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Abstract

We give sufficient conditions on a symmetric tensor $S \in S^d \mathbb{F}^n$ to satisfy the equality: the symmetric rank of S, denoted as srank S, is equal to the rank of S, denoted as rank S. This is done by considering the rank of the unfolded S viewed as a matrix A(S). The condition is: rank $S \in$ {rank A(S), rank A(S) + 1}. In particular, srank S = rank S for $S \in S^d \mathbb{C}^n$ for the cases $(d, n) \in \{(3, 2), (4, 2), (3, 3)\}$. We discuss the analogs of the above results for border rank and best approximations of symmetric tensors.

Keywords: tensors, symmetric tensors, rank of tensor, symmetric rank of symmetric tensor, border ranks, best k-approximation of tensors.

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1 Introduction

For a field \mathbb{F} let $\otimes^{d}\mathbb{F}^n \supset S^{d}\mathbb{F}^n$ denote *d*-mode tensors and the subspace of symmetric tensors on \mathbb{F}^n . Let $\mathcal{T} \in \otimes^{d}\mathbb{F}^n$. Denote by rank \mathcal{T} the rank of the tensor \mathcal{T} . That is, for $\mathcal{T} \neq 0$ rank \mathcal{T} is the minimal number *k* such that \mathcal{T} is a sum of *k* rank one tensors. (rank 0 = 0.) We say that \mathcal{T} has a unique decomposition as a sum rank \mathcal{T} rank one tensors if this decomposition is unique up to a permutations of the summands. Assume that $\mathcal{S} \in S^d\mathbb{F}^n \setminus \{0\}$. Suppose that $|\mathbb{F}| \geq d$, i.e. \mathbb{F} has at least *d* elements. Then it is known that \mathcal{S} is a sum of *k* symmetric rank one tensors [14, Proposition 7.2]. See [1] for the case $|\mathbb{F}| = \infty$, i.e. \mathbb{F} has an infinite number of elements. The minimal *k* is the symmetric rank of \mathcal{S} , denoted as srank \mathcal{S} . Clearly, rank $\mathcal{S} \leq \operatorname{srank} \mathcal{S}$. In what follows we assume that $d \geq 3$ unless stated otherwise. In [20, P15, page 5] P. Comon asked if rank $\mathcal{S} = \operatorname{srank} \mathcal{S}$ over $\mathbb{F} = \mathbb{R}, \mathbb{C}$. This problem is also raised in [7, end §4.1, p' 1263]. This problem is sometimes referred as Comon's conjecture. In [7] it is shown that this conjecture holds in the first nontrivial case: rank $\mathcal{S} = 2$.

For a finite field the situation is more complicated: Observe first that for $\mathbb{F} = \mathbb{Z}_2$ and the symmetric matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ we have the the inequality rank A = 2 <srank A = 3. (A is a sum of all three distinct symmetric rank one matrices in

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 $S^2\mathbb{Z}_2^2$.) Second, it is shown in [14, Proposition 7.1] that over a finite field there exist symmetric tensors that are not a sum symmetric rank one tensors.

To state our result we need the following notions: For $n \in \mathbb{N}$ denote $[n] = \{1, \ldots, n\}$. Let $S = [s_{i_1, \ldots, i_d}]_{i_1, \ldots, i_d \in [n]} \in S^d \mathbb{F}^n$. Denote by A(S) an $n \times n^{d-1}$ matrix with entries $b_{\alpha\beta}$ where $\alpha \in [n]$ and $\beta = (\beta_1, \ldots, \beta_{d-1}) \in [n]^{d-1}$. Then $b_{\alpha\beta} := s_{\alpha,\beta_1,\ldots,\beta_{d-1}}$. $(A(S) \in \mathbb{F}^{n \times n^{d-1}}$ is the unfolding of S in the direction 1. As S is symmetric, the unfolding in every direction $k \in [d]$ gives rise to the same matrix.) Hence rank $A(S) \leq n$. If $m := \operatorname{rank} A(S) < n$ it means that we can choose another basis so that S is represented as $S' \in S^d \mathbb{F}^m$. Recall that rank $S \geq \operatorname{rank} A(S)$. (See for example the arguments in [12] for d = 3.). Thus, to study Comon's conjecture we can assume without loss of generality that rank A(S) = n.

Denote by $\Sigma(n, d, \mathbb{F})$ and $\Sigma_s(d, n, \mathbb{F})$ the Segre variety of rank one tensors plus the zero tensor and the subvariety of symmetric tensors of at most rank one in $(\mathbb{F}^n)^{\otimes d}$.

Let $F_{d,n,k}: \Sigma(n,d,\mathbb{F})^k \to (\mathbb{F}^n)^{\otimes d}$ be the polynomial map:

$$F_{d,n,k}((\mathcal{T}_1,\ldots,\mathcal{T}_k)) := \sum_{j=1}^k \mathcal{T}_j.$$
(1.1)

Let $\mathcal{T} = F_{d,n,k}((\mathcal{T}_1, \ldots, \mathcal{T}_k))$. In what follows we say that the decomposition $\mathcal{T} = \sum_{j=1}^k \mathcal{T}_j$ is unique if rank $\mathcal{T} = k$ and any decomposition of \mathcal{T} to a sum of r rank one tensors is obtained by permuting the order of the summands in $\mathcal{T} = \sum_{j=1}^k \mathcal{T}_j$.

Denote by $G_{d,n,k}$ the restriction of the map $F_{d,n,k}$ to : $\Sigma_s(n,d,\mathbb{F})^k$. Thus $F_{d,n,k}(\Sigma(n,d,\mathbb{F})^k)$ and $G_{d,n,k}(\Sigma_s(n,d,\mathbb{F})^k)$ are the sets of d-mode tensors on \mathbb{F}^n tensors of at most rank k and of symmetric tensors of at most symmetric rank k.

Chevalley's theorem yields that $F_{d,n,k}(\Sigma(n,d,\mathbb{C})^k)$ and $G_{d,n,k}(\Sigma_s(n,d,\mathbb{C})^k)$ are constructible sets. Hence the dimension of $G_{d,n,k}(\Sigma_s(n,d,\mathbb{C})^k)$ is the maximal rank of the Jacobian of the map $G_{d,n,k}$.

 $\mathcal{S} \in \mathrm{S}^d \mathbb{C}^n$ is said to have a generic symmetric rank k if the following conditions hold: First, the dimension of the constructible set $G_{d,n,k}(\Sigma_s(n,d,\mathbb{C})^k)$ is greater than the dimension of $G_{d,n,k-1}(\Sigma_s(n,d,\mathbb{C})^{k-1})$. Second, there exists a strict subvariety $O \subset \Sigma(n,d,\mathbb{C})^k$, such that $\mathcal{S} \in G_{d,n,k}(\Sigma_s(n,d,\mathbb{C})^k \setminus O)$. Let

$$k_{n,d} := \frac{\binom{n+d-1}{d}}{n}.$$
 (1.2)

Chiantini, Ottaviani and Vannieuwenhoven showed recently [5] that if $S \in S^d \mathbb{C}^n$ has a generic symmetric rank $k < k_{n,d}$ then $k = \operatorname{rank} S$. It is much easier to establish this kind of result for smaller values of k using Kruskal's theorem. See [14, Theorem 7.6].

The aim of this paper is to establish a much weaker result on Comon's conjecture, which does not use the term *generic*. In particular we show that Comon's conjecture holds for symmetric tensors of at most rank 3 and for 3-symmetric tensors of at most rank 5 over \mathbb{C} .

Our main result is

Theorem 1.1 Let $d \geq 3$, $|\mathbb{F}| \geq 3$ and $\mathcal{S} \in S^d \mathbb{F}^n$. Suppose that rank $\mathcal{S} \leq \operatorname{rank} A(\mathcal{S}) + 1$. Then srank $\mathcal{S} = \operatorname{rank} \mathcal{S}$.

We now summarize briefly the content of this paper. In §2 we recall Kruskal's theorem on the rank of 3-tensor. In §3 we prove Theorem 1.1 for the case rank S =rank A(S). In §4 we show that each $S \in S^3 \mathbb{F}^2$, where $|\mathbb{F}| \geq 3$, satisfies srank S =rank S. In §5 we prove Theorem 1.1 in the case d = 3 and rank S = rank A(S) + 1. In §6 we prove Theorem 1.1 for $d \geq 4$. In §7 we summarize our results for $\mathbb{F} = \mathbb{C}$. In §8 we discuss two other closely related conjectures: The first one conjectures that it is possible to replace in Comon's conjecture the ranks with border ranks. We show that this is true if the border rank of S is two. The second one conjectures that a best k-approximation of symmetric tensor can be chosen symmetric. For k = 1, i.e. best rank one approximation, this conjecture holds and it is a consequence of Banach's theorem [2].

2 Kruskal's theorem

We recall Kruskal's theorem for 3-tensors and any field \mathbb{F} . For p vectors $\mathbf{x}_1, \ldots, \mathbf{x}_p \in \mathbb{F}^n$ denote by $[\mathbf{x}_1 \ \mathbf{x}_2 \ \cdots \ \mathbf{x}_p]$ the $n \times p$ matrix whose columns are $\mathbf{x}_1, \ldots, \mathbf{x}_p$. Kruskal's rank of $[\mathbf{x}_1 \ \mathbf{x}_2 \ \cdots \ \mathbf{x}_p]$, denoted as Krank $(\mathbf{x}_1, \ldots, \mathbf{x}_p)$ is the maximal k such that any k vectors in the set $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ are linearly independent. (If $\mathbf{x}_i = 0$ for some $i \in [p]$ then Krank $(\mathbf{x}_1, \ldots, \mathbf{x}_p) = -\infty$.)

Theorem 2.1 (Kruskal) Let \mathbb{F} be a field, $r \in \mathbb{N}$ and $\mathbf{x}_i \in \mathbb{F}^m$, $\mathbf{y}_i \in \mathbb{F}^n$, $\mathbf{z}_i \in \mathbb{F}^p$ for $i \in [r]$. Assume that

$$\mathcal{T} = \sum_{i=1}^{r} \mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i.$$
(2.1)

Suppose that

 $2r + 2 \leq \operatorname{Krank}\left(\mathbf{x}_{1}, \dots, \mathbf{x}_{r}\right) + \operatorname{Krank}\left(\mathbf{y}_{1}, \dots, \mathbf{y}_{r}\right) + \operatorname{Krank}\left(\mathbf{z}_{1}, \dots, \mathbf{z}_{r}\right).$ (2.2)

Then rank $\mathcal{T} = r$. Furthermore, the decomposition (2.1) is unique.

Note that max(Krank $(\mathbf{x}_1, \ldots, \mathbf{x}_r)$, Krank $(\mathbf{y}_1, \ldots, \mathbf{y}_r)$, Krank $(\mathbf{z}_1, \ldots, \mathbf{z}_r)$) $\leq r$. Hence (2.2) yields that

 $\min(\operatorname{Krank}(\mathbf{x}_1,\ldots,\mathbf{x}_r),\operatorname{Krank}(\mathbf{y}_1,\ldots,\mathbf{y}_r),\operatorname{Krank}(\mathbf{z}_1,\ldots,\mathbf{z}_r)) \ge 2.$ (2.3)

In particular, $\min(m, n, p) \ge 2$.

In what follows we need a following simple corollary of Kruskal's theorem:

Lemma 2.2 Let $3 \leq d \in \mathbb{N}$. Assume that $\mathbf{x}_{j,1}, \ldots, \mathbf{x}_{j,r} \in \mathbb{F}^{n_j}$ are linearly independent for each $j \in [d]$. Let

$$\mathcal{T} = \sum_{i=1}^{r} \otimes_{j=1}^{d} \mathbf{x}_{j,i}.$$
(2.4)

Then rank $\mathcal{T} = r$. Furthermore, the decomposition (2.4) is unique.

Proof. Observe first that $\bigotimes_{j=1}^{p} \mathbf{x}_{j,1}, \ldots, \bigotimes_{j=1}^{p} \mathbf{x}_{j,r}$ linearly independent for $p = 1, \ldots, d$. Clearly, this is true for p = 1 and p = 2. Use the induction to prove this statement for $p \ge 3$ by observing that $\bigotimes_{j=1}^{p} \mathbf{x}_{j,i} = (\bigotimes_{j=1}^{p-1} \mathbf{x}_{j,i}) \otimes \mathbf{x}_{p,i}$ for $p = 3, \ldots, d$.

Consider \mathcal{T} given by (2.4). Suppose first that r = 1. Then \mathcal{T} is a rank one tensor and its decomposition is unique. Assume that $r \geq 2$. Consider \mathcal{T} as a 3-tensor on the 3-tensor product $\mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes (\otimes_{j=3}^d \mathbb{F}^{n_j})$. Clearly

Krank
$$(\mathbf{x}_{1,1}, \dots, \mathbf{x}_{r,1}) =$$
 Krank $(\mathbf{x}_{1,2}, \dots, \mathbf{x}_{r,2}) =$ (2.5)
Krank $(\otimes_{j=3}^{d} \mathbf{x}_{j,1}, \dots, \otimes_{j=3}^{d} \mathbf{x}_{j,r}) = r.$

As $3r - 2 \ge 2r$, Kruskal's theorem yields that the rank of \mathcal{T} as 3-tensor is r. Hence rank \mathcal{T} as d tensor is r too. Furthermore the decomposition (2.4) of \mathcal{T} as a 3-tensor is unique. Hence the decomposition(2.4) is unique.

In what follows we need the following lemma.

Lemma 2.3 Let $d \ge 3, n \ge 2$ and $S \in S^d \mathbb{F}^n$. Assume that

$$\mathcal{S} = \sum_{i=1}^{k} \otimes_{j=1}^{d} \mathbf{x}_{j,i}.$$
(2.6)

Then $S = \sum_{i=1}^{k} \bigotimes_{j=1}^{d} \mathbf{x}_{\sigma(j),i}$ for any permutation σ of [d]. Suppose that the following inequality holds:

$$2k + 2 \le K(\mathbf{x}_{1,1}, \dots, \mathbf{x}_{1,k}) + K(\mathbf{x}_{2,1}, \dots, \mathbf{x}_{2,k}) + K(\bigotimes_{j=3}^{d} \mathbf{x}_{j,1}, \dots, \bigotimes_{j=3}^{d} \mathbf{x}_{j,k}).$$
(2.7)

Then rank $S = \operatorname{srank} S = k$, *i.e.* $\operatorname{span}(\mathbf{x}_{1,i}) = \ldots = \operatorname{span}(\mathbf{x}_{d,i})$ for each $i \in [k]$. Furthermore, the decomposition (2.6) is unique.

Proof. Assume that (2.6) holds. Since S symmetric we deduce that $S = \sum_{i=1}^{k} \otimes_{j=1}^{d} \mathbf{x}_{\sigma(j),i}$ for any permutation σ of [d]. Suppose that (2.7) holds. Kruskal's theorem yields that the decomposition of S as a 3-tensor on $\mathbb{F}^n \otimes \mathbb{F}^n \otimes (\otimes^{d-2}\mathbb{F}^n)$ is unique. In particular, the decomposition (2.6) is unique. Hence rank S = k. Let σ be the transposition on [d] satisfying $\sigma(1) = 2, \sigma(2) = 1$. Then $S = \sum_{i=1}^{k} \otimes_{j=1}^{d} \mathbf{x}_{\sigma(j),i}$. (2.3) yields that $K(\otimes_{j=3}^{d} \mathbf{x}_{j,1}, \ldots, \otimes_{j=3}^{d} \mathbf{x}_{j,k}) \geq 2$. That is, the rank one tensors $\otimes_{j=3}^{d} \mathbf{x}_{j,p}$ and $\otimes_{j=3}^{d} \mathbf{x}_{j,q}$ are linearly independent for p < q. The uniqueness of the decomposition (2.6), (up to a permutation of summands), yields that $\mathbf{x}_{1,i} \otimes \mathbf{x}_{2,i} \otimes (\otimes_{j=3}^{d} \mathbf{x}_{j,i}) = \mathbf{x}_{2,i} \otimes \mathbf{x}_{1,i} \otimes (\otimes_{j=3}^{d} \mathbf{x}_{j,i})$ for each $i \in [n]$. Hence $\mathbf{x}_{1,i} \otimes \mathbf{x}_{2,i} = \mathbf{x}_{2,i} \otimes \mathbf{x}_{1,i}$. Therefore $\mathbf{x}_{1,i}$ and $\mathbf{x}_{2,i}$ are linearly dependent nonzero vectors. Let σ be a transposition on [d] satisfying $\sigma(2) = p, \sigma(p) = 2$ for some $p \geq 3$. (2.3) yields that $K(\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{1,k}) \geq 2$. The uniqueness of the decomposition (2.6), (up to a permutation for each $i \in [n]$. Hence $\mathbf{x}_{1,i} \otimes \mathbf{x}_{2,i} \otimes (\otimes_{j=3}^{d} \mathbf{x}_{\sigma(j),i})$. Therefore $\mathbf{x}_{1,i}$ and $\mathbf{x}_{2,i}$ are linearly dependent nonzero vectors. Let σ be a transposition of summands), yields that $\mathbf{x}_{1,i} \otimes \mathbf{x}_{2,i} \otimes (\otimes_{j=3}^{d} \mathbf{x}_{\sigma(j),i})$. Therefore $\mathbf{x}_{2,i}$ and $\mathbf{x}_{p,i}$ are collinear for each $i \in [n]$. Hence $\mathbf{x}_{1,i} \otimes \mathbf{x}_{p,i} \otimes (\otimes_{j=3}^{d} \mathbf{x}_{\sigma(j),i})$. Therefore $\mathbf{x}_{2,i}$ and $\mathbf{x}_{p,i}$ are collinear for each $i \in [n]$. Hence $\mathrm{span}(\mathbf{x}_{1,i}) = \ldots = \mathrm{span}(\mathbf{x}_{d,i})$ for each $i \in [d]$. Thus the decomposition (2.6) is a decomposition to a sum of symmetric rank one tensors. Hence $\mathrm{srank} S = \mathrm{rank} S$. \Box

3 The case rank $S = \operatorname{rank} A(S)$

Theorem 3.1 Let $d \geq 3, n \geq 2$ and $S \in S(d, \mathbb{F}^n)$. Suppose that rank S = rank A(S). Then srank S = rank S. Furthermore, S has has a unique rank one decomposition.

Proof. We can assume without loss of generality that rank A(S) = n. So (2.6) holds for k = n. Clearly, $\mathbf{x}_{j,1}, \ldots, \mathbf{x}_{j,n}$ are linearly independent for each $j \in [d]$. Hence $K(\mathbf{x}_{j,1}, \ldots, \mathbf{x}_{j,n}) = n$ for $j \in [d]$. The proof of Lemma 2.2 yields that $K(\bigotimes_{j=3}^{d} \mathbf{x}_{j,1}, \ldots, \bigotimes_{j=3}^{d} \mathbf{x}_{j,n}) = n$. Therefore equality (2.5) holds for r = n. As $n \geq 2$ we deduce (2.7) for k = n. Lemma 2.3 yields the theorem.

The following corollary generalizes [7, Proposition 5.5] to any field \mathbb{F} :

Corollary 3.2 Let \mathbb{F} be a field, $S \in S^d \mathbb{F}^n \setminus \{0\}, d \geq 3$. Assume that rank $S \leq 2$. Then srank S = rank S.

Proof. Clearly, rank $A(S) \in \{1, 2\}$. If rank A(S) = 1 then $S = s \otimes^d \mathbf{u}$. Hence rank S = srank S = 1. If rank A(S) = 2 then rank S = 2 and we conclude the result from Theorem 3.1.

4 The case $S^3 \mathbb{F}^2$

Theorem 4.1 Let $S \in S^3 \mathbb{F}^2$. Assume that $|\mathbb{F}| \geq 3$. Then rank $S = \operatorname{srank} S \leq 3$.

For $\mathbb{F} = \mathbb{C}$ this result follows from the classical description of binary forms in two variables due Sylvester [23]. More generally, consult with [6] for results on the rank of tensors in $S^d \mathbb{F}^2$ for an algebraic closed field \mathbb{F} of characteristic zero.

Proof. In view of Corollary 3.2 it is enough to consider the case where rank $S \geq 3$. Let $S = [s_{i,j,k}]_{i,j,k \in [2]}$.

1. Assume that $s_{1,1,2}s_{1,2,2} \neq 0$. Let

$$S = \sum_{i=1}^{3} t_i \otimes^3 \mathbf{u}_i, \ \mathbf{u}_1 = (1, b)^{\top}, \quad \mathbf{u}_2 = (1, 0)^{\top}, \quad \mathbf{u}_3 = (0, 1)^{\top}.$$
(4.1)

Then

$$s_{1,1,2} = t_1 b, \quad s_{1,2,2} = t_1 b^2 \Rightarrow t_1 = \frac{s_{1,1,2}^2}{s_{1,2,2}}, \quad b = \frac{s_{1,2,2}}{s_{1,1,2}},$$

$$t_2 = s_{1,1,1} - t_1, \quad t_3 = s_{2,2,2} - t_1 b^3.$$

Hence rank $S \leq 3$. Our assumption yields that rank S = 3 and (4.1) is a minimal decomposition of S to rank one tensors. This decomposition shows that rank S = srank S.

- 2. Assume that $s_{1,1,2} = s_{1,2,2} = 0$ then $S = s_{1,1,1} \otimes^3 (1,0)^\top + s_{2,2,2} \otimes^3 (0,1)^\top$. This contradicts our assumption that rank $S \ge 3$.
- 3. It is left to discuss the case where rank $S \geq 3$ and $s_{1,1,2} = 0$ and $s_{1,2,2} \neq 0$. The homogeneous polynomial of degree 3 corresponding to S is

$$f(x_1, x_2) = s_{1,1,1}x_1^3 + s_{1,2,2}x_1x_2^2 + s_{2,2,2}x_2^3.$$

(a) Assume that the characteristic of \mathbb{F} is 3. Make the following change of variables: $x_1 = y_1, x_2 = y_1 + y_2$. The new tensor \mathcal{S}' satisfies 1.

- (b) Assume that the characteristic of \mathbb{F} is not 3.
 - i. Assume that $s_{1,1,1} \neq 0$. Make the following change of variables: $x_1 = y_1 + ay_2, x_2 = y_2$. Then

$$f(y_1, y_2) = \alpha y_1^3 + \beta y_1^2 y_2 + \gamma y_1 y_2^2 + \delta, \ \beta = 3as_{1,1,1}, \gamma = s_{1,2,2} + 3a^2 s_{1,1,1}, \beta = 3as_{1,1,1}, \beta = 3as_{1,1,1},$$

Then choose a nonzero a such that $s_{1,2,2} + 3a^2s_{1,1,1} \neq 0$. (This is always possible if $|\mathbb{F}| \geq 4$ as we assumed that $\mathbb{F} \neq \mathbb{Z}_3$ and $|\mathbb{F}| \geq 3$.) The new tensor S' satisfies 1.

- ii. Assume that $s_{1,1,1} = s_{2,2,2} = 0$. Make the following change of variables: $x_1 = y_1, x_2 = (y_1 + y_2)$. Then we are either in the case 1 if the characteristic of \mathbb{F} is not 2 or in the case $\beta(b)$ if the characteristic of \mathbb{F} is 2.
- iii. Assume that $s_{1,1,1} = 0$ and $s_{2,2,2} \neq 0$. Make the following change of variables: $y_2 = s_{1,2,2}x_1 + s_{2,2,2}x_2, y_2 = x_2$. Then we are in the case $\beta(\mathbf{b})$ ii.

Note that if $|\mathbb{F}| \gg 1$ then using the change of coordinates and then the above procedure we obtain that if rank $\mathcal{S} = 3$, rank $A(\mathcal{S}) = 2$ we have many presentation of \mathcal{S} as sum of three rank one symmetric tensors.

Observe next that for $\mathbb{F} = \mathbb{Z}_2$ not every symmetric tensor $S \in S^3 \mathbb{Z}_2^2$ is a sum of rank one symmetric tensors. The number of all symmetric tensors in $S^3 \mathbb{Z}_2^2$ is 2^4 . The number of all nonzero symmetric tensors which are sum of rank one symmetric tensors is $2^3 - 1$. Hence Theorem 4.1 does not hold for $\mathbb{F} = \mathbb{Z}_2$.

Corollary 4.2 Let $S \in S^{3}\mathbb{F}^{n}$. Assume that $|\mathbb{F}| \geq 3$ and rank S = 3. Then srank $S = \operatorname{rank} S$.

Proof. Clearly, rank $A(S) \in \{2,3\}$. If rank A(S) = 3 we deduce the corollary from Theorem 3.1. If rank A(S) = 2 we deduce the corollary from Theorem 4.1. \Box

5 The case d = 3 and rank $S = \operatorname{rank} A(S) + 1$

In this section we prove Theorem 1.1 for d = 3. In view of Theorem 3.1 it is enough to consider the case rank $S = \operatorname{rank} A(S) + 1$. Furthermore, in view of Theorem 4.1 it is enough to consider the case rank $A(S) \ge 3$. We first give the following obvious lemma:

Lemma 5.1 Let $|\mathbb{F}| \geq 3$ and $S \in S(3, \mathbb{F}^n)$. Suppose that rank $S = \operatorname{rank} A(S) + 1$. 1. Assume furthermore that there exists a decomposition of S to rank A(S) + 1rank one tensors such that at least one of them is symmetric, i.e. $s \otimes^3 \mathbf{u}$. Let $S' = S - s \otimes^3 \mathbf{u}$. Then rank $S' = \operatorname{rank} S - 1$ and rank $A(S') \in \{\operatorname{rank} A(S) - 1, \operatorname{rank} A(S)\}$. Furthermore:

- 1. If rank $A(S') = \operatorname{rank} A(S)$ then rank $S' = \operatorname{rank} A(S')$ and rank $S' = \operatorname{srank} S'$. Hence rank $S = \operatorname{srank} S$.
- 2. If rank $A(S') = \operatorname{rank} A(S) 1$ then rank $S' = \operatorname{rank} A(S') + 1$.

5.1 The case rank $A(\mathcal{S}) = 3$

We now discuss Theorem 1.1 where $S \in S(3, \mathbb{F}^n)$, where rank S = 4, rank A(S) = 3. Without loss of generality we can assume that n = 3. Then

$$S = \sum_{i=1}^{4} \mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i.$$
(5.1)

Suppose first that there is a decomposition (5.1) such that $\mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i$ is symmetric for some $i \in [4]$. Then we can use Lemma 5.1. Apply Theorems 3.1 and 4.1 to deduce that srank $S = \operatorname{rank} S = 4$.

Assume the Assumption: there no is a decomposition (5.1) such that $\mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i$ is symmetric for some $i \in [4]$. The first part of Lemma 2.3 yields:

$$0 = \sum_{i=1}^{4} \mathbf{x}_i \otimes (\mathbf{y}_i \otimes \mathbf{z}_i - \mathbf{z}_i \otimes \mathbf{y}_i).$$

As rank A(S) = 3 we can assume that $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly independent. Then $\mathbf{x}_4 = \sum_{j=1}^3 a_j \mathbf{x}_j$. Then the above equality yields:

$$0 = \sum_{j=1}^{3} \mathbf{x}_{j} \otimes (\mathbf{y}_{j} \otimes \mathbf{z}_{j} - \mathbf{z}_{j} \otimes \mathbf{y}_{j} + a_{j}(\mathbf{y}_{4} \otimes \mathbf{z}_{4} - \mathbf{z}_{4} \otimes \mathbf{y}_{4})).$$
(5.2)

As $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly independent it follows that

$$\mathbf{y}_j \otimes \mathbf{z}_j - \mathbf{z}_j \otimes \mathbf{y}_j = -a_j(\mathbf{y}_4 \otimes \mathbf{z}_4 - \mathbf{z}_4 \otimes \mathbf{y}_4) \text{ for } j \in [3].$$
(5.3)

Assume first that \mathbf{y}_4 and \mathbf{z}_4 are collinear. Then \mathbf{y}_j and \mathbf{z}_j are collinear for $j \in [3]$. Hence we w.l.o.g we can assume that $\mathbf{y}_i = \mathbf{z}_i = \mathbf{u}_i$ for $i \in [4]$. So $S = \sum_{i=1}^{4} \mathbf{x}_i \otimes \mathbf{u}_i \otimes \mathbf{u}_i$. Since S is symmetric we can we obtain that $S = \sum_{i=1}^{4} \mathbf{u}_i \otimes \mathbf{u}_i \otimes \mathbf{x}_i$. Renaming the vectors we can assume that in the original decomposition (5.1) we have that \mathbf{x}_i and \mathbf{y}_i are collinear for $i \in [4]$. Since we assumed that no rank one tensor $\mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i$ is not symmetric, we deduce that each pair $\mathbf{y}_i, \mathbf{z}_i$ in the original decomposition (5.1) is not collinear. In particular, it is enough to study the case where \mathbf{y}_4 and \mathbf{z}_4 are not collinear, i.e. $\mathbf{y}_4 \otimes \mathbf{z}_4 - \mathbf{z}_4 \otimes \mathbf{y}_4 \neq 0$. Let $\mathbf{U} := \operatorname{span}(\mathbf{y}_4, \mathbf{z}_4)$. Then dim $\mathbf{U} = 2$.

Suppose that $a_j \neq 0$ for some $j \in [3]$. Then (5.3) yields that \mathbf{y}_j and \mathbf{z}_j are not collinear and span $(\mathbf{y}_j, \mathbf{z}_j) = \mathbf{U}$. Assume that $a_j = 0$. Then \mathbf{y}_j and \mathbf{z}_j are collinear.

Assume first that $a_1a_2a_3 \neq 0$. Then the above arguments yields that $\operatorname{span}(\mathbf{y}_1, \ldots, \mathbf{y}_4) \subseteq \mathbf{U}$, which contradicts the assumption that $\operatorname{span}(\mathbf{y}_1, \ldots, \mathbf{y}_4) = \mathbb{F}^3$.

So we need to assume that at least one of $a_i = 0$. Assume first that exactly one $a_i = 0$. Without loss of generality we can assume that in (5.2) $a_1 = 0$ and $a_2a_3 \neq 0$. This yields that \mathbf{y}_1 and \mathbf{z}_1 are collinear. Our Assumption yields that \mathbf{x}_1 and \mathbf{y}_1 are not collinear. Furthermore $\operatorname{span}(\mathbf{y}_2, \mathbf{z}_2) = \operatorname{span}(\mathbf{y}_3, \mathbf{z}_3) = \mathbf{U}$. Hence $\operatorname{span}(\mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4) \subseteq \mathbf{U}$. As $\operatorname{span} \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4$ is the whole space, we deduce that $\mathbf{y}_1 \notin \operatorname{span}(\mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4)$. Similarly $\mathbf{y}_1 \notin \operatorname{span}(\mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4)$. Furthermore, dim $\operatorname{span}(\mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4) = 2$. Hence $\operatorname{span}(\mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4) = \mathbf{U}$. We now recall that $\mathcal{S} =$ $\sum_{i=1}^{4} \mathbf{y}_i \otimes \mathbf{x}_i \otimes \mathbf{z}_i$. Again, by renaming the indices 2, 3, 4 we can assume that $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$, are linearly independent. Since $\operatorname{span}(\mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4) = \mathbf{U}$ it follows that $\mathbf{y}_4 = b_2\mathbf{y}_2 + b_3\mathbf{y}_3$. Since S is symmetric we have the equality $S = \sum_{i=1}^{4} \mathbf{y}_i \otimes \mathbf{x}_i \otimes \mathbf{z}_i$. Permuting the last two factors we obtain the equality $0 = \sum_{i=1}^{4} \mathbf{y}_i \otimes (\mathbf{x}_i \otimes \mathbf{z}_i - \mathbf{z}_i \otimes \mathbf{y}_i)$. Hence we have an analogous equality to (5.2):

$$0 = \mathbf{y}_1 \otimes (\mathbf{x}_1 \otimes \mathbf{z}_1 - \mathbf{z}_1 \otimes \mathbf{x}_1) + \sum_{j=2}^3 \mathbf{y}_j \otimes (\mathbf{x}_j \otimes \mathbf{z}_j - \mathbf{z}_j \otimes \mathbf{x}_j + b_j (\mathbf{x}_4 \otimes \mathbf{z}_4 - \mathbf{z}_4 \otimes \mathbf{x}_4)).$$

Therefore $\mathbf{x}_1 \otimes \mathbf{z}_1 - \mathbf{z}_1 \otimes \mathbf{x}_1 = 0$. Thus \mathbf{x}_1 and \mathbf{z}_1 are collinear. Recall that we already showed that \mathbf{y}_1 and \mathbf{z}_1 are collinear. Hence $\mathbf{x}_1 \otimes \mathbf{y}_1 \otimes \mathbf{z}_1$ is a symmetric rank one tensor. So we have a contradiction to our Assumption.

Finally let us assume that $a_i = a_j = 0$ for some two distinct indices $i, j \in [3]$. W.l.o.g. we can assume that $\mathbf{x}_4 = \mathbf{x}_3$, i.e. $a_1 = a_2 = 0, a_3 = 1$. This implies that \mathbf{y}_i and \mathbf{z}_i are collinear for i = 1, 2. Furthermore

$$C := \mathbf{y}_3 \otimes \mathbf{z}_3 + \mathbf{y}_4 \otimes \mathbf{z}_4 = \mathbf{z}_3 \otimes \mathbf{y}_3 + \mathbf{z}_4 \otimes \mathbf{y}_4$$

So *C* is a symmetric matrix. Note that *C* is a rank two matrix. Otherwise $\mathbf{y}_3 \otimes \mathbf{z}_3$ and $\mathbf{y}_4 \otimes \mathbf{z}_4$ are collinear. Then $\mathbf{x}_3 \otimes \mathbf{y}_3 \otimes \mathbf{z}_3 + \mathbf{x}_3 \otimes \mathbf{y}_4 \otimes \mathbf{z}_4$ is a rank one tensor. So rank $S \leq 3$, contrary to our assumptions. Thus we can assume that $C = \mathbf{y}_3 \otimes \mathbf{y}_4 + \mathbf{y}_4 \otimes \mathbf{y}_3$ and $\mathbf{y}_3, \mathbf{y}_4$ are linearly independent. Hence we can assume that

$$\begin{aligned} \mathcal{S} &= & \mathbf{x}_1 \otimes \mathbf{y}_1 \otimes \mathbf{y}_1 + \mathbf{x}_2 \otimes \mathbf{y}_2 \otimes \mathbf{y}_2 + \mathbf{x}_3 \otimes (\mathbf{y}_3 \otimes \mathbf{y}_4 + \mathbf{y}_4 \otimes \mathbf{y}_3) = \\ & \mathbf{y}_1 \otimes \mathbf{x}_1 \otimes \mathbf{y}_1 + \mathbf{y}_2 \otimes \mathbf{x}_2 \otimes \mathbf{y}_2 + \mathbf{y}_3 \otimes \mathbf{x}_3 \otimes \mathbf{y}_4 + \mathbf{y}_4 \otimes \mathbf{x}_3 \otimes \mathbf{y}_3 = \\ & \mathbf{y}_1 \otimes \mathbf{y}_1 \otimes \mathbf{x}_1 + \mathbf{y}_2 \otimes \mathbf{y}_2 \otimes \mathbf{x}_2 + \mathbf{y}_3 \otimes \mathbf{y}_4 \otimes \mathbf{x}_3 + \mathbf{y}_4 \otimes \mathbf{y}_3 \otimes \mathbf{x}_3. \end{aligned}$$

Our Assumption yields that the pairs $\mathbf{x}_1, \mathbf{y}_1$ and $\mathbf{x}_2, \mathbf{y}_2$ are linearly independent. Hence $Q := \mathbf{x}_1 \otimes \mathbf{y}_1 - \mathbf{y}_1 \otimes \mathbf{x}_1 \neq 0$. Subtracting the third expression for S from the second one we deduce

$$\mathbf{y}_1 \otimes (\mathbf{x}_1 \otimes \mathbf{y}_1 - \mathbf{y}_1 \otimes \mathbf{x}_1) + \mathbf{y}_2 \otimes (\mathbf{x}_2 \otimes \mathbf{y}_2 - \mathbf{y}_2 \otimes \mathbf{x}_2) + \mathbf{y}_3 \otimes (\mathbf{x}_3 \otimes \mathbf{y}_4 - \mathbf{y}_4 \otimes \mathbf{x}_3) + \mathbf{y}_4 \otimes (\mathbf{x}_3 \otimes \mathbf{y}_3 - \mathbf{y}_3 \otimes \mathbf{x}_3) = 0$$

As $\mathbf{y}_3, \mathbf{y}_4$ are linearly independent, without loss in generality we may assume that $\mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4$ are linearly independent. So $\mathbf{y}_1 = b_2\mathbf{y}_2 + b_3\mathbf{y}_3 + b_4\mathbf{y}_4$. Substitute in the above equality this expression for \mathbf{y}_1 only for the \mathbf{y}_1 appearing in the left-hand side to obtain

$$\mathbf{y}_2 \otimes (\mathbf{x}_2 \otimes \mathbf{y}_2 - \mathbf{y}_2 \otimes \mathbf{x}_2 + b_2 Q_2) + \mathbf{y}_3 (\mathbf{x}_3 \otimes \mathbf{y}_4 - \mathbf{y}_4 \otimes \mathbf{x}_3 + b_3 Q) + \mathbf{y}_4 (\mathbf{x}_3 \otimes \mathbf{y}_3 - \mathbf{y}_3 \otimes \mathbf{x}_3 + b_4 Q) = 0$$

Hence

$$\mathbf{x}_2 \otimes \mathbf{y}_2 - \mathbf{y}_2 \otimes \mathbf{x}_2 + b_2 Q_2 = \mathbf{x}_3 \otimes \mathbf{y}_4 - \mathbf{y}_4 \otimes \mathbf{x}_3 + b_3 Q = \mathbf{x}_3 \otimes \mathbf{y}_3 - \mathbf{y}_3 \otimes \mathbf{x}_3 + b_4 Q = 0.$$

Note that our Assumption yields that $b_2 \neq 0$. Hence $\operatorname{span}(\mathbf{x}_2, \mathbf{y}_2) = \operatorname{span}(\mathbf{x}_1, \mathbf{y}_1)$. Suppose first that $b_3 \neq 0$. Then $\operatorname{span}(\mathbf{x}_3, \mathbf{y}_4) = \operatorname{span}(\mathbf{x}_1, \mathbf{y}_1)$. This contradicts the assumption that $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly independent. As $\mathbf{x}_3 = \mathbf{x}_4$, we get also a contradiction if $b_4 \neq 0$. Hence $b_3 = b_4 = 0$. So $\mathbf{y}_3, \mathbf{y}_4 \in \operatorname{span}(\mathbf{x}_3)$. This contradicts the assumption that \mathbf{y}_3 and \mathbf{y}_4 are linearly independent.

In conclusion we showed that our Assumption never holds. The proof of this case of Theorem 1.1 is concluded. $\hfill \Box$

5.2 Case rank $A(\mathcal{S}) \ge 4$

Proof. By induction on $r = \operatorname{rank} A(S) \ge 3$. For r = 3 the proof follows from the results above. Assume that Theorem holds for rank S = r+1. Assume now that rank A(S) = r + 1 and rank S = r + 2. Without loss of generality we can assume that n = r + 1. Suppose first that the assumptions of Lemma 5.1 hold. If we are in the case 1 then srank $S = \operatorname{rank} S$. If we are in the case 2. then we deduce from the induction hypothesis that srank $S = \operatorname{rank} S$.

As in the proof of the case rank S = 3 we assume the Assumption: There does not exist a decomposition of S to rank A(S) + 1 rank one tensors such that at least one of them is symmetric. We will show that we will obtain a contradiction.

Suppose $S = \sum_{i=1}^{n+1} \mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i$. The we have the equality

$$0 = \sum_{i=1}^{n+1} \mathbf{x}_i \otimes (\mathbf{y}_i \otimes \mathbf{z}_i - \mathbf{z}_i \otimes \mathbf{y}_i)$$

and the fact that span of all x's, y's and z's is \mathbb{F}^n .

Without loss of generality we may assume that $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent. So $\mathbf{x}_{n+1} = \sum_{i=1}^n a_i \mathbf{x}_i$. Hence

$$\sum_{i+1}^{n} \mathbf{x}_{i} \otimes (\mathbf{y}_{i} \otimes \mathbf{z}_{i} - \mathbf{z}_{i} \otimes \mathbf{y}_{i} + a_{i}(\mathbf{y}_{n+1} \otimes \mathbf{z}_{n+1} - \mathbf{z}_{n+1} \otimes \mathbf{y}_{n+1})) = 0.$$
(5.4)

As in the case n = 3 we can assume that \mathbf{y}_{n+1} and \mathbf{z}_{n+1} are not collinear. Thus if $a_i = 0$ we deduce that \mathbf{y}_i and \mathbf{z}_i are collinear. If $a_i \neq 0$ we deduce that $\operatorname{span}(\mathbf{y}_i, \mathbf{z}_i) = \operatorname{span}(\mathbf{y}_{n+1}, \mathbf{z}_{n+1})$. Since $\mathbf{y}_1, \ldots, \mathbf{y}_{n+1}$ span \mathbb{F}^n we can have at most two nonzero a_i . Since $\mathbf{x}_{n+1} \neq 0$ we must have at least one nonzero a_i . Assume first that n-1 out of $\{a_1, \ldots, a_n\}$ are zero. We may assume without loss of generality that $a_1 = \ldots = a_{n-1} = 0$ and $a_n = 1$. So $\mathbf{x}_{n+1} = \mathbf{x}_n$. Without loss of generality we may assume that

$$\mathcal{S} = \mathbf{x}_n \otimes (\mathbf{y}_n \otimes \mathbf{z}_n + \mathbf{y}_{n+1} \otimes \mathbf{z}_{n+1}) + \sum_{i=1}^{n-1} \mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{y}_i.$$

Since S is symmetric as in case n = 3 we deduce that $\mathbf{y}_n \otimes \mathbf{z}_n + \mathbf{y}_{n+1} \otimes \mathbf{z}_{n+1}$ is symmetric and has rank two. So we can assume that $\mathbf{z}_n = \mathbf{y}_{n+1}, \mathbf{z}_{n+1} = \mathbf{y}_n$ and dim span $(\mathbf{y}_n, \mathbf{y}_{n+1}) = 2$. We now repeat the arguments in the proof of this case for n = 3 to deduce the contradiction.

Suppose finally that exactly n-2 out of $\{a_1, \ldots, a_n\}$ are zero. We may assume without loss of generality that $a_1 = \ldots = a_{n-2} = 0$ and $a_{n-1}, a_n \neq 0$. So $\operatorname{span}(\mathbf{y}_{n-1}, \mathbf{z}_{n-1}) = \operatorname{span}(\mathbf{y}_n, \mathbf{z}_n) = \operatorname{span}(\mathbf{y}_{n+1}, \mathbf{z}_{n+1})$. Hence $\mathbf{y}_{n-1}, \mathbf{y}_n, \mathbf{y}_{n+1}$ are linearly dependent. Since $\mathbf{y}_1, \ldots, \mathbf{y}_{n+1}$ span the whole space we must have that dim $\operatorname{span}(\mathbf{y}_{n-1}, \mathbf{y}_n, \mathbf{y}_{n+1}) = 2$. Without loss of generality we may assume the following: First, $\mathbf{y}_n, \mathbf{y}_{n+1}$ are linearly independent and $\mathbf{y}_{n-1} = a\mathbf{y}_n + b\mathbf{y}_{n+1}$. Second $\mathbf{z}_k = \mathbf{y}_k$ for $k = 1, \ldots, n-2$. So we can assume that

$$\mathcal{S} = \sum_{j=1}^{n-2} \mathbf{x}_j \otimes \mathbf{y}_j \otimes \mathbf{y}_j + \sum_{j=n-1}^{n+1} \mathbf{x}_j \otimes \mathbf{y}_j \otimes \mathbf{z}_j = \sum_{j=1}^{n-2} \mathbf{y}_j \otimes \mathbf{x}_j \otimes \mathbf{y}_j + \sum_{j=n-1}^{n+1} \mathbf{y}_j \otimes \mathbf{x}_j \otimes \mathbf{z}_j.$$

Permuting the las two factors in the last part of the above identity we obtain:

$$0 = \sum_{j=1}^{n-2} \mathbf{y}_j \otimes (\mathbf{x}_j \otimes \mathbf{y}_j - \mathbf{y}_j \otimes \mathbf{x}_j) + \sum_{j=n-1}^{n+1} \mathbf{y}_j (\mathbf{x}_j \otimes \mathbf{z}_j - \mathbf{z}_j \otimes \mathbf{x}_j).$$

Substitute $\mathbf{y}_{n-1} = a\mathbf{y}_n + b\mathbf{y}_{n+1}$ and recall that $\mathbf{y}_1, \ldots, \mathbf{y}_{n-2}, \mathbf{y}_n, \mathbf{y}_{n+1}$ are linearly independent. Hence \mathbf{x}_i and \mathbf{y}_i are collinear for $i = 1, \ldots, n-2 \ge 2$. This contradicts our Assumption.

6 Theorem 1.1 for $d \ge 4$

In this section we show Theorem for 1.1 for $d \ge 4$. Theorem 3.1 yields that it is enough to consider the case where rank $S = \operatorname{rank} A(S) + 1$. We need the following lemma:

Lemma 6.1 Let $2 \leq d \in \mathbb{N}$. Assume that $\mathbf{x}_{j,1}, \ldots, \mathbf{x}_{j,n+1} \in \mathbb{F}^n \setminus \{\mathbf{0}\}$ and $\operatorname{span}(\mathbf{x}_{j,1}, \ldots, \mathbf{x}_{j,n+1}) = \mathbb{F}^n$ for $j \in [d]$. Consider the n + 1 rank one d-tensors $\otimes_{j=1}^d \mathbf{x}_{j,i}, i \in [n+1]$. Then either all of them are linearly independent or n of these tensors are linearly independent and the other one is a multiple of one of the nlinearly independent tensors.

Proof. It is enough to consider the case where the n + 1 rank one *d*-tensors $\otimes_{j=1}^{d} \mathbf{x}_{j,i}, i \in [n+1]$ are linearly dependent. Without loss of generality we may assume that $\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{1,n}$ are linearly independent. Hence the *n* tensors $\otimes_{j=1}^{d} \mathbf{x}_{j,i}, i \in [n]$ are linearly independent as rank one matrices $\mathbf{x}_{1,i} \otimes (\otimes_{j=2}^{d} \mathbf{x}_{j,i})$ for $i \in [n]$. (I.e., the corresponding unfolding of *n* tensors in mode 1 are linearly independent.) Assume that $\mathbf{x}_{1,n+1} = \sum_{j=1}^{n} a_j \mathbf{x}_{1,j}$ where not a_j are zero. Since we assumed that $\otimes_{j=1}^{d} \mathbf{x}_{j,i}, i \in [n+1]$ are linearly dependent it follows that $\otimes_{j=1}^{d} \mathbf{x}_{j,n+1} = \sum_{i=1}^{n} b_i \otimes_{j=1}^{d} \mathbf{x}_{j,i}$. So we obtain the identity $\sum_{i=1}^{n} \mathbf{x}_{1,i} \otimes \mathcal{T}_i = 0$. Here $\mathcal{T}_i \in \otimes^{d-1} \mathbb{F}^n$ is a tensor of at most rank 2. Since $\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{1,n}$ are linearly independent if follows that each \mathcal{T}_i is zero. Hence if $a_i \neq 0$ it follows that b_i is not zero and $\otimes_{j=2} \mathbf{x}_{j,n+1}$ and $\otimes_{j=2} \mathbf{x}_{j,i}$ are collinear. Therefore $\mathbf{x}_{j,i}$ and $\mathbf{x}_{j,n+1}$ are collinear for $j = 2, \ldots, d$. Since dim span $(\mathbf{x}_{j,1}, \ldots, \mathbf{x}_{j,n+1}) = n$, we can't have another $a_k \neq 0$. So $\otimes_{j=1}^{d} \mathbf{x}_{j,n+1}$ is collinear with $\otimes_{j=1} \mathbf{x}_{j,i}$ as we claimed.

Proof of Theorem 1.1 for $d \ge 4$ and rank $S = \operatorname{rank} A(S) + 1$. Without loss of generality we may assume that $n = \operatorname{rank} A(S) \ge 2$. Assume that $S = \sum_{i=1}^{n+1} \bigotimes_{j=1}^{d} \mathbf{x}_{j,i}$. Clearly, the assumptions of Lemma 6.1 holds. Consider the d-2rank one tensors $\bigotimes_{j \in [d] \setminus \{p,q\}} \mathbf{x}_{j,i}$ for fixed $p \ne q \in [d]$ and $i \in [n+1]$. Suppose that these n + 1 rank one tensors are linearly independent. We claim that $\mathbf{x}_{p,i}$ and $\mathbf{x}_{q,i}$ are collinear for each $i \in [n+1]$. Without loss of generality we may assume that p = 1, q = 2. By interchanging the first two factors in the representation of S as a rank n + 1 tensor we deduce:

$$\sum_{i=1}^{n+1} (\mathbf{x}_{1,i} \otimes \mathbf{x}_{2,i} - \mathbf{x}_{2,i} \otimes \mathbf{x}_{1,i}) \otimes (\otimes_{j=3}^{d} \mathbf{x}_{j,i}) = 0.$$

As $\otimes_{j=3}^{d} \mathbf{x}_{j,i}, i \in [n+1]$ are linearly independent we deduce that $\mathbf{x}_{1,i} \otimes \mathbf{x}_{2,i} - \mathbf{x}_{2,i} \otimes \mathbf{x}_{1,i} = 0$ for each $i \in [n+1]$. I.e., $\mathbf{x}_{1,i}$ and $\mathbf{x}_{2,i}$ are collinear for each $i \in [n+1]$.

Suppose first that for each pair of integers $1 \le p < q \le d \otimes_{j \in [d] \setminus \{p,q\}} \mathbf{x}_{j,i}, i \in [n+1]$ are linearly independent. Hence $\mathbf{x}_{j,i} \in \text{span}(\mathbf{x}_{1,i})$ for $j \in [d]$ and $i \in [n+1]$. Therefore $\otimes_{j=1}^{d} \mathbf{x}_{j,i}$ is a rank one symmetric tensor for each $i \in [n+1]$. Thus srank S = rank S.

Assume now, without loss of generality, that $\otimes_{j=3}^{d} \mathbf{x}_{j,i}, i \in [n+1]$ are linearly dependent. By applying Lemma 6.1 we can assume without loss of generality that $\otimes_{j=3}^{d} \mathbf{x}_{j,i}, i \in [n]$ are linearly independent and $\otimes_{j=3}^{d} \mathbf{x}_{j,n+1} = \otimes_{j=3}^{d} \mathbf{x}_{j,n}$. Without loss of generality we may assume that $\mathbf{x}_{j,n+1} = \mathbf{x}_{j,n}$ for $j \geq 3$. (We may need to rescale the vectors $\mathbf{x}_{2,n+1}, \ldots, \mathbf{x}_{d,n+1}$.) Hence $\mathbf{x}_{j,1}, \ldots, \mathbf{x}_{j,n}$ are linearly independent for each $j \geq 3$. Therefore we have the following decomposition of S as a 3-tensor in $(\otimes^2 \mathbb{F}^n) \otimes \mathbb{F}^n \otimes (\otimes^{d-3} \mathbb{F}^n)$:

$$S = (\mathbf{x}_{1,n} \otimes \mathbf{x}_{2,n} + \mathbf{x}_{1,n+1} \otimes \mathbf{x}_{2,n+1}) \otimes \mathbf{x}_{3,n} \otimes (\otimes_{j=4}^{d} \mathbf{x}_{j,n}) + (6.1)$$
$$\sum_{i=1}^{n-1} (\mathbf{x}_{1,i} \otimes \mathbf{x}_{2,i}) \otimes \mathbf{x}_{3,i} \otimes (\otimes_{j=4}^{d} \mathbf{x}_{j,i}) = \sum_{i=1}^{n} \mathcal{T}_{i} \otimes (\otimes_{j=4}^{d} \mathbf{x}_{j,i}).$$

Clearly, $\otimes_{j=4}^{d} \mathbf{x}_{j,1}, \ldots, \otimes_{j=4}^{d} \mathbf{x}_{j,n}$ are linearly independent. Since S is symmetric by interchanging every two distinct factors $p, q \in [3]$ in $\otimes^{d} \mathbb{F}^{n}$ we deduce that $\mathcal{T}_{1}, \ldots, \mathcal{T}_{n}$ are symmetric 3-tensors. Consider the symmetric tensor $\mathcal{T}_{n} = (\mathbf{x}_{1,n} \otimes \mathbf{x}_{2,n} + \mathbf{x}_{1,n+1} \otimes \mathbf{x}_{2,n+1}) \otimes \mathbf{x}_{3,n}$. As the rank of $A(\mathcal{T}_{n})$ in the the third coordinate is 1 it follows that rank A(S) = 1. Hence rank $\mathcal{T}_{n} = 1$. Therefore rank $S \leq n$ contrary to our assumptions.

Corollary 6.2 Let $|\mathbb{F}| \geq 3, d \geq 3, n \geq 2$. Assume that $\mathcal{S} \in S^d \mathbb{F}^n$. Then srank $\mathcal{S} = \operatorname{rank} \mathcal{S}$ under the following assumptions:

- 1. rank $S \leq 3$.
- 2. srank $\mathcal{S} \leq 4$.

Proof. It is enough to consider the case where $n = \operatorname{rank} A(S) \ge 2$. 1. Clearly, rank $S \in \{2, 3\}$. Theorem 1.1 yields that srank $S = \operatorname{rank} S$. 2. Assume to the contrary that rank $S < \operatorname{srank} S \le 4$. Then rank $S \le 3$. Part 1. implies the contradiction rank $S = \operatorname{srank} S$.

7 Symmetric tensors over \mathbb{C}

Recall the known maximal value of the symmetric rank in $S^d \mathbb{C}^n$, denoted as $\mu(d, n)$:

- 1. $\mu(d,2) = d$ [6], [3, §3.1];
- 2. $\mu(3,3) = 5$ [22, §96], [8] and [19];
- 3. $\mu(3,4) = 7$ [22, §97];
- 4. $\mu(3,5) \le 10$ [10];

5. $\mu(4,3) = 7$ [17, 9].

Theorem 7.1 Let $\mathbb{F} = \mathbb{C}$ and S be a symmetric tensor in $S^d \mathbb{C}^n$. Then srank $S = \operatorname{rank} S$ in the following cases:

- 1. $d \ge 3, n \ge 2$ and rank $\mathcal{S} \in \{ \operatorname{rank} A(\mathcal{S}), \operatorname{rank} A(\mathcal{S}) + 1 \}.$
- 2. For n = 2 and d = 3.
- 3. For n = 2 and d = 4
- 4. n = d = 3.
- 5. $\mathcal{S} \in S^3 \mathbb{C}^n$ and rank $\mathcal{S} \leq 5$.
- 6. $\mathcal{S} \in S^3 \mathbb{C}^n$ and srank $\mathcal{S} \leq 6$.

Proof. Assume that $\mathcal{S} \in S^d \mathbb{C}^n$. Clearly, it is enough to prove the theorem for the case rank $A(\mathcal{S}) \geq 2$. Furthermore, we can assume that $n = A(\mathcal{S})$. Thus it is enough to assume the following conditions:

$$2 \le n = \operatorname{rank} A(\mathcal{S}) \le \operatorname{rank} \mathcal{S} \le \operatorname{srank} \mathcal{S} \le \mu(d, n).$$
(7.1)

1. follows from Theorem 1.1.

2. Assume $\mathcal{S} \in S^3 \mathbb{C}^2$. As $\mu(3,2) = 3$ we deduce the theorem from 1.

3. Assume that $S \in S^4 \mathbb{C}^2$. Suppose that rank $S \in \{2,3\}$. Then 1. yields that srank $S = \operatorname{rank} S$.

Suppose that rank $S \ge 4$. As $\mu(4, 2) = 4$ in view of (7.1) it follows that srank S = rank S = 4.

4. Assume now that $S \in S^3 \mathbb{C}^3$. Suppose first that rank A(S) = 2. Then by changing a basis in \mathbb{C}^3 we can assume that $S \in S^3 \mathbb{C}^2$. Part 2. yields that srank $S = \operatorname{rank} S$.

Suppose that rank A(S) = 3. If rank $S \in \{3, 4\}$ then 1. yields that srank S =rank S. Suppose now that rank $S \ge 5$. (7.1) yields that srank $S \ge 5$. The equality $\mu(3,3) = 5$ yields that rank S = 5. Hence srank S =rank S = 5.

5. (7.1) yields rank $A(S) \leq 5$. If rank A(S) = 2 then 2. yields that srank S =rank S. If rank A(S) = 3 then 4. yields that srank S =rank S. If rank $A(S) \geq 4$ then 1. yields that srank S =rank S.

6. Assume to the contrary that rank S < srank S. So rank $S \leq 5$. 5. implies the contradiction rank S = srank S.

8 Two version of Comon's conjecture

In this section we assume that $\mathbb{F} = \mathbb{R}, \mathbb{C}$.

8.1 Border rank

Definition 8.1 Let $\mathcal{T} \in \otimes^d \mathbb{F}^n \setminus \{0\}$. Then the border of \mathcal{T} , denoted as $\operatorname{brank}_{\mathbb{F}} \mathcal{T}$, is $r \in \mathbb{N}$ if the following conditions hold

1. There exists a sequence $\mathcal{T}_k \in \otimes^d \mathbb{F}^n$, $k \in \mathbb{N}$ such that rank $\mathcal{T}_k = r$ for $k \in \mathbb{N}$ and $\lim_{k \to \infty} \mathcal{T}_k = \mathcal{T}$.

2. Assume that a sequence $\mathcal{T}_k \in \otimes^d \mathbb{F}^n, k \in \mathbb{N}$ converges to \mathcal{T} . Then $\liminf_{k \to \infty} \operatorname{rank} \mathcal{T}_k \geq r$.

Clearly, $\operatorname{brank}_{\mathbb{F}} \mathcal{T} \leq \operatorname{rank} \mathcal{T}$. For d = 2 it is well known that $\operatorname{brank}_{\mathbb{F}} \mathcal{T} = \operatorname{rank} \mathcal{T}$. Hence

$$\operatorname{rank} A(\mathcal{T}) \le \operatorname{brank}_{\mathbb{F}} \mathcal{T}.$$
(8.1)

For d > 2 one has examples where $\operatorname{brank}_{\mathbb{F}} \mathcal{T} < \operatorname{rank} \mathcal{T}$ [7]: Assume that $\mathbf{x}, \mathbf{y} \in \mathbb{F}^n$ are linearly independent. Let

$$S = \mathbf{x} \otimes \mathbf{x} \otimes \mathbf{y} + \mathbf{x} \otimes \mathbf{y} \otimes \mathbf{x} + \mathbf{y} \otimes \mathbf{x} \otimes \mathbf{x}, \quad S = \lim_{\epsilon \to 0} \frac{1}{\epsilon} (\otimes^3 (\mathbf{x} + \epsilon \mathbf{y}) - \otimes^3 \mathbf{x}). \quad (8.2)$$

It is straightforward to show that rank S = 3, $\operatorname{brank}_{\mathbb{F}} S = 2$. (See the proof of Theorem 8.3.)

Assume that $S \in S^d \mathbb{F}^n \setminus \{0\}$. Then the symmetric border rank of S, denoted as $\operatorname{sbrank}_{\mathbb{F}} S$, is $r \in \mathbb{N}$ if the following conditions hold

- 1. There exists a sequence $S_k \in S^d \mathbb{F}^n, k \in \mathbb{N}$ such that srank $S_k = r$ for $k \in \mathbb{N}$ and $\lim_{k\to\infty} S_k = S$.
- 2. Assume that a sequence $S_k \in S^d \mathbb{F}^n, k \in \mathbb{N}$ converges to S. Then $\liminf_{k\to\infty} \operatorname{srank} S_k \geq r$.

Clearly, srank $S \geq \operatorname{sbrank}_{\mathbb{F}} S$ and $\operatorname{sbrank}_{\mathbb{F}} S \geq \operatorname{brank}_{\mathbb{F}} S$. Thus we showed

$$\operatorname{rank} A(\mathcal{S}) \leq \operatorname{brank}_{\mathbb{F}} \mathcal{S} \leq \operatorname{sbrank}_{\mathbb{F}} \mathcal{S} \leq \operatorname{srank} \mathcal{S}.$$

$$(8.3)$$

The analog of Comon's conjecture is the equality $\operatorname{brank}_{\mathbb{F}} S = \operatorname{sbrank}_{\mathbb{F}} S$. See [4]. The analog of Theorem 1.1 will be the following conjecture:

Conjecture 8.2 Let $d \geq 3$, $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and $S \in S^d \mathbb{F}^n$. Suppose that $\operatorname{brank}_{\mathbb{F}} S < \operatorname{rank} S$ and $\operatorname{brank}_{\mathbb{F}} S \leq \operatorname{rank} A(S) + 1$. Then $\operatorname{sbrank}_{\mathbb{F}} S = \operatorname{brank}_{\mathbb{F}} S$.

The following theorem proves the first nontrivial case of this conjecture:

Theorem 8.3 Let $S \in S^d \mathbb{F}^n$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}, d \ge 3, n \ge 2$. Then $\operatorname{brank}_{\mathbb{F}} S = 2 < \operatorname{rank} S$ if and only if there exist two linearly independent $\mathbf{x}, \mathbf{y} \in \mathbb{F}^n$ and $a, b \in \mathbb{F}, b \ne 0$ such that

$$S = a \otimes^{d} \mathbf{x} + b \sum_{j=0}^{d-1} (\otimes^{j} \mathbf{x}) \otimes \mathbf{y} \otimes (\otimes^{d-j-1} \mathbf{x})$$
(8.4)

In particular brank_{\mathbb{F}} $\mathcal{S} = \operatorname{sbrank}_{\mathbb{F}} \mathcal{S}$.

8.2 Proof of Theorem 8.3

Lemma 8.4 Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and assume that $A = [a_{i,j}]_{i \in [M], j \in [N]} \in \mathbb{F}^{M \times N}$, r =rank A. Suppose that the sequence $A_k = \sum_{i=1}^q \mathbf{x}_{i,k} \mathbf{y}_{i,k}^\top$, $k \in \mathbb{N}$ satisfies the following conditions:

$$\lim_{k \to \infty} A_k = A, \quad \lim_{k \to \infty} \mathbf{x}_{i,k} = \mathbf{x}_i \text{ for } i \in [q].$$
(8.5)

1. Assume that q = r. Then there exists a positive integer K, such that for $k \ge K$ the two sets of vectors $\mathbf{x}_{1,k}, \ldots, \mathbf{x}_{r,k}$ and $\mathbf{y}_{1,k}, \ldots, \mathbf{y}_{r,k}$ are linearly independent.

2. Assume that $\mathbf{x}_1, \ldots, \mathbf{x}_q$ are linearly independent. Then

$$\lim_{k \to \infty} \mathbf{y}_{i,k} = \mathbf{y}_i \text{ for } i \in [q] \quad and \ A = \sum_{i=1}^q \mathbf{x}_i \mathbf{y}_i^\top.$$
(8.6)

Furthermore, dim span $(\mathbf{y}_1, \ldots, \mathbf{y}_q) = r$. In particular, if q = r then $\mathbf{y}_1, \ldots, \mathbf{y}_r$ are linearly independent.

Proof. Clearly, rank $A_k \leq q$. The first condition of (8.5) yields that $q \geq r$. 1. Suppose q = r. Hence rank $A_k = r$ for $k \geq K$. Hence the two sets of vectors $\mathbf{x}_{1,k}, \ldots, \mathbf{x}_{r,k}$ and $\mathbf{y}_{1,k}, \ldots, \mathbf{y}_{r,k}$ are linearly independent.

2. Complete $\mathbf{x}_1, \ldots, \mathbf{x}_q$ to a basis $\mathbf{x}_1, \ldots, \mathbf{x}_M$ in \mathbb{F}^M . Let $\mathbf{z}_1, \ldots, \mathbf{z}_N$ be a basis in \mathbb{F}^N . Hence $\mathbf{x}_i \mathbf{z}_j^\top, i \in [M], j \in [N]$ is a basis in $\mathbb{F}^{M \times N}$. Therefore $A_k = \sum_{i \in [M], j \in [N]} a_{ij,k} \mathbf{x}_i \mathbf{z}_j^\top$ for $k \in \mathbb{N}$. The first equality of (8.5) yields that $\lim_{k \to \infty} a_{ij,k} = a_{ij}$ for $i \in [M], j \in [N]$. The second equality of (8.5) yields that $\mathbf{x}_{1,k}, \ldots, \mathbf{x}_{q,k}, \mathbf{x}_{q+1}, \ldots, \mathbf{x}_M$ is a basis in \mathbb{F}^M for $k \geq K$. In what follows we assume that $k \geq K$. Let $Q_k \in \mathbf{GL}(M, \mathbb{F})$ be the transition matrix from the basis $[\mathbf{x}_1, \ldots, \mathbf{x}_q, \mathbf{x}_{q+1}, \ldots, \mathbf{x}_M]$ to the basis

 $[\mathbf{x}_{1,k},\ldots,\mathbf{x}_{q,k},\mathbf{x}_{q+1},\ldots,\mathbf{x}_M]$. Clearly, $\lim_{k\to\infty}Q_k=I_M$. Then

$$A_k = \sum_{i \in [M], j \in [N]} b_{ij,k} \mathbf{x}_{i,k} \mathbf{z}_j.$$

Compare this equality with the assumption that $A_k = \sum_{i=1}^q \mathbf{x}_{i,k} \mathbf{y}_{i,k}^{\top}$ to deduce that $b_{ij,k} = 0$ for i > q and $\mathbf{y}_{i,k} = \sum_{j \in [N]} b_{ij,k} \mathbf{z}_j$ for $i \in [q]$. Let $\tilde{A}_k = [a_{ij,k}], B_k = [b_{ij,k}] \in \mathbb{F}^{M \times N}$. Then $B_k = Q_k \tilde{A}_k$. Hence $\lim_{k \to \infty} B_k = \lim_{k \to \infty} Q_k A_k = \tilde{A} = [a_{ij}]$. This shows (8.6). Since rank A = r it follows that dim span $(\mathbf{y}_1, \ldots, \mathbf{y}_q) = r$. Thus if $q = r \mathbf{y}_1, \ldots, \mathbf{y}_r$ are linearly independent.

Assume the assumptions of Definition 8.1. Without loss of generality we can assume that

$$\mathcal{T}_{k} = \sum_{i=1}^{r} \otimes_{j=1}^{d} \mathbf{x}_{i,j,k}, \quad \|\mathbf{x}_{i,j,k}\| = 1, i \in [r], j \in [d-1], k \in \mathbb{N}.$$
(8.7)

By considering a subsequence of $k \in \mathbb{N}$ without loss of generality we can assume that

$$\lim_{k \to \infty} \mathbf{x}_{i,j,k} = \mathbf{x}_{i,j} \quad i \in [r], j \in [d-1].$$
(8.8)

(Here $\|\mathbf{x}\|$ is the Euclidean norm on \mathbb{F}^n .)

Lemma 8.5 Let $S \in S^{d}\mathbb{F}^{n}, d \geq 3, n \geq 2$. Assume that $1 < r = \operatorname{rank} A(S) = \operatorname{brank}_{\mathbb{F}} S < \operatorname{rank} S$. Let $\mathcal{T}_{k} \in \otimes^{d}\mathbb{F}^{n}, k \in \mathbb{N}$ be a sequence of the form (8.7) satisfying (8.8). Assume furthermore that $\lim_{k\to\infty} \mathcal{T}_{k} = S$. Then the tensors $\otimes_{j=1}^{d-1} \mathbf{x}_{1,j}, \ldots, \otimes_{j=1}^{d-1} \mathbf{x}_{r,j}$ are linearly dependent.

Proof. Assume to the contrary that the tensors $\otimes_{j=1}^{d-1} \mathbf{x}_{1,j}, \ldots, \otimes_{j=1}^{d-1} \mathbf{x}_{r,j}$ are linearly independent. Lemma 8.4 yields that $\lim_{k\to\infty} \mathbf{x}_{i,d,k} = \mathbf{x}_{i,d}$ for $i \in [r]$. Hence $\mathcal{S} = \sum_{i=1}^{r} \otimes_{j=1}^{d} \mathbf{x}_{i,j}$. Thus rank $\mathcal{S} \leq r$ which contradicts our assumptions. \Box

Lemma 8.6 Let $A \in \mathbb{F}^{M \times N}$ be a matrix of rank two. Assume that $A_k = \mathbf{a}_k \mathbf{b}_k^\top - \mathbf{c}_k \mathbf{d}_k^\top, k \in \mathbb{N}$ converges to A. Suppose furthermore that

$$\lim_{k \to \infty} \frac{1}{\|\mathbf{a}_k\|} \mathbf{a}_k = \mathbf{a}, \quad \lim_{k \to \infty} \frac{1}{\|\mathbf{c}_k\|} \mathbf{c}_k = \mathbf{c}, \quad \mathbf{c} = \alpha \mathbf{a} \text{ for } |\alpha| = 1, \quad (8.9)$$
$$\lim_{k \to \infty} \frac{1}{\|\mathbf{b}_k\|} \mathbf{b}_k = \mathbf{b}, \quad \lim_{k \to \infty} \frac{1}{\|\mathbf{d}_k\|} \mathbf{d}_k = \mathbf{d}.$$

Then

$$\mathbf{b} = \alpha \mathbf{d}, \ \lim_{k \to \infty} \|\mathbf{a}_k\| \|\mathbf{b}_k\| = \lim_{k \to \infty} \|\mathbf{c}_k\| \|\mathbf{d}_k\| = \infty, \ \lim_{k \to \infty} \frac{\|\mathbf{c}_k\| \|\mathbf{d}_k\|}{\|\mathbf{a}_k\| \|\mathbf{b}_k\|} = 1.$$
(8.10)

Furthermore $A = \mathbf{a}\mathbf{f}^{\top} + \mathbf{g}\mathbf{b}^{\top}$, where $\operatorname{span}(\mathbf{a}, \mathbf{g}) = \operatorname{Range} A$ and $\operatorname{span}(\mathbf{b}, \mathbf{f}) = \operatorname{Range} A^{\top}$. In particular, \mathbf{g} and \mathbf{f} are limits of linear combinations of $\mathbf{a}_k, \mathbf{c}_k$ and $\mathbf{b}_k, \mathbf{d}_k$ respectively.

Proof. Observe that

$$A_{k}^{\top} = (\frac{1}{\|\mathbf{b}_{k}\|}\mathbf{b}_{k})(\|\mathbf{b}_{k}\|\|\mathbf{a}_{k}\|)(\frac{1}{\|\mathbf{a}_{k}\|}\mathbf{a}_{k}^{\top}) - (\frac{1}{\|\mathbf{d}_{k}\|}\mathbf{d}_{k})(\|\mathbf{d}_{k}\|\|\mathbf{c}_{k}\|)(\frac{1}{\|\mathbf{c}_{k}\|}\mathbf{c}_{k}^{\top}).$$
(8.11)

Suppose first that span(**b**) \neq span(**d**). Then **b** and **d** are linearly independent. Lemma 8.4 yields that $A^{\top} = a\mathbf{b}\mathbf{a}^{\top} + c\mathbf{d}\mathbf{a}^{\top}$. Hence rank A = 1 which contradicts our assumptions. As $\|\mathbf{c}\| = \|\mathbf{d}\| = 1$ it follows that $\mathbf{b} = \beta \mathbf{d}$ for some scalar β of length 1.

We next observe that $\mathbf{a} \in \text{Range}(A)$ and $\mathbf{b} \in \text{Range} A^{\top}$. Indeed, without loss of generality, we can assume that rank $A_k = 2$ for $k \in \mathbb{N}$. Hence $\mathbf{a}_k \in \text{Range}(A_k), \mathbf{b}_k \in \text{Range}(A_k^{\top})$. As $\lim_{k\to\infty} A_k = A$ the assumptions (8.10) yield that $\mathbf{a} \in \text{Range} A, \mathbf{b} \in \text{Range} A^{\top}$.

Assume that the sequence $\{\|\mathbf{a}_k\|\|\|\mathbf{b}_k\|\}, k \in \mathbb{N}$ contains a bounded subsequence $\{n_k\}, k \in \mathbb{N}$. Since $\lim_{k\to\infty} A_k = A$ it follows the subsequence $\|\mathbf{c}_{n_k}\|\|\mathbf{d}_{n_k}\|, k \in \mathbb{N}$ is also bounded. Taking convergent subsequences of the above two subsequences we deduce that $A = \gamma \mathbf{a} \mathbf{b}^{\top}$. This contradicts our assumption that rank A = 2. Hence the second equality of (8.10) holds. Rewrite (8.11) as

$$A_{k} = \|\mathbf{a}_{k}\|\|\mathbf{b}_{k}\|((\frac{1}{\|\mathbf{a}_{k}\|}\mathbf{a}_{k})(\frac{1}{\|\mathbf{b}_{k}\|}\mathbf{b}_{k}^{\top}) - (\frac{\|\mathbf{c}_{k}\|\|\mathbf{d}_{k}\|}{\|\mathbf{a}_{k}\|\|\mathbf{b}_{k}\|})(\frac{1}{\|\mathbf{c}_{k}\|}\mathbf{c}_{k})(\frac{1}{\|\mathbf{d}_{k}\|}\mathbf{d}_{k}^{\top})).$$

Use the assumptions that $\lim_{k\to\infty} A_k = A$, where rank A = 2, the facts that $\|\mathbf{a}\| = \|\mathbf{b}\| = \|\mathbf{c}\| = \|\mathbf{d}\| = 1$ and $\mathbf{c} = \alpha \mathbf{a}, \mathbf{b} = \beta \mathbf{d}$ to deduce the third and the first part of (8.10).

It is left to show that $A = \mathbf{af}^{\top} + \mathbf{gb}^{\top}$. Choose orthonormal bases $\mathbf{x}_1, \ldots, \mathbf{x}_N$ and $\mathbf{y}_1, \ldots, \mathbf{y}_N$ in \mathbb{F}^M and \mathbb{F}^N respectively with the following properties:

$$\mathbf{x}_1 = \mathbf{a}$$
, span $(\mathbf{x}_1, \mathbf{x}_2) =$ Range A , $\mathbf{y}_1 = \mathbf{b}$, span $(\mathbf{y}_1, \mathbf{y}_2) =$ Range A^{\top} .

In what follows we assume that $k \gg 1$. Choose orthonormal bases $\mathbf{x}_{1,k}, \mathbf{x}_{2,k}$ and $\mathbf{y}_{1,k}, \mathbf{y}_{2,k}$ in Range A_k and Range A_k^{\top} respectively such that

$$\mathbf{x}_{1,k} = rac{1}{\|\mathbf{a}_k\|} \mathbf{a}_k, \ \mathbf{y}_{1,k} = rac{1}{\|\mathbf{b}_k\|} \mathbf{b}_k, \ \lim_{k \to \infty} \mathbf{x}_{2,k} = \mathbf{x}_2, \ \lim_{k \to \infty} \mathbf{y}_{2,k} = \mathbf{y}_2.$$

Observe next that $\{\mathbf{x}_{1,k}, \mathbf{x}_{2,k}, \mathbf{x}_3, \dots, \mathbf{x}_M\}$ and $\{\mathbf{y}_{1,k}, \mathbf{y}_{2,k}, \mathbf{y}_3, \dots, \mathbf{y}_N\}$ are bases in \mathbb{F}^M and \mathbb{F}^N which converge to bases $\mathbf{x}_1, \dots, \mathbf{x}_N$ and $\mathbf{y}_1, \dots, \mathbf{y}_N$ respectively.

In the bases $\{\mathbf{x}_{1,k}, \mathbf{x}_{2,k}, \mathbf{x}_3, \dots, \mathbf{x}_M\}$ and $\{\mathbf{y}_{1,k}, \mathbf{y}_{2,k}, \mathbf{y}_3, \dots, \mathbf{y}_N\}$ the rank one matrices $\mathbf{a}_k \mathbf{b}_k^\top, \mathbf{c}_k \mathbf{d}_k^\top$ are represented by the following block diagonal matrices: $\tilde{C}_k = C_k \oplus 0, \tilde{D}_k = D_k \oplus 0$ where

$$C_k = \begin{bmatrix} a_k & 0\\ 0 & 0 \end{bmatrix}, \quad D_k = \begin{bmatrix} b_k & c_k\\ d_k & e_k \end{bmatrix}$$

Note that $a_k = \|\mathbf{a}_k\| \|\mathbf{b}_k\|$. Hence $\lim_{k\to\infty} a_k = \infty$. As $\lim_{k\to\infty} A_k = A$ the arguments of the proof of Lemma 8.4 yield that $\lim_{k\to\infty} C_k - D_k = E = [e_{ij}] \in \mathbb{F}^{2\times 2}$. Hence $\lim_{k\to\infty} b_k = \infty$. Therefore $D_k = b_k \mathbf{u}_k \mathbf{v}_k^{\top}$, where $\mathbf{u}_k^{\top} = (1, u_k), \mathbf{v}_k^{\top} = (1, v_k)$. Furthermore

$$\lim_{k \to \infty} b_k u_k = e_{21} \Rightarrow \lim_{k \to \infty} u_k = 0, \quad \lim_{k \to \infty} b_k v_k = e_{12} \Rightarrow \lim_{k \to \infty} v_k = 0$$

Finally, observe that $e_{22} = \lim_{k \to \infty} b_k u_k v_k = 0$. This yields that $E = (1,0)^\top \mathbf{u}^\top + \mathbf{v}(1,0)$ for some choice of $\mathbf{u}, \mathbf{v} \in \mathbb{F}^2$. Hence $A = \mathbf{a}\mathbf{f}^\top + \mathbf{g}\mathbf{b}^\top$ as claimed.

As rank $A_k = 2$ and $\lim_{k\to\infty} A_k = A$ it follows that **g** and **f** are limits of linear combinations of $\mathbf{a}_k, \mathbf{c}_k$ and $\mathbf{b}_k, \mathbf{d}_k$ respectively.

Proof of Theorem 8.3. Assume first that S is of the form (8.4) where $b \neq 0$. Without loss of generality we can assume that b = 1. Clearly, Range A(S) =span (\mathbf{x}, \mathbf{y}) Hence rank A(S) = 2. Let $\mathcal{T}(\epsilon) := a \otimes^d \mathbf{x} + \frac{1}{\epsilon} (\otimes^d (\mathbf{x} + \epsilon \mathbf{y}) - \otimes^d \mathbf{x})$ for $\epsilon \neq 0$. Then rank $\mathcal{T}(\epsilon) = 2$ for $\epsilon^{-1} \neq a$. Clearly, $\lim_{\epsilon \to 0} \mathcal{T}(\epsilon) = S$. Hence $\operatorname{brank}_{\mathbb{F}} S = 2$. As $\mathcal{T}(\epsilon) \in S^d \mathbb{F}^n$ it follows that $\operatorname{sbrank}_{\mathbb{F}} S = 2$. We claim that rank S > 2. We can assume without loss of generality that n = 2 and $\mathbf{x} = \mathbf{e}_1 = (1, 0)^{\top}, \mathbf{y} = \mathbf{e}_2 = (0, 1)^{\top}$. Assume first that d = 3. So $S = [s_{i,j,k}]$ where

$$s_{1,1,1} = a, \ s_{1,1,2} = s_{1,2,1} = s_{2,1,1} = 1, \ s_{1,2,2} = s_{2,1,2} = s_{2,2,1} = s_{2,2,2} = 0.$$

Let

$$F = [s_{i,j,1}]_{i,j\in[2]} = \begin{bmatrix} a & 1\\ 1 & 0 \end{bmatrix}, \quad G = [s_{i,j,2}]_{i,j\in[2]} = \begin{bmatrix} 1 & 0\\ 0 & 0 \end{bmatrix}.$$

Then rank S = 2 if and only if the matrix GF^{-1} is diagonalizable, see e.g. [13]. Clearly, $GF^{-1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not diagonalizable. Hence rank S > 2. It is easy to show straightforward that rank S = 3.

Assume now that d > 3. Let $\phi : \mathbb{F}^2 \to \mathbb{F}$ be the linear functional such that $\phi(\mathbf{e}_1) = \phi(\mathbf{e}_2) = 1$. Consider the following map $\psi : (\mathbb{F}^2)^d \to \otimes^3 \mathbb{F}^2$: $\psi((\mathbf{u}_1, \ldots, \mathbf{u}_d)) = (\prod_{j=4}^d \phi(\mathbf{u}_j)) \otimes_{i=1}^3 \mathbf{u}_j$. Clearly, ψ is a multilinear map. The universal lifting property of the tensor product yields that ψ lifts to the linear map $\Psi : \otimes^d \mathbb{F}^2 \to \otimes^3 \mathbb{F}^2$ such that

$$\Psi(\otimes_{j=1}^{d}\mathbf{u}_{i}) = (\prod_{j=4}^{d}\phi(\mathbf{u}_{j}))\otimes_{i=1}^{3}\mathbf{u}_{j}.$$

Observe that a rank one tensor is mapped to either rank one tensor or zero tensor. Clearly, the image of a symmetric rank one tensor is a symmetric tensor of at most rank one. Hence $\Psi : S^d \mathbb{F}^2 \to S^3 \mathbb{F}^2$. Assume that S of the form (8.4), where $\mathbf{x} = \mathbf{e}_1$ and $\mathbf{y} = \mathbf{e}_2$. Then

$$\Psi(\mathcal{S}) = (a + (d - 3)b)\mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x} + b(\mathbf{y} \otimes \mathbf{x} \otimes \mathbf{x} + \mathbf{x} \otimes \mathbf{y} \otimes \mathbf{x} + \mathbf{x} \otimes \mathbf{x} \otimes \mathbf{y}).$$

Assume to the contrary that rank S = 2. Then rank $\Psi(S) \leq 2$. This contradicts our proof that rank $\Psi(S) = 3$. Hence rank $S \geq 3$.

Assume now that $S \in S^{d}\mathbb{F}^{n}$ and $2 = \operatorname{rank} A(S) = \operatorname{brank}_{\mathbb{F}} S < \operatorname{rank} S$. Let $\mathcal{T}_{k} \in \otimes^{d}\mathbb{F}^{n}$ be a sequence of tensors of rank two converging to S. So $\mathcal{T}_{k} = \otimes_{j=1}^{d} \mathbf{x}_{j,k} - \otimes_{j=1}^{d} \mathbf{y}_{j,k}$. Since rank A(S) = 2 we can assume without loss of generality: First, $\mathbf{x}_{j,k}$ and $\mathbf{y}_{j,k}$ are linearly independent for $j \in [d], k \in \mathbb{N}$. Second,

$$\lim_{k \to \infty} \frac{1}{\|\mathbf{x}_{j,k}\|} \mathbf{x}_{j,k} = \mathbf{x}_j, \ \lim_{k \to \infty} \frac{1}{\|\mathbf{y}_{j,k}\|} \mathbf{y}_{j,k} = \mathbf{y}_j \text{ for } j \in [d].$$

Lemma 8.5 yields that $\otimes_{j=1}^{d-1} \mathbf{x}_j$ and $\otimes_{j=1}^{d-1} \mathbf{y}_j$ are linearly dependent. Hence $\operatorname{span}(\mathbf{x}_j) = \operatorname{span}(\mathbf{y}_j)$ for $j \in [d-1]$. Lemma 8.6 yields that \mathbf{x}_d and \mathbf{y}_d linearly dependent. So $\operatorname{span}(\mathbf{x}_d) = \operatorname{span}(\mathbf{y}_d)$. Apply Lemma 8.6 to $A = A(\mathcal{S})^{\top}, A_k = A(\mathcal{T}_k)^{\top}, k \in \mathbb{N}$. It then follows that $\mathcal{S} = (\otimes_{j=1}^{d-1} \mathbf{x}_j) \otimes \mathbf{z} + \mathcal{F} \otimes \mathbf{x}_d$ for some $\mathcal{F} \in \otimes^{d-1} \mathbb{F}^n$. Furthermore $\operatorname{span}(\mathbf{x}_d, \mathbf{z}) = \operatorname{Range} A(\mathcal{S})$. As \mathbf{x}_d and \mathbf{z} are linearly independent and \mathcal{S} symmetric it follow that $\otimes_{j=1}^{d-1}, \mathcal{F} \in S^{d-1} \mathbb{F}^n$. Hence $\operatorname{span}(\mathbf{x}_1) = \cdots = \operatorname{span}(\mathbf{x}_{d-1}) = \operatorname{span}(\mathbf{x})$. Thus $\otimes_{j=1}^{d-1} \mathbf{x}_j = t \otimes^{d-1} \mathbf{x}$. By considering the unfolding of \mathcal{S} in another mode we deduce that $\operatorname{span}(\mathbf{x}_d) = \operatorname{span}(\mathbf{x})$. Observe next that $\operatorname{rank} \mathcal{F} > 1$. Otherwise $\operatorname{rank} \mathcal{S} \leq 2$ which contradicts our assumptions. Lemma 8.6 yields that \mathcal{F} is a limit of linear combinations of $\otimes_{j=1}^{d-1} \mathbf{x}_{j,k}$ and $\otimes_{j=1}^{d-1} \mathbf{y}_{j,k}$. Hence $\operatorname{brank}_{\mathbb{F}} \mathcal{F} \leq 2$. As $\operatorname{rank} \mathcal{F} > 1$ it follows that $\operatorname{brank}_{\mathbb{F}} \mathcal{F} = 2$. In summary we showed:

$$S = \otimes^{d-1} \mathbf{x} \otimes \mathbf{z} + \mathcal{F} \otimes \mathbf{x},$$
span(\mathbf{x}, \mathbf{z}) = Range $A(S), \ \mathcal{F} \in S^{d-1} \mathbb{F}^n, \ \text{brank}_{\mathbb{F}} \ \mathcal{F} = 2.$
(8.12)

We now prove the following claim: Assume that $S \in S^d \mathbb{F}^n$, rank A(S) = 2 <rank S. Suppose furthermore S is a limit of linear linear combinations of $\otimes_{j=1}^d \mathbf{x}_{j,k}, \otimes_{j=1}^d \mathbf{y}_{j,k}$, where the following limit exist and satisfy:

$$\lim_{k \to \infty} \frac{1}{\|\mathbf{x}_{j,k}\|} \mathbf{x}_{j,k}, \lim_{k \to \infty} \frac{1}{\|\mathbf{y}_{j,k}\|} \mathbf{y}_{j,k} \in \operatorname{span}(\mathbf{x}) \text{ for } j \in [d].$$
(8.13)

Then (8.4) holds.

We prove the claim by induction on d. Assume first that d = 3. Observe first that two dimensional subspace Range $A(S)^{\top} = \operatorname{span}(\mathbf{a}, \mathbf{c})$, as given by Lemma 8.6, is in $S^2 \mathbb{F}^n$. i.e. the space of symmetric matrices. Lemma 8.6 yields that \mathcal{F} is a limit of linear combinations of $\mathbf{x}_{1,k} \otimes \mathbf{x}_{2,k}$ and $\mathbf{y}_{1,k} \otimes \mathbf{y}_{2,k}$. As $\lim_{k\to\infty} \frac{1}{\|\mathbf{x}_{1,k}\|\|\mathbf{x}_{2,k}\|} \mathbf{x}_{1,k} \otimes \mathbf{x}_{2,k} =$ $t\mathbf{x} \otimes \mathbf{x}$ it follows that Range $A(S)^{\top}$ contains rank one matrix $\mathbf{x} \otimes \mathbf{x}$. Lemma 8.6 yields that Range $A(S)^{\top}$ contains a rank two matrix of the form $\mathbf{x} \otimes \mathbf{f} + \mathbf{g} \otimes \mathbf{x}$. Since this matrix is symmetric it is of the form $c\mathbf{x} \otimes \mathbf{x} + \mathbf{x} \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{x}$ for some scalar cand $\mathbf{u} \in \mathbb{F}^n$ which is linearly independent of \mathbf{x} . Hence $\mathcal{F} = d\mathbf{x} \otimes \mathbf{x} + \mathbf{x} \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{x}$ for $\mathbf{v} = d\mathbf{u}, d \neq 0$. As rank A(S) = 2 it follows that span $(\mathbf{x}, \mathbf{v}) = \operatorname{span}(\mathbf{x}, \mathbf{z})$. Therefore we showed that

$$\mathcal{S} = a\mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x} + b\mathbf{x} \otimes \mathbf{x} \otimes \mathbf{v} + c(\mathbf{x} \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{x}) \otimes \mathbf{x}.$$

Interchange the last two factors in S to deduce that $0 = (b-c)(\mathbf{x} \otimes \mathbf{x} \otimes \mathbf{v} - \mathbf{x} \otimes \mathbf{v} \otimes \mathbf{x})$. Hence b = c and (8.4) holds for d = 3.

Assume now that (8.4) holds for d = p and suppose that d = p + 1. Consider (8.12). Suppose first that $2 < \operatorname{rank} \mathcal{F}$. Then the induction hypothesis applies to \mathcal{F} . Hence \mathcal{F} is of the form (8.4) and

$$\mathcal{S} = \otimes^p \mathbf{x} \otimes \mathbf{z} + (a \otimes^p \mathbf{x} + b \sum_{j=0}^{p-1} \otimes^j \mathbf{x} \otimes \mathbf{y} \otimes (\otimes^{p-1-j} \mathbf{x})) \otimes \mathbf{x}, \ b \neq 0.$$

Note that Range $A(S) = \operatorname{span}(\mathbf{x}, \mathbf{y}) = \operatorname{span}(\mathbf{x}, \mathbf{z})$. Hence

$$\mathcal{S} = a' \otimes^{p+1} \mathbf{x} + c \otimes^{p} \mathbf{x} \otimes \mathbf{y} + b \sum_{j=0}^{p-1} \otimes^{j} \mathbf{x} \otimes \mathbf{y} \otimes (\otimes^{p-j} \mathbf{x}).$$

Interchange the last two factors in S to deduce that $(c-b) \otimes^{p-1} \mathbf{x} (\mathbf{x} \otimes \mathbf{y} - \mathbf{y} \otimes \mathbf{x}) = 0$. Hence b = c and S is of the form (8.4).

It is left to consider the case where \mathcal{F} is a symmetric tensor of rank two. So rank $\mathcal{F} = 2$. Hence rank $A(\mathcal{F}) = 2$. Theorem 3.1 yields that $\mathcal{F} = s \otimes^p \mathbf{u} + t \otimes^d \mathbf{v}$, were $s, t = \pm 1$, and this decomposition is unique. (The \pm are needed if $\mathbb{F} = \mathbb{R}$ and p is even.) Clearly, $\operatorname{span}(\mathbf{u}, \mathbf{v}) = \operatorname{Range} A(\mathcal{S})$. It is enough to assume that n = 2. Recall that $A(\mathcal{F})$ is a limit of a linear combinations of two rank one matrices: $\mathbf{x}_{1,k}(\otimes_{j=2}^p \mathbf{x}_{j,k})^{\top}, \mathbf{y}_{1,k}(\otimes_{j=2}^p \mathbf{y}_{j,k})^{\top}, k \in \mathbb{N}$. The assumption (8.13) implies that we can use Lemma 8.6. Hence $\mathcal{F} = \mathbf{x} \otimes \mathcal{G} + \mathbf{g} \otimes (\otimes^{p-1} \mathbf{x})$. Therefore $\otimes^{p-1} \mathbf{x} \in$ $\operatorname{span}(\mathbb{Q}^{p-1}\mathbf{u}, \otimes^{p-1}\mathbf{v})$. We claim that this possible if and only if either $\operatorname{span}(\mathbf{x}) =$ $\operatorname{span}(\mathbf{u})$ or $\operatorname{span}(\mathbf{x}) = \operatorname{span}(\mathbf{v})$. Suppose to the contrary that $\operatorname{span}(\mathbf{x}) \neq \operatorname{span}(\mathbf{u})$ or $\operatorname{span}(\mathbf{x}) \neq \operatorname{span}(\mathbf{v})$. So $\otimes^{p-1} \mathbf{x} = s \otimes^{p-1} \mathbf{u} + t \otimes^{p-1} \mathbf{v}$. Clearly $st \neq 0$. Let $\phi : \mathbb{F}^2 \to \mathbb{F}$ be a nonzero linear functional such that $\phi(\mathbf{x}) = 0$. Let $\Psi : \otimes^{p-1} \mathbb{F}^2 \to \otimes^{p-2} \mathbb{F}^2$ be the linear mapping of the form given above: $\Psi(\otimes_{j=1}^{p-1} \mathbf{w}_j) = \phi(\mathbf{w}_1) \otimes_{j=2}^{p-1} \mathbf{w}_j$. So

$$0 = \Psi(\otimes^{p-1} \mathbf{x}) = s\phi(\mathbf{u}) \otimes^{p-2} \mathbf{u} + t\phi(\mathbf{v}) \otimes^{p-2} \mathbf{v}.$$

This is impossible since \mathbf{u} and \mathbf{v} are linearly independent. Thus we can assume that $\operatorname{span}(\mathbf{v}) = \operatorname{span}(\mathbf{x}), \ \mathcal{T} = s \otimes^p \mathbf{u} + t' \otimes^p \mathbf{x} \text{ and } \mathbf{z} \in \operatorname{span}(\mathbf{u}, \mathbf{x}).$ Thus

$$\mathcal{S} = a \otimes^{p+1} \mathbf{x} + b \otimes^{p} \mathbf{x} \otimes \mathbf{u} + s \otimes^{p} \mathbf{u} \otimes \mathbf{x}.$$

As rank S = 3 it follows that $S' := S - a \otimes^{p+1} \mathbf{x}$ is a symmetric tensor of rank two. Theorem 3.1 claims that the decomposition $S' = b \otimes^p \mathbf{x} \otimes \mathbf{u} + s \otimes^p \mathbf{u} \otimes \mathbf{x}$ is unique and $\otimes^p \mathbf{x} \otimes \mathbf{u}, \otimes^p \mathbf{u} \otimes \mathbf{x}$ are symmetric tensors. So $\operatorname{span}(\mathbf{u}) = \operatorname{span}(\mathbf{x})$ which contradicts our assumption that \mathbf{u} and \mathbf{x} are linearly independent. \Box

8.3 Approximation of symmetric tensors

Define on $\otimes^{d} \mathbb{F}^{n}$ the standard inner product:

$$\langle \mathcal{P}, \mathcal{Q} \rangle := \sum_{i_j \in [n], j \in [d]} p_{i_1, \dots, i_d} \overline{q_{i_1, \dots, i_d}}, \quad \mathcal{P} = [p_{i_1, \dots, i_d}], \mathcal{Q} = [q_{i_1, \dots, i_p}] \in \otimes^d \mathbb{F}^n.$$

Observe that $\langle \otimes_{j=1}^{d} \mathbf{x}_{j}, \otimes_{j=1}^{d} \mathbf{y}_{j} \rangle = \prod_{j=1}^{d} \langle \mathbf{x}_{j}, \mathbf{y}_{j} \rangle$. Assume that $k \in [1, d-1]$. Denote by $\operatorname{Gr}(k, \mathbb{F}^{n})$ the Grasmannian manifold of k-dimensional subspace in \mathbb{F}^{n} . Let

$$\operatorname{Gr}(k,d,\mathbb{F}^n) := \{ \otimes_{j=1}^d \mathbf{U}_j, \quad \mathbf{U}_j \in \operatorname{Gr}(k,\mathbb{F}^n), j \in [d] \}.$$

For a given $\otimes_{j=1}^{d} \mathbf{U}_{j}$ denote by $P_{\otimes_{j=1}^{d} \mathbf{U}_{j}} : \otimes^{d} \mathbb{F}^{n} \to \otimes_{j=1}^{d} \mathbf{U}_{j}$ the orthogonal projection of $\otimes^{d} \mathbb{F}^{n}$ on $\otimes_{j=1}^{d} \mathbf{U}_{j}$. A best k-approximation of $\mathcal{T} \in \otimes^{d} \mathbb{F}^{n}$ is each tensor \mathcal{T}^{\star} satisfying

$$\min_{\otimes_{j=1}^{d} \mathbf{U}_{j} \in \operatorname{Gr}(k,d,\mathbb{F}^{n})} \|\mathcal{T} - P_{\otimes_{j=1}^{d} \mathbf{U}_{j}}(\mathcal{T})\| = \|\mathcal{T} - \mathcal{T}^{\star}\|, \ \mathcal{T}^{\star} = P_{\otimes_{j=1}^{d} \mathbf{U}_{j}^{\star}}(\mathcal{T}), \otimes_{j=1}^{d} \mathbf{U}_{j}^{\star} \in \operatorname{Gr}(k,d,\mathbb{F}^{n})$$

See [14, 16]. The results of [15] yield that \mathcal{T}^{\star} is unique for \mathcal{T} outside of a semialgebraic set of dimension less than the real dimension of $\otimes^{d} \mathbb{F}^{n}$. The analog of Comon's conjecture is:

Conjecture 8.7 Let $n - 1, d - 1 \in \mathbb{N}, k \in [d - 1]$ and $S \in S^d \mathbb{F}^n$. Then a best k-approximation of S can be chosen to be a symmetric tensor.

This conjecture is known to hold in the following cases: For d = 2 it is a consequence of Singular Value Decomposition. For k = 1 and d > 2 it follows from Banach's theorem [2]. See [13] for $\mathbb{F} = \mathbb{R}$. Similar arguments combined with Banach's theorem yield the case $\mathbb{F} = \mathbb{C}$. It is shown in [14] that for $\mathbb{F} = \mathbb{R}$ there is a semi-algebraic set in $S^d \mathbb{R}^n$ of dimension dim $S^d \mathbb{R}^n$ for which the conjecture holds.

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