Existence of solutions of the master equation in the smooth case

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Abstract

We give a different proof of a theorem of W. Gangbo and A. Swiech on the short time existence of solutions of the master equation.

Introduction

Mean Field Games are games with a continuum of players, each of which sees only the "mean field" generated by the other ones. They attracted the attention of a wider set of analysts after the lectures of P. L. Lions at the Collège de France, which are available in video streaming (see also the written presentation [11]). They can model a wide array of phenomena in physics and mathematical economics; we dwell a little on one aspect of the latter. Actually, the idea of considering a continuum of players came up naturally in mathematical economy, where it was used ([6], see also [14] for a more elementary presentation) to model the formation of prices in a market with perfect concurrence. Quoting from [6], "the essential idea of this notion is that the economy under consideration has a "very large" number of participants, and that the influence of each participant is "negligible"".

To be more precise, let us look at the situation of [15]: we have a probability measure μ_s on the d-dimensional torus $\mathbf{T}^d = \frac{\mathbf{R}^d}{\mathbf{Z}^d}$ which models the distribution of the players at time s; we fix an initial time $t < 0$, an initial distribution $\bar\mu$ and we suppose that μ_s evolves according to the continuity equation, forward in time,

$$\begin{cases} \partial_s\mu_s + \operatorname{div}(X\mu_s) = 0 \quad s > t \\ \mu_t = \bar\mu \end{cases} \tag{1}$$

where the vector field X is a control which we are free to choose in the following.

Let us call $\mathcal{P}(\mathbf{T}^d)$ the space of the Borel probability measures on $\mathbf{T}^d$, and let us suppose that we are given two potentials $\mathcal{F}, \mathcal{U}_0 \colon \mathcal{P}(\mathbf{T}^d) \to \mathbf{R}$. We would like the whole society to minimize the value function

$$\mathcal{V}(t,\bar\mu) \colon = \inf\left\{ \int_t^0 \mathrm{d}s \left[\int_{\mathbf{T}^d} \frac{1}{2}|X^2(s,x)|^2 \mathrm{d}\mu_s(x) - \mathcal{F}(\mu_s) \right] + \mathcal{U}_0(\mu_0) \right\} \tag{2}$$

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where the inf is over all curves which satisfy (1) and all controls X. It turns out that under suitable hypotheses on $\mathcal{F}$ and $\mathcal{U}_0$ the inf is a minimum: there is a vector field X minimizing in (2); by (1), we also have a minimal trajectory μ_s.

In (2), we minimize the cost for the whole society, but what about its members? One possible notion is that of Nash equilibrium: roughly, we are on a Nash equilibrium if no one can get a better deal by a unilateral change of strategy. It happens that, in our case, the optimum for the whole society is a Nash equilibrium. Actually, under suitable hypotheses on $\mathcal{F}$ and $\mathcal{U}_0$, we shall be able to define two functions $F(x,\mu)$ and $u_0(x,\mu)$ which, heuristically, are the "mean field" potentials felt by the particle placed at x, provided the other ones are distributed as μ. We shall see that the drift X in (1) optimal for the whole group is also best for the single particle; namely, $X(s,q) = -\partial_x v(s,q)$ where v solves the Hamilton-Jacobi equation with time reversed

$$\begin{cases} -\partial_t v(s,q) + \frac{1}{2}|\partial_q v(s,q)|^2 + F(q,\mu_s) = 0 \quad s \leq 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad v(0,q) = u_0(q,\mu_0). \end{cases} \tag{3}$$

Equivalently, the particle initially placed at q minimizes its cost:

$$\int_t^0 \frac{1}{2}[|\dot{q}(s)|^2 + F(q(s),\mu_s)]\mathrm{d}s + u_0(q(0),\mu_0)$$

if it follows the vector field X.

Since the value function $\mathcal{V}(t,\mu)$ of (2) is defined on the metric space $\mathcal{P}(\mathbf{T}^d)$, this approach calls for a study of the Hamilton-Jacobi equation in metric spaces; we refer the reader to [3], [16] and [20] for three definitions of viscosity solutions of H-J in metric spaces.

In this framework, the task is to solve the coupled equations (1) and (3); it turns out that, formally, these two equations are equivalent to the so-called master equation, i. e. formula (6) below. Heuristically, the solution of the master equation is a value function both for the single particle and the whole community. In [15] it is shown that, under suitable hypotheses on $\mathcal{F}$ and $\mathcal{U}$, the master equation has a smooth solution for t negative and small and that the master equation is equivalent (this time rigorously) to (1) and (3).

In this paper, we want to give a different proof of the results of [15]. Instead of working in $\mathcal{P}(\mathbf{T}^d)$, we take up a suggestion of [11] (see also [18], [19]) and work in the space of L^2 parametrizations of particles: a parametrization for μ will be a function $\sigma \in L^2([0,1)^d, \mathbf{R}^d)$ whose law, when projected on $\mathbf{T}^d$, is μ. In other words, we are choosing $[0,1)^d$ as parameter space.

We shall see that this approach is equivalent to that of [15]; as in [15], the implicit function theorem is at the core of our proof, but we are going to use it in a way that is closer to the original approach of [10].

We set $M = L^2([0,1)^d, \mathbf{R}^d)$ and denote by $AC([a,b], X)$ the set of the absolutely continuous functions from $[a,b]$ to a space X; throughout the paper, we shall denote by ∇, D and d the gradients of functions on $\mathbf{T}^d$, M and $\mathcal{P}(\mathbf{T}^d)$ respectively.

We want to prove the following.

Theorem 1. *Let $\hat{\mathcal{F}}, \hat{\mathcal{U}}_0 \colon M \to \mathbf{R}$ be respectively a potential and a final condition satisfying the hypotheses of section 2 below. Then, the following points hold.*

1) *There is $T > 0$ such that, if $t \in [-T, 0]$ and $\psi \in M$, the minimum*

$$\hat{\mathcal{U}}(t, \psi) := \min \left\{ \int_t^0 [\frac{1}{2}||\dot{\sigma}_s||_M^2 - \hat{\mathcal{F}}(\sigma_s)]\mathrm{d}s + \hat{\mathcal{U}}_0(\sigma_0) : \sigma \in AC([t, 0], M), \quad \sigma_t = \psi \right\} \tag{4}$$

is attained on a unique curve $\sigma^{(t,\psi)} \in AC([t, 0], M)$.

2) *The maps $: (t, \psi) \to \sigma^{(t,\psi)}$ and $: (t, \psi) \to \hat{\mathcal{U}}(t, \psi)$ are of class C^2; moreover, they are $L^2_{\mathbf{Z}}$ and H-equivariant in the last variable for the groups $L^2_{\mathbf{Z}}$ and H defined in section 1 below.*

3) *There are two functions of class C^3*

$$\hat{F}, \hat{u}_0 \colon \mathbf{T}^d \times M \to \mathbf{R}$$

such that, if we set

$$u(t, x, \psi) = \min \Big\{ \int_t^0 [\frac{1}{2}|\dot{q}(s)|^2 - \hat{F}(q(s), \sigma_s^{(t,\psi)})]\mathrm{d}s + \hat{u}_0(q(0), \sigma_0^{(t,\psi)}) :$$

$$q \in AC([t, 0], \mathbf{T}^d), \quad q(t) = x \Big\} \tag{5}$$

then u is of class C^2 in $[-T, 0] \times \mathbf{T}^d \times M$ and satisfies the master equation

$$-\partial_t u(t, q, \psi) + \frac{1}{2}|\nabla u(t, q, \psi)|^2 + F(q, \psi) + \langle \nabla u(t, \psi(\cdot), \psi), Du(t, q, \psi) \rangle_M = 0 \qquad \forall (t, x, \psi) \in [-T, 0] \times \mathbf{T}^d \times M \tag{6}$$

where $\langle \cdot, \cdot \rangle_M$ denotes the inner product in M. To districate the inner product above, we note that $Du(t, q, \psi) \in M$ because it is the gradient with respect to the M variable; moreover, $: x \to \nabla u(t, \psi(x), \psi)$ belongs to M since it is the C^2 function $u(t, \cdot, \psi)$ composed with ψ. The function u is $\mathbf{Z}^d$-equivariant in the second variable and $L^2_{\mathbf{Z}}$ and H-equivariant in the last one.

4) *Let the law of ψ be absolutely continuous with respect to the Lebesgue measure; then, for $s \in [-T, 0]$ the law of $\sigma_s^{(t,\psi)}$ is absolutely continuous too.*

5) *For $\mathcal{L}^d$ a. e. $x \in [0, 1)^d$ we have that, for all $s \in [-T, 0]$,*

$$\dot{\sigma}_s^{(t,x)}(x) = -\nabla u(s, \sigma_s^{(t,x)}(x), \sigma_s^{(t,x)}).$$

In other words, the orbit $q(s)$ minimal in (5) coincides with $\sigma_s^{(t,\psi)}(x)$ if they start at the same point of $\mathbf{T}^d$; equivalently, $: s \to \sigma_s^{(t,\psi)}(x)$ minimizes the one-particle problem (5) for $\mathcal{L}^d$ a. e. $x \in [0, 1)^d$.

Recently the master equation has been studied extensively, expecially from the stochastic viewpoint; we refer the reader to [7], [8], [9], [12] and [13].

The paper is organized as follows: section 1 contains the notation and a theorem of [11] about the relationship between differentiability on parametrizations and on measures; section 2 recalls the hypotheses used in [15] from section 6 onwards; in section 3 we recall the method of [10] for the minimum of (4), in section 4 we deal with the master equation (6).

§1
Preliminaries and notation

We denote by $\pi\colon \mathbf{R}^d \to \mathbf{T}^d\colon = \frac{\mathbf{R}^d}{\mathbf{Z}^d}$ the natural projection, and by $|\cdot|_{\mathbf{T}^d}$ the distance on $\mathbf{T}^d$ given by

$$|x-y|_{\mathbf{T}^d} = \min\{|\tilde{x}-\tilde{y}| : \pi(\tilde{x}) = x, \quad \pi(\tilde{y}) = y\}.$$

We let $\mathcal{P}(\mathbf{T}^d)$ be the space of Borel probability measures on $\mathbf{T}^d$; if $\mu_1, \mu_2 \in \mathcal{P}(\mathbf{T}^d)$, we denote by $\Gamma(\mu_1, \mu_2)$ the set of all the Borel probability measures on $\mathbf{T}^d \times \mathbf{T}^d$ whose first and second marginals are, respectively, μ_1 and μ_2. For $\lambda \geq 1$ we define the λ-Wasserstein distance on $\mathcal{P}(\mathbf{T}^d)$ by

$$\mathcal{W}_\lambda(\mu_1, \mu_2)^\lambda = \min_{\gamma \in \Gamma(\mu_1, \mu_2)} \int_{\mathbf{T}^d \times \mathbf{T}^d} |x-y|^\lambda_{\mathbf{T}^d} \mathrm{d}\gamma(x, y). \tag{1.1}$$

We refer the reader to [4] or [23] for the proof that the minimum is attained and that $(\mathcal{P}(\mathbf{T}^d), \mathcal{W}_\lambda)$ is a compact metric space.

When $\lambda = 2$ (which is the only case we consider in this paper) we denote by $\Gamma_o(\mu_1, \mu_2)$ the set of the minimizers in (1.1).

We want to parametrize $\mu \in \mathcal{P}(\mathbf{T}^d)$ with a map $\sigma \in M\colon = L^2([0,1)^d, \mathbf{R}^d)$. To do this, we begin to define $\mathcal{P}_2(\mathbf{R}^d)$ as the set of the Borel probability measures on $\mathbf{R}^d$ with finite second moment. Following [19], we push forward $\mu \in \mathcal{P}_2(\mathbf{R}^d)$ to $\tilde{\mu}\colon = \pi_\sharp \mu \in \mathcal{P}(\mathbf{T}^d)$. By the definition of push-forward, this is tantamount to

$$\int_{\mathbf{T}^d} f(x)\mathrm{d}\tilde{\mu}(x) = \int_{\mathbf{R}^d} f(x)\mathrm{d}\mu(x) \qquad \forall f \in C(\mathbf{T}^d, \mathbf{R})$$

where we have identified f with its lift to a periodic function on $\mathbf{R}^d$.

If $\pi_\sharp\mu_1 = \pi_\sharp\mu_2 = \tilde{\mu}$, we say with [19] that μ_1 and μ_2 are two representatives of $\tilde{\mu}$. By lemma 1.2 of [19], it is possible to lift any couple of measures on $\mathbf{T}^d$ to measures on $\mathbf{R}^d$ in such a way to preserve the 2-Wasserstein distance. More precisely, if $\tilde{\mu}_1, \tilde{\mu}_2 \in \mathcal{P}(\mathbf{T}^d)$, then there are two representatives $\mu_1, \mu_2 \in \mathcal{P}_2(\mathbf{R}^d)$ such that μ_1 is supported in $[0,1]^d$, μ_2 in $[-1,2]^d$ and

$$\mathcal{W}_2(\tilde{\mu}_1, \tilde{\mu}_2)^2 = W_2(\mu_1, \mu_2)^2\colon = \min_{\gamma \in \Gamma(\mu_1, \mu_2)} \int_{\mathbf{R}^d \times \mathbf{R}^d} |x-y|^2 \mathrm{d}\gamma(x, y) \tag{1.2}$$

where we have denoted by W_2 the 2-Wasserstein distance on $\mathcal{P}_2(\mathbf{R}^d)$.

Let $\mathcal{L}^d$ denote the d-dimensional Lebesgue measure on $[0,1)^d$ and let $\mu \in \mathcal{P}_2(\mathbf{R}^d)$; it is standard ([4] or [23]) that there is a map $\psi \in M$ (actually, ψ is the gradient of a convex function) such that $\psi_\sharp \mathcal{L}^d = \mu$. The trivial converse is that, if $\psi \in M$, then $\psi_\sharp \mathcal{L}^d \in \mathcal{P}_2(\mathbf{R}^d)$. The map ψ is called the Brenier map, or the parametrization of μ.

For completeness' sake, we give a well-known extension of lemma 6.4 of [11].

Lemma 1.1. *1) Let $\mu_1, \mu_2 \in \mathcal{P}_2(\mathbf{R}^d)$, let $\psi_1, \psi_2 \in M$ be two parametrizations of μ_1, μ_2 respectively and let $\gamma \in \Gamma(\mu_1, \mu_2)$. Then, there is a sequence of invertible, measure-preserving maps $h_n\colon [0,1)^d \to [0,1)^d$ such* *that* $(\psi_1 \circ h_n, \psi_2)_\sharp \mathcal{L}^d$ *converges weak∗ to* γ*. Moreover, for all functions* $f \in C(\mathbf{T}^d \times \mathbf{R}^d, \mathbf{R})$ *such that* $\frac{f(x,v)}{1+|v|^2}$ *is bounded, we have that*

$$\int_{\mathbf{R}^d \times \mathbf{R}^d} f(x, x-y) \mathrm{d}\gamma(x,y) = \lim_{n \to +\infty} \int_{[0,1)^d} f(\psi_1 \circ h_n(x), \psi_2(x) - \psi_1 \circ h_n(x)) \mathrm{d}x. \tag{1.3}$$

2) Let $\tilde{\mu}_1, \tilde{\mu}_2 \in \mathcal{P}(\mathbf{T}^d)$ *and let* $\mu_1, \mu_2 \in \mathcal{P}_2(\mathbf{R}^d)$ *be two representatives such that (1.2) holds. Let* $\psi_1, \psi_2 \in M$ *be as in point 1). Then,*

$$\mathcal{W}_2(\tilde{\mu}_1, \tilde{\mu}_2)^2 = W_2(\mu_1, \mu_2)^2 = \inf \int_{[0,1)^d} |\psi_1 \circ h(x) - \psi_2(x)|^2 \mathrm{d}x \tag{1.4}$$

where the inf *is over all invertible, measure-preserving maps* $h\colon [0,1)^d \to [0,1)^d$.

Proof. As for (1.4), the first equality comes from (1.2). For the second one, we note that, since $(\psi_1 \circ h, \psi_2)_\sharp \mathcal{L}^d \in \Gamma(\mu_1, \mu_2)$, we have that

$$W_2(\mu_1, \mu_2)^2 \le \inf_h \int_{[0,1)^d} |\psi_2(x) - \psi_1 \circ h(x)|^2 \mathrm{d}x.$$

The opposite inequality follows immediately from point 1), which we prove it in the steps below using a variation of the technique of [11].

Step 1. We begin to suppose that μ_1 and μ_2 are supported in a common cube, say $\tilde{Q}^l = [-l, l)^d$. We partition $\tilde{Q}^l$ into smaller cubes

$$Q_k = \frac{2kl}{2^n} + \frac{1}{2^n} \tilde{Q}^l$$

with $k = (k_1, \dots, k_d) \in \mathbf{Z}^d$ such that $-2^n + 1 \le k_i \le 2^n - 1$. Next, we relabel the Q_k to Q_i, with i in a finite set of $\mathbf{N}$.

In the step 3, 4 and 5 below we are going to find maps h_n such that

$$\mathcal{L}^d[(\psi_1 \circ h_n, \psi_2)^{-1}(Q_i \times Q_j)] = \gamma(Q_i \times Q_j) \quad \text{for all} \quad i, j. \tag{1.5}$$

Using the fact that the sides of Q_i have length $\frac{2l}{2^n}$ and that μ_1 and μ_2 are supported in $\tilde{Q}_l$, the formula above easily implies that $(\psi_1 \circ h_n, \psi_2)_\sharp \mathcal{L}^d$ converges to γ in the weak∗ topology. Formula (1.3) now follows because γ and $(\psi_1 \circ h_n, \psi_2)_\sharp \mathcal{L}^d$ are supported in $\tilde{Q}^l \times \tilde{Q}^l$, a compact set on which $: (x,y) \to f(x, y-x)$ is continuous.

Step 2. Before showing (1.5) for the case with bounded support, let us show how it implies (1.3) in the general case.

Let $h\colon [0,1)^d \to [0,1)^d$ be measure preserving. The equality below comes from the definition of push-forward; in the inequality, $\tilde{Q}^l$ is the cube of step 1.

$$\left| \int_{[0,1)^d} f(\psi_1 \circ h(x), \psi_2(x) - \psi_1 \circ h(x)) \mathrm{d}x - \int_{\mathbf{R}^d \times \mathbf{R}^d} f(x, y-x) \mathrm{d}\gamma(x,y) \right| =$$

$$\left| \int_{\mathbf{R}^d \times \mathbf{R}^d} f(x, y-x) \mathrm{d}(\psi_1 \circ h, \psi_2)_\sharp \mathcal{L}^d(x,y) - \int_{\mathbf{R}^d \times \mathbf{R}^d} f(x, y-x) \mathrm{d}\gamma(x,y) \right| \le$$

$$\int_{(\tilde{Q}^l\times\tilde{Q}^l)^c} |f(x,y-x)|\mathrm{d}(\psi_1\circ h,\psi_2)_\sharp\mathcal{L}^d(x,y)+ \tag{1.6$_a$}$$

$$\int_{(\tilde{Q}^l\times\tilde{Q}^l)^c} |f(x,y-x)|\mathrm{d}\gamma(x,y)+ \tag{1.6$_b$}$$

$$\left|\int_{(\tilde{Q}^l\times\tilde{Q}^l)} f(x,y)\mathrm{d}(\psi_1\circ h,\psi_2)_\sharp\mathcal{L}^p(x,y)-\int_{(\tilde{Q}^l\times\tilde{Q}^l)} f(x,y-x)\mathrm{d}\gamma(x,y)\right|. \tag{1.6$_c$}$$

Let $\epsilon>0$; from the formula above we see that (1.3) follows if we prove that we can find $l\in\mathbf{N}$ such that

$$(1.6)_a<\epsilon$$

for all measure-preserving h,

$$(1.6)_b\le\epsilon$$

and that, once l is fixed in this way, we can find a measure-preserving h such that

$$(1.6)_c\le\epsilon.$$

The last formula comes immediately from step 1; $(1.6)_b<\epsilon$ follows because the measure $|f(x,y-x)|\gamma$ is finite and $\cap_l(\tilde{Q}_l\times\tilde{Q}_l)^c=\emptyset$.

As for $(1.6)_a\le\epsilon$, it suffices to prove that $|f(x,y-x)|(\psi_1\circ h,\psi_2)_\sharp\mathcal{L}^d$ is a tight set of measures as h varies in the measure-preserving maps of $[0,1)^d$. By our hypotheses on f, this follows if we show that $(1+|y-x|^2)(\psi_1\circ h,\psi_2)_\sharp\mathcal{L}^d$ is tight. This is equivalent to say that $|\psi_1\circ h-\psi_2|^2$ is uniformly integrable as h varies among the measure-preserving maps, which follows if we prove that $|\psi_1\circ h|^2$ is uniformly integrable; we leave the easy proof of this to the reader.

Step 3. In this step, we define the pre-images of the cubes Q_i, which the map h_n of step 1 will permute in a Rubik cube fashion. We set

$$A_i=\psi_1^{-1}(Q_i)\subset[0,1)^d,\qquad B_i=\psi_2^{-1}(Q_i)\subset[0,1)^d.$$

The equalities on the left in the two formulas below follow since $\gamma\in\Gamma(\mu_1,\mu_2)$; those on the right come from the fact that $\mu_j=(\psi_j)_\sharp\mathcal{L}^d$ for $j=1,2$.

$$\gamma(Q_i\times[-l,l)^d)=\mu_1(Q_i)=\mathcal{L}^d(A_i),\qquad\gamma([-l,l)^d\times Q_i)=\mu_2(Q_i)=\mathcal{L}^d(B_i). \tag{1.7}$$

In the next two steps, we shall settle the first row of cubes, say $\{A_i\times B_1\}_i$. The idea is to partition B_1 into sets $B_{i,1}$ and to find sets $A_{i,1}\subset A_i$ such that $\mathcal{L}^d(A_{i,1})=\mathcal{L}^d(B_{i,1})=\gamma(Q_i\times Q_1)$; then, we shall send $A_{i,1}$ into $B_{i,1}$ by a measure-preserving map. We shall see that this yields (1.5) for $j=1$.

Step 4. We assert that we can find sets $A_{i,1}\subset A_i$ such that

$$\mathcal{L}^d(A_{i,1})=\gamma(Q_i\times Q_1)\quad\text{and}\quad\sum_i\mathcal{L}^d(A_{i,1})=\mathcal{L}^d(B_1). \tag{1.8}$$

Note that the sets $A_{i,1}$ are disjoint since the A_i are disjoint. Moreover, we can find sets $B_{i,1} \subset B_1$ such that

$$\begin{cases} \mathcal{L}^d(B_{i,1}) = \mathcal{L}^d(A_{i,1}) \\ \text{the } B_{i,1} \text{ are disjoint} \\ \mathcal{L}^d\left(B_1 \setminus \bigcup_i B_{i,1}\right) = 0 \\ B_{i,1} \supset A_{i,1} \cap B_1 \\ B_{i,1} \cap A_{j,1} = \emptyset \quad \text{if} \quad j \neq i. \end{cases} \tag{1.9}$$

We begin to show that the first equality of (1.8) implies the second one: the first equality below follows since the Q_i partition $[-l,l)^d$, the second one follows since γ has μ_2 as the second marginal, the third one since $(\psi_2)_\sharp \mathcal{L}^d = \mu_2$ and the fourth one from the definition of B_1.

$$\sum_i \gamma(Q_i \times Q_1) = \gamma([-l,l)^d \times Q_1) = \mu_2(Q_1) = \mathcal{L}^d(\psi_2^{-1}(Q_1)) = \mathcal{L}^d(B_1).$$

Thus, we only have to find sets $A_{i,1} \subset A_i$ which satisfy the first formula of (1.8); since $\mathcal{L}^d$ is non-atomic and, by (1.7),

$$\mathcal{L}^d(A_i) = \gamma(Q_i \times [-l,l)^d) \geq \gamma(Q_i \times Q_1)$$

this is standard.

Now, we find the sets $B_{i,1}$ which satisfy (1.9). First of all we note that, by (1.8),

$$\mathcal{L}^d(B_1 \setminus \bigcup_{i \geq 2} A_{i,1}) \geq \mathcal{L}^d(A_{1,1}).$$

Since the $A_{i,1}$ are disjoint,we also have that $B_1 \cap A_{1,1}$ does not intersect $A_{i,1}$ for $i \geq 2$; moreover, $\mathcal{L}^d(B_1 \cap A_{1,1}) \leq \mathcal{L}^d(A_{1,1})$. Thus, we can find $B_{1,1} \subset B_1$ such that

a) $B_{1,1} \supset A_{1,1} \cap B_1$,

b) $\mathcal{L}^d(B_{1,1}) = \mathcal{L}^d(A_{1,1})$,

c) $B_{1,1}$ is disjoint from $A_{i,1}$ for $i \geq 2$.

Point *c*) follows by the last formula: in $B_1 \setminus \bigcup_{i \geq 2} A_{i,1}$ there is enough space to accommodate a $B_{1,1}$ satisfying *b*).

We show the next step of the induction, namely how to find $B_{2,1}$. By (1.8) and the aforesaid,

$$\mathcal{L}^d\left(B_1 \setminus \left(B_{1,1} \cup \bigcup_{i \neq 2} A_{i,1}\right)\right) \geq \mathcal{L}^d(A_{2,1}).$$

Using this, we can find $B_{2,1} \subset B_1$ such that

a′) $B_{2,1} \supset A_{2,1} \cap B_1$,

b′) $\mathcal{L}^d(B_{2,1}) = \mathcal{L}^d(A_{2,1})$,

c′) $B_{2,1}$ is disjoint from $B_{1,1}$ and from $A_{i,1}$ for $i \neq 2$.

Iterating, we get the sets $B_{i,1}$; the first, second, fourth and fifth formulas of (1.9) follow by construction, the third one by the first formula of (1.9), (1.8) and the fact that the $B_{i,1}$ are disjoint.

Step 5. In this step, we define h_n on the first row of cubes: we want to find an invertible, bi-measurable map $\hat{h}_1$ which preserve Lebesgue measure and such that, for all i,

$$\begin{cases} \hat{h}_1(x) = x \quad \text{if} \quad x \notin \bigcup_i (A_{i,1} \cup B_{i,1}) \\ (\psi_1 \circ \hat{h}_1, \psi_2)^{-1}(Q_i \times Q_1) = B_{i,1}. \end{cases} \tag{1.10}$$

Before proving this, note that $\mathcal{L}^d(B_{i,1}) = \gamma(Q_i \times Q_1)$ by (1.8) and (1.9); this and (1.10) proves that (1.5) holds for the first row of cubes $\{Q_i \times Q_1\}_i$. The other rows will follow by induction, as we shall see in step 6.

We prove (1.10). First of all, there are invertible maps $\phi_i: B_{i,1} \to A_{i,1}$ which preserve Lebesgue measure and which are the identity on $A_{i,1} \cap B_{i,1}$. This is easy to do: we set $\phi_i(x) = x$ on $A_{i,1} \cap B_{i,1}$; then, we use theorem 15.5.16 of [22] to get an invertible, measure-preserving map ϕ_i from $B_{i,1} \setminus A_{i,1}$ to $A_{i,1} \setminus B_{i,1}$; recall that these sets have the same Lebesgue measure by the first one of (1.9).

Next, we glue together the maps ϕ_i in the following way:

$$\hat{h}_1(x) = \begin{cases} x & \text{if} \quad x \notin \bigcup_i (A_{i,1} \cup B_{i,1}) \\ \phi_i(x) & \text{if} \quad x \in B_{i,1} \\ \phi_i^{-1}(x) & \text{if} \quad x \in A_{i,1}. \end{cases}$$

The definition is well-posed: since by (1.9) the $B_{i,1}$ are disjoint, and since we saw above that the $A_{i,1}$ are disjoint, the only possible conflict is when $x \in B_{i,1} \cap A_{j,1}$. But then by (1.9) $j = i$; now on $B_{i,1} \cap A_{i,1}$ ϕ_i and ϕ_i^{-1} coincide, since both are the identity on this set.

To check (1.10), we begin to note that its first formula comes straight from the definition of $\hat{h}_1$. As for the second one, if $x \in (\psi_1 \circ \hat{h}_1, \psi_2)^{-1}(Q_i \times Q_1)$, then $x \in \psi_2^{-1}(Q_1) = B_1$ and $\hat{h}_1(x) \in \psi_1^{-1}(Q_i) = A_i$. Now B_1 is partitioned by the $B_{j,1}$ and the only $B_{j,1}$ which $\hat{h}_1$ sends to A_i is $B_{i,1}$. Thus, $x \in B_{i,1}$, proving that $(\psi_1 \circ \hat{h}_1, \psi_2)^{-1}(Q_i \times Q_1) = B_{i,1}$.

Step 6. We saw above that (1.5) follows if we show (1.10) for all the other rows; we do this by iteration. By the last step, the pre-image of $\cup_i (Q_i \times Q_1)$ by $(\psi_1 \circ \hat{h}_1, \psi_2)$ is B_1. We want to adjust the second row of cubes without touching B_1. To do this, we restrict $(\psi_1 \circ \hat{h}_1, \psi_2)$ to B_1^c; its image will fall in

$$\bigcup_{j \neq 1} (Q_i \times Q_j).$$

Now we apply the procedure of the first step to the second row, i. e. to $\{Q_i \times Q_2\}_i$ and to $(\psi_1 \circ \hat{h}_1, \psi_2)$. We get a map $\hat{h}_2$ from B_1^c to itself such that $(\psi_1 \circ \hat{h}_1 \circ \hat{h}_2, \psi_2)$ satisfies (1.5) for $j = 2$. Now we extend $\hat{h}_2$ to be the identity on B_1, and we get that $(\psi_1 \circ \hat{h}_1 \circ \hat{h}_2, \psi_2)$ satisfies (1.5) for $j = 1$ too. To close, it suffices to call h_n the last step of the iteration, the one in which all the rows are settled.

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We can look at $\mathcal{W}_2$ on $\mathcal{P}(\mathbf{T}^d)$ keeping track of the action of $\mathbf{R}^d$ on $\mathbf{T}^d$. Let us define

$$\pi_{\mathbf{T}^d}: \mathbf{T}^d \times \mathbf{R}^d \to \mathbf{T}^d$$

as the projection on the first coordinate, and let us set

$$\alpha: \mathbf{T}^d \times \mathbf{R}^d \to \mathbf{T}^d, \qquad \alpha: (x, v) \to x + v.$$

Let $\tilde{\mu}_1, \tilde{\mu}_2 \in \mathcal{P}(\mathbf{T}^d)$; we say that $\gamma \in \mathcal{P}_2(\mathbf{T}^d \times \mathbf{R}^d)$ belongs to $\Psi(\tilde{\mu}_1, \tilde{\mu}_2)$ if $(\pi_{\mathbf{T}^d})\sharp\gamma = \tilde{\mu}_1$ and $\alpha_\sharp\gamma = \tilde{\mu}_2$; we leave to the reader the simple proof that

$$\mathcal{W}_2^2(\tilde{\mu}_1, \tilde{\mu}_2) = \min_{\gamma \in \Psi(\tilde{\mu}_1, \tilde{\mu}_2)} \int_{\mathbf{T}^d \times \mathbf{R}^d} |v|^2 \mathrm{d}\gamma(x, v). \tag{1.11}$$

We denote by $\Psi_o(\tilde{\mu}_1, \tilde{\mu}_2)$ the set of minimals.

In the following, we shall denote by L^2_μ a space of L^2 functions for the measure μ; we shall omit the μ when it is the Lebesgue measure.

Let now $G: \mathcal{P}(\mathbf{T}^d) \to \mathbf{R}$ be a function; we say that G is differentiable at $\tilde{\mu} \in \mathcal{P}(\mathbf{T}^d)$ if there is a vector field $\xi \in L^2_{\tilde{\mu}}(\mathbf{T}^d, \mathbf{R}^d)$ such that

$$\left| G(\tilde{\nu}) - G(\tilde{\mu}) - \int_{\mathbf{T}^d \times \mathbf{R}^d} \langle \xi(x), v \rangle \mathrm{d}\gamma(x, v) \right| = o(\mathcal{W}_2(\tilde{\mu}, \tilde{\nu}))$$

for all $\tilde{\nu} \in \mathcal{P}(\mathbf{T}^d)$ and all $\gamma \in \Psi_o(\tilde{\mu}, \tilde{\nu})$; we have denoted by $\langle \cdot, \cdot \rangle$ the inner product in $\mathbf{R}^d$.

Following [15], we say that G is strongly differentiable at $\tilde{\mu}$ if there is $k > 0$ such that

$$\left| G(\tilde{\nu}) - G(\tilde{\mu}) - \int_{\mathbf{T}^d \times \mathbf{R}^d} \langle \xi(x), v \rangle \mathrm{d}\gamma(x, v) \right| \leq k \int_{\mathbf{T}^d \times \mathbf{R}^d} |v|^2 \mathrm{d}\gamma(x, v)$$

for all $\tilde{\nu} \in \mathcal{P}(\mathbf{T}^d)$ and all $\gamma \in \Psi(\tilde{\mu}, \tilde{\nu})$. Note that we don't restrict the transfer plan γ to be in $\Psi_o(\tilde{\mu}, \tilde{\nu})$; it is immediate that strong differentiability implies differentiability. Of course, there are parallel definitions of differentiability and strong differentiability in $\mathcal{P}_2(\mathbf{R}^d)$, which we forego to state.

If $G: \mathcal{P}(\mathbf{T}^d) \to \mathbf{R}$, we can define

$$\bar{G}: \mathcal{P}_2(\mathbf{R}^d) \to \mathbf{R}, \qquad \bar{G}(\mu) = G(\pi_\sharp\mu). \tag{1.12}$$

Lemma 1.2. *Let $G: \mathcal{P}(\mathbf{T}^d) \to \mathbf{R}$ be strongly differentiable at $\tilde{\mu}$ and let $\bar{G}: \mathcal{P}_2(\mathbf{R}^d) \to \mathbf{R}$ be defined as in (1.12). Then, $\bar{G}$ is strongly differentiable at any $\mu \in \mathcal{P}_2(\mathbf{R}^d)$ such that $\pi_\sharp\mu = \tilde{\mu}$.*

Conversely, if $\bar{G}: \mathcal{P}_2(\mathbf{R}^d) \to \mathbf{R}$ quotients to a map $G: \mathcal{P}(\mathbf{T}^d) \to \mathbf{R}$ and is strongly differentiable at μ, then G is strongly differentiable at $\tilde{\mu} = \pi_\sharp\mu$.

Proof. We begin with the direct statement. Let $\tilde{\xi} \in L^2(\mathbf{T}^d, \tilde{\mu})$ be the derivative of G at $\tilde{\mu}$; we define $\xi: \mathbf{R}^d \to \mathbf{R}^d$ by $\xi(y) = \tilde{\xi}(\pi(y))$. We assert that $\xi \in L^2(\mathbf{R}^d, \mu)$; indeed, since $\pi_\sharp\mu = \tilde{\mu}$ we get the equality below, while the inequality comes from the fact that $\tilde{\xi} \in L^2_{\tilde{\mu}}$.

$$\int_{\mathbf{R}^d} |\xi(x)|^2 \mathrm{d}\mu(x) = \int_{\mathbf{T}^d} |\tilde{\xi}(x)|^2 \mathrm{d}\tilde{\mu}(x) < +\infty.$$

We prove that ξ is the derivative of $\bar{G}$ at μ. Let $\nu \in \mathcal{P}_2(\mathbf{R}^d)$ project on $\tilde{\nu} \in \mathcal{P}(\mathbf{T}^d)$ and let $\gamma \in \Psi(\mu,\nu)$; if we define $\tilde{\gamma} = (\pi \times id)_\sharp \gamma$ we see easily that $\tilde{\gamma} \in \Psi(\tilde{\mu},\tilde{\nu})$. We disintegrate γ as $\mu \otimes \gamma_x$ and $\tilde{\gamma}$ as $\tilde{\mu} \otimes \tilde{\gamma}_q$, where γ_x and $\tilde{\gamma}_q$ are measures on $\mathbf{R}^d$. An easy check shows that, if $f \in C(\mathbf{T}^d \times \mathbf{R}^d)$ with $\frac{f(x,v)}{1+|v|^2}$ bounded, then

$$\int_{\mathbf{R}^d} \mathrm{d}\mu(x) \int_{\mathbf{R}^d} f(x,y) \mathrm{d}\gamma_x(y) = \int_{\mathbf{T}^d} \mathrm{d}\tilde{\mu}(q) \int_{\mathbf{R}^d} f(q,y) \mathrm{d}\tilde{\gamma}_q(y).$$

The first equality below comes from (1.12) and the disintegration of γ; the second one comes from the definition of ξ using the fact that $\tilde{\mu} = \pi_\sharp \mu$ and the formula above. The third equality comes from the disintegration of $\tilde{\gamma}$. The first inequality comes from the fact that G is strongly differentiable, while the last equality is obvious.

$$\left| \bar{G}(\nu) - \bar{G}(\mu) - \int_{\mathbf{R}^d \times \mathbf{R}^d} \langle \xi(x), v \rangle \mathrm{d}\gamma(x,v) \right| =$$

$$\left| G(\tilde{\nu}) - G(\tilde{\mu}) - \langle \int_{\mathbf{R}^d} \xi(x) \mathrm{d}\mu(x), \int_{\mathbf{R}^d} v \mathrm{d}\gamma_x(v) \rangle \right| =$$

$$\left| G(\tilde{\nu}) - G(\tilde{\mu}) - \langle \int_{\mathbf{T}^d} \tilde{\xi}(q) \mathrm{d}\tilde{\mu}(q), \int_{\mathbf{R}^d} v \mathrm{d}\tilde{\gamma}_q(v) \rangle \right| =$$

$$\left| G(\tilde{\nu}) - G(\tilde{\mu}) - \int_{\mathbf{T}^d \times \mathbf{R}^d} \langle \tilde{\xi}(q), v \rangle \mathrm{d}\tilde{\gamma}(q,v) \right| \leq$$

$$k \int_{\mathbf{T}^d \times \mathbf{R}^d} |v|^2 \mathrm{d}\tilde{\gamma}(x,v) = k \int_{\mathbf{R}^d \times \mathbf{R}^d} |v|^2 \mathrm{d}\gamma(x,v).$$

Since this is the definition of strong differentiability in $\mathcal{P}_2(\mathbf{R}^d)$, we are done.

We prove the converse.

Step 1. Let $\tilde{\mu}, \tilde{\nu} \in \mathcal{P}(\mathbf{T}^d)$, let $\mu \in \mathcal{P}_2(\mathbf{R}^d)$ be such that $\pi_\sharp \mu = \tilde{\mu}$ and let $\tilde{\gamma} \in \Psi(\tilde{\mu},\tilde{\nu})$. Recall that we have defined a map $\alpha\colon (x,v) \to x+v$. We assert that we can find $\gamma \in \mathcal{P}_2(\mathbf{R}^d \times \mathbf{R}^d)$ and $\nu \in \mathcal{P}_2(\mathbf{R}^d)$ such that

a) the first marginal of γ is μ,

b) $(\pi \times id)_\sharp \gamma = \tilde{\gamma}$ and

c) $\alpha_\sharp \gamma = \nu$ and $\pi_\sharp \nu = \tilde{\nu}$; in particular, $\gamma \in \Psi(\mu,\nu)$.

To find γ, we disintegrate μ as $\mu = \beta_q \otimes \tilde{\mu}$, with β_q a probability measure on the fiber $\{q + \mathbf{Z}^d\}$; in other words, $\beta_q(z) \geq 0$ and

$$\sum_{z \in \mathbf{Z}^d} \beta_q(z) = 1.$$

Then, we can define γ by

$$\int_{\mathbf{R}^d \times \mathbf{R}^d} f(x,v) \mathrm{d}\gamma(x,v) = \int_{\mathbf{T}^d \times \mathbf{R}^d} \left[\sum_{z \in \mathbf{Z}^d} \beta_q(z) f(q+z,v) \right] \mathrm{d}\tilde{\gamma}(q,v)$$

for all continuous functions $f\colon \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}$ such that $\frac{f(x,v)}{1+|v|^2}$ is bounded. Setting $\nu = \alpha_\sharp \gamma$ we easily check that γ and ν satisfy *a*), *b*) and *c*).

Step 2. Let ξ be the derivative of $\bar{G}$ at μ; we assert that $\xi = \tilde{\xi} \circ \pi$, where $\tilde{\xi}$ is a vector field on $\mathbf{T}^d$. This is easy to see: for instance, taking a vector field η supported in a small ball $B(x_0, r)$ of $\mathbf{R}^d$, considering

$\gamma_{\epsilon,z} = \mu \otimes (id + \epsilon\eta(\cdot + z))_\sharp \mathcal{L}^d$ for $z \in \mathbf{Z}^d$, setting $\nu_{\epsilon,z} = \alpha_\sharp \gamma_{\epsilon,z}$ and noting that $\bar{G}(\nu_{\epsilon,z})$, which quotients on $\mathcal{P}(\mathbf{T}^d)$, depends on z only through $\mu(B(z_0, r))$.

End of the proof. The two steps above yield the first equality below, while the inequality comes from the fact that $\bar{G}$ is strongly differentiable at μ.

$$\left| G(\tilde{\nu}) - G(\tilde{\mu}) - \int_{\mathbf{T}^d \times \mathbf{R}^d} \langle \tilde{\xi}(q), v \rangle \mathrm{d}\tilde{\gamma}(q, v) \right| =$$

$$\left| \bar{G}(\nu) - \bar{G}(\mu) - \int_{\mathbf{R}^d \times \mathbf{R}^d} \langle \xi(x), v \rangle \mathrm{d}\gamma(x, v) \right| \leq k \int_{\mathbf{R}^d \times \mathbf{R}^d} |v|^2 \mathrm{d}\gamma(x, v) = k \int_{\mathbf{T}^d \times \mathbf{R}^d} |v|^2 \mathrm{d}\tilde{\gamma}(q, v).$$

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We shall denote by H the group of all bi-measurable maps $h: [0,1)^d \to [0,1)^d$ which preserve Lebesgue measure; we also set $L^2_{\mathbf{Z}} := L^2([0,1)^d, \mathbf{Z}^d)$, which is a group under addition.

Given $G: \mathcal{P}(\mathbf{T}^d) \to \mathbf{R}$, we can define a function

$$\hat{G}: M \to \mathbf{R}, \qquad \hat{G}(\psi) = G(\pi_\sharp \circ \psi_\sharp \mathcal{L}^d). \tag{1.13}$$

Clearly, the map $\hat{G}$ defined above is H and $L^2_{\mathbf{Z}}$-equivariant, i. e.

$$\hat{G}(\psi \circ h + z) = \hat{G}(\psi) \qquad \forall (\psi, h, z) \in M \times H \times L^2_{\mathbf{Z}}. \tag{1.14}$$

Going in the opposite direction, if $\hat{G}: M \to \mathbf{R}$ is a continuous map such that (1.14) holds, we can define

$$\bar{G}: \mathcal{P}_2(\mathbf{R}^d) \to \mathbf{R}, \qquad \bar{G}(\mu) = \hat{G}(\psi) \tag{1.15}$$

where $\psi \in M$ is such that $\psi_\sharp \mathcal{L}^p = \mu$. We prove that $\bar{G}$ is well-defined on $\mathcal{P}_2(\mathbf{R}^d)$: actually, we are going to see that $\bar{G}$ quotients to a function G on $\mathcal{P}(\mathbf{T}^d)$. Indeed, if $\psi_1, \psi_2 \in M$ are such that $\pi_\sharp(\psi_i)_\sharp \mathcal{L}^p = \tilde{\mu} \in \mathcal{P}(\mathbf{T}^d)$ for $i = 1, 2$, then it is standard (lemma 6.4 of [11] or lemma 1.1 above) that there are $h_n \in H$ and $z_n \in L^2_{\mathbf{Z}}$ such that

$$||\psi_1 - \psi_2 \circ h_n - z_n||_M \to 0 \quad \text{as} \quad n \to +\infty.$$

The equality below comes from (1.14), while the limit comes from the formula above and the continuity of $\hat{G}$.

$$\hat{G}(\psi_1) - \hat{G}(\psi_2) = \hat{G}(\psi_1) - \hat{G}(\psi_2 \circ h_n + z_n) \to 0.$$

This proves that $\hat{G}$ is well defined; as for the differentiability of $\hat{G}$, we recall theorems 6.2 and 6.5 of [11].

Proposition 1.3. *Let $\hat{G}: M \to \mathbf{R}$ be continuous and let it satisfy (1.14). Then, the following happens.*
1) If $\hat{G}$ is differentiable at ψ, then $\hat{G}$ is differentiable at η for all $\eta \in M$ such that $\eta_\sharp \mathcal{L}^d = \psi_\sharp \mathcal{L}^d$. Moreover, the law of $D\hat{G}(\psi)$ does not depend on the choice of η.
2) Let us suppose that $\hat{G}$ is of class C^1 and let $\mu \in \mathcal{P}_2(\mathbf{R}^d)$. Then, there is $\xi \in L^2_\mu(\mathbf{R}^d, \mathbf{R}^d)$ such that, for all ψ satisfying $\psi_\sharp \mathcal{L}^d = \mu$, we have

$$D\hat{G}(\psi)(x) = \xi \circ \psi(x) \quad \textit{for } \mathcal{L}^p \text{ a. e. } x.$$

3) Let $\hat{G} \in C^2(M, \mathbf{R})$ with a bounded second derivative and let it satisfy (1.14); then, the function $\bar{G}: \mathcal{P}_2(\mathbf{R}^d) \to \mathbf{R}$ defined by (1.15) is strongly differentiable. By lemma 1.2 this implies that its quotient G on $\mathcal{P}(\mathbf{T}^d)$ is strongly differentiable.

Proof. Point 1) is theorem 6.2 of [11], point 2 theorem 6.5. We prove the easy consequence 3).

We want to show that $\bar{G}$ is strongly differentiable at any $\mu \in \mathcal{P}_2(\mathbf{R}^d)$. Thus, let $\nu \in \mathcal{P}_2(\mathbf{R}^d)$ and let $\psi, \eta \in M$ be such that $\psi_\sharp \mathcal{L}^p = \mu$, $\eta_\sharp \mathcal{L}^p = \nu$; let $\lambda \in \Psi(\mu, \nu)$ and let ξ be as in point 2) above. Let $\beta: (x, v) \to (x, x+v)$; since $\lambda \in \Psi(\mu, \nu)$ it is easy to check that $\gamma := \beta_\sharp \lambda$ belongs to $\Gamma(\mu, \nu)$. By formula (1.3) of lemma 1.1 we can find $h_n \in H$ such that

$$\int_{[0,1)^d} |\psi(x) - \eta \circ h_n(x)|^2 \mathrm{d}x \to \int_{\mathbf{R}^d \times \mathbf{R}^d} |q - q'|^2 \mathrm{d}\gamma(q, q')$$

or equivalently, setting $\lambda_n := (\psi, \eta \circ h_n - \psi)_\sharp \mathcal{L}^d$,

$$\int_{\mathbf{R}^d \times \mathbf{R}^d} |v|^2 \mathrm{d}\lambda_n(x, v) \to \int_{\mathbf{R}^d \times \mathbf{R}^d} |v|^2 \mathrm{d}\lambda(x, v). \tag{1.16}$$

We assert that

$$\int_{\mathbf{R}^d \times \mathbf{R}^d} \langle \xi(x), v \rangle \mathrm{d}\lambda_n(x, v) \to \int_{\mathbf{R}^d \times \mathbf{R}^d} \langle \xi(x), v \rangle \mathrm{d}\lambda(x, v). \tag{1.17}$$

Indeed, if ξ were continuous, this would follow from (1.3). In the general case, we can find a continuous vector field ξ' such that $||\xi - \xi'||_{L^2_\mu} < \epsilon$; the first inequalities in the two formulas below are Hölder while the second ones come from (1.16).

$$\left| \int_{\mathbf{R}^d \times \mathbf{R}^d} \langle \xi - \xi', v \rangle \mathrm{d}\lambda_n(x, v) \right| \le ||\xi - \xi'||_{L^2_\mu} \left[\int_{\mathbf{R}^d \times \mathbf{R}^d} |v|^2 \mathrm{d}\lambda_n(x, v) \right]^{\frac{1}{2}} \le M ||\xi - \xi'||_{L^2_\mu} \le M\epsilon,$$

$$\left| \int_{\mathbf{R}^d \times \mathbf{R}^d} \langle \xi - \xi', v \rangle \mathrm{d}\lambda(x, v) \right| \le ||\xi - \xi'||_{L^2_\mu} \left[\int_{\mathbf{R}^d \times \mathbf{R}^d} |v|^2 \mathrm{d}\lambda(x, v) \right]^{\frac{1}{2}} \le M ||\xi - \xi'||_{L^2_\mu} \le M\epsilon.$$

These two formulas imply the second inequality below; the third one follows from (1.3) taking n large enough.

$$\left| \int_{\mathbf{R}^d \times \mathbf{R}^d} \langle \xi(x), v \rangle \mathrm{d}(\lambda_n - \lambda)(x, v) \right| \le$$

$$\left| \int_{\mathbf{R}^d \times \mathbf{R}^d} \langle \xi - \xi', v \rangle \mathrm{d}(\lambda_n - \lambda) \right| + \left| \int_{\mathbf{R}^d \times \mathbf{R}^d} \langle \xi', v \rangle \mathrm{d}(\lambda_n - \lambda) \right| \le$$

$$2\epsilon M + \left| \int_{\mathbf{R}^d \times \mathbf{R}^d} \langle \xi', v \rangle \mathrm{d}(\lambda_n - \lambda) \right| \le 2\epsilon M + \epsilon.$$

This proves (1.17). By (1.17), there is $\epsilon_n \to 0$ such that the first inequality below holds. The second one follows if we take k to be the sup of $\frac{1}{2}||D^2\hat{G}||$, which is finite by hypothesis. The last inequality follows from (1.16).

$$\left| \bar{G}(\nu) - \bar{G}(\mu) - \int_{\mathbf{R}^d \times \mathbf{R}^d} \langle \xi(x), v \rangle \mathrm{d}\lambda(x, v) \right| \le$$

$$\left| \hat{G}(\eta \circ h_n) - \hat{G}(\psi) - \int_{[0,1)^d} \langle \xi(\psi(x)), \eta \circ h_n(x) - \psi(x) \rangle \mathrm{d}x \right| + \epsilon_n \le$$

$$k\int_{[0,1)^d}|\eta\circ h_n(x)-\psi(x)|^2\mathrm{d}x+\epsilon_n\le k\int_{\mathbf{T}^d\times\mathbf{R}^d}|v|^2\mathrm{d}\lambda(x,v)+2\epsilon_n.$$

Letting $n\to+\infty$, we recover the definition of strong differentiability at μ.

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In the opposite direction, we have the following.

Lemma 1.4. *Let $G:\mathcal{P}(\mathbf{T}^d)\to\mathbf{R}$ be a function and let $\hat G: M\to\mathbf{R}$ be defined as in (1.13). Let us suppose that G is strongly differentiable at $\tilde\mu\in\mathcal{P}(\mathbf{T}^d)$, let $\mu\in\mathcal{P}_2(\mathbf{R}^d)$ be a representative of $\tilde\mu$ and let $\psi\in M$ such that $\psi_\sharp\mathcal{L}^d=\mu$. Then, $\hat G$ is differentiable at $\psi\circ h+z$ for all $(h,z)\in H\times L^2_{\mathbf{Z}}$, and*

$$D\hat G(u\circ h+z)=D\hat G(u)\circ h. \tag{1.18}$$

Proof. We define $\bar G:\mathcal{P}_2(\mathbf{R}^d)\to\mathbf{R}$ as in (1.12); by lemma 1.2, $\bar G$ is strongly differentiable at any representative μ of $\tilde\mu$.

Let ξ be the derivative of $\bar G$ at μ and let $\psi\in M$ be such that $(\psi)_\sharp\mathcal{L}^p=\mu$. Let $\eta\in M$ and let us set $\nu=\eta_\sharp\mathcal{L}^p$. If we define $\lambda=(\psi,\eta-\psi)_\sharp\mathcal{L}^p$, we get the first equality below. Now $\lambda\in\Psi(\mu,\nu)$ and G is strongly differentiable at μ with differential ξ; for some $k>0$ this implies the inequality below, while the last equality comes from the definitions of $\hat G$ and λ.

$$k\int_{[0,1)^d}|\psi(x)-\eta(x)|^2\mathrm{d}x=k\int_{\mathbf{T}^p\times\mathbf{R}^d}|v|^2\mathrm{d}\lambda(x,v)\ge$$

$$\left|\bar G(\nu)-\bar G(\mu)-\int_{\mathbf{R}^d\times\mathbf{R}^d}\langle\xi(q),v\rangle\mathrm{d}\lambda(q,v)\right|=$$

$$\left|\hat G(\eta)-\hat G(\psi)-\int_{[0,1)^d}\langle\xi\circ\psi(x),\eta(x)-\psi(x)\rangle\mathrm{d}x\right|.$$

The last formula implies that $\hat G$ is differentiable at ψ.

As for point 2), this is a general property of equivariant functions: if T_h is a set of bounded linear operators from M to M having the group property

$$T_{h_1}\circ T_{h_2}=T_{h_1h_2}$$

then it is standard that

$$D\hat G(T_hu)=[T^T_{h^{-1}}D\hat G(u)]$$

where A^T denotes the adjoint operator of A. Setting $T_hu:=u\circ h$ and substituting, we get (1.18).

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§**2**

Assumptions on the potential and the final condition

We recall the assumptions used in [15] from section 6 onward.

We begin to suppose that we are given $U^0, U^1, \phi \in C^3(\mathbf{T}^d)$ such that the lifts of ϕ and U^1 to $\mathbf{R}^d$ are even.

Our potential is the function $\mathcal{F}\colon \mathcal{P}(\mathbf{T}^d) \to \mathbf{R}$ defined by

$$\mathcal{F}(\mu) = \frac{1}{2}\int_{\mathbf{T}^d} (\phi * \mu)(z) \mathrm{d}\mu(z) = \frac{1}{2}\int_{\mathbf{T}^d \times \mathbf{T}^d} \phi(z-z') \mathrm{d}\mu(z) \mathrm{d}\mu(z')$$

where the symbol $*$ denotes, as usual, convolution. The final condition is the function $\mathcal{U}_0\colon \mathcal{P}(\mathbf{T}^d) \to \mathbf{R}$ given by

$$\mathcal{U}_0(\mu) = \int_{\mathbf{T}^d} [U^0(z) + \frac{1}{2}(U^1 * \mu)(z)] \mathrm{d}\mu(z) =$$
$$\int_{\mathbf{T}^d \times \mathbf{T}^d} [U^0(z) + \frac{1}{2} U^1(z-z')] \mathrm{d}\mu(z) \mathrm{d}\mu(z').$$

It is shown in [15] that $\mathcal{F}$ and $\mathcal{U}$ are strongly differentiable.

We recall from the introduction that we denote by d the differential of functions on $\mathcal{P}(\mathbf{T}^d)$, by D and ∇ that of functions on M and on $\mathbf{R}^d$ respectively.

Always by [15], we have that

$$\mathrm{d}\mathcal{F}(\mu) = \nabla F(q,\mu) \quad \text{and} \quad \mathrm{d}\mathcal{U}_0(\mu) = \nabla u_0(q,\mu)$$

where

$$F(q,\mu) = (\phi * \mu)(q) \quad \text{and} \quad u_0(q,\mu) = U^0(q) + (U^1 * \mu)(q).$$

By (1.13), $\mathcal{F}$ and $\mathcal{U}$ induce functions $\hat{\mathcal{F}}$ and $\hat{\mathcal{U}}_0$ on M; by the definition of push-forward we see that, if $\sigma \in M$,

$$\hat{\mathcal{F}}(\sigma) = \frac{1}{2}\int_{[0,1)^d \times [0,1)^d} \phi[\sigma(x) - \sigma(y)] \mathrm{d}x \mathrm{d}y, \tag{2.1$_a$}$$

$$\hat{\mathcal{U}}_0(\sigma) = \int_{[0,1)^d \times [0,1)^d} \{U^0(\sigma(x)) + \frac{1}{2} U^1[\sigma(x) - \sigma(y)]\} \mathrm{d}x \mathrm{d}y. \tag{2.1$_b$}$$

Also the functions F and u_0 extend to parametrizations:

$$\hat{F}\colon \mathbf{R}^d \times M \to \mathbf{R}^d, \qquad \hat{F}(q,\sigma) = \int_{[0,1)^d} \phi[q - \sigma(x)] \mathrm{d}x, \tag{2.2$_a$}$$

$$\hat{u}_0\colon \mathbf{R}^d \times M \to \mathbf{R}^d, \qquad \hat{u}_0(q,\sigma) = U^0(q) + \int_{[0,1)^d} U^1[q - \sigma(x)] \mathrm{d}x. \tag{2.2$_b$}$$

We forego the proof of the following lemma, which follows from our hypotheses on ϕ, U^0, U^1 and standard facts about the Nemitsky operators (see for instance [2]).

Lemma 2.1. *Let $\hat{\mathcal{F}}, \hat{\mathcal{U}}_0\colon M \to \mathbf{R}$ be defined as in (2.1), let $\hat{F}, \hat{u}_0$ be as in (2.2). Then, $\hat{\mathcal{F}}$ and $\hat{\mathcal{U}}_0$ are functions of class C^3 on M. Denoting by $\langle\cdot,\cdot\rangle$ and by $\langle\cdot,\cdot\rangle_M$ the inner products in $\mathbf{R}^d$ and in M respectively, we have that*

$$D\hat{\mathcal{F}}(\sigma)\psi = \int_{[0,1)^d \times [0,1)^d} \langle \nabla\phi[\sigma(x) - \sigma(y)], \psi(x)\rangle \mathrm{d}x \mathrm{d}y = \langle \nabla \hat{F}(\sigma(\cdot), \sigma), \psi\rangle_M$$

and

$$D\hat{\mathcal{U}}_0(\sigma)\psi=\int_{[0,1)^d\times[0,1)^d}\langle\nabla U^0(\sigma(x))+\nabla U^1[\sigma(x)-\sigma(y)],\psi(x)\rangle\mathrm{d}x\mathrm{d}y=\langle\nabla\hat{u}_0(\sigma(\cdot),\sigma),\psi\rangle_M.$$

In other words, $D\hat{\mathcal{F}}(\sigma)$ is represented by the function $\nabla\hat{F}(\sigma(\cdot),\sigma)\in M$, $D\hat{\mathcal{U}}_0(\sigma)$ by the funtion $\nabla\hat{u}_0(\sigma(\cdot),\sigma)\in M$. The functions $\hat{F}$ and $\hat{u}_0$ are of class C^3 in both variables, with bounded first, second and third derivatives. Moreover, $\hat{F}$ and $\hat{u}_0$ are $\mathbf{Z}^d$-equivariant in the first variable; they are also $L^2_{\mathbf{Z}}$ and H-equivariant in the second one.

§3
Minima on short time intervals

In lemmas 3.2-3.5 below, we recall the method of [10] for the minimals of the value function; in lemma 3.1, we prove that the value functions on measures and on parametrizations coincide.

Definitions. Let $\mu\colon(t,0)\to\mathcal{P}(\mathbf{T}^d)$ be a curve of measures satisfying, in the weak sense (the precise definition is in the proof of lemma 3.1 below), the continuity equation

$$\partial_s\mu_s+\mathrm{div}(X\mu_s)=0 \tag{3.1}$$

for a drift $X\in L^2((t,0)\times\mathbf{T}^d,\mathcal{L}^1\otimes\mu_t)$. We define the augmented action of (μ_s,X) as

$$\mathcal{A}(t,\mu_s,X)=\int_t^0[\frac{1}{2}||X(s,\cdot)||^2_{L^2_{\mu_s}}-\mathcal{F}(\mu_s)]\mathrm{d}s+\mathcal{U}_0(\mu_0).$$

The value function on $\mathcal{P}(\mathbf{T}^d)$ is defined by

$$\mathcal{U}\colon(-\infty,0]\times\mathcal{P}(\mathbf{T}^d)\to\mathbf{R},\qquad\mathcal{U}(t,\bar{\mu})=\inf\mathcal{A}(t,\mu_s,X) \tag{3.2}$$

where the inf is over all paths (μ_s,X) which satisfy (3.1) and such that $\mu_t=\bar{\mu}$. We are not going to need this, but the inf is actually a minimum.

Augmented action and value function lift in a natural way to the space M. Given $t\le 0$ and a curve $\sigma\in AC((t,0),M)$, we can define

$$\hat{\mathcal{A}}(t,\sigma)=\int_t^0[\frac{1}{2}||\dot{\sigma}_s||^2_M-\hat{\mathcal{F}}(\sigma_s)]\mathrm{d}s+\hat{\mathcal{U}}_0(\sigma_0).$$

For $t\le 0$ and $\psi\in M$, we set

$$\hat{\mathcal{U}}(t,\psi)=\inf\{\hat{\mathcal{A}}(t,\sigma)\,:\,\sigma\in AC((t,0),M)\quad\text{and}\quad\sigma_t=\psi\}.$$

Lemma 3.1. *Let $\mathcal{U}$ and $\hat{\mathcal{U}}$ be defined as above. Then, the following holds.*
1) The function $\hat{\mathcal{U}}$ is continuous. Moreover, it is H and $L^2_{\mathbf{Z}}$-equivariant, i. e.

$$\hat{\mathcal{U}}(t,\psi)=\hat{\mathcal{U}}(t,\psi\circ h+z)\qquad\forall(t,\psi,h,z)\in(-\infty,0]\times M\times H\times L^2_{\mathbf{Z}}.$$

2) Let $\tilde\mu\in\mathcal{P}(\mathbf{T}^d)$ and let $\psi\in M$ be such that $(\pi\circ\psi)_\sharp\mathcal{L}^d=\tilde\mu$. Then,

$$\mathcal{U}(t,\tilde\mu)=\hat{\mathcal{U}}(t,\psi).$$

Proof. Point 1) is easy to dispatch, since continuity is standard; we follow [18] for equivariance. If σ_s is an AC curve with $\sigma_t=\psi$, $h\in H$ and $z\in L^2_{\mathbf{Z}}$, then $\tilde\sigma_s=\sigma_s\circ h+z$ is AC and satisfies $\tilde\sigma_t=\psi\circ h+z$; moreover, since the Lagrangian and $\hat{\mathcal{U}}_0$ are $L^2_{\mathbf{Z}}$ and H-equivariant, we see immediately that

$$\mathcal{A}(t,\sigma)=\mathcal{A}(t,\tilde\sigma).$$

Clearly, this implies that $\hat{\mathcal{U}}(t,\psi\circ h+z)\le\hat{\mathcal{U}}(t,\psi)$; the opposite inequality is similar.

As for point 2), we begin to prove that

$$\hat{\mathcal{U}}(t,\psi)\le\mathcal{U}(t,\tilde\mu). \tag{3.3}$$

We assert that this follows if we show that, for any curve (μ_s,X) satisfying (3.1) with $\mu_t=\tilde\mu$ we can find $\sigma\in AC([t,0],M)$ such that
i) $(\pi\circ\sigma_t)_\sharp\mathcal{L}^d=(\pi\circ\psi)_\sharp\mathcal{L}^d=\tilde\mu$,
ii) $\mathcal{A}(t,\mu_s,X)=\hat{\mathcal{A}}(t,\sigma)$.

Indeed, we saw after formula (1.15) that *i)* together with point 1) of this lemma implies that $\hat{\mathcal{U}}(t,\sigma_0)=\hat{\mathcal{U}}(t,\psi)$; since *ii)* implies that $\hat{\mathcal{U}}(t,\sigma_0)\le\mathcal{U}(t,\tilde\mu)$, formula (3.3) follows.

Thus, let (μ_s,X) be a weak solution of (3.1) with $\mu_t=\tilde\mu$. By proposition 4.21 of [5] (or theorem 8.2.1 of [4]) there is a measure Ξ on $C([t,0],\mathbf{T}^d)$ such that, denoting by $\eta_s\colon C([t,0],\mathbf{T}^d)\to\mathbf{T}^d$ the evaluation map $\eta_s\colon\gamma\to\gamma_s$, we have

$$(\eta_s)_\sharp\Xi=\mu_s\quad\text{for all}\quad s\in[t,0]. \tag{3.4}$$

Moreover, Ξ concentrates on absolutely continuous curves and

$$\int_{C([a,b],\mathbf{T}^d)}\mathrm{d}\Xi(\gamma)\int_t^0|\dot\gamma(s)|^2\mathrm{d}s=\int_t^0||X(s,x)||^2_{L^2_{\mu_s}}\mathrm{d}s. \tag{3.5}$$

It is standard (see for instance theorem 15.5.16 of [22]) that there is a Borel map $B\colon[0,1)^d\to C([t,0],\mathbf{T}^d)$ such that $\Xi=B_\sharp\mathcal{L}^d$. We set

$$\sigma_s(x)=B(x)(s)=\eta_s\circ B(x).$$

Now point *i)* follows from (3.4), since $(\sigma_t)_\sharp\mathcal{L}^d=(\eta_t\circ B)_\sharp\mathcal{L}^d=(\eta_t)_\sharp\Xi=\mu_t$. We prove point *ii)*.

The first equality below is the definition of $\mathcal{A}$, the second one is implied by (3.4) and (3.5) while the third one follows because $\Xi=B_\sharp\mathcal{L}^d$ and $(\eta_0)_\sharp\Xi=\mu_0=(\sigma_0)_\sharp\mathcal{L}^d$. The last equality is the definition of $\hat{\mathcal{A}}$.

$$\mathcal{A}(t,\mu_s,X)=\int_t^0\left[\frac{1}{2}||X(s,\cdot)||^2_{L^2_{\mu_s}}-\frac{1}{2}\int_{\mathbf{T}^d\times\mathbf{T}^d}\phi(q-q')\mathrm{d}\mu_s(q)\mathrm{d}\mu_s(q')\right]\mathrm{d}s+\mathcal{U}_0(\mu_0)=$$

$$\int_t^0\mathrm{d}s\left[\int_{C([a,b],\mathbf{T}^d)}\frac{1}{2}|\dot\gamma(s)|^2\mathrm{d}\Xi(\gamma)-\frac{1}{2}\int_{C([a,b],\mathbf{T}^d)\times C([a,b],\mathbf{T}^d)}\phi(\gamma(s)-\gamma'(s))\mathrm{d}\Xi(\gamma)\mathrm{d}\Xi(\gamma')\right]+$$

$$+\mathcal{U}_0((\eta_0)_\sharp\Xi) = \int_t^0 \left[\frac{1}{2}||\dot\sigma_s||^2_M \mathrm{d}s - \int_t^0 \hat{\mathcal{F}}(\sigma_s)\mathrm{d}s\right] + \hat{\mathcal{U}}_0(\sigma_0) = \hat{\mathcal{A}}(t,\sigma).$$

To prove the inequality opposite to (3.3), we let $\sigma \in AC((t,0),M)$ with $\sigma_0 = \psi$ and we define

$$\mu_s = (\pi \circ \sigma_s)_\sharp \mathcal{L}^d \quad \text{for} \quad s \in (t,0). \tag{3.6}$$

We want to show

a) that μ satisfies (3.1) for a suitable drift X and

b) that the augmented action of (μ_s, X) isn't larger than the augmented action of σ.

Clearly, a) and b) imply the inequality opposite to (3.3), from which the thesis follows. We begin with a): the idea is that $X(s,q)$ is the average of the velocities $\dot\sigma_s(x)$ of the curves which satisfy $\sigma_s(x) = q$.

The measure $\mathcal{L}^1 \otimes (\pi \circ \sigma_s, \dot\sigma_s)_\sharp \mathcal{L}^d$ on $[t,0] \times \mathbf{T}^d \times \mathbf{R}^d$ has marginal $\mathcal{L}^1 \otimes (\pi\circ\sigma_s)_\sharp\mathcal{L}^d$ on $[t,0]\times\mathbf{T}^d$; we disintegrate $\mathcal{L}^1 \otimes (\pi \circ \sigma_s, \dot\sigma_s)_\sharp \mathcal{L}^d = \mathcal{L}^1 \otimes (\pi\circ\sigma_s)_\sharp\mathcal{L}^d \otimes \nu_{s,q}$ where $\nu_{s,q}$ is a measure on $\mathbf{R}^d$, depending in a Borel way on $(s,q) \in [t,0]\times\mathbf{T}^d$. In other words, if $f \in C(\mathbf{T}^d \times \mathbf{R}^d)$ is such that $\frac{|f(x,v)|}{1+|v|^2}$ is bounded, then the first equality below holds for $\mathcal{L}^1$ a. e. $s \in [a,b]$; the second equality comes from (3.6).

$$\int_{[0,1)^d} f(\sigma_s(x), \dot\sigma_s(x))\mathrm{d}x = \int_{[0,1)^d}\mathrm{d}x \int_{\mathbf{R}^d} f(\sigma_s(x), v)\mathrm{d}\nu_{s,\sigma_s(x)}(v) = \int_{\mathbf{T}^d}\mathrm{d}\mu_s(q)\int_{\mathbf{R}^d} f(q,v)\mathrm{d}\nu_{s,q}(v). \tag{3.7}$$

We set

$$X(s,q) = \int_{\mathbf{R}^d} v\mathrm{d}\nu_{s,q}(v)$$

Let now $\phi \in C_c^\infty((t,0)\times\mathbf{T}^d)$; the first equality below comes from (3.6), the second one from the definition of X and the third one from (3.7). The last equality follows since ϕ has compact support in $(t,0)\times\mathbf{T}^d$.

$$\int_t^0 \mathrm{d}s \int_{\mathbf{T}^d} [\partial_s\phi(s,q) + \langle\nabla\phi(s,q), X(s,q)\rangle]\mathrm{d}\mu_s(q) =$$

$$\int_t^0 \mathrm{d}s \int_{[0,1)^d} [\partial_s\phi(s,\sigma_s(x)) + \langle\nabla\phi(s,\sigma_s(x)), X(s,\sigma_s(x))\rangle]\mathrm{d}x =$$

$$\int_t^0 \mathrm{d}s \int_{[0,1)^d} \left[\partial_s\phi(s,\sigma_s(x)) + \langle\nabla\phi(s,\sigma_s(x)), \int_{\mathbf{R}^d} v\mathrm{d}\nu_{s,\sigma_s(x)}(v)\rangle\right]\mathrm{d}x =$$

$$\int_t^0 \mathrm{d}s \int_{[0,1)^d} [\partial_s\phi(s,\sigma_s(x)) + \langle\nabla\phi(s,\sigma_s(x)), \dot\sigma_s(x)\rangle]\mathrm{d}x =$$

$$\int_t^0 \left[\frac{\mathrm{d}}{\mathrm{d}s}\int_{[0,1)^d}\phi(s,\sigma_s(x))\mathrm{d}x\right]\mathrm{d}s = 0.$$

This means that (μ_s, X) is a weak solution of (3.1), i. e. point a) holds.

As for b), it is the same calculation, up to the use of Jensen's inequality:

$$\int_t^0 \left[\frac{1}{2}\int_{\mathbf{T}^d}|X(s,q)|^2\mathrm{d}\mu_s(q) - \mathcal{F}(\mu_s)\right]\mathrm{d}s + \mathcal{U}_0(\mu_0) \leq$$

$$\int_t^0 \left[\frac{1}{2}\int_{\mathbf{R}^d}|v|^2\mathrm{d}\nu_{s,q}(v) - \hat{\mathcal{F}}(\sigma_s)\right]\mathrm{d}s + \hat{\mathcal{U}}(\sigma_0) = \int_t^0\left[\frac{1}{2}||\dot\sigma_s||^2_M - \hat{\mathcal{F}}(\sigma_s)\right]\mathrm{d}s + \hat{\mathcal{U}}(\sigma_0).$$

\\\

Secured by the last lemma, from now on we shall concentrate on $\hat{\mathcal{A}}$ and $\hat{\mathcal{U}}$.

Definition. By $H^1_M(t,0)$ we denote the space of the maps $\sigma \in AC((t,0),M)$ such that

$$||\sigma||^2_{H^1_M} := ||\sigma_t||^2_M + \int_t^0 ||\dot\sigma_s||^2_M \mathrm{d}s < +\infty.$$

It is standard ([1]) that this is a Hilbert space for the inner product

$$\langle \sigma, \eta \rangle_{H^1_M} := \langle \sigma_t, \eta_t \rangle_M + \int_t^0 \langle \dot\sigma_s, \dot\eta_s \rangle \mathrm{d}s.$$

We recall the Poincaré-Wirtinger inequality

$$\sup_{s\in(t,0)} ||\sigma_s||_M \le ||\sigma_t||_M + |t|^{\frac{1}{2}} \cdot ||\sigma||_{H^1_M}.$$

Lemma 3.2. *For $t<0$, let us consider the functional*

$$I : H^1_M(t,0) \to \mathbf{R}, \qquad I : \sigma \to \hat{\mathcal{A}}(t,\sigma)$$

where the augmented action $\hat{\mathcal{A}}$ has been defined at the beginning of this section. Then, the following points hold.

1) The functional I is of class C^1 on $H^1_M(t,0)$. For $\hat F$ and $\hat u_0$ defined as in (2.2), we have

$$I'(\sigma)(h) = \int_t^0 [\langle \dot\sigma_s, \dot h_s \rangle_M - \langle \nabla \hat F(\sigma_s(\cdot),\sigma_s), h_s \rangle_M] \mathrm{d}s + \langle \nabla \hat u(\sigma_0(\cdot),\sigma_0)), h_0 \rangle_M =$$

$$\int_t^0 \langle \dot\sigma_s, \dot h_s \rangle_M \mathrm{d}s - \int_t^0 \mathrm{d}s \int_{[0,1)^d\times[0,1)^d} \langle \nabla\phi(\sigma_s(x)-\sigma_s(y)), h_s(x) \rangle \mathrm{d}x\mathrm{d}y +$$

$$\int_{[0,1)^d\times[0,1)^d} \langle \nabla U^0(\sigma_0(x)) + \nabla U^1(\sigma_0(x)-\sigma_0(y)), h_0(x) \rangle \mathrm{d}x\mathrm{d}y. \tag{3.8}$$

To explain the notation, we recall that $\nabla \hat F(\cdot,\sigma_s)$ is a C^2 function from $\mathbf{T}^d$ to $\mathbf{R}^d$ and thus $\nabla\hat F(\sigma_s(\cdot),\sigma_s) \in M$.
2) Let $\sigma \in H^1_M(t,0)$ be minimal in the definition of $\hat{\mathcal{U}}(t,\psi)$; then, σ solves

$$\begin{cases} \ddot\sigma_s(x) = -(\nabla\phi * \mu_s)(\sigma_s(x)) = -\nabla\hat F(\sigma_s(x),\sigma_s) \quad \text{for} \quad s \in (t,0) \\ \sigma_t(x) = \psi(x) \\ \dot\sigma_0(x) = -\nabla U^0(\sigma_0(x)) - (\nabla U^1 * \mu_0)(\sigma_0(x)) = -\nabla \hat u_0(\sigma_0(x),\sigma_0) \end{cases} \tag{3.9}$$

where we have set $\mu_s = (\sigma_s)_\sharp \mathcal{L}^p$. The equalities are in the space M, i. e. they hold for a. e. $x \in [0,1)^d$.

Proof. Since the potential $\hat{\mathcal{F}}$ and the final condition $\hat{\mathcal{U}}$ are defined by (2.1), the proof of (3.8) is classical (see for instance [2]) and we forego it.

We recall the proof of point 2), which again is classical. Since I is of class C^1 by point 1), if σ minimizes I under the constraint $\sigma_t = \psi$, then we must have that

$$I'(\sigma)(h) = 0 \quad \text{for all} \quad h \in H^1_M(t,0) \quad \text{with} \quad h_t = 0.$$

Integrating by parts in (3.8), this implies that

$$\int_t^0 \langle -\ddot{\sigma}_s - (\nabla \hat{F}(\sigma_s(\cdot), \sigma_s), h_s \rangle_M \mathrm{d}s + \langle \dot{\sigma}_0, h_0 \rangle_M + \langle \nabla \hat{u}_0(\sigma_0(\cdot), \sigma_0), h_0 \rangle_M = 0$$

for all $h \in H^1_M(t,0)$ with $h_t = 0$. Clearly, this implies the first and third formulas of (3.9), while the second one comes from the boundary conditions on the minimal σ.

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Finding minima of I is a delicate proposition (see for instance [21]) because Tonelli's theorem does not apply to the infinite-dimensional space M. However, in our case the implicit function theorem comes to the rescue: in the next three lemmas we recall the approach of [10] in our situation. In the next lemma, we denote by $B_X(\psi, r)$ the ball in X of radius r and centered in ψ.

Lemma 3.3. *There are $T, r > 0$ such that the following holds. Let $t \in [-T, 0]$, and let $\psi \in M$; we shall denote by ψ both the element of M and the function of $H^1_M(t,0)$ constantly equal to ψ.*
1) There is a unique function $\sigma^{(t,\psi)} \in C^1([-T,0], M)$ such that
i) $\sigma_s^{(t,\psi)} \in B_M(\psi, r)$ for $s \in [-T, 0]$, and
ii) $\sigma^{(t,\psi)}$ satisfies (3.9).

By the Poincaré-Wirtinger inequality, this implies that (3.9) has a unique solution in $B_{H^1_M(-T,0)}(\psi, r')$ for some $r' > 0$.
2) The map

$$\Phi \colon [-T, 0] \times M \to H^1_M(-T, 0), \qquad \Phi \colon (t, \psi) \to \sigma^{(t,\psi)}$$

is of class C^2 and equivariant, i. e. $\sigma^{(t,\psi\circ h+z)} = \sigma^{(t,\psi)} \circ h + z$ for all $h \in H$ and $z \in L^2_{\mathbf{Z}}$.

Proof. Let us consider the map

$$\Sigma \colon [-T, 0] \times M \to M, \qquad \Sigma \colon (s, \tilde{\psi}) \to \sigma_s$$

where σ_s solves the Cauchy problem

$$\begin{cases} \ddot{\sigma}_s(x) = -\nabla \hat{F}(\sigma_s(x), \sigma_s) \\ \quad \sigma_0 = \tilde{\psi} \\ \dot{\sigma}_0(x) = -\nabla \hat{u}_0(\sigma_0(x), \sigma_0) = -\nabla \hat{u}_0(\tilde{\psi}(x), \psi) \end{cases} \tag{3.10}$$

for the functions $\hat{F}$ and $\hat{u}$ which have been defined in (2.2). Since these two functions are of class C^3 by lemma 2.1, their gradients are in C^2 and the map Σ is of class C^2 by the continuous dependence theorem.

Step 1. We assert that points 1) and 2) follow if we show that there is a C^2 function $\tilde{\psi}: [-T, 0] \times M \to M$ which is, for all $\psi \in M$, the unique solution in $B(\psi, r)$ of

$$\Sigma(t, \tilde{\psi}(t, \psi)) = \psi. \tag{3.11}$$

Indeed, if this holds we can set

$$\sigma_s^{(t,\psi)} = \Sigma(s, \tilde{\psi}(t, \psi)) \tag{3.12}$$

and (3.11) immediately implies that

$$\sigma_t^{(t,\psi)} = \psi$$

i. e. $\sigma^{(t,\psi)}$ satisfies the second equation of (3.9).

Moreover, the map $: (t, \psi, s) \to \sigma_s^{(t,\psi)}$ is of class C^2 because of (3.12) and the fact that Σ and $\tilde{\psi}$ are of class C^2; in particular, $\sigma^{(t,\psi)} \in H_M^1(-T, 0)$. The map $\sigma^{(t,\psi)}$ solves the first equation of (3.9) because $: s \to \Sigma(s, \tilde{\psi}(t, \psi))$ solves it by the definition of Σ. Finally, $\sigma^{(t,\psi)}$ satisfies the third equation of (3.9) simply because it satisfies the third equation of (3.10). Uniqueness follows because, if (3.9) had two different solutions in $B_M(\psi, r)$, then also (3.11) would have two different solutions in $B_M(\psi, r)$, and we are supposing that this is not the case.

We prove the last assertion of the lemma, equivariance. Recall that $\hat{F}$ and $\hat{u}_0$ are H and $L^2_{\mathbf{Z}}$-equivariant; in particular, if $\sigma^{(t,\psi)}$ satisfies (3.9) and $(h, z) \in H \times L^2_{\mathbf{Z}}$, then also $\sigma^{(t,\psi)} \circ h + z$ satisfies (3.9) for the initial condition $\psi \circ h + z$. By the uniqueness of point 1), this implies that $\sigma^{(t,\psi\circ h+z)} = \sigma^{(t,\psi)} + z$ for all $h \in H$ and $z \in L^2_{\mathbf{Z}}$.

Step 2. In this step and in the following ones, we check that we can apply the implicit function theorem to solve for ψ in (3.11).

First of all, we saw above that the map Σ is C^2. By definition, $\Sigma(0, \psi) = \psi$ for all $\psi \in M$, which implies that

$$D\Sigma(0, \psi_0) = Id \qquad \forall \psi_0 \in M.$$

Thus, the implicit function theorem yields the existence of a C^2 function $\tilde{\psi}(t, \psi)$ defined in $[-T_0, 0] \times B_M(\psi_0, r)$ which solves (3.1).

In step 3 below, we shall see that T_0 and r do not depend on ψ_0; in step 4, we shall use the monodromy theorem to glue the local solutions into a solution defined globally on $[-T_0, 0] \times M$.

Step 3. We prove that we can choose T_0 and r independent on ψ_0.

If we look at the proof of the implicit function theorem, we see that $T_0, r > 0$ must be chosen in order that the Lipschitz constant of $: \psi \to \Sigma(t, \psi) - \psi$ is smaller than, say, $\frac{1}{2}$ in $[-T_0, 0] \times B(\psi_0, r)$; by the Lagrange theorem, this follows if $||D\Sigma(t, \psi) - Id|| \leq \frac{1}{2}$ in $[-T_0, 0] \times B(\psi_0, r)$. This follows by a Taylor development, since we saw above that $D\Sigma(0, \psi) - Id = 0$ for all ψ and that $||\partial_t D\Sigma(t, \psi)||$ is bounded in $[-1, 0] \times M$.

Step 4. By the last step, in each neighbourhood $[-T_0, 0] \times B(\psi_0, r)$ we can define a function $\tilde{\psi}$ which satisfies (3.12); since M is simply connected, we can use the monodromy theorem (see for instance theorem 1.8 of chapter 3 of [2]) to define globally a function $\tilde{\psi}: [-T_0, 0] \times M \to M$ satisfying (3.11).

\\\

Definition. From now on, $\sigma_s^{(t,\psi)}$ will be defined as in the last lemma.

Since the map $:(t,\psi) \to \sigma^{(t,\psi)}$ is of class C^2, the next lemma reduces to a classical computation ([10]) which we are only going to sketch; we continue in our practice of denoting by D the derivative in the M variable.

Lemma 3.4. *We set*

$$\hat{\mathcal{V}}(t,\psi) = \int_t^0 [\frac{1}{2}||\dot{\sigma}_s^{(t,\psi)}||_M^2 - \hat{\mathcal{F}}(\sigma_s^{(t,\psi)})]\mathrm{d}s + \hat{\mathcal{U}}_0(\sigma_0^{(t,\psi)}). \tag{3.13}$$

Then, $\hat{\mathcal{V}} \in C^2([-T,0]\times M)$ *and we have*

$$\begin{cases} -\partial_t \hat{\mathcal{V}}(t,\psi) + \frac{1}{2}||D\hat{\mathcal{V}}(t,\psi)||_M^2 + \hat{\mathcal{F}}(\psi) = 0 \quad \textit{for} \quad (t,\psi) \in [-T,0]\times M \\ \hat{\mathcal{V}}(0,\psi) = \hat{\mathcal{U}}_0(\psi). \end{cases} \tag{3.14}$$

Moreover,

$$\dot{\sigma}_s^{(t,\psi)} = -D\hat{\mathcal{V}}(s,\sigma_s^{(t,\psi)}) \quad \textit{for all} \quad s,t \in [-T,0]. \tag{3.15}$$

Proof. First of all, $\hat{\mathcal{V}} \in C^2([-T,0]\times M)$ by point 2) of lemma 3.3. Next, we differentiate with respect to ψ both terms of (3.13); after using (3.8) and (3.9) we get that

$$\dot{\sigma}_t^{(t,\psi)} = -D\hat{\mathcal{V}}(t,\sigma_t^{(t,\psi)}) = -D\hat{\mathcal{V}}(t,\psi). \tag{3.16}$$

Now we differentiate in (3.13) with respect to t; after an integration by parts, we get that

$$\partial_t \hat{\mathcal{V}}(t,\psi) = -\frac{1}{2}||\dot{\sigma}_t^{(t,\psi)}||_M^2 + \hat{\mathcal{F}}(\sigma_t^{(t,\psi)}) +$$

$$\int_t^0 \langle -\ddot{\sigma}_s^{(t,\psi)} - D\hat{\mathcal{F}}(\sigma_s^{(t,\psi)}), \partial_t \sigma_t^{(s,\psi)} \rangle_M \mathrm{d}s +$$

$$\langle \dot{\sigma}_s^{(t,\psi)}, \partial_t \sigma_s^{(t,\psi)} \rangle_M |_{s=t}^{s=0} + \langle D\hat{\mathcal{U}}(\sigma_0^{(t,\psi)}), \partial_t \sigma_0^{(t,\psi)} \rangle_M.$$

We note that the integral term is zero by the first equation of (3.9). Since $\sigma_t^{(t,\psi)} = \psi$ for all t, differentiating we get that

$$\partial_t \sigma_s^{(t,\psi)}|_{s=t} = -\dot{\sigma}_t^{(t,\psi)}.$$

Together with the last equation of (3.9), the last two equations imply that

$$\partial_t \hat{\mathcal{V}}(t,\psi) = \frac{1}{2}||\dot{\sigma}_t^{(t,\psi)}||_M^2 + \hat{\mathcal{F}}(\sigma_t^{(t,\psi)}).$$

Bt (3.16), this implies (3.14).

Next, we assert that (3.15) follows from (3.16) if we show that, for all $t, s, \tau \in [-T, 0]$, we have that

$$\sigma_\tau^{(t,\psi)} = \sigma_\tau^{(s,\sigma_s^{(t,\psi)})}. \tag{3.17}$$

To show the assertion, we denote by the dot the derivative in the τ variable; now (3.17) implies the first equality below, (3.16) the second one.

$$\dot\sigma_\tau^{(t,\psi)}|_{\tau=s} = \dot\sigma_\tau^{(s,\sigma_s^{(t,\psi)})}|_{\tau=s} = -D\hat{\mathcal{V}}(s, \sigma_s^{(t,\psi)}).$$

To show (3.17), by the uniqueness of lemma 3.3 it suffices to show that $: \tau \to \sigma_\tau^{(t,\psi)}$ satisfies

$$\begin{cases} \ddot\sigma_\tau^{(t,\psi)}(x) = -\nabla\hat{F}(\sigma_\tau^{(t,\psi)}(x), \sigma_\tau^{(t,\psi)}) \\ \sigma_s^{(t,\psi)}(x) = \sigma_s^{(t,\psi)}(x) \\ \dot\sigma_0^{(t,\psi)}(x) = -\nabla\hat{u}_0(\sigma_0^{(t,\psi)}(x), \sigma_0^{(t,\psi)}) \end{cases}$$

which is obvious since $\sigma^{(t,\psi)}$ satisfies (3.9).

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Lemma 3.5. *Let $t \in [-T, 0]$ and let $\psi \in M$. Then,*
1) for all $s \in [-T, 0]$, $\sigma^{(t,\psi)}$ is the unique minimal in the definition of $\hat{\mathcal{U}}(s, \sigma_s^{(t,\psi)})$.
2) $\hat{\mathcal{U}}(t, \psi) = \hat{\mathcal{V}}(t, \psi)$ for $(t, \psi) \in [-T, 0] \times M$.

Proof. Point 2) follows immediately from point 1) and the definitions of $\hat{\mathcal{U}}$ and $\hat{\mathcal{V}}$; we recall the classical proof of [10] for point 1). Let $\hat{\mathcal{V}}$ be as in the last lemma and let us consider the functional

$$J_s\colon H^1_M(t, 0) \to \mathbf{R},$$

$$J_s\colon \sigma \to \int_s^0 [\frac{1}{2}||\dot\sigma_\tau||^2_M - \mathcal{F}(\sigma_\tau) + \partial_\tau\hat{\mathcal{V}}(\tau, \sigma_\tau) + \langle D\hat{\mathcal{V}}(\tau, \sigma_\tau), \dot\sigma_\tau\rangle_M]\mathrm{d}\tau. \tag{3.18}$$

Since $\hat{\mathcal{V}}$ is of class C^2 by lemma 3.4, we get the first equality below, while the second one follows from the second formula of (3.14) and the definition of $\hat{\mathcal{A}}$ at the beginning of this section.

$$J_s(\sigma) = \int_s^0 [\frac{1}{2}||\dot\sigma_\tau||^2_M - \mathcal{F}(\sigma_\tau)]\mathrm{d}\tau + \hat{\mathcal{V}}(0, \sigma_0) - \hat{\mathcal{V}}(s, \sigma_s) =$$

$$\hat{\mathcal{A}}(s, \sigma) - \hat{\mathcal{V}}(s, \sigma_s). \tag{3.19}$$

Thus, if we restrict to the curves $\sigma \in H^1_M(s, 0)$ with $\sigma_s = \sigma_s^{(t,\psi)}$, minimizing J_s is the same as minimizing $\hat{\mathcal{A}}(s, \sigma)$: the thesis follows if we check that $\sigma^{(t,\psi)}$ is minimal for J_s. Actually, we are going to show that the integrand of J_s is constantly equal to its minimum along $(\tau, \sigma_\tau^{(t,\psi)}, \dot\sigma_\tau^{(t,\psi)})$.

Clearly, for all $(\tau, \eta) \in [-T, 0] \times M$ the minimum of the Lagrangian of J_s

$$B_{\tau,\eta}\colon M \to \mathbf{R}$$

$$B_{\tau,\eta}\colon \dot\lambda \to \frac{1}{2}||\dot\lambda||^2_M - \mathcal{F}(\eta) + \partial_\tau\hat{\mathcal{V}}(\tau,\eta) + \langle D_\eta\hat{\mathcal{V}}(\tau,\eta),\dot\lambda\rangle_M$$

is attained at $\dot\lambda = -D_\eta\hat{\mathcal{V}}(\tau,\eta)$; substituting this value into the expression for $B_{\tau,\eta}$ we get the inequality below, while the equality is the first formula of (3.14).

$$B_{\tau,\eta}(\dot\lambda) \ge -\frac{1}{2}||D_\eta\hat{\mathcal{V}}(\tau,\eta)||^2_M - \mathcal{F}(\eta) + \partial_\tau\hat{\mathcal{V}}(\tau,\eta) = 0 \qquad \forall\dot\lambda\in M. \tag{3.20}$$

On the other side, (3.15) implies the second equality below, (3.14) the third one.

$$B_{\tau,\dot\sigma^{(t,\psi)}_\tau}(\dot\sigma^{(t,\psi)}_\tau) = \frac{1}{2}||\dot\sigma^{(t,\psi)}_\tau||^2_M - \mathcal{F}(\sigma^{(t,\psi)}_\tau) + \partial_\tau\hat{\mathcal{V}}(\tau,\sigma^{(t,\psi)}_\tau) + \langle D\hat{\mathcal{V}}(\tau,\sigma^{(t,\psi)}_\tau),\dot\sigma^{(t,\psi)}_\tau\rangle_M =$$

$$-\frac{1}{2}||D\hat{\mathcal{V}}(\tau,\sigma^{(t,\psi)}_\tau)||^2_M - \mathcal{F}(\sigma^{(t,\psi)}_\tau) + \partial_\tau\hat{\mathcal{V}}(\tau,\sigma^{(t,\psi)}_\tau) = 0.$$

The last two formulas imply that $:\tau \to \sigma^{(t,\psi)}_\tau$ minimizes J_s, as we wanted.

We prove uniqueness: by the aforesaid, if σ_τ minimizes, then the integrand of J_s must be zero along σ_τ. By (3.20), this implies that $\dot\sigma_\tau = -D\mathcal{V}(\tau,\sigma_\tau)$. By (3.15) this implies that σ_τ and $\sigma^{(t,\psi)}_\tau$ satisfy the same differential equation; we recall from lemma 3.4 that $-D\hat{\mathcal{V}}(t,\psi)$ is Lipschitz. Since $\sigma_s = \sigma^{(t,\psi)}_s$ by hypothesis, we get that $\sigma_\tau = \sigma^{(t,\psi)}_\tau$ for $\tau\in[-T,0]$ by the existence and uniqueness theorem.

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§4
The master equation

In this section, we are going to define the value function for the single particle; we shall see that it determines the movement of the whole pack and that it satisfies the master equation.

Definition. We define

$$v\colon [-T,0]\times\mathbf{T}^d\times[-T,0]\times M\to\mathbf{R},$$

$$v(s,q|t,\psi) = \min\left\{\int_s^0 [\frac{1}{2}|\dot y(\tau)|^2 - \hat F(y(\tau),\sigma^{(t,\psi)}_\tau)]\mathrm{d}\tau + \hat u_0(y(0),\sigma^{(t,\psi)}_0)\right\} \tag{4.1}$$

where the minimum (whose existence is guaranteed by Tonelli's theorem) is over all $y\in AC((s,0),\mathbf{T}^p)$ such that $y(s)=q$. In the notation for v we have inaugurated the practice of placing the "parameters", in this case (t,ψ), after the vertical slash. In other words, we are interested in the equation solved by v in the first two variables. If we freeze (t,ψ), then $v(s,q|t,\psi)$ is the value function of the particle q, given that the whole pack moves like $\sigma^{(t,\psi)}$. Thus, v solves, in its first two variables, the Hamilton-Jacobi equation.

Lemma 4.1. *Up to reducing T, the following holds.*
1) For $s,t\in[-T,0]$, the minimum in the definition of $v(s,q|t,\psi)$ is attained on a unique function

$$:\tau\to y(\tau|s,q,t,\psi).$$

Again, the parameters of the orbit (i. e. the initial conditions of the single particle and of the whole pack) are on the right of the vertical slash.

2) The map

$$: (\tau, s, q, t, \psi) \to y(\tau|s, q, t, \psi)$$

is of class C^2.

3) The value function

$$: (s, q, t, \psi) \to v(s, q|t, \psi)$$

is of class C^2 with bounded first and second derivatives. It is $\mathbf{Z}^d$-equivariant in the second variable, H and $L^2_{\mathbf{Z}}$-equivariant in the fourth one. For all $(t, \psi) \in [-T, 0] \times M$ it satisfies the Hamilton-Jacobi equation with time reversed

$$\begin{cases} -\partial_s v(s, q|t, \psi) + \frac{1}{2}|\nabla v(s, q|t, \psi)|^2 + \hat{F}(q, \sigma_s^{(t,\psi)}) = 0 \quad (s, q) \in [-T, 0] \times \mathbf{T}^d \\ v(0, q|t, \psi) = \hat{u}_0(q, \sigma_0^{(t,\psi)}) \end{cases} \tag{4.2}$$

in the classical sense. Recall that we denote the gradient in the $\mathbf{T}^p$ variable by ∇, in the M variable by D.

4) We have that, for $\mathcal{L}^p$ a. e. $x \in [0, 1)^d$ and all $t, s, \tau \in [-T, 0]$,

$$\dot{y}(\tau|s, \sigma_s^{(t,\psi)}(x), t, \psi) = \dot{\sigma}_\tau^{(t,\psi)}(x) = -\nabla v(\tau, y(\tau|s, \sigma_s^{(t,\psi)}(x), t, \psi)|t, \psi) = -D\hat{\mathcal{V}}(\tau, \sigma_\tau^{(t,\psi)})(x).$$

5) Let us define the function S as the flow of $-\nabla v$, i. e. as

$$S(s, q, \tau|t, \psi) = y(\tau)$$

where y solves

$$\begin{cases} \dot{y}(\tau) = -\nabla v(\tau, y(\tau)|t, \psi) \\ y(s) = q. \end{cases} \tag{4.3}$$

Then, up to reducing T, there is $D_2 > 0$ independent of $(s, q, \tau, t, \psi) \in [-T, 0] \times \mathbf{T}^d \times [-T, 0]^2 \times M$ such that

$$\frac{1}{D_2} \le \det \frac{\partial S(s, q, \tau|t, \psi)}{\partial q} \le D_2.$$

Proof. We fix (t, ψ) as the initial condition of the whole pack; we consider the time dependent Lagrangian

$$\mathcal{L}(s, q, \dot{q}) = \frac{1}{2}|\dot{q}|^2 - \hat{F}(q, \sigma_s^{(t,\psi)})$$

and the final condition

$$: q \to \hat{u}_0(q, \sigma_0^{(t,\psi)}).$$

Note that, by lemma 2.1, $\mathcal{L}$ is C^3 in $(s, q, \dot{q})$; it depends in a C^2 way on the parameters (t, ψ) by lemma 3.3. Analogously, $\hat{u}_0$ is C^3 in the variable q and C^2 in (t, ψ). Now points 1), 2) and 3) follow by the argument of [10], which we have seen in lemmas 3.3, 3.4 and 3.5 above.

As for point 4), formula (3.15) gives that, for all $\tau \in [-T, 0]$,

$$\dot{\sigma}_\tau^{(t,\psi)}(x) = -D\hat{\mathcal{V}}(\tau, \sigma_\tau^{(t,\psi)})(x) \quad \text{for } \mathcal{L}^p \text{ a. e.} \quad x \in [0,1)^d.$$

On the other side, with exactly the same proof we used for formula (3.15) we see that

$$\dot{y}(\tau | s, \sigma_s^{(t,\psi)}(x), t, \psi) = -\nabla v(\tau, y(\tau | s, \sigma_s^{(t,\psi)}(x), t, \psi) | t, \psi) \quad \text{for} \quad t, s, \tau \in [-T, 0].$$

Thus, it suffices to show the first equality of point 4). Classical Hamilton-Jacobi theory (which we recalled above in lemmas 3.3 to 3.5) implies that the minimizer

$$: \tau \to y(\tau | s, q, t, \psi)$$

satisfies

$$\begin{cases} \frac{\mathrm{d}^2}{\mathrm{d}\tau^2} y(\tau | s, q, t, \psi) = -\nabla \hat{F}(y(\tau | s, q, t, \psi), \sigma_\tau^{(t,\psi)}) \\ \quad y(s | s, q, t, \psi) = q \\ \quad \dot{y}(0 | s, q, t, \psi) = -\nabla \hat{u}_0(y(0 | s, q, t, \psi), \sigma_0^{(t,\psi)}). \end{cases}$$

If $q = \sigma_s^{(t,\psi)}(x)$ then, by (3.9), this is the same equation that is satisfied by $: \tau \to \sigma_\tau^{(t,\psi)}(x)$ for $\mathcal{L}^d$ a. e. $x \in [0,1)^d$; by the uniqueness of lemma 3.3 this implies the first equality of point 4).

We prove point 5). Since $S(s, q, s | t, \psi) = q$ by definition, we see that $\partial_q S(s, q, s | t, \psi) = Id$; thus, point 5) follows if we show that the map $: \tau \to \partial_q S(s, q, \tau | t, \psi)$ is Lipschitz uniformly in (s, q, τ, t, ψ); in other words, we have to show that the norm of $\partial_{q\tau}^2 S(s, q, \tau | t, \psi)$ is bounded. This follows easily by (4.3), the differentiable dependence theorem and point 3) of this lemma, which implies

$$|\partial_{q,q}^2 v(s, q | t, \psi)| \leq M \qquad \forall (s, q, t, \psi) \in [-T, 0] \times \mathbf{T}^d \times [-T, 0] \times M.$$

\\\

We can apply to the value function $v(s, q | t, \psi)$ a change of coordinates: namely, instead of seeing it as a function of $\sigma_t^{(t,\psi)} = \psi$, we can see it as a function of $\sigma_s^{(t,\psi)}$. In other words, we can define a function u as

$$u(s, q | \sigma_s^{(t,\psi)}) := v(s, q | t, \psi).$$

Equivalently, by (3.17) we get that, for $\psi \in M$, $\psi = \sigma_t^{(s, \sigma_s^{(t,\psi)})}$; setting $\eta = \sigma_s^{(t,\psi)}$ and substituting in the formula above, we get that

$$u(s, q | \eta) = v(s, q | t, \sigma_t^{(s,\eta)}) \quad \text{for all} \quad t \in [-T, 0], \quad \eta \in M \tag{4.4}$$

which incidentally proves that the definition of u is well posed. The first equality below comes from (4.4), since $\sigma_s^{(s,\psi)} = \psi$; the second one is (4.1).

$$u(s, q | \psi) = v(s, q | s, \psi) =$$

$$\min\Big\{\int_s^0[\frac{1}{2}|\dot y(\tau)|^2-\hat F(y(\tau),\sigma_\tau^{(s,\psi)})]\mathrm{d}\tau+\hat u_0(y(0),\sigma_0^{(s,\psi)})\,:\,y\in AC((s,0),\mathbf{T}^p),\quad y(s)=q\Big\}. \tag{4.5}$$

Lemma 4.2. *Let*

$$u\colon[-T,0]\times\mathbf{T}^d\times M\to\mathbf{R}$$

be defined as in (4.4) or as in (4.5), which is the same. Then, u is of class C^2 in all its variables and satisfies the master equation

$$-\partial_t u(t,q|\psi)+\frac{1}{2}|\nabla u(t,q|\psi)|^2+F(q,\psi)+\langle\nabla u(t,\psi(\cdot)|\psi),Du(t,q|\psi)\rangle_M=0.$$

Proof. By (4.4), lemma 4.1 and the chain rule we get that u is of class C^2 in all its variables. Since $\sigma_t^{(t,\psi)}=\psi$ for all t, differentiating we get that

$$\frac{\partial}{\partial s}\sigma_t^{(s,\psi)}|_{s=t}=-\dot\sigma_t^{(t,\psi)}. \tag{4.6}$$

The first equality of (4.5) implies the equalities below.

$$Du(t,q|\psi)=Dv(t,q|t,\psi),\qquad\nabla u(t,q|\psi)=\nabla v(t,q|t,\psi). \tag{4.7}$$

The first equality below is point 4) of lemma 4.1, the second one comes from (4.7).

$$\dot\sigma_t^{(t,\psi)}(x)=-\nabla v(t,\psi(x)|t,\psi)=-\nabla u(t,\psi(x)|\psi). \tag{4.8}$$

If we differentiate (4.4) in s, we get the first equality below; the second one comes from (4.2) and (4.6); the last one comes from (4.7) and (4.8).

$$\partial_s u(s,q|\psi)|_{s=t}=\partial_s v(s,q|t,\sigma_t^{(s,\psi)})|_{s=t}+\langle Dv(s,q|t,\sigma_t^{(s,\psi)}),\frac{\partial}{\partial s}\sigma_t^{(s,\psi)}\rangle_M|_{s=t}=$$

$$\frac{1}{2}|\nabla v(t,q|t,\psi)|^2+\hat F(q,\psi)-\langle Dv(t,q|t,\psi),\dot\sigma_t^{(t,\psi)}\rangle_M=$$

$$\frac{1}{2}|\nabla u(t,q|\psi)|^2+\hat F(q,\psi)+\langle Du(t,q|\psi),\nabla u(t,\psi(\cdot)|\psi)\rangle_M.$$

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End of the proof of theorem 1. Point 1) follows from lemma 3.5; point 2) is point 2) of lemma 3.3; point 3) is lemma 4.2; point 4) follows from point 5) of lemma 4.1; point 5) is point 4) of lemma 4.1 and (4.7).

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Remark. By the results of section 1, $u(t,q|\psi)$ quotients to a function on measures which is strongly differentiable, with continuous derivative; it satisfies the master equation in the classical sense, i. e. taking derivatives at their face value.

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