CONVEX INTEGRATION AND INFINITELY MANY WEAK SOLUTIONS TO THE PERONA-MALIK EQUATION IN ALL DIMENSIONS

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ABSTRACT. We prove that for all smooth nonconstant initial data the initial-Neumann boundary value problem for the Perona-Malik equation in image processing possesses infinitely many Lipschitz weak solutions on smooth bounded convex domains in all dimensions. Such existence results have not been known except for the one-dimensional problems. Our approach is motivated by reformulating the Perona-Malik equation as a nonhomogeneous partial differential inclusion with linear constraint and uncontrollable components of gradient. We establish a general existence result by a suitable Baire's category method under a pivotal density hypothesis. We finally fulfill this density hypothesis by convex integration based on certain approximations from an explicit formula of lamination convex hull of some matrix set involved.

1. INTRODUCTION

In this paper, we study the initial and Neumann boundary value problem:

$$
\begin{cases} u_t = \operatorname{div}\left(\frac{Du}{1+|Du|^2}\right) & \text{in } \Omega\times(0,T), \\ \partial u/\partial \mathbf{n} = 0 & \text{on } \partial\Omega\times(0,T), \\ u(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{1.1}
$$

where $\Omega \subset \mathbf{R}^n$ is a smooth bounded convex domain, $T > 0$ is a given number, $u = u(x,t)$ is the unknown function with u_t denoting its time-derivative and $Du = (u_{x_1}, \cdots, u_{x_n})$ its spatial gradient, $\mathbf{n}$ is outer unit normal on $\partial\Omega$, and $u_0(x)$ is a given smooth function satisfying

$$
Du_0 \not\equiv 0 \ \text{ in } \Omega, \quad \partial u_0/\partial\mathbf{n} = 0 \ \text{ on } \partial\Omega. \tag{1.2}
$$

Problem (1.1), especially when $n = 2$, is a famous *Perona-Malik model* in image processing introduced by Perona and Malik [27] for denoising and edge enhancement of a computer vision. In this model, $u(x,t)$ represents an improved version of the initial gray level $u_0(x)$ of a noisy picture. The anisotropic diffusion $\operatorname{div}(\frac{Du}{1+|Du|^2})$ is forward parabolic in the *subcritical* region where $|Du| < 1$ and backward parabolic in the *supercritical* region where $|Du| > 1$.

The expectation of the Perona-Malik model is that disturbances with small gradient in the subcritical region will be smoothed out by the forward parabolic diffusion, while sharp edges corresponding to large gradient in the supercritical region will be enhanced by the backward parabolic equation. Such expected phenomenology has

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been implemented and observed in some numerical experiments, showing the stability and effectiveness of the model. On the other hand, many analytical works have shown that the model is highly ill-posed when the initial datum u_0 is *transcritical* in Ω; namely, there are subregions in Ω where $|Du_0| < 1$ and where $|Du_0| > 1$, respectively. For transcritical initial data, due to the backward parabolicity, even a proper notion and the existence of well-posed solutions to (1.1) have remained largely unsettled. Most analytical works have focused on the study of singular perturbations, Young measure solutions, numerical scheme analyses, and examples and properties of certain classical solutions; see, e.g., [3, 6, 12, 13, 14, 17, 18].

The present paper addresses the analytical issue concerning the existence of certain exact weak solutions to problem (1.1). Let $\Omega_T = \Omega \times (0,T)$. We say that a Lipschitz function $u \in W^{1,\infty}(\Omega_T)$ is a *weak solution* to (1.1) provided for all $\zeta \in C^\infty(\bar{\Omega}_T)$ and $s \in [0,T]$,

$$\int_\Omega u(x,s)\zeta(x,s)dx + \int_0^s \int_\Omega (-u\zeta_t + \sigma(Du)\cdot D\zeta)\,dxdt = \int_\Omega u_0(x)\zeta(x,0)dx, \tag{1.3}$$

where $\sigma(p) = \frac{p}{1+|p|^2}$ $(p \in \mathbf{R}^n)$ is the *Perona-Malik function.* The first existence result on such weak solutions was established by K. Zhang [32] for the one-dimensional problem, whose pivotal idea is to reformulate the one-dimensional Perona-Malik equation as a differential inclusion with linear constraint and then prove the existence using a modified method of convex integration following the ideas of [20, 24]. Based on a similar approach of differential inclusion, we have recently proved in [19] that for all dimensions n if the domain Ω is a ball and the nonconstant initial function u_0 is smooth and radially symmetric then (1.1) admits infinitely many radially symmetric Lipschitz weak solutions.

The main purpose of this paper is to extend the results of [19, 32] to problem (1.1) on all n-dimensional smooth convex domains for all nonconstant smooth initial data.

Our main result of the paper is the following theorem.

Theorem 1.1. *Let $\Omega \subset \mathbf{R}^n$ be a bounded convex domain with $\partial\Omega$ of $C^{2+\alpha}$ and let $u_0 \in C^{2+\alpha}(\bar{\Omega})$ satisfy* (1.2) *for some constant $0 < \alpha < 1$. Then* (1.1) *possesses infinitely many weak solutions. Moreover, if $\|Du_0\|_{L^\infty(\Omega)} \geq 1$ and $\lambda > 0$, then these weak solutions u will satisfy the* almost gradient maximum principle:

$$\|Du\|_{L^\infty(\Omega_T)} \leq \|Du_0\|_{L^\infty(\Omega)} + \lambda.$$

This theorem asserts that the Perona-Malik problem (1.1) admits infinitely many Lipschitz weak solutions no matter whether the initial datum is subcritical, supercritical, or transcritical.

Existence of classical solutions to Problem (1.1) depends heavily on the initial data u_0. Kawohl & Kutev [17] showed that a classical solution exists in any dimension if u_0 is subcritical in $\bar{\Omega}$ (see also [18]). Later, Gobbino [15] showed that the problem cannot admit a global classical solution when $n = 1$ if u_0 is transcritical. Recently, Ghisi & Gobbino [13, 14] have studied the existence and properties of certain classical solutions of the Perona-Malik equation in the one-dimensional or n-dimensional radially symmetric cases with suitably chosen initial data; their initial values can be arbitrarily given in the subcritical region, but the values in the supercritical region must be *predetermined* by the subcritical initial values.

We remark that the convexity of the domain is needed to guarantee a *gradient maximum principle* for the classical solution to initial-Neumann boundary value problem of a class of quasilinear uniformly parabolic equations (see Theorem 2.1 below). This gradient maximum principle turns out to be crucial for the proof of main theorem, and an example in [1, Theorem 4.1] showed that such a gradient maximum principle may fail even for heat equation without the convexity of the domain. However, domain convexity seemed to be overlooked in [17, Theorem 6.1].

For the proof of Theorem 1.1, in what follows, we assume the initial function u_0 satisfies

$$\int_\Omega u_0(x)dx = 0, \tag{1.4}$$

since otherwise one can solve solution $\tilde{u}$ of (1.1) with new initial datum $\tilde{u}_0 = u_0 - \frac{1}{|\Omega|}\int_\Omega u_0 dx$; then $u = \tilde{u} + \frac{1}{|\Omega|}\int_\Omega u_0 dx$ will solve (1.1).

Our proof is based on a crucial generalization of the ideas of [19, 32, 33]. Let us discuss this generalization in some details because it exhibits several different features from the one-dimensional setup.

Assume $u \in W^{1,\infty}(\Omega_T)$ is a weak solution to (1.1) and suppose there exists a vector function $v \in W^{1,\infty}(\Omega_T; \mathbf{R}^n)$ such that $\operatorname{div} v = u$ and $v_t = \sigma(Du)$ a.e. in Ω_T. Let $w = (u, v)\colon \Omega_T \to \mathbf{R}^{1+n}$, with space-time Jacobian matrix denoted by

$$\nabla w = \begin{pmatrix} Du & u_t \\ Dv & v_t \end{pmatrix}$$

as an element in the matrix space $\mathbf{M}^{(1+n)\times(n+1)}$. Given $s \in \mathbf{R}$, define the set $K(s)$ in $\mathbf{M}^{(1+n)\times(n+1)}$ by

$$K(s) = \left\{ \begin{pmatrix} p & c \\ B & \sigma(p) \end{pmatrix} \,\Big|\, p \in \mathbf{R}^n,\ c \in \mathbf{R},\ B \in \mathbf{M}^{n\times n},\ \operatorname{tr} B = s \right\}. \tag{1.5}$$

Then $w = (u, v)$ solves the *nonhomogeneous partial differential inclusion*:

$$\nabla w(x,t) \in K(u(x,t)), \quad \text{a.e. } (x,t) \in \Omega_T.$$

Conversely, suppose we have found a function $\Phi = (u^*, v^*)$, where $u^* \in W^{1,\infty}(\Omega_T)$ and $v^* \in W^{1,\infty}(\Omega_T; \mathbf{R}^n)$, such that

$$\begin{cases} u^*(x,0) = u_0(x) \ (x \in \Omega), \\ \operatorname{div} v^* = u^* \ \text{ a.e. in } \Omega_T, \\ v^*(\cdot,t)\cdot \mathbf{n}|_{\partial\Omega} = 0 \ \ \forall\, t \in [0,T]. \end{cases} \tag{1.6}$$

Assume $w = (u, v) \in W^{1,\infty}(\Omega_T; \mathbf{R}^{1+n})$ solves the Dirichlet problem of nonhomogeneous differential inclusion:

$$\begin{cases} \nabla w(x,t) \in K(u(x,t)), & \text{a.e. } (x,t) \in \Omega_T, \\ w(x,t) = \Phi(x,t), & (x,t) \in \partial\Omega_T. \end{cases} \tag{1.7}$$

Then it can be verified that u is a weak solution to (1.1) (see Lemma 3.2).

The Dirichlet problem (1.7) falls into the framework of general nonhomogeneous partial differential inclusions studied by Dacorogna & Marcellini [10] using Baire's category method and by Müller & Sychev [26] using the convex integration method; see also [20]. Study of such differential inclusions has stemmed from the successful understanding of homogeneous differential inclusions of the form $Du(x) \in K$ first encountered in the study of crystal microstructure by Ball & James [2], Chipot & Kinderlehrer [7] and Müller & Šverák [24]. Recently, the method of differential inclusions has been successfully applied to other important problems ; see, e.g., [8, 11, 23, 25, 30, 31].

We point out that the existence result of [26] is not applicable to problem (1.7) even in dimension $n = 1$, as has already been noticed in [32, 33]. A key condition in the main existence theorem of [26], when applied to (1.7), would require that the boundary function Φ satisfy

$$\nabla\Phi(x,t) \in U(u^*(x,t)) \cup K(u^*(x,t)), \quad a.e. \ (x,t) \in \Omega_T,$$

where $U(s) \subset \mathbf{M}^{(1+n)\times(n+1)}$ $(s \in \mathbf{R})$ are bounded sets that are *reducible to* $K(s)$ in the sense that, for every $s_0 \in \mathbf{R}$, $\xi_0 \in U(s_0)$, $\epsilon > 0$, and bounded Lipschitz domain $G \subset \mathbf{R}^{n+1}$, there exist a piecewise affine function $w \in W_0^{1,\infty}(G; \mathbf{R}^{1+n})$ and a $\delta > 0$ satisfying, for a.e. $z = (x,t) \in G$,

$$\xi_0 + \nabla w(z) \in \bigcap_{|s-s_0|<\delta} U(s), \quad \int_G \operatorname{dist}(\xi_0 + \nabla w(z), K(s_0))\, dz < \epsilon |G|.$$

The second condition would imply $\operatorname{tr} B_0 = s_0$ for each $\xi_0 = \begin{pmatrix} p_0 & c_0 \\ B_0 & \beta_0 \end{pmatrix} \in U(s_0)$ and $s_0 \in \mathbf{R}$; but then $\cap_{|s-s_0|<\delta} U(s) = \emptyset$, which makes the first condition impossible.

However, certain geometric structures of the set $K(0)$ turn out still useful, especially when it comes to the relaxation of homogeneous differential inclusion $\nabla\omega(z) \in K(0)$ with $\omega = (\varphi, \psi)$. We explicitly compute the first-order lamination set $L(K(0))$ of $K(0)$ consisting of all $\xi \in \mathbf{M}^{(1+n)\times(n+1)} \setminus K(0)$ such that $\xi = \lambda\xi_1 + (1-\lambda)\xi_2$ for some $\lambda \in (0,1)$ and $\xi_1, \xi_2 \in K(0)$ with $\operatorname{rank}(\xi_1 - \xi_2) = 1$. We obtain the explicit formula (see Theorem 4.1)

$$L(K(0)) = \left\{ \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} \;\middle|\; \operatorname{tr} B = 0, \ |\beta|^2 + (p\cdot\beta)^2 - p\cdot\beta < 0 \right\},$$

which enables us to extract enough information on the diagonal components of differential inclusion $\nabla\omega(z) \in K(0)$ and establish a relaxation result on $(D\varphi, \psi_t)$ (see Theorem 4.6). Although for such relaxation we must have $\operatorname{div}\psi = 0$, the resulting φ_t can be arbitrarily small; this is important for the subsequent handling of the linear constraint $\operatorname{div} v = u$ in problem (1.7).

Another difficulty concerning problem (1.7) is that when $n = 1$, one can control $\|v_x\|_{L^\infty(\Omega_T)}$ in terms of $u = v_x$ (see [32]); however, for $n \geq 2$, it is impossible to control $\|Dv\|_{L^\infty(\Omega_T)}$ in terms of $u = \operatorname{div} v$. So, if $n \geq 2$, the space $W^{1,\infty}(\Omega_T; \mathbf{R}^n)$ is not suitable for the function v. It turns out that a suitable space for v is the space $W^{1,2}((0,T); L^2(\Omega; \mathbf{R}^n))$ of abstract functions (see Lemma 3.1); in this setting, the linear constraint $\operatorname{div} v = u$ must be understood in the sense of distributions.

We design a new approach to overcome the lack of control on Dv: instead of defining an admissible class for $w = (u, v)$, we define a suitable admissible class for only the functions $u \in W^{1,\infty}(\Omega_T)$, treating v as auxiliary functions. Of course, during all the relevant constructions, the linear constraint $\operatorname{div} v = u$ must be satisfied. In this regard, we need a linear operator $\mathcal{R}$ that serves as a (distributional) right inverse of the divergence operator: $\operatorname{div}\mathcal{R} = Id$. By the results of [4], such an operator may not exist as a bounded operator on certain spaces, but for our purpose, it suffices to construct such an operator $\mathcal{R}$ that is bounded from $L^\infty(Q \times I)$ to $L^\infty(Q \times I; \mathbf{R}^n)$ for the box domains $Q \times I$ in $\mathbf{R}^{n+1}$; this is achieved by following some construction in [4].

Finally we remark that although the result of this paper heavily relies on the explicit formula of $L(K(0))$, the method can handle some general forward-backward parabolic equations; however, we do not intend to discuss further results of this direction in the present paper.

The rest of the paper is organized as follows. In Section 2, we collect several necessary preliminary results, some of which cannot be found in the standard references. In Section 3, we set up a new general procedure for proving Theorem 1.1 under a pivotal density hypothesis of an admissible class $\mathcal{U}$; this setup is suitable for a Baire's category method and simplifies some of the arguments even for the one-dimensional problem. In Section 4, as the heart of the matter for fulfilling the density hypothesis and thus proving Theorem 1.1, we present the essential geometric considerations, including an explicit computation of the set $L(K(0))$ above and establishing a critical relaxation property (Theorem 4.6) by convex integration with linear constraint. In Section 5, we construct the suitable admissible class $\mathcal{U}$ after defining a specific boundary function $\Phi = (u^*, v^*)$. In Section 6, we fulfill the key density hypothesis for admissible class $\mathcal{U}$ (Theorem 6.1) and finally complete the proof of Theorem 1.1 according to the setup of Section 3.

2. Some preliminary results

2.1. **Uniformly parabolic quasilinear equations.** We refer to the standard references (e.g., [21, 22]) for general theory of parabolic equations, including some notation concerning functions and domains of class $C^{k+\alpha}$ for integer $k \geq 0$ and number $0 < \alpha < 1$.

Assume $f \in C^3([0, \infty))$ is a function satisfying

$$\theta \leq f(s) + 2sf'(s) \leq \Theta \quad \forall\, s \geq 0, \tag{2.1}$$

where $\Theta \geq \theta > 0$ are constants. This condition is equivalent to $\theta \leq (sf(s^2))' \leq \Theta$ for all $s \in \mathbf{R}$; hence, $\theta \leq f(s) \leq \Theta$ for all $s \geq 0$. Let

$$A(p) = f(|p|^2)p \quad (p \in \mathbf{R}^n).$$

Then we have

$$A^i_{p_j}(p) = f(|p|^2)\delta_{ij} + 2f'(|p|^2)p_i p_j \quad (i, j = 1, 2, \cdots, n;\ p \in \mathbf{R}^n)$$

and hence the *uniform ellipticity condition*:

$$\theta |q|^2 \leq \sum_{i,j=1}^n A^i_{p_j}(p) q_i q_j \leq \Theta |q|^2 \quad \forall\, p,\ q \in \mathbf{R}^n. \tag{2.2}$$

Theorem 2.1. *Let $\Omega \subset \mathbf{R}^n$ be a bounded convex domain with $\partial\Omega$ of $C^{2+\alpha}$ and $u_0 \in C^{2+\alpha}(\bar{\Omega})$ satisfy $Du_0 \cdot \mathbf{n} = 0$ on $\partial\Omega$. Then the initial-Neumann boundary value* *problem*

$$\begin{cases} u_t = \operatorname{div}(A(Du)) & \textit{in } \Omega_T, \\ \partial u/\partial \mathbf{n} = 0 & \textit{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \textit{for } x \in \Omega \end{cases} \tag{2.3}$$

has a unique solution $u \in C^{2+\alpha,\frac{2+\alpha}{2}}(\bar\Omega_T)$. *Moreover, the gradient maximum principle holds:*

$$\|Du\|_{L^\infty(\Omega_T)} = \|Du_0\|_{L^\infty(\Omega)}. \tag{2.4}$$

Proof. 1. As problem (2.3) is uniformly parabolic by (2.2), the existence of unique classical solution u in $C^{2+\alpha,\frac{2+\alpha}{2}}(\bar\Omega_T)$ follows from the standard theory; see [22, Theorem 13.24]. To prove the gradient maximum principle (2.4), note that, since $A \in C^3(\mathbf{R}^n)$, a standard bootstrap argument based on the regularity theory of linear parabolic equations [21, 22] shows that the solution u has all continuous partial derivatives $u_{x_ix_jx_k}$ and u_{x_it} within Ω_T for $1 \le i,j,k \le n$.

2. Let $v = |Du|^2$. Then, within Ω_T, we compute

$$\begin{aligned} \Delta v &= 2Du \cdot D(\Delta u) + 2|D^2u|^2, \\ u_t &= \operatorname{div}(A(Du)) = \operatorname{div}(f(v)Du) = f'(v)Dv \cdot Du + f(v)\Delta u, \\ Du_t &= f''(v)(Dv \cdot Du)Dv + f'(v)(D^2u)Dv \\ &\quad + f'(v)(D^2v)Du + f'(v)(\Delta u)Dv + f(v)D(\Delta u). \end{aligned}$$

Plugging these equations into $v_t = 2Du \cdot Du_t$, we obtain

$$v_t - \mathcal{L}(v) - B \cdot Dv = -2f(|Du|^2)|D^2u|^2 \le 0 \quad \text{in } \Omega_T, \tag{2.5}$$

where operator $\mathcal{L}(v)$ and coefficient B are defined by

$$\begin{aligned} \mathcal{L}(v) &= f(|Du|^2)\Delta v + 2f'(|Du|^2)Du \cdot (D^2v)Du, \\ B &= 2f''(v)(Dv \cdot Du)Du + 2f'(v)(D^2u)Du + 2f'(v)(\Delta u)Du. \end{aligned}$$

We write $\mathcal{L}(v) = \sum_{i,j=1}^n a_{ij}v_{x_ix_j}$, with coefficients $a_{ij} = a_{ij}(x,t)$ given by

$$a_{ij} = A^i_{p_j}(Du) = f(|Du|^2)\delta_{ij} + 2f'(|Du|^2)u_{x_i}u_{x_j} \quad (i,j = 1,\cdots,n).$$

Note that on $\bar\Omega_T$ all eigenvalues of the matrix (a_{ij}) lie in $[\theta,\Theta]$.

3. We show

$$\max_{(x,t)\in\bar\Omega_T} v(x,t) = \max_{x\in\bar\Omega} v(x,0),$$

which proves (2.4). We prove this by contradiction. Suppose

$$M := \max_{(x,t)\in\bar\Omega_T} v(x,t) > \max_{x\in\bar\Omega} v(x,0). \tag{2.6}$$

Let $(x_0,t_0) \in \bar\Omega_T$ be such that $v(x_0,t_0) = M$; then $t_0 > 0$. If $x_0 \in \Omega$, then the strong maximum principle applied to (2.5) would imply that v is constant on Ω_{t_0}, which yields $v(x,0) \equiv M$ on $\bar\Omega$, a contradiction to (2.6). Consequently $x_0 \in \partial\Omega$ and thus $v(x_0,t_0) = M > v(x,t)$ for all $(x,t) \in \Omega_T$. We can then apply Hopf's Lemma for parabolic equations [28] to (2.5) to deduce $\partial v(x_0,t_0)/\partial\mathbf{n} > 0$. However, a result of [1, Lemma 2.1] (see also [16, Theorem 2]) asserts that $\partial v/\partial\mathbf{n} \le 0$ on $\partial\Omega \times [0,T]$ (convexity of Ω is used and necessary here), which gives a desired contradiction. $\square$

2.2. **Modification of the Perona-Malik function.** We need to modify the Perona-Malik function $\sigma(p) = \frac{p}{1+|p|^2}$ to obtain a uniformly parabolic problem of type (2.3). For this purpose, let

$$\rho(s) = \frac{s}{1+s^2} \quad (s \geq 0)$$

and, for $0 < \delta < 1/2$, let $m = m_\pm(\delta)$ be the solutions of $\rho(m) = \delta$; that is,

$$m_\pm(\delta) = \frac{1 \pm \sqrt{1-4\delta^2}}{2\delta}. \tag{2.7}$$

The following result can be proved in a similar way as in [6, 32]; we omit the proof (see Figure 1).

Lemma 2.2. *Let $0 < \delta < 1/2$ and $1 < \Lambda < m_+(\delta)$. Then there exists a function $\rho^* \in C^3([0,\infty))$ satisfying that*

$$\begin{aligned} &\rho^*(s) = \rho(s) \quad \forall\, 0 \leq s \leq m_-(\delta), \\ &\rho^*(s) < \rho(s) \quad \forall\, m_-(\delta) < s \leq \Lambda, \\ &\theta \leq (\rho^*)'(s) \leq \Theta \quad \forall\, 0 \leq s < \infty \end{aligned}$$

for some constants $\Theta > \theta > 0$. Moreover, define $f(0) = 1$ and $f(s) = \rho^*(\sqrt{s})/\sqrt{s}$ for $s > 0$; then $f \in C^3([0,\infty))$ and (2.1) is fulfilled.

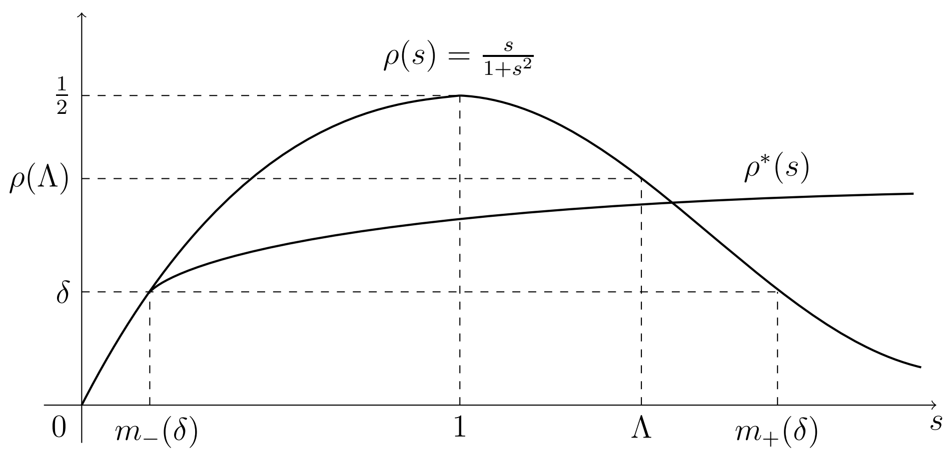

FIGURE 1. The graphs of function $\rho(s)$ and the modified function $\rho^*(s)$ in Lemma 2.2.

2.3. **Right inverse of the divergence operator.** To deal with the linear constraint $\operatorname{div} v = u$, we follow an argument of [4, Lemma 4] to construct a right inverse $\mathcal{R}$ of the divergence operator: $\operatorname{div} \mathcal{R} = Id$ (in the sense of distributions in Ω_T). For the purpose of this paper, the construction of $\mathcal{R}$ is restricted to the *box* domains, by which we mean domains given by $Q = J_1 \times J_2 \times \cdots \times J_n$, where $J_i = (a_i, b_i) \subset \mathbf{R}$ is a finite open interval.

Given such a box Q, we define a linear operator $\mathcal{R}_n \colon L^\infty(Q) \to L^\infty(Q;\mathbf{R}^n)$ inductively on dimension n. If $n = 1$, for $u \in L^\infty(J_1)$, we define $v = \mathcal{R}_1 u$ by

$$v(x_1) = \int_{a_1}^{x_1} u(s)ds \quad (x_1 \in J_1).$$

Assume $n = 2$. Let $u \in L^\infty(J_1 \times J_2)$. Set $\tilde u(x_1) = \int_{a_2}^{b_2} u(x_1, s)\, ds$ for $x_1 \in J_1$. Then $\tilde u \in L^\infty(J_1)$. Let $\tilde v = \mathcal{R}_1 \tilde u$; that is,

$$\tilde v(x_1) = \int_{a_1}^{x_1} \tilde u(s) ds = \int_{a_1}^{x_1} \int_{a_2}^{b_2} u(s, \tau)\, d\tau ds \quad (x_1 \in J_1).$$

Let $\rho_2 \in C_c^\infty(a_2, b_2)$ be such that $0 \le \rho_2(s) \le \frac{C_0}{b_2 - a_2}$ and $\int_{a_2}^{b_2} \rho_2(s) ds = 1$. Define $v = \mathcal{R}_2 u \in L^\infty(J_1 \times J_2; \mathbf{R}^2)$ by $v = (v^1, v^2)$ with $v^1(x_1, x_2) = \rho_2(x_2)\tilde v(x_1)$ and

$$v^2(x_1, x_2) = \int_{a_2}^{x_2} u(x_1, s) ds - \tilde u(x_1) \int_{a_2}^{x_2} \rho_2(s) ds.$$

Note that if $u \in W^{1,\infty}(J_1 \times J_2)$ then $\tilde u \in W^{1,\infty}(J_1)$; hence $v = \mathcal{R}_2 u \in W^{1,\infty}(J_1 \times J_2; \mathbf{R}^2)$ and $\operatorname{div} v = u$ a.e. in $J_1 \times J_2$. Moreover, if $u \in C^1(\overline{J_1 \times J_2})$ then v is in $C^1(\overline{J_1 \times J_2}; \mathbf{R}^2)$.

Assume that we have defined the operator $\mathcal{R}_{n-1}$. Let $u \in L^\infty(Q)$ with $Q = J_1 \times J_2 \times \cdots \times J_n$ and $x = (x', x_n) \in Q$, where $x' \in Q' = J_1 \times \cdots \times J_{n-1}$ and $x_n \in J_n$. Set $\tilde u(x') = \int_{a_n}^{b_n} u(x', s)\, ds$ for $x' \in Q'$. Then $\tilde u \in L^\infty(Q')$. By the assumption, $\tilde v = \mathcal{R}_{n-1}\tilde u \in L^\infty(Q'; \mathbf{R}^{n-1})$ is defined. Write $\tilde v(x') = (Z^1(x'), \cdots, Z^{n-1}(x'))$, and let $\rho_n \in C_c^\infty(a_n, b_n)$ be a function satisfying $0 \le \rho_n(s) \le \frac{C_0}{b_n - a_n}$ and $\int_{a_n}^{b_n} \rho_n(s) ds = 1$. Define $v = \mathcal{R}_n u \in L^\infty(Q; \mathbf{R}^n)$ as follows. For $x = (x', x_n) \in Q$, $v(x) = (v^1(x), v^2(x), \cdots, v^n(x))$ is defined by

$$\begin{aligned} v^k(x', x_n) &= \rho_n(x_n) Z^k(x') \quad (k = 1, 2, \cdots, n-1), \\ v^n(x', x_n) &= \int_{a_n}^{x_n} u(x', s) ds - \tilde u(x') \int_{a_n}^{x_n} \rho_n(s) ds. \end{aligned}$$

Then $\mathcal{R}_n \colon L^\infty(Q) \to L^\infty(Q; \mathbf{R}^n)$ is a well-defined linear operator; moreover,

$$\|\mathcal{R}_n u\|_{L^\infty(Q)} \le C_n \left(|J_1| + \cdots + |J_n|\right) \|u\|_{L^\infty(Q)}, \tag{2.8}$$

where $C_n > 0$ is a constant depending only on n.

As in the case $n = 2$, we see that if $u \in W^{1,\infty}(Q)$ then $v = \mathcal{R}_n u \in W^{1,\infty}(Q; \mathbf{R}^n)$ and $\operatorname{div} v = u$ a.e. in Q. Also, if $u \in C^1(\bar Q)$ then $v = \mathcal{R}_n u$ is in $C^1(\bar Q; \mathbf{R}^n)$. Moreover, if $u \in W_0^{1,\infty}(Q)$ satisfies $\int_Q u(x) dx = 0$, then one can easily show that $v = \mathcal{R}_n u \in W_0^{1,\infty}(Q; \mathbf{R}^n)$.

Let I be a finite open interval in $\mathbf{R}$. We now extend the operator $\mathcal{R}_n$ to an operator $\mathcal{R}$ on $L^\infty(Q \times I)$ by defining, for a.e. $(x, t) \in Q \times I$,

$$(\mathcal{R}u)(x, t) = (\mathcal{R}_n u(\cdot, t))(x) \quad \forall\, u \in L^\infty(Q \times I). \tag{2.9}$$

Then $\mathcal{R} \colon L^\infty(Q \times I) \to L^\infty(Q \times I; \mathbf{R}^n)$ is a bounded linear operator.

We have the following result.

Theorem 2.3. *Let $u \in W_0^{1,\infty}(Q \times I)$ satisfy $\int_Q u(x, t)\, dx = 0$ for all $t \in I$. Then $v = \mathcal{R}u \in W_0^{1,\infty}(Q \times I; \mathbf{R}^n)$, $\operatorname{div} v = u$ a.e. in $Q \times I$, and*

$$\|v_t\|_{L^\infty(Q \times I)} \le C_n \left(|J_1| + \cdots + |J_n|\right) \|u_t\|_{L^\infty(Q \times I)}, \tag{2.10}$$

where $Q = J_1 \times \cdots \times J_n$ and C_n is the same constant as in (2.8). Moreover, if $u \in C^1(\overline{Q \times I})$ then $v = \mathcal{R}u \in C^1(\overline{Q \times I}; \mathbf{R}^n)$.

Proof. Given $u \in W_0^{1,\infty}(Q \times I)$, let $v = \mathcal{R}u$. We easily verify that v is Lipschitz continuous in t and hence v_t exists. It also follows that $v_t = \mathcal{R}(u_t)$. Clearly, if $\int_Q u(x,t)dx = 0$ then $v(x,t) = 0$ whenever $t \in \partial I$ or $x \in \partial Q$. This proves $v \in W_0^{1,\infty}(Q \times I; \mathbf{R}^n)$ and the estimate (2.10) follows from (2.8). Finally, from the definition of $\mathcal{R}u$, we see that if $u \in C^1(\overline{Q \times I})$ then $v = \mathcal{R}u \in C^1(\overline{Q \times I}; \mathbf{R}^n)$. □

3. General setup for existence

In this section we set up the general procedure for proving our main theorem, Theorem 1.1.

3.1. **Sufficient conditions for weak solutions.**

Since our setup differs from the usual formulation of differential inclusions, we first prove the next two results to clarify some relevant issues, which are elementary but not too obvious.

Lemma 3.1. *Suppose $u \in W^{1,\infty}(\Omega_T)$ is such that $u(x,0) = u_0(x)$ $(x \in \Omega)$, there exists a vector function $v \in W^{1,2}((0,T); L^2(\Omega; \mathbf{R}^n))$ with weak time-derivative v_t satisfying $v_t = \sigma(Du)$ a.e. in Ω_T, and for each $\zeta \in C^\infty(\bar{\Omega}_T)$ and $t \in [0,T]$,*

$$\int_\Omega v(x,t) \cdot D\zeta(x,t)\, dx = -\int_\Omega u(x,t)\zeta(x,t)\, dx. \tag{3.1}$$

Then u is a weak solution to (1.1).

Proof. To verify (1.3), given any $\zeta \in C^\infty(\bar{\Omega}_T)$, let

$$g(t) = \int_\Omega u(x,t)\zeta(x,t)dx, \quad h(t) = \int_\Omega u(x,t)\zeta_t(x,t)dx \quad (0 \le t \le T).$$

Then for each $\psi \in C_c^\infty(0,T)$, by (3.1),

$$\begin{aligned} \int_0^T \psi_t(t) g(t)\, dt &= -\int_0^T \int_\Omega \psi_t(t) v(x,t) \cdot D\zeta(x,t)\, dxdt, \\ \int_0^T \psi(t) h(t)\, dt &= -\int_0^T \int_\Omega \psi(t) v(x,t) \cdot D\zeta_t(x,t)\, dxdt. \end{aligned}$$

Since $v \in W^{1,2}((0,T); L^2(\Omega; \mathbf{R}^n))$ and $v_t = \sigma(Du)$, one has

$$\int_0^T \int_\Omega (\psi(t) D\zeta(x,t))_t \cdot v(x,t) dxdt = -\int_0^T \int_\Omega \psi(t)\sigma(Du(x,t)) \cdot D\zeta(x,t)\, dxdt.$$

Now as $(\psi D\zeta)_t = \psi_t D\zeta + \psi D\zeta_t$, combining previous equations, we have

$$\int_0^T \psi_t(t) g(t)\, dt = \int_0^T \psi(t) \left(-h(t) + \int_\Omega \sigma(Du(x,t)) \cdot D\zeta(x,t)\, dx \right) dt;$$

this proves that g is weakly differentiable in $(0,T)$ with weak derivative

$$g'(t) = h(t) - \int_\Omega \sigma(Du(x,t)) \cdot D\zeta(x,t)\, dx \quad \text{a.e. } t \in (0,T).$$

From this, upon integrating, (1.3) follows for all $s \in [0,T]$. □

Let $\Phi = (u^*, v^*) \in W^{1,\infty}(\Omega_T; \mathbf{R}^{1+n})$ satisfy (1.6), and let $W^{1,\infty}_{u^*}(\Omega_T)$ and $W^{1,\infty}_{v^*}(\Omega_T; \mathbf{R}^n)$ denote the usual Dirichlet classes with boundary traces u^*, v^*, respectively.

Let $\mathcal{U}$ be *some* nonempty and bounded subset of $W^{1,\infty}_{u^*}(\Omega_T)$ such that for each $u \in \mathcal{U}$, there exists a vector function $v \in W^{1,\infty}_{v^*}(\Omega_T; \mathbf{R}^n)$ satisfying $\operatorname{div} v = u$ a.e. in Ω_T and $\|v_t\|_{L^\infty(\Omega_T)} \leq 1/2$. Any such set $\mathcal{U}$ is called an *admissible class.*

Given any $\epsilon > 0$, define $\mathcal{U}_\epsilon$ to be the set of $u \in \mathcal{U}$ such that there exists a vector function $v \in W^{1,\infty}_{v^*}(\Omega_T; \mathbf{R}^n)$ satisfying $\operatorname{div} v = u$ a.e. in Ω_T, $\|v_t\|_{L^\infty(\Omega_T)} \leq 1/2$, and

$$\int_{\Omega_T} |v_t(x,t) - \sigma(Du(x,t))| \, dxdt \leq \epsilon |\Omega_T|.$$

(Note that $\mathcal{U}_\epsilon = \mathcal{U}$ for all $\epsilon \geq 1$.)

Lemma 3.2. *Let $u \in \mathcal{U}$. Then any vector function $v \in W^{1,\infty}_{v^*}(\Omega_T; \mathbf{R}^n)$ determined above satisfies the integral identity* (3.1) *for each $\zeta \in C^\infty(\bar{\Omega}_T)$ and $t \in [0,T]$.*

Proof. Let $\zeta \in C^\infty(\bar{\Omega}_T)$ and define

$$h(t) = \int_\Omega (v(x,t) \cdot D\zeta(x,t) + u(x,t)\zeta(x,t)) \, dx.$$

Then h is continuous on $[0,T]$ and for each $\psi \in C^1[0,T]$,

$$\begin{aligned}
\int_0^T h(t)\psi(t)\, dt &= \int_0^T \int_\Omega \psi(t) \left(v(x,t) \cdot D\zeta(x,t) + u(x,t)\zeta(x,t)\right) dxdt \\
&= \int_0^T \int_\Omega \psi(t) \left(v(x,t) \cdot D\zeta(x,t) + \operatorname{div} v(x,t)\zeta(x,t)\right) dxdt \\
&= \int_0^T \int_\Omega \operatorname{div}(\zeta(x,t)\psi(t)v(x,t)) \, dxdt \\
&= \int_0^T \int_\Omega \operatorname{div}(\zeta(x,t)\psi(t)v^*(x,t)) \, dxdt = 0,
\end{aligned}$$

resulting from $v|_{\partial\Omega_T} = v^*|_{\partial\Omega_T}$ and $v^*(\cdot,t) \cdot \mathbf{n}|_{\partial\Omega} = 0$ for all $t \in [0,T]$. Hence $h \equiv 0$ on $[0,T]$. This completes the proof. $\square$

3.2. **General existence theorem by Baire's category method.** We prove a general existence theorem under a density hypothesis.

Theorem 3.3. *Let $\mathcal{U} \subset W^{1,\infty}_{u^*}(\Omega_T)$ be an admissible class. Assume, for each $\epsilon > 0$, $\mathcal{U}_\epsilon$ is dense in $\mathcal{U}$ under the L^∞-norm. Then, given any $\varphi \in \mathcal{U}$, for each $\eta > 0$, there exists a weak solution $u \in W^{1,\infty}_{u^*}(\Omega_T)$ to problem* (1.1) *satisfying $\|u - \varphi\|_{L^\infty(\Omega_T)} < \eta$. Furthermore, if $\mathcal{U}$ contains a function in $W^{1,\infty}_{u^*}(\Omega_T)$ that is not a weak solution to* (1.1), *then* (1.1) *admits infinitely many weak solutions.*

Proof. 1. Let $\mathcal{X}$ be the closure of $\mathcal{U}$ in the metric space $L^\infty(\Omega_T)$. Then $(\mathcal{X}, L^\infty)$ is a complete metric space. By assumption, $\mathcal{U}_\epsilon$ is a dense subset of $\mathcal{X}$. Furthermore, since $\mathcal{U}$ is bounded in $W^{1,\infty}_{u^*}(\Omega_T)$, we have $\mathcal{X} \subset W^{1,\infty}_{u^*}(\Omega_T)$.

2. Let $\mathcal{Y} = L^1(\Omega_T; \mathbf{R}^n)$. For $h > 0$, define $T_h \colon \mathcal{X} \to \mathcal{Y}$ as follows. Given any $u \in X$, write $u = u^* + w$ with $w \in W^{1,\infty}_0(\Omega_T)$ and define

$$T_h(u) = Du^* + D(\rho_h * w),$$

where $\rho_h(z) = h^{-N}\rho(z/h)$, with $z = (x,t)$ and $N = n+1$, is the standard mollifier on $\mathbf{R}^N$, and $\rho_h * w$ is the usual convolution on $\mathbf{R}^N$ with w extended to be zero outside Ω_T. Then, for each $h > 0$, the map $T_h\colon (\mathcal{X}, L^\infty) \to (\mathcal{Y}, L^1)$ is continuous, and for each $u \in \mathcal{X}$,

$$\lim_{h\to 0^+} \|T_h(u) - Du\|_{L^1(\Omega_T)} = \lim_{h\to 0^+} \|\rho_h * Dw - Dw\|_{L^1(\Omega_T)} = 0.$$

Therefore, the spatial gradient operator $D\colon \mathcal{X} \to \mathcal{Y}$ is the pointwise limit of a sequence of continuous functions $T_h\colon \mathcal{X} \to \mathcal{Y}$; hence $D\colon \mathcal{X} \to \mathcal{Y}$ is a *Baire-one function*. By Baire's category theorem (e.g., [5, Theorem 10.13]), there exists a *residual set* $\mathcal{G} \subset \mathcal{X}$ such that the operator D is continuous at each point of $\mathcal{G}$. Since $\mathcal{X} \setminus \mathcal{G}$ is of *first category*, the set $\mathcal{G}$ is *dense* in $\mathcal{X}$. Therefore, given any $\varphi \in \mathcal{X}$, for each $\eta > 0$, there exists a function $u \in \mathcal{G}$ such that $\|u - \varphi\|_{L^\infty(\Omega_T)} < \eta$.

3. We now prove that each function $u \in \mathcal{G}$ is a weak solution to (1.1). Let $u \in \mathcal{G}$ be given. By the density of $\mathcal{U}_\epsilon$ in $(\mathcal{X}, L^\infty)$, for each $j \in \mathbf{N}$, there exists a function $u_j \in \mathcal{U}_{1/j}$ such that $\|u_j - u\|_{L^\infty(\Omega_T)} < 1/j$. Since the operator $D\colon (\mathcal{X}, L^\infty) \to (\mathcal{Y}, L^1)$ is continuous at u, we have $Du_j \to Du$ in $L^1(\Omega_T; \mathbf{R}^n)$. Furthermore, from the definition of $\mathcal{U}_{1/j}$, there exists a vector function $v_j \in W^{1,\infty}_{v^*}(\Omega_T; \mathbf{R}^n)$ such that, for each $\zeta \in C^\infty(\bar{\Omega}_T)$ and $t \in [0,T]$,

$$\begin{gathered} \int_\Omega v_j(x,t) \cdot D\zeta(x,t)dx = -\int_\Omega u_j(x,t)\zeta(x,t)\,dx, \\ \|(v_j)_t\|_{L^\infty(\Omega_T)} \le \frac{1}{2}, \quad \int_0^T \int_\Omega |(v_j)_t - \sigma(Du_j)|\,dxdt \le \frac{1}{j}|\Omega_T|. \end{gathered} \tag{3.2}$$

Since $v_j(x,0) = v^*(x,0) \in W^{1,\infty}(\Omega; \mathbf{R}^n)$, from $\|(v_j)_t\|_{L^\infty(\Omega_T)} \le 1/2$, it follows that both sequences $\{v_j\}$ and $\{(v_j)_t\}$ are bounded in $L^2(\Omega_T; \mathbf{R}^n) \approx L^2((0,T); L^2(\Omega; \mathbf{R}^n))$. We may assume $v_j \rightharpoonup v$ and $(v_j)_t \rightharpoonup v_t$ weakly in $L^2((0,T); L^2(\Omega; \mathbf{R}^n))$ for some $v \in W^{1,2}((0,T); L^2(\Omega; \mathbf{R}^n))$. Upon taking the limit as $j \to \infty$ in (3.2) and noticing $v \in C([0,T]; L^2(\Omega; \mathbf{R}^n))$, we obtain that

$$\begin{gathered} \int_\Omega v(x,t)\cdot D\zeta(x,t)\,dx = -\int_\Omega u(x,t)\zeta(x,t)\,dx \quad (t \in [0,T]), \\ v_t(x,t) = \sigma(Du(x,t)) \quad \textit{a.e. } (x,t) \in \Omega_T. \end{gathered}$$

Consequently, by Lemma 3.1, u is a weak solution to (1.1).

4. Finally, assume $\mathcal{U}$ contains a function that is not a weak solution to (1.1); hence $\mathcal{G} \neq \mathcal{U}$. Then $\mathcal{G}$ cannot be a finite set since otherwise the L^∞-closure $\mathcal{X} = \overline{\mathcal{G}} = \overline{\mathcal{U}}$ would be a finite set, making $\mathcal{U} = \mathcal{G}$; therefore, in this case, (1.1) admits infinitely many weak solutions. This completes the proof. □

The rest of the paper is devoted to the construction of a suitable admissible class $\mathcal{U} \subset W^{1,\infty}(\Omega_T)$ fulfilling the *density property*:

$$\mathcal{U}_\epsilon \text{ is dense in } \mathcal{U} \text{ under the } L^\infty\text{-norm for each } \epsilon > 0. \tag{3.3}$$

4. Geometric considerations: Relaxation of $\nabla\omega(z) \in K(0)$

Let $K(s)$ be the matrix set defined by (1.5) above. Since $K(s)$ is a translation of set $K(0)$, we focus on the set $K_0 = K(0)$; that is,

$$K_0 = \left\{ \begin{pmatrix} p & c \\ B & \sigma(p) \end{pmatrix} \Big| \, p \in \mathbf{R}^n,\ c \in \mathbf{R},\ B \in \mathbf{M}^{n\times n},\ \operatorname{tr} B = 0 \right\},$$

where $\sigma(p) = \frac{p}{1+|p|^2}$ is the Perona-Malik function.

4.1. **Rank-one lamination of K_0.** We first compute certain rank-one structures of the set K_0.

Let $L(K_0)$ be the set of all matrices $\xi \in \mathbf{M}^{(1+n)\times(n+1)}$ that are not in K_0 but are representable by $\xi = \lambda\xi_1 + (1-\lambda)\xi_2$ for some $\lambda \in (0,1)$ and $\xi_1,\ \xi_2 \in K_0$ with $\operatorname{rank}(\xi_1 - \xi_2) = 1$, or equivalently,

$$L(K_0) = \{\xi \notin K_0 \mid \xi + t_\pm \eta \in K_0 \text{ for some } t_- < 0 < t_+ \text{ and } \operatorname{rank} \eta = 1\}.$$

Suppose $\xi = \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} \in L(K_0)$, with $\xi + t_\pm\eta \in K_0$, where $t_- < 0 < t_+$ and η is a rank-one matrix given by

$$\eta = \begin{pmatrix} a \\ \alpha \end{pmatrix} \otimes (q, b) = \begin{pmatrix} aq & ab \\ \alpha\otimes q & b\alpha \end{pmatrix}, \quad a^2 + |\alpha|^2 \neq 0,\ b^2 + |q|^2 \neq 0,$$

for some $a,\ b \in \mathbf{R}$ and $\alpha,\ q \in \mathbf{R}^n$; here $\alpha \otimes q$ denotes the rank-one or zero matrix $(\alpha^i q_j)$ in $\mathbf{M}^{n\times n}$.

Condition $\xi + t_\pm\eta \in K_0$ with $t_- < 0 < t_+$ is equivalent to the following:

$$\operatorname{tr} B = 0, \quad \alpha \cdot q = 0, \quad \sigma(p + t_\pm aq) = \beta + t_\pm b\alpha. \tag{4.1}$$

If $aq = 0$, then $\sigma(p) = \beta + tb\alpha$ has two different solutions of t only when $b\alpha = 0$, but then we would have $\sigma(p) = \beta$ and thus $\xi \in K_0$, a contradiction. Therefore, $aq \neq 0$. By rescaling η and $t_\pm$, we assume $a = 1$ and $|q| = 1$; namely,

$$\eta = \begin{pmatrix} q & b \\ \alpha\otimes q & b\alpha \end{pmatrix}, \quad |q| = 1, \quad \alpha\cdot q = 0.$$

Case 1. Assume $b\alpha = 0$. In this case, by (4.1), the equation $\sigma(p + tq) = \beta$ has two solutions of t of opposite signs and thus we must have $p = xq$ and $\beta = uq$, and $\sigma(xq + tq) = uq$ becomes a quadratic equation $x + t = u + u(x^2 + 2xt + t^2)$, which has two solutions $t = t_\pm$ of opposite signs if and only if $u \neq 0$ and $x^2 - \frac{x}{u} + 1 < 0$; this condition can be written as

$$|\beta|^2 + (p\cdot\beta)^2 - p\cdot\beta = u^2 + (xu)^2 - xu < 0.$$

Remark 4.1. In this case one can always select $\eta = \begin{pmatrix} q & b \\ 0 & 0 \end{pmatrix}$, $|q| = 1,\ b \in \mathbf{R}$. This is the case for the one-dimensional problems studied in [19, 32, 33], where the existence results are primarily proved based on the structure of such η's. However, if $n \geq 2$, such η's are not sufficient to characterize all the rank-one structures. *Case 2* below thus becomes pivotal.

Case 2. Assume $b\alpha \neq 0$; so $b \neq 0$ and $\alpha \neq 0$. In this case, we write

$$\eta = \begin{pmatrix} q & b \\ \frac{1}{b}\gamma \otimes q & \gamma \end{pmatrix}, \quad |q| = 1,\ \gamma \cdot q = 0,\ \gamma \neq 0,\ b \neq 0.$$

Since the equation $\sigma(p + tq) = \beta + t\gamma$ has two solutions $t = t_\pm$ of opposite signs, we must have $p = xq + y\gamma$ and $\beta = uq + v\gamma$, and the equation $\sigma(p+tq) = \beta + t\gamma$ becomes a system of two equations:

$$\begin{cases} x + t = u(1 + (x+t)^2 + |\gamma|^2 y^2), \\ y = (v+t)(1 + (x+t)^2 + |\gamma|^2 y^2). \end{cases} \tag{4.2}$$

This system has two solutions $t = t_\pm$ of opposite signs, and thus $u \neq 0$ and $y \neq 0$. So (4.2) is equivalent to a system of two *quadratic* equations:

$$\begin{cases} t^2 + (2x - \frac{1}{u})t + x^2 + |\gamma|^2 y^2 + 1 - \frac{x}{u} = 0, \\ t^2 + (v+x)t + xv - yu = 0. \end{cases} \tag{4.3}$$

The necessary and sufficient condition for (4.3) to have two solutions $t = t_\pm$ of opposite signs is that the two quadratic equations of t have the same coefficients and the constant terms are negative, which yields that

$$x = \frac{1}{u} + v, \quad x^2 + |\gamma|^2 y^2 + 1 - \frac{x}{u} = xv - yu < 0.$$

Here, if $v = 0$, then $x = \frac{1}{u}$, and taking this into the inequality, we have $1 + |\gamma|^2 y^2 < 0$, a contradiction. So $v \neq 0$. Therefore

$$uv = xu - 1, \quad |\gamma|^2 = \frac{1 - xu}{yv} - \frac{1}{y^2}, \tag{4.4}$$

and

$$xv - yu = \left(\frac{x}{u} - \frac{y}{v}\right)(xu - 1) < 0. \tag{4.5}$$

We now solve for x, y, u, v from (4.4) in terms of p and β. From $p = xq + y\gamma$, $\beta = uq + v\gamma$, it follows that

$$q = \frac{1}{xv - yu}(vp - y\beta), \quad \gamma = \frac{1}{xv - yu}(-up + x\beta). \tag{4.6}$$

By (4.4) and (4.6), we have

$$p \cdot \beta = xu + yv|\gamma|^2 = 1 - \frac{v}{y}, \quad \frac{x}{u} = \frac{\frac{v}{y}|p|^2 - p \cdot \beta}{\frac{v}{y} p \cdot \beta - |\beta|^2},$$

where $\frac{v}{y} p \cdot \beta - |\beta|^2 \neq 0$ by (4.5). Let $k = x/u$, $l = y/v$. Then

$$l = \frac{1}{1 - p \cdot \beta}, \quad k = \frac{(1 - p \cdot \beta)|p|^2 - p \cdot \beta}{(1 - p \cdot \beta) p \cdot \beta - |\beta|^2}. \tag{4.7}$$

Moreover,

$$k - l = \frac{x}{u} - \frac{y}{v} = \frac{|p|^2 - l p \cdot \beta}{p \cdot \beta - l|\beta|^2} - l = \frac{|p - l\beta|^2}{p \cdot \beta - l|\beta|^2}.$$

From $|q|=1$, we have $xv-yu=(k-l)uv=-|vp-lv\beta|$ and hence

$$u=-\operatorname{sgn}(v)\frac{|p-l\beta|}{k-l}=-\operatorname{sgn}(v)\frac{p\cdot\beta-l|\beta|^2}{|p-l\beta|}, \tag{4.8}$$

where $\operatorname{sgn}(v)=v/|v|$ is the sign of $v\neq 0$. We then obtain x, v, y by

$$x=ku,\quad v=x-\frac{1}{u}=ku-\frac{1}{u},\quad y=lv=lku-\frac{l}{u}. \tag{4.9}$$

In this way, we have solved x, y, u, v in terms of p, β, uniquely up to the sign change. We can check that both conditions in (4.4) are satisfied.

Let us consider inequality (4.5) for these solutions. Equation on $|\gamma|^2$ in (4.4) implies $\frac{y}{v}(1-xu)>1$ and hence the inequality (4.5) yields $\frac{x}{u}(1-xu)>\frac{y}{v}(1-xu)>1$. So $0<xu<1$ and thus $x/u>y/v>1$, i.e., $k>l>1$. Then we deduce the inequality

$$|\beta|^2+(p\cdot\beta)^2-p\cdot\beta<0. \tag{4.10}$$

4.2. **Exact formula of $L(K_0)$.** In fact, inequality (4.10) exactly characterizes the set $L(K_0)$. We have the following result.

Theorem 4.1.

$$L(K_0)=\left\{\begin{pmatrix} p & c\\ B & \beta\end{pmatrix}\;\middle|\; \operatorname{tr}B=0,\ |\beta|^2+(p\cdot\beta)^2-p\cdot\beta<0\right\}. \tag{4.11}$$

Moreover, given any $\xi\in L(K_0)$, there exist a rank-one matrix

$$\eta=\begin{pmatrix} q & b\\ \frac{1}{b}\gamma\otimes q & \gamma\end{pmatrix}$$

with $|q|=1,\ \gamma\cdot q=0,\ b\neq 0$ and two numbers $t_-<0<t_+$ such that

$$\xi+t_\pm\eta\in K_0,$$

where $|b|>0$ can be arbitrarily small.

Proof. Let S be the set defined on the right-hand side of (4.11). The previous calculations show that $L(K_0)\subseteq S$. To verify the reverse inclusion $S\subseteq L(K_0)$, let $\xi=\begin{pmatrix} p & c\\ B & \beta\end{pmatrix}\in S$. Then $|\beta|^2+(p\cdot\beta)^2-p\cdot\beta<0$. So $0<p\cdot\beta<1$ and $(1-p\cdot\beta)p\cdot\beta-|\beta|^2>0$, and hence we can define $l,\ k$ by (4.7), so that $l>0$, $k>0$. To fix the sign, we define u by (4.8) with $+$ sign:

$$u=\frac{p\cdot\beta-l|\beta|^2}{|p-l\beta|}=\frac{(1-p\cdot\beta)p\cdot\beta-|\beta|^2}{|(1-p\cdot\beta)p-\beta|}>0.$$

We now define x,v,y by (4.9). Then $x>0$, $v<0$, and $y<0$. After deducing that $xv-yu<0$, we finally define q,γ by (4.6). It is then straightforward to check the following:

$$p=xq+y\gamma,\quad \beta=uq+v\gamma,\quad |q|=1,\quad \gamma\cdot q=0,$$
$$x=\frac{1}{u}+v,\quad x^2+|\gamma|^2y^2+1-\frac{x}{u}=xv-yu<0.$$

In particular, equation $\sigma(p+tq)=\beta+t\gamma$ has two solutions $t=t_\pm$ with $t_-<0<t_+$. Now let $\eta=\begin{pmatrix} q & b\\ \frac{1}{b}\gamma\otimes q & \gamma\end{pmatrix}$, where $b\neq 0$ is arbitrary. Then $\xi+t_\pm\eta\in K_0$, and so $\xi\in L(K_0)$. The proof is now complete. $\square$

Remark 4.2. The quantities $q \in \mathbf{S}^{n-1}$, $\gamma \in \mathbf{R}^n$ and $t_\pm$ defined in the proof depend *continuously* on $(p, \beta) \in \mathbf{R}^{n+n}$ with $|\beta|^2 + (p\cdot\beta)^2 - p\cdot\beta < 0$. We may also take $b > 0$ (or $b < 0$) to be a continuous function of all such (p, β).

4.3. **The approximating sets $\mathcal{S}_\delta$ and S_δ.** Given any $0 \le \delta \le 1/2$, let

$$\mathcal{S}_\delta = \{(p,\beta) \in \mathbf{R}^{n+n} \mid \delta|(1-p\cdot\beta)p - \beta| + |\beta|^2 + (p\cdot\beta)^2 - p\cdot\beta < 0\}, \tag{4.12}$$

$$S_\delta = \left\{ \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} \;\middle|\; \operatorname{tr} B = 0,\ (p,\beta) \in \mathcal{S}_\delta \right\}.$$

Then $S_0 = L(K_0)$. Immediate properties of the open sets $\mathcal{S}_\delta$ are that

$$\mathcal{S}_{\delta_2} \subset \mathcal{S}_{\delta_1} \quad \text{for } 0 \le \delta_1 < \delta_2 \le 1/2, \quad \mathcal{S}_{1/2} = \emptyset,$$

$$\mathcal{S}_\delta \ne \emptyset \quad \text{for } 0 \le \delta < 1/2, \quad \mathcal{S}_0 = \bigcup_{0<\delta<1/2} \mathcal{S}_\delta.$$

In what follows we always assume $0 < \delta < 1/2$ unless otherwise stated.

Proposition 4.2. *Let $\xi \in S_\delta$ and $\xi_\pm \in K_0$ with $\operatorname{rank}(\xi_+ - \xi_-) = 1$ satisfy that ξ lies in the open line segment (ξ_-, ξ_+). Then $(\xi_-, \xi_+) \subset S_\delta$.*

Proof. Consider functions

$$F(\xi) = |(1-p\cdot\beta)p - \beta|, \;\; G(\xi) = |\beta|^2 + (p\cdot\beta)^2 - p\cdot\beta, \quad \forall\, \xi = \begin{pmatrix} p & c \\ B & \beta \end{pmatrix}.$$

Then both F and G vanish on set K_0. For the given ξ and $\xi_\pm$, let $f(\tau) = F(\xi^\tau)$ and $g(\tau) = G(\xi^\tau)$, where $\xi^\tau = \tau\xi_+ + (1-\tau)\xi_-$. The rank-one condition implies that the corresponding term $p^\tau \cdot \beta^\tau$ is linear in τ; hence $g(\tau)$ is a quadratic polynomial of τ and $f(\tau)$ is the length of a vector quadratic in τ. Since both $f(\tau)$ and $g(\tau)$ vanish when $\tau = 0$ and 1, we must have $g(\tau) = C_1\tau(1-\tau)$ and $f(\tau) = C_2\tau(1-\tau)$ for some constants C_1, C_2. Since $\xi = \xi^\lambda \in S_\delta$ for some $0 < \lambda < 1$, we have

$$\delta f(\lambda) + g(\lambda) = \lambda(1-\lambda)(\delta C_2 + C_1) < 0.$$

This implies the constant $\delta C_2 + C_1$ is negative. Hence $\delta f(\tau) + g(\tau) = \tau(1-\tau)(\delta C_2 + C_1) < 0$ for all $\tau \in (0,1)$, which proves exactly $\xi^\tau \in S_\delta$ for all $\tau \in (0,1)$. ☐

Remark 4.3. The *lamination convex hull* K^{lc} of a set $K \subset \mathbf{M}^{m\times n}$ is defined to be the smallest set $S \subset \mathbf{M}^{m\times n}$ containing K with the property that if $\xi_\pm \in S$ with $\operatorname{rank}(\xi_+ - \xi_-) = 1$ then $(\xi_-, \xi_+) \subset S$ (see [9]). As in the proof of Proposition 4.2, one can see that $K_0^{lc} = K_0 \cup L(K_0)$.

Lemma 4.3. *Let $m_\pm(\delta)$ be defined by* (2.7) *above. It follows that*

$$\sup_{(p,\beta)\in\mathcal{S}_\delta} |p| = m_+(\delta), \qquad \inf_{(p,\beta)\in\mathcal{S}_\delta} |p| = m_-(\delta).$$

Proof. Let $(p,\beta) \in \mathcal{S}_\delta$. Then $0 < p\cdot\beta < 1$ and $\beta\cdot((1-p\cdot\beta)p - \beta) > 0$. We write

$$p\cdot\beta = |p||\beta|\cos\theta, \;\; \beta\cdot((1-p\cdot\beta)p - \beta) = |\beta||(1-p\cdot\beta)p - \beta|\cos\theta'$$

for some $0 \le \theta,\ \theta' < \pi/2$. A simple geometry shows that $\theta \le \theta'$. As $|\beta|^2 + (p\cdot\beta)^2 - p\cdot\beta < 0$, we have

$$|\beta|^2 + |p|^2|\beta|^2\cos^2\theta - |p||\beta|\cos\theta < 0,$$

and so,

$$|\beta| < \frac{1}{2}, \quad \frac{1-\sqrt{1-4|\beta|^2}}{2|\beta|\cos\theta} < |p| < \frac{1+\sqrt{1-4|\beta|^2}}{2|\beta|\cos\theta}.$$

Condition $(p,\beta)\in\mathcal{S}_\delta$ becomes $\delta<|\beta|\cos\theta'$; so $|\beta|\geq|\beta|\cos\theta>\delta$ and

$$\frac{1-\sqrt{1-4\delta^2}}{2\delta} < \frac{1-\sqrt{1-4|\beta|^2}}{2|\beta|} \leq \frac{1-\sqrt{1-4|\beta|^2}}{2|\beta|\cos\theta} < |p| < \frac{1+\sqrt{1-4|\beta|^2}}{2|\beta|\cos\theta} < \frac{1+\sqrt{1-4\delta^2}}{2\delta}.$$

This proves

$$\frac{1-\sqrt{1-4\delta^2}}{2\delta} \leq \inf_{(p,\beta)\in\mathcal{S}_\delta}|p| \leq \sup_{(p,\beta)\in\mathcal{S}_\delta}|p| \leq \frac{1+\sqrt{1-4\delta^2}}{2\delta}. \tag{4.13}$$

Next, fix any $\beta\in\mathbf{R}^n$ with $\delta<|\beta|<1/2$. Let l,l' be any numbers satisfying

$$\frac{1-\sqrt{1-4|\beta|^2}}{2|\beta|} < l < l' < \frac{1+\sqrt{1-4|\beta|^2}}{2|\beta|}, \tag{4.14}$$

and $p=\frac{l}{|\beta|}\beta$, $p'=\frac{l'}{|\beta|}\beta$. Then (p,β), (p',β) are both in $\mathcal{S}_\delta$ with $|p|=l, |p'|=l'$; so

$$\inf_{(p,\beta)\in\mathcal{S}_\delta}|p| \leq l < l' \leq \sup_{(p,\beta)\in\mathcal{S}_\delta}|p|.$$

As l,l' are arbitrary and satisfy (4.14), we have

$$\inf_{(p,\beta)\in\mathcal{S}_\delta}|p| \leq \frac{1-\sqrt{1-4|\beta|^2}}{2|\beta|} < \frac{1+\sqrt{1-4|\beta|^2}}{2|\beta|} \leq \sup_{(p,\beta)\in\mathcal{S}_\delta}|p|.$$

Finally, taking $|\beta|\to\delta^+$ and combining with (4.13) complete the proof. □

As an immediate consequence of the previous lemma, we have

Corollary 4.4. $\mathcal{S}_\delta\subset\{(p,\beta)\mid m_-(\delta)<|p|<m_+(\delta),\ \delta<|\beta|<1/2\}$.

4.4. **A useful convex integration lemma.** The following result is important for convex integration with linear constraint. For a more general case, see [29, Lemma 2.1].

Lemma 4.5. *Let $\lambda_1,\lambda_2>0$ and $\eta_1=-\lambda_1\eta,\ \eta_2=\lambda_2\eta$ with*

$$\eta=\begin{pmatrix} q & b\\ \frac{1}{b}\gamma\otimes q & \gamma\end{pmatrix}, \quad |q|=1,\ \gamma\cdot q=0,\ b\neq 0.$$

Let $G\subset\mathbf{R}^{n+1}$ be a bounded domain. Then for each $\epsilon>0$, there exists a function $\omega=(\varphi,\psi)\in C_c^\infty(\mathbf{R}^{n+1};\mathbf{R}^{1+n})$ with $\operatorname{supp}(\omega)\subset\subset G$ that satisfies the following properties:

(a) $\operatorname{div}\psi=0$ *in G,*
(b) $|\{z\in G\mid\nabla\omega(z)\notin\{\eta_1,\ \eta_2\}\}|<\epsilon$,
(c) $\operatorname{dist}(\nabla\omega(z),[\eta_1,\eta_2])<\epsilon$ *for all $z\in G$,*
(d) $\|\omega\|_{L^\infty(G)}<\epsilon$,
(e) $\int_{\mathbf{R}^n}\varphi(x,t)\,dx=0$ *for each $t\in\mathbf{R}$.*

Proof. 1. The proof follows a simplified version of [29, Lemma 2.1]. Define a map $\mathcal{P}\colon C^1(\mathbf{R}^{n+1}) \to C^0(\mathbf{R}^{n+1};\mathbf{R}^{1+n})$ by setting $\mathcal{P}(h) = (u,v)$, where, for $h(x,t) \in C^1(\mathbf{R}^{n+1})$,

$$u(x,t) = q\cdot Dh(x,t), \quad v(x,t) = \frac{1}{b}(\gamma\otimes q - q\otimes\gamma)Dh(x,t).$$

We easily see that $\mathcal{P}(h) = (u,v) \in C_c^\infty(\mathbf{R}^{n+1};\mathbf{R}^{1+n})$, $\operatorname{supp}(\mathcal{P}(h)) \subset \operatorname{supp}(h)$, $\operatorname{div} v \equiv 0$, and $\int_{\mathbf{R}^n} u(x,t)\,dx = 0$ for all $t\in\mathbf{R}$, for all $h\in C_c^\infty(\mathbf{R}^{n+1})$. For $h(x,t) = f(q\cdot x + bt)$ with $f\in C^\infty(\mathbf{R})$, $w = (u,v) = \mathcal{P}(h)$ is given by $u(x,t) = f'(q\cdot x+bt)$ and $v(x,t) = f'(q\cdot x+bt)\frac{\gamma}{b}$, and hence $\nabla w(x,t) = f''(q\cdot x+bt)\eta$. We also note that $\mathcal{P}(gh) = g\mathcal{P}(h) + h\mathcal{P}(g)$ and hence

$$\nabla\mathcal{P}(gh) = g\nabla\mathcal{P}(h) + h\nabla\mathcal{P}(g) + \mathcal{B}(\nabla g,\nabla h) \quad \forall\, g,\ h\in C^\infty(\mathbf{R}^{n+1}), \tag{4.15}$$

where $\mathcal{B}(\nabla g,\nabla h)$ is a bilinear map of ∇g and ∇h; so $|\mathcal{B}(\nabla h,\nabla g)| \le C|\nabla h||\nabla g|$ for some constant $C>0$.

2. Let $G_\epsilon \subset\subset G$ be a smooth sub-domain such that $|G\setminus G_\epsilon| < \epsilon/2$, and let $\rho_\epsilon\in C_c^\infty(G)$ be a cut-off function satisfying $0\le\rho_\epsilon\le 1$ in G, $\rho_\epsilon = 1$ on G_ϵ. As G is bounded, $G\subset\{(x,t)\mid k<q\cdot x+bt<l\}$ for some numbers $k<l$. For each $\tau>0$, we can find a function $f_\tau\in C_c^\infty(k,l)$ satisfying

$$-\lambda_1\le f_\tau''\le\lambda_2,\ |\{s\in(k,l)\mid f_\tau''(s)\notin\{-\lambda_1,\ \lambda_2\}\}| < \tau,\ \|f_\tau\|_{L^\infty} + \|f_\tau'\|_{L^\infty} < \tau.$$

3. Define $\omega = (\varphi,\psi) = \mathcal{P}(\rho_\epsilon(x,t)h_\tau(x,t))$, where $h_\tau(x,t) = f_\tau(q\cdot x+bt)$. Then $\|h_\tau\|_{C^1}\le C\|f_\tau\|_{C^1}\le C\tau$, $\omega\in C_c^\infty(\mathbf{R}^{n+1};\mathbf{R}^{1+n})$, $\operatorname{supp}(\omega)\subset\operatorname{supp}(\rho_\epsilon)\subset\subset G$, and (a) and (e) are satisfied. Note that

$$|\omega| \le |\rho_\epsilon||\mathcal{P}(h_\tau)| + |h_\tau||\mathcal{P}(\rho_\epsilon)| \le C_\epsilon\tau,$$

where $C_\epsilon>0$ is a constant depending on $\|\rho_\epsilon\|_{C^1(G)}$. So we can choose a $\tau_1>0$ so small that (d) is satisfied for all $0<\tau<\tau_1$. Note also that

$$\{z\in G\mid \nabla\omega(z)\notin\{\eta_1,\eta_2\}\} \subseteq (G\setminus G_\epsilon)\cup\{z\in G_\epsilon\mid f_\tau''(q\cdot x+bt)\notin\{-\lambda_1,\ \lambda_2\}\}.$$

Since $|\{z\in G_\epsilon\mid f_\tau''(q\cdot x+bt)\notin\{-\lambda_1,\ \lambda_2\}| \le N|\{s\in(k,l)\mid f_\tau''(s)\notin\{-\lambda_1,\ \lambda_2\}\}|$ for some constant $N>0$ depending only on set G, there exists a $\tau_2>0$ such that

$$|\{z\in G\mid \nabla\omega(z)\notin\{\eta_1,\eta_2\}\}| \le \frac{\epsilon}{2} + N\tau < \epsilon$$

for all $0<\tau<\tau_2$. Therefore, (b) is satisfied. Finally, note that

$$\rho_\epsilon\nabla\mathcal{P}(h_\tau(x,t)) = \rho_\epsilon f_\tau''(q\cdot x+bt)\eta \in [\eta_1,\eta_2] \quad \text{in } G$$

and, by (4.15), for all $z=(x,t)\in G$,

$$|\nabla\omega(z) - \rho_\epsilon\nabla\mathcal{P}(h_\tau(x,t))| \le |h_\tau||\nabla\mathcal{P}(\rho_\epsilon)| + |\mathcal{B}(\nabla h_\tau,\nabla\rho_\epsilon)| \le C_\epsilon'\tau < \epsilon$$

for all $0<\tau<\tau_3$, where $C_\epsilon'>0$ is a constant depending on $\|\rho_\epsilon\|_{C^2(G)}$, and $\tau_3>0$ is another constant. Hence (c) is satisfied. Taking $0<\tau<\min\{\tau_1,\tau_2,\tau_3\}$, the proof is complete. □

4.5. **Relaxation of $\nabla\omega(z) \in K_0$.** We now study the relaxation of homogeneous inclusion$\nabla\omega(z) \in K_0$. We prove the following result in a form suitable for later use.

Theorem 4.6. *Let $\mathcal{K}$ be a compact subset of $\mathcal{S}_\delta$. Let $\tilde{Q} \times \tilde{I}$ be a box in $\mathbf{R}^{n+1}$. Then, given any $\epsilon > 0$, there exists a $\rho_0 > 0$ such that for each box $Q \times I \subset \tilde{Q} \times \tilde{I}$, point $(p, \beta) \in \mathcal{K}$, and number $\rho > 0$ sufficiently small, there exists a function $\omega = (\varphi, \psi) \in C_c^\infty(Q \times I; \mathbf{R}^{1+n})$ satisfying the following properties:*

(a) $\operatorname{div}\psi = 0$ *in* $Q \times I$,
(b) $(p' + D\varphi(z), \beta' + \psi_t(z)) \in \mathcal{S}_\delta$ *for all* $z \in Q \times I$ *and* $|(p', \beta') - (p, \beta)| \leq \rho_0$,
(c) $\|\omega\|_{L^\infty(Q\times I)} < \rho$,
(d) $\int_{Q\times I} |\beta + \psi_t(z) - \sigma(p + D\varphi(z))|dz < \epsilon|Q \times I|/|\tilde{Q} \times \tilde{I}|$,
(e) $\int_Q \varphi(x, t)dx = 0$ *for all* $t \in I$,
(f) $\|\varphi_t\|_{L^\infty(Q\times I)} < \rho$.

Proof. Let $\xi = \xi(p, \beta) = \begin{pmatrix} p & 0 \\ O & \beta \end{pmatrix}$ for $(p, \beta) \in \mathcal{K} \subset \mathcal{S}_\delta$, where O is the $n \times n$ zero matrix. We omit the dependence on (p, β) in the following whenever it is clear from the context. Since $\xi \in S_\delta \subset L(K_0)$ on $\mathcal{K}$, it follows from Theorem 4.1 and its remark that given any $\rho > 0$, there exist *continuous* functions $q : \mathcal{K} \to \mathbf{S}^{n-1}$, $\gamma : \mathcal{K} \to \mathbf{R}^n$, $t_\pm : \mathcal{K} \to \mathbf{R}$, and $b : \mathcal{K} \to (0, \infty)$ with $\gamma \cdot q = 0$, $t_- < 0 < t_+$ on $\mathcal{K}$ such that letting $\eta = \begin{pmatrix} q & b \\ \frac{1}{b}\gamma \otimes q & \gamma \end{pmatrix}$ on $\mathcal{K}$, we have

$$\xi + t_\pm\eta \in K_0, \quad 0 < b < \frac{\rho}{t_+ - t_-} \quad \text{on } \mathcal{K}.$$

Writing $\xi_\pm = \begin{pmatrix} p_\pm & c_\pm \\ B_\pm & \beta_\pm \end{pmatrix} = \xi + t_\pm\eta$ on $\mathcal{K}$, we have $\xi = \lambda\xi_+ + (1 - \lambda)\xi_-$, $\lambda = \frac{-t_-}{t_+ - t_-} \in (0, 1)$ on $\mathcal{K}$.

Proposition 4.2 implies that on $\mathcal{K}$, both $\xi_-^\tau = \tau\xi_+ + (1-\tau)\xi_-$ and $\xi_+^\tau = (1-\tau)\xi_+ + \tau\xi_-$ belong to S_δ for each $\tau \in (0, 1)$. Let $0 < \tau < \min_{\mathcal{K}} \min\{\lambda, 1 - \lambda\} \leq \frac{1}{2}$ be a small number to be selected later. Let $\lambda' = \frac{\lambda - \tau}{1 - 2\tau}$ on $\mathcal{K}$. Then $\lambda' \in (0, 1)$, $\xi = \lambda'\xi_+^\tau + (1-\lambda')\xi_-^\tau$ on $\mathcal{K}$. Moreover, on $\mathcal{K}$, $\xi_+^\tau - \xi_-^\tau = (1-2\tau)(\xi_+ - \xi_-)$ is rank-one, $[\xi_-^\tau, \xi_+^\tau] \subset (\xi_-, \xi_+) \subset S_\delta$, and

$$c\tau \leq |\xi_+^\tau - \xi_+| = |\xi_-^\tau - \xi_-| = \tau|\xi_+ - \xi_-| = \tau(t_+ - t_-)|\eta| \leq C\tau,$$

where $C = \max_{\mathcal{K}}(t_+ - t_-)|\eta| \geq \min_{\mathcal{K}}(t_+ - t_-)|\eta| = c > 0$. By continuity, $H_\tau = \cup_{(p,\beta)\in\mathcal{K}}[\xi_-^\tau(p, \beta), \xi_+^\tau(p, \beta)]$ is a compact subset of S_δ, where S_δ is open in

$$\Sigma_0 = \left\{ \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} \;\middle|\; \operatorname{tr}B = 0 \right\}.$$

So $d_\tau = \operatorname{dist}(H_\tau, \partial|_{\Sigma_0} S_\delta) > 0$, where $\partial|_{\Sigma_0}$ is the relative boundary in the space Σ_0.

Let $\eta_1 = -\lambda_1\eta = -\lambda'(1 - 2\tau)(t_+ - t_-)\eta$, $\eta_2 = \lambda_2\eta = (1 - \lambda')(1 - 2\tau)(t_+ - t_-)\eta$ on $\mathcal{K}$, where $\lambda_1 = \tau(-t_+) + (1 - \tau)(-t_-) > 0$, $\lambda_2 = (1 - \tau)t_+ + \tau t_- > 0$ on $\mathcal{K}$ with $\tau > 0$ sufficiently small. Applying Lemma 4.5 to matrices η_1, η_2 and set $G = Q \times I$, we obtain that for each $\rho > 0$, there exists a function $\omega = (\varphi, \psi) \in C_c^\infty(Q \times I; \mathbf{R}^{1+n})$

and an open set $G_\rho \subset\subset Q \times I$ satisfying the following conditions:

$$
(4.16) \qquad \begin{cases} (1) & \operatorname{div} \psi = 0 \text{ in } Q \times I, \\ (2) & |(Q \times I) \setminus G_\rho| < \rho; \;\; \xi + \nabla\omega(z) \in \{\xi^\tau_-,\ \xi^\tau_+\} \text{ for all } z \in G_\rho, \\ (3) & \xi + \nabla\omega(z) \in [\xi^\tau_-, \xi^\tau_+]_\rho \text{ for all } z \in Q \times I, \\ (4) & \|\omega\|_{L^\infty(Q\times I)} < \rho, \\ (5) & \int_Q \varphi(x,t)\,dx = 0 \text{ for all } t \in I, \\ (6) & \|\varphi_t\|_{L^\infty(Q\times I)} < 2\rho, \end{cases}
$$

where $[\xi^\tau_-, \xi^\tau_+]_\rho$ denotes the ρ-neighborhood of closed line segment $[\xi^\tau_-, \xi^\tau_+]$. From (4.16.3), (4.16.6) follows as

$$
|\varphi_t| < |c_+ - c_-| + \rho = (t_+ - t_-)|b| + \rho < 2\rho \quad \text{in } Q \times I.
$$

By (4.16.3), $|\beta + \psi_t(z)| \le C + \rho$ for $z \in Q \times I$; hence

$$
(4.17) \qquad \begin{aligned} &\int_{Q\times I} |\beta + \psi_t - \sigma(p + D\varphi)|dz \\ &\le \int_{G_\rho} |\beta + \psi_t - \sigma(p + D\varphi)|dz + (C + \rho + \frac{1}{2})\rho \\ &\le |Q \times I| \max\{|\beta^\tau_\pm - \sigma(p^\tau_\pm)|\} + (C + \rho + \frac{1}{2})\rho \\ &\le C|Q \times I|\tau + |Q \times I| \max\{|\sigma(p_\pm) - \sigma(p^\tau_\pm)|\} + (C + \rho + \frac{1}{2})\rho, \end{aligned}
$$

where $\xi^\tau_\pm = \begin{pmatrix} p^\tau_\pm & c^\tau_\pm \\ B^\tau_\pm & \beta^\tau_\pm \end{pmatrix}$.

Note (a), (c), (e), and (f) follow from (4.16), where 2ρ in (4.16.6) can be adjusted to ρ as in (f). By the uniform continuity of σ on the set $J = \{p' \in \mathbf{R}^n \,|\, |p'| \le m_+(\delta)+1\}$, we can find a $\rho' > 0$ such that $|\sigma(p') - \sigma(p'')| < \frac{\epsilon}{3|\tilde{Q}\times\tilde{I}|}$ whenever p', $p'' \in J$ and $|p' - p''| < \rho'$, where $m_+(\delta) > 0$ is the number defined in Lemma 4.3. We then choose a $\tau > 0$ so small that $C\tau < \rho'$ and $C|\tilde{Q} \times \tilde{I}|\tau < \frac{\epsilon}{3}$. Since $p_\pm$, $p^\tau_\pm \in J$ and $|p_\pm - p^\tau_\pm| \le C\tau < \rho'$, it follows from (4.17) that (d) holds for any choice of $\rho > 0$ with $(C + \rho + \frac{1}{2})\rho < \frac{\epsilon|Q\times I|}{3|\tilde{Q}\times\tilde{I}|}$. Next, we choose a $\rho_0 > 0$ such that $\rho_0 < \frac{d_\tau}{2}$. If $0 < \rho < \rho_0$, then by (4.16.1) and (4.16.3), for all $z \in Q \times I$ and $|(p', \beta') - (p, \beta)| \le \rho_0$,

$$
\xi(p', \beta') + \nabla\omega(z) \in \Sigma_0, \quad \operatorname{dist}(\xi(p', \beta') + \nabla\omega(z), H_\tau) < d_\tau,
$$

and so $\xi(p', \beta') + \nabla\omega(z) \in S_\delta$, that is, $(p' + D\varphi(z), \beta' + \psi_t(z)) \in \mathcal{S}_\delta$. Thus (b) holds.

The proof is now complete. □

5. Construction of admissible class $\mathcal{U}$

In this section, we define a suitable admissible class $\mathcal{U}$ as required in Section 3. Assume Ω and u_0 are as given in Theorem 1.1, with (1.4) fulfilled in addition. Let $M = \|Du_0\|_{L^\infty(\Omega)} > 0$.

5.1. **The modified uniformly parabolic problem.** We first apply Lemma 2.2 to construct the function $f \in C^3([0,\infty))$ with a suitable choice of $\delta \in (0,1/2)$ and $1 < \Lambda < m_+(\delta)$ according to the value of M as follows (see Figure 1).

(i) If $0 < M < 1$, we select $0 < \delta < \frac{M}{1+M^2}$ and arbitrary $1 < \Lambda < m_+(\delta)$.

(ii) If $M \geq 1$ and $\lambda > 0$, we select $\delta = \frac{M+\lambda}{1+(M+\lambda)^2}$ and arbitrary $\Lambda \in (M, M+\lambda)$; in this case, $m_+(\delta) = M + \lambda$.

Note that in both cases we have $M < \Lambda$. Once $f \in C^3([0,\infty))$ is constructed, we define

$$A(p) = f(|p|^2)p \quad (p \in \mathbf{R}^n).$$

By Lemma 2.2, equation $u_t = \operatorname{div} A(Du)$ is uniformly parabolic. We have the following result.

Lemma 5.1. *With δ selected above and $\mathcal{S}_\delta$ defined by* (4.12)*, one has*

$$(p, A(p)) \in \mathcal{S}_\delta \quad \forall\, m_-(\delta) < |p| \leq M.$$

Proof. From the definition of set $\mathcal{S}_\delta$, it follows that $(p, A(p)) \in \mathcal{S}_\delta$ if and only if

$$\frac{\delta}{|p|} < f(|p|^2) < \frac{1}{1+|p|^2}; \text{ namely, } \rho(m_-(\delta)) = \delta < \rho^*(|p|) < \rho(|p|).$$

By Lemma 2.2, this condition is satisfied if $m_-(\delta) < |p| \leq \Lambda$. $\square$

5.2. **The suitable boundary function Φ.** By Theorem 2.1, the initial-Neumann boundary value problem

$$\begin{cases} u^*_t = \operatorname{div}(A(Du^*)) & \text{in } \Omega_T, \\ \partial u^*/\partial \mathbf{n} = 0 & \text{on } \partial\Omega \times [0,T], \\ u^*(x,0) = u_0(x), & x \in \Omega \end{cases} \tag{5.1}$$

admits a unique classical solution $u^* \in C^{2+\alpha, \frac{2+\alpha}{2}}(\bar{\Omega}_T)$ satisfying

$$|Du^*(x,t)| \leq M, \quad (x,t) \in \Omega_T.$$

From conditions (1.2) and (1.4), we can find a function $h \in C^{2+\alpha}(\bar{\Omega})$ satisfying

$$\Delta h = u_0 \;\text{ in } \Omega, \quad \partial h/\partial \mathbf{n} = 0 \;\text{ on } \partial\Omega.$$

Now let $v_0 = Dh \in C^{1+\alpha}(\bar{\Omega}; \mathbf{R}^n)$ and define, for $(x,t) \in \Omega_T$,

$$v^*(x,t) = v_0(x) + \int_0^t A(Du^*(x,s))\, ds. \tag{5.2}$$

Define $\Phi = (u^*, v^*) \in C^1(\bar{\Omega}_T; \mathbf{R}^{1+n})$. Then it is easy to see that Φ satisfies condition (1.6) above; i.e.,

$$\begin{cases} u^*(x,0) = u_0(x) \ (x \in \Omega), \\ \operatorname{div} v^* = u^* \;\text{ in } \Omega_T, \\ v^*(\cdot,t) \cdot \mathbf{n}|_{\partial\Omega} = 0 \;\ \forall\, t \in [0,T]. \end{cases}$$

Lemma 5.2. *Let*

$$\mathcal{K}_\delta = \{(p, \sigma(p)) \mid |p| \leq m_-(\delta)\}.$$

Then

$$(Du^*(x,t), v^*_t(x,t)) \in \mathcal{S}_\delta \cup \mathcal{K}_\delta \quad \forall\, (x,t) \in \Omega_T.$$

Proof. Given $(x,t)\in\Omega_T$, let $p=Du^*(x,t)$; then $|p|\le M$. By (5.2), $v_t^*(x,t)=A(p)$. If $|p|\le m_-(\delta)$, then $A(p)=\sigma(p)$ and hence

$$(Du^*(x,t),v_t^*(x,t))=(p,A(p))=(p,\sigma(p))\in\mathcal{K}_\delta.$$

If $m_-(\delta)<|p|\le M$, then by Lemma 5.1

$$(Du^*(x,t),v_t^*(x,t))=(p,A(p))\in\mathcal{S}_\delta.$$

Hence $(Du^*,v_t^*)\in\mathcal{S}_\delta\cup\mathcal{K}_\delta$ in Ω_T. ☐

5.3. **The admissible class $\mathcal{U}$.** In what follows, we say that a function u is *piecewise* C^1 in Ω_T and write $u\in C^1_{pc}(\Omega_T)$ provided that there exists a sequence of disjoint open sets $\{G_j\}_{j=1}^\infty$ in Ω_T with $|\partial G_j|=0$ such that

$$u\in C^1(\bar G_j)\quad\forall\, j\in\mathbf{N}=\{1,2,\cdots\},\quad |\Omega_T\setminus\cup_{j=1}^\infty G_j|=0.$$

(For our purpose it is also acceptable to allow only *finitely* many pieces in this definition.)

Definition 5.1. Let $\mu=\|u_t^*\|_{L^\infty(\Omega_T)}+1$. We define the *admissible class*

$$\mathcal{U}=\Big\{u\in C^1_{pc}\cap W^{1,\infty}_{u^*}(\Omega_T)\;\Big|\;\|u_t\|_{L^\infty}<\mu;\ \exists v\in C^1_{pc}\cap W^{1,\infty}_{v^*}(\Omega_T;\mathbf{R}^n)\\ \text{such that }\operatorname{div}v=u\text{ and }(Du,v_t)\in\mathcal{S}_\delta\cup\mathcal{K}_\delta\text{ a.e. in }\Omega_T\Big\}. \tag{5.3}$$

For each $\epsilon>0$, let $\mathcal{U}_\epsilon$ be defined by

$$\mathcal{U}_\epsilon=\Big\{u\in\mathcal{U}\;\Big|\;\exists\, v\in C^1_{pc}\cap W^{1,\infty}_{v^*}(\Omega_T;\mathbf{R}^n)\text{ such that }\operatorname{div}v=u\text{ and}\\ (Du,v_t)\in\mathcal{S}_\delta\cup\mathcal{K}_\delta\text{ a.e. in }\Omega_T,\text{ and }\textstyle\int_{\Omega_T}|v_t-\sigma(Du)|dxdt\le\epsilon|\Omega_T|\Big\}.$$

Remark 5.2. Clearly $u^*\in\mathcal{U}$; so $\mathcal{U}$ is *nonempty.* Also $\mathcal{U}$ is a bounded subset of $W^{1,\infty}_{u^*}(\Omega_T)$ as $\mathcal{S}_\delta\cup\mathcal{K}_\delta$ is bounded. Moreover, by Corollary 4.4, for each $u\in\mathcal{U}$, its corresponding vector function v satisfies $\|v_t\|_{L^\infty(\Omega_T)}\le 1/2$. Finally, note that $m_-(\delta)<|Du^*|\le M$ on some nonempty open subset of Ω_T and $A(Du^*)\ne\sigma(Du^*)$ on this set; hence u^* itself is not a weak solution to (1.1).

6. Density property and Proof of Theorem 1.1

In this final section, we prove the density property of the sets $\mathcal{U}_\epsilon$ and then complete the proof of Theorem 1.1.

6.1. **The density property of $\mathcal{U}$.** Let $\mathcal{U}$ and $\mathcal{U}_\epsilon$ be as defined in Section 5. We establish the density property (3.3).

Theorem 6.1. *For each $\epsilon>0$, $\mathcal{U}_\epsilon$ is dense in $\mathcal{U}$ under the L^∞-norm.*

Proof. Let $u\in\mathcal{U}$, $\eta>0$. The goal is to show that there exists a function $\tilde u\in\mathcal{U}_\epsilon$ such that $\|\tilde u-u\|_{L^\infty(\Omega_T)}<\eta$. For clarity, we divide the proof into several steps.

1. Note $\|u_t\|_{L^\infty(\Omega_T)}<\mu-\tau_0$ for some $\tau_0>0$ and there exists a function $v\in C^1_{pc}\cap W^{1,\infty}_{v^*}(\Omega_T;\mathbf{R}^n)$ such that $\operatorname{div}v=u$ and $(Du,v_t)\in\mathcal{S}_\delta\cup\mathcal{K}_\delta$ a.e. in Ω_T. Since both u and v are piecewise C^1 in Ω_T, there exists a sequence of disjoint open sets $\{G_j\}_{j=1}^\infty$ in Ω_T with $|\partial G_j|=0\ \forall j\ge 1$ such that

$$u\in C^1(\bar G_j),\ v\in C^1(\bar G_j;\mathbf{R}^n)\ \ \forall j\ge 1,\quad |\Omega_T\setminus\cup_{j=1}^\infty G_j|=0.$$

2. Let $j \in \mathbf{N}$ be fixed. Note that $(Du(z), v_t(z)) \in \bar{\mathcal{S}}_\delta \cup \mathcal{K}_\delta$ for all $z \in G_j$ and that $H_j = \{z \in G_j \mid (Du(z), v_t(z)) \in \partial\mathcal{S}_\delta\}$ is a (relatively) closed set in G_j with measure zero. So $\tilde{G}_j = G_j \setminus H_j$ is an open subset of G_j with $|\tilde{G}_j| = |G_j|$, and $(Du(z), v_t(z)) \in \mathcal{S}_\delta \cup \mathcal{K}_\delta$ for all $z \in \tilde{G}_j$.

3. For each $\tau > 0$, let $\mathcal{G}_\tau = \{(p,\beta) \in \mathcal{S}_\delta \mid |\beta - \sigma(p)| > \tau,\ \mathrm{dist}((p,\beta), \partial\mathcal{S}_\delta) > \tau)\}$; then $\mathcal{G}_\tau \subset\subset \mathcal{S}_\delta$. We can find a $\tau_j > 0$ such that

$$\int_{F_j} |v_t(z) - \sigma(Du(z))|\, dz < \frac{\epsilon}{3 \cdot 2^j} |\Omega_T|, \tag{6.1}$$

where $z = (x,t)$ and $F_j = \{z \in \tilde{G}_j \mid (Du(z), v_t(z)) \notin \mathcal{G}_{\tau_j}\}$. To check this, note $F_j = F^1_{\tau_j} \cup F^2_{\tau_j}$, where $F^1_\tau = \{z \in \tilde{G}_j \mid |v_t(z) - \sigma(Du(z))| \le \tau\}$ and $F^2_\tau = \{z \in \tilde{G}_j \mid \mathrm{dist}((Du(z), v_t(z)), \partial\mathcal{S}_\delta) \le \tau\}$. Clearly, $\int_{F^1_\tau} |v_t - \sigma(Du)|dz \le \tau|\Omega_T| \to 0$ as $\tau \to 0^+$. Since F^2_τ is decreasing as $\tau \to 0^+$ and $v_t = \sigma(Du)$ on $\cap_{\tau>0} F^2_\tau$, it follows that $\int_{F^2_\tau} |v_t - \sigma(Du)|dz \to 0$ as $\tau \to 0^+$.

4. Let $O_j = \tilde{G}_j \setminus F_j$. Furthermore, we may require that the number τ_j be chosen in such a way that either O_j is empty or O_j is a nonempty open set with $|\partial O_j| = 0$ (see [32]). Let J be the set of all $j \in \mathbf{N}$ with $O_j \neq \emptyset$. Then for $j \notin J$, $F_j = \tilde{G}_j$.

5. We now fix a $j \in J$. Note that $O_j = \{z \in \tilde{G}_j \mid (Du(z), v_t(z)) \in \mathcal{G}_{\tau_j}\}$ and that $\mathcal{K}_j := \bar{\mathcal{G}}_{\tau_j}$ is a compact subset of $\mathcal{S}_\delta$. Let $\tilde{Q} \subset \mathbf{R}^n$ be an open box with $\Omega \subset \tilde{Q}$ and $\tilde{I} = (0,T)$. Applying Theorem 4.6 to box $\tilde{Q} \times \tilde{I}$, $\mathcal{K}_j$, and $\epsilon' = \frac{\epsilon|\Omega_T|}{12}$, we obtain a constant $\rho_j > 0$ that satisfies the conclusion of the theorem. By the uniform continuity of σ on compact subsets of $\mathbf{R}^n$, we can find a $\theta = \theta_{\epsilon,\delta} > 0$ such that

$$|\sigma(p) - \sigma(p')| < \frac{\epsilon}{12} \tag{6.2}$$

if $|p|$, $|p'| \le m_+(\delta)$ and $|p - p'| \le \theta$. Also by the uniform continuity of u, v, and their gradients on $\bar{G}_j$, there exists a $\nu_j > 0$ such that

$$\begin{aligned} &|u(z) - u(z')| + |\nabla u(z) - \nabla u(z')| + |v(z) - v(z')| \\ &\quad + |\nabla v(z) - \nabla v(z')| < \min\{\tfrac{\rho_j}{2}, \tfrac{\epsilon}{12}, \theta\} \end{aligned} \tag{6.3}$$

whenever $z, z' \in \bar{G}_j$ and $|z - z'| \le \nu_j$. We now cover O_j (up to measure zero) by a sequence of disjoint open cubes $\{Q^i_j \times I^i_j\}_{i=1}^\infty$ in Ω_T whose sides are parallel to the axes with center z^i_j and diameter $l^i_j < \nu_j$.

6. Fix $i \in \mathbf{N}$ and write $w = (u,v)$, $\xi = \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} = \nabla w(z^i_j) = \begin{pmatrix} Du(z^i_j) & u_t(z^i_j) \\ Dv(z^i_j) & v_t(z^i_j) \end{pmatrix}$.

By the choice of $\rho_j > 0$ in Step 5 via Theorem 4.6, since $Q^i_j \times I^i_j \subset \tilde{Q} \times \tilde{I}$ and $(p,\beta) \in \mathcal{K}_j$, for all sufficiently small $\rho > 0$, there exists a function $\omega^i_j = (\varphi^i_j, \psi^i_j) \in C^\infty_c(Q^i_j \times I^i_j; \mathbf{R}^{1+n})$ satisfying

(a) $\mathrm{div}\, \psi^i_j = 0$ in $Q^i_j \times I^i_j$,

(b) $(p' + D\varphi^i_j(z), \beta' + (\psi^i_j)_t(z)) \in \mathcal{S}_\delta$ for all $z \in Q^i_j \times I^i_j$ and all $|(p',\beta') - (p,\beta)| \le \rho_j$,

(c) $\|\omega^i_j\|_{L^\infty(Q^i_j \times I^i_j)} < \rho$,

(d) $\int_{Q^i_j \times I^i_j} |\beta + (\psi^i_j)_t(z) - \sigma(p + D\varphi^i_j(z))|dz < \epsilon' |Q^i_j \times I^i_j| / |\tilde{Q} \times \tilde{I}|$,

(e) $\int_{Q^i_j} \varphi^i_j(x,t)dx = 0$ for all $t \in I^i_j$,

(f) $\|(\varphi_j^i)_t\|_{L^\infty(Q_j^i\times I_j^i)} < \rho$.

Here, we let $0 < \rho \le \min\{\tau_0, \frac{\rho_j}{2C}, \frac{\epsilon}{12C}, \eta\}$, where C_n is the constant in Theorem 2.3 and C is the product of C_n and the sum of the lengths of all sides of $\tilde{Q}$. By (e), we can apply Theorem 2.3 to φ_j^i on $Q_j^i \times I_j^i$ to obtain a function $g_j^i = \mathcal{R}\varphi_j^i \in C^1(\overline{Q_j^i \times I_j^i};\mathbf{R}^n) \cap W_0^{1,\infty}(Q_j^i \times I_j^i;\mathbf{R}^n)$ such that $\operatorname{div} g_j^i = \varphi_j^i$ in $Q_j^i \times I_j^i$ and

$$\|(g_j^i)_t\|_{L^\infty(Q_j^i\times I_j^i)} \le C\|(\varphi_j^i)_t\|_{L^\infty(Q_j^i\times I_j^i)} \le \frac{\rho_j}{2}. \tag{6.4}$$

7. As $|v_t|, |\sigma(Du)| \le 1/2$ a.e. in Ω_T, we can select a finite index set $\mathcal{I} \subset J \times \mathbf{N}$ so that

$$\int_{\bigcup_{(j,i)\in(J\times\mathbf{N})\setminus\mathcal{I}} Q_j^i\times I_j^i} |v_t(z) - \sigma(Du(z))|dz \le \frac{\epsilon}{3}|\Omega_T|. \tag{6.5}$$

We finally define

$$(\tilde{u}, \tilde{v}) = (u, v) + \sum_{(j,i)\in\mathcal{I}} \chi_{Q_j^i\times I_j^i}(\varphi_j^i, \psi_j^i + g_j^i) \quad \text{in } \Omega_T.$$

As a side remark, note here that only *finitely* many functions $(\varphi_j^i, \psi_j^i + g_j^i)$ are disjointly patched to the original (u, v) to obtain a new function $(\tilde{u}, \tilde{v})$ towards the goal of the proof. The reason for using only finitely many pieces of gluing is due to the lack of control over the spatial gradients $D(\psi_j^i + g_j^i)$, and overcoming this difficulty is at the heart of this paper.

8. Let us now check that $\tilde{u}$ together with $\tilde{v}$ indeed gives the desired result. By construction, it is clear that $\tilde{u} \in C^1_{pc} \cap W^{1,\infty}_{u^*}(\Omega_T)$, $\tilde{v} \in C^1_{pc} \cap W^{1,\infty}_{v^*}(\Omega_T;\mathbf{R}^n)$. By the choice of ρ in (f) as $\rho \le \tau_0$, we have $\|\tilde{u}_t\|_{L^\infty(\Omega_T)} < \mu$. Next, let $(j, i) \in \mathcal{I}$, and observe that for $z \in Q_j^i \times I_j^i$, with $(p, \beta) = (Du(z_j^i), v_t(z_j^i)) \in \mathcal{G}_{\tau_j}$, since $|z - z_j^i| < l_j^i < \nu_j$, it follows from (6.3) and (6.4) that

$$|(Du(z), v_t(z) + (g_j^i)_t(z)) - (p, \beta)| \le \rho_j,$$

and so $(D\tilde{u}(z), \tilde{v}_t(z)) \in \mathcal{S}_\delta$, by (b) above. From (a) and $\operatorname{div} g_j^i = \varphi_j^i$, for $z \in Q_j^i \times I_j^i$,

$$\operatorname{div}\tilde{v}(z) = \operatorname{div}(v + \psi_j^i + g_j^i)(z) = u(z) + 0 + \varphi_j^i(z) = \tilde{u}(z).$$

Therefore, $\tilde{u} \in \mathcal{U}$. Next, observe

$$\begin{aligned}\int_{\Omega_T} |\tilde{v}_t - \sigma(D\tilde{u})|dz &= \int_{\cup_{j\in\mathbf{N}} F_j} |v_t - \sigma(Du)|dz \\ &+ \int_{\cup_{(j,i)\in(J\times\mathbf{N})\setminus\mathcal{I}} Q_j^i\times I_j^i} |v_t - \sigma(Du)|dz + \int_{\cup_{(j,i)\in\mathcal{I}} Q_j^i\times I_j^i} |\tilde{v}_t - \sigma(D\tilde{u})|dz \\ &= I_1 + I_2 + I_3.\end{aligned}$$

From (6.1) and (6.5), we have $I_1 + I_2 \le \frac{2\epsilon}{3}|\Omega_T|$. Note also that for $(j, i) \in \mathcal{I}$ and $z \in Q_j^i \times I_j^i$, from (6.3), (6.4), and (f),

$$\begin{aligned}|\tilde{v}_t(z) - \sigma(D\tilde{u}(z))| &= |v_t(z) + (\psi_j^i)_t(z) + (g_j^i)_t(z) - \sigma(Du(z) + D\varphi_j^i(z))| \\ &\le |v_t(z) - v_t(z_j^i)| + |v_t(z_j^i) + (\psi_j^i)_t(z) - \sigma(Du(z_j^i) + D\varphi_j^i(z))| \\ &\quad + |(g_j^i)_t(z)| + |\sigma(Du(z_j^i) + D\varphi_j^i(z)) - \sigma(Du(z) + D\varphi_j^i(z))|\end{aligned}$$

$$\le \frac{\epsilon}{6} + |v_t(z^i_j) + (\psi^i_j)_t(z) - \sigma(Du(z^i_j) + D\varphi^i_j(z))|$$
$$+|\sigma(Du(z^i_j) + D\varphi^i_j(z)) - \sigma(Du(z) + D\varphi^i_j(z))|.$$

Here, as $(D\tilde{u}(z), \tilde{v}_t(z)) \in \mathcal{S}_\delta$, we have $|Du(z) + D\varphi^i_j(z)| = |D\tilde{u}(z)| \le m_+(\delta)$, and by (6.3), $|Du(z^i_j) - Du(z)| < \theta$. From (6.2) we now have

$$|\sigma(Du(z^i_j) + D\varphi^i_j(z)) - \sigma(Du(z) + D\varphi^i_j(z))| < \frac{\epsilon}{12}.$$

Integrating the above inequality over $Q^i_j \times I^i_j$, we thus obtain from (d) that

$$\int_{Q^i_j \times I^i_j} |\tilde{v}_t(z) - \sigma(D\tilde{u}(z))| dz \le \frac{\epsilon}{4}|Q^i_j \times I^i_j| + \frac{\epsilon|\Omega_T|}{12} \frac{|Q^i_j \times I^i_j|}{|\tilde{Q} \times \tilde{I}|}$$
$$\le \frac{\epsilon}{3}|Q^i_j \times I^i_j|,$$

which yields that $I_3 \le \frac{\epsilon}{3}|\Omega_T|$. Hence $I_1 + I_2 + I_3 \le \epsilon|\Omega_T|$, and so $\tilde{u} \in \mathcal{U}_\epsilon$. Finally, from (c) with $\rho \le \eta$ and the definition of $\tilde{u}$, we have

$$\|\tilde{u} - u\|_{L^\infty(\Omega_T)} < \eta.$$

This completes the proof. □

6.2. **Completion of Proof of Theorem 1.1.** The existence of infinitely many weak solutions to problem (1.1) readily follows from a combination of Remark 5.2, Theorem 6.1, and Theorem 3.3.

To prove the last statement of Theorem 1.1, assume $M = \|Du_0\|_{L^\infty(\Omega)} \ge 1$ and $\lambda > 0$. Recall that in this case $\delta = \frac{M+\lambda}{1+(M+\lambda)^2}$. By the definition of $\mathcal{U}$ and Corollary 4.4, we have $|Du| \le m_+(\delta) = M + \lambda$ a.e. in Ω_T for all $u \in \mathcal{U}$. On the other hand, following Step 3 in the proof of Theorem 3.3, every weak solution $u \in \mathcal{G}$ is the L^∞-limit of some sequence $u_j \in \mathcal{U}$ such that $Du_j \to Du$ a.e. in Ω_T. So these weak solutions $u \in \mathcal{G}$ must satisfy $\|Du\|_{L^\infty(\Omega_T)} \le M + \lambda$.

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