} \times \boldsymbol{b}+\boldsymbol{h}) \cdot \phi \boldsymbol{z} d x d t
$$

Furthermore, thanks to (4.40) we obtain

$$
\lim _{m \rightarrow \infty} \frac{1}{2} \int_{Q_{r}}\left(\partial_{t} \phi+\Delta \phi\right)\left|\boldsymbol{z}_{m}\right|^{2} d x d t=\frac{1}{2} \int_{Q_{r}}\left(\partial_{t} \phi+\Delta \phi\right)|\boldsymbol{z}|^{2} d x d t
$$

Hence, we are in the position to carry out the passage to the limit $m \rightarrow+\infty$ in (4.35) to get

$$
\begin{align*}
& \lim _{m \rightarrow \infty} \int_{Q_{r}} \phi\left|\nabla \boldsymbol{z}_{m}\right|^{2} d x d t \\
& \quad=\frac{1}{2} \int_{Q_{r}}\left(\partial_{t} \phi+\Delta \phi\right)|\boldsymbol{z}|^{2} d x d t+\int_{Q_{r}}(\boldsymbol{V} \times \boldsymbol{b}+\boldsymbol{h}) \cdot \phi \boldsymbol{z} d x d t+\int_{Q_{r}}\left(\pi_{1}+\pi_{2}\right) \nabla \phi \cdot \boldsymbol{z} d x d t \tag{4.41}
\end{align*}
$$

Accordingly, we see that $\boldsymbol{z} \in V^{2}\left(Q_{r}\right)$ and

$$
\begin{equation*}
\partial_{t} \boldsymbol{z}-\Delta \boldsymbol{z}=-\nabla\left(\pi_{1}+\pi_{2}\right)-\boldsymbol{V} \times \boldsymbol{u}+\boldsymbol{h} \quad \text { in } \quad Q_{r} \tag{4.42}
\end{equation*}
$$

in sense of distributions. Taking into account that $\boldsymbol{z} \in L^{4}\left(Q_{r}\right)$, and $\boldsymbol{V} \times \boldsymbol{u} \in L^{4 / 3}(Q)$, we obtain the following energy equality

$$
\begin{align*}
& \int_{Q_{r}} \phi|\nabla \boldsymbol{z}|^{2} d x d t \\
& =\frac{1}{2} \int_{Q_{r}}\left(\partial_{t} \phi+\Delta \phi\right)|\boldsymbol{z}|^{2} d x d t+\int_{Q_{r}}(\boldsymbol{V} \times \boldsymbol{b}+\boldsymbol{h}) \cdot \phi \boldsymbol{z} d x d t \\
& \quad \quad+\int_{Q_{r}}\left(\pi_{1}+\pi_{2}\right) \nabla \phi \cdot \boldsymbol{z} d x d t \tag{4.43}
\end{align*}
$$

Thus, observing (4.25), combining (4.41) and (4.43) using a well-known liminf-limsup argument noticing that $\sqrt{\phi} \nabla \boldsymbol{z}_{m} \rightarrow \sqrt{\phi} \nabla \boldsymbol{z}$ weakly in $L^{2}\left(Q_{r}\right)$ we get

$$
\sqrt{\phi} \nabla \boldsymbol{z}_{m} \rightarrow \sqrt{\phi} \nabla \boldsymbol{z} \quad \text { strongly in } \quad L^{2}\left(Q_{r}\right) \quad \text { as } \quad m \rightarrow \infty .
$$

On the other hand, since $\pi_{\mathrm{hm}}$ is harmonic, thanks to (4.39) we get

$$
\sqrt{\phi} \nabla^{2} \pi_{m, \mathrm{hm}} \rightarrow \sqrt{\phi} \nabla^{2} \pi_{\mathrm{hm}} \quad \text { strongly in } \quad L^{2}\left(Q_{r}\right) \quad \text { as } \quad m \rightarrow \infty
$$

As $\nabla \boldsymbol{v}_{m}=\nabla \boldsymbol{z}_{m}-\nabla^{2} \pi_{m, \text { hm }}$ a.e. in $Q_{r}$, we arrive at

$$
\sqrt{\phi} \nabla \boldsymbol{v}_{m} \rightarrow \sqrt{\phi} \nabla \boldsymbol{v} \quad \text { strongly in } \quad L^{2}\left(Q_{r}\right) \quad \text { as } \quad m \rightarrow \infty .
$$

Hence, thanks to the embedding $V^{2}\left(Q_{r}\right) \hookrightarrow L^{4}\left(Q_{r}\right)$ along with (4.31) we get

$$
\sqrt{\phi} \boldsymbol{u}_{m} \rightarrow \sqrt{\phi} \boldsymbol{u} \quad \text { strongly in } \quad L^{4}\left(Q_{r}\right) \quad \text { as } \quad m \rightarrow \infty
$$

Since the above statement holds for any cylinder $Q_{r} \subset Q$, we get the first claim (4.26) of the lemma.

Now, it remains to verify (4.27). In fact, according to Lemma4.1, we have $\boldsymbol{V}_{m} \in$ $L^{4}\left(Q_{r}\right)$, which implies that $\frac{\boldsymbol{V}_{m}}{1+\delta_{m}\left|\boldsymbol{B}_{m}\right|} \times \boldsymbol{u}_{m} \in L^{2}\left(Q_{r}\right)$. This allows us to test (4.28) with $\theta^{2} \boldsymbol{V}_{m}$, where $\theta \in C_{\mathrm{c}}^{\infty}\left(B_{r} \times\left(t_{0}-r^{2}, t_{0}\right]\right)$. Arguing as in the proof of Lemma4.1, we obtain

$$
\begin{aligned}
& \frac{1}{2} \int_{B_{r}} \theta^{2}(t)\left|\boldsymbol{V}_{m}(t)\right|^{2} d x+\int_{t_{0}-r^{2}}^{t} \int_{B_{r}} \theta^{2}\left|\nabla \boldsymbol{V}_{m}\right|^{2} d x d s \\
& \quad=\frac{1}{2} \int_{t_{0}-r^{2}}^{t} \int_{B_{r}}\left(\partial_{t} \theta^{2}+\Delta \theta^{2}\right)\left|\boldsymbol{V}_{m}\right|^{2} d x d s \\
& \quad-\int_{t_{0}-r^{2}}^{t} \int_{B_{r}}\left(\frac{\boldsymbol{V}_{m}}{1+\delta_{m}\left|\boldsymbol{B}_{m}\right|} \times \boldsymbol{u}_{m}-\boldsymbol{h}\right) \cdot \nabla \times\left(\theta^{2} \boldsymbol{V}_{m}\right) d x d s
\end{aligned}
$$

for a.e. $t \in\left(t_{0}-r^{2}, t_{0}\right)$, which leads to

$$
\begin{align*}
& \left(\int_{Q_{r}} \theta^{4}\left|\boldsymbol{V}_{m}\right|^{4} d x d t\right)^{1 / 2} \\
& \quad \leq c r^{-2}\left(1+\left\|\boldsymbol{u}_{m}\right\|_{4}^{2}\right) C\left(\boldsymbol{u}_{0}, \ldots\right)+\hat{C}\left\|\boldsymbol{u}_{m}\right\|_{4, Q_{r}}\left(\int_{Q_{r}} \theta^{4}\left|\boldsymbol{V}_{m}\right|^{4} d x d t\right)^{1 / 2} . \tag{4.44}
\end{align*}
$$

Whence, the proof of (4.27) can be completed by a similar argument to the proof of Lemma4.1, by using the strong $L^{4}$ convergence (4.26).

## 5 Proof of Theorem 1.2

Let $\left(\boldsymbol{u}_{m}, p_{m}, \boldsymbol{B}_{m}\right) \in V_{\text {div }}^{2}(Q) \times L^{2}\left(0, T ; L_{\text {loc }}^{2}\right) \times V_{\text {div }}^{2}(Q)$ be a weak solution to the approximate system (4.1)-(4.4) such that $\nabla \boldsymbol{u}_{m} \in V_{\mathrm{loc}}^{2}(Q)(m \in \mathbb{N})$, which can be guaranteed by Lemma 4.1 (for the definition of $V_{\text {loc }}^{2}(Q)$ see Section 4).

In our discussion below we use the following notation. Let $X_{0}=\left(x_{0}, t_{0}\right) \in Q$.

$$
\begin{aligned}
& E_{m}(r)=E_{m}\left(r, X_{0}\right):=\left(f_{Q_{r}\left(X_{0}\right)}\left|\boldsymbol{B}_{m}-\left(\boldsymbol{B}_{m}\right)_{r, X_{0}}\right|^{4} d x d t\right)^{1 / 4} \\
& F_{m}(r)=F_{m}\left(r, X_{0}\right):=\left(r^{-2} \int_{Q_{r}\left(X_{0}\right)}\left|\nabla \boldsymbol{B}_{m}\right|^{2} d x d t\right)^{1 / 2} \\
& H_{m}(r)=H_{m}\left(r, X_{0}\right):=\left\|\boldsymbol{u}_{m}\right\|_{4, Q_{r}}+r^{-1}\|\boldsymbol{g}\|_{2, Q_{r}} \quad 0<r<\sqrt{t_{0}} .
\end{aligned}
$$

Next, we define the set of possible singularities of $B$ by means of $\Sigma(\boldsymbol{B})=\cup_{k=1}^{\infty} \Sigma_{k} \cup \Sigma_{\infty}$, where

$$
\begin{aligned}
\Sigma_{k} & :=\bigcup_{0<\rho<T} \bigcap_{0<r \leq \rho}\left\{X_{0} \in \mathbb{R}^{2} \times(r, T) \left\lvert\, \liminf _{m \rightarrow \infty} F_{m}\left(r, X_{0}\right) \geq \frac{1}{k}\right.\right\}, \quad k \in \mathbb{N}, \\
\Sigma_{\infty} & :=\left\{X_{0} \in Q\left|\sup _{0<r<\sqrt{t_{0}}}\right| \boldsymbol{B}_{r, X_{0}} \mid=+\infty\right\} .
\end{aligned}
$$

Let $Q_{r}=Q_{r}\left(X_{0}\right) \subset Q$ be any cylinder such that condition (2.13) is fulfilled for $\boldsymbol{B}=\boldsymbol{B}_{m}$ and $\boldsymbol{u}=\boldsymbol{u}_{m}$, i.e.

$$
\begin{equation*}
C_{1}\left\{F_{m}(r)^{2}+\left\|\boldsymbol{u}_{m}\right\|_{4, Q_{r}}^{2}\right\} \leq \frac{1}{2} \tag{5.1}
\end{equation*}
$$

As stated in Remark[2.4, the condition (2.13) implies (2.15). Thus, (5.1) implies

$$
\begin{equation*}
E_{m}(r / 2) \leq C_{2}\left(1+\left|\left(\boldsymbol{B}_{m}\right)_{r, X_{0}}\right|^{2}\right)\left\{F_{m}(r)+F_{m}(r)^{2}+H_{m}(r)+H_{m}(r)^{2}\right\} \tag{5.2}
\end{equation*}
$$

On the other hand, (2.16) with $\boldsymbol{B}=\boldsymbol{B}_{m}$ and $\boldsymbol{u}=\boldsymbol{u}_{m}$ reads

$$
\begin{equation*}
F_{m}(r / 2) \leq C_{3}\left(1+\left|\left(\boldsymbol{B}_{m}\right)_{r, X_{0}}\right|\right)\left\{E_{m}(r)+E_{m}(r)^{2}+H_{m}(r)+H_{m}(r)^{2}\right\} \tag{5.3}
\end{equation*}
$$

Let $X_{0} \in Q \backslash \Sigma(\boldsymbol{B})$ be fixed. Set $d_{0}=\sqrt{t_{0}} / 2$ and $K=\overline{Q_{d_{0}}}$. Appealing to Lemma4.2, and applying Sobolev's embedding theorem, we see that

$$
\begin{equation*}
\left\|\boldsymbol{u}_{m}\right\|_{8 /(4-\lambda), K} \leq L \quad \forall m \in \mathbb{N} \tag{5.4}
\end{equation*}
$$

where $L=$ const $>0$ depends on $d_{0}, \boldsymbol{u}_{0}, \boldsymbol{B}_{0}, \boldsymbol{f}$ and $\boldsymbol{g}$ only. Furthermore, we may choose $0<R_{1}<d_{0}$ such that

$$
\begin{equation*}
C_{1}\left\|\boldsymbol{u}_{m}\right\|_{4, Q_{R_{1}}}^{2} \leq \frac{1}{16} \quad \forall m \in \mathbb{N} \tag{5.5}
\end{equation*}
$$

where $C_{1}$ stands for the constant appearing in (5.1). Using Hölder's inequality, recalling the assumption on $\boldsymbol{g}$ along with (5.4), it follows that

$$
\begin{align*}
H_{m}\left(r, X_{0}\right) & \leq\left(\pi^{\lambda / 8-1 / 4}\left\|\boldsymbol{u}_{m}\right\|_{8 /(4-\lambda), K}+\|\boldsymbol{g}\|_{\mathcal{M}^{2, \lambda}(K)}\right) r^{(\lambda-2) / 2} \\
& \leq C_{4} r^{(\lambda-2) / 2} \quad \forall \quad 0<r \leq R_{1} \tag{5.6}
\end{align*}
$$

Next, we set

$$
M:=512 \sup _{0<r<d\left(X_{0} / 2\right)}(|\boldsymbol{B}|)_{r, X_{0}}+1<+\infty .
$$

Let $0<\alpha<\frac{2-\lambda}{2}$. We take $\tau>0$ such that

$$
\begin{equation*}
2 \tau^{1-\alpha} C_{0}\left(1+M^{5}\right) \leq \frac{1}{2} \quad \text { and } \quad \tau^{\alpha} \leq \frac{1}{2} \tag{5.7}
\end{equation*}
$$

(Recall, the constant $C_{0}>0$ has been defined in Lemma3.1).
Now, let $\varepsilon_{0}=\varepsilon_{0}(\tau, M, L, K, \alpha), R_{0}=R_{0}(\tau, M, L, K, \alpha)$ and $\delta_{0}=\delta_{0}(\tau, M, L, K, \alpha)$ denote the numbers according to Lemma 3.2. In addition, we define $\varepsilon_{1}>0$ by the relation

$$
\begin{equation*}
2 \tau^{-4} \varepsilon_{1}=1 \tag{5.8}
\end{equation*}
$$

Next we may choose $0<R_{2} \leq \min \left\{R_{0}, R_{1}\right\}$ such that the following conditions hold

$$
\begin{gather*}
C_{2}\left(1+M^{2}\right)\left(C_{4}+C_{4}^{2}\right) R_{2}^{(\lambda-2) / 2} \leq \frac{1}{8} \min \left\{\varepsilon_{0}, \varepsilon_{1}\right\},  \tag{5.9}\\
2 R_{2}^{\alpha} \leq \frac{1}{2} \min \left\{\varepsilon_{0}, \varepsilon_{1}\right\} \tag{5.10}
\end{gather*}
$$

Now, we take $k \in \mathbb{N}$ such that

$$
\begin{equation*}
C_{2}\left(1+M^{2}\right)\left\{\frac{1}{k}+\frac{1}{k^{2}}\right\} \leq \frac{1}{8} \min \left\{\varepsilon_{0}, \varepsilon_{1}\right\} \quad \text { and } \quad \frac{C_{1}}{k} \leq \frac{1}{4} . \tag{5.11}
\end{equation*}
$$

Owing to $X_{0} \in Q \backslash \Sigma_{k}$ eventually replacing $R_{2}$ by a smaller number we may also assume that $\lim _{\inf }^{m \rightarrow \infty} F_{m}\left(R_{2}, X_{0}\right)<\frac{1}{k}$. Accordingly we are able to select a subsequence $\left\{m_{j}\right\}$ such that

$$
\begin{equation*}
F_{m_{j}}\left(R_{2}, X_{0}\right)<\frac{1}{k} \quad \forall j \in \mathbb{N} . \tag{5.12}
\end{equation*}
$$

Since $\boldsymbol{B}_{m} \rightarrow \boldsymbol{B}$ in $L^{1}\left(Q_{R_{2}}\right)$ and $\delta_{m} \rightarrow 0$ as $m \rightarrow+\infty$, there exists $m_{0} \in \mathbb{N}$ with the property

$$
\begin{equation*}
\left(\left|\boldsymbol{B}_{m}\right|\right)_{R_{2}, X_{0}} \leq(|\boldsymbol{B}|)_{R_{2}, X_{0}}+\frac{1}{512} \leq \frac{M}{512} \quad \text { and } \quad \delta_{m} \leq \delta_{0} \quad \forall m \geq m_{0} \tag{5.13}
\end{equation*}
$$

Observing (5.12), (5.11) and (5.5), we have

$$
\begin{equation*}
C_{1}\left\{F_{m_{j}}\left(R_{2}, X_{0}\right)+4\left\|\boldsymbol{u}_{m_{j}}\right\|_{4, Q_{R_{2}}\left(X_{0}\right)}\right\} \leq \frac{1}{2} \quad \forall j \in \mathbb{N} \tag{5.14}
\end{equation*}
$$

As (5.14) implies (5.2), employing (5.13), (5.12) and (5.6), we get

$$
\begin{equation*}
E_{m_{j}}\left(R_{2} / 2, X_{0}\right) \leq C_{2}\left(1+M^{2}\right)\left\{\frac{1}{k}+\frac{1}{k^{2}}+\left(C_{4}+C_{4}^{2}\right) R_{2}^{(\lambda-2) / 2}\right\} \tag{5.15}
\end{equation*}
$$

for all $m_{j} \geq m_{0}$. In view of (5.11) and (5.9), (5.15) gives

$$
\begin{equation*}
E_{m_{j}}\left(R_{2}, X_{0}\right) \leq \frac{1}{4} \min \left\{\varepsilon_{0}, \varepsilon_{1}\right\} \quad \forall m_{j} \geq m_{0} \tag{5.16}
\end{equation*}
$$

Set $R_{3}=R_{2} / 2$. Let $Y \in Q_{R_{2}}\left(X_{0}\right)$. Clearly,

$$
\begin{align*}
& E_{m_{j}}\left(R_{3}, Y\right) \leq 2 E_{m_{j}}\left(R_{2}, X_{0}\right) \leq \frac{1}{2} \min \left\{\varepsilon_{0}, \varepsilon_{1}\right\},  \tag{5.17}\\
& \left|\left(\boldsymbol{B}_{m_{j}}\right)_{R_{3}, Y}\right| \leq 256 f_{Q_{R_{2}\left(X_{0}\right)}}\left|\boldsymbol{B}_{m_{j}}\right| d x d t \leq \frac{M}{2} \tag{5.18}
\end{align*}
$$

We claim that for every $i \in \mathbb{N} \cup\{0\}$, there holds

$$
\begin{align*}
& E_{m_{j}}\left(\tau^{i} R_{3}, Y\right) \leq 2^{-i} \tau^{\alpha i} E_{m_{j}}\left(R_{3}, Y\right)+\left(1-2^{-i}\right) \tau^{\alpha i} R_{3}^{\alpha}  \tag{5.19}\\
& \left|\left(\boldsymbol{B}_{m_{j}}\right)_{\tau^{i} R_{3}, Y}\right| \leq M-2^{-i+1} \tag{5.20}
\end{align*}
$$

In fact, for $i=0$, (5.19) is trivially fulfilled, while (5.20) holds in view of (5.18).
Now, we assume that both (5.19) and (5.20) are fulfilled for $i \in N \cup\{0\}$. Then (5.19) together with (5.17) and (5.10) implies

$$
\begin{equation*}
E_{m_{j}}\left(\tau^{i} R_{3}, Y\right)+\tau^{\alpha i} R_{3}^{\alpha} \leq \tau^{\alpha i}\left(E_{m_{j}}\left(R_{3}, Y\right)+2 R_{3}^{\alpha}\right) \leq \tau^{\alpha i} \min \left\{\varepsilon_{0}, \varepsilon_{1}\right\} \tag{5.21}
\end{equation*}
$$

In particular, observing (5.20) we have

$$
E_{m_{j}}\left(\tau^{i} R_{3}, Y\right)+\left(\tau^{i} R_{3}\right)^{\alpha} \leq \varepsilon_{0}, \quad\left|\left(\boldsymbol{B}_{m_{j}}\right)_{\tau^{i} R_{3}, Y}\right| \leq M
$$

Thus, we are in a position to apply Lemma 3.2 with $R=\tau^{i} R_{3}$. This together with (5.19) gives

$$
\begin{align*}
E_{m_{j}}\left(\tau^{i+1} R_{3}, Y\right) & \leq 2 \tau C_{0}\left(1+M^{5}\right)\left(E_{m_{j}}\left(\tau^{i} R_{3}, Y\right)+\tau^{\alpha i} R_{3}^{\alpha}\right) \\
& \leq \frac{1}{2} \tau^{\alpha} E_{m_{j}}\left(\tau^{i} R_{3}, Y\right)+\frac{1}{2} \tau^{\alpha(i+1)} R_{3}^{\alpha} \\
& \leq 2^{-(i+1)} \tau^{\alpha(i+1)} E_{m_{j}}\left(R_{3}, Y\right)+\left(1-2^{-(i+1)}\right) \tau^{\alpha(i+1)} R_{3}^{\alpha} \tag{5.22}
\end{align*}
$$

Consequently (5.19) holds true for $i+1$.
Now, it remains to show (5.20) for $i+1$. First, from (5.19) along with (5.17) and (5.10) we infer

$$
\begin{equation*}
E_{m_{j}}\left(\tau^{i} R_{3}, Y\right) \leq \tau^{\alpha i}\left(E_{m_{j}}\left(R_{3}, Y\right)+R_{3}^{\alpha}\right) \leq \tau^{\alpha i} \varepsilon_{1} \tag{5.23}
\end{equation*}
$$

Using the triangle inequality and Jensen's inequality, we find

$$
\begin{aligned}
\left|\left(\boldsymbol{B}_{m_{j}}\right)_{\tau^{i+1} R_{3}, Y}\right| & \leq\left|\left(\boldsymbol{B}_{m_{j}}\right)_{\tau^{i} R_{3}, Y}\right|+\left|\left(\boldsymbol{B}_{m_{j}}\right)_{\tau^{i+1} R_{3}, Y}-\left(\boldsymbol{B}_{m_{j}}\right)_{\tau^{i} R_{3}, Y}\right| \\
& \leq\left|\left(\boldsymbol{B}_{m_{j}}\right)_{\tau^{i} R_{3}, Y}\right|+2 \tau^{-4} E_{m_{j}}\left(\tau^{i} R_{3}, Y\right) .
\end{aligned}
$$

Estimating the first term on the right by using (5.20) and the second one by the aid of (5.23) together with (5.7) and (5.8), we obtain

$$
\begin{aligned}
\left|\left(\boldsymbol{B}_{m_{j}}\right)_{\tau^{i+1} R_{3}, Y}\right| & \leq M-2^{-i+1}+2 \tau^{-4} \tau^{\alpha i} \varepsilon_{1} \\
& \leq M-2^{-i+1}+2^{-i}=M-2^{-i}
\end{aligned}
$$

This completes the proof of (5.20) for $i+1$. Whence, the claim.
Since (5.19) holds true for every $Y \in \overline{Q_{R_{3}}\left(X_{0}\right)}$, by a standard iteration argument we get a constant $C_{5}>0$ such that

$$
\begin{equation*}
\left(f_{Q_{r}(Y)}\left|\boldsymbol{B}_{m_{j}}-\left(\boldsymbol{B}_{m_{j}}\right)_{r, Y}\right|^{4} d x\right)^{1 / 4} \leq C_{5} r^{\alpha} \quad \forall 0<r<R_{3}, \quad \forall Y \in \overline{Q_{R_{3}}\left(X_{0}\right)} \tag{5.24}
\end{equation*}
$$

Thus, by means of the lower semi continuity of the $L^{4}$-norm the above inequality remains true for $\boldsymbol{B}$. Using the well-known integral characterization of the Hölder continuity in the parabolic setting[17], we obtain

$$
\begin{equation*}
\left.\boldsymbol{B}\right|_{\overline{Q_{R_{3}}\left(X_{0}\right)}} \in C^{\alpha, \alpha / 2}\left(\overline{Q_{R_{3}}\left(X_{0}\right)}\right) \tag{5.25}
\end{equation*}
$$

(For the definition of $C^{\alpha, \alpha / 2}\left(\overline{Q_{R_{3}}\left(X_{0}\right)}\right)$ see appendix below). Clearly, (5.24) shows that

$$
\lim _{r \rightarrow 0^{+}} E_{m_{j}}(r, Y)=0 \quad \text { uniformly for } \quad Y \in Q_{R_{3}}\left(X_{0}\right) \quad \text { and } \quad j \in \mathbb{N} .
$$

Hence, in view of (5.3) we get $Y \notin \bigcup_{k=1}^{\infty} \Sigma_{k}$. Taking into account that $\boldsymbol{B}$ is Hölder continuous on $\overline{Q_{\rho_{0}}\left(X_{0}\right)}$, it follows that $Q_{\rho_{0}}\left(X_{0}\right) \subset Q \backslash \Sigma_{\infty}$ and thus

$$
Q_{R_{3}}\left(X_{0}\right) \subset Q \backslash \Sigma(\boldsymbol{B})
$$

Consequently, $\Sigma(\boldsymbol{B})$ is a closed set. This completes the proof of the main theorem.
Theorem 5.1. For the singular set constructed in the proof of Theorem 1.2 we have

$$
\begin{equation*}
d \mathcal{P}_{\beta}(\Sigma(\boldsymbol{B}))=0 \quad \forall \beta>2, \tag{5.26}
\end{equation*}
$$

where $d \mathcal{P}_{\beta}(\cdot)$ is the $\beta$-dimensional parabolic Hausdorff measure. In particular, the Hausdorff dimension of $\Sigma(\boldsymbol{B})$ satisfies $\operatorname{dim}_{\mathcal{H}}(\Sigma(\boldsymbol{B})) \leq 2$.
Proof Let $2<\beta \leq \lambda$ be arbitrarily chosen. First we show that

$$
d \mathcal{P}_{\beta}\left(\Sigma_{k}\right)=0 \quad \forall k \in \mathbb{N} .
$$

Let $X_{0} \in \Sigma_{k}$. Fix $\varepsilon>0$. Then there exists $0<r\left(X_{0}\right)<\varepsilon$ and $m\left(X_{0}\right) \in \mathbb{N}$, such that

$$
\begin{equation*}
r\left(X_{0}\right)^{-2} \int_{Q_{r\left(X_{0}\right)}\left(X_{0}\right)}\left|\nabla \boldsymbol{B}_{m}\right|^{2} d x d t \geq \frac{1}{2 k} \quad \forall m \geq m\left(X_{0}\right) \tag{5.27}
\end{equation*}
$$

Clearly, the family of cylinders $\left\{Q_{r\left(X_{0}\right)}\left(X_{0}\right)\right\}_{X_{0} \in \Sigma_{k}}$ forms a covering of $\Sigma_{k}$. Thanks to the Vitali covering lemma there exists a pairwise disjoint family $\left\{Q_{r_{i}}\left(X_{i}\right)\right\}_{i \in \mathbb{N}}\left(r_{i}:=r\left(X_{i}\right)\right)$ such that $\left\{Q_{3 r_{i}}\left(X_{i}\right)\right\}_{i \in \mathbb{N}}$ covers $\Sigma_{k}$. Let $N \in \mathbb{N}$ be arbitrarily chosen. Set

$$
m_{N}:=\max \left\{m\left(X_{1}\right), \ldots, m\left(X_{N}\right)\right\} .
$$

Then, from (5.27) with $X_{0}=X_{i}(i=1, \ldots, N)$ and $m=m_{N}$ we infer

$$
\begin{aligned}
\sum_{i=1}^{N} r_{i}^{\beta} & \leq \varepsilon^{\beta-2} \sum_{i=1}^{N} r_{i}^{2} \leq 2 \varepsilon^{\beta-2} k \sum_{i=1}^{N} \int_{Q_{r_{i}}\left(X_{i}\right)}\left|\nabla \boldsymbol{B}_{m_{N}}\right|^{2} d x d t \leq 2 \varepsilon^{\beta-2} k \int_{Q}\left|\nabla \boldsymbol{B}_{m_{N}}\right|^{2} \\
& \leq \varepsilon^{\beta-2} k C\left(\left\|\boldsymbol{u}_{0}\right\|_{2}, \ldots\right) .
\end{aligned}
$$

This shows that

$$
\begin{equation*}
\sum_{i=1}^{\infty} r_{i}^{\beta} \leq \varepsilon^{\beta-2} k C\left(\left\|\boldsymbol{u}_{0}\right\|_{2}, \ldots\right) \tag{5.28}
\end{equation*}
$$

Consequently, $d \mathcal{P}_{\beta}\left(\Sigma_{k}\right)=0$, which implies that $d \mathcal{P}_{\beta}\left(\bigcup_{k=1}^{\infty} \Sigma_{k}\right)=0$.
Now, it remains to prove that $d \mathcal{P}_{\beta}\left(\Sigma_{\infty}\right)=0$. As we will see below this follows easily from the following implication

$$
\begin{equation*}
\sup _{0<r<\sqrt{t_{0}}} r^{-\beta} \int_{Q_{r}\left(X_{0}\right)}|\nabla B|^{2} d x d t<+\infty \quad \Longrightarrow \quad X_{0} \notin \Sigma_{\infty}, \quad X_{0} \in Q . \tag{5.29}
\end{equation*}
$$

Indeed, let $X_{0} \in Q$ such that the condition on the left in (5.29) holds true. Choose $0<\rho_{0}<\sqrt{t_{0}}$ sufficiently small (specified below) and set $r_{i}=2^{-i} \rho_{0} \quad(i \in \mathbb{N})$.

Fix $i \in \mathbb{N}$. By using the parabolic Poincaré-type inequality (see Lemma A.1, appendix below), arguing as in the proof of (2.12), we estimate

$$
\begin{align*}
& \int_{Q_{r_{i}}}\left|\boldsymbol{B}-\boldsymbol{B}_{r_{i}, X_{0}}\right|^{2} d x d t \\
& \qquad \leq c\left(1+\left|\boldsymbol{B}_{r_{i}, X_{0}}\right|^{2}\right) r_{i}^{-2} \int_{Q_{r_{i}}}|\nabla \boldsymbol{B}|^{2} d x d t \\
& \quad+c\left(1+\left|\boldsymbol{B}_{r_{i}, X_{0}}\right|^{2}\right) r^{-2} \int_{Q_{r_{i}}}\left(|\boldsymbol{g}|^{2}+|\boldsymbol{u}|^{2}\right) d x d t \\
& \quad+C_{6}\left\{r_{i}^{-2} \int_{Q_{r_{i}}}|\nabla \boldsymbol{B}|^{2}+\left(\int_{Q_{r_{i}}}|\boldsymbol{u}|^{4} d x d t\right)^{1 / 2}\right\} f_{Q_{r_{i}}}\left|\boldsymbol{B}-\boldsymbol{B}_{r_{i}, X_{0}}\right|^{2} d x d t \tag{5.30}
\end{align*}
$$

for an absolute constant $C_{6}>0$. Due to $\boldsymbol{u} \in L^{4}(Q)$ and our assumption on $X_{0}$ we may choose $\rho_{0}$ sufficiently small such that the numerical value in $\{\ldots\}$ is less than $\frac{1}{2 C_{6}}$, which leads to

$$
\begin{align*}
& f_{Q_{r_{i}}}\left|\boldsymbol{B}-\boldsymbol{B}_{r_{i}, X_{0}}\right|^{2} d x d t \\
& \qquad \begin{aligned}
\leq 2 c(1 & \left.+\left|\boldsymbol{B}_{r_{i}, X_{0}}\right|^{2}\right) r_{i}^{-2} \int_{Q_{r_{i}}}|\nabla \boldsymbol{B}|^{2} d x d t \\
& +2 c\left(1+\left|\boldsymbol{B}_{r_{i}, X_{0}}\right|^{2}\right) r_{i}^{-2} \int_{Q_{r_{i}}}\left(|\boldsymbol{g}|^{2}+|\boldsymbol{u}|^{2}\right) d x d t .
\end{aligned}
\end{align*}
$$

Appealing to Lemma4.2, we see that $\boldsymbol{u} \in L_{\mathrm{loc}}^{q}(Q)$ for all $1 \leq q<+\infty$. In particular, $\boldsymbol{u} \in \mathcal{M}^{2, \lambda}\left(Q_{\sqrt{t_{0}} / 2}\right)$. Recalling that $\boldsymbol{g} \in \mathcal{M}^{2, \lambda}(Q)$ and $\beta \leq \lambda$ from (5.31), we deduce that

$$
\begin{equation*}
f_{Q_{r_{i}}}\left|\boldsymbol{B}-\boldsymbol{B}_{r_{i}, X_{0}}\right|^{2} d x d t \leq c\left(1+\left|\boldsymbol{B}_{r_{i}, X_{0}}\right|^{2}\right) r_{i}^{\beta-2} \tag{5.32}
\end{equation*}
$$

with a constant $c>0$ depending neither on $r_{i}$ nor on $\rho_{0}$. Using the triangle inequality and employing (5.32), it follows that

$$
\begin{equation*}
\left|\left|\boldsymbol{B}_{r_{i+1}, X_{0}}\right|-\left|\boldsymbol{B}_{r_{i}, X_{0}}\right|\right| \leq C_{7}\left(1+\left|\boldsymbol{B}_{r_{i}, X_{0}}\right|\right) r_{i}^{(\beta-2) / 2} \tag{5.33}
\end{equation*}
$$

where $C_{7}=$ const $>0$ is independent on $r_{i}$ and $\rho_{0}$. Thus, eventually replacing $\rho_{0}$ by a smaller one, we may assume that

$$
C_{7} \sum_{i=0}^{\infty} r_{i}^{(\beta-2) / 2}=C_{7} \rho_{0}^{(\beta-2) / 2} \frac{1}{1-2^{(\beta-2) / 2}} \leq \frac{1}{2}
$$

Then, with help of Lemma A. 2 (see appendix below) from (5.33) we conclude that

$$
\begin{equation*}
\left|\boldsymbol{B}_{r_{i}, X_{0}}\right| \leq 1+2\left|\boldsymbol{B}_{\rho_{0}, X_{0}}\right| \quad \forall i \in \mathbb{N} \tag{5.34}
\end{equation*}
$$

what completes the proof of (5.29).
Now, let $\varepsilon>0$ be arbitrarily chosen. According to (5.29) for every $X_{0} \in \Sigma_{\infty}$ we may choose $0<r=r\left(X_{0}\right) \leq \varepsilon$ such that

$$
r^{-\beta} \int_{Q_{r}\left(X_{0}\right)}|\nabla B|^{2} d x d t \geq \frac{1}{\varepsilon}
$$

Thus, by the Vitali covering lemma there exists a pairwise disjoint family $\left\{Q_{r_{i}}\left(X_{i}\right)\right\}$ $\left(r_{i}:=r\left(X_{i}\right)\right.$ such that $\left\{Q_{3 r_{i}}\left(X_{i}\right)\right\}$ covers $\Sigma_{\infty}$. Similarly to the above we conclude

$$
\sum_{i=1}^{\infty} r_{i}^{\beta} \leq c \varepsilon\|\nabla \boldsymbol{B}\|_{2}^{2}
$$

Thus, $d \mathcal{P}_{\beta}\left(\Sigma_{\infty}\right)=0$, and the proof of the theorem is complete.

## Acknowledgements

Chae was partially supported by NRF grants 2006-0093854 and 2009-0083521, while Wolf has been supported by the Brain Pool Project of the Korea Federation of Science and Technology Societies (141S-1-3-0022).

## A Appendix

For $X=(x, t), Y=(y, s) \in \mathbb{R}^{n+1}$ we define the parabolic metric

$$
d_{p}(X, Y)=\max \left\{|x-y|,|s-t|^{\frac{1}{2}}\right\}, \quad X, Y \in \mathbb{R}^{n+1}
$$

Let $Q=\Omega \times(a, b)$, where $\Omega \subset \mathbb{R}^{n}$ is a bounded domain, and $-\infty<a<b<+\infty$. Then, for $0<\gamma<1$ we define the space of Hölder continuous functions on $Q, C^{\gamma, \frac{\gamma}{2}}(\bar{Q})$ by functions $f: \bar{Q} \rightarrow \mathbb{R}$ such that

$$
[f]_{C^{\gamma}, \frac{\gamma}{2}}=\sup _{X, Y \in \bar{Q}, X \neq Y} \frac{|f(X)-f(Y)|}{d_{p}(X, Y)^{\gamma}}<+\infty .
$$

The following parabolic version of the Poincare inequality has been proved in [22, Lemma B.3]

Lemma A. 1 (Parabolic Poincaré-type inequality). Let $Q_{r}=Q_{r}\left(X_{0}\right) \subset \mathbb{R}^{n+1}(n \in \mathbb{N})$. Let $u \in L^{p}\left(Q_{r}\right)$ be such that $\nabla u \in L^{p}\left(Q_{r}\right)(1 \leq p<+\infty)$. In addition suppose that there exists $\boldsymbol{f} \in L^{1}\left(Q_{r}\right)^{n}$ such that $\partial_{t} u=\nabla \cdot \boldsymbol{f}$ in sense of distributions, i.e.

$$
\begin{equation*}
\int_{Q_{r}} u \partial_{t} \varphi d x d t=\int_{Q_{r}} \boldsymbol{f} \cdot \nabla \varphi d x d t \quad \forall \varphi \in C_{\mathrm{c}}^{\infty}\left(Q_{r}\right) \tag{A.1}
\end{equation*}
$$

Then

$$
\begin{equation*}
f_{Q_{r}}\left|u-u_{Q_{r}}\right|^{p} d x d t \leq c r^{p} f_{Q_{r}}|\nabla u|^{p} d x d t+c r^{p}\left(f_{Q_{r}}|\boldsymbol{f}| d x d t\right)^{p}, \tag{A.2}
\end{equation*}
$$

where $c=$ const $>0$, depending on $n$ and $p$ only, but not on $r, u$ or $\boldsymbol{f}$.
The following elementary algebraic lemma has been used in the proof of Theorem 5.1 ,
Lemma A.2. Let $\left\{M_{i}\right\}$ and $\left\{\lambda_{i}\right\}$ be sequences of positive numbers such that $\sum \lambda_{i} \leq \frac{1}{2}$, and

$$
\begin{equation*}
\left|M_{j+1}-M_{j}\right| \leq\left(1+M_{j}\right) \lambda_{j} \quad \forall j \in \mathbb{N} . \tag{A.3}
\end{equation*}
$$

Then,

$$
\begin{equation*}
M_{i} \leq 1+2 M_{1} \quad \forall i \in \mathbb{N} \tag{A.4}
\end{equation*}
$$

Proof We prove the statement of this lemma by induction. Cleary, for $i=1$ the assertion is trivially fulfilled. Assume, (A.4) holds for $j=1, \ldots, i$. Then, with help of of triangle inequality and (A.3) for $j=1, \ldots, i$ we get

$$
\begin{aligned}
M_{i+1} & \leq M_{1}+\left|M_{i+1}-M_{1}\right| \leq M_{1}+\sum_{j=1}^{i}\left|M_{j+1}-M_{j}\right| \\
& \leq M_{1}+\sum_{j=1}^{i}\left(1+M_{j}\right) \lambda_{j} \leq M_{1}+\left(2+2 M_{1}\right) \sum_{j=1}^{i} \lambda_{j} \leq 1+2 M_{1} .
\end{aligned}
$$

Whence, the claim is proved.

## References

[1] M. Acheritogaray, P. Degond, A. Frouvelle, and J.-G. Liu, Kinetic formulation and global existence for the Hall-magnetohydrodynamic system, Kinetic and Related Models, 4 (2011), pp. 901-918.
[2] L. Caffarelli, R. Kohn, and L. Nirenberg, Partial regularity of suitable weak solutions of the Navier-Stokes equations, Comm. Pure Appl. Math., 35 (1982), pp. 771-831.
[3] D. Chae, P. Degond, and J.-G. Liu, Well-posedness for Hallmagnetohydrodynamics, Ann. Inst. Henri Poincare-Analyse Nonlineaire, 31 (2014), pp. 555-565.
[4] D. Chae and J. Lee, On the blow-up criterion and small data global existence for the Hall-magnetohydrodynamics, J. Differential Equations, 256 (2014), pp. 38353858.
[5] D. Chae and M. Schonbek, On the temporal decay for the Hallmagnetohydrodynamic equations, J. Differential Equations, 255 (2013), pp. 39713982.
[6] D. Chae and J. Wolf, On partial regularity for the steady Hallmagnetohydrodynamics system, preprint, (2015).
[7] E. Dumas and F. Sueur, On the weak solutions to the Maxwell-Landau-Lifshitz equations and to the Hall-magnetohydrodynamic equations, Comm. Math. Phys., 330 (2014), pp. 1179-1225.
[8] J. Fan, S. Huang, and G. Nakamura., Well-posedness for the axisymmetric incompressible viscous Hall-magnetohydrodynamic equations, Appl. Math. Lett., 26 (2013), pp. 963-967.
[9] T. Forbes, Magnetic reconnection in solar flares, Geophys. Astrophys. Fluid Dyn., 62 (1991), pp. 15-36.
[10] M. Giaquinta, Multiple integrals in the calculus of variations and nonlinear elliptic systems, Ann. of Math. Studies, vol. 105, Princeton Univ. press, Princeton, New Jersey, 1983.
[11] H. Homann and R. Grauer, Bifurcation analysis of magnetic reconnection in Hall-MHD systems, Physica D, 208 (2005), pp. 59-72.
[12] O. A. Ladyzehnskaya and G. A. Seregin, On partial regularity of sutiable weak solutions to the three-dimensional Navier-Stokes equations, J. Math. Fluid Mech., 1 (1999), pp. 356-387.
[13] M. J. Lighthill, Studies on magnetohydrodynamic waves and other anisotropic wave motions, Philos. Trans. R. Soc. Lond., Ser. A (1960), pp. 397-430.
[14] F. H. Lin, A new proof of the Caffarelli-Kohn-Nirenberg theorem, Comm. Pure Appl. Math., 51 (1998), pp. 241-257.
[15] H. Miura and D. Hori, Hall effects on local structure in decaying MHD turbulence, J. Plasma Fusion Res., 8 (2009), pp. 73-76.
[16] J. M. Polygiannakis and X. Moussas, A review of magneto-vorticity induction in Hall- MHD plasmas, Plasma Phys. Control \& Fusion, 43 (2001), pp. 195-221.
[17] G. D. Prato, Spazi $\mathcal{L}^{p, \theta}(\Omega, \delta)$ e loro proprietà, Ann. Mat. Pura Appl., 69 (1965), pp. 383-392.
[18] V. Scheffer, Partial regularity of solutions to the Navier-Stokes equations, Pacific J. Math., 66 (1976), pp. 535-552.
[19] D. Shalybkov and V. Urpin, The Hall effect and the decay of magnetic fields, Astron. Astrophys., (1997), pp. 685-690.
[20] A. N. Simakov and L. Chacón, Quantitative, analytical model for magnetic reconnection in Hall-magnetohydrodynamics, Phys. Rev. Lett., 101 (2008).
[21] M. Wardle, Star formation and the Hall effect, Astrophys. Space Sci., 292 (2004), pp. 317-323.
[22] J. Wolf, "Regularität schwacher Lösungen elliptischer und parabolischer Systeme partieller Differentialgleichungen mit Entartung. Der fall $1<p<2$ ", Dissertation, Humboldt-Universität zu Berlin, Berlin (2001).
[23] J. Wolf, On the local regularity of suitable weak solutions to the generalized NavierStokes equations, Annali della Universita Ferrara, (Doi 10.1007/s11565-014-0203-6) (2014).
[24] J. Wolf, On the local pressure of the Navier-Stokes equations and related systems, submitted, (2015).

