Sliding mode control for a nonlinear phase-field system

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Abstract

In the present contribution the sliding mode control (SMC) problem for a phase-field model of Caginalp type is considered. First we prove the well-posedness and some regularity results for the phase-field type state systems modified by the state-feedback control laws. Then, we show that the chosen SMC laws force the system to reach within finite time the sliding manifold (that we chose in order that one of the physical variables or a combination of them remains constant in time). We study three different types of feedback control laws: the first one appears in the internal energy balance and forces a linear combination of the temperature and the phase to reach a given (space dependent) value, while the second and third ones are added in the phase relation and lead the phase onto a prescribed target. While the control law is non-local in space for the first two problems, it is local in the third one, i.e., its value at any point and any time just depends on the value of the state.

Key words: phase field system, nonlinear boundary value problems, phase transition, sliding mode control, state-feedback control law.

AMS (MOS) Subject Classification: 34B15, 82B26, 34H05, 93B52.

1 Introduction

Sliding mode control (SMC) has for many years been recognized as one of the fundamental approaches for the systematic design of robust controllers for nonlinear complex dynamic systems that operate under uncertainty. Moreover, SMC is nowadays considered a classical tool for the regulation of continuous - or discrete - time systems in finite-dimensional settings (cf., e.g., the monographs [1, 10, 11, 14, 19, 33, 34, 36]).

The main advantage of sliding mode control is that it allows the separation of the motion of the overall system in independent partial components of lower dimensions, and consequently it reduces the complexity of the control problem. The design of feedback control systems with sliding modes implies the design of suitable control functions enforcing motions along ad-hoc manifolds. Hence, the main idea behind this scheme is first to identify a manifold of lower dimension (called the sliding manifold) where the control goal is fulfilled and such that the original system restricted to this sliding manifold has a desired behavior, and then to act on the system through the control in order to constrain the evolution on it, that is, to design a SMC-law that forces the trajectories of the system to reach the sliding surface and maintains them on it.

Sliding mode controls, while being relatively easy to design, feature properties of both robustness with respect to unmodelled dynamics and insensitivity to external disturbances that are quite attractive in many applications. Hence, in the last years there has been a growing interest in the extension of the well developed methods for finite-dimensional systems described by ODEs (cf., e.g., [24–27]) to the control of infinite-dimensional dynamical systems. While in some early works [27–29] only special classes of evolutions were considered, the theoretical development in a general Hilbert space setting or for PDE systems has gained attention only in the last ten years. In this respect, we can quote the papers [6], [23], and [35] dealing with sliding modes control for semilinear PDEs. In particular, in [6] the stabilization problem of a one-dimensional unstable heat conduction system (rod) modeled by a parabolic partial differential equation, powered with a Dirichlet type actuator from one of the boundaries was considered. A delay-independent SMC strategy was proposed in [35] to control a class of quasi-linear parabolic PDE systems with time-varying delay, while in [23] the authors study a sliding mode control law for a class of parabolic systems where the control acts through a Neumann boundary condition and the control space is finite-dimensional.

In the present contribution we would like to employ – to the best of our knowledge for the first time in the literature – a SMC technique for a nonlinear PDE system of phase-field type. In particular, we consider the following rather simple version of the phase-field system of Caginalp type (see [5]):

$$\partial_t\big(\vartheta + \ell\varphi\big) - \kappa\Delta\vartheta = f \qquad \text{in } Q := (0,T)\times\Omega \tag{1.1}$$

$$\partial_t\varphi - \nu\Delta\varphi + F'(\varphi) = \gamma\vartheta \qquad \text{in } Q \tag{1.2}$$

where Ω is the three-dimensional domain in which the evolution takes place, T is some final time, ϑ denotes the relative temperature around some critical value that is taken to be 0 without loss of generality, and φ is the order parameter. Moreover, ℓ, κ, ν and γ are positive constants, f is a source term and F' represents the derivative of a double-well potential F. Typical examples are

$$F_{reg}(r) = \frac{1}{4}(r^2-1)^2, \quad r \in \mathbb{R} \tag{1.3}$$

$$F_{log}(r) = \big((1+r)\ln(1+r) + (1-r)\ln(1-r)\big) - c_0 r^2, \quad r \in (-1,1) \tag{1.4}$$

$$F_{obs}(r) = I(r) - c_0 r^2, \quad r \in \mathbb{R} \tag{1.5}$$

where $c_0 > 1$ in (1.4) in order to produce a double well, while c_0 is an arbitrary positive number in (1.5), and the function I in (1.5) is the indicator function of $[-1,1]$, i.e., it takes the values 0 or $+\infty$ according to whether or not r belongs to $[-1,1]$. The potential (1.3) and (1.4) are the usual classical regular potential and the so-called logarithmic potential, respectively. More generally, the potential F could be just the sum

$$F = \widehat{\beta} + \widehat{\pi},$$

where $\widehat{\beta}$ is a convex function that is allowed to take the value $+\infty$, and $\widehat{\pi}$ is a smooth perturbation (not necessarily concave). In such a case, $\widehat{\beta}$ is supposed to be proper and lower semicontinuous so that its subdifferential is well-defined and can replace the derivative which might not exist. This happens in the case (1.5) and equation (1.2) becomes a differential inclusion.

The above system is complemented by initial conditions like $\vartheta(0) = \vartheta_0$ and $\varphi(0) = \varphi_0$ and suitable boundary conditions. Concerning the latter, as very usual we take the homogeneous Neumann condition for both ϑ and φ, that is,

$$\partial_n \vartheta = 0 \quad \text{and} \quad \partial_n \varphi = 0 \quad \text{on } \Sigma := (0,T) \times \Gamma$$

where Γ is the boundary of Ω and ∂_n is the (say, outward) normal derivative.

Equations (1.1)–(1.2) yield a system of phase field type. Such systems have been introduced (cf. [5]) in order to include phase dissipation effects in the dynamics of moving interfaces arising in thermally induced phase transitions. In our case, we move from the following expression for the total free energy

$$\mathcal{F}(\vartheta,\varphi) = \int_\Omega \Big(-\frac{c_0}{2}\vartheta^2 - \gamma\vartheta\varphi + F(\varphi) + \frac{\nu}{2}|\nabla\varphi|^2 \Big) \tag{1.6}$$

where c_0 and γ stand for specific heat and latent heat coefficients, respectively, with a terminology motivated by earlier studies (see [9]) on the Stefan problem; we refer to the monography [13] which deals with phase change models as well. In this connection, let us introduce the enthalpy e by

$$e = -\frac{\delta\mathcal{F}}{\delta\vartheta} \quad (\text{– the variational derivative of } \mathcal{F} \text{ with respect to } \vartheta)$$

that is $e = c_0\vartheta + \gamma\varphi$. Then, the governing balance and phase equations are given by

$$\partial_t e + \operatorname{div} \mathbf{q} = \tilde{f} \tag{1.7}$$

$$\partial_t \varphi + \frac{\delta\mathcal{F}}{\delta\varphi} = 0 \tag{1.8}$$

where $\mathbf{q}$ denotes the thermal flux vector, $\tilde{f}$ represents some heat source and the variational derivative of $\mathcal{F}$ with respect to φ appears in (1.8). Hence, (1.8) reduces exactly to (1.2) along with the homogeneous Neumann boundary condition for φ. Moreover, if we assume the classical Fourier law $\mathbf{q} = -\tilde{\kappa}\,\nabla\vartheta$, then (1.7) is nothing but the usual energy balance equation of the Caginalp model [5]. By setting $\ell := \gamma/c_0$, $\kappa := \tilde{\kappa}/c_0$, $f := \tilde{f}/c_0$, we easily see that (1.1) follows from (1.7) and the Neumann boundary condition for ϑ is a consequence of the no-flux condition $\mathbf{q}\cdot\mathbf{n} = 0$ on the boundary. We also point out that the above phase field system has received a good deal of attention in the last decades and it can be deduced as a special gradient-flow problem (cf., e.g., [30] and references therein).

As already noticed, the well-posedness, the long-time behavior of solutions, and also the related optimal control problems have been widely studied in the literature. We refer, without any sake of completeness, e.g., to [4, 12, 16, 20, 22] and references therein for the well-posedness and long time behavior results and to [7, 8, 17, 18] for the related optimal control problems.

The present paper is also related to the control problems, but it goes in the direction of designing sliding mode controls for the above phase-field system. Indeed our main objective is to find out some state-feedback control laws $(\vartheta,\varphi) \mapsto u(\vartheta,\varphi)$ that can be inserted in one of the equations in order that the dynamics of the system modified in this way forces the value $(\vartheta(t),\varphi(t))$ of the solution to reach some manifold of the phase space in a finite time and then lie there with a sliding mode.

The first analytical difficulty consists in deriving the equations governing the sliding modes and the conditions for this motion to exist. The problem needs the development of special methods, since the conventional theorems regarding existence and uniqueness of solutions are not directly applicable. Moreover, we need to manipulate the system through the control in order to constrain the evolution on the desired sliding manifold. In particular, we study three cases.

In the first one, a feedback control is added to the internal energy balance equation (1.1) in order to force a linear relationship between ϑ and φ; in the second case, a prescribed distribution φ^* of the order parameter is forced by means of a feedback control added to the phase dynamics (1.2). Notice that both these choices can be considered physically meaningful in the framework of phase transition processes, since in both cases the quantities we are forcing to reach time-independent values may have a physical meaning. In the first problem, we can take the internal energy as a particular case, while the target φ^* we force for the phase parameter in the second problem could represent one of the so called pure phases (e.g., pure water or pure ice in a water-ice phase change process). Moreover, in both cases we have reduced the problem to a simplified dynamics involving only the evolution of φ in the first case and only of ϑ in the second one (cf. also Remark 2.8).

In each of the above problems, the control law we introduce is non-local in space, i.e., the value at (t,x) of the control depends on the whole state $(\vartheta(t,\cdot),\varphi(t,\cdot))$ at the time t and not only on the value $(\vartheta(t,x),\varphi(t,x))$. The objective of the third problem is to design a control law that reaches the same target as in the second one and is local at the same time. However, such a problem looks much more difficult and we can ensure the existence of the desired sliding mode only under a suitable compatibility condition on Ω.

The paper is organized as follows. In the next section, we list our assumptions, state the problem in a precise form and present our results. The last two sections are devoted to the corresponding proofs. Section 3 deals with well-posedness and regularity, while the existence of the sliding modes is proved in Section 4.

2 Statement of the problem and results

In this section, we describe the problem under study and present our results. As in the Introduction, Ω is the body where the evolution takes place. We assume

$$\Omega \subset \mathbb{R}^3 \quad \text{to be open, bounded, connected, and smooth}$$

and write $|\Omega|$ for its Lebesgue measure. Moreover, Γ and ∂_n still stand for the boundary of Ω and the outward normal derivative, respectively. Given a finite final time $T > 0$, we set for convenience $Q := (0, T) \times \Omega$. Now, we specify the assumptions on the structure of our system. We assume that

$$\ell,\, \kappa,\, \nu,\, \gamma \in (0, +\infty) \tag{2.1}$$

$$\widehat{\beta} : \mathbb{R} \to [0, +\infty] \quad \text{is convex, proper and l.s.c.} \quad \text{with} \quad \widehat{\beta}(0) = 0 \tag{2.2}$$

$$\widehat{\pi} : \mathbb{R} \to \mathbb{R} \quad \text{is a } C^1 \text{ function} \quad \text{and} \quad \widehat{\pi}\,' \quad \text{is uniformly Lipschitz.} \tag{2.3}$$

We set for brevity

$$\beta := \partial\widehat{\beta} \quad \text{and} \quad \pi := \widehat{\pi}\,' \tag{2.4}$$

and denote by $D(\beta)$ and $D(\widehat{\beta})$ the effective domains of β and $\widehat{\beta}$, respectively. Next, in order to simplify notations, we set

$$V := H^1(\Omega), \quad H := L^2(\Omega), \quad W := \{v \in H^2(\Omega) : \partial_n v = 0\} \tag{2.5}$$

and endow the spaces V and H with their standard norms $\|\cdot\|_V$ and $\|\cdot\|_H$. On the contrary, we write $\|\cdot\|_W$ for the norm in W defined by

$$\|v\|_W^2 = \|v\|_H^2 + |\Omega|^{4/3} \|\Delta v\|_H^2 \quad \text{for every } v \in W \tag{2.6}$$

and we term C_Ω the best constant realizing the inequality

$$\|v\|_\infty \leq C_\Omega\, \|v\|_W \quad \text{for every } v \in W. \tag{2.7}$$

The reason of this choice will be explained later on (see the forthcoming Remark 2.11). Now, we just notice that $\|\cdot\|_W$ is equivalent to the norm induced on W by the standard one in $H^2(\Omega)$ (thanks to the regularity theory of elliptic equations) and that the constant C_Ω actually exists due to the continuous embedding $H^2(\Omega) \subset C^0(\overline{\Omega})$ (since $\Omega \subset \mathbb{R}^3$ is bounded and smooth) and only depends on Ω (see, e.g., [15]). Finally, for the norms both in $L^\infty(\Omega)$ and in $L^\infty(Q)$ we use the same symbol $\|\cdot\|_\infty$ whenever no confusion can arise.

Furthermore, let

$$\operatorname{Sign} : H \to 2^H \quad \text{be the subdifferential of the map} \quad \|\cdot\|_H : H \to \mathbb{R} \tag{2.8}$$

i.e.,

$$\operatorname{Sign} v = \frac{v}{\|v\|_H} \quad \text{if } v \in H \text{ and } v \neq 0 \tag{2.9}$$

$$\operatorname{Sign} 0 \ \text{ is the closed unit ball of } H. \tag{2.10}$$

Thus, β and Sign are maximal monotone operators on $\mathbb{R}$ and H, respectively (see, e.g., [2, Thm. 2.8, p. 47]). In the sequel, we use the same symbol β to denote the maximal monotone operator induced on L^2-spaces.

Yosida regularizations of β and Sign. Let us introduce the Yosida regularization $\beta_\varepsilon : \mathbb{R} \to \mathbb{R}$ and $\mathrm{Sign}_\varepsilon : H \to H$ at level $\varepsilon > 0$ (see, e.g., [2, formulas (2.26), p. 37]) as well as the Moreau regularization of $\|\cdot\|_H$ (see, e.g., [2, formula (2.38), p. 48])

$$\|v\|_{H,\varepsilon} := \min_{w\in H}\bigl\{\tfrac{1}{2\varepsilon}\|w-v\|_H^2 + \|w\|_H\bigr\} = \int_0^{\|v\|_H} \min\{s/\varepsilon, 1\}\, ds \quad \text{for } v \in H. \tag{2.11}$$

For the reader's convenience, we sketch the justification of the last equality of (2.11). We write $\|\cdot\|$ instead of $\|\cdot\|_H$ for simplicity. For $w \in H$ and $y \geq 0$ we set

$$G(w) := \frac{1}{2\varepsilon}\,\|w-v\|^2 + \|w\| \quad \text{and} \quad g(y) := \frac{1}{2\varepsilon}(y - \|v\|)^2 + y$$

and observe that the triangle inequality $\bigl|\|w\| - \|v\|\bigr| \leq \|w - v\|$ yields $G(w) \geq g(\|w\|)$ for every $w \in H$. Now, from one side, one easily checks that

$$\min_{y\geq 0} g(y) = \frac{1}{2\varepsilon}\,\|v\|^2 \quad \text{if } \|v\| \leq \varepsilon \quad \text{and} \quad \min_{y\geq 0} g(y) = \|v\| - \frac{\varepsilon}{2} \quad \text{if } \|v\| > \varepsilon.$$

This means that $\min_{y\geq 0} g(y)$ coincides with the right-hand side of (2.11). On the other hand, we have

$$G(0) = \frac{1}{2\varepsilon}\,\|v\|^2 \quad \text{in any case,} \quad \text{and} \quad G\bigl((1-\varepsilon/\|v\|)v\bigr) = \|v\| - \frac{\varepsilon}{2} \quad \text{if } \|v\| > \varepsilon.$$

Thus, $\min_{w\in H} G(w) = \min_{y\geq 0} g(y)$ and (2.11) is proved. Next, we recall that β_ε and $\mathrm{Sign}_\varepsilon$ are monotone and that (see, e.g., [2, Prop. 2.2 (ii), p. 38] and [2, Thm. 2.9, p. 48] for some of these properties)

$$\mathrm{Sign}_\varepsilon\, v \ \text{ is the gradient at } v \text{ of the } C^1 \text{ functional } \ \|\cdot\|_{H,\varepsilon} \tag{2.12}$$

$$\mathrm{Sign}_\varepsilon\, v = \frac{v}{\max\{\varepsilon, \|v\|_H\}} \quad \text{for every } v \in H \tag{2.13}$$

$$(\mathrm{Sign}_\varepsilon\, v, v)_H \geq \|v\|_H - \frac{\varepsilon}{4} \quad \text{for every } v \in H \tag{2.14}$$

$$|\beta_\varepsilon(r)| \leq |\beta^\circ(r)| \quad \text{for every } r \in D(\beta), \quad \text{where} \\ \beta^\circ(r) \text{ is the element of } \beta(r) \text{ having minimum modulus.} \tag{2.15}$$

We point out that the Young inequality has been used to derive (2.14).

At this point, we describe the state system modified by the state-feedback control law and we study two cases. In the first one, a feedback control is added to the first equation (1.1) in order to force a linear relationship between ϑ and φ; in the second case, a prescribed distribution of the order parameter is forced by means of a feedback control that is added to equation (1.2). In principle, for the data, we require that

$$f \in L^2(Q), \quad \vartheta_0 \in V, \quad \varphi_0 \in V \quad \text{and} \quad \widehat{\beta}(\varphi_0) \in L^1(\Omega). \tag{2.16}$$

Given $\rho > 0$ and some target that depends on the case we want to consider, we look for a quadruplet $(\vartheta, \varphi, \xi, \sigma)$ satisfying at least the regularity requirements

$$\vartheta, \varphi \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \tag{2.17}$$

$$\xi \in L^2(0,T;H) \quad \text{and} \quad \sigma \in L^\infty(0,T;H), \tag{2.18}$$

and solving the related system we introduce at once. We notice that the homogeneous Neumann boundary conditions for both ϑ and φ are contained in (2.17) (see the definition (2.5) of W). The problems corresponding to the cases sketched above are the following.

Given $\eta^* \in W$ and $\alpha \in \mathbb{R}$, the first system is

$$\partial_t\bigl(\vartheta + \ell\varphi\bigr) - \kappa\Delta\vartheta = f - \rho\sigma \quad \text{a.e. in } Q \tag{2.19}$$

$$\partial_t\varphi - \nu\Delta\varphi + \xi + \pi(\varphi) = \gamma\vartheta \quad \text{a.e. in } Q \tag{2.20}$$

$$\xi \in \beta(\varphi) \quad \text{a.e. in } Q \tag{2.21}$$

$$\sigma(t) \in \mathrm{Sign}(\vartheta(t) + \alpha\varphi(t) - \eta^*) \quad \text{for a.a. } t \in (0,T) \tag{2.22}$$

$$\vartheta(0) = \vartheta_0 \quad \text{and} \quad \varphi(0) = \varphi_0\,. \tag{2.23}$$

In the sequel, we also term such problem *Problem (A)*.

The second problem, which we call *Problem (B)*, depends on a given $\varphi^* \in W$ and consists in the equations

$$\partial_t\bigl(\vartheta + \ell\varphi\bigr) - \kappa\Delta\vartheta = f \quad \text{a.e. in } Q \tag{2.24}$$

$$\partial_t\varphi - \nu\Delta\varphi + \xi + \pi(\varphi) = \gamma\vartheta - \rho\sigma \quad \text{a.e. in } Q \tag{2.25}$$

$$\xi \in \beta(\varphi) \quad \text{a.e. in } Q \tag{2.26}$$

$$\sigma(t) \in \mathrm{Sign}(\varphi(t) - \varphi^*) \quad \text{for a.a. } t \in (0,T) \tag{2.27}$$

$$\vartheta(0) = \vartheta_0 \quad \text{and} \quad \varphi(0) = \varphi_0\,. \tag{2.28}$$

The last case, termed *Problem (C)*, is the same as the previous one with the following difference: the non-local operator Sign is replaced by the local sign : $\mathbb{R} \to 2^{\mathbb{R}}$ defined by

$$\mathrm{sign}\, r := \frac{r}{|r|} \quad \text{if } r \neq 0 \quad \text{and} \quad \mathrm{sign}\, 0 := [-1,1]. \tag{2.29}$$

Notice that sign is the subdifferential of the real function $r \mapsto |r|$ and thus is maximal monotone. For the sake of clarity, we write Problem (C), explicitly. Given $\varphi^* \in W$, we look for a quadruplet $(\vartheta, \varphi, \xi, \sigma)$ satisfying

$$\partial_t\bigl(\vartheta + \ell\varphi\bigr) - \kappa\Delta\vartheta = f \quad \text{a.e. in } Q \tag{2.30}$$

$$\partial_t\varphi - \nu\Delta\varphi + \xi + \pi(\varphi) = \gamma\vartheta - \rho\sigma \quad \text{a.e. in } Q \tag{2.31}$$

$$\xi \in \beta(\varphi) \quad \text{a.e. in } Q \tag{2.32}$$

$$\sigma \in \mathrm{sign}(\varphi - \varphi^*) \quad \text{a.e. in } Q \tag{2.33}$$

$$\vartheta(0) = \vartheta_0 \quad \text{and} \quad \varphi(0) = \varphi_0\,. \tag{2.34}$$

Here are our results on the well-posedness of the above problems.

Theorem 2.1. *Assume* (2.1)–(2.3), (2.16),

$$\eta^* \in W \quad \textit{and} \quad \alpha \in \mathbb{R}. \tag{2.35}$$

Then, for every $\rho > 0$, Problem (A) has at least a solution $(\vartheta, \varphi, \xi, \sigma)$ satisfying (2.17)–(2.18) *and the estimates*

$$\|\vartheta\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\varphi\|_{H^1(0,T;H)\cap L^2(0,T;W)} + \|\xi\|_{L^2(0,T;H)} + \|\sigma\|_{L^\infty(0,T;H)} \leq C_1 \tag{2.36}$$

$$\|\vartheta\|_{H^1(0,T;H)\cap L^2(0,T;W)} \leq C_2\bigl(\rho^{1/2} + 1\bigr) \tag{2.37}$$

where C_1 *and* C_2 *depend only on the quantities involved in assumptions* (2.1)–(2.3), (2.16) *and* (2.35)*. Moreover, the solution is unique if* $\alpha = \ell$*. Furthermore, if in addition*

$$\varphi_0 \in W \quad \textit{and} \quad \beta^\circ(\varphi_0) \in H \tag{2.38}$$

then, there exists a solution that also satisfies

$$\varphi \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W) \quad \textit{and} \quad \xi \in L^\infty(0,T;H) \tag{2.39}$$

$$\|\varphi\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^\infty(0,T;W)} \leq C_3\bigl(\rho^{1/2}+1\bigr) \tag{2.40}$$

where C_3 *depends on the norms related to* (2.38) *as well. In particular,* φ *is bounded. Finally, the component* ϑ *of any solution satisfying all the above regularity requirements is bounded whenever* $\vartheta_0 \in V \cap L^\infty(\Omega)$ *and* $f \in L^\infty(0,T;H)$*.*

Theorem 2.2. *Assume* (2.1)–(2.3), (2.16)*, as well as*

$$\varphi^* \in W \quad \textit{and} \quad \beta^\circ(\varphi^*) \in H\,. \tag{2.41}$$

Then, for every $\rho > 0$*, Problem (B) has at least a solution* $(\vartheta, \varphi, \xi, \sigma)$ *satisfying* (2.17)–(2.18) *and the estimates*

$$\|\vartheta\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\varphi\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\sigma\|_{L^\infty(0,T;H)} \leq C_4 \tag{2.42}$$

$$\|\vartheta\|_{H^1(0,T;H)\cap L^2(0,T;W)} + \|\varphi\|_{H^1(0,T;H)\cap L^2(0,T;W)} + \|\xi+\rho\sigma\|_{L^2(0,T;H)} \leq C_5\bigl(\rho^{1/2}+1\bigr) \tag{2.43}$$

where C_4 *and* C_5 *depend only on the quantities involved in assumptions* (2.1)–(2.3), (2.16) *and* (2.41)*. Furthermore, the components* ϑ *and* φ *of the solution are uniquely determined, and* ξ *and* σ *are uniquely determined as well if* β *is single-valued.*

A similar result holds for Problem (C). We present the corresponding statement in a more accurate form for a reason that will be clear later on.

Theorem 2.3. *Assume* (2.1)–(2.3), (2.16) *and* (2.41)*. Then, for every* $\rho > 0$*, Problem (C) has at least a solution* $(\vartheta, \varphi, \xi, \sigma)$ *satisfying* (2.17)–(2.18)*. Furthermore, the components* ϑ *and* φ *of the solution are uniquely determined, and* ξ *and* σ *are uniquely determined as well if* β *is single-valued. Finally, if the conditions*

$$f \in H^1(0,T;H), \quad \vartheta_0 \in W, \quad \varphi_0 \in W \quad \textit{and} \quad \beta^\circ(\varphi_0) \in H \tag{2.44}$$

are assumed in addition, then (2.39) *holds as well as*

$$\vartheta \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W). \tag{2.45}$$

In particular, both φ *and* ϑ *are bounded. Moreover, the estimates*

$$\|\varphi - \varphi^*\|_\infty \leq \rho\, C_{str}\, C_\Omega |\Omega|^{7/6} + C_6 \tag{2.46}$$

$$\|\vartheta\|_\infty \leq \rho\, C_{str}\, C_\Omega |\Omega|^{7/6} + C_7 \tag{2.47}$$

hold true with a structural constant C_{str} *depending only on the physical parameters* ℓ, κ, ν *and* γ*, the constant* C_Ω *given by* (2.7) *and some constants* C_6 *and* C_7 *depending on the structure of the systems,* Ω*,* T *and on the norms of the data involved.*

Remark 2.4. The above results are quite general. In particular, both potentials (1.3) and (1.4) are certainly allowed and the multi-valued potential (1.5) has to be excluded just in the parts of Theorems 2.2 and 2.3 regarding uniqueness for the pair (ξ,σ), which might be not uniquely determined, in general. Concerning the constant C_{str} of Theorem 2.3, we will prove that we can take

$$C_{str} = 2\max\Bigl\{\frac{6^{1/2}}{\nu}\,,\ \frac{\ell}{\kappa^{1/2}\nu^{1/2}} + \frac{4\ell}{\kappa}\Bigr\}. \tag{2.48}$$

However, no sharpness is guaranteed at all.

For each of the first two problems, the existence of the desired sliding mode is ensured for ρ large enough. For every $T>0$ we have indeed

Theorem 2.5. *Assume* (2.1)–(2.3), (2.16), (2.35), (2.38) *and* $f \in L^\infty(0,T;H)$. *Then, for some* $\rho^*>0$ *and for every* $\rho>\rho^*$, *there exist a solution* $(\vartheta,\varphi,\xi,\sigma)$ *to problem* (2.19)–(2.23) *and a time* $T^* \in [0,T)$ *such that*

$$\vartheta(t) + \alpha\varphi(t) = \eta^* \quad a.e.\ in\ \Omega \quad for\ every\ t\in[T^*,T]. \tag{2.49}$$

Theorem 2.6. *Assume* (2.1)–(2.3), (2.16) *and* (2.41). *Then, for some* $\rho^*>0$ *and for every* $\rho>\rho^*$, *there exist a solution* $(\vartheta,\varphi,\xi,\sigma)$ *to problem* (2.24)–(2.28) *and a time* $T^*\in[0,T)$ *such that*

$$\varphi(t) = \varphi^* \quad a.e.\ in\ \Omega \quad for\ every\ t\in[T^*,T]. \tag{2.50}$$

Remark 2.7. In the proof we give in Section 4, we compute possible values of ρ^* and T^* that fit the conclusions of our results. For Problems (A) and (B), we can take respectively

$$\rho^* := C_A^2 + 2C_A + \frac{2}{T}\,\|\vartheta_0+\alpha\varphi_0-\eta^*\|_H \quad \text{and} \quad T^* := \frac{2\|\vartheta_0+\alpha\varphi_0-\eta^*\|_H}{\rho - C_A^2 - 2C_A}$$
$$\rho^* := 2C_B + \frac{2}{T}\,\|\varphi_0-\varphi^*\|_H \quad \text{and} \quad T^* := \frac{2\|\varphi_0-\varphi^*\|_H}{\rho-2C_B}$$

where the constants C_A and C_B are constructed in the proofs of Theorems 2.1 and 2.2 in order that

$$\|f - (\ell-\alpha)\partial_t\varphi - \kappa\alpha\Delta\varphi - \kappa\Delta\eta^*\|_{L^\infty(0,T;H)} \le C_A\bigl(\rho^{1/2}+1\bigr)$$
$$\|\gamma\vartheta + \nu\Delta\varphi^* - \beta^\circ(\varphi^*) - \pi(\varphi)\|_{L^\infty(0,T;H)} \le C_B\,.$$

More precisely, we refer to (4.6)–(4.8) and (4.11)–(4.13) and we notice that our starting point in those proofs is the validity of the analogous estimates for the solutions to the approximating problems obtained by replacing the monotone operators by their Yosida regularizations. It follows that the above values of ρ^* and T^* depend continuously on the potentials and on the physical parameters of the systems. We also observe that the time T^* is roughly proportional to $1/\rho$ in both cases, whence it tends to zero as ρ tends to infinity, i.e., the sliding mode can be forced to start as soon as one desires by prescribing a sufficiently big factor ρ in front of the feedback control.

Remark 2.8. The minimal value of T^* of the first statement (if it is positive) also satisfies the following property: the function $t\mapsto \|\vartheta(t)+\alpha\varphi(t)-\eta^*\|_H$ is strictly decreasing on $[0,T^*]$. A similar remark holds for the function $t\mapsto\|\varphi(t)-\varphi^*\|_H$ in the second statement (and in the next one, at least under some reinforcement of the assumptions, as shown in Remark 4.2). In each case, the dynamics of the system is simpler after the time T^*, since one of the unknowns can be eliminated by using the sliding mode condition. For instance, in the second situation, the evolution of ϑ after T^* is ruled just by the heat equation.

The situation for Problem (C) is different, since we can ensure the existence of the desired sliding mode for ρ large enough only if further conditions are fulfilled. Namely, we need a restriction involving the structure of the system and the domain Ω (that is why we have written the statement of Theorem 2.3 in that form). Our result only involves the component φ of the solution, and we recall that φ is uniquely determined.

Theorem 2.9. *Assume* (2.1)–(2.3), (2.16), (2.41), (2.44) *and*

$$\Delta\varphi^* \in L^\infty(\Omega) \quad \textit{and} \quad \beta^\circ(\varphi^*) \in L^\infty(\Omega). \tag{2.51}$$

Let C_{str} and C_Ω be the constants appearing in (2.47) *and in* (2.7)*, respectively, and assume that*

$$\gamma\, C_{str} C_\Omega |\Omega|^{7/6} < 1. \tag{2.52}$$

Then, for some $\rho^* > 0$ and for every $\rho > \rho^*$, the following is true: if $(\vartheta, \varphi, \xi, \sigma)$ is a solution to problem (2.30)–(2.34)*, there exists a time $T^* \in [0, T)$ such that*

$$\varphi(t) = \varphi^* \quad \textit{a.e. in } \Omega \quad \textit{for every } t \in [T^*, T]. \tag{2.53}$$

Remark 2.10. Assume that the constants C_{str}, C_Ω and C_7 realize the inequalities (2.47) and (2.52) (i.e., in contrast with the situation of Remark 2.7, just such inequalities are required as a starting point). Then, as shown in the proof we perform in the last section, possible values of ρ^* and T^* that fit the conclusion of the above theorem are given by (here L is the Lipschitz constant of π)

$$\rho^* := \frac{\gamma C_7 + \nu\|\Delta\varphi^*\|_\infty + \|\xi^*\|_\infty + M_\pi^* + M_0/T}{1 - \gamma\, C_{str} C_\Omega |\Omega|^{7/6}} \quad \text{and} \quad T^* := \frac{M_0}{\rho - A(\rho)}$$
$$\text{where} \quad M_\pi^* := L(M_0 + \|\varphi^*\|_\infty) + |\pi(0)|\,, \quad M_0 := \|\varphi_0 - \varphi^*\|_\infty \quad \text{and}$$
$$A(\rho) := \gamma\big(C_{str} C_\Omega |\Omega|^{7/6}\rho + C_7\big) + \nu\|\Delta\varphi^*\|_\infty + \|\xi^*\|_\infty + M_\pi^*\,.$$

In particular, the last two sentences of Remark 2.7 also apply to the present case.

Remark 2.11. In order to understand the meaning of (2.52), let us assume that the structure of the system is chosen, so that the physical constants are fixed, and let us think of a class of open sets having the same shape. Precisely, we fix an open set Ω_0 of measure 1 and assume that $\Omega = x_0 + \lambda R\,\Omega_0$ for some $x_0 \in \mathbb{R}^3$, $\lambda > 0$ and some rotation $R \in SO(3)$. Then $\lambda = |\Omega|^{1/3}$ and one easily checks that our definition (2.6) of $\|\cdot\|_W$ yields $C_\Omega = C_{\Omega_0}\,|\Omega|^{-1/2}$, since the H-norms of v and of Δv are properly balanced in the norm $\|v\|_W$ under a rescaling of a function v. Then, the smallness condition (2.52) means that $|\Omega|$ is small enough. Indeed, the left-hand side of (2.52) becomes $\gamma C_{str} C_{\Omega_0} |\Omega|^{2/3}$ in the chosen class of domains.

In performing our a priori estimates in the remainder of the paper, we often account for the Hölder inequality and the elementary inequalities (for arbitrary $a, b \geq 0$)

$$(a+b)^{1/2} \leq a^{1/2} + b^{1/2}, \quad (a+b)^2 \leq 2a^2 + 2b^2 \quad \text{and} \quad ab \leq \delta a^2 + \frac{1}{4\delta}\, b^2 \tag{2.54}$$

where $\delta > 0$ in the latter (Young's inequality). Moreover, we repeatedly use the notation

$$Q_t := (0,t) \times \Omega\,. \tag{2.55}$$

For simplicity, we usually omit dx, ds, etc. in integrals. More precisely, we explicitly write, e.g., ds only if the variable s actually appears in the function under the integral sign. Finally, while a particular care is taken in computing some constants, we follow a general rule to denote less important ones, in order to avoid boring calculations. The small-case symbol c stands for different constants independent of ρ but depending on Ω, the final time T, the shape of the nonlinearities and on the constants and the norms of the functions involved in the assumptions of our statements. The dependence on ρ will be always written explicitly, indeed. Hence, the meaning of c might change from line to line and even in the same chain of equalities or inequalities. On the contrary, we mark precise constants which we can refer to by using different symbols, e.g., capital letters, mainly with indices, like in (2.7).

3 Proof of the well-posedness results

This section is devoted to the proof of Theorems 2.1–2.3. However, as far as existence is concerned, we confine ourselves to derive the formal a priori estimates that lead to the desired regularity and just sketch how a completely rigorous proof could be performed.

3.1 Proof of Theorem 2.1

We start with problem (2.19)–(2.23) and transform it into an equivalent system in new unknown functions. In order to argue in terms of the variable which the operator Sign applies to, we set

$$\eta := \vartheta + \alpha\varphi - \eta^* \tag{3.1}$$

then, η has to satisfy the analog of (2.17) and the new problem is the following

$$\partial_t\bigl(\eta + (\ell - \alpha)\varphi\bigr) - \kappa\Delta\eta + \kappa\alpha\Delta\varphi = f + \kappa\Delta\eta^* - \rho\sigma \quad \text{a.e. in } Q \tag{3.2}$$
$$\partial_t\varphi - \nu\Delta\varphi + \xi + \pi(\varphi) = \gamma(\eta - \alpha\varphi + \eta^*) \quad \text{a.e. in } Q \tag{3.3}$$
$$\xi \in \beta(\varphi) \quad \text{a.e. in } Q \tag{3.4}$$
$$\sigma(t) \in \mathrm{Sign}(\eta(t)) \quad \text{for a.a. } t \in (0,T) \tag{3.5}$$
$$\eta(0) = \vartheta_0 + \alpha\varphi_0 - \eta^* \quad \text{and} \quad \varphi(0) = \varphi_0\,. \tag{3.6}$$

First a priori estimate. We multiply (3.2) and (3.3) by η and $\partial_t\varphi$, respectively, sum up and integrate over Q_t with an arbitrary $t \in (0,T]$. Then, we add $\nu\int_{Q_t}\varphi\partial_t\varphi =$

$(\nu/2)\int_\Omega(|\varphi(t)|^2-|\varphi_0|^2)$ to both sides. With the help of (2.16) and (2.8), we infer that

$$\begin{aligned}&\frac{1}{2}\int_\Omega|\eta(t)|^2+\kappa\int_{Q_t}|\nabla\eta|^2-\kappa\alpha\int_{Q_t}\nabla\varphi\cdot\nabla\eta+\rho\int_0^t\|\eta(s)\|_H\,ds\\&\quad+\int_{Q_t}|\partial_t\varphi|^2+\frac{\nu}{2}\int_\Omega|\nabla\varphi(t)|^2+\int_\Omega\widehat{\beta}(\varphi(t))+\frac{\nu}{2}\int_\Omega|\varphi(t)|^2\\&\leq c-(\ell-\alpha)\int_{Q_t}\partial_t\varphi\,\eta+\int_{Q_t}\bigl(f+\kappa\Delta\eta^*\bigr)\eta\\&\quad-\int_{Q_t}\pi(\varphi)\,\partial_t\varphi+\gamma\int_{Q_t}\bigl(\eta-\alpha\varphi+\eta^*\bigr)\partial_t\varphi+\nu\int_{Q_t}\varphi\partial_t\varphi.\end{aligned}$$

Now, it is straightforward to use the linear growth of π that follows from Lipschitz continuity, the Young and Hölder inequalities, (2.16), (2.35), and the Gronwall lemma to deduce that

$$\|\eta\|_{L^\infty(0,T;H)\cap L^2(0,T;V)}+\|\varphi\|_{H^1(0,T;H)\cap L^\infty(0,T;V)}+\|\widehat{\beta}(\varphi)\|_{L^\infty(0,T;L^1(\Omega))}\leq c\,. \tag{3.7}$$

Second a priori estimate. We write (3.3) as

$$-\nu\Delta\varphi(t)+\xi(t)=g_1(t)\quad\text{and}\quad\xi(t)\in\beta(\varphi(t))\quad\text{for a.a. }t\in(0,T)$$

with an obvious meaning of g_1 and treat t as a parameter. We formally multiply by $\Delta\varphi(t)$ (the correct proof deals with the regularized problem) and find $\|\Delta\varphi(t)\|_H\leq\|g_1(t)\|_H$ for a.a. $t\in(0,T)$. Then, we use (3.7), (2.3), (2.35) (which imply $\|g_1\|_{L^2(0,T;H)}\leq c$), elliptic regularity and a comparison in the above equation, in order to conclude that

$$\|\varphi\|_{L^2(0,T;W)}+\|\xi\|_{L^2(0,T;H)}\leq c\,. \tag{3.8}$$

Third a priori estimate. We write (3.2) as

$$\partial_t\eta-\kappa\Delta\eta+\rho\sigma=g_2\quad\text{with}\quad\|g_2\|_{L^2(0,T;H)}\leq c \tag{3.9}$$

where we used (3.7)–(3.8), (2.16) and (2.35) once more. Then, we multiply by $\partial_t\eta$ and integrate over Q_t. Thanks to the chain rule property (stated, e.g., in [3, Lemme 3.3, p. 73]) and to the fact that $\eta(0)\in V$, we obtain

$$\int_{Q_t}|\partial_t\eta|^2+\frac{\kappa}{2}\int_\Omega|\nabla\eta(t)|^2+\rho\|\eta(t)\|_H=c(1+\rho)+\int_{Q_t}g_2\,\partial_t\eta$$

whence immediately

$$\|\partial_t\eta\|_{L^2(0,T;H)}+\|\eta\|_{L^\infty(0,T;V)}\leq c\bigl(\rho^{1/2}+1\bigr). \tag{3.10}$$

Fourth a priori estimate. We behave as we did for (3.8). From (3.9) we have

$$-\kappa\Delta\eta(t)+\rho\sigma(t)=g_3(t):=g_2(t)-\partial_t\eta(t)\quad\text{for a.a. }t\in(0,T).$$

Then, we formally multiply by $-\Delta\eta(t)$ and notice that $\nabla\sigma(t)\cdot\nabla\eta(t)\geq0$ a.e. in Ω (at least formally; the inequality we need if Sign were replaced by $\mathrm{Sign}_\varepsilon$ would immediately follow from (2.13)). Hence, we get $\kappa^{1/2}\|\Delta\eta(t)\|_H \le \|g_3(t)\|_H$ for a.a. $t \in (0,T)$. By owing to (3.9), (3.10) and elliptic regularity, we deduce that

$$\|\eta\|_{L^2(0,T;W)} \le c\bigl(\rho^{1/2}+1\bigr). \tag{3.11}$$

Consequence. Estimates (3.7)–(3.11) and assumption (2.35) imply for $\vartheta = \eta - \alpha\varphi + \eta^*$

$$\|\vartheta\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le c \quad\text{and}\quad \|\vartheta\|_{H^1(0,T;H)\cap L^2(0,T;W)} \le c\bigl(\rho^{1/2}+1\bigr). \tag{3.12}$$

Existence for Problem (A). The above a priori estimates are rigorous for the solution to the approximating problem obtained by replacing β and Sign by the corresponding Yosida regularizations. Namely, one writes

$$\xi = \beta_\varepsilon(\varphi) \quad \text{a.e. in } Q \quad\text{and}\quad \sigma(t) = \mathrm{Sign}_\varepsilon(\eta(t)) \quad \text{for a.a. } t\in(0,T) \tag{3.13}$$

in place of (3.4)–(3.5). The approximating problem is more regular and has a solution $(\eta_\varepsilon, \varphi_\varepsilon, \xi_\varepsilon, \sigma_\varepsilon)$. To see that, one can rewrite the approximating problem by eliminating the time derivative $\partial_t\varphi$ in (3.2) on accout of (3.3). One obtains the Cauchy problem for a system of the form

$$\partial_t(\eta,\varphi) + \mathcal{A}(\eta,\varphi) + \mathcal{B}_\varepsilon(\eta,\varphi) = \mathcal{F}$$

where $\mathcal{A}$ is an unbounded operator in $\mathcal{H} := H \times H$, $\mathcal{B}_\varepsilon : \mathcal{H} \to \mathcal{H}$ is a Lipschitz continuous perturbation and $\mathcal{F}$ is a source term. Namely, $\mathcal{A}$ acts as follows

$$\begin{aligned}&\mathcal{A} : (\eta,\varphi) \mapsto \bigl(-\kappa\Delta\eta + \lambda\Delta\varphi, -\nu\Delta\varphi\bigr) \quad \text{for } (\eta,\varphi)\in D(\mathcal{A}) := W\times W\\ &\text{where} \quad \lambda := \kappa\alpha + (\ell-\alpha)\nu.\end{aligned}$$

Now, let us introduce the following inner product in $\mathcal{H}$

$$\bigl((\eta,\varphi),(\tilde\eta,\tilde\varphi)\bigr)_{\mathcal{H}} := \int_\Omega \eta\tilde\eta + \Bigl(\frac{\lambda^2}{\kappa\nu}+1\Bigr)\int_\Omega \varphi\tilde\varphi.$$

Then, we have for $(\eta,\varphi)\in D(\mathcal{A})$

$$\begin{aligned}&\bigl(\mathcal{A}(\eta,\varphi),(\eta,\varphi)\bigr)_{\mathcal{H}} = \int_\Omega \Bigl(\kappa|\nabla\eta|^2 - \lambda\nabla\eta\cdot\nabla\varphi + \frac{\lambda^2}{\kappa}|\nabla\varphi|^2\Bigr) + \nu\int_\Omega |\nabla\varphi|^2\\ &\ge \frac{\kappa}{2}\int_\Omega|\nabla\eta|^2 + \frac{\lambda^2}{2\kappa}\int_\Omega|\nabla\varphi|^2 + \nu\int_\Omega|\nabla\varphi|^2 \ge \frac{\kappa}{2}\int_\Omega|\nabla\eta|^2 + \nu\int_\Omega|\nabla\varphi|^2.\end{aligned}$$

This shows that $\mathcal{A}$ is monotone in $\mathcal{H}$ with respect to that inner product. Then, maximal monotonicy follows since the range of $\mathcal{A}+\mathrm{Id}_{\mathcal{H}}$ is the whole of $\mathcal{H}$ due to the Lax-Milgram theorem and elliptic regularity. Therefore, the approximating problem has a solution (see, e.g., [31, Cor. 4.1 p. 181]). So, by starting from the analogs of the above formal a priori estimates (that is, from the rigorous ones, for which properties (2.12)–(2.14) have to be used) and owing to standard weak, weakstar and strong compactness results (see, e.g., [32, Sect. 8, Cor. 4]), we have for a subsequence at least

$$\begin{aligned}&\eta_\varepsilon \to \eta \quad \text{weakly star in } H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)\\ &\qquad\quad \text{and strongly in } C^0([0,T];H) &(3.14)\\ &\varphi_\varepsilon \to \varphi \quad \text{weakly star in } H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)\\ &\qquad\quad \text{and strongly in } C^0([0,T];H) &(3.15)\\ &\xi_\varepsilon \to \xi \quad \text{weakly in } L^2(0,T;H) &(3.16)\\ &\sigma_\varepsilon \to \sigma \quad \text{weakly star in } L^\infty(0,T;H). &(3.17)\end{aligned}$$

We stress that $\xi_\varepsilon := \beta_\varepsilon(\varphi_\varepsilon)$ and $\sigma_\varepsilon := \mathrm{Sign}_\varepsilon(\eta_\varepsilon)$, i.e., the same as in (3.13), where the subscripts ε were omitted for convenience. Here, ξ and σ have the meaning given by (3.16)–(3.17). Clearly, the limits φ, ξ and σ and the function ϑ computed from (3.1) satisfy the regularity requirements and the estimates of the statement (see also (3.12)). Moreover, it follows that $\pi(\varphi_\varepsilon)$ converges to $\pi(\varphi)$ strongly in $L^2(Q)$ and that ξ and σ satisfy (3.4)–(3.5) (because β and Sign induce maximal monotone operators on $L^2(Q)$ and $L^2(0,T;H)$, respectively, and then they are weakly-strongly closed; see, e.g., [2, Cor. 2.4, p. 41]). Hence, $(\eta,\varphi,\xi,\sigma)$ solves the original problem (3.2)–(3.6).

Uniqueness for Problem (A). We assume $\alpha=\ell$ and show that the solution is unique. Let $(\eta_i,\varphi_i,\xi_i,\sigma_i)$, $i=1,2$, be two solutions. We write equations (3.2)–(3.3) for both of them and take the differences. If we set $\eta:=\eta_1-\eta_2$ and analogously define φ, ξ and σ, we obtain

$$\partial_t\eta-\kappa\Delta\eta+\kappa\ell\Delta\varphi+\rho\sigma=0 \tag{3.18}$$

$$\partial_t\varphi-\nu\Delta\varphi+\xi=\gamma(\eta-\ell\varphi)+\pi(\varphi_2)-\pi(\varphi_1). \tag{3.19}$$

Now, we multiply these equations by η and $(\kappa\ell^2/\nu)\varphi$, respectively, sum up and integrate over Q_t. As π is Lipschitz continuous, we have

$$\frac12\int_\Omega|\eta(t)|^2+\frac{\kappa\ell^2}{2\nu}\int_\Omega|\varphi(t)|^2+\kappa\int_{Q_t}\bigl(|\nabla\eta|^2-\ell\nabla\varphi\cdot\nabla\eta+\ell^2|\nabla\varphi|^2\bigr)$$
$$+\rho\int_0^t\bigl(\sigma(s),\eta(s)\bigr)_H\,ds+\frac{\kappa\ell^2}{\nu}\int_{Q_t}\xi\varphi\le c\int_{Q_t}\bigl(|\eta|^2+|\varphi|^2\bigr).$$

The last two terms on the left-hand side are nonnegative by monotonicity and the integral involving the gradients is estimated from below this way

$$\int_{Q_t}\bigl(|\nabla\eta|^2-\ell\,\nabla\varphi\cdot\nabla\eta+\ell^2|\nabla\varphi|^2\bigr)\ge\frac12\int_{Q_t}\bigl(|\nabla\eta|^2+\ell^2|\nabla\varphi|^2\bigr). \tag{3.20}$$

At this point, we combine and apply the Gronwall lemma. We conclude that $\eta=0$ and $\varphi=0$, i.e., $\eta_1=\eta_2$ and $\varphi_1=\varphi_2$. By comparison in (3.2) and (3.3) written for both solutions, we deduce that $\sigma_1=\sigma_2$ and $\xi_1=\xi_2$, respectively.

Further regularity. We assume (2.38) and prove (2.39). To this end, it suffices to perform the estimate corresponding to (2.39) on the component φ_ε of the solution to the approximating problem sketched above, uniformly with respect to ε. This can be done by a heavy calculation involving difference quotients. Therefore, we confine ourselves to derive a formal a priori estimate. We write equations (3.3)–(3.4) by replacing β by its Yosida regularization β_ε in the latter, and formally differentiate with respect to time. By writing φ instead of φ_ε for simplicity, we have (see (3.7), (3.10), and (2.3))

$$\partial_t^2\varphi-\nu\Delta\partial_t\varphi+\beta_\varepsilon'(\varphi)\partial_t\varphi=g_3\quad\text{with}\quad\|g_3\|_{L^2(0,T;H)}\le c\bigl(\rho^{1/2}+1\bigr). \tag{3.21}$$

Now, we multiply by $\partial_t\varphi$ and integrate over Q_t. We obtain

$$\frac12\int_\Omega|\partial_t\varphi(t)|^2+\nu\int_{Q_t}|\nabla\partial_t\varphi|^2+\int_{Q_t}\beta_\varepsilon'(\varphi)\,|\partial_t\varphi|^2=\int_{Q_t}g_3\,\partial_t\varphi+\frac12\int_\Omega|\partial_t\varphi(0)|^2.$$

As β_ε' is nonnegative by monotonicity, the only term that needs some treatment is the last one on the right-hand side. We formally have from (3.3), the modified (3.4) and the initial conditions

$$\partial_t\varphi(0)=\nu\Delta\varphi_0-\beta_\varepsilon(\varphi_0)-\pi(\varphi_0)+\gamma\vartheta_0\,. \tag{3.22}$$

On the other hand, (2.15) implies that $\|\beta_\varepsilon(\varphi_0)\|_H \le \|\beta^\circ(\varphi_0)\|_H$. Therefore, on account of (2.38), $\|\partial_t\varphi(0)\|_H$ remains bounded and the estimate

$$\|\partial_t\varphi\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le c\bigl(\rho^{1/2}+1\bigr)$$

follows uniformly with respect to ε. Thus, the same estimate holds for the limiting φ. At this point, by comparison in (3.3), we get a bound for the sum $-\nu\Delta\varphi+\xi$ in $L^\infty(0,T;H)$ and the argument used to derive (3.8) (where t is just a parameter) completes the regularity (2.39) and the estimate (2.40). In order to conclude the proof of Theorem 2.1, we have to prove that the component ϑ of any solution satisfying all the regularity requirements of the statement is bounded whenever we assume that $\vartheta_0 \in L^\infty(\Omega)$ and $f \in L^\infty(0,T;H)$, in addition. To this end, it suffices to write (2.19) in the form

$$\partial_t\vartheta - \kappa\Delta\vartheta = f - \rho\sigma - \ell\partial_t\varphi$$

and observe that the right-hand side of this equation belongs to $L^\infty(0,T;H)$. Then, we can argue, e.g., as in [21, Thm. 7.1, p. 181] with $r=\infty$ and $q=2$, where the case of the Dirichlet boundary conditions is treated in detail: by a careful check, the reader can make the necessary modifications to adapt the procedure to the case of the homogeneous Neumann boundary conditions. □

3.2 Proof of Theorem 2.2

As the argument is quite similar to the previous one, we proceed quickly. Also in this case, we introduce new unknowns and transform the problem. Let us recall the assumption (2.41) on φ^* and set

$$\eta := \vartheta + \ell\varphi, \quad \chi := \varphi - \varphi^* \quad \text{and} \quad \xi^* := \beta^\circ(\varphi^*). \tag{3.23}$$

Then, η and χ have to satisfy the analog of (2.17) and the new problem is the following

$$\partial_t\eta - \kappa\Delta\eta + \kappa\ell\Delta\chi = f - \kappa\ell\Delta\varphi^* \quad \text{a.e. in } Q \tag{3.24}$$

$$\partial_t\chi - \nu\Delta\chi + \xi - \xi^* + \pi(\chi+\varphi^*)$$
$$= \gamma(\eta - \ell\chi - \ell\varphi^*) + \nu\Delta\varphi^* - \xi^* - \rho\sigma \quad \text{a.e. in } Q \tag{3.25}$$

$$\xi \in \beta(\chi+\varphi^*) \quad \text{a.e. in } Q \tag{3.26}$$

$$\sigma(t) \in \mathrm{Sign}(\chi(t)) \quad \text{for a.a. } t\in(0,T) \tag{3.27}$$

$$\eta(0) = \vartheta_0 + \ell\varphi_0 \quad \text{and} \quad \chi(0) = \varphi_0 - \varphi^*. \tag{3.28}$$

First a priori estimate. We multiply (3.24) by η and (3.25) by $(\kappa\ell^2/\nu)\chi$, integrate over Q_t and sum up. Then, we rearrange a little and use the Lipschitz continuity of π and the Young inequality. Using also (2.41), we obtain

$$\frac12\int_\Omega|\eta(t)|^2 + \frac{\kappa\ell^2}{2\nu}\int_\Omega|\chi(t)|^2 + \kappa\int_{Q_t}\Bigl(|\nabla\eta|^2 - \ell\,\nabla\eta\cdot\nabla\chi + \ell^2|\nabla\chi|^2\Bigr)$$
$$+\frac{\kappa\ell^2}{\nu}\int_{Q_t}(\xi-\xi^*)\chi + \frac{\kappa\ell^2\rho}{\nu}\int_0^t\bigl(\sigma(s),\chi(s)\bigr)_H\,ds$$
$$\le c\int_{Q_t}\bigl(|\eta|^2+|\chi|^2+1\bigr).$$

Now, we observe that (3.20) can be applied and that the last two terms on the above left-hand side are nonnegative by monotonicity. Thus, by applying the Gronwall lemma, we conclude that

$$\|\eta\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\chi\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le c\,. \tag{3.29}$$

Second a priori estimate. We observe that (3.25) looks like

$$\partial_t\chi - \nu\Delta\chi + \xi + \rho\sigma = g_1 \quad \text{with} \quad \|g_1\|_{L^2(0,T;H)} \le c\,.$$

Therefore, multiplication by $\partial_t\chi$ and integration over Q_t yield

$$\begin{aligned}&\int_{Q_t} |\partial_t\chi|^2 + \frac{\nu}{2}\int_\Omega |\nabla\chi(t)|^2 + \int_\Omega \widehat{\beta}(\chi(t)+\varphi^*) + \rho\|\chi(t)\|_H \\ &\le c(1+\rho) + \int_{Q_t} g_1\partial_t\chi \le c\,(1+\rho) + \frac{1}{2}\int_{Q_t}|\partial_t\chi|^2.\end{aligned}$$

We immediately deduce that

$$\|\chi\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le c\big(\rho^{1/2}+1\big). \tag{3.30}$$

Further a priori estimates. We want to obtain

$$\|\eta\|_{H^1(0,T;H)\cap L^2(0,T;W)} + \|\chi\|_{H^1(0,T;H)\cap L^2(0,T;W)} + \|\xi+\rho\sigma\|_{L^2(0,T;H)} \le c\big(\rho^{1/2}+1\big). \tag{3.31}$$

To this end, we argue as we did for (3.8)–(3.11) with just one modification of our argument concerning the pointwise estimate of $\|\Delta\chi(t)\|_H$. We still multiply by $\Delta\chi(t)$. However, since φ^* is not supposed to be a constant, this requires some care and cannot be as simple as for (3.8). In order to be more precise on this delicate point, we consider the solution to the ε-problem obtained by replacing β and Sign with their Yosida regularizations β_ε and $\mathrm{Sign}_\varepsilon$. For simplicity, we avoid stressing the time t for a while. We write the regularized (3.25) in the form

$$\begin{aligned}&-\Delta\chi + \frac{1}{\nu}\beta_\varepsilon(\chi+\varphi^*) + \frac{\rho}{\nu}\,\mathrm{Sign}_\varepsilon\,\chi = g_2 \quad \text{with} \\ &g_2 := \frac{1}{\nu}\big(-\partial_t\chi - \pi(\chi+\varphi^*) + \gamma(\eta - \ell\chi - \ell\varphi^*) + \nu\Delta\varphi^*\big)\end{aligned}$$

and read $-\Delta\chi$ as $-\Delta(\chi+\varphi^*)+\Delta\varphi^*$ when multiplying the second term of the equation by $-\Delta\chi$. Owing to (2.13) (which also implies $\|\mathrm{Sign}_\varepsilon\,\chi\|_H \le 1$) and to the elementary inequalities (2.54), we obtain

$$\begin{aligned}&\|\Delta\chi\|_H^2 + \frac{1}{\nu}\int_\Omega \beta_\varepsilon'(\chi+\varphi^*)|\nabla(\chi+\varphi^*)|^2 + \frac{\rho}{\nu}\int_\Omega \frac{|\nabla\chi|^2}{\max\{\varepsilon,\|\chi\|_H\}} \\ &= -\int_\Omega g_2\Delta\chi - \frac{1}{\nu}\int_\Omega \beta_\varepsilon(\chi+\varphi^*)\Delta\varphi^* \\ &= -\int_\Omega g_2\Delta\chi + \int_\Omega\Big(-\Delta\chi + \frac{\rho}{\nu}\,\mathrm{Sign}_\varepsilon\,\chi - g_2\Big)\Delta\varphi^* \\ &= -\int_\Omega (g_2+\Delta\varphi^*)\Delta\chi + \int_\Omega\Big(-g_2 + \frac{\rho}{\nu}\,\mathrm{Sign}_\varepsilon\,\chi\Big)\Delta\varphi^*\end{aligned}$$

$$\le (\|g_2\|_H + \|\Delta\varphi^*\|_H)\|\Delta\chi\|_H + \|g_2\|_H\,\|\Delta\varphi^*\|_H + \frac{\rho}{\nu}\,\|\Delta\varphi^*\|_H$$
$$\le \frac{1}{2}\,\|\Delta\chi\|_H^2 + \frac{3}{2}\,\|g_2\|_H^2 + c\,(\rho+1)\,.$$

By recalling the meaning of g_2, we conclude that we have for a.a. $t \in (0,T)$

$$\|\Delta\chi(t)\|_H^2 \le \frac{3}{\nu^2}\,\|-\partial_t\chi(t) - \pi(\chi(t)+\varphi^*) + \gamma(\eta(t) - \ell\chi(t) - \ell\varphi^*) + \nu\Delta\varphi^*\|_H^2 + c\,(\rho+1)\,. \tag{3.32}$$

Thus, the right bound for $\Delta\chi$ in $L^2(0,T;H)$ follows from (3.29)–(3.30). Then, $\xi + \rho\sigma$ is estimated in $L^2(0,T;H)$ by comparison in (3.25) and the complete (3.31) can be achieved like in the previous proof, as said at the beginning.

Existence for Problem (B). One can proceed as for Problem (A). Indeed, we have proved quite similar estimates (notice that (3.31) also yields $\|\xi\|_{L^2(0,T;H)} \le c(\rho+1)$ since $\|\sigma\|_{L^\infty(0,T;H)} \le 1$ by the definition of Sign) which are completely rigorous when performed on the solution to the approximating problem obtained by replacing β and Sign by their Yosida regularizations. Moreover, the proof of the existence of a solution to the approximating problem is similar to the one performed for Problem (A).

Uniqueness for Problem (B). Let $(\eta_i, \varphi_i, \xi_i, \sigma_i)$, $i = 1, 2$, be two solutions and define η_i and χ_i according to (3.23). By proceeding in the same way as we did for Problem (A), we easily obtain $\eta_1 = \eta_2$ and $\chi_1 = \chi_2$, whence $\vartheta_1 = \vartheta_2$ and $\varphi_1 = \varphi_2$, i.e., the first sentence of Theorem 2.2 about uniqueness. Now, assume β to be single-valued. Then, $\xi_1 = \xi_2$ since $\varphi_1 = \varphi_2$. Finally, by comparison in (2.25), we also deduce that $\sigma_1 = \sigma_2$. This concludes the proof of Theorem 2.2. □

3.3 Proof of Theorem 2.3

As we did for Problem (B), we introduce the new unknowns η and χ by means of (3.23) and deal with the following new problem:

$$\partial_t\eta - \kappa\Delta\eta + \kappa\ell\Delta\chi = f - \kappa\ell\Delta\varphi^* \quad \text{a.e. in } Q \tag{3.33}$$
$$\partial_t\chi - \nu\Delta\chi + \xi - \xi^* + \pi(\chi+\varphi^*) = \gamma(\eta - \ell\chi - \ell\varphi^*) + \nu\Delta\varphi^* - \xi^* - \rho\sigma \quad \text{a.e. in } Q \tag{3.34}$$
$$\xi \in \beta(\chi + \varphi^*) \quad \text{a.e. in } Q \tag{3.35}$$
$$\sigma \in \operatorname{sign}\chi \quad \text{a.e. in } Q \tag{3.36}$$
$$\eta(0) = \vartheta_0 + \ell\varphi_0 \quad \text{and} \quad \chi(0) = \varphi_0 - \varphi^*. \tag{3.37}$$

Existence and uniqueness for Problem (C). This problem differs from Problem (B) just in (3.36), where sign appears in place of the non-local operator Sign. Therefore, both existence and partial uniqueness can be obtained by the same argument.

It remains to prove the regularity part of Theorem 2.3 and the estimates. For the regularity of φ and ξ, one could combine the techniques used for Problems (A) and (B), while the regularity of ϑ is classical, on account of (2.44) and of the regularity of $\partial_t\varphi$ already proved. However, the forthcoming argument also shows the desired regularity. What needs much more care is the control of the constants entering (2.46)–(2.47). This forces us to perform a number of a priori estimates which we derive just formally, for brevity.

First a priori estimate. As we did for (3.29), we multiply (3.33) by η and (3.34) by $(\kappa\ell^2/\nu)\chi$, integrate over Q_t and sum up. Then, we owe to (3.20), the Lipschitz continuity of π, the Young inequality and the Gronwall lemma. We obtain

$$\|\eta\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\chi\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le c\,. \tag{3.38}$$

Second a priori estimate. We write (3.34) as

$$\partial_t\chi - \nu\Delta\chi + \xi + \rho\sigma = g_1 \quad \text{with} \quad \|g_1\|_{L^2(0,T;H)} \le c$$

and multiply it by $\partial_t\chi$. As for (3.30), we have

$$\|\chi\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le c\big(\rho^{1/2}+1\big). \tag{3.39}$$

Third a priori estimate. There holds

$$\begin{aligned} \|\Delta\chi(t)\|_H^2 \le \frac{3}{\nu^2}\,&\|-\partial_t\chi(t) - \pi(\chi(t)+\varphi^*) + \gamma(\eta(t)-\ell\chi(t)-\ell\varphi^*) + \nu\Delta\varphi^*\|_H^2 \\ &+ c\,(\rho+1) \end{aligned} \tag{3.40}$$

i.e., the same as (3.32). Inequality (3.40) can be proved with the same calculations that led to (3.32) with obvious changes in the proof (like $\|\mathrm{sign}_\varepsilon\,\chi\|_H \le |\Omega|^{1/2}$ in place of $\|\mathrm{Sign}_\varepsilon\,\chi\|_H \le 1$, whence just a different value of the final c). From this and the previous estimates, we deduce that

$$\|\Delta\chi\|_{L^2(0,T;H)} \le c\big(\rho^{1/2}+1\big). \tag{3.41}$$

Fourth a priori estimate. Now, we multiply (3.33) by $\partial_t\eta$, integrate over Q_t and get

$$\int_{Q_t}|\partial_t\eta|^2 + \frac{\kappa}{2}\int_\Omega|\nabla\eta(t)|^2 = \frac{\kappa}{2}\int_\Omega|\nabla\eta(0)|^2 + \int_{Q_t}(f-\kappa\ell\Delta\chi-\kappa\ell\Delta\varphi^*)\partial_t\eta.$$

From (2.16) and (3.40), we immediately infer that

$$\|\eta\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le c\big(\rho^{1/2}+1\big). \tag{3.42}$$

Fifth a priori estimate. We start from (3.34) and smooth the monotone nonlinearities by replacing them with their Yosida approximations. By differentiating with respect to time, we have

$$\partial_t^2\chi - \nu\Delta\partial_t\chi + \big\{\beta_\varepsilon'(\chi+\varphi^*) + \rho\,\mathrm{sign}_\varepsilon'(\chi)\big\}\partial_t\chi = g_3 \tag{3.43}$$

where $g_3 := \gamma(\partial_t\eta - \ell\partial_t\chi) - \pi'(\chi+\varphi^*)\partial_t\chi$, and we can read the initial value

$$\partial_t\chi(0) = \nu\Delta\varphi_0 - \beta_\varepsilon(\varphi_0) - \pi(\varphi_0) + \gamma\vartheta_0 - \rho\,\mathrm{sign}_\varepsilon(\varphi_0-\varphi^*). \tag{3.44}$$

Notice that

$$\|g_3\|_{L^2(0,T;H)} \le c\big(\rho^{1/2}+1\big) \quad \text{and} \quad \|\partial_t\chi(0)\|_H \le \rho\,|\Omega|^{1/2} + c \tag{3.45}$$

thanks to (2.38). Thus, by multiplying (3.43) by $\partial_t \chi$, integrating over Q_t, observing that β'_ε and $\mathrm{sign}'_\varepsilon$ are nonnegative, and using (3.39) for $\partial_t \chi$ and (3.45), we obtain

$$\frac{1}{2}\int_\Omega |\partial_t \chi(t)|^2 + \nu \int_{Q_t} |\nabla \partial_t \chi|^2 \leq \int_{Q_t} |g_3|\,|\partial_t \chi| + \frac{1}{2}\int_\Omega |\partial_t \chi(0)|^2 \leq \frac{\rho^2 |\Omega|}{2} + c\,\rho + c\,.$$

We deduce the following estimates for $\partial_t \chi$

$$\|\partial_t \chi\|^2_{L^\infty(0,T;H)} \leq \rho^2 |\Omega| + c\,\rho + c \quad \text{and} \quad \|\nabla \partial_t \chi\|^2_{L^2(0,T;H)} \leq \frac{\rho^2 |\Omega|}{2\nu} + c\,\rho + c\,. \tag{3.46}$$

Sixth a priori estimate. We use (3.38), (3.40) and (3.46). We deduce that

$$\|\Delta \chi\|^2_{L^\infty(0,T;H)} \leq \frac{6\rho^2\,|\Omega|}{\nu^2} + c\,\rho + c\,.$$

Now, we recall the definition (2.6) of $\|\cdot\|_W$. Thus, we also have

$$\|\chi\|^2_{L^\infty(0,T;W)} \leq \frac{6\rho^2\,|\Omega|^{7/3}}{\nu^2} + c\,\rho + c\,.$$

Finally, we apply (2.7). We conclude that $\chi \in L^\infty(Q)$ and that

$$\|\chi\|^2_\infty \leq \frac{6\rho^2\,C_\Omega^2\,|\Omega|^{7/3}}{\nu^2} + c\,\rho + c$$

whence also (by the first elementary inequality (2.54))

$$\|\chi\|_\infty \leq \frac{6^{1/2}\,\rho\,C_\Omega\,|\Omega|^{7/6}}{\nu} + c\bigl(\rho^{1/2} + 1\bigr) \leq 2\,\frac{6^{1/2}\,\rho\,C_\Omega\,|\Omega|^{7/6}}{\nu} + c\,. \tag{3.47}$$

Hence, if we choose the last value of c as C_6, we see that (2.46) holds with C_{str} as in (2.48).

Seventh a priori estimate. On account of the regularity of f in (2.44), we formally differentiate (3.33) with respect to time and test the resulting equation by $\partial_t \eta$. As $\partial_t \eta(0)$, which is recovered from (3.33), is bounded in H by a constant due to (2.44), we obtain by (3.42)

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\partial_t \eta(t)|^2 + \kappa \int_{Q_t} |\nabla \partial_t \eta|^2 = \frac{1}{2}\int_\Omega |\partial_t \eta(0)|^2 + \kappa\ell \int_{Q_t} \nabla \partial_t \eta \cdot \nabla \partial_t \chi + \int_{Q_t} \partial_t f\,\partial_t \eta \\
&\leq c + \kappa \int_{Q_t} |\nabla \partial_t \eta|^2 + \frac{\kappa \ell^2}{4}\int_Q |\nabla \partial_t \chi|^2.
\end{aligned}$$

Owing to the second of (3.46), we infer that

$$\|\partial_t \eta\|^2_{L^\infty(0,T;H)} \leq \frac{\kappa \ell^2 \rho^2 |\Omega|}{4\nu} + c\,\rho + c\,. \tag{3.48}$$

Eighth a priori estimate. By recalling that $\vartheta = \eta - \ell\chi - \ell\varphi^*$ by (3.23), the first inequality in (3.46) and estimate (3.48) yield

$$\|\partial_t \vartheta\|_{L^\infty(0,T;H)} \leq \widehat{C}\,\rho\,|\Omega|^{1/2} + c\rho^{1/2} + c \quad \text{where} \quad \widehat{C} := \frac{\kappa^{1/2}\ell}{2\nu^{1/2}} + \ell\,. \tag{3.49}$$

Once such an estimate is obtained, we can recover a bound for $\Delta\vartheta$ from (2.30) and repeat for ϑ what we have done for χ. Here is the quick sequence of deductions. First, we have

$$\|\Delta\vartheta\|_{L^\infty(0,T;H)} \leq \frac{1}{\kappa}\bigl(\|f\|_{L^\infty(0,T;H)} + \|\partial_t\vartheta\|_{L^\infty(0,T;H)} + \ell\|\partial_t\chi\|_{L^\infty(0,T;H)}\bigr)$$
$$\leq \frac{\widehat{C}+\ell}{\kappa}\,\rho|\Omega|^{1/2} + c\rho^{1/2} + c$$

and we derive

$$\|\vartheta\|^2_{L^\infty(0,T;W)} \leq \|\vartheta\|^2_{L^\infty(0,T;H)} + |\Omega|^{4/3}\|\Delta\vartheta\|^2_{L^\infty(0,T;H)}$$
$$\leq 4\,\frac{(\widehat{C}+\ell)^2}{\kappa^2}\,\rho^2|\Omega|^{7/3} + c(\rho+1)\,.$$

Therefore

$$\|\vartheta\|_\infty \leq C_\Omega\, 2\,\frac{\widehat{C}+\ell}{\kappa}\,\rho|\Omega|^{7/6} + c\bigl(\rho^{1/2}+1\bigr) \leq C_\Omega\, 4\,\frac{\widehat{C}+\ell}{\kappa}\,\rho|\Omega|^{7/6} + c$$

so that (2.47) holds with the last value of c as C_7 and C_{str} as in (2.48) (recall the value of $\widehat{C}$ in (3.49)).

4 Existence of sliding modes

This section is devoted to the proof of Theorems 2.5, 2.6 and 2.9. The argument we use to prove the existence of sliding modes in the first two cases relies on the following lemma, which ensures the existence of an extinction time T^* for a real function.

Lemma 4.1. *Let $a_0, b_0, \psi_0, \rho \in \mathbb{R}$ be such that*

$$a_0,\ b_0,\ \psi_0 \geq 0 \quad \textit{and} \quad \rho > a_0^2 + 2b_0 + 2\,\frac{\psi_0}{T} \tag{4.1}$$

and let $\psi : [0,T] \to [0,+\infty)$ be an absolutely continuous function satisfying $\psi(0) = \psi_0$ and

$$\psi' + \rho \leq a_0\,\rho^{1/2} + b_0 \quad \textit{a.e. in the set} \;\; P := \{t \in (0,T) :\ \psi(t) > 0\}. \tag{4.2}$$

Then, the following conclusions hold true.

i) If $\psi_0 = 0$, then ψ vanishes identically.

ii) If $\psi_0 > 0$, there exists $T^* \in (0,T)$ satisfying $T^* \leq 2\psi_0/(\rho - a_0^2 - 2b_0)$ such that ψ is strictly decreasing in $(0,T^*)$ and ψ vanishes in $[T^*,T]$.

Proof. Assumption (4.2) and the Young inequality imply that

$$\psi' \leq -s_0 \quad \text{a.e. in } P, \quad \text{where} \quad s_0 := \frac{1}{2}\,\rho - \frac{1}{2}\,a_0^2 - b_0 \tag{4.3}$$

and we notice that (4.1) implies

$$s_0 > \frac{\psi_0}{T}\,. \tag{4.4}$$

In particular, $s_0 > 0$. Moreover, if $0 \le t_1 < t_2 \le T$ and $(t_1, t_2) \subseteq P$, then

$$\psi(t_1) = \psi(t_2) - \int_{t_1}^{t_2} \psi'(t)\, dt \ge \psi(t_2) + s_0(t_2 - t_1) \ge s_0(t_2 - t_1) > 0. \tag{4.5}$$

Now, we prove the lemma.

i) By contradiction, let P be non-empty. So, we can pick a connected component of it. This is an open interval (a, b) and we can apply (4.5) to obtain $\psi(a) > 0$. Thus, $a > 0$ since $\psi_0 = 0$, whence $\psi > 0$ also in $(a', a]$ for some $a' < a$. This contradicts the definition of connected component.

ii) As $\psi_0 > 0$, we can define the strictly positive number T^* by setting

$$T^* := \sup\{t \in (0, T) : \ \psi(s) > 0 \text{ for every } s \in (0, t)\}.$$

By (4.5) with $t_1 = 0$ and $t_2 = T^*$ and (4.4), we have $\psi_0 \ge s_0 T^*$, whence $T^* \le \psi_0/s_0 < T$, i.e., the first conditions of the statement. Furthermore, $\psi' \le -s_0 < 0$ in $(0, T^*)$, so that ψ is strictly decreasing in this interval. Finally, we have to show that ψ vanishes in $[T^*, T]$ and we argue by contradiction by assuming that $P \cap (T^*, T) \ne \emptyset$ and picking a connected component of this set. This is an open interval (a, b), with $T^* \le a < b \le T$, in principle. However, $a = T^*$ would contradict the definition of T^*, whence $a > T^*$. Therefore, by applying (4.5) with $t_1 = a$ and $t_2 = b$, we obtain $\psi(a) > 0$ and the definition of connected component is contradicted as in the previous case. $\square$

Proof of Theorem 2.5. Let $(\vartheta, \varphi, \xi, \sigma)$ be a solution to problem (2.19)–(2.23) given by (3.14)–(3.17). We show that this solution fulfills the requirements of the statement. First of all, we observe that estimates (2.36)–(2.37) and (2.40) hold for the approximating solution, by construction. Moreover, $f \in L^\infty(0, T; H)$ by assumption. Hence, we can write the modified (3.2) in the form

$$\partial_t \eta_\varepsilon - \kappa \Delta \eta_\varepsilon + \rho \sigma_\varepsilon = g_\varepsilon := f - (\ell - \alpha)\partial_t \varphi_\varepsilon - \kappa\alpha\Delta\varphi_\varepsilon + \kappa\Delta\eta^* \tag{4.6}$$

$$\|g_\varepsilon\|_{L^\infty(0,T;H)} \le C\big(\rho^{1/2} + 1\big) \tag{4.7}$$

where C depends only on the structure and the data involved in the statement. At this point, we set

$$\rho^* := C^2 + 2C + \frac{2}{T}\, \|\vartheta_0 + \alpha\varphi_0 - \eta^*\|_H \tag{4.8}$$

and assume $\rho > \rho^*$. We also set

$$\psi(t) := \|\eta(t)\|_H \quad \text{and} \quad \psi_\varepsilon(t) := \|\eta_\varepsilon(t)\|_H \quad \text{for } t \in [0, T]. \tag{4.9}$$

Now, by assuming $h \in (0, T)$ and $t \in (0, T - h)$, we multiply (4.6) by σ_ε and integrate over $(t, t + h) \times \Omega$. We obtain

$$\int_t^{t+h} \big(\partial_t \eta_\varepsilon(s), \sigma_\varepsilon(s)\big)_H\, ds + \kappa \int_t^{t+h} \int_\Omega \nabla\eta_\varepsilon \cdot \nabla\sigma_\varepsilon + \rho \int_t^{t+h} \|\sigma_\varepsilon(s)\|_H^2\, ds$$
$$= \int_t^{t+h} \big(g_\varepsilon(s), \sigma_\varepsilon(s)\big)_H\, ds. \tag{4.10}$$

As (2.11) and (2.12) imply that

$$\big(\partial_t \eta_\varepsilon(t), \sigma_\varepsilon(t)\big)_H = \frac{d}{dt} \int_0^{\psi_\varepsilon(t)} \min\{s/\varepsilon, 1\}\, ds \quad \text{for a.a. } t \in (0, T)$$

we have for the first term of (4.10)

$$\int_t^{t+h} \big(\partial_t \eta_\varepsilon(s), \sigma_\varepsilon(s)\big)_H \, ds = \int_{\psi_\varepsilon(t)}^{\psi_\varepsilon(t+h)} \min\{s/\varepsilon, 1\} \, ds.$$

The second integral in (4.10) is nonnegative. Indeed, (2.13) implies

$$\nabla \eta_\varepsilon(t) \cdot \nabla \sigma_\varepsilon(t) = \frac{|\nabla \eta_\varepsilon(t)|^2}{\max\{\varepsilon, \|\eta_\varepsilon(t)\|_H\}} \ge 0 \quad \text{a.e. in } \Omega, \ \text{ for a.a. } t \in (0,T).$$

As $\|\sigma_\varepsilon(s)\|_H \le 1$ for every s and (4.7) holds, we deduce from (4.10)

$$\int_{\psi_\varepsilon(t)}^{\psi_\varepsilon(t+h)} \min\{s/\varepsilon, 1\} \, ds + \rho \int_t^{t+h} \|\sigma_\varepsilon(s)\|_H^2 \, ds \le h \, C\big(\rho^{1/2} + 1\big).$$

At this point, we let ε tend to zero. As we are assuming that (3.14) and (3.17) hold at least for a subsequence, we infer that

$$\begin{aligned} &\psi(t+h) - \psi(t) + \rho \int_t^{t+h} \|\sigma(s)\|_H^2 \, ds \\ &\le \lim_{\varepsilon \searrow 0} \int_{\psi_\varepsilon(t)}^{\psi_\varepsilon(t+h)} \min\{s/\varepsilon, 1\} \, ds + \rho \liminf_{\varepsilon \searrow 0} \int_t^{t+h} \|\sigma_\varepsilon(s)\|_H^2 \, ds \le h \, C\big(\rho^{1/2} + 1\big) \end{aligned}$$

for every $h \in (0,T)$ and $t \in (0, T-h)$. This implies that

$$\psi'(t) + \rho \|\sigma(t)\|_H^2 \le C\big(\rho^{1/2} + 1\big) \quad \text{for a.a. } t \in (0,T).$$

As $\|\sigma(t)\|_H = 1$ if $\|\eta(t)\| > 0$ by (2.9), we can apply the lemma with $a_0 = b_0 = C$ and we observe that our condition $\rho > \rho^*$ completely fits the assumptions by (4.8). Thus, we find $T^* \in [0,T)$ such that $\eta(t) = 0$ for every $t \in [T^*, T]$, i.e., (2.49). □

Proof of Theorem 2.6. By arguing as in the previous proof, we pick a solution $(\vartheta, \varphi, \xi, \sigma)$ to problem (2.24)–(2.28) obtained as the limit of the solution $(\vartheta_\varepsilon, \varphi_\varepsilon, \xi_\varepsilon, \sigma_\varepsilon)$ of the corresponding approximating problem and show that all the requirements of the statement are fulfilled. We introduce the functions η and χ defined by (3.23) and the analogs η_ε and χ_ε, and owe to (2.42)–(2.43) for the approximating solution. Therefore, we can rewrite the equation approximating (3.25) in the form

$$\begin{aligned} &\partial_t \chi_\varepsilon - \nu \Delta \chi_\varepsilon + \beta_\varepsilon(\chi_\varepsilon + \varphi^*) - \beta_\varepsilon(\varphi^*) + \rho \sigma_\varepsilon \\ &= g_\varepsilon := \gamma(\eta_\varepsilon - \ell \chi_\varepsilon - \ell \varphi^*) + \nu \Delta \varphi^* - \beta_\varepsilon(\varphi^*) - \pi(\chi_\varepsilon + \varphi^*) \end{aligned} \tag{4.11}$$

with

$$\|g_\varepsilon\|_{L^\infty(0,T;H)} \le C \tag{4.12}$$

where C depends only on the structure and the data involved in the statement. At this point, we set

$$\rho^* := 2C + \frac{2}{T} \, \|\varphi_0 - \varphi^*\|_H \tag{4.13}$$

and assume $\rho > \rho^*$. We also set

$$\psi(t) := \|\chi(t)\|_H \quad \text{and} \quad \psi_\varepsilon(t) := \|\chi_\varepsilon(t)\|_H \quad \text{for } t \in [0,T]. \tag{4.14}$$

Now, we multiply (4.11) by σ_ε and integrate over $(t,t+h)\times\Omega$ as before. We obtain

$$\begin{aligned}&\int_t^{t+h}\bigl(\partial_t\chi_\varepsilon(s),\sigma_\varepsilon(s)\bigr)_H\,ds+\nu\int_t^{t+h}\int_\Omega\nabla\chi_\varepsilon\cdot\nabla\sigma_\varepsilon\\&\quad+\int_t^{t+h}\bigl(\beta_\varepsilon(\chi_\varepsilon(s)+\varphi^*)-\beta_\varepsilon(\varphi^*),\mathrm{Sign}_\varepsilon(\chi_\varepsilon(s))\bigr)_H\,ds+\rho\int_t^{t+h}\|\sigma_\varepsilon(s)\|_H^2\,ds\\&=\int_t^{t+h}\bigl(g_\varepsilon(s),\sigma_\varepsilon(s)\bigr)_H\,ds.\end{aligned}\tag{4.15}$$

The first two terms and the left-hand side can be dealt with as in the previous proof. The third integral on the left-hand side is nonnegative since the two factors of the product have the same sign. Therefore, by arguing as above and then letting ε tend to zero, we obtain

$$\psi'(t)+\rho\|\sigma(t)\|_H^2\le C.$$

As $\|\sigma(t)\|_H=1$ if $\|\chi(t)\|>0$ by (2.9), we can apply the lemma with $a_0=0$ and $b_0=C$ since $\rho>\rho^*$ (see (4.13)). Thus, we find $T^*\in[0,T)$ such that $\chi(t)=0$ for every $t\in[T^*,T]$. This condition coincides with (2.50). □

Proof of Theorem 2.9. For Problem (C) we use a different argument since we cannot apply Lemma 4.1. Our method relies on a comparison technique on $\chi:=\varphi-\varphi^*$, where $(\vartheta,\varphi,\xi,\sigma)$ is the solution we are dealing with, by introducing the solution w of an ordinary Cauchy problem with a well-chosen right-hand side. The function χ has the same regularity of φ and satisfies

$$\partial_t\chi-\nu\Delta\chi+\xi-\xi^*+\pi(\chi+\varphi^*)+\rho\,\sigma=\gamma\vartheta+\nu\Delta\varphi^*-\xi^*\quad\text{a.e. in }Q\tag{4.16}$$

$$\text{where}\quad\xi^*:=\beta^\circ(\varphi^*)\tag{4.17}$$

$$\xi\in\beta(\chi+\varphi^*)\quad\text{and}\quad\sigma\in\operatorname{sign}\chi\quad\text{a.e. in }Q\tag{4.18}$$

$$\partial_n\chi=0\quad\text{a.e. on }\Sigma\quad\text{and}\quad\chi(0)=\chi_0:=\varphi_0-\varphi^*.\tag{4.19}$$

Our starting point is just estimate (2.47), i.e., we only suppose that the constants C_{str}, C_Ω and C_7 satisfy it and do not require that they are constructed as in the proof of Theorem 2.3. In order to introduce the ingredients of the Cauchy problem mentioned above, we set for convenience

$$M_0:=\|\chi_0\|_\infty\quad\text{and}\quad M_\pi^*:=L(M_0+\|\varphi^*\|_\infty)+|\pi(0)|\tag{4.20}$$

$$M(\rho):=\rho\,C_{str}C_\Omega|\Omega|^{7/6}+C_7\tag{4.21}$$

$$A(\rho):=\gamma M(\rho)+\nu\|\Delta\varphi^*\|_\infty+\|\xi^*\|_\infty+M_\pi^*\tag{4.22}$$

where L is the Lipschitz constant of π. We observe that (cf. (2.47))

$$\|\vartheta\|_\infty\le M(\rho)\quad\text{and}\quad|\pi(\varphi^*\pm r)|\le M_\pi^*\quad\text{a.e. in }\Omega\quad\text{for every }r\in[0,M_0].\tag{4.23}$$

We assume (2.52) and define ρ^* as the solution to $\rho=A(\rho)+M_0/T$, i.e.,

$$\rho^*:=\frac{\gamma C_7+\nu\|\Delta\varphi^*\|_\infty+\|\xi^*\|_\infty+M_\pi^*+M_0/T}{1-\gamma\,C_{str}C_\Omega|\Omega|^{7/6}}\,.\tag{4.24}$$

We claim that ρ^* fulfills the properties of the statement. So, we fix $\rho>\rho^*$ and consider any solution of the transformed problem according to (4.16)–(4.19). We observe that our assumption $\rho>\rho^*$ implies

$$\rho>A(\rho)+\frac{M_0}{T}\,,\quad\text{whence also}\quad\rho>A(\rho)\tag{4.25}$$

and we can set

$$T^* := \frac{M_0}{\rho - A(\rho)} \,. \tag{4.26}$$

Hence, (4.25) ensures that the definition of T^* is meaningful and that $T^* \geq 0$. More precisely, $T^* = 0$ if $M_0 = 0$, i.e., $\varphi_0 = \varphi^*$, and $T^* > 0$ otherwise. The first inequality in (4.25) implies that $T^* < T$. The rest of the proof is devoted to prove that $\chi(t) = 0$ for every $t \in [T^*, T]$. This is done by comparison arguments, as mentioned at the beginning. We introduce the ordinary Cauchy problem

$$w' + \rho\zeta = A(\rho), \quad \zeta \in \operatorname{sign} w \quad \text{and} \quad w(0) = M_0 \,. \tag{4.27}$$

As $A(\rho)/\rho \in [0, 1) \subset \operatorname{sign} 0$ by (4.25), one checks that its unique solution is given by

$$w(t) = \bigl(M_0 - (\rho - A(\rho))t\bigr)^+ \quad \text{for } t \in [0, T]. \tag{4.28}$$

Notice that $0 \leq w \leq M_0$ and that w vanishes on $[T^*, T]$ by the definition (4.26) of T^*. Thus, by also reading w as a space independent function defined in Q rather than in $(0, T)$, it suffices to prove that $|\chi| \leq w$ a.e. in Q. To this end, we observe that w trivially satisfies the homogeneous Neumann boundary condition and write (4.27) in the following forms

$$\partial_t w - \nu\Delta w + \pi(\varphi^* + w) + \rho\zeta = A(\rho) + \pi(\varphi^* + w) \tag{4.29}$$

$$\partial_t w - \nu\Delta w - \pi(\varphi^* - w) - \rho(-\zeta) = A(\rho) - \pi(\varphi^* - w) \tag{4.30}$$

$$\text{with} \quad \zeta \in \operatorname{sign} w \quad \text{or, equivalently,} \quad -\zeta \in \operatorname{sign}(-w) \,.$$

We set $\psi := (\chi - w)^+$, the positive part of $\chi - w$, and multiply the difference between (4.16) and (4.29) by ψ. By accounting for (4.23) and the definition (4.22) of $A(\rho)$, we have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\psi(t)|^2 + \nu\int_{Q_t} |\nabla\psi|^2 + \int_{Q_t} (\xi - \xi^*)\psi + \rho\int_{Q_t} (\sigma - \zeta)\psi \\
&\quad + \int_{Q_t} \bigl(\pi(\varphi^* + \chi) - \pi(\varphi^* + w)\bigr)\psi \\
&= \int_{Q_t} \bigl(\gamma\vartheta + \nu\Delta\varphi^* - \xi^* - A(\rho) - \pi(\varphi^* + w)\bigr)\psi \\
&\leq \int_{Q_t} \bigl(\gamma M(\rho) + \nu\|\Delta\varphi^*\|_\infty + \|\xi^*\|_\infty - A(\rho) + M_\pi^*\bigr)\psi = 0.
\end{aligned}$$

Now, we observe that the integrals on the left-hand side involving ξ and σ are nonnegative: indeed, where $\psi \neq 0$, we have $\psi > 0$ and $\chi > w$, whence $\varphi^* + \chi > \varphi^* + w \geq \varphi^*$, so that $\xi \geq \xi^*$ and $\sigma \geq \zeta$. On the other hand, we have

$$\int_{Q_t} \bigl(\pi(\varphi^* + \chi) - \pi(\varphi^* + w)\bigr)\psi \geq -L\int_{Q_t} |\chi - w|\psi = -L\int_{Q_t} |\psi|^2.$$

Therefore, we deduce that

$$\int_\Omega |\psi(t)|^2 \leq L\int_{Q_t} |\psi|^2. \tag{4.31}$$

By applying the Gronwall lemma, we conclude that $\psi = 0$, i.e., $\chi \leq w$. Now, we set $\psi := (\chi + w)^-$, the negative part of $\chi + w$, add equations (4.16) and (4.30) to each other and multiply the resulting equality by $-\psi$. By accounting for (4.23) and the definition (4.22) of $A(\rho)$ once more, we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega |\psi(t)|^2 + \nu\int_{Q_t} |\nabla\psi|^2 + \int_{Q_t} (\xi - \xi^*)(-\psi) + \rho\int_{Q_t} \bigl(\sigma - (-\zeta)\bigr)(-\psi) \\
&\quad + \int_{Q_t} \bigl(\pi(\varphi^* + \chi) - \pi(\varphi^* - w)\bigr)(-\psi) \\
&= \int_{Q_t} \bigl(-\gamma\vartheta - \nu\Delta\varphi^* + \xi^* - A(\rho) + \pi(\varphi^* - w)\bigr)\psi \\
&\leq \int_{Q_t} \bigl(\gamma M(\rho) + \nu\|\Delta\varphi^*\|_\infty + \|\xi^*\|_\infty - A(\rho) + M^*_\pi\bigr)\psi = 0.
\end{aligned}
$$

Also in this case, the integrals on the left-hand side involving ξ and σ are nonnegative: indeed, where $\psi \neq 0$, we have $\psi > 0$ and $\chi < -w$, whence $\varphi^* + \chi < \varphi^* - w \leq \varphi^*$, so that $\xi \leq \xi^*$ and $\sigma \leq -\zeta$. On the other hand, we have

$$
\int_{Q_t} \bigl(\pi(\varphi^* + \chi) - \pi(\varphi^* - w)\bigr)(-\psi) \geq -L\int_{Q_t} |\chi + w|\psi = -L\int_{Q_t} |\psi|^2.
$$

Hence, we deduce (4.31) with the new meaning of ψ and apply the Gronwall lemma. We obtain $\psi = 0$, i.e., $-\chi \leq w$. Therefore, we have proved that $|\chi| \leq w$, and this implies that $\chi(t) = 0$ for every $t \in [T^*, T]$. $\square$

Remark 4.2. As announced in Remark 2.8, we can show that the function $\|\chi(\,\cdot\,)\|_H$ is strictly decreasing while positive provided that ρ is large enough, at least under a reinforcement of assumption (2.52). Indeed, with the notations of (2.46)–(2.47), we have to require that

$$
\rho > (\gamma + L)C_{str}C_\Omega|\Omega|^{7/6}\rho + \widetilde{C} \tag{4.32}
$$

where we have set

$$
\widetilde{C} := \gamma C_7 + LC_6 + \nu\|\Delta\varphi^*\|_\infty + \|\xi^*\|_\infty + L\|\varphi^*\|_\infty + |\pi(0)|.
$$

Notice that (4.32) is true provided that

$$
(\gamma + L)C_{str}C_\Omega|\Omega|^{7/6} < 1 \quad \text{and} \quad \rho > \frac{\widetilde{C}}{1 - (\gamma + L)C_{str}C_\Omega|\Omega|^{7/6}}\,.
$$

We multiply (4.16) written at the time t by $\chi(t)$ and integrate over Ω. By ignoring some nonnegative terms on the left-hand side, observing that $\sigma\chi = |\chi|$ by the definition of sign, and owing to (2.46)–(2.47), we easily obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\chi(t)\|_H^2 + \rho\int_\Omega |\chi(t)| \leq \int_\Omega \bigl(\gamma\vartheta(t) + \nu\Delta\varphi^* + \xi^* - \pi(\chi(t) + \varphi^*)\bigr)\chi(t) \\
&\leq \bigl(\gamma\|\vartheta\|_\infty + \nu\|\Delta\varphi^*\|_\infty + \|\xi^*\|_\infty + L\|\chi\|_\infty + L\|\varphi^*\|_\infty + |\pi(0)|\bigr)\int_\Omega |\chi(t)|. \\
&\leq \bigl((\gamma + L)C_{str}C_\Omega|\Omega|^{7/6}\rho + \widetilde{C}\bigr)\int_\Omega |\chi(t)|.
\end{aligned}
$$

Therefore, on account of (4.32), we conclude that $(d/dt)\|\chi(t)\|_H^2 < 0$ while $\|\chi(t)\|_1 > 0$, or equivalently $\|\chi(t)\|_H > 0$.

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