# Degenerate SDE with Hölder-Dini Drift and Non-Lipschitz Noise Coefficient * 

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#### Abstract

The existence-uniqueness and stability of strong solutions are proved for a class of degenerate stochastic differential equations, where the noise coeffcicient might be non-Lipschitz, and the drift is locally Dini continuous in the component with noise (i.e. the second component) and locally Hölder-Dini continuous of order $\frac{2}{3}$ in the first component. Moreover, the weak uniqueness is proved under weaker conditions on the noise coefficient. Furthermore, if the noise coefficient is $C^{1+\varepsilon}$ for some $\varepsilon>0$ and the drift is Hölder continuous of order $\alpha \in\left(\frac{2}{3}, 1\right)$ in the first component and order $\beta \in(0,1)$ in the second, the solution forms a $C^{1}$-stochastic diffeormorphism flow. To prove these results, we present some new characterizations of Hölder-Dini space by using the heat semigroup and slowly varying functions.


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## 1 Introduction

Consider the following ordinary differential equation (abbreviated as ODE):

$$
\dot{x}(t)=b(x(t)), x(0)=x_{0} .
$$

It is classical that the equation is well-posed for Lipschitz $b$ but usually ill-posed if $b$ is only Hölder continuous. For instance, for $b(x):=|x|^{\alpha}$ with $\alpha \in(0,1)$ and $x_{0}=0$, the above ODE has two solutions: $x(t) \equiv 0$ and $x(t)=(1-\alpha) t^{1 /(1-\alpha)}, t \geqslant 0$. However, if the above ODE

[^0]is perturbed by a strong enough noise (e.g. the Browian motion), the equation might be well-posed for very singular $b$. For instance, consider the following SDE on $\mathbb{R}^{d}$ :
$$
\mathrm{d} X_{t}=b_{t}\left(X_{t}\right) \mathrm{d} t+\sigma \mathrm{d} W_{t}, \quad X_{0}=x
$$
where $W_{t}$ is a $d$-dimensional standard Brownian motion on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$, $\sigma$ is an invertible matrix. If $b$ is a bounded measurable function, Veretennikov [22] proved that the above SDE admits a unique strong solution, which extended an earlier result of Zvonkin [32] in the case of $d=1$. More recent results about the above SDE can be found in [9, 14, 30] and references therein for further development in this direction.

It is worthy noticing that all the well-posedness results mentioned above are done only for the time-white noise, which means that the noise is a distribution of the time variable. In this work, we are concerning with the following problem: Is it possible to prove the well-posedness of the ODE with singular $b$ perturbed by an absolutely continuous Gaussian process? More concretely, consider the following random ODE:

$$
\begin{equation*}
\mathrm{d} X_{t}=\left[b_{t}\left(X_{t}\right)+\sigma W_{t}\right] \mathrm{d} t, \quad X_{0}=x \tag{1.1}
\end{equation*}
$$

We aim to find minimal conditions on $b$ and $\sigma$ ensuring the well-posedness of this random ODE. By regarding $X_{t}$ as the first component process $X_{t}^{(1)}$ and introducing $X_{t}^{(2)}:=\sigma W_{t}$, this problem is reduced to the study of the following more general degenerate SDE for $X_{t}:=\left(X_{t}^{(1)}, X_{t}^{(2)}\right)$ on $\mathbb{R}^{d_{1}+d_{2}}=\mathbb{R}^{d_{1}} \times \mathbb{R}^{d_{2}}$ :

$$
\begin{equation*}
\mathrm{d} X_{t}=b_{t}\left(X_{t}\right) \mathrm{d} t+\left(0, \sigma_{t}\left(X_{t}\right) \mathrm{d} W_{t}\right), \quad X_{0}=x=\left(x^{(1)}, x^{(2)}\right) \in \mathbb{R}^{d_{1}+d_{2}} \tag{1.2}
\end{equation*}
$$

where, for $\mathbb{R}_{+}:=(0, \infty)$, the maps $\sigma: \mathbb{R}_{+} \times \mathbb{R}^{d_{1}+d_{2}} \rightarrow \mathbb{R}^{d_{2}} \otimes \mathbb{R}^{d_{2}}$ and $b=\left(b^{(1)}, b^{(2)}\right):$ $\mathbb{R}_{+} \times \mathbb{R}^{d_{1}+d_{2}} \rightarrow \mathbb{R}^{d_{1}+d_{2}}$ are measurable and locally bounded. This model is known as the stochastic Hamiltonian system with potential $H$ if $b=\nabla H$, which includes the kinetic Fokker-Planck equation as a typical example (see [23]).

In the following, we will use $\nabla^{(1)}$ and $\nabla^{(2)}$ to denote the gradient operators on the first space $\mathbb{R}^{d_{1}}$ and the second space $\mathbb{R}^{d_{2}}$ respectively. Thus, for every $(t, x) \in \mathbb{R}_{+} \times \mathbb{R}^{d_{1}+d_{2}}$, $\nabla^{(2)} b_{t}^{(1)}(x) \in \mathbb{R}^{d_{2}} \otimes \mathbb{R}^{d_{1}}$ with $\left(\nabla^{(2)} b_{t}^{(1)}(x)\right) h:=\nabla_{h}^{(2)} b_{t}^{(1)}(x) \in \mathbb{R}^{d_{1}}, h \in \mathbb{R}^{d_{2}}$. By Itô's formula, the infinitesimal generator associated to (1.2) is given by

$$
\begin{equation*}
\mathscr{L}_{t}^{\Sigma, b} u=\operatorname{tr}\left(\Sigma_{t} \cdot \nabla^{(2)} \nabla^{(2)} u\right)+b_{t} \cdot \nabla u \tag{1.3}
\end{equation*}
$$

where $\Sigma_{t}(x):=\frac{1}{2} \sigma_{t}(x) \sigma_{t}^{*}(x)$ and $\operatorname{tr}(\cdot)$ denotes the trace of a matrix.
Let $|\cdot|$ denote the Euclidiean norm and let $\|\cdot\|$ denote the operator norm. We introduce below the notion of Hölder-Dini continuity.

Definition 1.1. An increasing function $\phi: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$is called a Dini function if

$$
\begin{equation*}
\int_{0}^{1} \frac{\phi(t)}{t} \mathrm{~d} t<\infty \tag{1.4}
\end{equation*}
$$

A measurable function $\phi: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$is called a slowly varying function at zero if for any $\lambda>0$,

$$
\begin{equation*}
\lim _{t \rightarrow 0} \frac{\phi(\lambda t)}{t}=1 \tag{1.5}
\end{equation*}
$$

A function $f$ on the Euclidiean space is called Hölder-Dini continuous of order $\alpha \in[0,1)$ if

$$
|f(x)-f(y)| \leqslant|x-y|^{\alpha} \phi(|x-y|), \quad|x-y| \leqslant 1
$$

holds for some Dini function $\phi$, and is called Dini-continuous if this condition holds for $\alpha=0$.
Let $\mathscr{D}_{0}$ be the set of all Dini functions, and $\mathscr{S}_{0}$ the set of all slowly varying functions that are bounded from 0 and $\infty$ on $[\varepsilon, \infty)$ for any $\varepsilon>0$. Notice that the typical examples in $\mathscr{D}_{0} \cap \mathscr{S}_{0}$ are $\phi(t):=\left(\log \left(1+t^{-1}\right)\right)^{-\beta}$ for $\beta>1$.

Roughly speaking, for the existence and uniqueness of the solutions to (1.2), we will need $b^{(1)}\left(\cdot, x^{(2)}\right)$ and $b^{(2)}\left(\cdot, x^{(2)}\right)$ and $\nabla^{(2)} b^{(2)}\left(x^{(1)}, \cdot\right)$ with fixed $x^{(2)}$ to be locally Hölder-Dini continuous of order $\frac{2}{3}$, and $b^{(2)}\left(x^{(1)}, \cdot\right)$ with fixed $x^{(1)}$ to be merely Dini continuous. These coincide with the continuity conditions used in [25] for infinite-dimensional degenerate systems with linear $b^{(1)}$.

Moreover, it is known that (1.2) is well-posed if $\sigma$ and $b$ are "almostly Lipschitz continuous", see e.g. [27, 8, 19]. In this paper we show that, under the above mentioned much weaker conditions on $b$, such a non-Lipschitz condition on $\sigma$ still implies the well-posedness. To characterize this condition, we introduce the class

$$
\mathscr{C}:=\left\{\gamma \in C^{1}\left(\mathbb{R}_{+} ; \mathbb{R}_{+}\right): \int_{0}^{1} \frac{1}{t \gamma(t)} \mathrm{d} t=\infty, \liminf _{t \downarrow 0}\left(\frac{\gamma(t)}{4}+t \gamma^{\prime}(t)\right)>0\right\}
$$

where $\int_{0}^{1} \frac{1}{t \gamma(t)} \mathrm{d} t=\infty$ is the key condition, and $\liminf _{t \downarrow 0}\left(\frac{\gamma(t)}{4}+t \gamma^{\prime}(t)\right)>0$ comes from our calculations in the present framework, which is weaker than the following condition used in [8, Theorem B]:

$$
\lim _{t \downarrow 0} \gamma(t)=\infty, \quad \lim _{t \downarrow 0} \frac{t \gamma^{\prime}(t)}{\gamma(t)}=0
$$

Typical functions in $\mathscr{C}$ include

$$
\gamma_{1}(t):=\log \left(1+t^{-1}\right), \quad \gamma_{2}(t):=\gamma_{1}(t) \log \log \left(\mathrm{e}+t^{-1}\right), \gamma_{3}(t):=\gamma_{2}(t) \log \log \log \left(\mathrm{e}^{2}+t^{-1}\right) \ldots
$$

In the following four subsections, we state our main results on the weak solutions, the strong solutions, the stability of solutions with respect to coefficients, and the $C^{1}$-stochastic diffeormorphism flows respectively.

### 1.1 Weak solutions

We introduce the following assumptions for some $\phi \in \mathscr{D}_{0} \cap \mathscr{S}_{0}$ and some increasing function $C: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$:
(C1) (Hypoellipticity) $\sigma_{t}(x)$ and $\left[\nabla^{(2)} b_{t}^{(1)}(x)\right]\left[\nabla^{(2)} b_{t}^{(1)}(x)\right]^{*}$ are invertible with

$$
\left\|\nabla^{(2)} b_{t}^{(1)}\right\|_{\infty}+\left\|\left(\left[\nabla^{(2)} b_{t}^{(1)}\right]\left[\nabla^{(2)} b_{t}^{(1)}\right]^{*}\right)^{-1}\right\|_{\infty}+\left\|\sigma_{t}\right\|_{\infty}+\left\|\sigma_{t}^{-1}\right\|_{\infty} \leqslant C(t), \quad t \geqslant 0 .
$$

(C2) (Regularity of $b^{(1)}$ ) For any $x, y \in \mathbb{R}^{d_{1}+d_{2}}$ with $|x-y| \leqslant 1$ and $t \geqslant 0$,

$$
\begin{aligned}
& \left|b_{t}^{(1)}(x)-b_{t}^{(1)}(y)\right| \leqslant C(t)\left|x^{(1)}-y^{(1)}\right|^{\frac{2}{3}} \phi\left(\left|x^{(1)}-y^{(1)}\right|\right), \quad \text { if } x^{(2)}=y^{(2)}, \\
& \left\|\nabla^{(2)} b_{t}^{(1)}(x)-\nabla^{(2)} b_{t}^{(1)}(y)\right\| \leqslant C(t) \phi\left(\left|x^{(2)}-y^{(2)}\right|\right), \quad \text { if } x^{(1)}=y^{(1)}
\end{aligned}
$$

(C3) (Regularity of $b^{(2)}, \sigma$ ) Either

$$
\left\{\begin{array}{l}
\left|b_{t}^{(2)}(x)-b_{t}^{(2)}(y)\right| \leqslant C(t)\left\{\left|x^{(1)}-y^{(1)}\right|^{\frac{2}{3}} \phi\left(\left|x^{(1)}-y^{(1)}\right|\right)+\phi^{\frac{7}{2}}\left(\left|x^{(2)}-y^{(2)}\right|\right)\right\},  \tag{1.6}\\
\left\|\sigma_{t}(x)-\sigma_{t}(y)\right\| \leqslant C(t)|x-y|^{\frac{2}{3}} \phi(|x-y|), \quad t \geqslant 0,|x-y| \leqslant 1 ;
\end{array}\right.
$$

or for $t \geqslant 0,|x-y| \leqslant 1$, there hold $\left\|\nabla^{(2)} \sigma_{t}\right\|_{\infty} \leqslant C(t)$ and

$$
\left\{\begin{array}{l}
\left|b_{t}^{(2)}(x)-b_{t}^{(2)}(y)\right| \leqslant C(t)\left\{\left|x^{(1)}-y^{(1)}\right|^{\frac{2}{3}} \phi\left(\left|x^{(1)}-y^{(1)}\right|\right)+\phi\left(\left|x^{(2)}-y^{(2)}\right|\right)\right\}  \tag{1.7}\\
\left\|\nabla^{(2)} \sigma_{t}\left(x^{(1)}, x^{(2)}\right)-\nabla^{(2)} \sigma_{t}\left(y^{(1)}, x^{(2)}\right)\right\| \leqslant C(t)\left|x^{(1)}-y^{(1)}\right|^{\frac{1}{9}} \phi\left(\left|x^{(1)}-y^{(1)}\right|\right) \\
\left\|\sigma_{t}\left(x^{(1)}, x^{(2)}\right)-\sigma_{t}\left(y^{(1)}, x^{(2)}\right)\right\| \leqslant C(t)\left|x^{(1)}-y^{(1)}\right|^{\frac{2}{3}} \phi\left(\left|x^{(1)}-y^{(1)}\right|\right) .
\end{array}\right.
$$

Intuitively, there should be a balance between the regularities of $b^{(2)}$ and $\sigma$; that is, with a stronger condition on $\sigma$ we will only need a weaker regularity of $b^{(2)}$. Conditions (1.6) and (1.7), as well as (1.8) and (1.9) below, are introduced in this spirit.

Theorem 1.1. Assume that (C1)-(C3) hold for some $\phi \in \mathscr{D}_{0} \cap \mathscr{S}_{0}$ and increasing function $C: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$. Then (1.2) has a unique weak solution.

Remark 1.1. In [16], Menozzi showed that the weak uniqueness holds for (1.2) under the assumptions that $\sigma$ is Hölder continuous and $b$ is Lipschitz continuous. In [17, Priola showed that there is a unique weak solution to (1.2) when $\sigma_{t}(x)=\sigma(x)$ is bounded continuous, $b^{(1)}(x)=x^{(2)}$ and $b^{(2)}(x)$ is bounded measurable. Although our assumptions on $b^{(2)}$ and $\sigma$ are stronger, we allow $b^{(1)}(x)$ to be merely Hölder-Dini continuous in $x^{(1)}$. In fact, this is the main source of the difficulty in our study, since due to the singularity of $b^{(1)}(x)$ in $x^{(1)}$ we have to carefully estimate the regularization of the noise transported from the second component to the first, see Lemma 3.1 below.

### 1.2 Strong solutions

By a localization argument, we will take the following local conditions on $\sigma$ and $b$.
(A) For any $n \in \mathbb{N}$, there exist a constant $C_{n} \in \mathbb{R}_{+}$, some $\phi_{n} \in \mathscr{D}_{0} \cap \mathscr{S}_{0}$ and $\gamma_{n} \in \mathscr{C}$ such that the following conditions hold for all $t \in[0, n]$ :
(A1) (Hypoellipticity) $\sigma_{t}(x)$ and $\left[\nabla^{(2)} b_{t}^{(1)}(x)\right]\left[\nabla^{(2)} b_{t}^{(1)}(x)\right]^{*}$ are invertible and locally bounded with

$$
\sup _{x \in \mathbb{R}^{d_{1}+d^{2}},\left|x^{(1)}\right| \leqslant n}\left\|\left(\left[\nabla^{(2)} b_{t}^{(1)}\right]\left[\nabla^{(2)} b_{t}^{(1)}\right]^{*}\right)^{-1}\right\|(x)+\sup _{|x| \leqslant n}\left\|\sigma_{t}^{-1}\right\|(x) \leqslant C_{n} .
$$

(A2) (Regularity of $b^{(1)}$ ) For any $x, y \in \mathbb{R}^{d_{1}+d_{2}}$ with $|x| \vee|y| \leqslant n$,

$$
\begin{aligned}
& \left|b_{t}^{(1)}(x)-b_{t}^{(1)}(y)\right| \leqslant\left|x^{(1)}-y^{(1)}\right|^{\frac{2}{3}} \phi_{n}\left(\left|x^{(1)}-y^{(1)}\right|\right), \quad \text { if } x^{(2)}=y^{(2)}, \\
& \left\|\nabla^{(2)} b_{t}^{(1)}(x)-\nabla^{(2)} b_{t}^{(1)}(y)\right\| \leqslant \phi_{n}\left(\left|x^{(2)}-y^{(2)}\right|\right), \quad \text { if } x^{(1)}=y^{(1)} .
\end{aligned}
$$

$$
\left\{\begin{array}{l}
\left|b_{t}^{(2)}(x)-b_{t}^{(2)}(y)\right| \leqslant\left\{\left|x^{(1)}-y^{(1)}\right|^{\frac{2}{3}} \phi_{n}\left(\left|x^{(1)}-y^{(1)}\right|\right)+\phi_{n}^{\frac{7}{2}}\left(\left|x^{(2)}-y^{(2)}\right|\right)\right\},  \tag{1.8}\\
\left\|\sigma_{t}(x)-\sigma_{t}(y)\right\| \leqslant|x-y| \sqrt{\gamma_{n}(|x-y|)}, \quad|x| \vee|y| \leqslant n
\end{array}\right.
$$

or $\sup _{|x| \leqslant n}\left\|\nabla^{(2)} \sigma_{t}(x)\right\|_{\infty} \leqslant C_{n}$ and for $|x| \vee|y| \leqslant n$,

$$
\left\{\begin{array}{l}
\left|b_{t}^{(2)}(x)-b_{t}^{(2)}(y)\right| \leqslant\left\{\left|x^{(1)}-y^{(1)}\right|^{\frac{2}{3}} \phi_{n}\left(\left|x^{(1)}-y^{(1)}\right|\right)+\phi_{n}\left(\left|x^{(2)}-y^{(2)}\right|\right)\right\}  \tag{1.9}\\
\left\|\nabla^{(2)} \sigma_{t}\left(x^{(1)}, x^{(2)}\right)-\nabla^{(2)} \sigma_{t}\left(y^{(1)}, x^{(2)}\right)\right\| \leqslant\left|x^{(1)}-y^{(1)}\right| \sqrt{\gamma_{n}\left(\left|x^{(1)}-y^{(1)}\right|\right)} \\
\left\|\sigma_{t}\left(x^{(1)}, x^{(2)}\right)-\sigma_{t}\left(y^{(1)}, x^{(2)}\right)\right\| \leqslant\left|x^{(1)}-y^{(1)}\right| \sqrt{\gamma_{n}\left(\left|x^{(1)}-y^{(1)}\right|\right)}
\end{array}\right.
$$

Theorem 1.2. (1) Under assumption (A), for any $x \in \mathbb{R}^{d_{1}+d_{2}}$, SDE (1.2) has a unique solution $X_{t}(x)$ up to the explosion time $\zeta(x)$.
(2) If, in particular, $b_{t}(x)$ and $\sigma_{t}(x)$ do not depend on $x^{(1)}$, then the above assertion follows provided for any $n \in \mathbb{N}$ there exists $\phi_{n} \in \mathscr{D}_{0} \cap \mathscr{S}_{0}$ and $\gamma_{n} \in \mathscr{C}$ such that (A1) and

$$
\left\{\begin{array}{l}
\left\|\sigma_{t}(x)-\sigma_{t}(y)\right\| \leqslant\left|x^{(2)}-y^{(2)}\right| \sqrt{\gamma_{n}\left(\left|x^{(2)}-y^{(2)}\right|\right)},  \tag{1.10}\\
\left|b_{t}^{2}(x)-b_{t}^{(2)}(y)\right|+\left\|\nabla^{(2)} b_{t}^{(1)}(x)-\nabla^{(2)} b_{t}^{(1)}(y)\right\| \leqslant \phi_{n}\left(\left|x^{(2)}-y^{(2)}\right|\right)
\end{array}\right.
$$

hold for all $t,|x|,|y| \leqslant n$.
(3) If there exists $H \in C^{2}\left(\mathbb{R}^{d_{1}+d_{2}}\right)$ such that

$$
\begin{equation*}
H \geqslant 1, \quad \lim _{|x| \rightarrow \infty} H(x)=\infty, \quad\left|\nabla^{(2)} H\right|^{2} \leqslant C H^{2-\varepsilon}, \quad \mathscr{L}_{t}^{\Sigma, b} H \leqslant \Phi(t) H, \quad t \geqslant 0 \tag{1.11}
\end{equation*}
$$

holds for some constant $\varepsilon \in(0,1]$ and positive increasing function $\Phi$, then the solution to (1.2) is non-explosive and for any $\varepsilon^{\prime} \in[0, \varepsilon)$,

$$
\begin{equation*}
\mathbb{E} \exp \left[\sup _{t \in[0, T]} H\left(X_{t}(x)\right)^{\varepsilon^{\prime}}\right] \leqslant \Psi(T) \exp \left[H(x)^{\varepsilon}\right], \quad T>0, x \in \mathbb{R}^{d_{1}+d_{2}} \tag{1.12}
\end{equation*}
$$

holds for some increasing function $\Psi:[0, \infty) \rightarrow(0, \infty)$.
Remark 1.2. (1) When $b^{(1)}$ is linear, an infinite-dimensional version of the well-posedness has been proved in [25] by following the line of [24] for non-degenerate SPDEs, see [4, 5, ,6, (7) for discussions on the pathwise uniqueness of SPDEs with Hölder continuous drifts and non-degenerate additive noises.
(2) When $m=d$, the well-posedness was also proved in [2] under a stronger assumption where $\sigma$ is Lipschitz continuous, $b(x)$ is Hölder continuous of order $\alpha \in\left(\frac{3}{2}, 1\right)$ in $x^{(1)}$ and order $\beta \in(0,1)$ in $x^{(2)}$, and $\nabla^{(2)} b^{(1)}$ is Hölder continuous. In fact, we will show in Theorem 1.7 below that under this assumption and that $\sigma \in C^{1+\varepsilon}$ for some $\varepsilon>0$ the solutions to (1.2) form $C^{1}$-stochastic diffeomorphism flows. Notice that the proofs given in [2] strongly depend
on the explicit form of the fundamental solutions of linear degenerate Kolmogorov's operators, while our proof is based on explicit probability formulas of the semigroup associated to the linear stochastic Hamiltonian system (see Section 2.4 below).

To illustrate Theorem [1.2, we present below three direct consequences, where the first generalizes to (1.1), the second includes a class of SDEs with unbounded time-delay which are interesting by themselves, and the last presents a new well-posedness result for nondegenerate SDEs.

Corollary 1.3. The following stochastic differential-integral equation on $\mathbb{R}^{d}$ admits a unique strong solution up to life time:

$$
\mathrm{d} X_{t}=\left(b_{t}\left(X_{t}\right)+\int_{0}^{t} \sigma_{s}\left(X_{s}\right) \mathrm{d} W_{s}\right) \mathrm{d} t,
$$

where $W_{t}$ is a d-dimensional Brownian motion, $b: \mathbb{R}_{+} \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}, \sigma: \mathbb{R}_{+} \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d} \otimes \mathbb{R}^{d}$ are measurable such that $b, \sigma$ and $\sigma^{-1}$ are locally bounded, and for any $n \geqslant 1$ there exist $\phi_{n} \in \mathscr{D}_{0} \cap \mathscr{S}_{0}$ and $\gamma_{n} \in \mathscr{C}$ such that for all $t,|x|,|y| \leqslant n$,

$$
\begin{align*}
& \left|b_{t}(x)-b_{t}(y)\right| \leqslant|x-y|^{\frac{2}{3}} \phi_{n}(|x-y|),  \tag{1.13}\\
& \left|\sigma_{t}(x)-\sigma_{t}(y)\right| \leqslant|x-y| \sqrt{\gamma_{n}(|x-y|)}
\end{align*}
$$

Proof. Let $\tilde{X}_{t}^{(1)}=X_{t}, \tilde{X}_{t}^{(2)}=\int_{0}^{t} \sigma\left(X_{s}\right) \mathrm{d} W_{s}$. Then the equation reduces to (1.2) on $\mathbb{R}^{d+d}$ with

$$
\tilde{b}_{t}^{(1)}(\tilde{x}):=b_{t}\left(\tilde{x}^{(1)}\right)+\tilde{x}^{(2)}, \quad \tilde{b}_{t}^{(2)}:=0, \quad \tilde{\sigma}_{t}(\tilde{x}):=\sigma_{t}\left(\tilde{x}^{(1)}\right) .
$$

Obviously, the local boundedness of $b, \sigma$ and $\sigma^{-1}$ as well as (1.13) imply (A) with (1.8) for $(\tilde{b}, \tilde{\sigma})$. Then the proof is finished by Theorem 1.2(1).
Corollary 1.4. Let $b$ and $\sigma$ satisfy $(\mathbf{A})$ and let $b_{t}^{(1)}(x)=b_{t}^{(1)}\left(x^{(2)}\right)$ not depend on $x^{(1)}$. Then for any $Y_{0}=y \in \mathbb{R}^{d_{2}}$, the following $S D E$ with unbounded time-delay has a unique solution up to life time:

$$
\mathrm{d} Y_{t}=b_{t}^{(2)}\left(\int_{0}^{t} b_{s}^{(1)}\left(Y_{s}\right) \mathrm{d} s, Y_{t}\right) \mathrm{d} t+\sigma\left(\int_{0}^{t} b_{s}^{(1)}\left(Y_{s}\right) \mathrm{d} s, Y_{t}\right) \mathrm{d} W_{t}, \quad Y_{0}=y
$$

Proof. Let $X_{t}^{(1)}=\int_{0}^{t} b_{s}^{(1)}\left(Y_{s}\right) \mathrm{d} s$ and $X_{t}^{(2)}=Y_{t}$. Then the SDE reduces to (1.2) with $X_{0}=$ $(0, y) \in \mathbb{R}^{d_{1}+d_{2}}$. So, the desired assertion follows from Theorem 1.2.

Finally, since existing well-posedness results for non-degenerated SDEs at least assumed that $\sigma$ is weakly differentiable (see [9, 30] and references within), the following result is new even in the non-degenerate setting.
Corollary 1.5. The following SDE on $\mathbb{R}^{d}$ admits a unique strong solution up to life time:

$$
\mathrm{d} X_{t}=b_{t}\left(X_{t}\right)+\sigma_{t}\left(X_{t}\right) \mathrm{d} W_{t},
$$

where $W_{t}$ is a d-dimensional Brownian motion, $b: \mathbb{R}_{+} \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}, \sigma: \mathbb{R}_{+} \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d} \otimes \mathbb{R}^{d}$ are measurable such that $b, \sigma$ and $\sigma^{-1}$ are locally bounded, and for any $n \geqslant 1$ there exist $\phi_{n} \in \mathscr{D}_{0} \cap \mathscr{S}_{0}$ and $\gamma_{n} \in \mathscr{C}$ such that for all $t,|x|,|y| \leqslant n$,

$$
\begin{equation*}
\left|b_{t}(x)-b_{t}(y)\right| \leqslant \phi_{n}(|x-y|),\left|\sigma_{t}(x)-\sigma_{t}(y)\right| \leqslant C(t)|x-y| \sqrt{\gamma_{n}(|x-y|)} \tag{1.14}
\end{equation*}
$$

Proof. Let $\tilde{X}_{t}^{(1)}=\int_{0}^{t} X_{s} \mathrm{~d} s, \tilde{X}_{t}^{(2)}=X_{t}$. Then the equation reduces to (1.2) on $\mathbb{R}^{d+d}$ with

$$
\tilde{b}_{t}^{(1)}(\tilde{x}):=\tilde{x}^{(2)}, \quad \tilde{b}_{t}^{(2)}(\tilde{x})=b_{t}\left(\tilde{x}^{(2)}\right), \quad \tilde{\sigma}_{t}(\tilde{x})=\sigma_{t}\left(\tilde{x}^{(2)}\right) .
$$

Obviously, the local boundedness of $b, \sigma$ and $\sigma^{-1}$, together with (1.14), implies that (A1) and (1.10) for $(\tilde{b}, \tilde{\sigma})$. Then the proof is finished by Theorem 1.2(2).

### 1.3 Stability of solutions with respect to coefficients

About the continuous dependence of strong solutions with respect to the coefficients $(b, \sigma)$, we have

Theorem 1.6. Let $\left(b^{k}, \sigma^{k}\right)_{k \in \mathbb{N}_{\infty}}$ be a sequence of functions satisfying (A1), (A2) and

$$
\left\{\begin{array}{l}
\left|\left(b^{k}\right)_{t}^{(2)}(x)-\left(b^{k}\right)_{t}^{(2)}(y)\right| \leqslant\left\{\left|x^{(1)}-y^{(1)}\right|^{\frac{2}{3}} \phi_{n}\left(\left|x^{(1)}-y^{(1)}\right|\right)+\phi_{n}^{\frac{7}{2}}\left(\left|x^{(2)}-y^{(2)}\right|\right)\right\},  \tag{1.15}\\
\left\|\sigma_{t}^{k}(x)-\sigma_{t}^{k}(y)\right\| \leqslant C_{n}|x-y|, \quad t \leqslant n,|x| \vee|y| \leqslant n
\end{array}\right.
$$

with the same localization constants $C_{n}$ and $\phi_{n} \in \mathscr{D}_{0} \cap \mathscr{S}_{0}$. Assume that $\left(b^{k}, \sigma^{k}\right)$ satisfies (1.11) with the same $H$ and $C$, and for each $t, x$,

$$
\lim _{k \rightarrow \infty}\left\|\sigma_{t}^{k}(x)-\sigma_{t}^{\infty}(x)\right\|+\left|b_{t}^{k}(x)-b_{t}^{\infty}(x)\right|=0
$$

Let $X_{t}^{k}(x)$ be the unique solution of (1.2) corresponding to $\left(b^{k}, \sigma^{k}\right)$ for each $k \in \mathbb{N}_{\infty}$. Then for each $\varepsilon, T>0$ and $x \in \mathbb{R}^{d}$,

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \mathbb{P}\left(\sup _{t \in[0, T]}\left|X_{t}^{k}(x)-X_{t}^{\infty}(x)\right| \geqslant \varepsilon\right)=0 \tag{1.16}
\end{equation*}
$$

Moreover, if for some $p>d$ and for all $T, R>0$,

$$
\begin{equation*}
\sup _{k \in \mathbb{N}_{\infty}} \sup _{|x| \leqslant R} \mathbb{E}\left(\sup _{t \in[0, T]}\left|\nabla X_{t}^{k}(x)\right|^{p}\right)<\infty, \tag{1.17}
\end{equation*}
$$

then for each $\varepsilon, R, T>0$,

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \mathbb{P}\left(\sup _{t \in[0, T]} \sup _{|x| \leqslant R}\left|X_{t}^{k}(x)-X_{t}^{\infty}(x)\right| \geqslant \varepsilon\right)=0 . \tag{1.18}
\end{equation*}
$$

Remark 1.3. See Theorem 1.7] below for sufficient conditions of (1.17). According to [26, Theorem 2.3], condition (1.17) can be replaced with the following weaker one: for some $p>d$ and for all $T, R>0$,

$$
\sup _{k \in \mathbb{N}_{\infty}} \mathbb{E}\left(\sup _{t \in[0, T]}\left|X_{t}^{k}(x)-X_{t}^{k}(y)\right|^{p}\right) \leqslant C|x-y|^{p}, \quad|x| \vee|y| \leqslant R .
$$

## $1.4 \quad C^{1}$-stochastic diffeormorphism flow

In order to show the $C^{1}$-diffeomorphism flow property of $X_{t}(x)$, we need stronger conditions as shown in the following result.

Theorem 1.7. Assume (C1) and that for some constant $\beta \in\left(0, \frac{1}{3}\right)$ and increasing function $C:[0, \infty) \rightarrow \mathbb{R}_{+}$the conditions

$$
\begin{aligned}
& \left|b_{t}^{(1)}(x)-b_{t}^{(1)}(y)\right| \leqslant C(t)\left|x^{(1)}-y^{(1)}\right|^{\beta+\frac{2}{3}}, \quad \text { if } x^{(2)}=y^{(2)}, \\
& \left\|\nabla^{(2)} b_{t}^{(1)}(x)-\nabla^{(2)} b_{t}^{(1)}(y)\right\| \leqslant C(t)\left|x^{(2)}-y^{(2)}\right|^{\beta}, \quad \text { if } x^{(1)}=y^{(1)}, \\
& \left|b_{t}^{(2)}(x)-b_{t}^{(2)}(y)\right| \leqslant C(t)\left(\left|x^{(1)}-y^{(1)}\right|^{\beta+\frac{2}{3}}+\left|x^{(2)}-y^{(2)}\right|^{\beta}\right) \\
& \left\|\nabla \sigma_{t}\right\|_{\infty} \leqslant C(t), \quad\left\|\nabla \sigma_{t}(x)-\nabla \sigma_{t}(y)\right\| \leqslant C(t)|x-y|^{\beta}
\end{aligned}
$$

hold for any $|x-y| \leqslant 1, t \geqslant 0$. Then the unique strong solution $\left\{X_{t}(\cdot)\right\}_{t \geqslant 0}$ to (1.2) is a $C^{1}$-stochastic diffeomorphism flow, and

$$
\begin{equation*}
\sup _{x \in \mathbb{R}^{d_{1}+d_{2}}} \mathbb{E}\left(\sup _{t \in[0, T]}\left\|\nabla X_{t}(x)\right\|^{p}\right)<\infty, \quad T>0, p \geqslant 1 . \tag{1.19}
\end{equation*}
$$

In the above result, $b$ has at most linear growth. The following result shows that by making perturbations to $b$, it is possible to prove the $C^{1}$-stochastic diffeomorphism flow property for $b$ of high order polynomial growth.

Theorem 1.8. Keep the same assumptions of Theorem 1.7. Let $a: \mathbb{R}_{+} \times \mathbb{R}^{d_{1}+d_{2}} \rightarrow \mathbb{R}^{d_{2}}$ be a measurable function such that $\nabla a_{t}$ is locally Hölder continuous uniformly in $t \in[0, T]$ for any $T>0$. Suppose also that for some $H \in C^{2}\left(\mathbb{R}^{d_{1}+d_{2}}\right), \varepsilon \in(0,1], \delta_{1}, \delta_{2}, C_{1}, C_{2}, C_{3}>0$ and positive increasing function $\Phi$, and for all $t \geqslant 0, x \in \mathbb{R}^{d_{1}+d_{2}}$,

$$
\begin{equation*}
C_{1}\left(1+|x|^{\delta_{1}}\right) \leqslant H(x) \leqslant C_{2}\left(1+|x|^{\delta_{2}}\right),\left|\nabla^{(2)} H\right|^{2} \leqslant C_{3} H^{2-\varepsilon},\left|\mathscr{L}_{t}^{\Sigma, b+a} H\right| \leqslant \Phi(t) H \tag{1.20}
\end{equation*}
$$

and for some $\varepsilon^{\prime} \in[0, \varepsilon)$ and positive increasing function $\Phi^{\prime}$, and for all $t \geqslant 0$ and $x, x^{\prime} \in$ $\mathbb{R}^{d_{1}+d_{2}}$,

$$
\begin{equation*}
\left|a_{t}(x)\right| \leqslant \Phi^{\prime}(t) H(x)^{\varepsilon^{\prime}},\left|a_{t}(x)-a_{t}\left(x^{\prime}\right)\right| \leqslant \Phi^{\prime}(t)\left(H(x)^{\varepsilon^{\prime}}+H\left(x^{\prime} \varepsilon^{\varepsilon^{\prime}}\right)\left|x-x^{\prime}\right|\right. \tag{1.21}
\end{equation*}
$$

Then the SDE

$$
\begin{equation*}
\mathrm{d} X_{t}=\left[a_{t}\left(X_{t}\right)+b_{t}\left(X_{t}\right)\right] \mathrm{d} t+\left(0, \sigma_{t}\left(X_{t}\right) \mathrm{d} W_{t}\right), \quad X_{0}=x \in \mathbb{R}^{d_{1}+d_{2}} \tag{1.22}
\end{equation*}
$$

has a unique strong solution $X_{t}(x)$ such that $\left\{X_{t}(\cdot)\right\}_{t \geqslant 0}$ forms a $C^{1}$-stochastic diffeomorphism flow, and for any $T>0$ and $p \geqslant 1$, there exists a constant $C>0$ such that

$$
\begin{equation*}
\mathbb{E}\left(\sup _{t \in[0, T]}\left\|\nabla X_{t}(x)\right\|^{p}\right) \leqslant C \mathrm{e}^{H(x)^{\varepsilon}}, \quad x \in \mathbb{R}^{d_{1}+d_{2}} \tag{1.23}
\end{equation*}
$$

Below is a simple example illustrating Theorem 1.8, where the drift is neither local Lipschitz nor of linear growth.

Example 1.1. Let $d_{1}=d_{2}=d, \alpha \in\left(\frac{2}{3}, 1\right], m \in \mathbb{N}$ and $c_{1}, c_{2}>0$. Take

$$
H(x)=1+\frac{1}{2}\left|x^{(2)}\right|^{2}+c_{1}\left|x^{(1)}\right|^{\alpha+1}+c_{2}\left|x^{(1)}\right|^{m+1} .
$$

Let $\sigma$ be an invertible $d \times d$-matrix. Consider the following SDE

$$
\mathrm{d}\left(X_{t}^{(1)}, X_{t}^{(2)}\right)=\left(X_{t}^{(2)},-\nabla^{(1)} H\left(X_{t}\right)\right) \mathrm{d} t+\left(0, \sigma \mathrm{~d} W_{t}\right)
$$

It is easy to see that Theorem 1.8 applies to

$$
b(x)=\left(x^{(2)},-c_{1}(\alpha+1) x^{(1)}\left|x^{(1)}\right|^{\alpha-1}\right), \quad a(x)=\left(0,-c_{2}(m+1) x^{(1)}\left|x^{(1)}\right|^{m-1}\right) .
$$

In the spirit of [32, 22], the key point of the study is to construct a time-dependent diffeomorphism on $\mathbb{R}^{d_{1}+d_{2}}$ which transforms (1.2) into an equation with regular enough coefficients ensuring the desired assertions. To this end, we take a freezing coefficient argument, which is different from the one used in [2], so that the construction is reduced to solve an parabolic equation associated to a linear stochastic Hamiltonian system. To figure out the minimal conditions on $b$ and $\sigma$ for the required estimates on solutions to this parabolic equation, we introduce some techniques in Section 2, in particular, some characterizations of the continuity using the heat semigroup. Moreover, in Section 2 we also present gradient estimates on the semigroup of the linear stochastic Hamiltonian system. With these preparations, in Section 3 we investigate the parabolic equation associated to the generator $\mathscr{L}_{t}^{\Sigma, b}$ (see (3.1) below), which in turn provides the desired diffeomorphism on $\mathbb{R}^{d_{1}+d_{2}}$. Finally, in Section 4 we present complete proofs of the above theorems.

## 2 Preparations

This section contains some results which will be used to construct the regularization transform in the proof of the main results. We first present a Volterra-Gronwall type inequality associated to a Dini function, then characterize the continuity of functions using the heat semigroup, and finally introduce derivative formula and gradient estimates on linear stochastic Hamiltonian systems.

Throughout the paper, the letter $C$ with or without subscripts will denote a positive constant whose value may change from one appearance to another. For two real functions $f$ and $g$, we write $f \preceq g$ if $f \leqslant C_{0} g$ for some $C_{0}>0$; and $f \asymp g$ if $C_{1} g \leqslant f \leqslant C_{2} g$ for some $C_{1}, C_{2}>0$.

### 2.1 Volterra-Gronwall inequality associated to a Dini function

Lemma 2.1. Let $\phi: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$be a Dini function. For any $T>0$, there exists a constant $C=C(\phi, T)>0$ such that if $\lambda \geqslant 0$ and bounded measurable functions $f, h: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$ satisfy

$$
h(t) \leqslant \int_{0}^{t} \mathrm{e}^{-\lambda(t-s)} \frac{\phi(t-s)}{t-s}(h(s)+f(s)) \mathrm{d} s, \quad t \in(0, T]
$$

then

$$
h(t) \leqslant C \int_{0}^{t} \mathrm{e}^{-\lambda(t-s)} \frac{\phi(t-s)}{t-s} f(s) \mathrm{d} s, \quad t \in(0, T] .
$$

Proof. Let $a_{1}(t)=\frac{\phi(t)}{t}$ and define

$$
a_{n+1}(t)=\int_{0}^{t} a_{n}(t-s) a_{1}(s) \mathrm{d} s, \quad t \in(0, T], \quad n \in \mathbb{N}
$$

Since $\int_{0}^{T} \frac{\phi(t)}{t} \mathrm{~d} t<\infty$, by [27, Theorem 1] with $k(t, s):=\frac{\phi(t-s)}{t-s} 1_{\{s<t\}}$ (see also [28, Lemma 2.1]), we have

$$
a(t):=\sum_{n=1}^{\infty} a_{n}(t) \in L^{1}([0, T])
$$

and

$$
\begin{equation*}
a(t)=a_{1}(t)+\int_{0}^{t} a(t-s) a_{1}(s) \mathrm{d} s \tag{2.1}
\end{equation*}
$$

Letting

$$
g(t)=\int_{0}^{t} \mathrm{e}^{-\lambda(t-s)} a_{1}(t-s) f(s) \mathrm{d} s
$$

then by [28, Lemma 2.2], we have

$$
h(t) \leqslant g(t)+\int_{0}^{t} \mathrm{e}^{-\lambda(t-s)} a(t-s) g(s) \mathrm{d} s .
$$

Combining this with (2.1) and using Fubini's theorem, we obtain

$$
\begin{aligned}
h(t) & \leqslant g(t)+\int_{0}^{t} \mathrm{e}^{-\lambda(t-s)} a(t-s)\left(\int_{0}^{s} \mathrm{e}^{-\lambda(s-r)} a_{1}(s-r) f(r) \mathrm{d} r\right) \mathrm{d} s \\
& =g(t)+\int_{0}^{t}\left(\int_{r}^{t} \mathrm{e}^{-\lambda(t-s)} a(t-s) \mathrm{e}^{-\lambda(s-r)} a_{1}(s-r) \mathrm{d} s\right) f(r) \mathrm{d} r \\
& =g(t)+\int_{0}^{t} \mathrm{e}^{-\lambda(t-r)} f(r) \mathrm{d} r \int_{0}^{t-r} a(t-r-s) a_{1}(s) \mathrm{d} s \\
& \leqslant g(t)+\int_{0}^{t} \mathrm{e}^{-\lambda(t-r)} a(t-r) f(r) \mathrm{d} r .
\end{aligned}
$$

So, it remains to prove

$$
\begin{equation*}
a(t) \leqslant C a_{1}(t), \quad t \in(0, T] \tag{2.2}
\end{equation*}
$$

for some constant $C>0$. By the increasing property of $\phi$, we have

$$
a_{1}(r t)=\frac{\phi(r t)}{r t} \leqslant \frac{\phi(t)}{r t}=\frac{a_{1}(t)}{r}, r \in(0,1), t \in(0, T] .
$$

By the standard induction argument, this implies

$$
\begin{equation*}
a_{n}(r t) \leqslant \frac{a_{n}(t)}{r}, \quad r \in(0,1), t \in(0, T], n \in \mathbb{N} . \tag{2.3}
\end{equation*}
$$

Indeed, by the change of variables and induction hypothesis, we have

$$
\begin{aligned}
a_{n+1}(r t) & =\int_{0}^{r t} a_{n}(r t-s) a_{1}(s) \mathrm{d} s=r \int_{0}^{t} a_{n}(r(t-s)) a_{1}(r s) \mathrm{d} s \\
& \leqslant \frac{1}{r} \int_{0}^{t} a_{n}(t-s) a_{1}(s) \mathrm{d} s=\frac{a_{n+1}(t)}{r}
\end{aligned}
$$

Thus, for any $\varepsilon \in(0,1)$ and $t \in(0, T]$, by (2.3) we have

$$
\begin{aligned}
& \int_{0}^{t} a(t-s) a_{1}(s) \mathrm{d} s=\sum_{n=1}^{\infty} \int_{0}^{t} a_{n}(t-s) a_{1}(s) \mathrm{d} s \\
& \leqslant \sum_{n=1}^{\infty} \int_{\varepsilon t}^{t} a_{n}(t-s) \frac{a_{1}(t)}{s / t} \mathrm{~d} s+\sum_{n=1}^{\infty} \int_{0}^{\varepsilon t} \frac{a_{n}(t)}{(t-s) / t} a_{1}(s) \mathrm{d} s \\
& \leqslant \frac{a_{1}(t)}{\varepsilon} \int_{0}^{T} a(s) \mathrm{d} s+\frac{a(t)}{1-\varepsilon} \int_{0}^{\varepsilon T} a_{1}(s) \mathrm{d} s
\end{aligned}
$$

Letting $\varepsilon \in(0,1)$ be small enough such that $\frac{1}{1-\varepsilon} \int_{0}^{\varepsilon T} a_{1}(s) \mathrm{d} s \leqslant \frac{1}{2}$, and combining this with (2.1), we obtain

$$
a(t) \leqslant 2 a_{1}(t)\left(1+\frac{1}{\varepsilon} \int_{0}^{T} a(t) \mathrm{d} t\right), \quad t \in(0, T] .
$$

This implies (2.2) since $a \in L^{1}([0, T])$.

### 2.2 Slowly varying functions

We first recall some important properties of slowly varying functions (cf. [1, Theorem 1.5.6 (ii) and Theorem 1.5.11]).

Proposition 2.2. For any $\phi \in \mathscr{S}_{0}$, the following assertions hold:
(i) For any $\delta>0$, there is a constant $C=C(\delta) \geqslant 1$ such that for all $t, s>0$,

$$
\frac{\phi(t)}{\phi(s)} \leqslant C \max \left\{\left(\frac{t}{s}\right)^{\delta},\left(\frac{t}{s}\right)^{-\delta}\right\} .
$$

(ii) For any $\beta>-1$, as $t \rightarrow 0$, we have

$$
\int_{0}^{t} s^{\beta} \phi(s) \mathrm{d} s \sim \frac{t^{\beta+1} \phi(t)}{\beta+1}, \quad \int_{t}^{1} s^{-\beta-2} \phi(s) \mathrm{d} s \sim \frac{t^{-\beta-1} \phi(t)}{\beta+1} .
$$

The following lemma is simple.

Lemma 2.3. For any bounded measurable function $\psi:(0,1] \rightarrow \mathbb{R}_{+}$, we have

$$
\begin{equation*}
[f]_{\psi}:=\sup _{|x-y| \leqslant 1} \frac{|f(x)-f(y)|}{\psi(|x-y|)}=\sup _{x \neq y} \frac{|f(x)-f(y)|}{\psi_{[0]}(|x-y|)}, \tag{2.4}
\end{equation*}
$$

where $\psi_{[0]}(t):=\psi(t) 1_{t \leqslant 1}+\psi_{*}(1) t 1_{t>1}$ and $\psi_{*}(1):=\sup _{s \in(0,1]} \psi(s)$.
Proof. Clearly, it suffices to prove that

$$
|f(x)-f(y)| \leqslant[f]_{\psi} \psi_{*}(1)|x-y|,|x-y| \geqslant 1
$$

Suppose that $n<|x-y| \leqslant n+1$ for some $n \in \mathbb{N}$. Let $x=x_{0}, x_{1}, \cdots, x_{n}, x_{n+1}=y$ be $n+2$-points in $\mathbb{R}^{d}$ so that

$$
\left|x_{i}-x_{i-1}\right|=1, i=1, \cdots, n,|x-y|=n+\left|x_{n+1}-x_{n}\right| .
$$

Then we have

$$
|f(x)-f(y)| \leqslant \sum_{i=1}^{n+1}\left|f\left(x_{i}\right)-f\left(x_{i-1}\right)\right| \leqslant[f]_{\psi} \psi_{*}(1)\left(n+\left|x_{n+1}-x_{n}\right|\right)=[f]_{\psi} \psi_{*}(1)|x-y|
$$

The proof is finished.
Due to the above lemma and also for later use, we introduce

$$
\begin{equation*}
\mathscr{R}_{\alpha}:=\left\{\phi_{[\alpha]}(t):=t^{\alpha} \phi(t) 1_{t \leqslant 1}+c_{\alpha} t 1_{t>1}: \phi \in \mathscr{S}_{0} \text { with } c_{\alpha}=\sup _{s \in(0,1]}\left(s^{\alpha} \phi(s)\right)<\infty\right\} \tag{2.5}
\end{equation*}
$$

for $\alpha \in[0,1]$, and let

$$
\mathscr{R}=\cup_{\alpha \in[0,1]} \mathscr{R}_{\alpha} .
$$

The function $\phi_{[\alpha]}$ with $\alpha \in[0,1]$ and $\phi \in \mathscr{D}_{0}$ not only characterizes the Hölder-Dini modulus, but also reduces the study to functions with linear growth. Notice that by (i) of Proposition 2.2, $c_{\alpha}$ in (2.5) is automatically finite for $\alpha \in(0,1]$.

Below we list the main properties of $\psi \in \mathscr{R}_{\alpha}$ for later use, which are easy consequences of Proposition 2.2.
Proposition 2.4. For $\alpha \in[0,1]$, let $\psi \in \mathscr{R}_{\alpha}$.
(i) For any $\delta>0$, there is a constant $C=C(\delta) \geqslant 1$ such that for all $t, s>0$,

$$
\begin{equation*}
\frac{\psi(t)}{\psi(s)} \leqslant C \max \left\{\left(\frac{t}{s}\right)^{\alpha+\delta},\left(\frac{t}{s}\right)^{\alpha-\delta}\right\} \tag{2.6}
\end{equation*}
$$

In particular, if $\alpha \in[0,1)$, then for all $t \geqslant s>0$,

$$
\begin{equation*}
\frac{s}{\psi(s)} \leqslant C \frac{t}{\psi(t)} \tag{2.7}
\end{equation*}
$$

(ii) If $\alpha \in(0,1)$, then there is a constant $C>0$ such that for all $t \in(0,1]$,

$$
\begin{equation*}
\int_{0}^{t} s^{-1} \psi(s) \mathrm{d} s \leqslant C \psi(t), \quad \int_{t}^{1} s^{-2} \psi(s) \mathrm{d} s \leqslant C t^{-1} \psi(t) \tag{2.8}
\end{equation*}
$$

(iii) There is a constant $C>0$ such that for all $s, t>0$,

$$
\begin{equation*}
\psi(s+t) \leqslant C(\psi(s)+\psi(t)) \tag{2.9}
\end{equation*}
$$

### 2.3 Characterization of continuity by using heat semigroup

Let $\mathscr{B}_{p}\left(\mathbb{R}^{d}\right)$ be the set of all measurable functions on $\mathbb{R}^{d}$ with polynomial growth. We will investigate the continuity of $f \in \mathscr{B}_{p}\left(\mathbb{R}^{d}\right)$ on $\mathbb{R}^{d}$ by using the standard heat semigroup

$$
\begin{equation*}
\mathbf{P}_{\theta} f(x)=\int_{\mathbb{R}^{d}} f(y) p_{\theta}(x-y) \mathrm{d} y, \theta>0 \tag{2.10}
\end{equation*}
$$

where

$$
p_{\theta}(x):=\frac{1}{(2 \pi \theta)^{d / 2}} \mathrm{e}^{-\frac{|x|^{2}}{2 \theta}} .
$$

Notice that by elementary calculus,

$$
\begin{equation*}
\left|\nabla^{k} \partial_{\theta}^{j} p_{\theta}(x)\right| \preceq \frac{|x|^{k}\left(\theta+|x|^{2}\right)^{j}}{\theta^{k+2 j}} p_{\theta}(x), \quad \theta>0, x \in \mathbb{R}^{d}, k, j=0,1 . \tag{2.11}
\end{equation*}
$$

For any measurable function $\psi:[0,1] \rightarrow \mathbb{R}_{+}$and $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$, define

$$
[f]_{\psi}:=\sup _{|x-y| \leqslant 1} \frac{|f(x)-f(y)|}{\psi(|x-y|)},\|f\|_{\infty}:=\sup _{x \in \mathbb{R}^{d}}|f(x)|,\|f\|_{\psi}:=[f]_{\psi}+\|f\|_{\infty} .
$$

It should be noticed by (2.4) and (2.5) that for any $\psi \in \mathscr{R}$,

$$
\begin{equation*}
|f(x)-f(y)| \leqslant \psi(|x-y|)[f]_{\psi}, \quad x, y \in \mathbb{R}^{d} \tag{2.12}
\end{equation*}
$$

and if $\psi_{1}(s) \leqslant C \psi_{2}(s), s \in(0,1]$ for some $C>0$, then

$$
[f]_{\psi_{2}} \leqslant C[f]_{\psi_{1}}
$$

We first present the following simple lemma.
Lemma 2.5. For any $\psi \in \mathscr{R}$ and $\beta \geqslant 0$, there exists a constant $C>0$ such that for all $\theta>0$,

$$
\begin{gather*}
\int_{\mathbb{R}^{d}}|z|^{\beta} \psi(|z|) p_{\theta}(z) \mathrm{d} z \leqslant C \theta^{\frac{\beta}{2}} \psi\left(\theta^{\frac{1}{2}}\right),  \tag{2.13}\\
\left\|\nabla^{k} \partial_{\theta}^{j} \mathbf{P}_{\theta} f\right\|_{\infty} \leqslant C[f]_{\psi} \theta^{-\frac{k}{2}-j} \psi\left(\theta^{\frac{1}{2}}\right), k, j=0,1 . \tag{2.14}
\end{gather*}
$$

Proof. Let $\psi \in \mathscr{R}_{\alpha}$ for some $\alpha \in[0,1]$. By the change of variables and (2.6), for any $\delta \in(0,1)$, we have

$$
\begin{aligned}
& \int_{\mathbb{R}^{d}}|z|^{\beta} \psi(|z|) p_{\theta}(z) \mathrm{d} z=\theta^{\frac{\beta}{2}} \int_{\mathbb{R}^{d}}|z|^{j} \psi\left(\theta^{\frac{1}{2}}|z|\right) p_{1}(z) \mathrm{d} z \\
& \preceq \theta^{\frac{\beta}{2}} \psi\left(\theta^{\frac{1}{2}}\right) \int_{\mathbb{R}^{d}}|z|^{\beta}\left(|z|^{\alpha+\delta} \vee|z|^{\alpha-\delta}\right) p_{1}(z) \mathrm{d} z
\end{aligned}
$$

which gives (2.13).
Next, for any $x \in \mathbb{R}^{d}$, let $f_{x}=f-f(x)$. By (2.11), (2.12) and (2.13) we obtain

$$
\begin{aligned}
\left|\nabla^{k} \partial_{\theta}^{j} \mathbf{P}_{\theta} f\right|(x) & =\left|\nabla^{k} \partial_{\theta}^{j} \mathbf{P}_{\theta} f_{x}\right|(x) \leqslant \int_{\mathbb{R}^{d}}\left|f_{x}(x+z)\right|\left|\nabla^{k} \partial_{\theta}^{j} p_{\theta}(z)\right| \mathrm{d} z \\
& \preceq[f]_{\psi} \int_{\mathbb{R}^{d}} \frac{|z|^{k}\left(\theta+|z|^{2}\right)^{j} \psi(|z|)}{\theta^{k+2 j}} p_{\theta}(z) \mathrm{d} z \preceq[f]_{\psi} \theta^{-\frac{k}{2}-j} \psi\left(\theta^{\frac{1}{2}}\right) .
\end{aligned}
$$

This proves (2.14).

We have the following commutator estimate result. A similar version for the Cauchy semigroup can be found in [3]. As an advantage of the present result, it applies to $f \in$ $\mathscr{B}_{p}\left(\mathbb{R}^{d}\right)$, the class of measurable functions with polynomial growth.

Lemma 2.6. Let $\psi \in \mathscr{R}_{\alpha}$ for some $\alpha \in[0,1]$ and $\phi: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$be increasing so that $\psi \phi$ satisfies for some $C>0$,

$$
\begin{equation*}
(\psi \phi)(t+s) \leqslant C((\psi \phi)(t)+(\psi \phi)(s)), \quad t, s>0 \tag{2.15}
\end{equation*}
$$

Suppose also that $\psi(t)$ is increasing on $[0,1]$ if $\alpha=0$, and $t^{-1} \psi(t)$ is decreasing on $[0,1]$ if $\alpha=1$. Then there exists a constant $C>0$ such that for any $f \in \mathscr{B}_{p}\left(\mathbb{R}^{d}\right)$ and $g \in \mathscr{B}_{b}\left(\mathbb{R}^{d}\right)$,

$$
\begin{equation*}
\left[\partial_{\theta} \mathbf{P}_{\theta}(f g)-f \partial_{\theta} \mathbf{P}_{\theta} g\right]_{\psi} \leqslant C[f]_{\psi \phi}\|g\|_{\infty} \theta^{-1} \phi\left(\theta^{\frac{1}{2}}\right), \quad \theta \in(0,1] \tag{2.16}
\end{equation*}
$$

Proof. By definition (2.10), we have

$$
\begin{equation*}
F_{\theta}(x):=\partial_{\theta} \mathbf{P}_{\theta}(f g)(x)-f(x) \partial_{\theta} \mathbf{P}_{\theta} g(x)=\int_{\mathbb{R}^{d}}(f(z)-f(x)) g(z) \partial_{\theta} p_{\theta}(x-z) \mathrm{d} z \tag{2.17}
\end{equation*}
$$

which, by (2.12), (2.11) and (2.13), implies that for all $\theta>0$,

$$
\begin{equation*}
\left\|F_{\theta}\right\|_{\infty} \preceq[f]_{\psi \phi}\|g\|_{\infty} \int_{\mathbb{R}^{d}}(\psi \phi)(|x-z|)\left|\partial_{\theta} p_{\theta}(z-x)\right| \mathrm{d} z \preceq[f]_{\psi \phi}\|g\|_{\infty} \theta^{-1}(\psi \phi)\left(\theta^{\frac{1}{2}}\right) . \tag{2.18}
\end{equation*}
$$

Thus, when $1 \geqslant|x-y|^{2} \geqslant \theta$, by (2.6) for $\alpha \in(0,1]$ and by the increasing property of $\psi$ for $\alpha=0$, we have

$$
\begin{equation*}
\left|F_{\theta}(x)-F_{\theta}(y)\right| \leqslant 2\left\|F_{\theta}\right\|_{\infty} \preceq[f]_{\psi \phi}\|g\|_{\infty} \psi(|x-y|) \theta^{-1} \phi\left(\theta^{\frac{1}{2}}\right) . \tag{2.19}
\end{equation*}
$$

On the other hand, by (2.17) we have

$$
\begin{align*}
F_{\theta}(x)-F_{\theta}(y) & =\int_{\mathbb{R}^{d}}(f(z)-f(x)) g(z)\left(\partial_{\theta} p_{\theta}(x-z)-\partial_{\theta} p_{\theta}(y-z)\right) \mathrm{d} z  \tag{2.20}\\
& +\int_{\mathbb{R}^{d}}(f(y)-f(x)) g(z) \partial_{\theta} p_{\theta}(y-z) \mathrm{d} z=: I_{1}+I_{2}
\end{align*}
$$

When $|x-y|^{2} \leqslant \theta \leqslant 1$, by (2.12), (2.15), (2.11) and (2.13), we have

$$
\begin{align*}
& \left|I_{1}\right| \preceq[f]_{\psi \phi}\|g\|_{\infty}|x-y| \int_{\mathbb{R}^{d} \times[0,1]}(\psi \phi)(|x-z|)\left|\nabla \partial_{\theta} p_{\theta}(x-z+r(y-x))\right| \mathrm{d} z \mathrm{~d} r \\
& \preceq[f]_{\psi \phi}\|g\|_{\infty}|x-y| \int_{\mathbb{R}^{d} \times[0,1]}[(\psi \phi)(|x-z+r(y-x)|)+(\psi \phi)(|x-y|)]  \tag{2.21}\\
& \quad \times\left|\nabla \partial_{\theta} p_{\theta}(x-z+r(y-x))\right| \mathrm{d} z \mathrm{~d} r \\
& \preceq[f]_{\psi \phi}\|g\|_{\infty}|x-y|(\psi \phi)\left(\theta^{\frac{1}{2}}\right) \theta^{-\frac{3}{2}} \preceq[f]_{\psi \phi}\|g\|_{\infty} \psi(|x-y|) \theta^{-1} \phi\left(\theta^{\frac{1}{2}}\right),
\end{align*}
$$

where the last step is due to (2.7) for $\alpha \in[0,1)$ and the decreasing property of $t^{-1} \psi(t)$ for $\alpha=1$. Moreover, since $\phi$ is increasing, when $|x-y|^{2} \leqslant \theta$, it follows from (2.12), (2.13) that

$$
\left|I_{2}\right| \preceq[f]_{\psi \phi}\|g\|_{\infty}(\psi \phi)(|x-y|) \theta^{-1} \leqslant[f]_{\psi \phi}\|g\|_{\infty} \psi(|x-y|) \theta^{-1} \phi\left(\theta^{\frac{1}{2}}\right)
$$

Combining this with (2.19), (2.20) and (2.21), we obtain (2.16).

We are now able to characterize a Hölder-Dini continuous function by using the heat semigroup (see [20] for the characterization of Hölder space by using Poisson integrals).

Lemma 2.7. For any $\phi \in \mathscr{R}$ with $\int_{0}^{1} \frac{\phi(s)}{s} \mathrm{~d} s<\infty$, letting

$$
\begin{equation*}
\bar{\phi}(t)=t+t \int_{t}^{1} \frac{\phi(s)}{s^{2}} \mathrm{~d} s+\int_{0}^{t} \frac{\phi(s)}{s} \mathrm{~d} s, t \in(0,1) \tag{2.22}
\end{equation*}
$$

then we have

$$
\begin{equation*}
\|f\|_{\bar{\phi}} \preceq\|f\|_{\infty}+\sup _{\theta \in(0,1]}\left(\frac{\left\|\theta \partial_{\theta} \mathbf{P}_{\theta} f\right\|_{\infty}}{\phi\left(\theta^{\frac{1}{2}}\right)}\right), \quad f \in \mathscr{B}_{b}\left(\mathbb{R}^{d}\right) . \tag{2.23}
\end{equation*}
$$

In particular, if $\phi \in \mathscr{R}_{\alpha}$ for some $\alpha \in(0,1)$, then

$$
\begin{equation*}
\|f\|_{\phi} \asymp\|f\|_{\infty}+\sup _{\theta \in(0,1]}\left(\frac{\left\|\theta \partial_{\theta} \mathbf{P}_{\theta} f\right\|_{\infty}}{\phi\left(\theta^{\frac{1}{2}}\right)}\right), \quad f \in \mathscr{B}_{b}\left(\mathbb{R}^{d}\right) . \tag{2.24}
\end{equation*}
$$

Proof. Notice that

$$
f(x)=\mathbf{P}_{\theta} f(x)-\int_{0}^{\theta} \partial_{s} \mathbf{P}_{s} f(x) \mathrm{d} s
$$

Since $\left\|\nabla \mathbf{P}_{s} f\right\|_{\infty} \preceq\|f\|_{\infty} / \sqrt{s}$ for $s>0$ and $\partial_{s} \nabla \mathbf{P}_{s} f(x)=\nabla \mathbf{P}_{s / 2}\left(\partial_{r} \mathbf{P}_{r}\right)_{r=s / 2} f(x)$, we have

$$
\nabla \mathbf{P}_{\theta} f(x)=\int_{\theta}^{\infty} \partial_{s} \nabla \mathbf{P}_{s} f(x) \mathrm{d} s=\int_{\theta}^{\infty} \nabla \mathbf{P}_{s / 2}\left(\partial_{r} \mathbf{P}_{r}\right)_{r=s / 2} f(x) \mathrm{d} s
$$

which, by (2.14), implies that for $\theta \in(0,1]$,

$$
\left\|\nabla \mathbf{P}_{\theta} f\right\|_{\infty} \preceq \ell(f)\left(\int_{1}^{\infty} s^{-\frac{3}{2}} \mathrm{~d} s+\int_{\theta}^{1} s^{-\frac{3}{2}} \phi\left(s^{\frac{1}{2}}\right) \mathrm{d} s\right) \preceq \ell(f)\left(1+\int_{\sqrt{\theta}}^{1} s^{-2} \phi(s) \mathrm{d} s\right),
$$

where $\ell(f)$ is the quantity of the right hand side of (2.23). Hence,

$$
\begin{aligned}
|f(x)-f(y)| & \leqslant\left\|\nabla \mathbf{P}_{\theta} f\right\|_{\infty}|x-y|+2 \int_{0}^{\theta}\left\|\partial_{s} \mathbf{P}_{s} f\right\|_{\infty} \mathrm{d} s \\
& \preceq \ell(f)\left(|x-y|+|x-y| \int_{\sqrt{\theta}}^{1} s^{-2} \phi(s) \mathrm{d} s+\int_{0}^{\theta} s^{-1} \phi\left(s^{\frac{1}{2}}\right) \mathrm{d} s\right),
\end{aligned}
$$

which in turn implies that by letting $\theta=|x-y|^{2} \leqslant 1$,

$$
|f(x)-f(y)| \preceq \ell(f) \bar{\phi}(|x-y|)
$$

where $\bar{\phi}$ is defined by (2.22). If $\alpha \in(0,1)$, by (2.7) and (2.8), we have $\bar{\phi}(t) \preceq \phi(t)$. Thus, (2.24) follows by (2.23) and (2.18) with $g=1$ and $\psi=1$.

Next, we consider the product space $\mathbb{R}^{d_{1}+d_{2}}$. For any $\psi_{1}, \psi_{2}: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$and $f \in$ $C\left(\mathbb{R}^{d_{1}+d_{2}}\right)$, set

$$
\begin{array}{cc}
{[f]_{\psi_{1}, \infty}:=\sup _{x^{(2)} \in \mathbb{R}^{d_{2}}}\left[f\left(\cdot, x^{(2)}\right)\right]_{\psi_{1}},} & {[f]_{\infty, \psi_{2}}:=\sup _{x^{(1)} \in \mathbb{R}^{d_{1}}}\left[f\left(x^{(1)}, \cdot\right)\right]_{\psi_{2}},} \\
{[f]_{\psi_{1}, \psi_{2}}:=[f]_{\psi_{1}, \infty}+[f]_{\infty, \psi_{2}},} & \|f\|_{\psi_{1}, \psi_{2}}:=[f]_{\psi_{1}, \psi_{2}}+\|f\|_{\infty},
\end{array}
$$

and for simplicity,

$$
[f]_{\psi}:=[f]_{\psi, \psi}, \quad\|f\|_{\psi}:=\|f\|_{\psi, \psi}
$$

Let $\mathbf{P}_{\theta}^{(i)}$ be the heat semigroup on $\mathbb{R}^{d_{i}}$, we set for $x=\left(x^{(1)}, x^{(2)}\right) \in \mathbb{R}^{d_{1}+d_{2}}$,

$$
\begin{equation*}
\mathbf{P}_{\theta}^{(1)} f(x)=\left\{\mathbf{P}_{\theta}^{(1)} f\left(\cdot, x^{(2)}\right)\right\}\left(x^{(1)}\right), \quad \mathbf{P}_{\theta}^{(2)} f(x)=\left\{\mathbf{P}_{\theta}^{(2)} f\left(x^{(1)}, \cdot\right)\right\}\left(x^{(2)}\right) \tag{2.25}
\end{equation*}
$$

Obviously, Lemmas 2.6 and 2.7 apply to both $\left(\|\cdot\|_{\phi, \infty}, \mathbf{P}_{\theta}^{(1)}\right)$ and $\left(\|\cdot\|_{\infty, \phi}, \mathbf{P}_{\theta}^{(2)}\right)$. For instance, letting $\mathbf{P}_{\theta}=\mathbf{P}_{\theta}^{(1)} \mathbf{P}_{\theta}^{(2)}$ be the Gaussian heat semigroup on $\mathbb{R}^{d_{1}+d_{2}}$, by the contractivity of $\mathbf{P}_{\theta}^{(i)}$ under the uniform norm, Lemma 2.6 implies the following result.
Lemma 2.8. Let $\psi_{1}, \psi_{2} \in \mathscr{R}$ and $\phi: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$be increasing such that $\psi_{i}, i=1,2$ and $\phi$ satisfy the same assumptions as in Lemma 2.6. Then there exists a constant $C>0$ such that

$$
\begin{equation*}
\left[\partial_{\theta} \mathbf{P}_{\theta}(f g)-f \partial_{\theta} \mathbf{P}_{\theta} g\right]_{\psi_{1}, \psi_{2}} \leqslant C[f]_{\psi_{1} \phi, \psi_{2} \phi}\|g\|_{\infty} \theta^{-1} \phi\left(\theta^{\frac{1}{2}}\right), \quad \theta>0 \tag{2.26}
\end{equation*}
$$

Finally, the following result characterizes $\|\cdot\|_{\infty, \phi}$ by using $\mathbf{P}^{(1)}$, and the same holds for $\left(\|\cdot\|_{\phi, \infty}, \mathbf{P}_{\theta}^{(2)}\right)$.
Lemma 2.9. For any $\psi_{1}, \psi_{2} \in \mathscr{R}$ and $\phi: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$, there exists a constant $C>0$ such that

$$
\left[\partial_{\theta} \mathbf{P}_{\theta}^{(1)}(f g)-f \partial_{\theta} \mathbf{P}_{\theta}^{(1)} g\right]_{\infty, \phi} \leqslant C[f]_{\psi_{1}, \psi_{2}}\|g\|_{\infty, \psi_{2}} \sup _{s \in(0,1]}\left(\frac{\psi_{1}\left(\theta^{\frac{1}{2}}\right) \psi_{2}(s)+\psi_{1}\left(\theta^{\frac{1}{2}}\right) \wedge \psi_{2}(s)}{\phi(s)}\right) \theta^{-1}
$$

holds for all $\theta \in(0,1]$ and measurable functions $f, g$ on $\mathbb{R}^{d_{1}+d_{2}}$.
Proof. By definition, we have

$$
F_{\theta}(x)=\partial_{\theta} \mathbf{P}_{\theta}^{(1)}(f g)(x)-f(x) \partial_{\theta} \mathbf{P}_{\theta}^{(1)} g(x)=\int_{\mathbb{R}^{d_{1}}} G\left(z^{(1)}, x^{(1)}, x^{(2)}\right) \partial_{\theta} p_{\theta}\left(x^{(1)}-z^{(1)}\right) \mathrm{d} z^{(1)}
$$

where

$$
G\left(z^{(1)}, x^{(1)}, x^{(2)}\right):=\left(f\left(z^{(1)}, x^{(2)}\right)-f\left(x^{(1)}, x^{(2)}\right)\right) g\left(z^{(1)}, x^{(2)}\right)
$$

Clearly, by (2.12) we have

$$
\begin{aligned}
&\left|G\left(z^{(1)}, x^{(1)}, x^{(2)}\right)-G\left(z^{(1)}, x^{(1)}, y^{(2)}\right)\right| \leqslant \psi_{1}\left(\left|x^{(1)}-z^{(1)}\right|\right)[f]_{\psi_{1}, \infty} \psi_{2}\left(\left|x^{(2)}-y^{(2)}\right|\right)[g]_{\infty, \psi_{2}} \\
&+2\left(\left(\psi_{1}\left(\left|x^{(1)}-z^{(1)}\right|\right)[f]_{\psi_{1}, \infty}\right)\right.\left.\wedge\left(\psi_{2}\left(\left|x^{(2)}-y^{(2)}\right|\right)[f]_{\infty, \psi_{2}}\right)\right)\|g\|_{\infty} .
\end{aligned}
$$

Hence, for $x, y \in \mathbb{R}^{d_{1}+d_{2}}$ with $x^{(1)}=y^{(1)}$, by (2.13), we obtain

$$
\begin{aligned}
\left|F_{\theta}(x)-F_{\theta}(y)\right| & \preceq[f]_{\psi_{1}, \infty}[g]_{\infty, \psi_{2}} \psi_{1}\left(\theta^{\frac{1}{2}}\right) \psi_{2}\left(\left|x^{(2)}-y^{(2)}\right|\right) \theta^{-1} \\
& +[f]_{\psi_{1}, \psi_{2}}\|g\|_{\infty}\left(\psi_{1}\left(\theta^{\frac{1}{2}}\right) \wedge \psi_{2}\left(\left|x^{(2)}-y^{(2)}\right|\right)\right) \theta^{-1}
\end{aligned}
$$

which in turn gives the desired estimate by dividing both sides by $\phi\left(\left|x^{(2)}-y^{(2)}\right|\right)$ and then taking supremum for $\left|x^{(2)}-y^{(2)}\right| \leqslant 1$.

### 2.4 Gradient estimates for linear stochastic Hamiltonian system

Let $B: \mathbb{R}_{+} \rightarrow \mathbb{R}^{d_{1}} \otimes \mathbb{R}^{d_{2}}, \sigma: \mathbb{R}_{+} \rightarrow \mathbb{R}^{d_{2}} \otimes \mathbb{R}^{d_{2}}$ be measurable such that $B_{r} B_{r}^{*}$ and $\sigma_{r}$ are invertible with

$$
\begin{equation*}
\kappa:=\sup _{r \in \mathbb{R}_{+}}\left(\left|B_{r}\right|+\left|\sigma_{r}\right|+\left|\left(B_{r} B_{r}^{*}\right)^{-1}\right|+\left|\sigma_{r}^{-1}\right|\right)<\infty . \tag{2.27}
\end{equation*}
$$

For $x=\left(x^{(1)}, x^{(2)}\right) \in \mathbb{R}^{d_{1}+d_{2}}$ and $0 \leqslant s \leqslant t$, define

$$
\begin{equation*}
X_{s, t}(x)=\left(x^{(1)}+\Gamma_{s, t} x^{(2)}+\int_{s}^{t} B_{r} \mathrm{~d} r \int_{s}^{r} \sigma_{r^{\prime}} \mathrm{d} W_{r^{\prime}}, x^{(2)}+\int_{s}^{t} \sigma_{r} \mathrm{~d} W_{r}\right) \tag{2.28}
\end{equation*}
$$

where $\left(W_{r}\right)_{r \geqslant 0}$ is a $d_{2}$-dimensional standard Brownian motion, and

$$
\begin{equation*}
\Gamma_{s, t}=\int_{s}^{t} B_{r} \mathrm{~d} r \tag{2.29}
\end{equation*}
$$

Clearly, $X_{s, t}(x)=\left(X_{s, t}^{(1)}, X_{s, t}^{(2)}\right)$ solves the following degenerate linear equation for $t \geqslant s$ :

$$
\begin{cases}\mathrm{d} X_{s, t}^{(1)}=B_{t} X_{s, t}^{(2)} \mathrm{d} t, & X_{s, s}^{(1)}=x^{(1)},  \tag{2.30}\\ \mathrm{d} X_{s, t}^{(2)}=\sigma_{t} \mathrm{~d} W_{t}, & X_{s, s}^{(2)}=x^{(2)} .\end{cases}
$$

Let $P_{s, t}$ be the Markov operator associated with $X_{s, t}(x)$, i.e.,

$$
P_{s, t} f(x)=\mathbb{E} f\left(X_{s, t}(x)\right), f \in \mathscr{B}_{p}\left(\mathbb{R}^{d_{1}+d_{2}}\right)
$$

We first investigate the derivative estimates of $P_{s, t} f$. To this end, we collect some frequently used notations here.

- For a smooth function $f$ on $\mathbb{R}^{d_{1}+d_{2}}, \nabla^{(1)} f$ and $\nabla^{(2)} f$ denotes the gradient of $f$ with respect to the variables $x^{(1)}$ and $x^{(2)}$ respectively. In particular, by (2.28) we have

$$
\begin{equation*}
\nabla^{(1)} P_{s, t} f=P_{s, t} \nabla^{(1)} f, \quad P_{s, t} \nabla^{(2)} f=\nabla^{(2)} P_{s, t} f-\Gamma_{s, t} \nabla^{(1)} P_{s, t} f . \tag{2.31}
\end{equation*}
$$

- For $h=\left(h^{(1)}, h^{(2)}\right) \in \mathbb{R}^{d_{1}+d_{2}}$, we also write

$$
\nabla f:=\left(\nabla^{(1)} f, \nabla^{(2)} f\right), \nabla_{h} f:=\langle\nabla f, h\rangle=\nabla_{h^{(1)}}^{(1)} f+\nabla_{h^{(2)}}^{(2)} f .
$$

- Let $\mathscr{U}$ be the set of all increasing functions $\phi: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$with the property

$$
\begin{equation*}
\phi(r t) \leqslant C r^{\delta} \phi(t), \quad r \geqslant 1, \quad t>0 \tag{2.32}
\end{equation*}
$$

for some $C, \delta>0$. Notice that by (2.6),

$$
\mathscr{D}_{0} \cap \mathscr{R} \subset \mathscr{U} .
$$

To estimate the derivatives of $P_{s, t} f$, we first present a Bismut type derivative formula which can be found in [28], [11] and [25]. For readers' convenience we state the formula in details and present a simple proof.

Fix $0 \leqslant s \leqslant t$ and define

$$
Q_{s, t}=\int_{s}^{t}(t-r)(r-s) B_{r} B_{r}^{*} \mathrm{~d} r \in \mathbb{R}^{d_{1}} \otimes \mathbb{R}^{d_{1}}
$$

By (2.27), it holds that for some $C>0$,

$$
\left|Q_{s, t}^{-1}\right| \leqslant C(t-s)^{-3}, \quad t>s
$$

For $h=\left(h^{(1)}, h^{(2)}\right) \in \mathbb{R}^{d_{1}+d_{2}}$, define for $r \in[s, t]$,

$$
\begin{equation*}
\Phi_{s, t}^{h}(r)=\frac{h^{(2)}}{t-s}+(t+s-2 r) B_{s}^{*} Q_{s, t}^{-1}\left[h^{(1)}+\int_{s}^{t} \frac{t-r^{\prime}}{t-s} B_{r^{\prime}} h^{(2)} \mathrm{d} r^{\prime}\right] . \tag{2.33}
\end{equation*}
$$

Obviously, by (2.27), there exists a constant $C>0$ such that

$$
\begin{equation*}
\left|\Phi_{s, t}^{h}(r)\right| \leqslant C\left(\frac{\left|h^{(2)}\right|}{t-s}+\frac{\left|h^{(1)}\right|}{(t-s)^{2}}\right), \quad 0 \leqslant s<t<\infty, r \in[s, t] . \tag{2.34}
\end{equation*}
$$

Theorem 2.10. For $n \in \mathbb{N}$, $s=s_{0}<s_{1} \cdots<s_{n}=t$ and $h_{1}, \cdots, h_{n} \in \mathbb{R}^{d_{1}+d_{2}}$, let

$$
\begin{equation*}
\breve{h}_{i}=\left(h_{i}^{(1)}+\Gamma_{s, s_{i-1}} h_{i}^{(2)}, h_{i}^{(2)}\right), \quad \xi_{s_{i-1}, s_{i}}^{\breve{h}_{i}}=\int_{s_{i-1}}^{s_{i}}\left\langle\sigma_{r}^{-1} \Phi_{s, t}^{\breve{h}_{i}}(r), \mathrm{d} W_{r}\right\rangle, \tag{2.35}
\end{equation*}
$$

where $\Gamma_{s, s_{i-1}}$ is defined by (2.29) and $i=1, \cdots, n$. Then for any $f \in \mathscr{B}_{p}\left(\mathbb{R}^{d_{1}+d_{2}}\right)$, we have

$$
\begin{equation*}
\nabla_{h_{1}} \cdots \nabla_{h_{n}} P_{s, t} f(x)=\mathbb{E}\left[f\left(X_{s, t}(x)\right) \prod_{i=1}^{n} \xi_{s_{i-1}, s_{i}}^{h_{i}}\right], x \in \mathbb{R}^{d_{1}+d_{2}} . \tag{2.36}
\end{equation*}
$$

Proof. (i) First of all, we consider the case of $n=1$. For $\varepsilon \in(0,1)$, define

$$
W_{r}^{\varepsilon}=W_{r}-\varepsilon \int_{s}^{r} \sigma_{r^{\prime}}^{-1} \Phi_{s, t}^{h}\left(r^{\prime}\right) \mathrm{d} r^{\prime}, \quad r \in[s, t]
$$

By Camaron-Martin's theorem, $\left(W_{r}^{\varepsilon}\right)_{r \in[s, t]}$ is still a Brownian motion under the probability measure $\mathrm{dP}_{\varepsilon}:=R_{\varepsilon} \mathrm{dP}$, where

$$
\begin{equation*}
R_{\varepsilon}:=\exp \left[\varepsilon \int_{s}^{t}\left\langle\sigma_{r}^{-1} \Phi_{s, t}^{h}(r), \mathrm{d} W_{r}\right\rangle-\frac{\varepsilon^{2}}{2} \int_{s}^{t}\left|\sigma_{r}^{-1} \Phi_{s, t}^{h}(r)\right|^{2} \mathrm{~d} r\right] . \tag{2.37}
\end{equation*}
$$

Thus, if we write

$$
X_{s, t}^{\varepsilon}(x):=\left(x^{(1)}+\varepsilon h^{(1)}+\int_{s}^{t} B_{r}\left[x^{(2)}+\varepsilon h^{(2)}+\int_{s}^{r} \sigma_{r^{\prime}} \mathrm{d} W_{r^{\prime}}^{\varepsilon}\right] \mathrm{d} r, x^{(2)}+\varepsilon h^{(2)}+\int_{s}^{t} \sigma_{r} \mathrm{~d} W_{r}^{\varepsilon}\right)
$$

then the law of $X_{s, t}(x+\varepsilon h)$ under $\mathbb{P}$ is the same as the law of $X_{s, t}^{\varepsilon}(x)$ under $\mathbb{P}_{\varepsilon}$, that is,

$$
P_{s, t} f(x+\varepsilon h)=\mathbb{E} f\left(X_{s, t}(x+\varepsilon h)\right)=\mathbb{E}\left(R_{\varepsilon} f\left(X_{s, t}^{\varepsilon}(x)\right)\right) .
$$

On the other hand, by definition (2.33), it is easy to see that
$X_{s, t}^{\varepsilon}(x)=X_{s, t}(x)+\varepsilon\left(h^{(1)}+\int_{s}^{t} B_{r}\left[h^{(2)}-\int_{s}^{r} \Phi_{s, t}^{h}\left(r^{\prime}\right) \mathrm{d} r^{\prime}\right] \mathrm{d} r, h^{(2)}-\int_{s}^{t} \Phi_{s, t}^{h}(r) \mathrm{d} r\right)=X_{s, t}(x)$.
Hence,

$$
\nabla_{h} P_{s, t} f(x)=\lim _{\varepsilon \neq 0} \frac{1}{\varepsilon} \mathbb{E}\left[f\left(X_{s, t}(x+\varepsilon h)\right)-f\left(X_{s, t}(x)\right)\right]=\lim _{\varepsilon \notin 0} \mathbb{E}\left[\frac{R_{\varepsilon}-1}{\varepsilon} f\left(X_{s, t}(x)\right)\right],
$$

which together with (2.37) yields (2.36) for $n=1$.
(ii) Assuming that (2.36) holds for $n=k \in \mathbb{N}$, we intend to prove (2.36) for $n=k+1$. Noticing that $P_{s, t} f=P_{s, s_{k}} P_{s_{k}, t} f$ and by definition (2.28),

$$
\nabla_{h_{k+1}} X_{s, s_{k}}=\left(h_{k+1}^{(1)}+\Gamma_{s, s_{k}} h_{k+1}^{(2)}, h_{k+1}^{(2)}\right)=\breve{h}_{k+1},
$$

by induction hypothesis, we have

$$
\begin{aligned}
\nabla_{h_{k+1}} \nabla_{h_{k}} \cdots \nabla_{h_{1}} P_{s, t} f(x) & =\nabla_{h_{k+1}} \mathbb{E}\left[\left(P_{s_{k}, t} f\right)\left(X_{s, s_{k}}(x)\right) \prod_{i=1}^{k} \xi_{s_{i-1}, s_{i}}^{\breve{h}_{i}}\right] \\
& =\mathbb{E}\left[\nabla_{h_{k+1}}\left(P_{s_{k}, t} f\left(X_{s, s_{k}}(x)\right)\right) \prod_{i=1}^{k} \xi_{s_{i-1}, s_{i}}^{\breve{h}_{i}}\right] \\
& =\mathbb{E}\left[\left(\nabla_{\breve{h}_{k+1}} P_{s_{k}, t} f\right)\left(X_{s, s_{k}}(x)\right) \prod_{i=1}^{k} \xi_{s_{i-1}, s_{i}}^{\breve{h}_{i}}\right] \\
& =\mathbb{E}\left[f\left(X_{s, t}(x)\right) \prod_{i=1}^{k+1} \xi_{s_{i-1}, s_{i}} \breve{h}_{i}\right]
\end{aligned}
$$

where in the last step we have used the independence of $\left\{X_{s, s_{k}}(x), \xi_{s_{i-1}, s_{i}}^{h_{i}}, i=1, \cdots, k\right\}$ and $\left\{X_{s_{k}, t}(x), \xi_{s_{k}, s_{k+1}}^{\breve{h}_{k+1}}\right\}$. The proof is complete.

Lemma 2.11. For any $p \geqslant 1$ and $\phi \in \mathscr{U}$, there is a constant $C=C(\phi, p, \kappa)>0$, where $\kappa$ is given in (2.27), such that for all $0 \leqslant s<t<\infty$,

$$
\begin{equation*}
\left\|\phi\left(\left|X_{s, t}^{(1)}(0)\right|\right)\right\|_{p} \leqslant C \phi\left((t-s)^{\frac{3}{2}}\right), \quad\left\|\phi\left(\left|X_{s, t}^{(2)}(0)\right|\right)\right\|_{p} \leqslant C \phi\left((t-s)^{\frac{1}{2}}\right) \tag{2.38}
\end{equation*}
$$

where $\|\cdot\|_{p}:=\left(\mathbb{E}|\cdot|^{p}\right)^{\frac{1}{p}}$.

Proof. First of all, by (2.28) and Burkholder's inequality, for any $p \geqslant 1$ there is a constant $C=C(p, \kappa)>0$ such that for all $0 \leqslant s<t<\infty$,

$$
\begin{equation*}
\left\|X_{s, t}^{(1)}(0)\right\|_{p} \leqslant C(t-s)^{\frac{3}{2}}, \quad\left\|X_{s, t}^{(2)}(0)\right\|_{p} \leqslant C(t-s)^{\frac{1}{2}} \tag{2.39}
\end{equation*}
$$

On the other hand, since $\phi \in \mathscr{U}$ is increasing, by (2.32) we obtain

$$
\begin{aligned}
\left\|\phi\left(\left|X_{s, t}^{(1)}(0)\right|\right)\right\|_{p} & =\left\|\phi\left((t-s)^{\frac{3}{2}}\left|(t-s)^{-\frac{3}{2}} X_{s, t}^{(1)}(0)\right|\right)\right\|_{p} \\
& \leqslant C \phi\left((t-s)^{\frac{3}{2}}\right)\left\|1+\left|(t-s)^{-\frac{3}{2}} X_{s, t}^{(1)}(0)\right|^{\delta}\right\|_{p} .
\end{aligned}
$$

Combining this with (2.39) we prove the first estimate. Similarly, we can prove the second estimate.

Below we present a simple consequence of the above formula, which will play crucial roles in the next section. In particular, as in [3], the pointwise estimate results given below allow us to borrow the Hölder regularity of $b^{(1)}$ to compensate the singularity along the first direction induced by the degeneracy.
Corollary 2.12. Let $\phi, \psi \in \mathscr{U}$. For any $T>0$ and $m, k \in \mathbb{N}_{0}=:\{0\} \cup \mathbb{N}$, there exists a constant $C>0$ such that for any $0 \leqslant s<t \leqslant T$ and any constants $K_{1}, K_{2} \geqslant 0$,

$$
\begin{equation*}
\left\|\left(\nabla^{(1)}\right)^{\otimes m}\left(\nabla^{(2)}\right)^{\otimes k} P_{s, t} f\right\|(0) \leqslant C\left(K_{1} \phi\left((t-s)^{\frac{3}{2}}\right)+K_{2} \psi\left((t-s)^{\frac{1}{2}}\right)\right)(t-s)^{-\frac{3 m}{2}-\frac{k}{2}} \tag{2.40}
\end{equation*}
$$

holds for any measurable function $f$ on $\mathbb{R}^{d_{1}+d_{2}}$ satisfying

$$
\begin{equation*}
|f(x)| \leqslant K_{1} \phi\left(\left|x^{(1)}\right|\right)+K_{2} \psi\left(\left|x^{(2)}\right|\right) \tag{2.41}
\end{equation*}
$$

Consequently, for any $m \in \mathbb{N}, k \in \mathbb{N}_{0}$ and any measurable function $f$ on $\mathbb{R}^{d_{1}+d_{2}}$,

$$
\begin{gather*}
\left\|\left(\nabla^{(1)}\right)^{\otimes m}\left(\nabla^{(2)}\right)^{\otimes k} P_{s, t} f\right\|_{\infty} \leqslant C[f]_{\phi, \infty} \phi\left((t-s)^{\frac{3}{2}}\right)(t-s)^{-\frac{3 m}{2}-\frac{k}{2}}  \tag{2.42}\\
\left\|\left(\nabla^{(2)}\right)^{\otimes k} P_{s, t} f\right\|_{\infty} \leqslant C\left([f]_{\phi, \infty} \phi\left((t-s)^{\frac{3}{2}}\right)+[f]_{\infty, \psi} \psi\left((t-s)^{\frac{1}{2}}\right)\right)(t-s)^{-\frac{k}{2}} \tag{2.43}
\end{gather*}
$$

Proof. We introduce the following notations:

$$
\breve{\xi}_{s, t}^{h}=\xi_{s, t}^{\breve{h}}, h \in \mathbb{R}^{d_{1}+d_{2}}, \breve{\xi}_{s, t}^{(\cdot, 0)}=\left(\breve{\xi}_{s, t}^{\left(e_{i}, 0\right)}\right)_{i=1, \cdots, d_{1}},
$$

where $\xi_{s, t}^{\breve{h}}$ is defined by (2.35), and $\left(e_{i}\right)_{i=1, \cdots, d_{1}}$ is the standard basis of $\mathbb{R}^{d_{1}}$. Similarly, we can define $\breve{\xi}_{s, t}^{(0, \cdot)} \in \mathbb{R}^{d_{2}}$. By (2.34), (2.35) and Burkholder's inequality, we have for any $T>0$ and $p \geqslant 1$,

$$
\begin{equation*}
\left\|\breve{\xi}_{s, t}^{(\cdot, 0)}\right\|_{p} \preceq(t-s)^{-\frac{3}{2}},\left\|\breve{\xi}_{s, t}^{(0, \cdot)}\right\|_{p} \preceq(t-s)^{-\frac{1}{2}}, 0 \leqslant s<t \leqslant T \tag{2.44}
\end{equation*}
$$

where $\|\cdot\|_{p}:=\left(\mathbb{E}|\cdot|^{p}\right)^{\frac{1}{p}}$.
Let $s_{i}=s+(t-s) i /(m+k), i=0,1, \cdots, m+k$ be the uniform partition of $[s, t]$. Using the above notations, by (2.36) we have

$$
\left\|\left(\nabla^{(1)}\right)^{\otimes m}\left(\nabla^{(2)}\right)^{\otimes k} P_{s, t} f\right\|(0) \leqslant \mathbb{E}\left\{\left|f\left(X_{s, t}(0)\right)\right| \cdot\left\|\prod_{i=1}^{m} \breve{\xi}_{s_{i-1}, s_{i}}^{(\cdot, 0)} \cdot \prod_{j=m+1}^{m+k} \breve{\xi}_{s_{j-1}, s_{j}}^{(0, \cdot)}\right\|\right\}
$$

Estimate (2.40) follows by Hölder's inequality and (2.41), (2.38), (2.44).
In general, for fixed $x_{0} \in \mathbb{R}^{d_{1}+d_{2}}$, let

$$
\begin{aligned}
& g_{x_{0}}(x):=f\left(x_{0}^{(1)}+\Gamma_{s, t} x_{0}^{(2)}, x^{(2)}+x_{0}^{(2)}\right), \\
& f_{x_{0}}(x):=f\left(x^{(1)}+x_{0}^{(1)}+\Gamma_{s, t} x_{0}^{(2)}, x^{(2)}+x_{0}^{(2)}\right)-g_{x_{0}}(x) .
\end{aligned}
$$

Noticing that $\nabla^{(1)} P_{s, t} g_{x_{0}} \equiv 0$, we have

$$
\left(\nabla^{(1)}\right)^{\otimes m}\left(\nabla^{(2)}\right)^{\otimes k} P_{s, t} f\left(x_{0}\right)=\left(\nabla^{(1)}\right)^{\otimes m}\left(\nabla^{(2)}\right)^{\otimes k} P_{s, t} f_{x_{0}}(0), \quad m \neq 0 .
$$

Thus, (2.42) follows from (2.40) with $K_{2}=0$. As for (2.43), it follows by (2.40).

## 3 A study for degenerate parabolic equations

Throughout this section, we fix $T, \lambda>0$ and consider the following degenerate parabolic equation with Hölder coefficients:

$$
\begin{equation*}
\partial_{t} u_{t}=\mathscr{L}_{t}^{\Sigma, b} u_{t}-\lambda u_{t}+f_{t}, u_{0}=0, \quad t \in[0, T], \tag{3.1}
\end{equation*}
$$

where $\mathscr{L}_{t}^{\Sigma, b}$ is defined by (1.3) and $f:[0, T] \times \mathbb{R}^{d_{1}+d_{2}} \rightarrow \mathbb{R}$ is measurable. The solution will be used in Section 4 to construct the diffeomorphism on $\mathbb{R}^{d_{1}+d_{2}}$ which transforms the original (1.2) into an equation with regular enough coefficients so that the existence and uniqueness of solutions are proved.

Before studying equation (3.1), we first estimate the gradients on $P_{s, t}\left(H \cdot \nabla^{(i)} f\right), i=1,2$, which are nontrivial consequences of Corollary 2.12, and will play a crucial role in estimating derivatives of $u_{t}$ in terms of the formula (3.32) below. For fixed $\phi \in \mathscr{D}_{0} \cap \mathscr{S}_{0}$, let

$$
\begin{equation*}
\Lambda_{\lambda}^{\phi}(t)=\mathrm{e}^{-\lambda t} t^{-1} \phi\left(t^{\frac{1}{2}}\right), \quad t \in(0, T] . \tag{3.2}
\end{equation*}
$$

### 3.1 Gradient estimates on $P_{s, t}\left(H \cdot \nabla^{(i)} f\right)$

Below all the constants appearing in $\preceq$ only depends on $T, d_{1}, d_{2}$ and $\phi$.
Lemma 3.1. Let $f \in C^{1}\left(\mathbb{R}^{d_{1}+d_{2}}\right)$ and $H \in C^{1}\left(\mathbb{R}^{d_{1}+d_{2}} ; \mathbb{R}^{d_{2}}\right)$ with $H(0)=0$. For $0 \leqslant s<t \leqslant$ $T$ and $k=0,1$, recalling the definition of $\phi_{[\alpha]}$ in (2.5), we have

$$
\begin{align*}
& \left\|\left(\nabla^{(2)}\right)^{\otimes(k+1)} P_{s, t}\left(H \cdot \nabla^{(2)} f\right)\right\|(0) \preceq[H]_{\phi}\left\|\nabla^{(2)} f\right\|_{\infty} \Lambda_{0}^{\phi}(t-s),  \tag{3.3}\\
& \left\|\nabla^{(1)}\left(\nabla^{(2)}\right)^{\otimes k} P_{s, t}\left(H \cdot \nabla^{(2)} f\right)\right\|(0) \preceq\|H\|_{\phi_{[(k+1) / 3]}, \infty}\left\|\nabla^{(2)} f\right\|_{\phi_{[(k+1) / 3]}, \infty} \Lambda_{0}^{\phi}(t-s),  \tag{3.4}\\
& \left\|\nabla^{(1)}\left(\nabla^{(2)}\right)^{\otimes k} P_{s, t}\left(H \cdot \nabla^{(2)} f\right)\right\|(0) \preceq[H]_{\left.\phi_{[2 / 3]}\right]}\left([f]_{1_{[(k+2) / 3]}, \infty}+\left\|\nabla^{(2)} f\right\|_{\infty}\right) \Lambda_{0}^{\phi}(t-s),  \tag{3.5}\\
& \left|\nabla^{(1)} P_{s, t}\left(H \cdot \nabla^{(2)} f\right)\right|(0) \preceq\left([H]_{1_{[2 / 3]}}+\left\|\nabla^{(2)} H\right\|_{\phi_{[1 / 9]}, \infty}\right)[f]_{1_{[2 / 3]}} \Lambda_{0}^{\phi}(t-s) . \tag{3.6}
\end{align*}
$$

Moreover, if for some $K>0$,

$$
\left|H\left(0, x^{(2)}\right)\right| \leqslant K\left|x^{(2)}\right| \phi\left(\left|x^{(2)}\right|\right),
$$

then

$$
\begin{align*}
& \left\|\left(\nabla^{(2)}\right)^{\otimes(k+1)} P_{s, t}\left(H \cdot \nabla^{(1)} f\right)\right\|(0) \preceq\left([H]_{\phi_{[1 / 3]}, \infty}+K\right)\left\|\nabla^{(1)} f\right\|_{\infty}(t-s)^{-\frac{k}{2}},  \tag{3.7}\\
& \left\|\nabla^{(1)}\left(\nabla^{(2)}\right)^{\otimes k} P_{s, t}\left(H \cdot \nabla^{(1)} f\right)\right\|(0) \preceq\left([H]_{\phi_{[(k+1) / 3]}, \phi}+K\right)\left\|\nabla^{(1)} f\right\|_{1_{[k / 3]}, \infty} \Lambda_{0}^{\phi}(t-s) . \tag{3.8}
\end{align*}
$$

Proof. (1) Since $H(0)=0$, recalling definition (2.5) and (2.12), we have

$$
|H(x)| \leqslant[H]_{\phi} \phi_{[0]}(|x|), \quad\left\|H \cdot \nabla^{(2)} f\right\|_{\phi_{[(k+1) / 3]}} \leqslant\|H\|_{\phi_{[(k+1) / 3]}}\left\|\nabla^{(2)} f\right\|_{\phi_{[(k+1) / 3]}}
$$

So, (3.3) follows by (2.40), and (3.4) follows by (2.42).
(2) To prove (3.5), we introduce

$$
f_{1}(x):=f\left(x^{(1)}, 0\right), f_{2}(x):=f\left(0, x^{(2)}\right), \widehat{f_{i}}(x):=f(x)-f_{i}(x), \quad i=1,2 .
$$

Moreover, for $\mathbf{P}_{\theta}^{(2)}$ being the heat semigroup on $\mathbb{R}^{d_{2}}$, let

$$
H_{2}^{\theta}=\mathbf{P}_{\theta}^{(2)} H_{2}-\mathbf{P}_{\theta}^{(2)} H_{2}(0), \quad \widehat{H}_{2}^{\theta}=H_{2}-H_{2}^{\theta}, \quad \theta>0 .
$$

We have

$$
\begin{equation*}
H \cdot \nabla^{(2)} f=\widehat{H}_{2} \cdot \nabla^{(2)} f+\widehat{H}_{2}^{\theta} \cdot \nabla^{(2)} f+H_{2}^{\theta} \cdot \nabla^{(2)} f, \quad \theta \in(0,1] . \tag{3.9}
\end{equation*}
$$

Below we investigate these three terms respectively.
(2a) Observing from (2.12) that

$$
\left|\widehat{H}_{2} \cdot \nabla^{(2)} f\right|(x) \leqslant[H]_{\phi_{[(k+1) / 3]}, \infty}\left\|\nabla^{(2)} f\right\|_{\infty} \phi_{[(k+1) / 3]}\left(\left|x^{(1)}\right|\right),
$$

by (2.40) we obtain

$$
\begin{equation*}
\left\|\nabla^{(1)}\left(\nabla^{(2)}\right)^{\otimes k} P_{s, t}\left(\widehat{H}_{2} \cdot \nabla^{(2)} f\right)\right\|(0) \preceq[H]_{\phi_{[(k+1) / 3]}, \infty}\left\|\nabla^{(2)} f\right\|_{\infty} \Lambda_{0}^{\phi}(t-s) . \tag{3.10}
\end{equation*}
$$

(2b) Since by (2.31) we have $\nabla^{(1)} P_{s, g} g=0$ for $g$ depending only on $x^{(2)}$, it follows that

$$
\nabla^{(1)} P_{s, t}\left(H_{2}^{\theta} \cdot \nabla^{(2)} f\right)=\nabla^{(1)} P_{s, t}\left(H_{2}^{\theta} \cdot \nabla^{(2)} \widehat{f_{2}}\right)=\nabla^{(1)} P_{s, t} \operatorname{div}^{(2)}\left(\widehat{f}_{2} H_{2}^{\theta}\right)-\nabla^{(1)} P_{s, t}\left(\widehat{f}_{2} \operatorname{div}^{(2)} H_{2}^{\theta}\right)
$$

Noting that Lemma 2.3 and (2.5) imply

$$
\begin{align*}
& \left|\widehat{f}_{2} H_{2}^{\theta}\right|(x) \leqslant\left\|\nabla^{(2)} H_{2}^{\theta}\right\|_{\infty}[f]_{1_{[2 / 3]}, \infty}\left|x^{(2)}\right|\left(\left|x^{(1)}\right|+\left|x^{(1)}\right|^{\frac{2}{3}}\right), \\
& \left|\widehat{f}_{2} \operatorname{div}^{(2)} H_{2}^{\theta}\right|(x) \leqslant\left\|\nabla^{(2)} H_{2}^{\theta}\right\|_{\infty}[f]_{1_{[2 / 3]}, \infty}\left(\left|x^{(1)}\right|+\left|x^{(1)}\right|^{\frac{2}{3}}\right), \tag{3.11}
\end{align*}
$$

from (2.31) and Corollary 2.12 we obtain

$$
\begin{align*}
\| \nabla^{(1)} P_{s, t} & \left(H_{2}^{\theta} \cdot \nabla^{(2)} f\right)\|(0) \leqslant\| \nabla^{(1)} P_{s, t} \operatorname{div}{ }^{(2)}\left(\widehat{f}_{2} H_{2}^{\theta}\right)\|(0)+\| \nabla^{(1)} P_{s, t}\left(\widehat{f}_{2} \operatorname{div}{ }^{(2)} H_{2}^{\theta}\right) \|(0) \\
& \leqslant\left\|\nabla^{(1)} \nabla^{(2)} P_{s, t}\left(\widehat{f}_{2} H_{2}^{\theta}\right)\right\|(0)+\left\|\Gamma_{s, t}\right\| \cdot\left\|\nabla^{(1)} \nabla^{(1)} P_{s, t}\left(\widehat{f}_{2} H_{2}^{\theta}\right)\right\|(0)  \tag{3.12}\\
& +\left\|\nabla^{(1)} P_{s, t}\left(\widehat{f}_{2} \operatorname{div}{ }^{(2)} H_{2}^{\theta}\right)\right\|(0) \preceq[f]_{1_{[2 / 3]}, \infty}\left\|\nabla^{(2)} H_{2}^{\theta}\right\|_{\infty}(t-s)^{-\frac{1}{2}} .
\end{align*}
$$

Similarly, using

$$
\left|\widehat{f}_{2} H_{2}^{\theta}\right|(x) \leqslant\left\|\nabla^{(2)} H_{2}^{\theta}\right\|_{\infty}[f]_{1_{[1]}, \infty}\left|x^{(2)}\right| \cdot\left|x^{(1)}\right|,
$$

$$
\left|\widehat{f}_{2} \operatorname{div}^{(2)} H_{2}^{\theta}\right|(x) \leqslant\left\|\nabla^{(2)} H_{2}^{\theta}\right\|_{\infty}[f]_{1_{[1]}, \infty}\left|x^{(1)}\right|
$$

to replace (3.11), we have

$$
\begin{equation*}
\left\|\nabla^{(1)} \nabla^{(2)} P_{s, t}\left(H_{2}^{\theta} \cdot \nabla^{(2)} f\right)\right\|(0) \preceq[f]_{1_{[1]}, \infty}\left\|\nabla^{(2)} H_{2}^{\theta}\right\|_{\infty}(t-s)^{-\frac{1}{2}} \tag{3.13}
\end{equation*}
$$

Moreover, by (2.14),

$$
\left\|\nabla^{(2)} H_{2}^{\theta}\right\|_{\infty} \preceq\left[H_{2}\right]_{\phi_{[2 / 3]}} \theta^{-\frac{1}{6}} \phi\left(\theta^{\frac{1}{2}}\right), \quad \theta \in(0,1] .
$$

Then (3.12) and (3.13) yield

$$
\begin{equation*}
\left\|\nabla^{(1)}\left(\nabla^{(2)}\right)^{\otimes k} P_{s, t}\left(H_{2}^{\theta} \cdot \nabla^{(2)} f\right)\right\|(0) \preceq[f]_{1_{[(k+2) / 3]}, \infty}[H]_{\infty, \phi[2 / 3]} \theta^{-\frac{1}{6}} \phi\left(\theta^{\frac{1}{2}}\right)(t-s)^{-\frac{1}{2}} \tag{3.14}
\end{equation*}
$$

(2c) Since $H(0)=0$, by (2.14) we have

$$
\begin{aligned}
\left|\widehat{H}_{2}^{\theta}(x)\right| & =\left|\int_{0}^{\theta}\left(\partial_{r} \mathbf{P}_{r}^{(2)} H_{2}(0)-\partial_{r} \mathbf{P}_{r}^{(2)} H_{2}(x)\right) \mathrm{d} r\right| \\
& \leqslant 2 \int_{0}^{\theta}\left\|\partial_{r} \mathbf{P}_{r}^{(2)} H_{2}\right\|_{\infty} \mathrm{d} r \preceq[H]_{\infty, \phi}[2 / 3] \\
& \int_{0}^{\theta} r^{-\frac{2}{3}} \phi\left(r^{\frac{1}{2}}\right) \mathrm{d} r \\
& \preceq[H]_{\infty, \phi[2 / 3]} \theta^{\frac{1}{3}} \phi\left(\theta^{\frac{1}{2}}\right), \quad \theta \in(0,1] .
\end{aligned}
$$

Thus, it follows from Corollary 2.12 that for $\theta \in(0,1]$,

Taking $\theta=(t-s)^{3}$, by combining (3.9) with (3.10), (3.14) and (3.15), we prove (3.5).
(3) We now prove (3.6). Since $\nabla^{(2)} f_{1}=0$ and $\nabla^{(1)} P_{s, t}\left(H_{2} \cdot \nabla^{(2)} f_{2}\right)=0$, we have

$$
\begin{align*}
& \nabla^{(1)} P_{s, t}\left(H \cdot \nabla^{(2)} f\right)=\nabla^{(1)} P_{s, t}\left(H_{2} \cdot \nabla^{(2)} \widehat{f}_{2}\right)+\nabla^{(1)} P_{s, t}\left(\widehat{H}_{2} \cdot \nabla^{(2)} \widehat{f}_{1}\right)  \tag{3.16}\\
& =\nabla^{(1)} P_{s, t} \operatorname{div}^{(2)}\left(\widehat{f}_{2} H_{2}+\widehat{f}_{1} \widehat{H}_{2}\right)-\nabla^{(1)} P_{s, t}\left(\widehat{f}_{2} \operatorname{div}^{(2)} H_{2}+\widehat{f}_{1} \operatorname{div}^{(2)} \widehat{H}_{2}\right)
\end{align*}
$$

Below we estimate these two terms respectively.
Firstly, by $\operatorname{div}^{(2)} \widehat{H}_{2}(x)=\operatorname{div}^{(2)} H(x)-\operatorname{div}^{(2)} H\left(0, x^{(2)}\right)$, we have

$$
\begin{aligned}
& \left|\widehat{f}_{2} \operatorname{div}^{(2)} H_{2}\right|(x) \preceq[f]_{1_{\left[\frac{2}{3}\right]}, \infty}\left\|\nabla^{(2)} H\right\|_{\infty}\left(\left|x^{(1)}\right|^{\frac{2}{3}}+\left|x^{(1)}\right|\right), \\
& \left.\left.\left|\widehat{f}_{1} \operatorname{div}^{(2)} \widehat{H}_{2}\right|(x) \preceq[f]_{\infty, 1\left[\frac{2}{3}\right]}\left[\nabla^{(2)} H\right]_{\left.\phi_{\left[\frac{1}{9}\right]}\right], \infty}\left(\left|x^{(2)}\right|^{\frac{2}{3}}+\left|x^{(2)}\right|\right) \phi_{\left[\frac{1}{9}\right]}| | x^{(1)} \right\rvert\,\right) .
\end{aligned}
$$

So, Corollary 2.12 implies

$$
\begin{align*}
& \left|\nabla^{(1)} P_{s, t}\left(\widehat{f}_{2} \operatorname{div}^{(2)} H_{2}\right)\right|(0) \preceq[f]_{1_{[2 / 3]}, \infty}| | \nabla^{(2)} H \|_{\infty}(t-s)^{-\frac{1}{2}},  \tag{3.17}\\
& \left|\nabla^{(1)} P_{s, t}\left(\widehat{f}_{1} \operatorname{div}^{(2)} \widehat{H}_{2}\right)\right|(0) \preceq[f]_{\infty, 1[2 / 3]}\left[\nabla^{(2)} H\right]_{\phi[1 / 9], \infty} \Lambda_{0}^{\phi}(t-s) .
\end{align*}
$$

Next, since

$$
\left|\widehat{f}_{2} H_{2}\right|(x)+\left|\widehat{f}_{1} \widehat{H}_{2}\right|(x) \leqslant[H]_{1_{[2 / 3]}}[f]_{\left.1_{[2 / 3]}\right]}\left(\left|x^{(2)}\right|+\left|x^{(2)}\right|^{\frac{2}{3}}\right)\left(\left|x^{(1)}\right|^{\frac{2}{3}}+\left|x^{(1)}\right|\right)
$$

it follows from (2.31) and Corollary 2.12 that

$$
\begin{aligned}
& \left|\nabla^{(1)} P_{s, t} \operatorname{div}^{(2)}\left(\widehat{f}_{2} H_{2}\right)\right|(0)+\left|\nabla^{(1)} P_{s, t} \operatorname{div}^{(2)}\left(\widehat{f}_{1} \widehat{H}_{2}\right)\right|(0) \\
& \preceq\left\|\nabla^{(1)} \nabla^{(2)} P_{s, t}\left(\widehat{f}_{2} H_{2}+\widehat{f}_{1} \widehat{H}_{2}\right)\right\|(0)+(t-s)\left\|\nabla^{(1)} \nabla^{(1)} P_{s, t}\left(\widehat{f}_{2} H_{2}+\widehat{f}_{1} \widehat{H}_{2}\right)\right\|(0) \\
& \preceq[H]_{1_{[2 / 3]}}[f]_{1_{[2 / 3]}}(t-s)^{-\frac{2}{3}} \preceq[H]_{1_{[2 / 3]}}[f]_{1_{[2 / 3]}} \Lambda_{0}^{\phi}(t-s) .
\end{aligned}
$$

Combining this with (3.16) and (3.17), we prove (3.6).
(4) Noticing that

$$
\left|H \cdot \nabla^{(1)} f\right|(x) \leqslant\left([H]_{\phi_{[1 / 3]}, \infty} \phi_{[1 / 3]}\left(\left|x^{(1)}\right|\right)+K\left|x^{(2)}\right| \phi\left(\left|x^{(2)}\right|\right)\right)\left\|\nabla^{(1)} f\right\|_{\infty},
$$

by Corollary 2.12, we obtain (3.7). Let $g:=H \cdot \nabla^{(1)} f$. Observing that

$$
\begin{aligned}
\left|\widehat{g}_{2}\right|(x) \leqslant & K\left[\nabla^{(1)} f\right]_{1_{[k / 3]}, \infty}\left|x^{(2)}\right| \phi\left(\left|x^{(2)}\right|\right)\left(\left|x^{(1)}\right|+\left|x^{(1)}\right| \frac{k}{3}\right) \\
& +[H]_{\left.\phi_{[(k+1) / 3]}\right)}\left\|\nabla^{(1)} f\right\|_{\infty} \phi_{[(k+1) / 3]}\left(\left|x^{(1)}\right|\right),
\end{aligned}
$$

and $\nabla^{(1)} g_{2} \equiv 0$, by Corollary 2.12 again, we have

$$
\begin{aligned}
\| \nabla^{(1)} & \left(\nabla^{(2)}\right)^{\otimes k} P_{s, t}\left(H \cdot \nabla^{(1)} f\right)\|(0)=\| \nabla^{(1)}\left(\nabla^{(2)}\right)^{\otimes k} P_{s, t} \widehat{g}_{2} \|(0) \\
& \preceq\left([H]_{\phi[(k+1) / 3], \phi}+K\right)\left\|\nabla^{(1)} f\right\|_{1_{[k / 3]}, \infty} \Lambda_{0}^{\phi}(t-s) .
\end{aligned}
$$

The proof is complete.

### 3.2 Smooth solutions and apriori estimates

In this subsection, we study the key apriori estimates for the smooth solutions of equation (3.1). To this aim we assume that

$$
\begin{equation*}
\sup _{t \in[0, T]}\left(\left\|\nabla^{\otimes k} b_{t}\right\|_{\infty}+\left\|\nabla^{\otimes k} f_{t}\right\|_{\infty}+\left\|\nabla^{\otimes k} \sigma_{t}\right\|_{\infty}+\left\|\sigma_{t}\right\|_{\infty}+\left\|\sigma_{t}^{-1}\right\|_{\infty}\right)<\infty, \quad k \in \mathbb{N} . \tag{3.18}
\end{equation*}
$$

For fixed $\phi \in \mathscr{D}_{0} \cap \mathscr{S}_{0}$, we introduce the following quantities for later use:

$$
\begin{align*}
\overline{\mathscr{Q}}_{\phi}:=\sup _{t \in[0, T]}\{ & {\left[b_{t}^{(1)}\right]_{\phi[2 / 3], \infty}+\left\|\nabla^{(2)} b_{t}^{(1)}\right\|_{\infty, \phi}+\left\|\left(\left[\nabla^{(2)} b_{t}^{(1)}\right]\left[\nabla^{(2)} b_{t}^{(1)}\right]^{*}\right)^{-1}\right\|_{\infty} }  \tag{3.19}\\
& \left.+\left\|\sigma_{t}^{-1}\right\|_{\infty}+\left\|\sigma_{t}\right\|_{\phi[2 / 3]}+\left[b_{t}^{(2)}\right]_{\phi[2 / 3], \phi}\right\},
\end{align*}
$$

and

$$
\begin{equation*}
\mathscr{Q}_{\phi}:=\overline{\mathscr{Q}}_{\phi}+\sup _{t \in[0, T]}\left[b_{t}^{(2)}\right]_{\phi[2 / 3], \phi^{7 / 2}}, \quad \mathscr{Q}_{\phi}^{\prime}:=\overline{\mathscr{Q}}_{\phi}+\sup _{t \in[0, T]}\left\|\nabla^{(2)} \sigma_{t}\right\|_{\phi_{[1 / 9]}, \infty}, \tag{3.20}
\end{equation*}
$$

where $\phi_{[\alpha]}$ is defined in (2.5). By (3.18), these quantities are all finite.
The main result of this section is the following, which is the key in the proofs of Theorems 1.1-1.8

Theorem 3.2. Under (3.18), (3.1) has a unique smooth solution u such that for all $t \in[0, T]$,

$$
\begin{align*}
\left\|\nabla u_{t}\right\|_{1_{[1 / 3]}, \infty}+ & \left\|\nabla^{(1)} \nabla^{(2)} u_{t}\right\|_{\infty}+\left\|\nabla^{(2)} \nabla^{(2)} u_{t}\right\|_{\phi^{3 / 2}} \\
& \leqslant C \int_{0}^{t} \mathrm{e}^{-\lambda(t-s)} \frac{\phi\left((t-s)^{\frac{1}{2}}\right)}{t-s}\left[f_{s}\right]_{\phi_{[2 / 3]}, \phi^{7 / 2}} \mathrm{~d} s,  \tag{3.21}\\
\left\|\nabla u_{t}\right\|_{\left[_{[1 / 3]}, \infty\right.}+ & \left\|\nabla \nabla^{(2)} u_{t}\right\|_{\infty} \leqslant C^{\prime} \int_{0}^{t} \mathrm{e}^{-\lambda(t-s)} \frac{\phi\left((t-s)^{\frac{1}{2}}\right)}{t-s}\left[f_{s}\right]_{\phi_{[2 / 3]}, \phi} \mathrm{d} s, \tag{3.22}
\end{align*}
$$

where $C=C\left(\phi, \mathscr{Q}_{\phi}\right)$ and $C^{\prime}=C^{\prime}\left(\phi, \mathscr{Q}_{\phi}^{\prime}\right)$ are increasing in $\mathscr{Q}_{\phi}$ and $\mathscr{Q}_{\phi}^{\prime}$ respectively.

Remark 3.1. We emphasize that the constants in Theorem 3.2 are increasing in $\mathscr{Q}_{\phi}$ or $\mathscr{Q}_{\phi}^{\prime}$, since this property enables us to make smooth approximations of relevant functionals in the proof of the main results without changing the constants.

We first prove the existence and uniqueness of $u$.
Lemma 3.3. Assume (3.18). Then (3.1) has a unique smooth solution u such that

$$
\begin{equation*}
\sup _{t \in[0, T]}\left\|\nabla^{k} u_{t}\right\|_{\infty}<\infty, \quad k \in \mathbb{N}, \quad \sup _{(t, x) \in[0, T] \times \mathbb{R}^{d_{1}+d_{2}}} \frac{\left|u_{t}(x)\right|}{1+|x|} \leqslant C \lambda^{-1}, \tag{3.23}
\end{equation*}
$$

holds for some constant $C$ increasing in $\sup _{t \in[0, T]}\left(\left\|\frac{\left|b_{t}\right|+\left|f_{t}\right|}{1+|\cdot|}\right\|_{\infty}+\left\|\sigma_{t}\right\|_{\infty}\right)$.
Proof. Let $X_{t, s}(x)=X_{t, s}$ solve the following SDE:

$$
\mathrm{d} X_{t, s}=b_{T-s}\left(X_{t, s}\right) \mathrm{d} s+\left(0, \sigma_{T-s}\left(X_{t, s}\right) \mathrm{d} W_{s}\right), \quad X_{t, t}=x \in \mathbb{R}^{d_{1}+d_{2}}, \quad s \in[t, T] .
$$

Notice that $u_{T-t}(x)$ solves the following backward equation:

$$
\partial_{t} u_{T-t}+\mathscr{L}_{T-t}^{b} u_{T-t}-\lambda u_{T-t}+f_{T-t}=0 .
$$

It is well-known that $u_{T-t}(x)$ has the following probabilistic representation (for example, see [31, Theorem 4.4]),

$$
u_{T-t}(x)=\int_{t}^{T} \mathrm{e}^{\lambda(t-s)} \mathbb{E} f_{T-s}\left(X_{t, s}(x)\right) \mathrm{d} s
$$

By (3.18), we have

$$
\sup _{s \in[0, T]}\left(\left\|\nabla^{k} f_{T-s}\right\|_{\infty}+\|\mathbb{E}\| \nabla^{k} X_{t, s}(\cdot)\| \|_{\infty}\right)<\infty, \quad k \geqslant 1
$$

Then $u_{t}$ has bounded derivatives uniformly in $t \in[0, T]$. Moreover, by the linear growth of $b$ and $f$, it is easy to derive the second inequality in (3.23).

In order to prove (3.21) and (3.22), we need the following three lemmas, which will be proved in the next subsection.

Lemma 3.4. Assume (3.18).
(1) There exists a constant $\bar{C}=\bar{C}\left(\phi, \overline{\mathscr{Q}}_{\phi}\right)$ increasing in $\overline{\mathscr{Q}}_{\phi}$ such that for any $0 \leqslant s<t \leqslant$ $T$,

$$
\begin{equation*}
\left\|\nabla^{(2)} u_{t}\right\|_{\infty}+\left\|\nabla^{(2)} \nabla^{(2)} u_{t}\right\|_{\infty} \leqslant \bar{C} \int_{0}^{t} \Lambda_{\lambda}^{\phi}(t-s)\left(\left\|\nabla u_{s}\right\|_{\infty}+\left[f_{s}\right]_{\phi}\right) \mathrm{d} s \tag{3.24}
\end{equation*}
$$

and for $k=0,1$,

$$
\begin{align*}
\left\|\nabla^{(1)}\left(\nabla^{(2)}\right)^{\otimes k} u_{t}\right\|_{\infty} & \leqslant \bar{C} \int_{0}^{t} \Lambda_{\lambda}^{\phi}(t-s)\left(\left\|\nabla^{(1)} u_{s}\right\|_{1_{[k / 3]}, \infty}\right.  \tag{3.25}\\
& \left.+\left\|\nabla^{(2)} u_{s}\right\|_{1_{[(k+2) / 3]}, \infty}+\left[f_{s}\right]_{\phi[(k+1) / 3], \phi}\right) \mathrm{d} s
\end{align*}
$$

(3) There exists a constant $C^{\prime}=C^{\prime}\left(\phi, \mathscr{Q}_{\phi}^{\prime}\right)$ increasing in $\mathscr{Q}_{\phi}^{\prime}$ such that for any $0 \leqslant s<$ $t \leqslant T$,

$$
\begin{equation*}
\left\|\nabla^{(1)} u_{t}\right\|_{\infty} \leqslant C^{\prime} \int_{0}^{t} \Lambda_{\lambda}^{\phi}(t-s)\left(\left\|\nabla^{(2)} u_{s}\right\|_{1_{[2 / 3]}}+\left[f_{s}\right]_{\phi_{[1 / 3]}, \infty}\right) \mathrm{d} s \tag{3.26}
\end{equation*}
$$

Lemma 3.5. Assume (3.18). There exist constants $C=C\left(\phi, \mathscr{Q}_{\phi}\right)$ and $C^{\prime}=C^{\prime}\left(\phi, \mathscr{Q}_{\phi}^{\prime}\right)$ which are increasing in $\mathscr{Q}_{\phi}$ and $\mathscr{Q}_{\phi}^{\prime}$ respectively, such that for all $0 \leqslant s<t \leqslant T$,

$$
\begin{equation*}
\left\|\nabla^{(1)} u_{t}\right\|_{1_{\left[\frac{1}{3}\right]}, \infty} \leqslant C \int_{0}^{t} \Lambda_{\lambda}^{\phi}(t-s)\left(\left\|\nabla \nabla^{(2)} u_{s}\right\|_{\infty}+\left\|\nabla^{(2)} \nabla^{(2)} u_{s}\right\|_{\infty, \phi^{3 / 2}}+\left[f_{s}\right]_{\phi_{[2 / 3]}, \phi^{2}}\right) \mathrm{d} s \tag{3.27}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|\nabla^{(1)} u_{t}\right\|_{1_{\left[\frac{1}{3}\right]}, \infty} \leqslant C^{\prime} \int_{0}^{t} \Lambda_{\lambda}^{\phi}(t-s)\left(\left\|\nabla \nabla^{(2)} u_{s}\right\|_{\infty}+\left[f_{s}\right]_{[2 / 3], \phi}\right) \mathrm{d} s \tag{3.28}
\end{equation*}
$$

Lemma 3.6. Assume (3.18). There exists a constant $C=C\left(\phi, \mathscr{Q}_{\phi}\right)$ increasing in $\mathscr{Q}_{\phi}$ such that for any $0 \leqslant s<t \leqslant T$,

$$
\begin{equation*}
\left\|\nabla^{(2)} \nabla^{(2)} u_{t}\right\|_{\phi^{3 / 2}} \leqslant C \int_{0}^{t} \Lambda_{\lambda}^{\phi}(t-s)\left(\left\|\nabla u_{s}\right\|_{\phi^{5 / 2}}+\left[f_{s}\right]_{\phi^{7 / 2}}\right) \mathrm{d} s \tag{3.29}
\end{equation*}
$$

Now we can give
Proof of Theorem 3.2. Letting

$$
h(t):=\left\|\nabla u_{t}\right\|_{1_{[1 / 3]}, \infty}+\left\|\nabla^{(1)} \nabla^{(2)} u_{t}\right\|_{\infty}+\left\|\nabla^{(2)} \nabla^{(2)} u_{t}\right\|_{\phi^{3 / 2}}
$$

and combining (3.24), (3.25), (3.27) and (3.29), we obtain

$$
\begin{aligned}
h(t) & \preceq \int_{0}^{t} \Lambda_{\lambda}^{\phi}(t-s)\left(h(s)+\left[f_{s}\right]_{\phi[2 / 3], \phi^{7 / 2}}\right) \mathrm{d} s \\
& =\int_{0}^{t} \mathrm{e}^{-\lambda(t-s)} \frac{\phi\left((t-s)^{\frac{1}{2}}\right)}{t-s}\left(h(s)+\left[f_{s}\right]_{\phi_{[2 / 3]}, \phi^{7 / 2}}\right) \mathrm{d} s,
\end{aligned}
$$

which yields (3.21) by Lemma 2.1.
Similarly, (3.22) follows by combining (3.25), (3.26) and (3.28).

### 3.3 Proofs of Lemmas 3.4 3.6 by using freezing equations and Duhamel's representation

To prove Lemmas 3.4 3.6 by using results presented in Section 2, we need to represent $u$ by using $P_{s, t}$. To this end, we introduce the following scheme of freezing coefficients at a fixed point $x_{0}=\left(x_{0}^{(1)}, x_{0}^{(2)}\right) \in \mathbb{R}^{d_{1}+d_{2}}$.

Let $y_{t}$ be the unique solution of the following ODE:

$$
\begin{equation*}
\frac{\mathrm{d} y_{t}}{\mathrm{~d} t}=-b_{t}\left(y_{t}\right), \quad y_{0}=x_{0} \in \mathbb{R}^{d_{1}+d_{2}} \tag{3.30}
\end{equation*}
$$

Since $b$ is smooth and has bounded derivatives due to (3.18),

$$
\begin{equation*}
\theta_{t}: x_{0} \mapsto y_{t} \text { is a diffeomorphism on } \mathbb{R}^{d_{1}+d_{2}} \tag{3.31}
\end{equation*}
$$

Let $\mathscr{L}_{t}^{x_{0}}$ be the freezing operator defined by

$$
\mathscr{L}_{t}^{x_{0}} u=\operatorname{tr}\left(A_{t} \cdot \nabla^{(2)} \nabla^{(2)} u\right)+\left(B_{t} x^{(2)}\right) \cdot \nabla^{(1)} u
$$

where $A_{t}:=\Sigma_{t}\left(y_{t}\right)$ and $B_{t}:=\left(\nabla^{(2)} b_{t}^{(1)}\right)\left(y_{t}\right)$. Set

$$
\tilde{u}_{t}(x)=u_{t}\left(x+y_{t}\right), \tilde{f}_{t}(x)=f_{t}\left(x+y_{t}\right), \tilde{\Sigma}_{t}(x)=\Sigma_{t}\left(x+y_{t}\right)-\Sigma_{t}\left(y_{t}\right)
$$

and

$$
\tilde{b}_{t}^{(2)}(x)=b_{t}^{(2)}\left(x+y_{t}\right)-b_{t}^{(2)}\left(y_{t}\right), \tilde{b}_{t}^{(1)}(x)=b_{t}^{(1)}\left(x+y_{t}\right)-b_{t}^{(1)}\left(y_{t}\right)-\nabla^{(2)} b_{t}^{(1)}\left(y_{t}\right) x^{(2)} .
$$

From (3.1) and (3.30) it is easy to see that $\tilde{u}$ satisfies

$$
\partial_{t} \tilde{u}=\mathscr{L}_{t}^{x_{0}} \tilde{u}-\lambda \tilde{u}+\operatorname{tr}\left(\tilde{\Sigma}_{t} \cdot \nabla^{(2)} \nabla^{(2)} \tilde{u}\right)+\tilde{b} \cdot \nabla \tilde{u}+\tilde{f}, \tilde{u}_{0}=0 .
$$

Let $P_{s, t}$ be the semigroup generated by $\mathscr{L}_{t}^{x_{0}}$. By Duhamel's formula, we have

$$
\begin{equation*}
\tilde{u}_{t}=\int_{0}^{t} \mathrm{e}^{-\lambda(t-s)} P_{s, t}\left(\operatorname{tr}\left(\tilde{\Sigma}_{s} \cdot \nabla^{(2)} \nabla^{(2)} \tilde{u}_{s}\right)+\tilde{b}_{s} \cdot \nabla \tilde{u}_{s}+\tilde{f}_{s}\right) \mathrm{d} s \tag{3.32}
\end{equation*}
$$

Note from the definition of $\tilde{b}_{t}^{(1)}(x)$ that

$$
\begin{align*}
\left|\tilde{b}_{t}^{(1)}\left(0, x^{(2)}\right)\right| & =\left|b_{t}^{(1)}\left(y_{t}^{(1)}, x^{(2)}+y_{t}^{(2)}\right)-b_{t}^{(1)}\left(y_{t}\right)-\nabla^{(2)} b_{t}^{(1)}\left(y_{t}\right) x^{(2)}\right| \\
& \leqslant\left|x^{(2)}\right| \int_{0}^{1}\left|\nabla^{(2)} b_{t}^{(1)}\left(y_{t}^{(1)}, r x^{(2)}+y_{t}^{(2)}\right)-\nabla^{(2)} b_{t}^{(1)}\left(y_{t}\right)\right| \mathrm{d} r  \tag{3.33}\\
& \leqslant C\left[\nabla^{(2)} b_{t}^{(1)}\right]_{\infty, \phi}\left|x^{(2)}\right| \phi_{[0]}\left(\left|x^{(2)}\right|\right)
\end{align*}
$$

Combining this with (3.20) and (3.19), we are able to apply (3.3), (3.4), (3.7) and (3.8) to derive the following lemma.

Lemma 3.7. Assume (3.18). There exist constants $\bar{C}=\bar{C}\left(\phi, \overline{\mathscr{Q}}_{\phi}\right)$ increasing in $\overline{\mathscr{D}}_{\phi}$, such that for all $0 \leqslant s<t \leqslant T$ and $k=0,1$,

$$
\begin{gather*}
\left\|\nabla^{(1)}\left(\nabla^{(2)}\right)^{\otimes k} P_{s, t}\left(\tilde{b}_{s} \cdot \nabla \tilde{u}_{s}\right)\right\|(0) \leqslant \bar{C} \Lambda_{0}^{\phi}(t-s)\left(\left\|\nabla^{(1)} u_{s}\right\|_{1_{\left[\frac{k}{3}\right]}, \infty}+\left\|\nabla^{(2)} u_{s}\right\|_{\phi_{\left[\frac{k+1}{3}\right]}, \infty}\right),  \tag{3.34}\\
\left\|\left(\nabla^{(2)}\right)^{\otimes(k+1)} P_{s, t}\left(\tilde{b}_{s} \cdot \nabla \tilde{u}_{s}\right)\right\|(0) \leqslant \bar{C} \Lambda_{0}^{\phi}(t-s)\left\|\nabla u_{s}\right\|_{\infty} . \tag{3.35}
\end{gather*}
$$

The following lemma is an easy consequence of (2.42) and (2.43).
Lemma 3.8. There is a constant $C=C(\phi, T)>0$ such that for all $0 \leqslant s<t \leqslant T$ and $k=0,1$,

$$
\begin{align*}
\left\|\nabla^{(1)}\left(\nabla^{(2)}\right)^{\otimes k} P_{s, t} \tilde{f}_{s}\right\|_{\infty} & \leqslant C \Lambda_{0}^{\phi}(t-s)\left[f_{s}\right]_{\phi(k+1) / 3], \infty},  \tag{3.36}\\
\left\|\left(\nabla^{(2)}\right)^{\otimes(k+1)} P_{s, t} \tilde{f}_{s}\right\|_{\infty} & \leqslant C \Lambda_{0}^{\phi}(t-s)\left[f_{s}\right]_{\phi} \tag{3.37}
\end{align*}
$$

Moreover, by (3.3), (3.5) and (3.6), we have
Lemma 3.9. Assume (3.18). There exist constants $\bar{C}=\bar{C}\left(\phi, \overline{\mathscr{Q}}_{\phi}\right)$ and $C^{\prime}=C^{\prime}\left(\phi, \tilde{\mathscr{Q}}_{\phi}\right)$ which are increasing in $\overline{\mathscr{Q}}_{\phi}$ and $\mathscr{Q}_{\phi}^{\prime}$ respectively, such that for all $0 \leqslant s<t \leqslant T$ and $k=0,1$,

$$
\begin{align*}
& \left\|\nabla^{(1)}\left(\nabla^{(2)}\right)^{\otimes k} P_{s, t}\left(\operatorname{tr}\left(\tilde{\Sigma}_{s} \cdot \nabla^{(2)} \nabla^{(2)} \tilde{u}_{s}\right)\right)\right\|(0) \\
& \quad \leqslant \bar{C}\left(\left\|\nabla^{(2)} u_{s}\right\|_{1_{[(k+2) / 3]}, \infty}+\left\|\nabla^{(2)} \nabla^{(2)} u_{s}\right\|_{\infty}\right) \Lambda_{0}^{\phi}(t-s)  \tag{3.38}\\
& \left\|\left(\nabla^{(2)}\right)^{\otimes(k+1)} P_{s, t}\left(\operatorname{tr}\left(\tilde{\Sigma}_{s} \cdot \nabla^{(2)} \nabla^{(2)} \tilde{u}_{s}\right)\right)\right\|(0) \leqslant \bar{C}\left\|\nabla^{(2)} \nabla^{(2)} u_{s}\right\|_{\infty} \Lambda_{0}^{\phi}(t-s) \tag{3.39}
\end{align*}
$$

and

$$
\begin{equation*}
\left|\nabla^{(1)} P_{s, t}\left(\operatorname{tr}\left(\tilde{\Sigma}_{s} \cdot \nabla^{(2)} \nabla^{(2)} \tilde{u}_{s}\right)\right)\right|(0) \leqslant C^{\prime}\left\|\nabla^{(2)} u_{s}\right\|_{1_{[2 / 3]}} \Lambda_{0}^{\phi}(t-s) . \tag{3.40}
\end{equation*}
$$

Now we are in a position to give the proofs of Lemmas 3.4]3.6.
Proof of Lemma 3.4. Now, substituting estimates in Lemmas 3.7 3.9) into (3.32), and noting that $\tilde{u}_{t}=u_{t}\left(\cdot+y_{t}\right)$ where, according to (3.31), $y_{t}$ runs all over $\mathbb{R}^{d_{1}+d_{2}}$ as $x_{0}$ does, and by Lemma 2.1 and (3.32), estimate (3.24) follows from (3.35), (3.37) and (3.39); estimate (3.25) follows from (3.34), (3.36), (3.38) and (3.24); and finally, estimate (3.26) follows from (3.34), (3.36) and (3.40).

Proof of Lemma 3.5. For simplicity, constants $C$ and $C^{\prime}$ below are corresponding to $\mathscr{Q}_{\phi}$ and $\mathscr{Q}_{\phi}^{\prime}$ respectively as in the statement, which may vary from line to line.
(1) Let $\mathbf{P}_{\theta}^{(1)}$ be defined by (2.25). Let $w_{t}^{\theta}(x):=\partial_{\theta} \mathbf{P}_{\theta}^{(1)} u_{t}(x)$ and

$$
\begin{aligned}
g_{t}^{\theta}(x) & :=\partial_{\theta} \mathbf{P}_{\theta}^{(1)}\left(b_{t} \cdot \nabla u_{t}\right)(x)-\left(b_{t} \cdot \nabla \partial_{\theta} \mathbf{P}_{\theta}^{(1)} u_{t}\right)(x)+\partial_{\theta} \mathbf{P}_{\theta}^{(1)} f_{t}(x) \\
& +\operatorname{tr}\left(\partial_{\theta} \mathbf{P}_{\theta}^{(1)}\left(\Sigma_{t} \cdot \nabla^{(2)} \nabla^{(2)} u_{t}\right)-\Sigma_{t} \cdot \partial_{\theta} \mathbf{P}_{\theta}^{(1)} \nabla^{(2)} \nabla^{(2)} u_{t}\right)(x)
\end{aligned}
$$

By equation (3.1), we have

$$
\partial_{t} w_{t}^{\theta}=\mathscr{L}_{t}^{\Sigma, b} w_{t}^{\theta}-\lambda w_{t}^{\theta}+g_{t}^{\theta} .
$$

By (3.25) with $k=0$, we have

$$
\begin{equation*}
\left\|\nabla^{(1)} w_{t}^{\theta}\right\|_{\infty} \leqslant \bar{C} \int_{0}^{t} \Lambda_{\lambda}^{\phi}(t-s)\left(\left[\nabla^{(2)} w_{s}^{\theta}\right]_{[[2 / 3]}, \infty+\left[g_{s}^{\theta}\right]_{\phi_{[1 / 3]}, \phi}\right) \mathrm{d} s \tag{3.41}
\end{equation*}
$$

By the definition of $w_{t}^{\theta}$ and using (2.16) for $g=1, \psi=1_{[2 / 3]}$ and $\phi=1_{[1 / 3]}$, we obtain

$$
\begin{equation*}
\left[\nabla^{(2)} w_{s}^{\theta}\right]_{[2 / 3], \infty} \preceq\left\|\nabla^{(1)} \nabla^{(2)} u_{s}\right\|_{\infty} \theta^{-\frac{5}{6}} \tag{3.42}
\end{equation*}
$$

Next, by Lemma [2.6 with $\psi=\phi_{\left[\frac{1}{3}\right]}$ and $\phi=1_{[1 / 3]}$, we obtain

$$
\begin{equation*}
\left[g_{t}^{\theta}\right]_{\phi_{\left[\frac{1}{3}\right]}, \infty} \leqslant C\left(\left[b_{t}\right]_{\phi_{\left[\frac{2}{3}\right]}, \infty}\left\|\nabla u_{t}\right\|_{\infty}+\left[\Sigma_{t}\right]_{\phi_{\left[\frac{2}{3}\right]}, \infty}\left\|\nabla^{(2)} \nabla^{(2)} u_{t}\right\|_{\infty}+\left[f_{t}\right]_{\left.\phi_{\left[\frac{2}{3}\right.}\right], \infty}\right) \theta^{-\frac{5}{6}} . \tag{3.43}
\end{equation*}
$$

Moreover, by Lemma 2.9 for $\psi_{1}=1_{\left[\frac{2}{3}\right]}$ and $\psi_{2}=\phi^{2}$, and using $a \wedge c \leqslant a^{\frac{1}{2}} c^{\frac{1}{2}}$ for $a, c>0$, we obtain

$$
\left[\partial_{\theta} \mathbf{P}_{\theta}^{(1)}\left(b_{t} \cdot \nabla u_{t}\right)-b_{t} \cdot \nabla \partial_{\theta} \mathbf{P}_{\theta}^{(1)} u_{t}\right]_{\infty, \phi} \leqslant C\left[b_{t}\right]_{1_{\left[\frac{2}{3}\right]}, \phi^{2}}\left\|\nabla u_{t}\right\|_{\infty, \phi^{2}} \theta^{-\frac{5}{6}}
$$

and

$$
\left[\partial_{\theta} \mathbf{P}_{\theta}^{(1)} f_{t}\right]_{\infty, \phi} \leqslant C\left[f_{t}\right]_{\left[\frac{2}{3}\right]}, \phi^{2}-\frac{\theta^{-\frac{5}{6}}}{} \leqslant C\left[f_{t}\right]_{\phi_{\left[\frac{2}{3}\right]}, \phi^{2}} \theta^{-\frac{5}{6}} .
$$

Finally, by Lemma 2.9 for $\psi_{1}=1_{[1]}$ and $\psi_{2}=\phi^{\frac{3}{2}}$, we obtain

$$
\left[\partial_{\theta} \mathbf{P}_{\theta}^{(1)}\left(\Sigma_{t} \cdot \nabla^{(2)} \nabla^{(2)} u_{t}\right)-\Sigma_{t} \cdot \partial_{\theta} \mathbf{P}_{\theta}^{(1)} \nabla^{(2)} \nabla^{(2)} u_{t}\right]_{\infty, \phi} \leqslant C\left\|\nabla^{(2)} \nabla^{(2)} u_{t}\right\|_{\infty, \phi^{3 / 2}} \theta^{-\frac{5}{6}}
$$

Therefore,

$$
\left[g_{t}^{\theta}\right]_{\infty, \phi} \leqslant C\left(\left\|\nabla u_{t}\right\|_{\infty, \phi^{2}}+\left\|\nabla^{(2)} \nabla^{(2)} u_{t}\right\|_{\infty, \phi^{3 / 2}}+\left[f_{t}\right]_{1_{\left[\frac{2}{3}\right]}, \phi^{2}}\right) \theta^{-\frac{5}{6}} .
$$

Combining this with (3.41), (3.43) and (3.42), and using (2.24), we obtain (3.27).
(2) We now prove (3.28) in the same way. By (3.26) for $\left(w^{\theta}, g^{\theta}\right)$ in place of $(u, f)$, we have

$$
\begin{equation*}
\left\|\nabla w_{t}^{\theta}\right\|_{\infty} \leqslant C^{\prime} \int_{0}^{t} \Lambda_{\lambda}^{\phi}(t-s)\left(\left\|\nabla^{(2)} w_{s}^{\theta}\right\|_{1_{\left[\frac{2}{3}\right]}}+\left[g_{s}^{\theta}\right]_{\phi_{\left[\frac{1}{3}\right]}, \infty}\right) \mathrm{d} s \tag{3.44}
\end{equation*}
$$

Due to (3.42) and(3.43), we only need to estimate $\left\|\nabla^{(2)} w_{s}^{\theta}\right\|_{\infty, 1_{[2 / 3]}}$. By Lemma 2.9 for $g=1$, $\psi_{1}=1_{[1]}, \psi_{2}=1_{[1]}$ and $\phi=1_{[2 / 3]}$, we have

$$
\left\|\nabla^{(2)} w_{s}^{\theta}\right\|_{\infty, 1_{[2 / 3]}} \preceq\left\|\nabla \nabla^{(2)} u_{t}\right\|_{\infty} \theta^{-\frac{5}{6}} .
$$

This, together with (3.44), (3.43) and (3.42), yields

$$
\sup _{\theta \in(0,1)}\left\|\theta^{\frac{5}{6}} \partial_{\theta} \mathbf{P}_{\theta}^{(1)} \nabla u_{t}\right\|_{\infty} \leqslant C^{\prime} \int_{0}^{t} \Lambda_{\lambda}^{\phi}(t-s)\left(\left\|\nabla \nabla^{(2)} u_{s}\right\|_{\infty}+\left\|\nabla u_{s}\right\|_{\infty}+\left[f_{s}\right]_{\phi_{\left[\frac{2}{3}\right]}, \infty}\right) \mathrm{d} s
$$

By Lemma 2.7 for $\phi(s)=s^{\frac{1}{3}}$, this implies (3.28).

Proof of Lemma 3.6. Let $\mathbf{P}_{\theta}$ be the semigroup on $\mathbb{R}^{d_{1}+d_{2}}$. Let $w_{t}^{\theta}=\partial_{\theta} \mathbf{P}_{\theta} u_{t}$ and

$$
\begin{aligned}
g_{t}^{\theta}(x) & =\partial_{\theta} \mathbf{P}_{\theta}\left(b_{t} \cdot \nabla u_{t}\right)(x)-\left(b_{t} \cdot \nabla \partial_{\theta} \mathbf{P}_{\theta} u_{t}\right)(x)+\partial_{\theta} \mathbf{P}_{\theta} f_{t}(x) \\
& +\operatorname{tr}\left(\partial_{\theta} \mathbf{P}_{\theta}\left(\Sigma_{t} \cdot \nabla^{(2)} \nabla^{(2)} u_{t}\right)-\Sigma_{t} \cdot \partial_{\theta} \mathbf{P}_{\theta} \nabla^{(2)} \nabla^{(2)} u_{t}\right)(x)
\end{aligned}
$$

By equation (3.1) we have

$$
\partial_{t} w_{t}^{\theta}=\mathscr{L}_{t}^{\Sigma, b} w_{t}^{\theta}-\lambda w_{t}^{\theta}+g_{t}^{\theta} .
$$

Thus, by (3.24) we have

$$
\begin{equation*}
\left\|\nabla^{(2)} \nabla^{(2)} w_{t}^{\theta}\right\|_{\infty} \preceq \int_{0}^{t} \Lambda_{\lambda}^{\phi}(t-s)\left(\left\|\nabla w_{s}^{\theta}\right\|_{\infty}+\left[g_{s}^{\theta}\right]_{\phi}\right) \mathrm{d} s \tag{3.45}
\end{equation*}
$$

On the other hand, by (2.14), we have

$$
\left\|\nabla w_{t}^{\theta}\right\|_{\infty}=\left\|\partial_{\theta} \mathbf{P}_{\theta} \nabla u_{t}\right\|_{\infty} \preceq \theta^{-1} \phi^{5 / 2}\left(\theta^{\frac{1}{2}}\right)\left\|\nabla u_{t}\right\|_{\phi^{5 / 2}}
$$

and by (2.16),

$$
\left[g_{t}^{\theta}\right]_{\phi} \preceq \theta^{-1} \phi^{5 / 2}\left(\theta^{\frac{1}{2}}\right)\left(\left[b_{t}\right]_{\phi^{7 / 2}}\left\|\nabla u_{t}\right\|_{\infty}+\left[f_{t}\right]_{\phi^{7 / 2}}+\left[\Sigma_{t}\right]_{\phi^{7 / 2}}\left\|\nabla^{(2)} \nabla^{(2)} u_{t}\right\|_{\infty}\right) .
$$

Substituting these two estimates into (3.45) and noticing that by (ii) of Proposition 2.2,

$$
\int_{0}^{t} s^{-1} \phi^{5 / 2}(s) \mathrm{d} s+t \int_{t}^{1} s^{-2} \phi^{5 / 2}(s) \mathrm{d} s \preceq \phi^{3 / 2}(t), t \in(0,1],
$$

by (2.23), we obtain

$$
\left\|\nabla^{(2)} \nabla^{(2)} u_{t}\right\|_{\phi^{3 / 2}} \preceq \int_{0}^{t} \Lambda_{\lambda}^{\phi}(t-s)\left(\left\|\nabla u_{s}\right\|_{\phi^{5 / 2}}+\left\|\nabla^{(2)} \nabla^{(2)} u_{s}\right\|_{\infty}+\left[f_{s}\right]_{\phi^{7 / 2}}\right) \mathrm{d} s
$$

which gives the desired estimate by Lemma 2.1.

### 3.4 Classical solutions of (3.1)

In this subsection we prove the existence and stability of classical solutions to equation (3.1).
Theorem 3.10. Assume $\mathscr{Q}_{\phi}<\infty$. For any $f:[0, T] \times \mathbb{R}^{d} \rightarrow \mathbb{R}$ with

$$
\sup _{s \in[0, T]}\left[f_{s}\right]_{\phi_{[2 / 3]}, \phi^{7 / 2}}<\infty
$$

there exist a unique classical solution $u$ to (3.1) such that for all $t \in[0, T]$,

$$
\begin{equation*}
\left\|\nabla u_{t}\right\|_{1_{[1 / 3]}, \infty}+\left\|\nabla^{(2)} \nabla^{(2)} u_{t}\right\|_{\phi^{3 / 2}} \leqslant C \int_{0}^{t} \mathrm{e}^{-\lambda(t-s)} \frac{\phi\left((t-s)^{\frac{1}{2}}\right)}{t-s}\left[f_{s}\right]_{\phi_{[2 / 3]}, \phi^{7 / 2}} \mathrm{~d} s \tag{3.46}
\end{equation*}
$$

Moreover, let $\left(b^{k}, \sigma^{k}, f^{k}\right)_{k \in \mathbb{N}_{\infty}}$ be a sequence of functions. Let $\mathscr{Q}_{\phi}^{k}$ be defined as in (3.20) in terms of $\left(b^{k}, \sigma^{k}\right)$. Assume that

$$
\sup _{k}\left(\mathscr{Q}_{\phi}^{k}+\sup _{s \in[0, T]}\left[f_{s}^{k}\right]_{\phi_{[2 / 3]}, \phi^{7 / 2}}\right)<\infty,
$$

and for each $t>0, x \in \mathbb{R}^{d_{1}+d_{2}}$,

$$
\lim _{k \rightarrow \infty}\left\|\sigma_{t}^{k}(x)-\sigma_{t}^{\infty}(x)\right\|+\left|b_{t}^{k}(x)-b_{t}^{\infty}(x)\right|+\left|f_{t}^{k}(x)-f_{t}^{\infty}(x)\right|=0
$$

Let $u_{t}^{k}(x)$ be the unique classical solution of (3.1) corresponding to $\left(b^{k}, \sigma^{k}, f^{k}\right)$ for each $k \in \mathbb{N}_{\infty}$. Then for each $T, R>0$,

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \sup _{t \in[0,1],|x| \leqslant R}\left(\left|u_{t}^{k}-u_{t}^{\infty}\right|+\left|\nabla\left(u_{t}^{k}-u_{t}^{\infty}\right)\right|+\left\|\nabla^{(2)} \nabla^{(2)}\left(u_{t}^{k}-u_{t}^{\infty}\right)\right\|\right)(x)=0 . \tag{3.47}
\end{equation*}
$$

Proof. (1) Let $\varrho$ be a non-negative smooth function with compact support in $\mathbb{R}^{d}$ having

$$
\int_{\mathbb{R}^{d}} \varrho(x) \mathrm{d} x=1 .
$$

For $n \in \mathbb{N}$, define $\varrho_{n}(x)=n^{d} \varrho(n x)$ and

$$
\begin{equation*}
b_{t}^{n}=\varrho_{n} * b_{t}, \quad \sigma_{t}^{n}=\varrho_{n} * \sigma_{t}, \quad f_{t}^{n}:=\varrho_{n} * f_{t} . \tag{3.48}
\end{equation*}
$$

Clearly, $b^{n}, \sigma^{n}$ and $f^{n}$ satisfy (3.18). Let $\mathscr{Q}_{\phi}^{n}$ be defined by (3.20) corresponding to $b^{n}, \sigma^{n}$. It is easy to see that for some $n_{0}$ large enough and all $n \geqslant n_{0}$,

$$
\mathscr{Q}_{\phi}^{n} \leqslant 2 \mathscr{Q}_{\phi} .
$$

Let $u^{n}$ be the unique smooth solution of the following equation

$$
\begin{equation*}
\partial_{t} u_{t}^{n}=\mathscr{L}_{t}^{\Sigma^{n}, b^{n}} u_{t}^{n}-\lambda u_{t}^{n}+f_{t}^{n}, \quad u_{0}^{n}=0, t \in[0, T], \tag{3.49}
\end{equation*}
$$

which enjoys the following uniform estimate:

$$
\begin{align*}
\left\|\nabla u_{t}^{n}\right\|_{1_{[1 / 3]}, \infty} & +\left\|\nabla^{(1)} \nabla^{(2)} u_{t}^{n}\right\|_{\infty}+\left\|\nabla^{(2)} \nabla^{(2)} u_{t}^{n}\right\|_{\phi^{3 / 2}} \\
& \leqslant C \int_{0}^{t} \mathrm{e}^{-\lambda(t-s)} \frac{\phi\left((t-s)^{\frac{1}{2}}\right)}{t-s}\left[f_{s}\right]_{\phi_{[2 / 3]}, \phi^{7 / 2}} \mathrm{~d} s, \tag{3.50}
\end{align*}
$$

So, Ascoli-Arzela's theorem implies the existence of $u$ such that, up to a subsequence,

$$
\lim _{n \rightarrow \infty} \sup _{t \in[0,1],|x| \leqslant R}\left(\left|u_{t}^{n}-u_{t}\right|+\left|\nabla\left(u_{t}^{n}-u_{t}\right)\right|+\left\|\nabla^{(2)} \nabla^{(2)}\left(u_{t}^{n}-u_{t}\right)\right\|\right)(x)=0, \quad R>0 .
$$

By taking limits for (3.49) and inequality (3.50), we obtain the existence of classical solutions of (3.1) as well as the estimate (3.46).
(2) We use a contradiction argument. Suppose that (3.47) does not hold. Then there is a subsequence $k_{m}$ such that

$$
\underline{\lim _{m \rightarrow \infty}} \sup _{t \in[0,1],|x| \leqslant R}\left(\left|u_{t}^{k_{m}}-u_{t}^{\infty}\right|+\left|\nabla\left(u_{t}^{k_{m}}-u_{t}^{\infty}\right)\right|+\left\|\nabla^{(2)} \nabla^{(2)}\left(u_{t}^{k_{m}}-u_{t}^{\infty}\right)\right\|\right)(x)>0 .
$$

On the other hand, repeating the proof in step (1), since $u^{\infty}$ is the unique solution of (3.1) corresponding to $\left(b^{\infty}, \sigma^{\infty}, f^{\infty}\right)$, there is a subsubsequence $k_{m}^{\prime}$ such that

$$
\lim _{m \rightarrow \infty} \sup _{t \in[0,1],|x| \leqslant R}\left(\left|u_{t}^{k_{m}^{\prime}}-u_{t}^{\infty}\right|+\left|\nabla\left(u_{t}^{k_{m}^{\prime}}-u_{t}^{\infty}\right)\right|+\left\|\nabla^{(2)} \nabla^{(2)}\left(u_{t}^{k_{m}^{\prime}}-u_{t}^{\infty}\right)\right\|\right)(x)=0 .
$$

Thus, we obtain a contradiction, and so, (3.47) holds.

We also have the following existence of Hölder classical solutions under Hölder assumptions.

Theorem 3.11. Assume for some $\alpha \in\left(\frac{2}{3}, 1\right), \beta \in\left(0, \frac{1}{2}\right)$,

$$
\begin{align*}
\mathscr{Q}_{\alpha, \beta}:=\sup _{t \in[0, T]}\{ & {\left[b_{t}^{(1)}\right]_{1_{[\alpha]}, \infty}+\left\|\nabla^{(2)} b_{t}^{(1)}\right\|_{\infty, 1_{[\beta]}}+\left\|\left(\left[\nabla^{(2)} b_{t}^{(1)}\right]\left[\nabla^{(2)} b_{t}^{(1)}\right]^{*}\right)^{-1}\right\|_{\infty} }  \tag{3.51}\\
& \left.+\left\|b_{t}^{(2)}\right\|_{1_{[\alpha]}, 1_{[\beta]}}+\left\|\sigma_{t}\right\|_{\infty}+\left\|\sigma_{t}^{-1}\right\|_{\infty}+\left\|\sigma_{t}\right\|_{1_{[\alpha]}}\right\}<\infty
\end{align*}
$$

Then for any $\varepsilon \in\left(0, \beta \wedge\left(\alpha-\frac{2}{3}\right)\right)$, there exist a unique solution $u$ to (3.1) and constants $\delta \in(0,1)$ depending only on $\alpha, \beta$, and $C=C\left(\mathscr{Q}_{\alpha, \beta}, \varepsilon, \delta\right)>0$, which is increasing in $\mathscr{Q}_{\alpha, \beta}$, and such that for all $t \in[0, T]$ and $\lambda \geqslant 0$,

$$
\begin{equation*}
\left\|\nabla u_{t}\right\|_{1_{[1 / 3]}}+\left\|\nabla \nabla^{(2)} u_{t}\right\|_{1_{[\varepsilon]}, 1_{[\varepsilon]}} \leqslant C \int_{0}^{t} \mathrm{e}^{-\lambda(t-s)}(t-s)^{-\delta}\left[f_{s}\right]_{1_{[\alpha]}, 1_{[\beta]}} \mathrm{d} s \tag{3.52}
\end{equation*}
$$

Proof. First of all, we assume (3.18). Following the proof of Lemma 3.6, by Lemma 2.8, we have for any $\varepsilon \in\left(0, \beta \wedge\left(\alpha-\frac{2}{3}\right)\right)$,

$$
\left[g_{t}^{\theta}\right]_{1_{[\alpha-\varepsilon]}, 1_{[\beta-\varepsilon]}} \preceq\left(\left[b_{t}\right]_{\left.1_{[\alpha]}, 1_{[\beta]}\right]}\left\|\nabla u_{t}\right\|_{\infty}+\left[f_{t}\right]_{1_{[\alpha]}, 1_{[\beta]}}+\left[\Sigma_{t}\right]_{\left.1_{[\alpha-\varepsilon]}, 1_{[\beta-\varepsilon]}\right]}\left\|\nabla^{(2)} \nabla^{(2)} u_{t}\right\|_{\infty}\right) \theta^{\frac{\varepsilon}{2}-1} .
$$

Noticing that for any $\phi \in \mathscr{S}_{0}$,

$$
\phi_{[2 / 3]}(s)=s^{3 / 2} \phi(s) \preceq 1_{[\alpha-\varepsilon]}(s)=s^{\alpha-\varepsilon}, \quad \phi^{7 / 2}(s) \preceq 1_{[\beta-\varepsilon]}(s)=s^{\beta-\varepsilon}, \quad s \in[0,1],
$$

by (3.21), we obtain that for some $\delta \in(0,1)$,

$$
\begin{aligned}
& \left\|\nabla w_{t}^{\theta}\right\|_{1_{[1 / 3]}, \infty}+\left\|\nabla \nabla^{(2)} w_{t}^{\theta}\right\|_{\infty} \preceq \int_{0}^{t} \mathrm{e}^{-\lambda(t-s)}(t-s)^{-\delta}\left[g_{s}^{\theta}\right]_{1_{[\alpha-\varepsilon]}, 1_{[\beta-\varepsilon]}} \mathrm{d} s \\
& \preceq \theta^{\frac{\varepsilon}{2}-1} \int_{0}^{t} \mathrm{e}^{-\lambda(t-s)}(t-s)^{-\delta}\left(\left\|\nabla u_{s}\right\|_{\infty}+\left\|\nabla^{(2)} \nabla^{(2)} u_{s}\right\|_{\infty}+\left[f_{s}\right]_{[\alpha \alpha]}, 1_{[\beta]}\right) \mathrm{d} s,
\end{aligned}
$$

which in turn gives (3.52) by Lemma 2.7 and (3.21). In general, we can follow the same approximation as done in Theorem 3.10.

## 4 Proofs of Main Results

Proof of Theorem [1.1. The existence of weak solution is well known, see e.g. [13, Theorem 2.2 and Remark 2.1, Chapter IV] and [21]. So, we only prove the uniqueness. Let $\left(\Omega, \mathscr{F}, \mathbb{P} ; X_{t}, W_{t}\right)$ and $\left(\Omega^{\prime}, \mathscr{F}^{\prime}, \mathbb{P}^{\prime} ; X_{t}^{\prime}, W_{t}^{\prime}\right)$ be two weak solutions of SDE (1.2) with $X_{0}=$ $X_{0}^{\prime}=x \in \mathbb{R}^{d_{1}+d_{2}}$. Fix $T>0$ and $f \in C_{b}^{\infty}\left([0, T] \times \mathbb{R}^{d_{1}+d_{2}}\right)$. For any $n \geqslant 1$, let $\sigma^{n}$ and $b^{n}$ be in (3.48), and let $\mathscr{Q}_{n}$ and $\mathscr{Q}_{n}^{\prime}$ be the numbers defined in (3.20) and (3.19) for $\left(b^{n}, \sigma^{n}\right)$ in place of $(b, \sigma)$. It is easy to see that for some $n_{0}$ large enough and all $n \geqslant n_{0}$,

$$
\mathscr{Q}_{n} \leqslant 2 \mathscr{Q}, \quad \mathscr{Q}_{n}^{\prime} \leqslant 2 \mathscr{Q}^{\prime}
$$

By Theorem 3.2 for $\left(\mathscr{L}_{T-t}^{\Sigma^{n}, b^{n}}, f_{T-t}\right)$ in place of $\left(\mathscr{L}_{t}^{\Sigma, b}, f_{t}\right)$, for any $\lambda \geqslant 0$ the equation

$$
\begin{equation*}
\partial_{t} u_{t}^{n}=\mathscr{L}_{T-t}^{\Sigma^{n}, b^{n}} u_{t}^{n}-\lambda u_{t}^{n}+f_{T-t}, \quad u_{0}^{n}=0, t \in[0, T] \tag{4.1}
\end{equation*}
$$

has a unique solution $u^{n}:[0, T] \times \mathbb{R}^{d_{1}+d_{2}} \rightarrow \mathbb{R}$ such that

$$
\begin{equation*}
\sup _{t \in[0, T], n \geqslant 1}\left(\left\|\nabla u_{t}^{n}\right\|_{1_{\left[\frac{1}{3}\right]}, \infty}+\left\|\nabla \nabla^{(2)} u_{t}^{n}\right\|_{\infty}\right) \leqslant \varepsilon(\lambda):=C \int_{0}^{T} \mathrm{e}^{-\lambda(t-s)} \frac{\phi\left((t-s)^{\frac{1}{2}}\right)}{t-s} \mathrm{~d} s \tag{4.2}
\end{equation*}
$$

for some constant $C>0$. So, Ascoli-Arzela's theorem implies the existence of

$$
u:[0, T] \times \mathbb{R}^{d_{1}+d_{2}} \rightarrow \mathbb{R}
$$

such that, up to a subsequence,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \sup _{t \in[0, T],|x| \leqslant R}\left(\left|u_{t}^{n}-u_{t}\right|+\left\|\nabla^{(2)}\left(u_{t}^{n}-u_{t}\right)\right\|\right)(x)=0, \quad R>0, \tag{4.3}
\end{equation*}
$$

and, moreover,

$$
\begin{equation*}
\sup _{t \in[0, T]}\left(\left[u_{t}\right]_{1_{[1]}}+\left[\nabla^{(2)} u_{t}\right]_{\left[{ }_{[1]}\right]}\right) \leqslant \varepsilon(\lambda) . \tag{4.4}
\end{equation*}
$$

Now, due to (1.2) and (4.1) with $\lambda=0$, Itô's formula for $u_{T-t}^{n}(x)$ implies

$$
\begin{aligned}
0 & =u_{T}^{n}(x)+\int_{0}^{T} \mathbb{E}\left\{\left(\partial_{s}+\mathscr{L}_{s}^{\Sigma, b}\right) u_{T-s}^{n}\left(X_{s}\right)\right\} \mathrm{d} s \\
& =u_{T}^{n}(x)+\mathbb{E} \int_{0}^{T}\left\{\operatorname{tr}\left[\left(\Sigma_{s}-\Sigma_{s}^{n}\right) \nabla^{(2)} \nabla^{(2)} u_{T-s}^{n}\right]+\left(b_{s}-b_{s}^{n}\right) \cdot \nabla u_{T-s}^{n}-f_{s}\right\}\left(X_{s}\right) \mathrm{d} s .
\end{aligned}
$$

So, according to (4.2), (4.3) and noting that $\left\{\left|b_{t}-b_{t}^{n}\right|+\left\|\sigma_{t}-\sigma_{t}^{n}\right\|\right\}_{n \geqslant 1}$ is bounded uniformly in $t \in[0, T]$ and converges to 0 as $n \rightarrow \infty$, by the dominated convergence theorem, letting $n \rightarrow \infty$ we obtain

$$
u_{T}(x)=\int_{0}^{T} \mathbb{E} f_{s}\left(X_{s}\right) \mathrm{d} s
$$

By the same reason, we also have

$$
u_{T}(x)=\int_{0}^{T} \mathbb{E}^{\prime} f_{s}\left(X_{s}^{\prime}\right) \mathrm{d} s
$$

Hence,

$$
\int_{0}^{T} \mathbb{E} f_{s}\left(X_{s}\right) \mathrm{d} s=\int_{0}^{T} \mathbb{E}^{\prime} f_{s}\left(X_{s}^{\prime}\right) \mathrm{d} s, \quad f \in C_{b}^{\infty}\left([0, T] \times \mathbb{R}^{d_{1}+d_{2}}\right)
$$

By [20, Corollary 6.2.4], this implies the weak uniqueness.

Proof of Theorem 1.2. If (1.11) holds, then the non-explosion and estimate (1.12) follows by [29, Lemma 2.2]. So, we only prove the existence and uniqueness of local solutions.
(1) We first assume that (A) holds for some $C_{n}=C, \phi_{n}=\phi$ and $\gamma_{n}=\gamma$ independent of $n \geqslant 1$. Noting that $\gamma \in \mathscr{C}$ implies $\gamma(r) \leqslant c r^{-\frac{1}{4}}$ for some $c>0$ and all $r \in(0,1]$, in this case we have either $\mathscr{Q}_{\phi}<\infty$ or $\mathscr{Q}_{\phi}^{\prime}<\infty$. Due to the existence of the weak solution as explained in the proof of Theorem 1.2, by the Yamada-Watanabe principle [27], we only need to prove the pathwise uniqueness.

Let $b^{n}, \sigma^{n}$ be defined as in (3.48). As in the proof of Theorem 1.1, by Theorem 3.2 for $\left(\mathscr{L}_{T-t}^{\Sigma^{n}, b^{n}}, b_{T-t}^{n}\right)$ in place of $\left(\mathscr{L}_{t}^{\Sigma, b}, f_{t}\right)$, the equation

$$
\begin{equation*}
\partial_{t} \mathbf{u}_{t}^{n}=\mathscr{L}_{T-t}^{\Sigma^{n}, b^{n}} \mathbf{u}_{t}^{n}-\lambda \mathbf{u}_{t}^{n}+b_{T-t}^{n}, \quad \mathbf{u}_{0}^{n}=0, t \in[0, T] \tag{4.5}
\end{equation*}
$$

has a unique solution $\mathbf{u}^{n}:[0, T] \times \mathbb{R}^{d_{1}+d_{2}} \rightarrow \mathbb{R}^{d_{1}+d_{2}}$ such that (4.2)-(4.4) hold for $\mathbf{u}^{n}$ and some $\mathbf{u}:[0, T] \times \mathbb{R}^{d_{1}+d_{2}} \rightarrow \mathbb{R}^{d_{1}+d_{2}}$ in place of $u^{n}$ and $u$. Let

$$
\Phi_{t}(x)=x+\mathbf{u}_{T-t}(x), \quad t \in[0, T], x \in \mathbb{R}^{d_{1}+d_{2}}
$$

Then for large enough $\lambda>0, \Phi_{t}$ is a homeomorphism on $\mathbb{R}^{d_{1}+d_{2}}$ such that

$$
\begin{equation*}
\sup _{t \in[0, T]}\left(\left[\Phi_{t}\right]_{1_{[1]}}+\left[\Phi_{t}^{-1}\right]_{1_{[1]}}\right)<\infty ; \tag{4.6}
\end{equation*}
$$

that is, both $\Phi_{t}$ and $\Phi_{t}^{-1}$ are Lipschitz continuous uniformly in $t \in[0, T]$.
Now, if $X_{t}$ solves (1.2) up to a stopping time $\tau \leqslant T$, then by Itô's formula and (4.5), we have

$$
\begin{aligned}
X_{t} & +\mathbf{u}_{T-t}^{n}\left(X_{t}\right)-X_{0}-\mathbf{u}_{T}\left(X_{0}\right) \\
= & \int_{0}^{t}\left\{\lambda \mathbf{u}_{T-s}^{n}+\operatorname{tr}\left[\left(\Sigma_{s}-\Sigma_{s}^{n}\right) \nabla^{(2)} \nabla^{(2)} \mathbf{u}_{T-s}^{n}\right]+\left(b_{s}-b_{s}^{n}\right) \cdot \nabla \mathbf{u}_{T-s}^{n}+b_{s}-b_{s}^{n}\right\}\left(X_{s}\right) \mathrm{d} s \\
& +\int_{0}^{t}\left(0, \sigma_{s} \mathrm{~d} W_{s}\right)+\int_{0}^{t}\left(\nabla_{\sigma_{s} \mathrm{~d} W_{s}}^{(2)} \mathbf{u}_{T-s}^{n}\right)\left(X_{s}\right), \quad t \in[0, \tau], \mathbb{P} \text {-a.s. }
\end{aligned}
$$

So, as explained in the proof of Theorem 1.1, by letting $n \rightarrow \infty$, we obtain for $t \in[0, \tau]$,

$$
\Phi_{t}\left(X_{t}\right)=\Phi_{0}\left(X_{0}\right)+\int_{0}^{t} \lambda \mathbf{u}_{T-s}\left(X_{s}\right) \mathrm{d} s+\int_{0}^{t}\left(0, \sigma_{s}\left(X_{s}\right) \mathrm{d} W_{s}\right)+\int_{0}^{t}\left(\nabla_{\sigma_{s} \mathrm{~d} W_{s}}^{(2)} \mathbf{u}_{T-s}\right)\left(X_{s}\right) .
$$

Therefore, if $\left(X_{t}\right)_{t \in[0, \tau]}$ solves (1.2), then $Y_{t}:=\Phi_{t}\left(X_{t}\right)$ solves the following SDE for $t \in[0, \tau]$ :

$$
\begin{equation*}
\mathrm{d} Y_{t}=\lambda\left(\mathbf{u}_{T-t} \circ \Phi_{t}^{-1}\right)\left(Y_{t}\right) \mathrm{d} t+\left\{\left(\nabla_{\sigma_{t} W_{t}}^{(2)} \Phi_{t}\right) \circ \Phi_{t}^{-1}\right\}\left(Y_{t}\right) . \tag{4.7}
\end{equation*}
$$

Since by (4.4) and (4.6), both $\mathbf{u}_{T-t} \circ \Phi_{t}^{-1}$ and $\left(\nabla^{(2)} \Phi_{t}\right) \circ \Phi_{t}^{-1}$ are Lipschitz continuous uniformly in $t \in[0, T]$, from the condition (1.8) or (1.9) on $\sigma$ we see that (4.7) has a unique solution up to time $T$ (see [19, Theorem 4.1]). So, the pathwise uniqueness of (1.2) holds up to any stopping time less than $T$. By the arbitrary of $T>0$ we conclude that (1.2) has a unique solution for all $t \geqslant 0$.
(2) Next, if $\sigma(x)$ and $b(x)$ do not depend on $x^{(1)}$, then so does $\mathbf{u}^{n}(x)$. In this case, if (1.10) holds with $\phi_{n}$ and $\gamma_{n}$ uniformly in $n \geqslant 1$, then $\overline{\mathscr{Q}}_{\phi}<\infty$ for some $\phi \in \mathscr{D}_{0} \cap \mathscr{S}_{0}$, so that by (3.24) we may repeat the above argument to prove the pathwise uniqueness.
(3) In general, by a localization argument as in [25, Proof of Theorem 1.1], we obtain the local existence and uniqueness of SDE (1.2) up to explosion time $\zeta$. More precisely, for any $m \geqslant 1$, let $\theta_{m} \in C_{0}^{\infty}\left(\mathbb{R}^{d_{1}+d_{2}} ; \mathbb{R}^{d_{1}+d_{2}}\right)$ be such that $\theta_{m}(x)=x$ for $|x| \leqslant m$. Define

$$
\begin{equation*}
\sigma_{m}(x)=\sigma \circ \theta_{m}(x), \quad b_{m}^{(2)}(x)=b^{(2)} \circ \theta_{m}(x), \quad b_{m}^{(1)}(x)=b^{(1)}\left(\theta_{m}^{(1)}(x), x^{(2)}\right) . \tag{4.8}
\end{equation*}
$$

Here and below, for simplicity of notation, we shall drop the time variables in $b$ and $\sigma$ since it does not play any role in the proof. If ( $\mathbf{A}$ ) or (1.10) holds, then for any $m \in \mathbb{N}, \sigma_{m}$ and $b_{m}$ satisfy the same assumption for some uniform $C, \phi$ and $\gamma$. For fixed $X_{0} \in \mathbb{R}^{d_{1}+d_{2}}$, let $X_{t}^{m}$ with $X_{0}^{m}=X_{0}$ be the unique solution to (1.2) for $\left(\sigma_{m}, b_{m}\right)$ in place of $(\sigma, b)$. Since $b_{m}(x)=b(x), \sigma_{m}(x)=\sigma(x)$ for $|x| \leqslant m, X_{t}^{m}$ solves the original equation (1.2) up to the stopping time

$$
\tau_{m}:=\inf \left\{t \geqslant 0:\left|X_{t}^{m}\right| \geqslant m\right\} .
$$

By step (1), we have $X_{t}^{n}=X_{t}^{m}$ for $t \leqslant \tau_{n} \wedge \tau_{m}$, and $\tau_{n}$ is increasing in $n$. Letting $\zeta=$ $\lim _{n \rightarrow \infty} \tau_{n}$, we see that

$$
X_{t}:=\sum_{t \in\left[\tau_{n-1}, \tau_{n}\right)} X_{t}^{n}, \quad \tau_{0}:=0, \quad t<\zeta
$$

is the unique solution to (1.2) with life time $\zeta$, i.e. $\lim \sup _{t \rightarrow \zeta}\left|X_{t}\right|=\infty$ holds a.s. on $\{\zeta<\infty\}$.
Proof of Theorem 1.6. (1) First of all, we assume that the global conditions in the theorem hold for $\left(b^{k}, \sigma^{k}\right)_{k \in \mathbb{N}}$. In this case, let $\mathbf{u}^{k}$ be the unique classical solution of (3.1) in Theorem 3.10 corresponding to $\left(b^{k}, \sigma^{k}, b^{k}\right)$. Define

$$
\Phi_{t}^{k}(x)=x+\mathbf{u}_{T-t}^{k}(x), \quad t \in[0, T], x \in \mathbb{R}^{d_{1}+d_{2}} .
$$

As in the proof of Theorem [1.2, for large enough $\lambda>0$, and for each $k \in \mathbb{N}_{\infty}$, $\Phi_{t}^{k}$ is a homeomorphism on $\mathbb{R}^{d_{1}+d_{2}}$ such that

$$
\sup _{k \in \mathbb{N}_{\infty}} \sup _{t \in[0, T]}\left(\left[\Phi_{t}^{k}\right]_{1_{[1]}}+\left[\left(\Phi_{t}^{k}\right)^{-1}\right]_{1_{[1]}}\right)<\infty ;
$$

By Itô's formula, $Y_{t}^{k}:=\Phi_{t}^{k}\left(X_{t}^{k}\right)$ solves the following SDE for $t \in[0, T]$,

$$
\mathrm{d} Y_{t}^{k}=g_{t}^{k}\left(Y_{t}^{k}\right) \mathrm{d} t+\Theta_{t}^{k}\left(Y_{t}^{k}\right) \mathrm{d} W_{t}, \quad Y_{0}^{k}=\Phi_{0}^{k}(x)
$$

where

$$
g_{t}^{k}:=\lambda \mathbf{u}_{T-t}^{k} \circ\left(\Phi^{k}\right)_{t}^{-1}, \quad \Theta_{t}^{k}:=\left(\nabla_{\sigma_{t}^{k}}^{(2)} \Phi_{t}^{k}\right) \circ\left(\Phi_{t}^{k}\right)^{-1}
$$

Moreover, by (3.47), it is easy to see that for each $t, x \in \mathbb{R}^{d}$,

$$
\lim _{k \rightarrow \infty}\left(\left|g_{t}^{k}(x)-g_{t}^{\infty}(x)\right|+\left\|\Theta_{t}^{k}(x)-\Theta_{t}^{\infty}(x)\right\|+\left|\Phi_{t}^{k}(x)-\Phi_{t}^{\infty}(x)\right|\right)=0
$$

and by (3.46), for all $x, y \in \mathbb{R}^{d_{1}+d_{2}}$,

$$
\sup _{k \in \mathbb{N}_{\infty}} \sup _{t \in[0, T]}\left(\left|g_{t}^{k}(x)-g_{t}^{k}(y)\right|+\left\|\Theta_{t}^{k}(x)-\Theta_{t}^{\infty}(y)\right\|\right) \leqslant C|x-y|
$$

Hence, by [18, Theorem 15, p.271], we have for each $T, \varepsilon>0$,

$$
\lim _{k \rightarrow \infty} \mathbb{P}\left(\sup _{t \in[0, T]}\left|Y_{t}^{k}-Y_{t}^{\infty}\right| \geqslant \varepsilon\right)=0
$$

which in turn implies (1.16).
(2) In general, by the assumption and (1.12), we have the following uniform estimate:

$$
\begin{equation*}
\sup _{k} \mathbb{E} \exp \left[\sup _{t \in[0, T]} H\left(X_{t}^{k}(x)\right)^{\varepsilon^{\prime}}\right] \leqslant \Psi(T) \exp \left[H(x)^{\varepsilon}\right], \quad T>0, x \in \mathbb{R}^{d_{1}+d_{2}} . \tag{4.9}
\end{equation*}
$$

For each $m \in \mathbb{N}$, let $\theta_{m} \in C_{0}^{\infty}\left(\mathbb{R}^{d_{1}+d_{2}} ; \mathbb{R}^{d_{1}+d_{2}}\right)$ be such that $\theta_{m}(x)=x$ for $H(x) \leqslant m$. Let $\sigma_{m}^{k}$ and $b_{m}^{k}$ be defined as in (4.8), and let $X_{t}^{k, m}(x)$ be the solution of SDE (1.2) corresponding to $\left(\sigma_{m}^{k}, b_{m}^{k}\right)$. Define the stopping times

$$
\tau_{m}^{k}:=\inf \left\{t>0: H\left(X_{t}^{k}(x)\right) \wedge H\left(X_{t}^{\infty}(x)\right) \geqslant m\right\}
$$

Then by (4.9), we have

$$
\begin{equation*}
\sup _{k} \mathbb{P}\left(\tau_{m}^{k}<T\right) \leqslant \sup _{k} \mathbb{E}\left(\sup _{t \in[0, T]} H\left(X_{t}^{k}(x)\right) \wedge H\left(X_{t}^{\infty}(x)\right)\right) / m \rightarrow 0, \quad m \rightarrow \infty . \tag{4.10}
\end{equation*}
$$

On the other hand, we have

$$
\begin{aligned}
\mathbb{P}\left(\sup _{t \in[0, T]}\left|X_{t}^{k}-X_{t}^{\infty}\right| \geqslant \varepsilon\right) & \leqslant \mathbb{P}\left(\sup _{t \in[0, T]}\left|X_{t}^{k}-X_{t}^{\infty}\right| \geqslant \varepsilon ; \tau_{m}^{k} \geqslant T\right)+\mathbb{P}\left(\tau_{m}^{k}<T\right) \\
& \leqslant \mathbb{P}\left(\sup _{t \in[0, T]}\left|X_{t}^{k, m}-X_{t}^{\infty, m}\right| \geqslant \varepsilon\right)+\mathbb{P}\left(\tau_{m}^{k}<T\right),
\end{aligned}
$$

which together with step (1) and (4.10) gives the desired estimate (1.16).
(3) Let $\varphi: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$be a bounded smooth function with $\varphi(r)=r$ for $|r| \leqslant 1$. Let $\xi_{t}^{k}(x):=\left|X_{t}^{k}(x)-X_{t}^{\infty}(x)\right|^{2}$. For fixed $R>0$, let $\chi_{R}: \mathbb{R}^{d} \rightarrow[0,1]$ be a smooth function with $\chi_{R}(x)=1$ for $|x| \leqslant R$ and $\chi_{R}(x)=0$ for $|x| \geqslant 2 R$. By Gagliado-Nirenberg's inequality and (1.17) for some $p>d$, we have

$$
\begin{aligned}
& \mathbb{E}\left[\sup _{t \in[0, T]}\left\|\varphi\left(\xi_{t}^{k}\right) \chi_{R}\right\|_{\infty}\right] \\
& \leqslant C \mathbb{E}\left[\sup _{t \in[0, T]}\left\|\varphi\left(\xi_{t}^{k}\right) \chi_{R}\right\|_{L^{p}}^{1-\frac{d}{p}}\left(\left\|\varphi\left(\xi_{t}^{k}\right) \chi_{R}\right\|_{L^{p}}+\left\|\chi_{R} \nabla\left(\varphi\left(\xi_{t}^{k}\right)\right)\right\|_{L^{p}}+1\right)^{\frac{d}{p}}\right] \\
& \leqslant C\left\{\mathbb { E } \left[\sup _{t \in[0, T]}\left\|\varphi\left(\xi_{t}^{k}\right) \chi_{R}\right\|_{\left.\left.p^{\frac{p(p-d)}{p^{2}-d}}\right]\right\}^{1-d / p^{2}} \rightarrow 0, n \rightarrow \infty,} \rightarrow \infty, n\right.\right.
\end{aligned}
$$

due to (1.16) and the dominated convergence theorem. So, (1.18) holds.

Proof of Theorem 1.7. By Theorem 1.2, for each $x \in \mathbb{R}^{d_{1}+d_{2}}$, there is a unique global solution $\left\{X_{t}(x), t \geqslant 0\right\}$ for SDE (1.2). Let $\mathbf{u}_{t}(x)$ be the unique solution of equation (3.1) in Theorem 3.11 corresponding to $(\sigma, b)$ and $f=b$. By (3.52), we have

$$
\begin{equation*}
\left\|\nabla \mathbf{u}_{t}\right\|_{1_{[1 / 3]}}+\left\|\nabla \nabla^{(2)} \mathbf{u}_{t}\right\|_{1_{[\varepsilon]}} \leqslant C \int_{0}^{t} \mathrm{e}^{-\lambda(t-s)}(t-s)^{-\delta} \mathrm{d} s \tag{4.11}
\end{equation*}
$$

As in the proof of Theorem 1.2, let

$$
\Phi_{t}(x)=x+\mathbf{u}_{T-t}(x)
$$

By (4.11), for large enough $\lambda>0, \Phi_{t}$ is a diffeomorphism on $\mathbb{R}^{d_{1}+d_{2}}$ such that

$$
\begin{equation*}
\sup _{t \in[0, T]}\left(\left\|\nabla \Phi_{t}\right\|_{1_{[\varepsilon]}}+\left\|\nabla \Phi_{t}^{-1}\right\|_{1_{[\varepsilon]}}\right)<\infty \tag{4.12}
\end{equation*}
$$

Moreover, as shown in the proof of Theorem 1.2 that if $X_{t}(x)$ solves SDE (1.2) then $Y_{t}=$ $\Phi_{t}\left(X_{t}\right)$ solves (4.7). By (4.12) and the condition on $\sigma$ in Theorem 1.7, we have

$$
\begin{equation*}
\sup _{t \in[0, T]}\left(\left\|\nabla\left(\mathbf{u}_{T-t} \circ \Phi_{t}^{-1}\right)\right\|_{1_{[\varepsilon]}}+\left\|\nabla\left\{\left(\nabla_{\sigma_{t}}^{(2)} \Phi_{t}\right) \circ \Phi_{t}^{-1}\right\}\right\|_{1_{[\varepsilon]}}\right)<\infty, \tag{4.13}
\end{equation*}
$$

for some $\varepsilon>0$. So, by [15, Theorem 4.6.5], $\left\{Y_{t}(\cdot)\right\}_{t \in[0, T]}$ forms a $C^{1}$-stochastic diffeomorphism flow, and so does $\left\{X_{t}(\cdot):=\Phi_{t}^{-1}\left(Y_{t}(\cdot)\right)\right\}_{t \in[0, T]}$. Finally, it is easy to prove (1.19) from (4.7), (4.12) and $\sup _{t \in[0, T]}\left\|\nabla \sigma_{t}\right\|_{\infty}<\infty$.

Proof of Theorem 1.8. As shown in the proof of Theorem 1.2 that SDE (1.22) admits a unique global strong solution $X_{t}(x)$ and $Y_{t}:=\Phi_{t}\left(X_{t}\right)$ solves (see (4.7))

$$
\begin{equation*}
\mathrm{d} Y_{t}=g_{t}^{a}\left(Y_{t}\right) \mathrm{d} t+\Theta_{t}\left(Y_{t}\right) \mathrm{d} W_{t}, \quad Y_{0}=y=: \Phi_{0}(x) \tag{4.14}
\end{equation*}
$$

where

$$
\begin{equation*}
g_{t}^{a}:=\left(\lambda \mathbf{u}_{T-t}+a_{t} \cdot \nabla^{(2)} \Phi_{t}\right) \circ \Phi_{t}^{-1}, \quad \Theta_{t}:=\left(\nabla_{\sigma_{t}}^{(2)} \Phi_{t}\right) \circ \Phi_{t}^{-1} \tag{4.15}
\end{equation*}
$$

By (1.21), (4.11), (4.12) and $\sigma_{t} \in C_{b}^{1}$ uniformly in $t \in[0, T]$, there is a constant $C>0$ such that for all $t \in[0, T]$ and $y, y^{\prime} \in \mathbb{R}^{d}$,

$$
\begin{equation*}
\left|g_{t}^{a}(y)-g_{t}^{a}\left(y^{\prime}\right)\right| \leqslant C\left(H^{\varepsilon^{\prime}} \circ \Phi_{t}^{-1}(y)+H^{\varepsilon^{\prime}} \circ \Phi_{t}^{-1}\left(y^{\prime}\right)\right)\left|y-y^{\prime}\right|, \quad\left\|\nabla \Theta_{t}\right\|_{\infty} \leqslant C \tag{4.16}
\end{equation*}
$$

On the other hand, by (1.12) and $\varepsilon^{\prime}<\varepsilon$, for any $K>0$, there exists $C_{K}>0$ such that

$$
\begin{equation*}
\mathbb{E} \exp \left[K \sup _{t \in[0, T]}\left(H \circ \Phi_{t}^{-1}\left(Y_{t}\right)\right)^{\varepsilon^{\prime}}\right] \leqslant C_{K} \exp \left[H(x)^{\varepsilon}\right] . \tag{4.17}
\end{equation*}
$$

In order to show the diffeomorphism property of $x \mapsto X_{t}(x)$, we shall use Kunita's argument. More precisely, we want to show the following estimates: for any $p \in \mathbb{R}$ and $T>0$, there are constants $C_{1}, C_{2}>0$ such that for all $x, x^{\prime} \in \mathbb{R}^{d_{1}+d_{2}}$ and $t \in[0, T]$,

$$
\begin{equation*}
\mathbb{E}\left|X_{t}(x)-X_{t}\left(x^{\prime}\right)\right|^{2 p} \leqslant C_{1}\left(\mathrm{e}^{H(x)^{\varepsilon}}+\mathrm{e}^{H\left(x^{\prime}\right)^{\varepsilon}}\right)\left|x-x^{\prime}\right|^{2 p} \tag{4.18}
\end{equation*}
$$

$$
\begin{equation*}
\mathbb{E}\left(1+\left|X_{t}(x)\right|^{\delta_{2}}\right)^{p} \leqslant C_{2}\left(1+|x|^{\delta_{1}}\right)^{p}, \quad p<0 ; \tag{4.19}
\end{equation*}
$$

and for any $p \geqslant 1$ and $T>0$, there is a constant $C_{3}>0$ such that for all $x \in \mathbb{R}^{d_{1}+d_{2}}$ and $t, s \in[0, T]$,

$$
\begin{equation*}
\mathbb{E}\left|X_{t}(x)-X_{s}(x)\right|^{2 p} \leqslant C_{3} \mathrm{e}^{H(x)^{\varepsilon}}|t-s|^{p} . \tag{4.20}
\end{equation*}
$$

Estimate (4.20) is direct by the assumptions, (1.22) and (4.17). Let us show (4.18). Set

$$
Z_{t}:=Y_{t}(y)-Y_{t}\left(y^{\prime}\right), \quad y=\Phi_{0}(x), y^{\prime}=\Phi_{0}\left(x^{\prime}\right)
$$

and

$$
G_{t}:=g_{t}^{a}\left(Y_{t}(y)\right)-g_{t}^{a}\left(Y_{t}\left(y^{\prime}\right)\right), \quad U_{t}:=\Theta_{t}\left(Y_{t}(y)\right)-\Theta_{t}\left(Y_{t}\left(y^{\prime}\right) .\right.
$$

By Itô's formula, we have

$$
\mathrm{d}\left|Z_{t}\right|^{2}=\left[2\left\langle Z_{t}, G_{t}\right\rangle+\operatorname{tr}\left(U_{t}^{*} U_{t}\right)\right] \mathrm{d} t+2\left\langle Z_{t}, U_{t} \mathrm{~d} W_{t}\right\rangle=\left|Z_{t}\right|^{2} \mathrm{~d}\left(N_{t}+M_{t}\right),
$$

where

$$
N_{t}:=\int_{0}^{t}\left|Z_{s}\right|^{-2}\left[2\left\langle Z_{s}, G_{s}\right\rangle+\operatorname{tr}\left(U_{s}^{*} U_{s}\right)\right] \mathrm{d} s, M_{t}:=2 \int_{0}^{t}\left|Z_{s}\right|^{-2}\left\langle Z_{s}, U_{s} \mathrm{~d} W_{s}\right\rangle
$$

Here we use the convention $\frac{0}{0}=0$. Notice that by (4.16),

$$
\begin{equation*}
\left|G_{t}\right| \leqslant C\left(H^{\varepsilon^{\prime}} \circ \Phi_{t}^{-1}\left(Y_{t}(y)\right)+H^{\varepsilon^{\prime}} \circ \Phi_{t}^{-1}\left(Y_{t}\left(y^{\prime}\right)\right)\right)\left|Z_{t}\right|, \quad\left|U_{t}\right| \leqslant C\left|Z_{t}\right| \tag{4.21}
\end{equation*}
$$

Hence, by (4.17), $N_{t}+M_{t}$ is a continuous semimartingale, and

$$
\left|Z_{t}\right|^{2}=\left|Z_{0}\right|^{2} \exp \left\{M_{t}-\frac{1}{2}\langle M\rangle_{t}+N_{t}\right\}
$$

Since for any $q \in \mathbb{R}, t \mapsto \exp \left\{q M_{t}-\frac{q^{2}}{2}\langle M\rangle_{t}\right\}$ is an exponential martingale, by (4.21), (4.17) and using Hölder's inequality, we have for any $p \in \mathbb{R}$,

$$
\mathbb{E}\left|Z_{t}\right|^{2 p}=\left|Z_{0}\right|^{2 p} \mathbb{E} \exp \left\{p M_{t}-\frac{p}{2}\langle M\rangle_{t}+p N_{t}\right\} \leqslant C\left(\mathrm{e}^{H(x)^{\varepsilon}}+\mathrm{e}^{H\left(x^{\prime}\right)^{\varepsilon}}\right)\left|Z_{0}\right|^{2 p}
$$

which in turn gives (4.18).
Next comes to (2.42). By Itô's formula and (1.20), we have

$$
\begin{aligned}
\mathbb{E} H\left(X_{t}(x)\right)^{p}= & H(x)^{p}+p \mathbb{E} \int_{0}^{t} H\left(X_{s}(x)\right)^{p-1}\left(\mathscr{L}_{s}^{\Sigma, b+a} H\right)\left(X_{s}(x)\right) \mathrm{d} s \\
& +\frac{p(p-1)}{2} \mathbb{E} \int_{0}^{t} H\left(X_{s}(x)\right)^{p-2}\left|\sigma_{t} \cdot \nabla^{(2)} H\left(X_{s}(x)\right)\right|^{2} \mathrm{~d} s \\
\leqslant & H(x)^{p}+C \mathbb{E} \int_{0}^{t} H\left(X_{s}(x)\right)^{p} \mathrm{~d} s
\end{aligned}
$$

which in turn gives (4.19) by Gronwall's inequality and (1.20).
Finally, by (4.18)-(4.20), as in the proof of Kunita [15, p.159-160] (see also [30, Theorem 3.4]), there is a full set $\Omega_{0}$ such that for all $\omega \in \Omega_{0}$ and $t>0, x \mapsto X_{t}(x, \omega)$ is a homeomorphism. On the other hand, since the coefficients of SDE (4.14) are $C^{1+\varepsilon}$, by [15, Theorem 4.7.2], $\left\{Y_{t}(\cdot)\right\}_{t \geqslant 0}$ defines a local $C^{1}$-diffeomorphism flow, so does $\left\{X_{t}(\cdot)\right\}_{t \geqslant 0}$. This together with the homeomorphism property implies the global $C^{1}$-diffeomorphism property of $\left\{X_{t}(\cdot)\right\}_{t \geqslant 0}$. Finally, (1.23) follows from (4.18) and [26, Lemma 2.1].

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