# Remarks on the symmetric rank of symmetric tensors 

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#### Abstract

We give sufficient conditions on a symmetric tensor $\mathcal{S} \in \mathrm{S}^{d} \mathbb{F}^{n}$ to satisfy the equality: the symmetric rank of $\mathcal{S}$, denoted as $\operatorname{srank} \mathcal{S}$, is equal to the rank of $\mathcal{S}$, denoted as $\operatorname{rank} \mathcal{S}$. This is done by considering the rank of the unfolded $\mathcal{S}$ viewed as a matrix $A(\mathcal{S})$. The condition is: $\operatorname{rank} \mathcal{S} \in$ $\{\operatorname{rank} A(\mathcal{S}), \operatorname{rank} A(\mathcal{S})+1\}$. In particular, srank $\mathcal{S}=\operatorname{rank} \mathcal{S}$ for $\mathcal{S} \in \mathrm{S}^{d} \mathbb{C}^{n}$ for the cases $(d, n) \in\{(3,2),(4,2),(3,3)\}$. We discuss the analogs of the above results for border rank and best approximations of symmetric tensors.


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## 1 Introduction

For a field $\mathbb{F}$ let $\otimes^{d} \mathbb{F}^{n} \supset \mathrm{~S}^{d} \mathbb{F}^{n}$ denote $d$-mode tensors and the subspace of symmetric tensors on $\mathbb{F}^{n}$. Let $\mathcal{T} \in \otimes^{d} \mathbb{F}^{n}$. Denote by rank $\mathcal{T}$ the rank of the tensor $\mathcal{T}$. That is, for $\mathcal{T} \neq 0 \operatorname{rank} \mathcal{T}$ is the minimal number $k$ such that $\mathcal{T}$ is a sum of $k$ rank one tensors. (rank $0=0$.) We say that $\mathcal{T}$ has a unique decomposition as a sum rank $\mathcal{T}$ rank one tensors if this decomposition is unique up to a permutations of the summands. Assume that $\mathcal{S} \in \mathrm{S}^{d} \mathbb{F}^{n} \backslash\{0\}$. Suppose that $|\mathbb{F}| \geq d$, i.e. $\mathbb{F}$ has at least $d$ elements. Then it is known that $\mathcal{S}$ is a sum of $k$ symmetric rank one tensors [14, Proposition 7.2]. See [1] for the case $|\mathbb{F}|=\infty$, i.e. $\mathbb{F}$ has an infinite number of elements. The minimal $k$ is the symmetric rank of $\mathcal{S}$, denoted as srank $\mathcal{S}$. Clearly, $\operatorname{rank} \mathcal{S} \leq \operatorname{srank} \mathcal{S}$. In what follows we assume that $d \geq 3$ unless stated otherwise. In [20, P15, page 5] P . Comon asked if $\operatorname{rank} \mathcal{S}=\operatorname{srank} \mathcal{S}$ over $\mathbb{F}=\mathbb{R}, \mathbb{C}$. This problem is also raised in [7, end $\S 4.1$, p’ 1263]. This problem is sometimes referred as Comon's conjecture. In [7] it is shown that this conjecture holds in the first nontrivial case: $\operatorname{rank} \mathcal{S}=2$.

For a finite field the situation is more complicated: Observe first that for $\mathbb{F}=\mathbb{Z}_{2}$ and the symmetric matrix $A=\left[\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right]$ we have the the inequality $\operatorname{rank} A=2<$ srank $A=3$. ( $A$ is a sum of all three distinct symmetric rank one matrices in

[^0]$S^{2} \mathbb{Z}_{2}^{2}$.) Second, it is shown in [14, Proposition 7.1] that over a finite field there exist symmetric tensors that are not a sum symmetric rank one tensors.

To state our result we need the following notions: For $n \in \mathbb{N}$ denote $[n]=$ $\{1, \ldots, n\}$. Let $\mathcal{S}=\left[s_{i_{1}, \ldots, i_{d}}\right]_{i_{1}, \ldots, i_{d} \in[n]} \in \mathrm{S}^{d} \mathbb{F}^{n}$. Denote by $A(\mathcal{S})$ an $n \times n^{d-1}$ matrix with entries $b_{\alpha \boldsymbol{\beta}}$ where $\alpha \in[n]$ and $\boldsymbol{\beta}=\left(\beta_{1}, \ldots, \beta_{d-1}\right) \in[n]^{d-1}$. Then $b_{\alpha \boldsymbol{\beta}}:=$ $s_{\alpha, \beta_{1}, \ldots, \beta_{d-1}} .\left(A(\mathcal{S}) \in \mathbb{F}^{n \times n^{d-1}}\right.$ is the unfolding of $\mathcal{S}$ in the direction 1 . As $\mathcal{S}$ is symmetric, the unfolding in every direction $k \in[d]$ gives rise to the same matrix.) Hence $\operatorname{rank} A(\mathcal{S}) \leq n$. If $m:=\operatorname{rank} A(\mathcal{S})<n$ it means that we can choose another basis so that $\mathcal{S}$ is represented as $\mathcal{S}^{\prime} \in \mathrm{S}^{d} \mathbb{F}^{m}$. Recall that $\operatorname{rank} \mathcal{S} \geq \operatorname{rank} A(\mathcal{S})$. (See for example the arguments in [12] for $d=3$.). Thus, to study Comon's conjecture we can assume without loss of generality that $\operatorname{rank} A(\mathcal{S})=n$.

Denote by $\Sigma(n, d, \mathbb{F})$ and $\Sigma_{s}(d, n, \mathbb{F})$ the Segre variety of rank one tensors plus the zero tensor and the subvariety of symmetric tensors of at most rank one in $\left(\mathbb{F}^{n}\right)^{\otimes d}$.

Let $F_{d, n, k}: \Sigma(n, d, \mathbb{F})^{k} \rightarrow\left(\mathbb{F}^{n}\right)^{\otimes d}$ be the polynomial map:

$$
\begin{equation*}
F_{d, n, k}\left(\left(\mathcal{T}_{1}, \ldots, \mathcal{T}_{k}\right)\right):=\sum_{j=1}^{k} \mathcal{T}_{j} . \tag{1.1}
\end{equation*}
$$

Let $\mathcal{T}=F_{d, n, k}\left(\left(\mathcal{T}_{1}, \ldots, \mathcal{T}_{k}\right)\right)$. In what follows we say that the decomposition $\mathcal{T}=$ $\sum_{j=1}^{k} \mathcal{T}_{j}$ is unique if rank $\mathcal{T}=k$ and any decomposition of $\mathcal{T}$ to a sum of $r$ rank one tensors is obtained by permuting the order of the summands in $\mathcal{T}=\sum_{j=1}^{k} \mathcal{T}_{j}$.

Denote by $G_{d, n, k}$ the restriction of the map $F_{d, n, k}$ to : $\Sigma_{s}(n, d, \mathbb{F})^{k}$. Thus $F_{d, n, k}\left(\Sigma(n, d, \mathbb{F})^{k}\right)$ and $G_{d, n, k}\left(\Sigma_{s}(n, d, \mathbb{F})^{k}\right)$ are the sets of of $d$-mode tensors on $\mathbb{F}^{n}$ tensors of at most rank $k$ and of symmetric tensors of at most symmetric rank $k$.

Chevalley's theorem yields that $F_{d, n, k}\left(\Sigma(n, d, \mathbb{C})^{k}\right)$ and $G_{d, n, k}\left(\Sigma_{s}(n, d, \mathbb{C})^{k}\right)$ are constructible sets. Hence the dimension of $G_{d, n, k}\left(\Sigma_{s}(n, d, \mathbb{C})^{k}\right)$ is the maximal rank of the Jacobian of the map $G_{d, n, k}$.
$\mathcal{S} \in \mathrm{S}^{d} \mathbb{C}^{n}$ is said to have a generic symmetric rank $k$ if the following conditions hold: First, the dimension of the constructible set $G_{d, n, k}\left(\Sigma_{s}(n, d, \mathbb{C})^{k}\right)$ is greater than the dimension of $G_{d, n, k-1}\left(\Sigma_{s}(n, d, \mathbb{C})^{k-1}\right)$. Second, there exists a strict subvariety $O \subset \Sigma(n, d, \mathbb{C})^{k}$, such that $\mathcal{S} \in G_{d, n, k}\left(\Sigma_{s}(n, d, \mathbb{C})^{k} \backslash O\right)$. Let

$$
\begin{equation*}
k_{n, d}:=\frac{\binom{n+d-1}{d}}{n} . \tag{1.2}
\end{equation*}
$$

Chiantini, Ottaviani and Vannieuwenhoven showed recently [5] that if $\mathcal{S} \in \mathrm{S}^{d} \mathbb{C}^{n}$ has a generic symmetric rank $k<k_{n, d}$ then $k=\operatorname{rank} S$. It is much easier to establish this kind of result for smaller values of $k$ using Kruskal's theorem. See [14, Theorem 7.6].

The aim of this paper is to establish a much weaker result on Comon's conjecture, which does not use the term generic. In particular we show that Comon's conjecture holds for symmetric tensors of at most rank 3 and for 3 -symmetric tensors of at most rank 5 over $\mathbb{C}$.

Our main result is
Theorem 1.1 Let $d \geq 3,|\mathbb{F}| \geq 3$ and $\mathcal{S} \in \mathrm{S}^{d} \mathbb{F}^{n}$. Suppose that $\operatorname{rank} \mathcal{S} \leq$ $\operatorname{rank} A(\mathcal{S})+1$. Then $\operatorname{srank} \mathcal{S}=\operatorname{rank} \mathcal{S}$.

We now summarize briefly the content of this paper. In $\S 2$ we recall Kruskal's theorem on the rank of 3 -tensor. In $\S 3$ we prove Theorem 1.1 for the case rank $\mathcal{S}=$ $\operatorname{rank} A(\mathcal{S})$. In $\S 4$ we show that each $\mathcal{S} \in \mathrm{S}^{3} \mathbb{F}^{2}$, where $|\mathbb{F}| \geq 3$, satisfies srank $\mathcal{S}=$ $\operatorname{rank} \mathcal{S}$. In $\S 5$ we prove Theorem 1.1 in the case $d=3$ and $\operatorname{rank} \mathcal{S}=\operatorname{rank} A(\mathcal{S})+1$. In $\S 6$ we prove Theorem 1.1 for $d \geq 4$. In $\S 7$ we summarize our results for $\mathbb{F}=\mathbb{C}$. In $\S 8$ we discuss two other closely related conjectures: The first one conjectures that it is possible to replace in Comon's conjecture the ranks with border ranks. We show that this is true if the border rank of $\mathcal{S}$ is two. The second one conjectures that a best $k$-approximation of symmetric tensor can be chosen symmetric. For $k=1$, i.e. best rank one approximation, this conjecture holds and it is a consequence of Banach's theorem [2].

## 2 Kruskal's theorem

We recall Kruskal's theorem for 3 -tensors and any field $\mathbb{F}$. For $p$ vectors $\mathbf{x}_{1}, \ldots, \mathbf{x}_{p} \in$ $\mathbb{F}^{n}$ denote by $\left[\mathbf{x}_{1} \mathbf{x}_{2} \cdots \mathbf{x}_{p}\right]$ the $n \times p$ matrix whose columns are $\mathbf{x}_{1}, \ldots, \mathbf{x}_{p}$. Kruskal's rank of $\left[\mathbf{x}_{1} \mathbf{x}_{2} \cdots \mathbf{x}_{p}\right]$, denoted as $\operatorname{Krank}\left(\mathbf{x}_{1}, \ldots, \mathbf{x}_{p}\right)$ is the maximal $k$ such that any $k$ vectors in the set $\left\{\mathbf{x}_{1}, \ldots, \mathbf{x}_{p}\right\}$ are linearly independent. (If $\mathbf{x}_{i}=0$ for some $i \in[p]$ then $\operatorname{Krank}\left(\mathbf{x}_{1}, \ldots, \mathbf{x}_{p}\right)=-\infty$.)

Theorem 2.1 (Kruskal) Let $\mathbb{F}$ be a field, $r \in \mathbb{N}$ and $\mathbf{x}_{i} \in \mathbb{F}^{m}, \mathbf{y}_{i} \in \mathbb{F}^{n}, \mathbf{z}_{i} \in \mathbb{F}^{p}$ for $i \in[r]$. Assume that

$$
\begin{equation*}
\mathcal{T}=\sum_{i=1}^{r} \mathbf{x}_{i} \otimes \mathbf{y}_{i} \otimes \mathbf{z}_{i} . \tag{2.1}
\end{equation*}
$$

Suppose that

$$
\begin{equation*}
2 r+2 \leq \operatorname{Krank}\left(\mathbf{x}_{1}, \ldots, \mathbf{x}_{r}\right)+\operatorname{Krank}\left(\mathbf{y}_{1}, \ldots, \mathbf{y}_{r}\right)+\operatorname{Krank}\left(\mathbf{z}_{1}, \ldots, \mathbf{z}_{r}\right) \tag{2.2}
\end{equation*}
$$

Then $\operatorname{rank} \mathcal{T}=r$. Furthermore, the decomposition (2.1) is unique.
Note that max $\left(\operatorname{Krank}\left(\mathbf{x}_{1}, \ldots, \mathbf{x}_{r}\right), \operatorname{Krank}\left(\mathbf{y}_{1}, \ldots, \mathbf{y}_{r}\right), \operatorname{Krank}\left(\mathbf{z}_{1}, \ldots, \mathbf{z}_{r}\right)\right) \leq r$. Hence (2.2) yields that

$$
\begin{equation*}
\min \left(\operatorname{Krank}\left(\mathbf{x}_{1}, \ldots, \mathbf{x}_{r}\right), \operatorname{Krank}\left(\mathbf{y}_{1}, \ldots, \mathbf{y}_{r}\right), \operatorname{Krank}\left(\mathbf{z}_{1}, \ldots, \mathbf{z}_{r}\right)\right) \geq 2 \tag{2.3}
\end{equation*}
$$

In particular, $\min (m, n, p) \geq 2$.
In what follows we need a following simple corollary of Kruskal's theorem:
Lemma 2.2 Let $3 \leq d \in \mathbb{N}$. Assume that $\mathbf{x}_{j, 1}, \ldots, \mathbf{x}_{j, r} \in \mathbb{F}^{n_{j}}$ are linearly independent for each $j \in[d]$. Let

$$
\begin{equation*}
\mathcal{T}=\sum_{i=1}^{r} \otimes_{j=1}^{d} \mathbf{x}_{j, i} . \tag{2.4}
\end{equation*}
$$

Then $\operatorname{rank} \mathcal{T}=r$. Furthermore, the decomposition (2.4) is unique.
Proof. Observe first that $\otimes_{j=1}^{p} \mathbf{x}_{j, 1}, \ldots, \otimes_{j=1}^{p} \mathbf{x}_{j, r}$ linearly independent for $p=$ $1, \ldots, d$. Clearly, this is true for $p=1$ and $p=2$. Use the induction to prove this statement for $p \geq 3$ by observing that $\otimes_{j=1}^{p} \mathbf{x}_{j, i}=\left(\otimes_{j=1}^{p-1} \mathbf{x}_{j, i}\right) \otimes \mathbf{x}_{p, i}$ for $p=3, \ldots, d$.

Consider $\mathcal{T}$ given by (2.4). Suppose first that $r=1$. Then $\mathcal{T}$ is a rank one tensor and its decomposition is unique. Assume that $r \geq 2$. Consider $\mathcal{T}$ as a 3 -tensor on the 3-tensor product $\mathbb{F}^{n_{1}} \otimes \mathbb{F}^{n_{2}} \otimes\left(\otimes_{j=3}^{d} \mathbb{F}^{n_{j}}\right)$. Clearly

$$
\begin{align*}
& \operatorname{Krank}\left(\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{r, 1}\right)=\operatorname{Krank}\left(\mathbf{x}_{1,2}, \ldots, \mathbf{x}_{r, 2}\right)=  \tag{2.5}\\
& \operatorname{Krank}\left(\otimes_{j=3}^{d} \mathbf{x}_{j, 1}, \ldots, \otimes_{j=3}^{d} \mathbf{x}_{j, r}\right)=r .
\end{align*}
$$

As $3 r-2 \geq 2 r$, Kruskal's theorem yields that the $\operatorname{rank}$ of $\mathcal{T}$ as 3 -tensor is $r$. Hence $\operatorname{rank} \mathcal{T}$ as $d$ tensor is $r$ too. Furthermore the decomposition (2.4) of $\mathcal{T}$ as a 3-tensor is unique. Hence the decomposition(2.4) is unique.

In what follows we need the following lemma.
Lemma 2.3 Let $d \geq 3, n \geq 2$ and $\mathcal{S} \in \mathrm{S}^{d} \mathbb{F}^{n}$. Assume that

$$
\begin{equation*}
\mathcal{S}=\sum_{i=1}^{k} \otimes_{j=1}^{d} \mathbf{x}_{j, i} . \tag{2.6}
\end{equation*}
$$

Then $\mathcal{S}=\sum_{i=1}^{k} \otimes_{j=1}^{d} \mathbf{x}_{\sigma(j), i}$ for any permutation $\sigma$ of $[d]$. Suppose that the following inequality holds:

$$
\begin{align*}
& 2 k+2 \leq K\left(\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{1, k}\right)+K\left(\mathbf{x}_{2,1}, \ldots, \mathbf{x}_{2, k}\right)+ \\
& K\left(\otimes_{j=3}^{d} \mathbf{x}_{j, 1}, \ldots, \otimes_{j=3}^{d} \mathbf{x}_{j, k}\right) . \tag{2.7}
\end{align*}
$$

Then $\operatorname{rank} \mathcal{S}=\operatorname{srank} \mathcal{S}=k$, i.e. $\operatorname{span}\left(\mathbf{x}_{1, i}\right)=\ldots=\operatorname{span}\left(\mathbf{x}_{d, i}\right)$ for each $i \in[k]$. Furthermore, the decomposition (2.6) is unique.

Proof. Assume that (2.6) holds. Since $\mathcal{S}$ symmetric we deduce that $\mathcal{S}=$ $\sum_{i=1}^{k} \otimes_{j=1}^{d} \mathbf{x}_{\sigma(j), i}$ for any permutation $\sigma$ of [d]. Suppose that (2.7) holds. Kruskal's theorem yields that the decomposition of $\mathcal{S}$ as a 3 -tensor on $\mathbb{F}^{n} \otimes \mathbb{F}^{n} \otimes\left(\otimes^{d-2} \mathbb{F}^{n}\right)$ is unique. In particular, the decomposition (2.6) is unique. Hence $\operatorname{rank} \mathcal{S}=k$. Let $\sigma$ be the transposition on [d] satisfying $\sigma(1)=2, \sigma(2)=1$. Then $\mathcal{S}=$ $\sum_{i=1}^{k} \otimes_{j=1}^{d} \mathbf{x}_{\sigma(j), i}$. (2.3) yields that $K\left(\otimes_{j=3}^{d} \mathbf{x}_{j, 1}, \ldots, \otimes_{j=3}^{d} \mathbf{x}_{j, k}\right) \geq 2$. That is, the rank one tensors $\otimes_{j=3}^{d} \mathbf{x}_{j, p}$ and $\otimes_{j=3}^{d} \mathbf{x}_{j, q}$ are linearly independent for $p<q$. The uniqueness of the decomposition (2.6), (up to a permutation of summands), yields that $\mathbf{x}_{1, i} \otimes \mathbf{x}_{2, i} \otimes\left(\otimes_{j=3}^{d} \mathbf{x}_{j, i}\right)=\mathbf{x}_{2, i} \otimes \mathbf{x}_{1, i} \otimes\left(\otimes_{j=3}^{d} \mathbf{x}_{j, i}\right)$ for each $i \in[n]$. Hence $\mathbf{x}_{1, i} \otimes \mathbf{x}_{2, i}=\mathbf{x}_{2, i} \otimes \mathbf{x}_{1, i}$. Therefore $\mathbf{x}_{1, i}$ and $\mathbf{x}_{2, i}$ are linearly dependent nonzero vectors. Let $\sigma$ be a transposition on $[d]$ satisfying $\sigma(2)=p, \sigma(p)=2$ for some $p \geq 3$. (2.3) yields that $K\left(\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{1, k}\right) \geq 2$. The uniqueness of the decomposition (2.6), (up to a permutation of summands), yields that $\mathbf{x}_{1, i} \otimes \mathbf{x}_{2, i} \otimes\left(\otimes_{j=3}^{d} \mathbf{x}_{j, i}\right)=$ $\mathbf{x}_{1, i} \otimes \mathbf{x}_{p, i} \otimes\left(\otimes_{j=3}^{d} \mathbf{x}_{\sigma(j), i}\right)$. Therefore $\mathbf{x}_{2, i}$ and $\mathbf{x}_{p, i}$ are collinear for each $i \in[n]$. Hence $\operatorname{span}\left(\mathbf{x}_{1, i}\right)=\ldots=\operatorname{span}\left(\mathbf{x}_{d, i}\right)$ for each $i \in[d]$. Thus the decomposition (2.6) is a decomposition to a sum of symmetric rank one tensors. Hence srank $\mathcal{S}=\operatorname{rank} \mathcal{S}$.

## 3 The case $\operatorname{rank} \mathcal{S}=\operatorname{rank} A(\mathcal{S})$

Theorem 3.1 Let $d \geq 3, n \geq 2$ and $\mathcal{S} \in \mathrm{S}\left(d, \mathbb{F}^{n}\right)$. Suppose that $\operatorname{rank} \mathcal{S}=$ $\operatorname{rank} A(\mathcal{S})$. Then $\operatorname{srank} \mathcal{S}=\operatorname{rank} \mathcal{S}$. Furthermore, $\mathcal{S}$ has has a unique rank one decomposition.

Proof. We can assume without loss of generality that $\operatorname{rank} A(\mathcal{S})=n$. So (2.6) holds for $k=n$. Clearly, $\mathbf{x}_{j, 1}, \ldots, \mathbf{x}_{j, n}$ are linearly independent for each $j \in[d]$. Hence $K\left(\mathbf{x}_{j, 1}, \ldots, \mathbf{x}_{j, n}\right)=n$ for $j \in[d]$. The proof of Lemma 2.2 yields that $K\left(\otimes_{j=3}^{d} \mathbf{x}_{j, 1}, \ldots, \otimes_{j=3}^{d} \mathbf{x}_{j, n}\right)=n$. Therefore equality (2.5) holds for $r=n$. As $n \geq 2$ we deduce (2.7) for $k=n$. Lemma 2.3 yields the theorem.

The following corollary generalizes [7, Proposition 5.5] to any field $\mathbb{F}$ :
Corollary 3.2 Let $\mathbb{F}$ be a field, $\mathcal{S} \in \mathrm{S}^{d} \mathbb{F}^{n} \backslash\{0\}, d \geq 3$. Assume that $\operatorname{rank} \mathcal{S} \leq 2$. Then $\operatorname{srank} \mathcal{S}=\operatorname{rank} \mathcal{S}$.

Proof. Clearly, $\operatorname{rank} A(S) \in\{1,2\}$. If $\operatorname{rank} A(S)=1$ then $\mathcal{S}=s \otimes^{d} \mathbf{u}$. Hence $\operatorname{rank} \mathcal{S}=\operatorname{srank} \mathcal{S}=1$. If $\operatorname{rank} A(\mathcal{S})=2$ then $\operatorname{rank} \mathcal{S}=2$ and we conclude the result from Theorem 3.1.

## 4 The case $S^{3} \mathbb{F}^{2}$

Theorem 4.1 Let $\mathcal{S} \in \mathrm{S}^{3} \mathbb{F}^{2}$. Assume that $|\mathbb{F}| \geq 3$. Then $\operatorname{rank} \mathcal{S}=\operatorname{srank} \mathcal{S} \leq 3$.
For $\mathbb{F}=\mathbb{C}$ this result follows from the classical description of binary forms in two variables due Sylvester [23]. More generally, consult with [6] for results on the rank of tensors in $S^{d} \mathbb{F}^{2}$ for an algebraic closed field $\mathbb{F}$ of characteristic zero.

Proof. In view of Corollary 3.2 it is enough to consider the case where $\operatorname{rank} \mathcal{S} \geq 3$. Let $\mathcal{S}=\left[s_{i, j, k}\right]_{i, j, k \in[2]}$.

1. Assume that $s_{1,1,2} s_{1,2,2} \neq 0$. Let

$$
\begin{equation*}
\mathcal{S}=\sum_{i=1}^{3} t_{i} \otimes^{3} \mathbf{u}_{i}, \mathbf{u}_{1}=(1, b)^{\top}, \quad \mathbf{u}_{2}=(1,0)^{\top}, \quad \mathbf{u}_{3}=(0,1)^{\top} . \tag{4.1}
\end{equation*}
$$

Then

$$
\begin{gathered}
s_{1,1,2}=t_{1} b, \quad s_{1,2,2}=t_{1} b^{2} \Rightarrow t_{1}=\frac{s_{1,1,2}^{2}}{s_{1,2,2}}, \quad b=\frac{s_{1,2,2}}{s_{1,1,2}} \\
t_{2}=s_{1,1,1}-t_{1}, \quad t_{3}=s_{2,2,2}-t_{1} b^{3}
\end{gathered}
$$

Hence $\operatorname{rank} \mathcal{S} \leq 3$. Our assumption yields that $\operatorname{rank} \mathcal{S}=3$ and (4.1) is a minimal decomposition of $\mathcal{S}$ to rank one tensors. This decomposition shows that $\operatorname{rank} \mathcal{S}=\operatorname{srank} \mathcal{S}$.
2. Assume that $s_{1,1,2}=s_{1,2,2}=0$ then $\mathcal{S}=s_{1,1,1} \otimes^{3}(1,0)^{\top}+s_{2,2,2} \otimes^{3}(0,1)^{\top}$. This contradicts our assumption that $\operatorname{rank} \mathcal{S} \geq 3$.
3. It is left to discuss the case where $\operatorname{rank} \mathcal{S} \geq 3$ and $s_{1,1,2}=0$ and $s_{1,2,2} \neq 0$. The homogeneous polynomial of degree 3 corresponding to $\mathcal{S}$ is

$$
f\left(x_{1}, x_{2}\right)=s_{1,1,1} x_{1}^{3}+s_{1,2,2} x_{1} x_{2}^{2}+s_{2,2,2} x_{2}^{3} .
$$

(a) Assume that the characteristic of $\mathbb{F}$ is 3 . Make the following change of variables: $x_{1}=y_{1}, x_{2}=y_{1}+y_{2}$. The new tensor $\mathcal{S}^{\prime}$ satisfies 1 .
(b) Assume that the characteristic of $\mathbb{F}$ is not 3 .
i. Assume that $s_{1,1,1} \neq 0$. Make the following change of variables: $x_{1}=y_{1}+a y_{2}, x_{2}=y_{2}$. Then
$f\left(y_{1}, y_{2}\right)=\alpha y_{1}^{3}+\beta y_{1}^{2} y_{2}+\gamma y_{1} y_{2}^{2}+\delta, \beta=3 a s_{1,1,1}, \gamma=s_{1,2,2}+3 a^{2} s_{1,1,1}$.
Then choose a nonzero $a$ such that $s_{1,2,2}+3 a^{2} s_{1,1,1} \neq 0$. (This is always possible if $|\mathbb{F}| \geq 4$ as we assumed that $\mathbb{F} \neq \mathbb{Z}_{3}$ and $|\mathbb{F}| \geq 3$.) The new tensor $\mathcal{S}^{\prime}$ satisfies 1 .
ii. Assume that $s_{1,1,1}=s_{2,2,2}=0$. Make the following change of variables: $x_{1}=y_{1}, x_{2}=\left(y_{1}+y_{2}\right)$. Then we are either in the case 1 if the characteristic of $\mathbb{F}$ is not 2 or in the case $3(\mathrm{~b}) \mathrm{i}$ if the characteristic of $\mathbb{F}$ is 2 .
iii. Assume that $s_{1,1,1}=0$ and $s_{2,2,2} \neq 0$. Make the following change of variables: $y_{2}=s_{1,2,2} x_{1}+s_{2,2,2} x_{2}, y_{2}=x_{2}$. Then we are in the case 3(b)ii.

Note that if $|\mathbb{F}| \gg 1$ then using the change of coordinates and then the above procedure we obtain that if $\operatorname{rank} \mathcal{S}=3, \operatorname{rank} A(\mathcal{S})=2$ we have many presentation of $\mathcal{S}$ as sum of three rank one symmetric tensors.

Observe next that for $\mathbb{F}=\mathbb{Z}_{2}$ not every symmetric tensor $\mathcal{S} \in S^{3} \mathbb{Z}_{2}^{2}$ is a sum of rank one symmetric tensors. The number of all symmetric tensors in $S^{3} \mathbb{Z}_{2}^{2}$ is $2^{4}$. The number of all nonzero symmetric tensors which are sum of rank one symmetric tensors is $2^{3}-1$. Hence Theorem 4.1 does not hold for $\mathbb{F}=\mathbb{Z}_{2}$.

Corollary 4.2 Let $\mathcal{S} \in \mathrm{S}^{3} \mathbb{F}^{n}$. Assume that $|\mathbb{F}| \geq 3$ and $\operatorname{rank} \mathcal{S}=3$. Then $\operatorname{srank} \mathcal{S}=\operatorname{rank} \mathcal{S}$.

Proof. Clearly, $\operatorname{rank} A(\mathcal{S}) \in\{2,3\}$. If $\operatorname{rank} A(\mathcal{S})=3$ we deduce the corollary from Theorem 3.1. If $\operatorname{rank} A(\mathcal{S})=2$ we deduce the corollary from Theorem 4.1.

## 5 The case $d=3$ and $\operatorname{rank} \mathcal{S}=\operatorname{rank} A(\mathcal{S})+1$

In this section we prove Theorem 1.1 for $d=3$. In view of Theorem 3.1 it is enough to consider the case $\operatorname{rank} \mathcal{S}=\operatorname{rank} A(\mathcal{S})+1$. Furthermore, in view of Theorem 4.1 it is enough to consider the case $\operatorname{rank} A(\mathcal{S}) \geq 3$. We first give the following obvious lemma:

Lemma 5.1 Let $|\mathbb{F}| \geq 3$ and $\mathcal{S} \in \mathrm{S}\left(3, \mathbb{F}^{n}\right)$. Suppose that $\operatorname{rank} \mathcal{S}=\operatorname{rank} A(\mathcal{S})+$ 1. Assume furthermore that there exists a decomposition of $\mathcal{S}$ to $\operatorname{rank} A(\mathcal{S})+1$ rank one tensors such that at least one of them is symmetric, i.e. $s \otimes^{3} \mathbf{u}$. Let $\mathcal{S}^{\prime}=$ $\mathcal{S}-s \otimes^{3} \mathbf{u}$. Then $\operatorname{rank} \mathcal{S}^{\prime}=\operatorname{rank} \mathcal{S}-1$ and $\operatorname{rank} A\left(\mathcal{S}^{\prime}\right) \in\{\operatorname{rank} A(\mathcal{S})-1, \operatorname{rank} A(\mathcal{S})\}$. Furthermore:

1. If $\operatorname{rank} A\left(\mathcal{S}^{\prime}\right)=\operatorname{rank} A(\mathcal{S})$ then $\operatorname{rank} \mathcal{S}^{\prime}=\operatorname{rank} A\left(\mathcal{S}^{\prime}\right)$ and $\operatorname{rank} \mathcal{S}^{\prime}=\operatorname{srank} \mathcal{S}^{\prime}$. Hence $\operatorname{rank} \mathcal{S}=\operatorname{srank} \mathcal{S}$.
2. If $\operatorname{rank} A\left(\mathcal{S}^{\prime}\right)=\operatorname{rank} A(\mathcal{S})-1$ then $\operatorname{rank} \mathcal{S}^{\prime}=\operatorname{rank} A\left(\mathcal{S}^{\prime}\right)+1$.

### 5.1 The case $\operatorname{rank} A(\mathcal{S})=3$

We now discuss Theorem 1.1 where $\mathcal{S} \in \mathrm{S}\left(3, \mathbb{F}^{n}\right)$, where $\operatorname{rank} \mathcal{S}=4$, $\operatorname{rank} A(\mathcal{S})=3$. Without loss of generality we can assume that $n=3$. Then

$$
\begin{equation*}
\mathcal{S}=\sum_{i=1}^{4} \mathbf{x}_{i} \otimes \mathbf{y}_{i} \otimes \mathbf{z}_{i} \tag{5.1}
\end{equation*}
$$

Suppose first that there is a decomposition (5.1) such that $\mathbf{x}_{i} \otimes \mathbf{y}_{i} \otimes \mathbf{z}_{i}$ is symmetric for some $i \in[4]$. Then we can use Lemma 5.1. Apply Theorems 3.1 and 4.1 to deduce that $\operatorname{srank} \mathcal{S}=\operatorname{rank} \mathcal{S}=4$.

Assume the Assumption: there no is a decomposition (5.1) such that $\mathbf{x}_{i} \otimes \mathbf{y}_{i} \otimes \mathbf{z}_{i}$ is symmetric for some $i \in[4]$. The first part of Lemma 2.3 yields:

$$
0=\sum_{i=1}^{4} \mathbf{x}_{i} \otimes\left(\mathbf{y}_{i} \otimes \mathbf{z}_{i}-\mathbf{z}_{i} \otimes \mathbf{y}_{i}\right)
$$

As $\operatorname{rank} A(\mathcal{S})=3$ we can assume that $\mathbf{x}_{1}, \mathbf{x}_{2}, \mathbf{x}_{3}$ are linearly independent. Then $\mathbf{x}_{4}=\sum_{j=1}^{3} a_{j} \mathbf{x}_{j}$. Then the above equality yields:

$$
\begin{equation*}
0=\sum_{j=1}^{3} \mathbf{x}_{j} \otimes\left(\mathbf{y}_{j} \otimes \mathbf{z}_{j}-\mathbf{z}_{j} \otimes \mathbf{y}_{j}+a_{j}\left(\mathbf{y}_{4} \otimes \mathbf{z}_{4}-\mathbf{z}_{4} \otimes \mathbf{y}_{4}\right)\right) \tag{5.2}
\end{equation*}
$$

As $\mathbf{x}_{1}, \mathbf{x}_{2}, \mathbf{x}_{3}$ are linearly independent it follows that

$$
\begin{equation*}
\mathbf{y}_{j} \otimes \mathbf{z}_{j}-\mathbf{z}_{j} \otimes \mathbf{y}_{j}=-a_{j}\left(\mathbf{y}_{4} \otimes \mathbf{z}_{4}-\mathbf{z}_{4} \otimes \mathbf{y}_{4}\right) \text { for } j \in[3] \tag{5.3}
\end{equation*}
$$

Assume first that $\mathbf{y}_{4}$ and $\mathbf{z}_{4}$ are collinear. Then $\mathbf{y}_{j}$ and $\mathbf{z}_{j}$ are collinear for $j \in[3]$. Hence we w.l.o.g we can assume that $\mathbf{y}_{i}=\mathbf{z}_{i}=\mathbf{u}_{i}$ for $i \in[4]$. So $\mathcal{S}=$ $\sum_{i=1}^{4} \mathbf{x}_{i} \otimes \mathbf{u}_{i} \otimes \mathbf{u}_{i}$. Since $\mathcal{S}$ is symmetric we can we obtain that $\mathcal{S}=\sum_{i=1}^{4} \mathbf{u}_{i} \otimes \mathbf{u}_{i} \otimes \mathbf{x}_{i}$. Renaming the vectors we can assume that in the original decomposition (5.1) we have that $\mathbf{x}_{i}$ and $\mathbf{y}_{i}$ are collinear for $i \in[4]$. Since we assumed that no rank one tensor $\mathbf{x}_{i} \otimes \mathbf{y}_{i} \otimes \mathbf{z}_{i}$ is not symmetric, we deduce that each pair $\mathbf{y}_{i}, \mathbf{z}_{i}$ in the original decomposition (5.1) is not collinear. In particular, it is enough to study the case where $\mathbf{y}_{4}$ and $\mathbf{z}_{4}$ are not collinear, i.e. $\mathbf{y}_{4} \otimes \mathbf{z}_{4}-\mathbf{z}_{4} \otimes \mathbf{y}_{4} \neq 0$. Let $\mathbf{U}:=\operatorname{span}\left(\mathbf{y}_{4}, \mathbf{z}_{4}\right)$. Then $\operatorname{dim} \mathbf{U}=2$.

Suppose that $a_{j} \neq 0$ for some $j \in[3]$. Then (5.3) yields that $\mathbf{y}_{j}$ and $\mathbf{z}_{j}$ are not collinear and $\operatorname{span}\left(\mathbf{y}_{j}, \mathbf{z}_{j}\right)=\mathbf{U}$. Assume that $a_{j}=0$. Then $\mathbf{y}_{j}$ and $\mathbf{z}_{j}$ are collinear.

Assume first that $a_{1} a_{2} a_{3} \neq 0$. Then the above arguments yields that $\operatorname{span}\left(\mathbf{y}_{1}, \ldots, \mathbf{y}_{4}\right)$ $\subseteq \mathbf{U}$, which contradicts the assumption that $\operatorname{span}\left(\mathbf{y}_{1}, \ldots, \mathbf{y}_{4}\right)=\mathbb{F}^{3}$.

So we need to assume that at least one of $a_{i}=0$. Assume first that exactly one $a_{i}=0$. Without loss of generality we can assume that in (5.2) $a_{1}=0$ and $a_{2} a_{3} \neq 0$. This yields that $\mathbf{y}_{1}$ and $\mathbf{z}_{1}$ are collinear. Our Assumption yields that $\mathbf{x}_{1}$ and $\mathbf{y}_{1}$ are not collinear. Furthermore $\operatorname{span}\left(\mathbf{y}_{2}, \mathbf{z}_{2}\right)=\operatorname{span}\left(\mathbf{y}_{3}, \mathbf{z}_{3}\right)=\mathbf{U}$. Hence $\operatorname{span}\left(\mathbf{y}_{2}, \mathbf{y}_{3}, \mathbf{y}_{4}\right) \subseteq \mathbf{U}$. As span $\mathbf{y}_{1}, \mathbf{y}_{2}, \mathbf{y}_{3}, \mathbf{y}_{4}$ is the whole space, we deduce that $\mathbf{y}_{1} \notin \operatorname{span}\left(\mathbf{y}_{2}, \mathbf{y}_{3}, \mathbf{y}_{4}\right)$. Similarly $\mathbf{y}_{1} \notin \operatorname{span}\left(\mathbf{z}_{2}, \mathbf{z}_{3}, \mathbf{z}_{4}\right)$. Furthermore, $\operatorname{dim} \operatorname{span}\left(\mathbf{y}_{2}, \mathbf{y}_{3}, \mathbf{y}_{4}\right)=2$. Hence $\operatorname{span}\left(\mathbf{y}_{2}, \mathbf{y}_{3}, \mathbf{y}_{4}\right)=\mathbf{U}$. We now recall that $\mathcal{S}=$ $\sum_{i=1}^{4} \mathbf{y}_{i} \otimes \mathbf{x}_{i} \otimes \mathbf{z}_{i}$. Again, by renaming the indices $2,3,4$ we can assume that $\mathbf{y}_{1}, \mathbf{y}_{2}, \mathbf{y}_{3}$, are linearly independent. Since $\operatorname{span}\left(\mathbf{y}_{2}, \mathbf{y}_{3}, \mathbf{y}_{4}\right)=\mathbf{U}$ it follows that $\mathbf{y}_{4}=b_{2} \mathbf{y}_{2}+b_{3} \mathbf{y}_{3}$.

Since $\mathcal{S}$ is symmetric we have the equality $\mathcal{S}=\sum_{i=1}^{4} \mathbf{y}_{i} \otimes \mathbf{x}_{i} \otimes \mathbf{z}_{i}$. Permuting the last two factors we obtain the equality $0=\sum_{i=1}^{4} \mathbf{y}_{i} \otimes\left(\mathbf{x}_{i} \otimes \mathbf{z}_{i}-\mathbf{z}_{i} \otimes \mathbf{y}_{i}\right)$. Hence we have an analogous equality to (5.2):
$0=\mathbf{y}_{1} \otimes\left(\mathbf{x}_{1} \otimes \mathbf{z}_{1}-\mathbf{z}_{1} \otimes \mathbf{x}_{1}\right)+\sum_{j=2}^{3} \mathbf{y}_{j} \otimes\left(\mathbf{x}_{j} \otimes \mathbf{z}_{j}-\mathbf{z}_{j} \otimes \mathbf{x}_{j}+b_{j}\left(\mathbf{x}_{4} \otimes \mathbf{z}_{4}-\mathbf{z}_{4} \otimes \mathbf{x}_{4}\right)\right)$.
Therefore $\mathbf{x}_{1} \otimes \mathbf{z}_{1}-\mathbf{z}_{1} \otimes \mathbf{x}_{1}=0$. Thus $\mathbf{x}_{1}$ and $\mathbf{z}_{1}$ are collinear. Recall that we already showed that $\mathbf{y}_{1}$ and $\mathbf{z}_{1}$ are collinear. Hence $\mathbf{x}_{1} \otimes \mathbf{y}_{1} \otimes \mathbf{z}_{1}$ is a symmetric rank one tensor. So we have a contradiction to our Assumption.

Finally let us assume that $a_{i}=a_{j}=0$ for some two distinct indices $i, j \in[3]$. W.l.o.g. we can assume that $\mathbf{x}_{4}=\mathbf{x}_{3}$, i.e. $a_{1}=a_{2}=0, a_{3}=1$. This implies that $\mathbf{y}_{i}$ and $\mathbf{z}_{i}$ are collinear for $i=1,2$. Furthermore

$$
C:=\mathbf{y}_{3} \otimes \mathbf{z}_{3}+\mathbf{y}_{4} \otimes \mathbf{z}_{4}=\mathbf{z}_{3} \otimes \mathbf{y}_{3}+\mathbf{z}_{4} \otimes \mathbf{y}_{4}
$$

So $C$ is a symmetric matrix. Note that $C$ is a rank two matrix. Otherwise $\mathbf{y}_{3} \otimes \mathbf{z}_{3}$ and $\mathbf{y}_{4} \otimes \mathbf{z}_{4}$ are collinear. Then $\mathbf{x}_{3} \otimes \mathbf{y}_{3} \otimes \mathbf{z}_{3}+\mathbf{x}_{3} \otimes \mathbf{y}_{4} \otimes \mathbf{z}_{4}$ is a rank one tensor. So rank $\mathcal{S} \leq$ 3 , contrary to our assumptions. Thus we can assume that $C=\mathbf{y}_{3} \otimes \mathbf{y}_{4}+\mathbf{y}_{4} \otimes \mathbf{y}_{3}$ and $\mathbf{y}_{3}, \mathbf{y}_{4}$ are linearly independent. Hence we can assume that

$$
\begin{aligned}
\mathcal{S}= & \mathbf{x}_{1} \otimes \mathbf{y}_{1} \otimes \mathbf{y}_{1}+\mathbf{x}_{2} \otimes \mathbf{y}_{2} \otimes \mathbf{y}_{2}+\mathbf{x}_{3} \otimes\left(\mathbf{y}_{3} \otimes \mathbf{y}_{4}+\mathbf{y}_{4} \otimes \mathbf{y}_{3}\right)= \\
& \mathbf{y}_{1} \otimes \mathbf{x}_{1} \otimes \mathbf{y}_{1}+\mathbf{y}_{2} \otimes \mathbf{x}_{2} \otimes \mathbf{y}_{2}+\mathbf{y}_{3} \otimes \mathbf{x}_{3} \otimes \mathbf{y}_{4}+\mathbf{y}_{4} \otimes \mathbf{x}_{3} \otimes \mathbf{y}_{3}= \\
& \mathbf{y}_{1} \otimes \mathbf{y}_{1} \otimes \mathbf{x}_{1}+\mathbf{y}_{2} \otimes \mathbf{y}_{2} \otimes \mathbf{x}_{2}+\mathbf{y}_{3} \otimes \mathbf{y}_{4} \otimes \mathbf{x}_{3}+\mathbf{y}_{4} \otimes \mathbf{y}_{3} \otimes \mathbf{x}_{3} .
\end{aligned}
$$

Our Assumption yields that the pairs $\mathbf{x}_{1}, \mathbf{y}_{1}$ and $\mathbf{x}_{2}, \mathbf{y}_{2}$ are linearly independent. Hence $Q:=\mathbf{x}_{1} \otimes \mathbf{y}_{1}-\mathbf{y}_{1} \otimes \mathbf{x}_{1} \neq 0$. Subtracting the third expression for $\mathcal{S}$ from the second one we deduce

$$
\begin{aligned}
& \mathbf{y}_{1} \otimes\left(\mathbf{x}_{1} \otimes \mathbf{y}_{1}-\mathbf{y}_{1} \otimes \mathbf{x}_{1}\right)+\mathbf{y}_{2} \otimes\left(\mathbf{x}_{2} \otimes \mathbf{y}_{2}-\mathbf{y}_{2} \otimes \mathbf{x}_{2}\right)+ \\
& \mathbf{y}_{3} \otimes\left(\mathbf{x}_{3} \otimes \mathbf{y}_{4}-\mathbf{y}_{4} \otimes \mathbf{x}_{3}\right)+\mathbf{y}_{4} \otimes\left(\mathbf{x}_{3} \otimes \mathbf{y}_{3}-\mathbf{y}_{3} \otimes \mathbf{x}_{3}\right)=0
\end{aligned}
$$

As $\mathbf{y}_{3}, \mathbf{y}_{4}$ are linearly independent, without loss in generality we may assume that $\mathbf{y}_{2}, \mathbf{y}_{3}, \mathbf{y}_{4}$ are linearly independent. So $\mathbf{y}_{1}=b_{2} \mathbf{y}_{2}+b_{3} \mathbf{y}_{3}+b_{4} \mathbf{y}_{4}$. Substitute in the above equality this expression for $\mathbf{y}_{1}$ only for the $\mathbf{y}_{1}$ appearing in the left-hand side to obtain
$\mathbf{y}_{2} \otimes\left(\mathbf{x}_{2} \otimes \mathbf{y}_{2}-\mathbf{y}_{2} \otimes \mathbf{x}_{2}+b_{2} Q_{2}\right)+\mathbf{y}_{3}\left(\mathbf{x}_{3} \otimes \mathbf{y}_{4}-\mathbf{y}_{4} \otimes \mathbf{x}_{3}+b_{3} Q\right)+\mathbf{y}_{4}\left(\mathbf{x}_{3} \otimes \mathbf{y}_{3}-\mathbf{y}_{3} \otimes \mathbf{x}_{3}+b_{4} Q\right)=0$.

## Hence

$\mathbf{x}_{2} \otimes \mathbf{y}_{2}-\mathbf{y}_{2} \otimes \mathbf{x}_{2}+b_{2} Q_{2}=\mathbf{x}_{3} \otimes \mathbf{y}_{4}-\mathbf{y}_{4} \otimes \mathbf{x}_{3}+b_{3} Q=\mathbf{x}_{3} \otimes \mathbf{y}_{3}-\mathbf{y}_{3} \otimes \mathbf{x}_{3}+b_{4} Q=0$.
Note that our Assumption yields that $b_{2} \neq 0$. Hence $\operatorname{span}\left(\mathbf{x}_{2}, \mathbf{y}_{2}\right)=\operatorname{span}\left(\mathbf{x}_{1}, \mathbf{y}_{1}\right)$. Suppose first that $b_{3} \neq 0$. Then $\operatorname{span}\left(\mathbf{x}_{3}, \mathbf{y}_{4}\right)=\operatorname{span}\left(\mathbf{x}_{1}, \mathbf{y}_{1}\right)$. This contradicts the assumption that $\mathbf{x}_{1}, \mathbf{x}_{2}, \mathbf{x}_{3}$ are linearly independent. As $\mathbf{x}_{3}=\mathbf{x}_{4}$, we get also a contradiction if $b_{4} \neq 0$. Hence $b_{3}=b_{4}=0$. So $\mathbf{y}_{3}, \mathbf{y}_{4} \in \operatorname{span}\left(\mathbf{x}_{3}\right)$. This contradicts the assumption that $\mathbf{y}_{3}$ and $\mathbf{y}_{4}$ are linearly independent.

In conclusion we showed that our Assumption never holds. The proof of this case of Theorem 1.1 is concluded.

### 5.2 Case $\operatorname{rank} A(\mathcal{S}) \geq 4$

Proof. By induction on $r=\operatorname{rank} A(\mathcal{S}) \geq 3$. For $r=3$ the proof follows from the results above. Assume that Theorem holds for $\operatorname{rank} \mathcal{S}=r+1$. Assume now that $\operatorname{rank} A(\mathcal{S})=r+1$ and $\operatorname{rank} \mathcal{S}=r+2$. Without loss of generality we can assume that $n=r+1$. Suppose first that the assumptions of Lemma 5.1 hold. If we are in the case 1 then $\operatorname{srank} \mathcal{S}=\operatorname{rank} \mathcal{S}$. If we are in the case 2. then we deduce from the induction hypothesis that $\operatorname{srank} \mathcal{S}=\operatorname{rank} \mathcal{S}$.

As in the proof of the case rank $\mathcal{S}=3$ we assume the Assumption: There does not exist a decomposition of $\mathcal{S}$ to $\operatorname{rank} A(\mathcal{S})+1 \operatorname{rank}$ one tensors such that at least one of them is symmetric. We will show that we will obtain a contradiction.

Suppose $\mathcal{S}=\sum_{i=1}^{n+1} \mathbf{x}_{i} \otimes \mathbf{y}_{i} \otimes \mathbf{z}_{i}$. The we have the equality

$$
0=\sum_{i=1}^{n+1} \mathbf{x}_{i} \otimes\left(\mathbf{y}_{i} \otimes \mathbf{z}_{i}-\mathbf{z}_{i} \otimes \mathbf{y}_{i}\right)
$$

and the fact that span of all x's, y's and z's is $\mathbb{F}^{n}$.
Without loss of generality we may assume that $\mathbf{x}_{1}, \ldots, \mathbf{x}_{n}$ are linearly independent. So $\mathbf{x}_{n+1}=\sum_{i=1}^{n} a_{i} \mathbf{x}_{i}$. Hence

$$
\begin{equation*}
\sum_{i+1}^{n} \mathbf{x}_{i} \otimes\left(\mathbf{y}_{i} \otimes \mathbf{z}_{i}-\mathbf{z}_{i} \otimes \mathbf{y}_{i}+a_{i}\left(\mathbf{y}_{n+1} \otimes \mathbf{z}_{n+1}-\mathbf{z}_{n+1} \otimes \mathbf{y}_{n+1}\right)\right)=0 \tag{5.4}
\end{equation*}
$$

As in the case $n=3$ we can assume that $\mathbf{y}_{n+1}$ and $\mathbf{z}_{n+1}$ are not collinear. Thus if $a_{i}=0$ we deduce that $\mathbf{y}_{i}$ and $\mathbf{z}_{i}$ are collinear. If $a_{i} \neq 0$ we deduce that $\operatorname{span}\left(\mathbf{y}_{i}, \mathbf{z}_{i}\right)=$ $\operatorname{span}\left(\mathbf{y}_{n+1}, \mathbf{z}_{n+1}\right)$. Since $\mathbf{y}_{1}, \ldots, \mathbf{y}_{n+1}$ span $\mathbb{F}^{n}$ we can have at most two nonzero $a_{i}$. Since $\mathbf{x}_{n+1} \neq 0$ we must have at least one nonzero $a_{i}$. Assume first that $n-1$ out of $\left\{a_{1}, \ldots, a_{n}\right\}$ are zero. We may assume without loss of generality that $a_{1}=\ldots=a_{n-1}=0$ and $a_{n}=1$. So $\mathbf{x}_{n+1}=\mathbf{x}_{n}$. Without loss of generality we may assume that

$$
\mathcal{S}=\mathbf{x}_{n} \otimes\left(\mathbf{y}_{n} \otimes \mathbf{z}_{n}+\mathbf{y}_{n+1} \otimes \mathbf{z}_{n+1}\right)+\sum_{i=1}^{n-1} \mathbf{x}_{i} \otimes \mathbf{y}_{i} \otimes \mathbf{y}_{i}
$$

Since $\mathcal{S}$ is symmetric as in case $n=3$ we deduce that $\mathbf{y}_{n} \otimes \mathbf{z}_{n}+\mathbf{y}_{n+1} \otimes \mathbf{z}_{n+1}$ is symmetric and has rank two. So we can assume that $\mathbf{z}_{n}=\mathbf{y}_{n+1}, \mathbf{z}_{n+1}=\mathbf{y}_{n}$ and $\operatorname{dim} \operatorname{span}\left(\mathbf{y}_{n}, \mathbf{y}_{n+1}\right)=2$. We now repeat the arguments in the proof of this case for $n=3$ to deduce the contradiction.

Suppose finally that exactly $n-2$ out of $\left\{a_{1}, \ldots, a_{n}\right\}$ are zero. We may assume without loss of generality that $a_{1}=\ldots=a_{n-2}=0$ and $a_{n-1}, a_{n} \neq 0$. So $\operatorname{span}\left(\mathbf{y}_{n-1}, \mathbf{z}_{n-1}\right)=\operatorname{span}\left(\mathbf{y}_{n}, \mathbf{z}_{n}\right)=\operatorname{span}\left(\mathbf{y}_{n+1}, \mathbf{z}_{n+1}\right)$. Hence $\mathbf{y}_{n-1}, \mathbf{y}_{n}, \mathbf{y}_{n+1}$ are linearly dependent. Since $\mathbf{y}_{1}, \ldots, \mathbf{y}_{n+1}$ span the whole space we must have that $\operatorname{dim} \operatorname{span}\left(\mathbf{y}_{n-1}, \mathbf{y}_{n}, \mathbf{y}_{n+1}\right)=2$. Without loss of generality we may assume the following: First, $\mathbf{y}_{n}, \mathbf{y}_{n+1}$ are linearly independent and $\mathbf{y}_{n-1}=a \mathbf{y}_{n}+b \mathbf{y}_{n+1}$. Second $\mathbf{z}_{k}=\mathbf{y}_{k}$ for $k=1, \ldots, n-2$. So we can assume that

$$
\mathcal{S}=\sum_{j=1}^{n-2} \mathbf{x}_{j} \otimes \mathbf{y}_{j} \otimes \mathbf{y}_{j}+\sum_{j=n-1}^{n+1} \mathbf{x}_{j} \otimes \mathbf{y}_{j} \otimes \mathbf{z}_{j}=\sum_{j=1}^{n-2} \mathbf{y}_{j} \otimes \mathbf{x}_{j} \otimes \mathbf{y}_{j}+\sum_{j=n-1}^{n+1} \mathbf{y}_{j} \otimes \mathbf{x}_{j} \otimes \mathbf{z}_{j} .
$$

Permuting the las two factors in the last part of the above identity we obtain:

$$
0=\sum_{j=1}^{n-2} \mathbf{y}_{j} \otimes\left(\mathbf{x}_{j} \otimes \mathbf{y}_{j}-\mathbf{y}_{j} \otimes \mathbf{x}_{j}\right)+\sum_{j=n-1}^{n+1} \mathbf{y}_{j}\left(\mathbf{x}_{j} \otimes \mathbf{z}_{j}-\mathbf{z}_{j} \otimes \mathbf{x}_{j}\right) .
$$

Substitute $\mathbf{y}_{n-1}=a \mathbf{y}_{n}+b \mathbf{y}_{n+1}$ and recall that $\mathbf{y}_{1}, \ldots, \mathbf{y}_{n-2}, \mathbf{y}_{n}, \mathbf{y}_{n+1}$ are linearly independent. Hence $\mathbf{x}_{i}$ and $\mathbf{y}_{i}$ are collinear for $i=1, \ldots, n-2 \geq 2$. This contradicts our Assumption.

## 6 Theorem 1.1 for $d \geq 4$

In this section we show Theorem for 1.1 for $d \geq 4$. Theorem 3.1 yields that it is enough to consider the case where $\operatorname{rank} \mathcal{S}=\operatorname{rank} A(\mathcal{S})+1$. We need the following lemma:

Lemma 6.1 Let $2 \leq d \in \mathbb{N}$. Assume that $\mathbf{x}_{j, 1}, \ldots \mathbf{x}_{j, n+1} \in \mathbb{F}^{n} \backslash\{\mathbf{0}\}$ and $\operatorname{span}\left(\mathbf{x}_{j, 1}, \ldots, \mathbf{x}_{j, n+1}\right)=\mathbb{F}^{n}$ for $j \in[d]$. Consider the $n+1$ rank one $d$-tensors $\otimes_{j=1}^{d} \mathbf{x}_{j, i}, i \in[n+1]$. Then either all of them are linearly independent or $n$ of these tensors are linearly independent and the other one is a multiple of one of the $n$ linearly independent tensors.

Proof. It is enough to consider the case where the $n+1$ rank one $d$-tensors $\otimes_{j=1}^{d} \mathbf{x}_{j, i}, i \in[n+1]$ are linearly dependent. Without loss of generality we may assume that $\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{1, n}$ are linearly independent. Hence the $n$ tensors $\otimes_{j=1}^{d} \mathbf{x}_{j, i}, i \in[n]$ are linearly independent as rank one matrices $\mathbf{x}_{1, i} \otimes\left(\otimes_{j=2}^{d} \mathbf{x}_{j, i}\right)$ for $i \in[n]$. (I.e., the corresponding unfolding of $n$ tensors in mode 1 are linearly independent.) Assume that $\mathbf{x}_{1, n+1}=\sum_{j=1}^{n} a_{j} \mathbf{x}_{1, j}$ where not $a_{j}$ are zero. Since we assumed that $\otimes_{j=1}^{d} \mathbf{x}_{j, i}, i \in$ $[n+1]$ are linearly dependent it follows that $\otimes_{j=1}^{d} \mathbf{x}_{j, n+1}=\sum_{i=1}^{n} b_{i} \otimes_{j=1}^{d} \mathbf{x}_{j, i}$. So we obtain the identity $\sum_{i=1}^{n} \mathbf{x}_{1, i} \otimes \mathcal{T}_{i}=0$. Here $\mathcal{T}_{i} \in \otimes^{d-1} \mathbb{F}^{n}$ is a tensor of at most rank 2. Since $\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{1, n}$ are linearly independent if follows that each $\mathcal{T}_{i}$ is zero. Hence if $a_{i} \neq 0$ it follows that $b_{i}$ is not zero and $\otimes_{j=2} \mathbf{x}_{j, n+1}$ and $\otimes_{j=2} \mathbf{x}_{j, i}$ are collinear. Therefore $\mathbf{x}_{j, i}$ and $\mathbf{x}_{j, n+1}$ are collinear for $j=2, \ldots, d$. Since dim $\operatorname{span}\left(\mathbf{x}_{j, 1}, \ldots, \mathbf{x}_{j, n+1}\right)=n$, we can't have another $a_{k} \neq 0$. So $\otimes_{j=1}^{d} \mathbf{x}_{j, n+1}$ is collinear with $\otimes_{j=1} \mathbf{x}_{j, i}$ as we claimed.

Proof of Theorem 1.1 for $d \geq 4$ and $\operatorname{rank} \mathcal{S}=\operatorname{rank} A(\mathcal{S})+1$. Without loss of generality we may assume that $n=\operatorname{rank} A(\mathcal{S}) \geq 2$. Assume that $\mathcal{S}=$ $\sum_{i=1}^{n+1} \otimes_{j=1}^{d} \mathbf{x}_{j, i}$. Clearly, the assumptions of Lemma 6.1 holds. Consider the $d-2$ rank one tensors $\otimes_{j \in[d] \backslash\{p, q\}} \mathbf{x}_{j, i}$ for fixed $p \neq q \in[d]$ and $i \in[n+1]$. Suppose that these $n+1$ rank one tensors are linearly independent. We claim that $\mathbf{x}_{p, i}$ and $\mathbf{x}_{q, i}$ are collinear for each $i \in[n+1]$. Without loss of generality we may assume that $p=1, q=2$. By interchanging the first two factors in the representation of $\mathcal{S}$ as a rank $n+1$ tensor we deduce:

$$
\sum_{i=1}^{n+1}\left(\mathbf{x}_{1, i} \otimes \mathbf{x}_{2, i}-\mathbf{x}_{2, i} \otimes \mathbf{x}_{1, i}\right) \otimes\left(\otimes_{j=3}^{d} \mathbf{x}_{j, i}\right)=0
$$

As $\otimes_{j=3}^{d} \mathbf{x}_{j, i}, i \in[n+1]$ are linearly independent we deduce that $\mathbf{x}_{1, i} \otimes \mathbf{x}_{2, i}-\mathbf{x}_{2, i} \otimes$ $\mathbf{x}_{1, i}=0$ for each $i \in[n+1]$. I.e., $\mathbf{x}_{1, i}$ and $\mathbf{x}_{2, i}$ are collinear for each $i \in[n+1]$.

Suppose first that for each pair of integers $1 \leq p<q \leq d \otimes_{j \in[d] \backslash\{p, q\}} \mathbf{x}_{j, i}, i \in[n+1]$ are linearly independent. Hence $\mathbf{x}_{j, i} \in \operatorname{span}\left(\mathbf{x}_{1, i}\right)$ for $j \in[d]$ and $i \in[n+1]$. Therefore $\otimes_{j=1}^{d} \mathbf{x}_{j, i}$ is a rank one symmetric tensor for each $i \in[n+1]$. Thus srank $S=\operatorname{rank} S$.

Assume now, without loss of generality, that $\otimes_{j=3}^{d} \mathbf{x}_{j, i}, i \in[n+1]$ are linearly dependent. By applying Lemma 6.1 we can assume without loss of generality that $\otimes_{j=3}^{d} \mathbf{x}_{j, i}, i \in[n]$ are linearly independent and $\otimes_{j=3}^{d} \mathbf{x}_{j, n+1}=\otimes_{j=3}^{d} \mathbf{x}_{j, n}$. Without loss of generality we may assume that $\mathbf{x}_{j, n+1}=\mathbf{x}_{j, n}$ for $j \geq 3$. (We may need to rescale the vectors $\mathbf{x}_{2, n+1}, \ldots, \mathbf{x}_{d, n+1}$.) Hence $\mathbf{x}_{j, 1}, \ldots, \mathbf{x}_{j, n}$ are linearly independent for each $j \geq 3$. Therefore we have the following decomposition of $\mathcal{S}$ as a 3 -tensor in $\left(\otimes^{2} \mathbb{F}^{n}\right) \otimes \mathbb{F}^{n} \otimes\left(\otimes^{d-3} \mathbb{F}^{n}\right):$

$$
\begin{align*}
\mathcal{S}= & \left(\mathbf{x}_{1, n} \otimes \mathbf{x}_{2, n}+\mathbf{x}_{1, n+1} \otimes \mathbf{x}_{2, n+1}\right) \otimes \mathbf{x}_{3, n} \otimes\left(\otimes_{j=4}^{d} \mathbf{x}_{j, n}\right)+  \tag{6.1}\\
& \sum_{i=1}^{n-1}\left(\mathbf{x}_{1, i} \otimes \mathbf{x}_{2, i}\right) \otimes \mathbf{x}_{3, i} \otimes\left(\otimes_{j=4}^{d} \mathbf{x}_{j, i}\right)=\sum_{i=1}^{n} \mathcal{T}_{i} \otimes\left(\otimes_{j=4}^{d} \mathbf{x}_{j, i}\right) .
\end{align*}
$$

Clearly, $\otimes_{j=4}^{d} \mathbf{x}_{j, 1}, \ldots, \otimes_{j=4}^{d} \mathbf{x}_{j, n}$ are linearly independent. Since $\mathcal{S}$ is symmetric by interchanging every two distinct factors $p, q \in[3]$ in $\otimes^{d} \mathbb{F}^{n}$ we deduce that $\mathcal{T}_{1}, \ldots, \mathcal{T}_{n}$ are symmetric 3 -tensors. Consider the symmetric tensor $\mathcal{T}_{n}=\left(\mathbf{x}_{1, n} \otimes \mathbf{x}_{2, n}+\mathbf{x}_{1, n+1} \otimes\right.$ $\left.\mathbf{x}_{2, n+1}\right) \otimes \mathbf{x}_{3, n}$. As the rank of $A\left(\mathcal{T}_{n}\right)$ in the the third coordinate is 1 it follows that $\operatorname{rank} A(S)=1$. Hence $\operatorname{rank} \mathcal{T}_{n}=1$. Therefore $\operatorname{rank} \mathcal{S} \leq n$ contrary to our assumptions.

Corollary 6.2 Let $|\mathbb{F}| \geq 3, d \geq 3, n \geq 2$. Assume that $\mathcal{S} \in \mathrm{S}^{d} \mathbb{F}^{n}$. Then srank $\mathcal{S}=\operatorname{rank} \mathcal{S}$ under the following assumptions:

1. $\operatorname{rank} \mathcal{S} \leq 3$.
2. $\operatorname{srank} \mathcal{S} \leq 4$.

Proof. It is enough to consider the case where $n=\operatorname{rank} A(\mathcal{S}) \geq 2$.

1. Clearly, $\operatorname{rank} \mathcal{S} \in\{2,3\}$. Theorem 1.1 yields that $\operatorname{srank} \mathcal{S}=\operatorname{rank} \mathcal{S}$.
2. Assume to the contrary that $\operatorname{rank} \mathcal{S}<\operatorname{srank} \mathcal{S} \leq 4$. Then $\operatorname{rank} \mathcal{S} \leq 3$. Part 1. implies the contradiction $\operatorname{rank} \mathcal{S}=\operatorname{srank} \mathcal{S}$.

## 7 Symmetric tensors over $\mathbb{C}$

Recall the known maximal value of the symmetric rank in $\mathrm{S}^{d} \mathbb{C}^{n}$, denoted as $\mu(d, n)$ :

1. $\mu(d, 2)=d[6],[3, \S 3.1]$;
2. $\mu(3,3)=5[22, \S 96],[8]$ and [19];
3. $\mu(3,4)=7[22, \S 97]$;
4. $\mu(3,5) \leq 10[10]$;
5. $\mu(4,3)=7[17,9]$.

Theorem 7.1 Let $\mathbb{F}=\mathbb{C}$ and $\mathcal{S}$ be a symmetric tensor in $\mathrm{S}^{d} \mathbb{C}^{n}$. Then $\operatorname{srank} \mathcal{S}=$ rank $\mathcal{S}$ in the following cases:

1. $d \geq 3, n \geq 2$ and $\operatorname{rank} \mathcal{S} \in\{\operatorname{rank} A(\mathcal{S}), \operatorname{rank} A(\mathcal{S})+1\}$.
2. For $n=2$ and $d=3$.
3. For $n=2$ and $d=4$
4. $n=d=3$.
5. $\mathcal{S} \in \mathrm{S}^{3} \mathbb{C}^{n}$ and $\operatorname{rank} \mathcal{S} \leq 5$.
6. $\mathcal{S} \in \mathrm{S}^{3} \mathbb{C}^{n}$ and $\operatorname{srank} \mathcal{S} \leq 6$.

Proof. Assume that $\mathcal{S} \in \mathrm{S}^{d} \mathbb{C}^{n}$. Clearly, it is enough to prove the theorem for the case $\operatorname{rank} A(\mathcal{S}) \geq 2$. Furthermore, we can assume that $n=A(\mathcal{S})$. Thus it is enough to assume the following conditions:

$$
\begin{equation*}
2 \leq n=\operatorname{rank} A(\mathcal{S}) \leq \operatorname{rank} \mathcal{S} \leq \operatorname{srank} \mathcal{S} \leq \mu(d, n) \tag{7.1}
\end{equation*}
$$

1. follows from Theorem 1.1.
2. Assume $\mathcal{S} \in \mathrm{S}^{3} \mathbb{C}^{2}$. As $\mu(3,2)=3$ we deduce the theorem from 1 .
3. Assume that $\mathcal{S} \in \mathrm{S}^{4} \mathbb{C}^{2}$. Suppose that $\operatorname{rank} \mathcal{S} \in\{2,3\}$. Then 1. yields that $\operatorname{srank} \mathcal{S}=\operatorname{rank} \mathcal{S}$.

Suppose that $\operatorname{rank} \mathcal{S} \geq 4$. As $\mu(4,2)=4$ in view of (7.1) it follows that srank $\mathcal{S}=$ $\operatorname{rank} \mathcal{S}=4$.
4. Assume now that $\mathcal{S} \in \mathrm{S}^{3} \mathbb{C}^{3}$. Suppose first that $\operatorname{rank} A(\mathcal{S})=2$. Then by changing a basis in $\mathbb{C}^{3}$ we can assume that $\mathcal{S} \in \mathrm{S}^{3} \mathbb{C}^{2}$. Part 2. yields that $\operatorname{srank} \mathcal{S}=\operatorname{rank} \mathcal{S}$.

Suppose that $\operatorname{rank} A(\mathcal{S})=3$. If $\operatorname{rank} \mathcal{S} \in\{3,4\}$ then 1. yields that $\operatorname{srank} \mathcal{S}=$ $\operatorname{rank} \mathcal{S}$. Suppose now that $\operatorname{rank} \mathcal{S} \geq 5$. (7.1) yields that $\operatorname{srank} \mathcal{S} \geq 5$. The equality $\mu(3,3)=5$ yields that $\operatorname{rank} \mathcal{S}=5$. Hence srank $\mathcal{S}=\operatorname{rank} \mathcal{S}=5$.
5. (7.1) yields $\operatorname{rank} A(\mathcal{S}) \leq 5$. If $\operatorname{rank} A(\mathcal{S})=2$ then 2. yields that srank $\mathcal{S}=$ $\operatorname{rank} \mathcal{S}$. If $\operatorname{rank} A(\mathcal{S})=3$ then 4. yields that $\operatorname{srank} \mathcal{S}=\operatorname{rank} \mathcal{S}$. If $\operatorname{rank} A(\mathcal{S}) \geq 4$ then 1. yields that $\operatorname{srank} \mathcal{S}=\operatorname{rank} \mathcal{S}$.
6. Assume to the contrary that $\operatorname{rank} \mathcal{S}<\operatorname{srank} \mathcal{S}$. So $\operatorname{rank} \mathcal{S} \leq 5$. 5. implies the contradiction rank $\mathcal{S}=\operatorname{srank} \mathcal{S}$.

## 8 Two version of Comon's conjecture

In this section we assume that $\mathbb{F}=\mathbb{R}, \mathbb{C}$.

### 8.1 Border rank

Definition 8.1 Let $\mathcal{T} \in \otimes^{d} \mathbb{F}^{n} \backslash\{0\}$. Then the border of $\mathcal{T}$, denoted as $\operatorname{brank}_{\mathbb{F}} \mathcal{T}$, is $r \in \mathbb{N}$ if the following conditions hold

1. There exists a sequence $\mathcal{T}_{k} \in \otimes^{d} \mathbb{F}^{n}, k \in \mathbb{N}$ such that $\operatorname{rank} \mathcal{T}_{k}=r$ for $k \in \mathbb{N}$ and $\lim _{k \rightarrow \infty} \mathcal{T}_{k}=\mathcal{T}$.
2. Assume that a sequence $\mathcal{T}_{k} \in \otimes^{d} \mathbb{F}^{n}, k \in \mathbb{N}$ converges to $\mathcal{T}$. Then $\liminf \operatorname{inco}_{k \rightarrow} \operatorname{rank} \mathcal{T}_{k} \geq r$.

Clearly, $\operatorname{brank}_{\mathbb{F}} \mathcal{T} \leq \operatorname{rank} \mathcal{T}$. For $d=2$ it is well known that $\operatorname{brank}_{\mathbb{F}} \mathcal{T}=\operatorname{rank} \mathcal{T}$. Hence

$$
\begin{equation*}
\operatorname{rank} A(\mathcal{T}) \leq \operatorname{brank}_{\mathbb{F}} \mathcal{T} \tag{8.1}
\end{equation*}
$$

For $d>2$ one has examples where $\operatorname{brank}_{\mathbb{F}} \mathcal{T}<\operatorname{rank} \mathcal{T}[7]:$ Assume that $\mathbf{x}, \mathbf{y} \in \mathbb{F}^{n}$ are linearly independent. Let

$$
\begin{equation*}
\mathcal{S}=\mathbf{x} \otimes \mathbf{x} \otimes \mathbf{y}+\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{x}+\mathbf{y} \otimes \mathbf{x} \otimes \mathbf{x}, \quad \mathcal{S}=\lim _{\epsilon \rightarrow 0} \frac{1}{\epsilon}\left(\otimes^{3}(\mathbf{x}+\epsilon \mathbf{y})-\otimes^{3} \mathbf{x}\right) \tag{8.2}
\end{equation*}
$$

It is straightforward to show that $\operatorname{rank} \mathcal{S}=3, \operatorname{brank}_{\mathbb{F}} \mathcal{S}=2$. (See the proof of Theorem 8.3.)

Assume that $\mathcal{S} \in \mathrm{S}^{d} \mathbb{F}^{n} \backslash\{0\}$. Then the symmetric border rank of $\mathcal{S}$, denoted as $\operatorname{sbrank}_{\mathbb{F}} \mathcal{S}$, is $r \in \mathbb{N}$ if the following conditions hold

1. There exists a sequence $\mathcal{S}_{k} \in \mathrm{~S}^{d} \mathbb{F}^{n}, k \in \mathbb{N}$ such that $\operatorname{srank} \mathcal{S}_{k}=r$ for $k \in \mathbb{N}$ and $\lim _{k \rightarrow \infty} \mathcal{S}_{k}=\mathcal{S}$.
2. Assume that a sequence $\mathcal{S}_{k} \in \mathrm{~S}^{d} \mathbb{F}^{n}, k \in \mathbb{N}$ converges to $\mathcal{S}$. Then $\liminf _{k \rightarrow \infty} \operatorname{srank} \mathcal{S}_{k} \geq r$.

Clearly, $\operatorname{srank} \mathcal{S} \geq \operatorname{sbrank}_{\mathbb{F}} \mathcal{S}$ and $\operatorname{sbrank}_{\mathbb{F}} \mathcal{S} \geq \operatorname{brank}_{\mathbb{F}} \mathcal{S}$. Thus we showed

$$
\begin{equation*}
\operatorname{rank} A(\mathcal{S}) \leq \operatorname{brank}_{\mathbb{F}} \mathcal{S} \leq \operatorname{sbrank}_{\mathbb{F}} \mathcal{S} \leq \operatorname{srank} \mathcal{S} \tag{8.3}
\end{equation*}
$$

The analog of Comon's conjecture is the equality brank ${ }_{\mathbb{F}} \mathcal{S}=\operatorname{sbrank}_{\mathbb{F}} \mathcal{S}$. See [4]. The analog of Theorem 1.1 will be the following conjecture:

Conjecture 8.2 Let $d \geq 3, \mathbb{F}=\mathbb{R}, \mathbb{C}$ and $\mathcal{S} \in \mathrm{S}^{d} \mathbb{F}^{n}$. Suppose that brank $_{\mathbb{F}} \mathcal{S}<$ $\operatorname{rank} \mathcal{S}$ and $\operatorname{brank}_{\mathbb{F}} \mathcal{S} \leq \operatorname{rank} A(\mathcal{S})+1$. Then $\operatorname{sbrank}_{\mathbb{F}} \mathcal{S}=\operatorname{brank}_{\mathbb{F}} \mathcal{S}$.

The following theorem proves the first nontrivial case of this conjecture:
Theorem 8.3 Let $\mathcal{S} \in \mathrm{S}^{d} \mathbb{F}^{n}$ for $\mathbb{F}=\mathbb{R}, \mathbb{C}, d \geq 3, n \geq 2$. Then brank $_{\mathbb{F}} \mathcal{S}=2<$ rank $\mathcal{S}$ if and only if there exist two linearly independent $\mathbf{x}, \mathbf{y} \in \mathbb{F}^{n}$ and $a, b \in \mathbb{F}, b \neq 0$ such that

$$
\begin{equation*}
\mathcal{S}=a \otimes^{d} \mathbf{x}+b \sum_{j=0}^{d-1}\left(\otimes^{j} \mathbf{x}\right) \otimes \mathbf{y} \otimes\left(\otimes^{d-j-1} \mathbf{x}\right) \tag{8.4}
\end{equation*}
$$

In particular $\operatorname{brank}_{\mathbb{F}} \mathcal{S}=\operatorname{sbrank}_{\mathbb{F}} \mathcal{S}$.

### 8.2 Proof of Theorem 8.3

Lemma 8.4 Let $\mathbb{F}=\mathbb{R}, \mathbb{C}$ and assume that $A=\left[a_{i, j}\right]_{i \in[M], j \in[N]} \in \mathbb{F}^{M \times N}, r=$ rank $A$. Suppose that the sequence $A_{k}=\sum_{i=1}^{q} \mathbf{x}_{i, k} \mathbf{y}_{i, k}^{\top}, k \in \mathbb{N}$ satisfies the following conditions:

$$
\begin{equation*}
\lim _{k \rightarrow \infty} A_{k}=A, \quad \lim _{k \rightarrow \infty} \mathbf{x}_{i, k}=\mathbf{x}_{i} \text { for } i \in[q] \tag{8.5}
\end{equation*}
$$

1. Assume that $q=r$. Then there exists a positive integer $K$, such that for $k \geq K$ the two sets of vectors $\mathbf{x}_{1, k}, \ldots, \mathbf{x}_{r, k}$ and $\mathbf{y}_{1, k}, \ldots, \mathbf{y}_{r, k}$ are linearly independent.
2. Assume that $\mathbf{x}_{1}, \ldots, \mathbf{x}_{q}$ are linearly independent. Then

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \mathbf{y}_{i, k}=\mathbf{y}_{i} \text { for } i \in[q] \quad \text { and } A=\sum_{i=1}^{q} \mathbf{x}_{i} \mathbf{y}_{i}^{\top} . \tag{8.6}
\end{equation*}
$$

Furthermore, dim $\operatorname{span}\left(\mathbf{y}_{1}, \ldots, \mathbf{y}_{q}\right)=r$. In particular, if $q=r$ then $\mathbf{y}_{1}, \ldots, \mathbf{y}_{r}$ are linearly independent.

Proof. Clearly, rank $A_{k} \leq q$. The first condition of (8.5) yields that $q \geq r$.

1. Suppose $q=r$. Hence rank $A_{k}=r$ for $k \geq K$. Hence the two sets of vectors $\mathbf{x}_{1, k}, \ldots, \mathbf{x}_{r, k}$ and $\mathbf{y}_{1, k}, \ldots, \mathbf{y}_{r, k}$ are linearly independent.
2. Complete $\mathbf{x}_{1}, \ldots, \mathbf{x}_{q}$ to a basis $\mathbf{x}_{1}, \ldots, \mathbf{x}_{M}$ in $\mathbb{F}^{M}$. Let $\mathbf{z}_{1}, \ldots, \mathbf{z}_{N}$ be a basis in $\mathbb{F}^{N}$. Hence $\mathbf{x}_{i} \mathbf{z}_{j}^{\top}, i \in[M], j \in[N]$ is a basis in $\mathbb{F}^{M \times N}$. Therefore $A_{k}=$ $\sum_{i \in[M], j \in[N]} a_{i j, k} \mathbf{x}_{i} \mathbf{z}_{j}^{\top}$ for $k \in \mathbb{N}$. The first equality of (8.5) yields that $\lim _{k \rightarrow \infty} a_{i j, k}=$ $a_{i j}$ for $i \in[M], j \in[N]$. The second equality of (8.5) yields that $\mathbf{x}_{1, k}, \ldots, \mathbf{x}_{q, k}, \mathbf{x}_{q+1}, \ldots, \mathbf{x}_{M}$ is a basis in $\mathbb{F}^{M}$ for $k \geq K$. In what follows we assume that $k \geq K$. Let $Q_{k} \in \mathbf{G L}(M, \mathbb{F})$ be the transition matrix from the basis $\left[\mathbf{x}_{1}, \ldots, \mathbf{x}_{q}, \mathbf{x}_{q+1}, \ldots, \mathbf{x}_{M}\right]$ to the basis
$\left[\mathbf{x}_{1, k}, \ldots, \mathbf{x}_{q, k}, \mathbf{x}_{q+1}, \ldots, \mathbf{x}_{M}\right]$. Clearly, $\lim _{k \rightarrow \infty} Q_{k}=I_{M}$. Then

$$
A_{k}=\sum_{i \in[M], j \in[N]} b_{i j, k} \mathbf{x}_{i, k} \mathbf{z}_{j} .
$$

Compare this equality with the assumption that $A_{k}=\sum_{i=1}^{q} \mathbf{x}_{i, k} \mathbf{y}_{i, k}^{\top}$ to deduce that $b_{i j, k}=0$ for $i>q$ and $\mathbf{y}_{i, k}=\sum_{j \in[N]} b_{i j, k} \mathbf{z}_{j}$ for $i \in[q]$. Let $\tilde{A}_{k}=\left[a_{i j, k}\right], B_{k}=\left[b_{i j, k}\right] \in$ $\mathbb{F}^{M \times N}$. Then $B_{k}=Q_{k} \tilde{A}_{k}$. Hence $\lim _{k \rightarrow \infty} B_{k}=\lim _{k \rightarrow \infty} Q_{k} A_{k}=\tilde{A}=\left[a_{i j}\right]$. This shows (8.6). Since rank $A=r$ it follows that $\operatorname{dim} \operatorname{span}\left(\mathbf{y}_{1}, \ldots, \mathbf{y}_{q}\right)=r$. Thus if $q=r \mathbf{y}_{1}, \ldots, \mathbf{y}_{r}$ are linearly independent.

Assume the assumptions of Definition 8.1. Without loss of generality we can assume that

$$
\begin{equation*}
\mathcal{T}_{k}=\sum_{i=1}^{r} \otimes_{j=1}^{d} \mathbf{x}_{i, j, k}, \quad\left\|\mathbf{x}_{i, j, k}\right\|=1, i \in[r], j \in[d-1], k \in \mathbb{N} . \tag{8.7}
\end{equation*}
$$

By considering a subsequence of $k \in \mathbb{N}$ without loss of generality we can assume that

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \mathbf{x}_{i, j, k}=\mathbf{x}_{i, j} \quad i \in[r], j \in[d-1] . \tag{8.8}
\end{equation*}
$$

(Here $\|\mathbf{x}\|$ is the Euclidean norm on $\mathbb{F}^{n}$.)
Lemma 8.5 Let $\mathcal{S} \in \mathrm{S}^{d} \mathbb{F}^{n}, d \geq 3, n \geq 2$. Assume that $1<r=\operatorname{rank} A(\mathcal{S})=$ $\operatorname{brank}_{\mathbb{F}} \mathcal{S}<\operatorname{rank} \mathcal{S}$. Let $\mathcal{T}_{k} \in \otimes^{d} \mathbb{F}^{n}, k \in \mathbb{N}$ be a sequence of the form (8.7) satisfying (8.8). Assume furthermore that $\lim _{k \rightarrow \infty} \mathcal{T}_{k}=\mathcal{S}$. Then the tensors $\otimes_{j=1}^{d-1} \mathbf{x}_{1, j}, \ldots, \otimes_{j=1}^{d-1} \mathbf{x}_{r, j}$ are linearly dependent.

Proof. Assume to the contrary that the tensors $\otimes_{j=1}^{d-1} \mathbf{x}_{1, j}, \ldots, \otimes_{j=1}^{d-1} \mathbf{x}_{r, j}$ are linearly independent. Lemma 8.4 yields that $\lim _{k \rightarrow \infty} \mathbf{x}_{i, d, k}=\mathbf{x}_{i, d}$ for $i \in[r]$. Hence $\mathcal{S}=\sum_{i=1}^{r} \otimes_{j=1}^{d} \mathbf{x}_{i, j}$. Thus rank $\mathcal{S} \leq r$ which contradicts our assumptions.

Lemma 8.6 Let $A \in \mathbb{F}^{M \times N}$ be a matrix of rank two. Assume that $A_{k}=\mathbf{a}_{k} \mathbf{b}_{k}^{\top}-$ $\mathbf{c}_{k} \mathbf{d}_{k}^{\top}, k \in \mathbb{N}$ converges to $A$. Suppose furthermore that

$$
\begin{align*}
& \lim _{k \rightarrow \infty} \frac{1}{\left\|\mathbf{a}_{k}\right\|} \mathbf{a}_{k}=\mathbf{a}, \lim _{k \rightarrow \infty} \frac{1}{\left\|\mathbf{c}_{k}\right\|} \mathbf{c}_{k}=\mathbf{c}, \mathbf{c}=\alpha \mathbf{a} \text { for }|\alpha|=1  \tag{8.9}\\
& \lim _{k \rightarrow \infty} \frac{1}{\left\|\mathbf{b}_{k}\right\|} \mathbf{b}_{k}=\mathbf{b}, \lim _{k \rightarrow \infty} \frac{1}{\left\|\mathbf{d}_{k}\right\|} \mathbf{d}_{k}=\mathbf{d}
\end{align*}
$$

Then

$$
\begin{equation*}
\mathbf{b}=\alpha \mathbf{d}, \lim _{k \rightarrow \infty}\left\|\mathbf{a}_{k}\right\|\left\|\mathbf{b}_{k}\right\|=\lim _{k \rightarrow \infty}\left\|\mathbf{c}_{k}\right\|\left\|\mathbf{d}_{k}\right\|=\infty, \lim _{k \rightarrow \infty} \frac{\left\|\mathbf{c}_{k}\right\|\left\|\mathbf{d}_{k}\right\|}{\left\|\mathbf{a}_{k}\right\|\left\|\mathbf{b}_{k}\right\|}=1 \tag{8.10}
\end{equation*}
$$

Furthermore $A=\mathbf{a f}^{\top}+\mathbf{g b}^{\top}$, where $\operatorname{span}(\mathbf{a}, \mathbf{g})=$ Range $A$ and $\operatorname{span}(\mathbf{b}, \mathbf{f})=$ Range $A^{\top}$. In particular, $\mathbf{g}$ and $\mathbf{f}$ are limits of linear combinations of $\mathbf{a}_{k}, \mathbf{c}_{k}$ and $\mathbf{b}_{k}, \mathbf{d}_{k}$ respectively.

Proof. Observe that

$$
\begin{equation*}
A_{k}^{\top}=\left(\frac{1}{\left\|\mathbf{b}_{k}\right\|} \mathbf{b}_{k}\right)\left(\left\|\mathbf{b}_{k}\right\|\left\|\mathbf{a}_{k}\right\|\right)\left(\frac{1}{\left\|\mathbf{a}_{k}\right\|} \mathbf{a}_{k}^{\top}\right)-\left(\frac{1}{\left\|\mathbf{d}_{k}\right\|} \mathbf{d}_{k}\right)\left(\left\|\mathbf{d}_{k}\right\|\left\|\mathbf{c}_{k}\right\|\right)\left(\frac{1}{\left\|\mathbf{c}_{k}\right\|} \mathbf{c}_{k}^{\top}\right) . \tag{8.11}
\end{equation*}
$$

Suppose first that $\operatorname{span}(\mathbf{b}) \neq \operatorname{span}(\mathbf{d})$. Then $\mathbf{b}$ and $\mathbf{d}$ are linearly independent. Lemma 8.4 yields that $A^{\top}=a \mathbf{b a}^{\top}+c \mathbf{d a}^{\top}$. Hence rank $A=1$ which contradicts our assumptions. As $\|\mathbf{c}\|=\|\mathbf{d}\|=1$ it follows that $\mathbf{b}=\beta \mathbf{d}$ for some scalar $\beta$ of length 1.

We next observe that $\mathbf{a} \in \operatorname{Range}(A)$ and $\mathbf{b} \in$ Range $A^{\top}$. Indeed, without loss of generality, we can assume that rank $A_{k}=2$ for $k \in \mathbb{N}$. Hence $\mathbf{a}_{k} \in \operatorname{Range}\left(A_{k}\right), \mathbf{b}_{k} \in$ Range ( $A_{k}^{\top}$ ). As $\lim _{k \rightarrow \infty} A_{k}=A$ the assumptions (8.10) yield that $\mathbf{a} \in$ Range $A, \mathbf{b} \in$ Range $A^{\top}$.

Assume that the sequence $\left\{\left\|\mathbf{a}_{k}\right\|\left\|\mathbf{b}_{k}\right\|\right\}, k \in \mathbb{N}$ contains a bounded subsequence $\left\{n_{k}\right\}, k \in \mathbb{N}$. Since $\lim _{k \rightarrow \infty} A_{k}=A$ it follows the subsequence $\left\|\mathbf{c}_{n_{k}}\right\|\left\|\mathbf{d}_{n_{k}}\right\|, k \in \mathbb{N}$ is also bounded. Taking convergent subsequences of the above two subsequences we deduce that $A=\gamma \mathbf{a b}^{\top}$. This contradicts our assumption that rank $A=2$. Hence the second equality of (8.10) holds. Rewrite (8.11) as

$$
A_{k}=\left\|\mathbf{a}_{k}\right\|\left\|\mathbf{b}_{k}\right\|\left(\left(\frac{1}{\left\|\mathbf{a}_{k}\right\|} \mathbf{a}_{k}\right)\left(\frac{1}{\left\|\mathbf{b}_{k}\right\|} \mathbf{b}_{k}^{\top}\right)-\left(\frac{\left\|\mathbf{c}_{k}\right\|\left\|\mathbf{d}_{k}\right\|}{\left\|\mathbf{a}_{k}\right\|\left\|\mathbf{b}_{k}\right\|}\right)\left(\frac{1}{\left\|\mathbf{c}_{k}\right\|} \mathbf{c}_{k}\right)\left(\frac{1}{\left\|\mathbf{d}_{k}\right\|} \mathbf{d}_{k}^{\top}\right)\right) .
$$

Use the assumptions that $\lim _{k \rightarrow \infty} A_{k}=A$, where rank $A=2$, the facts that $\|\mathbf{a}\|=$ $\|\mathbf{b}\|=\|\mathbf{c}\|=\|\mathbf{d}\|=1$ and $\mathbf{c}=\alpha \mathbf{a}, \mathbf{b}=\beta \mathbf{d}$ to deduce the third and the the first part of (8.10).

It is left to show that $A=\mathbf{a f}^{\top}+\mathbf{g b}^{\top}$. Choose orthonormal bases $\mathbf{x}_{1}, \ldots, \mathbf{x}_{N}$ and $\mathbf{y}_{1}, \ldots, \mathbf{y}_{N}$ in $\mathbb{F}^{M}$ and $\mathbb{F}^{N}$ respectively with the following properties:

$$
\mathbf{x}_{1}=\mathbf{a}, \operatorname{span}\left(\mathbf{x}_{1}, \mathbf{x}_{2}\right)=\text { Range } A, \mathbf{y}_{1}=\mathbf{b}, \operatorname{span}\left(\mathbf{y}_{1}, \mathbf{y}_{2}\right)=\text { Range } A^{\top} .
$$

In what follows we assume that $k \gg 1$. Choose orthonormal bases $\mathbf{x}_{1, k}, \mathbf{x}_{2, k}$ and $\mathbf{y}_{1, k}, \mathbf{y}_{2, k}$ in Range $A_{k}$ and Range $A_{k}^{\top}$ respectively such that

$$
\mathbf{x}_{1, k}=\frac{1}{\left\|\mathbf{a}_{k}\right\|} \mathbf{a}_{k}, \mathbf{y}_{1, k}=\frac{1}{\left\|\mathbf{b}_{k}\right\|} \mathbf{b}_{k}, \lim _{k \rightarrow \infty} \mathbf{x}_{2, k}=\mathbf{x}_{2}, \lim _{k \rightarrow \infty} \mathbf{y}_{2, k}=\mathbf{y}_{2} .
$$

Observe next that $\left\{\mathbf{x}_{1, k}, \mathbf{x}_{2, k}, \mathbf{x}_{3}, \ldots, \mathbf{x}_{M}\right\}$ and $\left\{\mathbf{y}_{1, k}, \mathbf{y}_{2, k}, \mathbf{y}_{3}, \ldots, \mathbf{y}_{N}\right\}$ are bases in $\mathbb{F}^{M}$ and $\mathbb{F}^{N}$ which converge to bases $\mathbf{x}_{1}, \ldots, \mathbf{x}_{N}$ and $\mathbf{y}_{1}, \ldots, \mathbf{y}_{N}$ respectively.

In the bases $\left\{\mathbf{x}_{1, k}, \mathbf{x}_{2, k}, \mathbf{x}_{3}, \ldots, \mathbf{x}_{M}\right\}$ and $\left\{\mathbf{y}_{1, k}, \mathbf{y}_{2, k}, \mathbf{y}_{3}, \ldots, \mathbf{y}_{N}\right\}$ the rank one matrices $\mathbf{a}_{k} \mathbf{b}_{k}^{\top}, \mathbf{c}_{k} \mathbf{d}_{k}^{\top}$ are represented by the following block diagonal matrices: $\tilde{C}_{k}=$ $C_{k} \oplus 0, \tilde{D}_{k}=D_{k} \oplus 0$ where

$$
C_{k}=\left[\begin{array}{cc}
a_{k} & 0 \\
0 & 0
\end{array}\right], \quad D_{k}=\left[\begin{array}{cc}
b_{k} & c_{k} \\
d_{k} & e_{k}
\end{array}\right] .
$$

Note that $a_{k}=\left\|\mathbf{a}_{k}\right\|\left\|\mathbf{b}_{k}\right\|$. Hence $\lim _{k \rightarrow \infty} a_{k}=\infty$. As $\lim _{k \rightarrow \infty} A_{k}=A$ the arguments of the proof of Lemma 8.4 yield that $\lim _{k \rightarrow \infty} C_{k}-D_{k}=E=\left[e_{i j}\right] \in \mathbb{F}^{2 \times 2}$. Hence $\lim _{k \rightarrow \infty} b_{k}=\infty$. Therefore $D_{k}=b_{k} \mathbf{u}_{k} \mathbf{v}_{k}^{\top}$, where $\mathbf{u}_{k}^{\top}=\left(1, u_{k}\right), \mathbf{v}_{k}^{\top}=\left(1, v_{k}\right)$. Furthermore

$$
\lim _{k \rightarrow \infty} b_{k} u_{k}=e_{21} \Rightarrow \lim _{k \rightarrow \infty} u_{k}=0, \quad \lim _{k \rightarrow \infty} b_{k} v_{k}=e_{12} \Rightarrow \lim _{k \rightarrow \infty} v_{k}=0
$$

Finally, observe that $e_{22}=\lim _{k \rightarrow \infty} b_{k} u_{k} v_{k}=0$. This yields that $E=(1,0)^{\top} \mathbf{u}^{\top}+$ $\mathbf{v}(1,0)$ for some choice of $\mathbf{u}, \mathbf{v} \in \mathbb{F}^{2}$. Hence $A=\mathbf{a f}^{\top}+\mathbf{g b}^{\top}$ as claimed.

As rank $A_{k}=2$ and $\lim _{k \rightarrow \infty} A_{k}=A$ it follows that $\mathbf{g}$ and $\mathbf{f}$ are limits of linear combinations of $\mathbf{a}_{k}, \mathbf{c}_{k}$ and $\mathbf{b}_{k}, \mathbf{d}_{k}$ respectively.

Proof of Theorem 8.3. Assume first that $\mathcal{S}$ is of the form (8.4) where $b \neq 0$. Without loss of generality we can assume that $b=1$. Clearly, Range $A(\mathcal{S})=$ $\operatorname{span}(\mathbf{x}, \mathbf{y})$ Hence $\operatorname{rank} A(\mathcal{S})=2$. Let $\mathcal{T}(\epsilon):=a \otimes^{d} \mathbf{x}+\frac{1}{\epsilon}\left(\otimes^{d}(\mathbf{x}+\epsilon \mathbf{y})-\otimes^{d} \mathbf{x}\right)$ for $\epsilon \neq 0$. Then $\operatorname{rank} \mathcal{T}(\epsilon)=2$ for $\epsilon^{-1} \neq a$. Clearly, $\lim _{\epsilon \rightarrow 0} \mathcal{T}(\epsilon)=\mathcal{S}$. Hence $\operatorname{brank}_{\mathbb{F}} \mathcal{S}=2$. As $\mathcal{T}(\epsilon) \in \mathrm{S}^{d} \mathbb{F}^{n}$ it follows that $\operatorname{sbrank}_{\mathbb{F}} \mathcal{S}=2$. We claim that rank $\mathcal{S}>2$. We can assume without loss of generality that $n=2$ and $\mathbf{x}=\mathbf{e}_{1}=(1,0)^{\top}, \mathbf{y}=\mathbf{e}_{2}=(0,1)^{\top}$. Assume first that $d=3$. So $\mathcal{S}=\left[s_{i, j, k}\right]$ where

$$
s_{1,1,1}=a, s_{1,1,2}=s_{1,2,1}=s_{2,1,1}=1, s_{1,2,2}=s_{2,1,2}=s_{2,2,1}=s_{2,2,2}=0
$$

Let

$$
F=\left[s_{i, j, 1}\right]_{i, j \in[2]}=\left[\begin{array}{ll}
a & 1 \\
1 & 0
\end{array}\right], \quad G=\left[s_{i, j, 2}\right]_{i, j \in[2]}=\left[\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right] .
$$

Then $\operatorname{rank} \mathcal{S}=2$ if and only if the matrix $G F^{-1}$ is diagonalizable, see e.g. [13]. Clearly, $G F^{-1}=\left[\begin{array}{ll}0 & 1 \\ 0 & 0\end{array}\right]$ is not diagonalizable. Hence $\operatorname{rank} \mathcal{S}>2$. It is easy to show straightforward that $\operatorname{rank} \mathcal{S}=3$.

Assume now that $d>3$. Let $\phi: \mathbb{F}^{2} \rightarrow \mathbb{F}$ be the linear functional such that $\phi\left(\mathbf{e}_{1}\right)=\phi\left(\mathbf{e}_{2}\right)=1$. Consider the following map $\psi:\left(\mathbb{F}^{2}\right)^{d} \rightarrow \otimes^{3} \mathbb{F}^{2}: \psi\left(\left(\mathbf{u}_{1}, \ldots, \mathbf{u}_{d}\right)\right)=$ $\left(\prod_{j=4}^{d} \phi\left(\mathbf{u}_{j}\right)\right) \otimes_{i=1}^{3} \mathbf{u}_{j}$. Clearly, $\psi$ is a multilinear map. The universal lifting property of the tensor product yields that $\psi$ lifts to the linear map $\Psi: \otimes^{d} \mathbb{F}^{2} \rightarrow \otimes^{3} \mathbb{F}^{2}$ such that

$$
\Psi\left(\otimes_{j=1}^{d} \mathbf{u}_{i}\right)=\left(\prod_{j=4}^{d} \phi\left(\mathbf{u}_{j}\right)\right) \otimes_{i=1}^{3} \mathbf{u}_{j}
$$

Observe that a rank one tensor is mapped to either rank one tensor or zero tensor. Clearly, the image of a symmetric rank one tensor is a symmetric tensor of at most
rank one. Hence $\Psi: \mathrm{S}^{d} \mathbb{F}^{2} \rightarrow \mathrm{~S}^{3} \mathbb{F}^{2}$. Assume that $\mathcal{S}$ of the form (8.4), where $\mathbf{x}=\mathbf{e}_{1}$ and $\mathbf{y}=\mathbf{e}_{2}$. Then

$$
\Psi(\mathcal{S})=(a+(d-3) b) \mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x}+b(\mathbf{y} \otimes \mathbf{x} \otimes \mathbf{x}+\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{x}+\mathbf{x} \otimes \mathbf{x} \otimes \mathbf{y}) .
$$

Assume to the contrary that $\operatorname{rank} \mathcal{S}=2$. Then $\operatorname{rank} \Psi(\mathcal{S}) \leq 2$. This contradicts our proof that $\operatorname{rank} \Psi(\mathcal{S})=3$. Hence $\operatorname{rank} \mathcal{S} \geq 3$.

Assume now that $\mathcal{S} \in \mathrm{S}^{d} \mathbb{F}^{n}$ and $2=\operatorname{rank} A(\mathcal{S})=\operatorname{brank}_{\mathbb{F}} \mathcal{S}<\operatorname{rank} \mathcal{S}$. Let $\mathcal{T}_{k} \in \otimes^{d} \mathbb{F}^{n}$ be a sequence of tensors of rank two converging to $\mathcal{S}$. So $\mathcal{T}_{k}=\otimes_{j=1}^{d} \mathbf{x}_{j, k}-$ $\otimes_{j=1}^{d} \mathbf{y}_{j, k}$. Since rank $A(\mathcal{S})=2$ we can assume without loss of generality: First, $\mathbf{x}_{j, k}$ and $\mathbf{y}_{j, k}$ are linearly independent for $j \in[d], k \in \mathbb{N}$. Second,

$$
\lim _{k \rightarrow \infty} \frac{1}{\left\|\mathbf{x}_{j, k}\right\|} \mathbf{x}_{j, k}=\mathbf{x}_{j}, \quad \lim _{k \rightarrow \infty} \frac{1}{\left\|\mathbf{y}_{j, k}\right\|} \mathbf{y}_{j, k}=\mathbf{y}_{j} \text { for } j \in[d] .
$$

Lemma 8.5 yields that $\otimes_{j=1}^{d-1} \mathbf{x}_{j}$ and $\otimes_{j=1}^{d-1} \mathbf{y}_{j}$ are linearly dependent. Hence span $\left(\mathbf{x}_{j}\right)=$ $\operatorname{span}\left(\mathbf{y}_{j}\right)$ for $j \in[d-1]$. Lemma 8.6 yields that $\mathbf{x}_{d}$ and $\mathbf{y}_{d}$ linearly dependent. So $\operatorname{span}\left(\mathbf{x}_{d}\right)=\operatorname{span}\left(\mathbf{y}_{d}\right)$. Apply Lemma 8.6 to $A=A(\mathcal{S})^{\top}, A_{k}=A\left(\mathcal{T}_{k}\right)^{\top}, k \in \mathbb{N}$. It then follows that $\mathcal{S}=\left(\otimes_{j=1}^{d-1} \mathbf{x}_{j}\right) \otimes \mathbf{z}+\mathcal{F} \otimes \mathbf{x}_{d}$ for some $\mathcal{F} \in \otimes^{d-1} \mathbb{F}^{n}$. Furthermore $\operatorname{span}\left(\mathbf{x}_{d}, \mathbf{z}\right)=$ Range $A(\mathcal{S})$. As $\mathbf{x}_{d}$ and $\mathbf{z}$ are linearly independent and $\mathcal{S}$ symmetric it follow that $\otimes_{j=1}^{d-1}, \mathcal{F} \in \mathrm{~S}^{d-1} \mathbb{F}^{n}$. Hence $\operatorname{span}\left(\mathrm{x}_{1}\right)=\cdots=\operatorname{span}\left(\mathrm{x}_{d-1}\right)=\operatorname{span}(\mathbf{x})$. Thus $\otimes_{j=1}^{d-1} \mathbf{x}_{j}=t \otimes^{d-1} \mathbf{x}$. By considering the unfolding of $\mathcal{S}$ in another mode we deduce that $\operatorname{span}\left(\mathbf{x}_{d}\right)=\operatorname{span}(\mathbf{x})$. Observe next that $\operatorname{rank} \mathcal{F}>1$. Otherwise rank $\mathcal{S} \leq 2$ which contradicts our assumptions. Lemma 8.6 yields that $\mathcal{F}$ is a limit of linear combinations of $\otimes_{j=1}^{d-1} \mathbf{x}_{j, k}$ and $\otimes_{j=1}^{d-1} \mathbf{y}_{j, k}$. Hence $\operatorname{brank}_{\mathbb{F}} \mathcal{F} \leq 2$. As rank $\mathcal{F}>1$ it follows that $\operatorname{brank}_{\mathbb{F}} \mathcal{F}=2$. In summary we showed:

$$
\begin{align*}
& \mathcal{S}=\otimes^{d-1} \mathbf{x} \otimes \mathbf{z}+\mathcal{F} \otimes \mathbf{x},  \tag{8.12}\\
& \operatorname{span}(\mathbf{x}, \mathbf{z})=\operatorname{Range} A(\mathcal{S}), \mathcal{F} \in \mathrm{S}^{d-1} \mathbb{F}^{n}, \operatorname{brank}_{\mathbb{F}} \mathcal{F}=2
\end{align*}
$$

We now prove the following claim: Assume that $\mathcal{S} \in \mathrm{S}^{d} \mathbb{F}^{n}, \operatorname{rank} A(\mathcal{S})=2<$ $\operatorname{rank} \mathcal{S}$. Suppose furthermore $\mathcal{S}$ is a limit of linear linear combinations of $\otimes_{j=1}^{d} \mathbf{x}_{j, k}, \otimes_{j=1}^{d} \mathbf{y}_{j, k}$, where the the following limit exist and satisfy:

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \frac{1}{\left\|\mathbf{x}_{j, k}\right\|} \mathbf{x}_{j, k}, \lim _{k \rightarrow \infty} \frac{1}{\left\|\mathbf{y}_{j, k}\right\|} \mathbf{y}_{j, k} \in \operatorname{span}(\mathbf{x}) \text { for } j \in[d] . \tag{8.13}
\end{equation*}
$$

Then (8.4) holds.
We prove the claim by induction on $d$. Assume first that $d=3$. Observe first that two dimensional subspace Range $A(\mathcal{S})^{\top}=\operatorname{span}(\mathbf{a}, \mathbf{c})$, as given by Lemma 8.6, is in $\mathrm{S}^{2} \mathbb{F}^{n}$. i.e. the space of symmetric matrices. Lemma 8.6 yields that $\mathcal{F}$ is a limit of linear combinations of $\mathbf{x}_{1, k} \otimes \mathbf{x}_{2, k}$ and $\mathbf{y}_{1, k} \otimes \mathbf{y}_{2, k}$. As $\lim _{k \rightarrow \infty} \frac{1}{\left\|\mathbf{x}_{1, k}\right\|\left\|\mathbf{x}_{2, k}\right\|} \mathbf{x}_{1, k} \otimes \mathbf{x}_{2, k}=$ $t \mathbf{x} \otimes \mathbf{x}$ it follows that Range $A(\mathcal{S})^{\top}$ contains rank one matrix $\mathbf{x} \otimes \mathbf{x}$. Lemma 8.6 yields that Range $A(\mathcal{S})^{\top}$ contains a rank two matrix of the form $\mathbf{x} \otimes \mathbf{f}+\mathbf{g} \otimes \mathbf{x}$. Since this matrix is symmetric it is of the form $c \mathbf{x} \otimes \mathbf{x}+\mathbf{x} \otimes \mathbf{u}+\mathbf{u} \otimes \mathbf{x}$ for some scalar $c$ and $\mathbf{u} \in \mathbb{F}^{n}$ which is linearly independent of $\mathbf{x}$. Hence $\mathcal{F}=d \mathbf{x} \otimes \mathbf{x}+\mathbf{x} \otimes \mathbf{v}+\mathbf{v} \otimes \mathbf{x}$ for $\mathbf{v}=d \mathbf{u}, d \neq 0$. As rank $A(\mathcal{S})=2$ it follows that $\operatorname{span}(\mathbf{x}, \mathbf{v})=\operatorname{span}(\mathbf{x}, \mathbf{z})$. Therefore we showed that

$$
\mathcal{S}=a \mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x}+b \mathbf{x} \otimes \mathbf{x} \otimes \mathbf{v}+c(\mathbf{x} \otimes \mathbf{v}+\mathbf{v} \otimes \mathbf{x}) \otimes \mathbf{x}
$$

Interchange the last two factors in $\mathcal{S}$ to deduce that $0=(b-c)(\mathbf{x} \otimes \mathbf{x} \otimes \mathbf{v}-\mathbf{x} \otimes \mathbf{v} \otimes \mathbf{x})$. Hence $b=c$ and (8.4) holds for $d=3$.

Assume now that (8.4) holds for $d=p$ and suppose that $d=p+1$. Consider (8.12). Suppose first that $2<\operatorname{rank} \mathcal{F}$. Then the induction hypothesis applies to $\mathcal{F}$. Hence $\mathcal{F}$ is of the form (8.4) and

$$
\mathcal{S}=\otimes^{p} \mathbf{x} \otimes \mathbf{z}+\left(a \otimes^{p} \mathbf{x}+b \sum_{j=0}^{p-1} \otimes^{j} \mathbf{x} \otimes \mathbf{y} \otimes\left(\otimes^{p-1-j} \mathbf{x}\right)\right) \otimes \mathbf{x}, b \neq 0
$$

Note that Range $A(\mathcal{S})=\operatorname{span}(\mathbf{x}, \mathbf{y})=\operatorname{span}(\mathbf{x}, \mathbf{z})$. Hence

$$
\mathcal{S}=a^{\prime} \otimes^{p+1} \mathbf{x}+c \otimes^{p} \mathbf{x} \otimes \mathbf{y}+b \sum_{j=0}^{p-1} \otimes^{j} \mathbf{x} \otimes \mathbf{y} \otimes\left(\otimes^{p-j} \mathbf{x}\right)
$$

Interchange the last two factors in $\mathcal{S}$ to deduce that $(c-b) \otimes^{p-1} \mathbf{x}(\mathbf{x} \otimes \mathbf{y}-\mathbf{y} \otimes \mathbf{x})=0$. Hence $b=c$ and $\mathcal{S}$ is of the form (8.4).

It is left to consider the case where $\mathcal{F}$ is a symmetric tensor of rank two. So $\operatorname{rank} \mathcal{F}=2$. Hence $\operatorname{rank} A(\mathcal{F})=2$. Theorem 3.1 yields that $\mathcal{F}=s \otimes^{p} \mathbf{u}+$ $t \otimes^{d} \mathbf{v}$, were $s, t= \pm 1$, and this decomposition is unique. (The $\pm$ are needed if $\mathbb{F}=\mathbb{R}$ and $p$ is even.) Clearly, $\operatorname{span}(\mathbf{u}, \mathbf{v})=$ Range $A(\mathcal{S})$. It is enough to assume that $n=2$. Recall that $A(\mathcal{F})$ is a limit of a linear combinations of two rank one matrices: $\mathbf{x}_{1, k}\left(\otimes_{j=2}^{p} \mathbf{x}_{j, k}\right)^{\top}, \mathbf{y}_{1, k}\left(\otimes_{j=2}^{p} \mathbf{y}_{j, k}\right)^{\top}, k \in \mathbb{N}$. The assumption (8.13) implies that we can use Lemma 8.6. Hence $\mathcal{F}=\mathbf{x} \otimes \mathcal{G}+\mathbf{g} \otimes\left(\otimes^{p-1} \mathbf{x}\right)$. Therefore $\otimes^{p-1} \mathbf{x} \in$ $\operatorname{span}\left(\otimes^{p-1} \mathbf{u}, \otimes^{p-1} \mathbf{v}\right)$. We claim that this possible if and only if either $\operatorname{span}(\mathbf{x})=$ $\operatorname{span}(\mathbf{u})$ or $\operatorname{span}(\mathbf{x})=\operatorname{span}(\mathbf{v})$. Suppose to the contrary that $\operatorname{span}(\mathbf{x}) \neq \operatorname{span}(\mathbf{u})$ or $\operatorname{span}(\mathbf{x}) \neq \operatorname{span}(\mathbf{v})$. So $\otimes^{p-1} \mathbf{x}=s \otimes^{p-1} \mathbf{u}+t \otimes^{p-1} \mathbf{v}$. Clearly st $\neq 0$. Let $\phi: \mathbb{F}^{2} \rightarrow \mathbb{F}$ be a nonzero linear functional such that $\phi(\mathbf{x})=0$. Let $\Psi: \otimes^{p-1} \mathbb{F}^{2} \rightarrow \otimes^{p-2} \mathbb{F}^{2}$ be the linear mapping of the form given above: $\Psi\left(\otimes_{j=1}^{p-1} \mathbf{w}_{j}\right)=\phi\left(\mathbf{w}_{1}\right) \otimes_{j=2}^{p-1} \mathbf{w}_{j}$. So

$$
0=\Psi\left(\otimes^{p-1} \mathbf{x}\right)=s \phi(\mathbf{u}) \otimes^{p-2} \mathbf{u}+t \phi(\mathbf{v}) \otimes^{p-2} \mathbf{v}
$$

This is impossible since $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. Thus we can assume that $\operatorname{span}(\mathbf{v})=\operatorname{span}(\mathbf{x}), \mathcal{T}=s \otimes^{p} \mathbf{u}+t^{\prime} \otimes^{p} \mathbf{x}$ and $\mathbf{z} \in \operatorname{span}(\mathbf{u}, \mathbf{x})$. Thus

$$
\mathcal{S}=a \otimes^{p+1} \mathbf{x}+b \otimes^{p} \mathbf{x} \otimes \mathbf{u}+s \otimes^{p} \mathbf{u} \otimes \mathbf{x}
$$

As rank $\mathcal{S}=3$ it follows that $\mathcal{S}^{\prime}:=\mathcal{S}-a \otimes^{p+1} \mathbf{x}$ is a symmetric tensor of rank two. Theorem 3.1 claims that the decomposition $\mathcal{S}^{\prime}=b \otimes^{p} \mathbf{x} \otimes \mathbf{u}+s \otimes^{p} \mathbf{u} \otimes \mathbf{x}$ is unique and $\otimes^{p} \mathbf{x} \otimes \mathbf{u}, \otimes^{p} \mathbf{u} \otimes \mathbf{x}$ are symmetric tensors. So $\operatorname{span}(\mathbf{u})=\operatorname{span}(\mathbf{x})$ which contradicts our assumption that $\mathbf{u}$ and $\mathbf{x}$ are linearly independent.

### 8.3 Approximation of symmetric tensors

Define on $\otimes^{d} \mathbb{F}^{n}$ the standard inner product:

$$
\langle\mathcal{P}, \mathcal{Q}\rangle:=\sum_{i_{j} \in[n], j \in[d]} p_{i_{1}, \ldots, i_{d}} \overline{q_{i_{1}, \ldots, i_{d}}}, \quad \mathcal{P}=\left[p_{\left.i_{1}, \ldots, i_{d}\right]}\right], \mathcal{Q}=\left[q_{i_{1}, \ldots, i_{p}}\right] \in \otimes^{d} \mathbb{F}^{n}
$$

Observe that $\left\langle\otimes_{j=1}^{d} \mathbf{x}_{j}, \otimes_{j=1}^{d} \mathbf{y}_{j}\right\rangle=\prod_{j=1}^{d}\left\langle\mathbf{x}_{j}, \mathbf{y}_{j}\right\rangle$. Assume that $k \in[1, d-1]$. Denote by $\operatorname{Gr}\left(k, \mathbb{F}^{n}\right)$ the Grasmannian manifold of $k$-dimensional subspace in $\mathbb{F}^{n}$. Let

$$
\operatorname{Gr}\left(k, d, \mathbb{F}^{n}\right):=\left\{\otimes_{j=1}^{d} \mathbf{U}_{j}, \quad \mathbf{U}_{j} \in \operatorname{Gr}\left(k, \mathbb{F}^{n}\right), j \in[d]\right\} .
$$

For a given $\otimes_{j=1}^{d} \mathbf{U}_{j}$ denote by $P_{\otimes_{j=1}^{d} \mathbf{U}_{j}}: \otimes^{d} \mathbb{F}^{n} \rightarrow \otimes_{j=1}^{d} \mathbf{U}_{j}$ the orthogonal projection of $\otimes^{d} \mathbb{F}^{n}$ on $\otimes_{j=1}^{d} \mathbf{U}_{j}$. A best $k$-approximation of $\mathcal{T} \in \otimes^{d} \mathbb{F}^{n}$ is each tensor $\mathcal{T}^{\star}$ satisfying
$\min _{\otimes_{j=1}^{d} \mathbf{U}_{j} \in \operatorname{Gr}\left(k, d, \mathbb{F}^{n}\right)}\left\|\mathcal{T}-P_{\otimes_{j=1}^{d} \mathbf{U}_{j}}(\mathcal{T})\right\|=\left\|\mathcal{T}-\mathcal{T}^{\star}\right\|, \mathcal{T}^{\star}=P_{\otimes_{j=1}^{d} \mathbf{U}_{j}^{\star}}(\mathcal{T}), \otimes_{j=1}^{d} \mathbf{U}_{j}^{\star} \in \operatorname{Gr}\left(k, d, \mathbb{F}^{n}\right)$.
See [14, 16]. The results of [15] yield that $\mathcal{T}^{\star}$ is unique for $\mathcal{T}$ outside of a semialgebraic set of dimension less than the real dimension of $\otimes^{d} \mathbb{F}^{n}$. The analog of Comon's conjecture is:

Conjecture 8.7 Let $n-1, d-1 \in \mathbb{N}, k \in[d-1]$ and $\mathcal{S} \in \mathrm{S}^{d} \mathbb{F}^{n}$. Then a best $k$-approximation of $\mathcal{S}$ can be chosen to be a symmetric tensor.

This conjecture is known to hold in the following cases: For $d=2$ it is a consequence of Singular Value Decomposition. For $k=1$ and $d>2$ it follows from Banach's theorem [2]. See [13] for $\mathbb{F}=\mathbb{R}$. Similar arguments combined with Banach's theorem yield the case $\mathbb{F}=\mathbb{C}$. It is shown in $[14]$ that for $\mathbb{F}=\mathbb{R}$ there is a semi-algebraic set in $S^{d} \mathbb{R}^{n}$ of dimension $\operatorname{dim} S^{d} \mathbb{R}^{n}$ for which the conjecture holds.

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