# Kyle-Back Equilibrium Models and Linear Conditional Mean-field SDEs 

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#### Abstract

In this paper we study the Kyle-Back strategic insider trading equilibrium model in which the insider has an instantaneous information on an asset, assumed to follow an Ornstein-Uhlenback-type dynamics that allows possible influence by the market price. Such a model exhibits some further interplay between insider's information and the market price, and it is the first time being put into a rigorous mathematical framework of the recently developed conditional mean-field stochastic differential equation (CMFSDEs). With the help of the "reference probability measure" concept in filtering theory, we shall first prove a general well-posedness result for a class of linear CMFSDEs, which is new in the literature of both filtering theory and mean-field SDEs, and will be the foundation for the underlying strategic equilibrium model. Assuming some further Gaussian structures of the model, we find a closed form of optimal intensity of trading strategy as well as the dynamic pricing rules. We shall also substantiate the well-posedness of the resulting optimal closed-loop system, whence the existence of Kyle-Back equilibrium. Our result recovers many existing results as special cases.


Keywords. Strategic insider trading, Kyle-Back equilibrium, conditional mean-field SDEs, reference measures, optimal closed-loop system.

2000 AMS Mathematics subject classification: 60H10, 91G80; 60G35, 93E11.

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## 1 Introduction

In his seminal paper, A.S. Kyle [21] first proposed a sequential equilibrium model of asset pricing with asymmetric information. The model was then extended by K. Back [2] to the continuous time version, and has since known as the Kyle-Back strategic insider trading equilibrium model. Roughly speaking, in such a model it is assumed that there are two types of traders in a risk neutral market: one informed (insider) trader vs. many uninformed (noise) traders. The insider "sees" both (possibly future) value of the fundamental asset as well as its market value, priced by the market makers, and acts strategically in a non-competitive manner. The noise traders, on the other hand, act independently with only market information of the asset. Finally, the market makers set the price of the asset, in a Bertrand competition fashion, based on the historical information of the collective market actions of all traders, without knowing identify the insider. The so-called Kyle-Back equilibrium is a closed-loop system in which the insider maximizes his/her expected return in a market efficient manner (i.e., following the given market pricing rule).

There has been a large number of literature on this topic. We refer to, e.g., [2,4,9, 14, 15, 17, 21 ] and the references therein for both discrete and continuous time models. It is noted, however, that in most of these works only the case of static information is considered, that is, it is assumed that information that the insider could observe is "time-invariant", often as the fundamental price at a given future moment. Mathematically, this amounts to saying that the insider has the knowledge of a given random variable whose value cannot be detected from the market information at current time. It is often assumed that the system has a certain Gaussian structure (e.g., the future price is a Gaussian random variable), so that the optimal strategies can be calculated explicitly. The situation will become more complicated when the fundamental price progresses as a stochastic process $\left\{v_{t}, t \geq 0\right\}$ and the insider is able to observe the price process dynamically in a "nonanticipative" manner. The asymmetric information nature of the problem has conceivably led to the use of filtering techniques in the study of the Kyle-Back model, and we refer to, e.g., [1] and [6] for the static information case, and to, e.g., 9 and [17] for the dynamic information case. It is noted that in [6] it is further assumed that the actions of noise traders may have some "memory", so that the observation process in the filtering problem is driven by a fractional Brownian motion, adding technical difficulties in a different aspect. We also note that the Kyle-Back model has been continuously extended in various directions. For example, in a static information setting, [16] recently considered the case when noise trading volatility is a stochastic process, and in the dynamic information case [10-12] studied the Kyle-Back equilibrium for the defaultable underlying asset via dynamic Markov bridges, exhibiting further theoretical potential of the problem.

In this paper we are interested in a generalized Kyle-Back equilibrium model in a dynamic in-
formation setting, in which the asset dynamics is of the form of an Ornstein-Uhlenback SDE whose drift also contains market sentiment (e.g., supply and demand, earning base, etc.), quantified by the market price. The problem is then naturally imbedded into a (linear) filtering problem in which both state and observation dynamics contain the filtered signal process (see $\S 2$ for details). We note that such a structure is not covered by the existing filtering theory, and thus it is interesting in its own right. In fact, under the setting of this paper the signal-observation dynamics form a "coupled" (linear) conditional mean-field stochastic differential equations (CMFSDEs, for short) whose well-posedness, to the best of our knowledge, is new.

The main objective of this paper is thus two-fold. First, we shall look for a rigorous framework on which the well-posedness of the underlying CMFSDE can be established. The main device of our approach is the "reference probability measure" that is often seen in the nonlinear filtering theory (see, e.g., [24]). Roughly speaking, we give the observable market movement a "prior" probability distribution so that it is a Brownian motion that is independent of the martingale representing the aggregated trading actions of the noisy traders, and we then prove that the original signal-observation SDEs have a weak solution. More importantly, we shall prove that the uniqueness holds among all weak solutions whose laws are absolutely continuous with respect to the reference probability measure. We should note that such a uniqueness in particular resolves a long-standing issue on the Kyle-Back equilibrium model: the identification of the total traded volume movements and the innovation process of the corresponding filtering problem, which has been argued either only heuristically or by economic instinct in the literature (see, e.g., [1]). The second goal of this paper is to identify the Kyle-Back equilibrium, that is, the optimal closed-loop system for this new type of partially observable optimization problem. Assuming a Gaussian structure and the linearity of the CMFSDEs, we give the explicit solutions to the insider's trading intensity and justify the well-posedness of the closed-loop system (whence the "existence" of the equilibrium). These solutions in particular cover many existing results as special cases.

The rest of the paper is organized as follows. In section 2 we give the preliminaries of the Kyle-Back equilibrium model and formulate the strategic insider trading problem. In section 3 we formulate a general form of linear CMFSDE and introduce the notion of its solutions and their uniqueness. We state the main well-posedness result, calculate its explicit solutions in the case of deterministic coefficients, and discuss an important extension to the unbounded coefficients case. Section 4 will be devoted to the proof of the main well-posedness theorem. In section 5 we characterize the optimal trading strategy, and give a first order necessary condition for the optimal intensity, and finally in section 6 we focus on the synthesis analysis, and validate the Kyle-Back equilibrium. In some spacial cases, we give the closed-form solutions, and compare them to the existing results.

## 2 Problem Formulation

In this section we describe a continuous time Kyle-Back equilibrium model that will be investigated in this paper, as well as related technical settings.

We begin by assuming that all randomness of the market comes from a common complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which is defined 2-dimensional Brownian motion $B=\left(B^{v}, B^{z}\right)$, where $B^{v}=\left\{B_{t}^{v}: t \geq 0\right\}$ represents the noise of the fundamental value dynamics, and $B^{z}=$ $\left\{B_{t}^{z}: t \geq 0\right\}$ represents the collective action of the noise traders. For notational clarity, we denote $\mathbb{F}^{v}=\left\{\mathcal{F}_{t}^{B^{v}}: t \geq 0\right\}$ and $\mathbb{F}^{z} \triangleq\left\{\mathcal{F}_{t}^{B^{z}}: t \geq 0\right\}$ to be the filtrations generated by $B^{v}$ and $B^{z}$, respectively, and denote $\mathbb{F}=\mathbb{F}^{v} \vee \mathbb{F}^{z}$, with the usual $\mathbb{P}$-augmentation such that it satisfies the usual hypotheses (cf. e.g., [25]).

Further, throughout the paper we will denote, for a generic Euclidean space $\mathbb{X}$, regardless of its dimension, $\langle\cdot, \cdot\rangle$ and $|\cdot|$ to be its inner product and norm, respectively. We denote the space of continuous functions defined on $[0, T]$ with the usual sup-norm by $\mathbb{C}([0, T] ; \mathbb{X})$, and we shall make use of the following notations:

- For any sub- $\sigma$-field $\mathcal{G} \subseteq \mathcal{F}_{T}$ and $1 \leq p<\infty, L^{p}(\mathcal{G} ; \mathbb{X})$ denotes the space of all $\mathbb{X}$-valued, $\mathcal{G}$-measurable random variables $\xi$ such that $\mathbb{E}|\xi|^{p}<\infty$. As usual, $\xi \in L^{\infty}(\mathcal{G} ; \mathbb{X})$ means that it is $\mathcal{G}$-measurable and bounded.
- For $1 \leq p<\infty, L_{\mathbb{F}}^{p}([0, T] ; \mathbb{X})$ denotes the space of all $\mathbb{X}$-valued, $\mathbb{F}$-progressively measurable processes $\xi$ satisfying $\mathbb{E} \int_{0}^{T}\left|\xi_{t}\right|^{p} d t<\infty$. The meaning of $L_{\mathbb{F}}^{\infty}([0, T] ; \mathbb{X})$ is defined similarly.

Throughout this paper we assume that all the processes are 1-dimensional, but higher dimensional cases can be easily deduced without substantial difficulties. Therefore, we will often drop $\mathbb{X}(=\mathbb{R})$ from the notation. Also, throughout the paper we shall denote all " $L^{p}$-norms" by $\|\cdot\|_{p}$, regardless its being $L^{p}(\mathcal{G})$ or $L_{\mathbb{F}}^{p}([0, T])$, when the context is clear.

Consider a given stock whose fundamental value (or its return) is $V=\left\{V_{t}: t \geq 0\right\}$ traded on a finite time interval $[0, T]$. There are three types of agents in the market:
(i) The insider, who directly observes the realization of the value $V_{t}$ at any time $t \in[0, T]$ and submits his order $X_{t}$ at $t \in[0, T]$.
(ii) The noise traders, who have no direct information of the given asset, and (collectively) submit an order $Z_{t}$ at $t \in[0, T]$ in the form

$$
\begin{equation*}
Z_{t}=\int_{0}^{t} \sigma_{t}^{z} d B_{t}^{z}, \quad t \geq 0 \tag{2.1}
\end{equation*}
$$

where $\sigma^{z}=\left\{\sigma_{t}^{z}: t \geq 0\right\}$ is a given continuous deterministic function satisfying $\sigma_{t}^{z}>0$.
(iii) The market makers, who observe only the total traded volume

$$
\begin{equation*}
Y_{t}=X_{t}+Z_{t}, \quad t \geq 0 \tag{2.2}
\end{equation*}
$$

and set the market price of the underlying asset at each time $t$, denoted by $P_{t}$, based on the observed information $\mathcal{F}_{t}^{Y} \triangleq \sigma\left\{Y_{s}, s \leq t\right\}$. We denote $S_{t} \triangleq \mathbb{E}\left[\left(V_{t}-P_{t}\right)^{2}\right]$ to be the measure of the discrepancy between market price and fundamental value of the asset.

We now give a more precise description of the two main ingredients in the model above: the dynamics of market price $P$ and the fundamental value $V$. First, in the same spirit of the original Kyle-Back model, we can assume that the market price $P$ is the result of a Bertrandtype competition among the market makers (cf. e.g., [4]), and therefore should be taken as $P_{t}=\mathbb{E}\left[V_{t} \mid \mathcal{F}_{t}^{Y}\right]$, at each time $t \geq 0$. Mathematically speaking, this amounts to saying that the market price is set to minimize the error $S_{t}$, among all $\mathcal{F}_{t}^{Y}$-measurable random variables in $L^{2}(\Omega)$, hence the "best estimator" that the market maker is able to choose given the information $\mathcal{F}_{t}^{Y}$. It is easy to see that under such a choice one must have $\mathbb{E}\left[\left(V_{t}-P_{t}\right) Y_{t}\right]=0$, that is, the market makers should expect a zero profit at each time $t \in[0, T]$. It is worth noting that, however, unlike the static case (i.e., $V_{t} \equiv v$ ), the process $P$ is no longer a $\left(\mathbb{P}, \mathbb{F}^{Y}\right)$-martingale in general.

Next, in this paper we shall also assume that the dynamics of the value of the stock $V=\left\{V_{t}\right\}$ takes the form of an Itô process: $d V_{t}=F_{t} d t+\sigma_{t}^{v} d B_{t}^{v}, t \geq 0$ (this would easily be the case if, e.g., the interest rate is non-zero). Furthermore, we shall assume that the drift $F_{t}=F\left(t, V_{t}, P_{t}\right), t \geq 0$. Here, the dependence of $F$ on the market price $P_{t}$ is based on the following rationale: the value of the stock is often affected by factors such as supply and demand, the earnings base (cash flow per share), or more generally, the market sentiment, which all depend on the market price of the stock. Consequently, taking the Gaussian structure into consideration, in what follows we shall assume that the process $V$ satisfies the following linear SDE:

$$
\left\{\begin{array}{l}
d V_{t}=\left(f_{t} V_{t}+g_{t} P_{t}+h_{t}\right) d t+\sigma_{t}^{v} d B_{t}^{v}=\left(f_{t} V_{t}+g_{t} \mathbb{E}\left[V_{t} \mid \mathcal{F}_{t}^{Y}\right]+h_{t}\right) d t+\sigma_{t}^{v} d B_{t}^{v}, \quad t \geq 0  \tag{2.3}\\
V_{0} \sim N\left(v_{0}, s_{0}\right)
\end{array}\right.
$$

where the functions $f_{t}, g_{t}, h_{t}$ and $\sigma_{t}^{v}$ are all deterministic continuous differentiable functions with respect to time $t \in[0, T]$, and $N\left(v_{0}, s_{0}\right)$ is a normal random variable with mean $v_{0}$ and standard deviation $s_{0}$.

Continuing, given the Gaussian structure of the dynamics, it is reasonable to assume that the insider's optimal trading strategy (in terms of "number of shares") is of the form (see, e.g., [1, $2,2,4,17,21]$ ):

$$
\begin{equation*}
d X_{t}=\beta_{t}\left(V_{t}-P_{t}\right) d t, \quad t \geq 0 \tag{2.4}
\end{equation*}
$$

where $\beta_{t}>0$ is a deterministic continuous differentiable function with respect to time $t$ in $[0, \mathrm{~T})$, known as the insider trading intensity. Consequently, it follows from (2.2) that the total traded volume process observed by the market makers can be expressed as

$$
\begin{equation*}
d Y_{t}=\beta_{t}\left(V_{t}-P_{t}\right) d t+\sigma_{t}^{z} d B_{t}^{z}=\beta_{t}\left(V_{t}-\mathbb{E}\left[V_{t} \mid \mathcal{F}_{t}^{Y}\right]\right) d t+\sigma_{t}^{z} d B_{t}^{z}, \quad t \geq 0 \tag{2.5}
\end{equation*}
$$

We note that SDEs (2.3) and (2.5) form a (linear) conditional mean-field SDE (CMFSDE), which is beyond the scope of the traditional filtering theory. Such SDE have been studied in [8] and [13] in general nonlinear forms, but none of them covers the one in this form. In fact, if we further allow the function $h$ in (2.3) to be an $\mathbb{F}^{Y}$-adapted process, as many stochastic control problems do, then (2.3) and (2.5) would become a fully convoluted CMFSDE whose wellposedness, to the best of our knowledge, has not been studied in the literature, even in the linear form. We should mention that the equation (2.5) in the case when $V_{t} \equiv v$ was already noted in [1] and [6, but without addressing the uniqueness of the solution. In the next sections we shall establish a mathematical framework so these SDEs can be studied rigorously.

Given the dynamics (2.3) and (2.5), our main purpose now is to find an optimal trading intensity $\beta$ for the insider to maximize his/her expected wealth, whence the Kyle-Back equilibrium. More specifically, denote the wealth process of the insider by $W=\left\{W_{t}: t \geq 0\right\}$, and assume that the strategy is self-financing (cf. e.g., [7), then the total wealth of the insider over time duration $[0, T]$, based on the market price made by the market makers, should be

$$
\begin{equation*}
W_{T}=\int_{0}^{T} X_{t} d P_{t}=X_{T} P_{T}-\int_{0}^{T} \beta_{t}\left(V_{t}-P_{t}\right) P_{t} d t=\int_{0}^{T} \beta_{t}\left(V_{t}-P_{t}\right)\left(P_{T}-P_{t}\right) d t . \tag{2.6}
\end{equation*}
$$

Here in the above we used a simple integration by parts and definition (2.4). Thus the optimization problems can be described as

$$
\begin{equation*}
\sup _{\beta} \mathbb{E}\left[W_{T}\right] \triangleq \sup _{\beta} J(\beta)=\sup _{\beta} \int_{0}^{T} \beta_{t} \mathbb{E}\left[\left(V_{t}-P_{t}\right)\left(P_{T}-P_{t}\right)\right] d t . \tag{2.7}
\end{equation*}
$$

Remark 2.1. We should remark that the simple form of optimization problem (2.7) is due largely to the linearity of the dynamics (2.3) and (2.5), as well as the Gaussian assumption on the initial state $v$. These lead to a Gaussian structure, whence the trading strategy (2.4). The general nonlinear and/or non-Gaussian Kyle-Back model requires further study of CMFSDE and associated filtering problem, and one should seek optimal control from a larger class of "admissible controls". In that case the first order condition studied in this paper will become a Pontryagin type stochastic maximum principle (see, for example, [8]), and the solution is expected to be much more involved. We will address such general problems in our future publications.

We end this section by noting that the main idea for solving the CMFSDE is to introduce the so-called reference probability space in nonlinear filtering literature (see, e.g., [24), which can be described as follows.

Assumption 2.2. There exists a probability space $\left(\Omega^{0}, \mathcal{F}^{0}, \mathbb{Q}^{0}\right)$ on which the process $\left(B_{t}^{v}, Y_{t}\right)$, $t \in[0, T]$, is a 2-dimensional continuous martingale, where $B^{v}$ is a standard Brownian motion and $Y$ is the observation process with quadratic variation $\langle Y\rangle_{t}=\int_{0}^{t}\left(\sigma_{s}^{z}\right)^{2} d s$. The probability measure $\mathbb{Q}^{0}$ will be referred to as the reference measure.

We remark that Assumption 2.2 amounts to saying that we are giving a prior distribution to the price process $Y=\left\{Y_{t}: t \geq 0\right\}$ that the market maker is observing, which is not unusual in statistical modeling, and will facilitate the discussion greatly. A natural example is the canonical space: $\Omega^{0} \triangleq C_{0}\left([0, T] ; \mathbb{R}^{2}\right)$, the space of all 2-dimensional continuous functions null at zero; $\mathcal{F}^{0} \triangleq \mathscr{B}\left(\Omega^{0}\right) ; \mathcal{F}_{t}^{0} \triangleq \mathscr{B}_{t}\left(\Omega^{0}\right)=\sigma\left\{\omega(\cdot \wedge t): \omega \in \Omega^{0}\right\}, t \in[0, T] ;$ and $\left(B^{v}, Y\right)$ is the canonical process. In the case $\sigma^{z} \equiv 1, \mathbb{Q}^{0}$ is the Wiener measure.

## 3 The Linear Conditional Mean-field SDEs

In this section we study the linear conditional mean-field SDEs (CMFSDE) (2.3) and (2.5) that play an important role in this paper. In fact, let us consider a slightly more general case that is useful in applications:

$$
\left\{\begin{array}{l}
d X_{t}=\left\{f_{t} X_{t}+g_{t} \mathbb{E}\left[X_{t} \mid \mathcal{F}_{t}^{Y}\right]+h_{t}\right\} d t+\sigma_{t}^{1} d B_{t}^{1}, \quad X_{0}=v ;  \tag{3.1}\\
d Y_{t}=\left\{H_{t} X_{t}+G_{t} \mathbb{E}\left[X_{t} \mid \mathcal{F}_{t}^{Y}\right]\right\} d t+\sigma_{t}^{2} d B_{t}^{2}, \quad Y_{0}=0,
\end{array}\right.
$$

where $B \triangleq\left(B^{1}, B^{2}\right)$ is a standard Brownian motion defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $v \sim N\left(v_{0}, s_{0}\right)$ is independent of $B$. In light of Assumption [2.2, throughout this section we shall assume the following:

Assumption 3.1. (i) The coefficients $f, g, \sigma^{1}, \sigma^{2}, G$, and $H$ are all deterministic, continuous functions; and $\sigma_{t}^{i}>0, i=1,2$, for all $t \in[0, T]$;
(ii) there exists a probability space $\left(\Omega^{0}, \mathcal{F}^{0}, \mathbb{Q}^{0}\right)$ on which the process $\left(B_{t}^{1}, Y_{t}\right), t \in[0, T]$, is a 2-dimensional continuous martingale, such that $B^{1}$ is a standard $\mathbb{Q}^{0}$-Brownian motion, and $\langle Y\rangle_{t}=\int_{0}^{t}\left|\sigma_{s}^{2}\right|^{2} d s, t \in[0, T], \mathbb{Q}^{0}$-a.s.;
(iii) the coefficient $h$ is an $\mathbb{F}^{Y}$-adapted, continuous process, such that

$$
\mathbb{E}^{\mathbb{Q}^{0}}\left[\sup _{0 \leq t \leq T}\left|h_{t}\right|^{2}\right]<\infty .
$$

Remark 3.2. The Assumption 3.1f(iii) amounts to saying that the process $h$ is defined on the reference probability space $\left(\Omega^{0}, \mathcal{F}^{0}, \mathbb{Q}^{0}\right)$, and adapted to the Brownian filtration $\mathbb{F}^{Y}$, as we often see in the stochastic control with partial observations (cf. [5]).

### 3.1 The General Result

To simplify notations in what follows we shall assume that $\sigma^{1}=\sigma^{2} \equiv 1$. We first introduce two definitions of the solution to CMFSDE (3.1). Let $\mathscr{P}(\mathbb{R})$ denote all probability measures on $\left(\mathbb{R}, \mathscr{B}(\mathbb{R})\right.$ ), where $\mathscr{B}(\mathbb{R})$ is the Borel $\sigma$-field of $\mathbb{R}$, and $\mu \sim N\left(v_{0}, s_{0}\right) \in \mathscr{P}(\mathbb{R})$ denotes the normal distribution with mean $v_{0}$ and variance $s_{0}$.

Definition 3.3. Let $\mu \in \mathscr{P}(\mathbb{R})$ be given. An eight-tuple $\left(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P} ; X, Y, B^{1}, B^{2}\right)$ is called a weak solution to CMFSDE (3.1) with initial distribution $\mu$ if
(i) $\left(B^{1}, B^{2}\right)$ is an $\mathbb{F}$-Brownian motion under $\mathbb{P}$;
(ii) $\left(X, Y, B^{1}, B^{2}\right)$ satisfies (3.1), $\mathbb{P}$-a.s.;
(iii) $X_{0} \sim \mu$; and is independent of $\left(B^{1}, B^{2}\right)$ under $\mathbb{P}$.

Definition 3.4. A weak solution $\left(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P} ; X, Y, B^{1}, B^{2}\right)$ is called a $\mathbb{Q}^{0}$-weak solution to CMFSDE (3.1) if
(i) there exists a probability measure $\mathbb{P}^{0}$ on $\left(\Omega^{0}, \mathcal{F}^{0}\right)$, and processes $\left(X^{0}, Y^{0}, B^{1,0}, B^{2,0}\right)$ defined on $\left(\Omega^{0}, \mathcal{F}^{0}, \mathbb{P}^{0}\right)$, whose law under $\mathbb{P}^{0}$ is the same as that of $\left(X, Y, B^{1}, B^{2}\right)$ under $\mathbb{P}$; and
(ii) $\mathbb{P}^{0} \sim \mathbb{Q}^{0}$.

In what follows for any $\mathbb{Q}^{0}$-weak solution, we shall consider only its copy on the reference measurable space $\left(\Omega^{0}, \mathcal{F}^{0}\right)$, and we shall still denote the solution by $\left(X, Y, B^{1}, B^{2}\right)$.

The uniqueness of the solutions to CMFSDE (3.1) is a more delicate issue. In fact, even the weak uniqueness (in the usual sense) for CMFSDE (2.3) and (2.5) is not clear. However, we have a much better hope, at least in the linear case, for $\mathbb{Q}^{0}$-solutions. We first introduce the following " $\mathbb{Q}^{0}$-pathwise uniqueness".

Definition 3.5. The CMFSDE (3.1) is said to have " $\mathbb{Q}$ "-pathwise uniqueness" if for any two $\mathbb{Q}^{0}$-weak solutions $\left(\Omega^{0}, \mathcal{F}^{0}, \mathbb{F}^{0}, \mathbb{P}^{i} ; X^{i}, Y^{i}, B^{1, i}, B^{2, i}\right), i=1,2$, such that
(i) $X_{0}^{1}=X_{0}^{2}$; and
(ii) $\mathbb{Q}^{0}\left\{\left(B_{t}^{1,1}, Y_{t}^{1}\right)=\left(B_{t}^{1,2}, Y_{t}^{2}\right), \forall t \in[0, T]\right\}=1$,
then it holds that $\mathbb{Q}^{0}\left\{\left(X_{t}^{1}, B_{t}^{2,1}\right)=\left(X_{t}^{2}, B_{t}^{2,2}\right), \forall t \in[0, T]\right\}=1$, and $\mathbb{P}^{1}=\mathbb{P}^{2}$.

Theorem 3.6. Assume that Assumption 3.1 is in force, and further that $h$ is bounded. Let $\mu \sim N\left(v_{0}, s_{0}\right)$ be given. Then CMFSDE (3.1) possesses a weak solution with initial distribution $\mu$, denoted by $\left(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P} ; X, Y, B^{1}, B^{2}\right)$.

Moreover, if we denote $P_{t}=\mathbb{E}^{\mathbb{P}}\left[X_{t} \mid \mathcal{F}_{t}^{Y}\right], t \in[0, T]$, then $P$ satisfies the following SDE:

$$
\left\{\begin{align*}
d P_{t} & =\left[\left(f_{t}+g_{t}\right) P_{t}+h_{t}\right] d t+S_{t} H_{t}\left\{d Y_{t}-\left[H_{t}+G_{t}\right] P_{t} d t\right\}, \quad t \in[0, T]  \tag{3.2}\\
P_{0} & =v_{0} .
\end{align*}\right.
$$

where $S_{t}=\operatorname{Var}\left(P_{t}\right)$ satisfies the Riccati equation:

$$
\begin{equation*}
d S_{t}=\left[1+2 f_{t} S_{t}-H_{t}^{2} S_{t}^{2}\right] d t, \quad S_{0}=s_{0} \tag{3.3}
\end{equation*}
$$

Furthermore, the weak solution can be chosen as $\mathbb{Q}^{0}$-weak solution, and the $\mathbb{Q}^{0}$-pathwise uniqueness holds.

We remark that Theorem [3.6 does not imply that CMFSDE (3.1) has a strong solution, as not every weak solution is a $\mathbb{Q}^{0}$-weak solution. The proof of Theorem 3.6 is a bit lengthy, we shall defer it to next section. We nevertheless present a lemma below, which will be frequently used in our discussion, so as to facilitate the argument in the next section.

To begin with, we consider any filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ on which is defined a standard Brownian Motion $\left(B_{t}^{1}, B_{t}^{2}\right)$. We assume that $\mathbb{F}=\mathbb{F}^{\left(B^{1}, B^{2}\right)}$. For any $\eta \in L_{\mathbb{F}}^{2}([0, T])$ we define $L^{\eta}$ to be the solution to the following SDE,

$$
\begin{equation*}
d L_{t}=L_{t} \eta_{t} d B_{t}^{2}, \quad t \geq 0, \quad L_{0}=1 \tag{3.4}
\end{equation*}
$$

In other words, $L^{\eta}$ is a local martingale in the form of the Doléans-Dade stochastic exponential:

$$
\begin{equation*}
L_{t}^{\eta}=\exp \left\{\int_{0}^{t} \eta_{s} d B_{s}^{2}-\frac{1}{2} \int_{0}^{t}\left|\eta_{s}\right|^{2} d s\right\} . \tag{3.5}
\end{equation*}
$$

Next let $\alpha \in L_{\mathbb{F}}^{2}([0, T])$ and consider the following SDE:

$$
\begin{equation*}
d Y_{t}=\left(\alpha_{t}+h(Y)_{t}\right) d t+d B_{t}^{2}, \quad Y_{0}=0 \tag{3.6}
\end{equation*}
$$

where $h:[0, T] \times \mathbb{C}([0, T]) \mapsto \mathbb{R}$ is "progressively measurable" in the sense that, it is a measurable function such that for each $t \in[0, T], h(y)_{t}=h(y \cdot \wedge t)_{t}$ for $y \in \mathbb{C}([0, T])$. (A simple case would be $h(y)_{t}=\tilde{h}\left(y_{t}\right)$, where $\tilde{h}$ is a measurable function.) We should note that in general the wellposedness of SDE (3.6) is non-trivial without any specific conditions on $h$, but in what follows we shall assume a priori that (3.6) has a (weak) solution on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that $h \in L_{\mathbb{F}^{Y}}^{2}([0, T])$ if $h_{t}=h(Y)_{t}, t \in[0, T]$, such that $\mathbb{E} \int_{0}^{T}\left|h(Y)_{t}\right|^{2} d t<\infty$. We have the following lemma.

Lemma 3.7. Suppose that the SDE (3.6) has a solution $Y_{t}, t \in[0, T]$, for given $\alpha_{t} \in L_{\mathbb{F}^{1}}^{2}([0, T])$ and $h \in L_{\mathbb{F}^{Y}}^{2}([0, T])$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\beta$ be given by

$$
\begin{equation*}
d \beta_{t}=\alpha_{t} d t+d B_{t}^{2}, \quad t \geq 0, \quad \beta_{0}=0 \tag{3.7}
\end{equation*}
$$

Assume further that $L^{-(\alpha+h)}$, the solution to (3.4) with $\eta=-(\alpha+h)$, is an $(\mathbb{F}, \mathbb{P})$-martingale. Then, for any $t \in[0, T]$, it holds that

$$
\begin{equation*}
\mathbb{E}^{\mathbb{P}}\left[\alpha_{t} \mid \mathcal{F}_{t}^{Y}\right]=\mathbb{E}^{\mathbb{P}}\left[\alpha_{t} \mid \mathcal{F}_{t}^{\beta}\right], \quad \forall t \in[0, T], \quad \mathbb{P} \text {-a.s. } \tag{3.8}
\end{equation*}
$$

Proof. Clearly, it suffices to prove $\mathbb{E}^{\mathbb{P}}\left[\alpha_{T} \mid \mathcal{F}_{T}^{Y}\right]=\mathbb{E}^{\mathbb{P}}\left[\alpha_{T} \mid \mathcal{F}_{T}^{\beta}\right]$, as the cases for $t<T$ are analogous. To this end, we define a new probability measure $\mathbb{Q}$ on $\left(\Omega, \mathcal{F}_{T}\right)$ by

$$
\left.\frac{d \mathbb{Q}}{d \mathbb{P}}\right|_{\mathcal{F}_{T}}=L_{T}^{-(\alpha+h)}
$$

where $L^{-(\alpha+h)}$ is the solution to the SDE (3.4) with $\eta=-(\alpha+h)$, and it is a true martingale on $[0, T]$ by assumption. By Girsanov Theorem, the process $\left(B^{1}, Y\right)$ is a standard Brownian motion on $[0, T]$ under $\mathbb{Q}$.

Now define $\bar{L}_{t}=1 / L_{t}^{-(\alpha+h)}$, then $\bar{L}$ satisfies the following $\operatorname{SDE}$ on $\left(\Omega, \mathcal{F}_{T}, \mathbb{Q}\right)$ :

$$
\begin{equation*}
d \bar{L}_{t}=\bar{L}_{t}\left(\alpha_{t}+h_{t}\right) d Y_{t}, \quad t \in[0, T], \quad \bar{L}_{0}=1 \tag{3.9}
\end{equation*}
$$

Furthermore, by the Kallianpur-Striebel formula, we have

$$
\begin{equation*}
\mathbb{E}^{\mathbb{P}}\left[\alpha_{T} \mid \mathcal{F}_{T}^{Y}\right]=\frac{\mathbb{E}^{\mathbb{Q}}\left[\alpha_{T} \bar{L}_{T} \mid \mathcal{F}_{T}^{Y}\right]}{\mathbb{E}^{\mathbb{Q}}\left[\bar{L}_{T} \mid \mathcal{F}_{T}^{Y}\right]} \tag{3.10}
\end{equation*}
$$

On the other hand, $\bar{L}$ has the explicit form:

$$
\begin{align*}
\bar{L}_{T} & =\exp \left\{\int_{0}^{T}\left[\alpha_{t}+h_{t}\right] d Y_{t}-\frac{1}{2} \int_{0}^{T}\left[\alpha_{t}+h_{t}\right]^{2} d t\right\}  \tag{3.11}\\
& =\exp \left\{\int_{0}^{T} h_{t} d Y_{t}-\frac{1}{2} \int_{0}^{T}\left|h_{t}\right|^{2} d t+\int_{0}^{T} \alpha_{t} d Y_{t}-\frac{1}{2} \int_{0}^{T}\left[\left|\alpha_{t}\right|^{2}+2 h_{t} \alpha_{t}\right] d t\right\} \\
& \triangleq \bar{L}_{T}^{0} \Lambda_{T}
\end{align*}
$$

where

$$
\bar{L}_{T}^{0} \triangleq \exp \left\{\int_{0}^{T} h_{t} d Y_{t}-\frac{1}{2} \int_{0}^{T}\left|h_{t}\right|^{2} d t\right\} ; \Lambda_{T} \triangleq \exp \left\{\int_{0}^{T} \alpha_{t} d Y_{t}-\frac{1}{2} \int_{0}^{T}\left[\left[\alpha_{t}\right]^{2}+2 h_{t} \alpha_{t}\right] d t\right\}
$$

Note that $h$ is $\mathbb{F}^{Y}$-adapted, so is $\bar{L}_{T}^{0}$. We derive from (3.10) that

$$
\begin{equation*}
\mathbb{E}^{\mathbb{P}}\left[\alpha_{T} \mid \mathcal{F}_{T}^{Y}\right]=\frac{\mathbb{E}^{\mathbb{Q}}\left[\alpha_{T} \Lambda_{T} \mid \mathcal{F}_{T}^{Y}\right]}{\mathbb{E}^{\mathbb{Q}}\left[\Lambda_{T} \mid \mathcal{F}_{T}^{Y}\right]} \tag{3.12}
\end{equation*}
$$

Now define $Y_{t}^{1}=\int_{0}^{t} h_{s} d s$. Since $h$ is $\mathbb{F}^{Y}$-adapted, so is $Y^{1}$, and consequently $\beta_{t}=Y_{t}-Y_{t}^{1}$, $t \geq 0$ is $\mathbb{F}^{Y}$-adapted. Moreover, since $\left(B^{1}, Y\right)$ is a standard Brownian motion under $\mathbb{Q}$, and $\alpha$ is $\mathbb{F}^{B^{1}}$-adapted, we conclude that $\alpha_{t}$ is independent of $Y_{t}$ under $\mathbb{Q}$. Therefore, using integration by parts we obtain that

$$
\begin{align*}
\Lambda_{T} & =\exp \left\{\int_{0}^{T} \alpha_{t} d \beta_{t}-\frac{1}{2} \int_{0}^{T}\left|\alpha_{t}\right|^{2} d t\right\}  \tag{3.13}\\
& =\exp \left\{\alpha_{T} \beta_{T}-\int_{0}^{T} \beta_{t} d \alpha_{t}-\frac{1}{2} \int_{0}^{T}\left|\alpha_{t}\right|^{2} d t\right\}
\end{align*}
$$

Since $\alpha$ is independent of $Y$ under $\mathbb{Q}$, and $\beta_{t}, t \in[0, T]$ is $\mathcal{F}_{T}^{Y}$-measurable, a Monotone Class argument shows that $\mathbb{E}^{\mathbb{Q}}\left[\Lambda_{T} \mid \mathcal{F}_{T}^{Y}\right]$ is $\mathcal{F}_{T}^{\beta}$ measurable; and similarly, $\mathbb{E}^{\mathbb{Q}}\left[\alpha_{T} \Lambda_{T} \mid \mathcal{F}_{T}^{Y}\right]$ is also $\mathcal{F}_{T}^{\beta}$ measurable. Consequently $\mathbb{E}^{\mathbb{P}}\left[\alpha_{T} \mid \mathcal{F}_{T}^{Y}\right]$ is $\mathcal{F}_{T}^{\beta}$ measurable, thanks to (3.10).

Finally, noting $\mathbb{F}^{\beta} \subseteq \mathbb{F}^{Y}$ we have

$$
\begin{equation*}
\mathbb{E}^{\mathbb{P}}\left[\alpha_{T} \mid \mathcal{F}_{T}^{Y}\right]=\mathbb{E}^{\mathbb{P}}\left\{\mathbb{E}^{\mathbb{P}}\left[\alpha_{T} \mid \mathcal{F}_{T}^{Y}\right] \mid \mathcal{F}_{T}^{\beta}\right\}=\mathbb{E}^{\mathbb{P}}\left[\alpha_{T} \mid \mathcal{F}_{T}^{\beta}\right], \tag{3.14}
\end{equation*}
$$

proving the lemma.

### 3.2 Deterministic Coefficient Cases

An important special case is when all the coefficients in the linear CMFSDE (3.1) are deterministic. In this case we expect that the solution $(X, Y)$ is Gaussian, and it can be solved in a much more explicit way. The following linear CMFSDE will be useful in the study of insider trading equilibrium model in the latter half of the paper.

$$
\left\{\begin{array}{l}
d X_{t}=\left[f_{t} X_{t}+g_{t} \mathbb{E}\left[X_{t} \mid \mathcal{F}_{t}^{Y}\right]+h_{t}\right] d t+\sigma_{t}^{1} d B_{t}^{1}, \quad X_{0}=v  \tag{3.15}\\
d Y_{t}=H_{t}\left(X_{t}-\mathbb{E}\left[X_{t} \mid \mathcal{F}_{t}^{Y}\right]\right) d t+\sigma_{t}^{2} d B_{t}^{2}, \quad Y_{0}=0,
\end{array}\right.
$$

where $v \sim N\left(v_{0}, s_{0}\right)$ and is independent of $\left(B^{1}, B^{2}\right)$ and all the coefficients are assumed to be deterministic.

### 3.2.1 Bounded Coefficients Case

In light of Theorem 3.6 let us introduce the following functions:

$$
\begin{equation*}
k_{t}=\frac{H_{t}^{2} S_{t}}{\left|\sigma_{t}^{2}\right|^{2}}, \quad l_{t}=\frac{H_{t} S_{t}}{\sigma_{t}^{2}}, \quad t \geq 0 \tag{3.16}
\end{equation*}
$$

where $S$ is the solution to the following Riccati equation

$$
\begin{equation*}
\frac{d S_{t}}{d t}=\left(\sigma_{t}^{1}\right)^{2}+2 f_{t} S_{t}-l_{t}^{2}, \quad t \geq 0, \quad S_{0}=s_{0} \tag{3.17}
\end{equation*}
$$

We have the following result.

Proposition 3.8. Let Assumption 3.1 be in force, and assume further that the process $h$ in (3.1) is also a deterministic and continuous function. Let $(X, Y)$ be the solution of (3.15) on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and denote $P_{t}=\mathbb{E}^{\mathbb{P}}\left[X_{t} \mid \mathcal{F}_{t}^{Y}\right], t \geq 0$. Then $X$ and $P$ have the following explicit forms respectively: for $t \geq 0$, it holds $\mathbb{P}$-a.s. that

$$
\begin{align*}
X_{t}= & P_{t}+\phi_{1}(t, 0)\left[v-v_{0}+\int_{0}^{t} \phi_{1}(0, r)\left(\sigma_{r}^{1} d B_{r}^{1}-l_{r} d Y_{r}\right)\right]  \tag{3.18}\\
P_{t}= & \phi_{2}(t, 0)\left\{v_{0}+\int_{0}^{t} \phi_{2}(0, r) h_{r} d r+\left(v-v_{0}\right) \phi_{3}(t, 0)\right.  \tag{3.19}\\
& \left.+\int_{0}^{t} \sigma_{r}^{1} \phi_{1}(0, r) \phi_{3}(t, r) d B_{r}^{1}+\int_{0}^{t}\left[\phi_{2}(0, r) l_{r}-\phi_{1}(0, r) \phi_{3}(t, r) l_{r}\right] d B_{r}^{2}\right\},
\end{align*}
$$

where, for $0 \leq r \leq t$,

$$
\left\{\begin{array}{l}
\phi_{1}(t, r)=\exp \left\{\int_{r}^{t}\left(f_{u}-k_{u}\right) d u\right\} ; \quad \phi_{2}(t, r)=\exp \left\{\int_{r}^{t}\left(f_{u}+g_{u}\right) d u\right\}  \tag{3.20}\\
\phi_{3}(t, r)=\int_{r}^{t} \phi_{1}(u, 0) \phi_{2}(0, u) k_{u} d u
\end{array}\right.
$$

Proof. We first note that the SDE (3.15) is a special case of (3.1) with $G=-H$. Then, following the same argument of Theorem 3.6] one can show that when $\sigma^{1}>0$ and $\sigma^{2}>0$ are not equal to 1 , the $\operatorname{SDE}(3.2)$ for the process $P_{t}=\mathbb{E}^{\mathbb{P}}\left[X_{t} \mid \mathcal{F}_{t}^{Y}\right]$ reads

$$
\begin{equation*}
d P_{t}=\left[\left(f_{t}+g_{t}\right) P_{t}+h_{t}\right] d t+\frac{H_{t} S_{t}}{\left(\sigma_{t}^{2}\right)^{2}} d Y_{t}, \quad P_{0}=v_{0} \tag{3.21}
\end{equation*}
$$

and $S_{t}$ satisfies a Riccati equation

$$
\begin{equation*}
\frac{d S_{t}}{d t}=\left(\sigma_{t}^{2}\right)^{2}+2 f_{t} S_{t}-\left[\frac{H_{t} S_{t}}{\sigma_{t}^{2}}\right]^{2}, \quad S_{0}=s_{0} \tag{3.22}
\end{equation*}
$$

Now applying the Girsanov transformation we can define a new probability measure $\mathbb{Q}$ under which $\left(B^{1}, Y\right)$ is a continuous martingale, such that $B^{1}$ is a standard Brownian motion, and $d\langle Y\rangle_{t}=\left|\sigma_{t}^{2}\right|^{2} d t$. Then, under $\mathbb{Q}$, the dynamic of $V_{t} \triangleq X_{t}-P_{t}$ can be written as

$$
\begin{aligned}
d V_{t} & =\left[f_{t}-\frac{H_{t}^{2} S_{t}}{\left(\sigma_{t}^{2}\right)^{2}}\right] V_{t} d t+\sigma_{t}^{1} d B_{t}^{1}-\frac{H_{t} S_{t}}{\sigma_{t}^{2}} d Y_{t} \\
& =\left[f_{t}-k_{t}\right] V_{t} d t+\sigma_{t}^{1} d B_{t}^{1}-l_{t} d Y_{t}, \quad X_{0}-P_{0}=v-v_{0} .
\end{aligned}
$$

It then follows that the identity (3.18) holds $\mathbb{Q}$-almost surely, and hence $\mathbb{P}$-almost surely.
Similarly, applying the constant variation formula for the linear SDE (3.21) and noting (3.1) we obtain that, with $\phi_{2}(t, r)=\exp \left(\int_{r}^{t}\left(f_{u}+g_{u}\right) d u\right)$, for $0 \leq r, t \leq T$,

$$
\begin{align*}
P_{t} & =\phi_{2}(t, 0)\left\{v_{0}+\int_{0}^{t} \phi_{2}(0, r) h_{r} d r+\int_{0}^{t} \phi_{2}(0, r) \frac{H_{r} S_{r}}{\left(\sigma_{r}^{2}\right)^{2}} d Y_{r}\right\}  \tag{3.23}\\
& =\phi_{2}(t, 0)\left\{v_{0}+\int_{0}^{t} \phi_{2}(0, r) h_{r} d r+\int_{0}^{t} \phi_{2}(0, r) \frac{H_{r} S_{r}}{\left(\sigma_{r}^{2}\right)^{2}}\left[H_{r}\left(X_{r}-P_{r}\right) d r+\sigma_{r}^{2} d B_{r}^{2}\right]\right\} .
\end{align*}
$$

Now plugging (3.18) into (3.23), and applying Fubini, we have

$$
\begin{align*}
P_{t}= & \phi_{2}(t, 0)\left\{v_{0}+\int_{0}^{t} \phi_{2}(0, r) h_{r} d r+\left(v-v_{0}\right) \int_{0}^{t} \phi_{1}(r, 0) \phi_{2}(0, r) k_{r} d r\right. \\
& +\int_{0}^{t} \phi_{1}(0, r) \sigma_{r}^{1} \int_{r}^{t} \phi_{1}(u, 0) \phi_{2}(0, u) k_{u} d u d B_{r}^{1} \\
& \left.+\int_{0}^{t}\left[\phi_{2}(0, r) l_{r}-\phi_{1}(0, r) l_{r} \int_{r}^{t} \phi_{1}(u, 0) \phi_{2}(0, u) k_{u} d u\right] d B_{r}^{2}\right\}  \tag{3.24}\\
= & \phi_{2}(t, 0)\left\{v_{0}+\int_{0}^{t} \phi_{2}(0, r) h_{r} d r+\left(v-v_{0}\right) \phi_{3}(t, 0)\right. \\
& \left.+\int_{0}^{t} \phi_{1}(0, r) \sigma_{r}^{1} \phi_{3}(t, r) d B_{r}^{1}+\int_{0}^{t}\left[\phi_{2}(0, r) l_{r}-\phi_{1}(0, r) \phi_{3}(t, r) l_{r}\right] d B_{r}^{2}\right\},
\end{align*}
$$

where $\phi_{3}(t, r) \triangleq \int_{r}^{t} \phi_{1}(u, 0) \phi_{2}(0, u) k_{u} d u$. This proves (3.19), whence the proposition.

### 3.2.2 Unbounded Coefficients Case

We note that Theorem 3.6 as well as the discussion so far rely heavily on the assumption that all the coefficients are bounded, especially $H$ and $G$ (see Assumption 3.1). However, in our applications we will see that the coefficients $H=-G=\beta$, where $\beta$ is the insider trading intensity which, at least in the optimal case, will satisfy $\lim _{t \rightarrow T^{-}} \beta_{t}=+\infty$, violating Assumption 3.1. In other words, the closed-loop system will exhibit a certain Brownian "bridge" nature (see also, e.g., [4, 10, 11]), for which the well-posedness result of Theorem 3.6 actually does not apply.

To overcome such a conflict, we introduce the following relaxed version of Assumption 3.1.
Assumption 3.9. There exists a sequence $\left\{T_{n}\right\}_{n \geq 1}$, with $0<T_{n} \nearrow T$, and a sequence of probability measures $\left\{\mathbb{Q}^{n}\right\}_{n \geq 1}$ on $\left(\Omega^{0}, \mathcal{F}^{0}\right)$, satisfying
(i) Assumption 3.1 holds for each $\left(\Omega^{0}, \mathcal{F}^{0}, \mathbb{Q}^{n}\right)$ over $\left[0, T_{n}\right], n \geq 1$
(ii) $\left.\mathbb{Q}^{n+1}\right|_{\mathcal{F}_{T_{n}}^{0}}=\mathbb{Q}^{n}, n \geq 1$.

We shall refer to the sequence of probability measures $\mathcal{Q}^{0}:=\left\{\mathbb{Q}^{n}\right\}_{n \geq 1}$ as the reference family of probability measures, and the associated sequence $\left\{T_{n}\right\}_{n \geq 1}$ as the announcing sequence. Clearly, if the reference measure $\mathbb{Q}^{0}$ exists, then $\mathbb{Q}^{n}=\left.\mathbb{Q}^{0}\right|_{\mathcal{F}_{T_{n}}^{0}}, n \geq 1$. It is known, however, that in the dynamic observation case the Kyle-Back equilibrium may only exist on $[0, T$ ) (see, e.g. [12] and the references cited therein). In such a case the reference family would play a fundamental role. A reasonable extension of the notion of $\mathbb{Q}^{0}$-weak solution over $[0, T)$ is as follows.

Definition 3.10. Let $\mathcal{Q}^{0}$ be a reference family of probability measures, with announcing sequence $\left\{T_{n}\right\}$. A sequence $\left\{\left(\Omega^{0}, \mathcal{F}^{0}, \mathbb{P}^{n}, X^{n}, Y^{n}, B^{1, n}, B^{2, n}\right)\right\}_{n \geq 1}$ is called a $\mathcal{Q}^{0}$-weak solution of (3.1) on $[0, T)$ if for each $n \geq 1,\left(\Omega^{0}, \mathcal{F}^{0}, \mathbb{P}^{n}, X^{n}, Y^{n}, B^{1, n}, B^{2, n}\right)$ is a $\mathbb{Q}^{n}$-weak solution on $\left[0, T_{n}\right]$.

It is worth noting that if the coefficients of CMFSDE (3.1) satisfy Assumption 3.1 on each sub-interval $\left[0, T_{n}\right]$, then one can apply Theorem 3.6 for each $n$ to get a $\mathcal{Q}^{0}$-solution. Furthermore, since the solutions will be pathwisely unique under each $\mathbb{Q}^{n}$ over $\left[0, T_{n}\right]$, it is easy to check that $\left(X_{t}^{n+1}, Y_{t}^{n+1}, B_{t}^{1, n+1}, B_{t}^{2, n+1}\right)=\left(X_{t}^{n}, Y_{t}^{n}, B_{t}^{1, n}, B_{t}^{2, n}\right), t \in\left[0, T_{n}\right], \mathbb{Q}^{n}$-a.s. We can then define a process $\left(X, Y, B^{1}, B^{2}\right)$ on $[0, T)$ by simply setting $\left(X_{t}, Y_{t}, B_{t}^{1}, B_{t}^{2}\right)=\left(X_{t}^{n}, Y_{t}^{n}, B_{t}^{1, n}, B_{t}^{2, n}\right)$, for $t \in\left[0, T_{n}\right], n \geq 1$, and we shall refer to such a process as the $\mathcal{Q}^{0}$-solution on $[0, T)$. The $\mathcal{Q}^{0}$ pathwise uniqueness on $[0, T)$ can be defined in an obvious way. We have the following extension of Theorem 3.6, whose proof is left for the interested reader.

Theorem 3.11. Assume that Assumption 3.9 is in force, and let $\mathcal{Q}^{0}$ be the family of reference measures with announcing sequence $\left\{T_{n}\right\}$. Assume further that Assumption 3.1 holds for each $\mathbb{Q}^{n}$ on $\left[0, T_{n}\right]$. Then CMFSDE (3.1) possesses a $\mathcal{Q}^{0}$-weak solution on $[0, T)$, and it is $\mathcal{Q}^{0}$-pathwisely unique on $[0, T)$.

## 4 Proof of Theorem 3.6

In this section we prove Theorem 3.6. We begin by making the following reduction: it suffices to consider the SDE (3.1) where the initial state $X_{0}=v \equiv v_{0}$ is deterministic, that is, $s_{0}=0$. Indeed, suppose that $\left(X^{x}, Y^{x}\right)$ is a weak solution of (3.1) along with some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{P}$-Brownian motion $\left(B^{1}, B^{2}\right)$, and $v$ is any random variable defined on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, with normal distribution $\mu \triangleq N\left(v_{0}, s_{0}\right)$, we define the product space

$$
\tilde{\Omega} \triangleq \Omega \otimes \mathbb{R}, \quad \tilde{\mathcal{F}} \triangleq \mathcal{F} \otimes \mathscr{B}(\mathbb{R}), \quad \tilde{\mathbb{P}} \triangleq \mathbb{P} \otimes \mu
$$

and write generic element of $\tilde{\omega} \in \tilde{\Omega}$ as $\tilde{\omega}=(\omega, x)$. Then for each $t \geq 0$, the mapping $\tilde{\omega} \mapsto$ $X_{t}^{x}(\omega)$ defines a random variable on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, and $x \mapsto X_{0}^{x} \triangleq v(x)$ is a normal random variable with distribution $N\left(v_{0}, s_{0}\right)$ and is independent of $\left(B^{1}, B^{2}\right)$, by definition. Bearing this in mind, throughout the section we shall assume that the initial state $X_{0}=x$ is deterministic.

### 4.1 Existence

Our main idea to prove the existence of the weak solution is to "decouple" the state and observation equations in (3.1) by considering the dynamics of the filtered state process $P_{t} \triangleq \mathbb{E}^{\mathbb{P}}\left[X_{t} \mid \mathcal{F}_{t}^{Y}\right]$, $t \geq 0$, which is known to satisfy an SDE, thanks to linear (Kalman-Bucy) filtering theory.

To be more precise, we consider the following system of SDEs on the reference probability
space $\left(\Omega, \mathcal{F}, \mathbb{Q}^{0}\right)$, on which $\left(B^{1}, Y\right)$ is a Brownian motion:

$$
\begin{cases}d X_{t}=\left[f_{t} X_{t}+g_{t} P_{t}+h_{t}\right] d t+d B_{t}^{1}, & X_{0}=x  \tag{4.1}\\ d B_{t}^{2}=d Y_{t}-\left[H_{t} X_{t}+G_{t} P_{t}\right] d t, & B_{0}^{2}=0 \\ d P_{t}=\left[\left(f_{t}+g_{t}\right) P_{t}+h_{t}\right] d t+S_{t} H_{t}\left\{d Y_{t}-\left[H_{t}+G_{t}\right] P_{t} d t\right\}, & P_{0}=x \\ d S_{t}=\left[2 f_{t} S_{t}-H_{t}^{2} S_{t}^{2}+1\right] d t, & S_{0}=0\end{cases}
$$

We note that by Assumption 3.1, all coefficients $f, g, H, G$ are deterministic and $h \in L_{\mathbb{F}^{Y}}^{2}(C([0, T]))$, it is easy to see that the linear system (4.1) has a (pathwisely) unique solution $\left(X_{t}, B_{t}^{2}, P_{t}\right)$ on $\left(\Omega, \mathcal{F}, \mathbb{Q}^{0}\right)$.

Now let $L=\left\{L_{t}\right\}_{t \geq 0}$ be the solution to the SDE:

$$
\begin{equation*}
d L_{t}=L_{t}\left(H_{t} X_{t}+G_{t} P_{t}\right) d Y_{t}, \quad L_{0}=1 \tag{4.2}
\end{equation*}
$$

Then $L$ is a positive $\mathbb{Q}^{0}$-local martingale, hence a $\mathbb{Q}^{0}$-supermartingale with $\mathbb{E}^{\mathbb{Q}^{0}}\left[L_{t}\right] \leq L_{0}=1$. Furthermore, $L$ can be written as the Doléans-Dade exponential:

$$
\begin{equation*}
L_{t}=\exp \left\{\int_{0}^{t}\left(H_{s} X_{s}+G_{s} P_{s}\right) d Y_{s}-\frac{1}{2} \int_{0}^{t}\left|H_{s} X_{s}+G_{s} P_{s}\right|^{2} d s\right\}, \quad t \in[0, T] \tag{4.3}
\end{equation*}
$$

We have the following lemma.
Lemma 4.1. Assume that Assumption 3.1 holds, and further that $h$ is bounded. Then the process $L=\left\{L_{t} ; t \geq 0\right\}$ is a true $\left(\mathbb{F}, \mathbb{Q}^{0}\right)$-martingale on $[0, T]$.

Proof. We follow the idea of that in [5]. Since $L$ is a supermartingale with $\mathbb{E}^{\mathbb{Q}^{0}}\left[L_{t}\right] \leq 1$, we need only show that $\mathbb{E}^{\mathbb{Q}^{0}}\left[L_{t}\right]=1$, for all $t \geq 0$. To this end, we define, for any $\varepsilon>0$,

$$
L_{t}^{\varepsilon} \triangleq \frac{L_{t}}{1+\varepsilon L_{t}}, \quad t \in[0, T] .
$$

Then clearly $0 \leq L_{t}^{\varepsilon} \leq L_{t} \wedge \frac{1}{\varepsilon}$, and an easy application of Itô's formula shows that

$$
\begin{equation*}
d L_{t}^{\varepsilon}=-\frac{\varepsilon L_{t}^{2}\left[H_{t} X_{t}+G_{t} P_{t}\right]^{2}}{\left(1+\varepsilon L_{t}\right)^{3}} d t+\frac{L_{t}\left[H_{t} X_{t}+G_{t} P_{t}\right]}{\left(1+\varepsilon L_{t}\right)^{2}} d Y_{t}, \quad t \geq 0 ; \quad L_{0}^{\varepsilon}=\frac{1}{1+\varepsilon} . \tag{4.4}
\end{equation*}
$$

Since for each fixed $\varepsilon>0$,

$$
\left|\frac{L_{t}\left[H_{t} X_{t}+G_{t} P_{t}\right]}{\left(1+\varepsilon L_{t}\right)^{2}}\right|^{2}=\left|\frac{\varepsilon L_{t}\left[H_{t} X_{t}+G_{t} P_{t}\right]}{\varepsilon\left(1+\varepsilon L_{t}\right)^{2}}\right|^{2} \leq \frac{\left[H_{t} X_{t}+G_{t} P_{t}\right]^{2}}{\varepsilon},
$$

we see that the stochastic integral on the right hand side of (4.4) is a true martingale. It then follows that

$$
\begin{equation*}
\mathbb{E}^{\mathbb{Q}^{0}}\left[L_{t}^{\varepsilon}\right]=\frac{1}{1+\varepsilon}-\mathbb{E}^{\mathbb{Q}^{0}}\left[\int_{0}^{t} \frac{\varepsilon L_{t}^{2}\left[H_{t} X_{t}+G_{t} P_{t}\right]^{2}}{\left(1+\varepsilon L_{t}\right)^{3}} d t\right] \tag{4.5}
\end{equation*}
$$

Next, we observe that $L_{t}>0$, and

$$
0 \leq \frac{\varepsilon L_{t}^{2}\left[H_{t} X_{t}+G_{t} P_{t}\right]^{2}}{\left(1+\varepsilon L_{t}\right)^{3}}=\frac{\left(\varepsilon L_{t}\right) L_{t}\left[H_{t} X_{t}+G_{t} P_{t}\right]^{2}}{\left(1+\varepsilon L_{t}\right)^{3}} \leq L_{t}\left[H_{t} X_{t}+G_{t} P_{t}\right]^{2}
$$

Note that $L^{\varepsilon}$ is bounded. By sending $\varepsilon \rightarrow 0$ on both sides of (4.5) and applying Dominated Convergence Theorem we can then conclude that $\mathbb{E}^{\mathbb{Q}^{0}}\left[L_{t}\right]=1$, provided

$$
\begin{equation*}
\mathbb{E}^{\mathbb{Q}^{0}}\left[\int_{0}^{T} L_{t}\left[H_{t} X_{t}+G_{t} P_{t}\right]^{2} d t\right]<\infty \tag{4.6}
\end{equation*}
$$

It remains to check (4.6). To this end, let us define $X_{t}=X_{t}^{1}+\alpha_{t}$, where

$$
\begin{cases}d \alpha_{t}=f_{t} \alpha_{t} d t+d B_{t}^{1}, & \alpha_{0}=x  \tag{4.7}\\ d X_{t}^{1}=\left[f_{t} X_{t}^{1}+g_{t} P_{t}+h_{t}\right] d t, & X_{0}^{1}=0\end{cases}
$$

By Gronwall's inequality, it is readily seen that

$$
\begin{equation*}
\left|X_{t}^{1}\right| \leq C \int_{0}^{t}\left|g_{s} P_{s}+h_{s}\right| d s \leq C\left[1+\int_{0}^{t}\left|P_{s}\right| d s\right], \quad t \in[0, T] . \tag{4.8}
\end{equation*}
$$

Here and in the sequel $C>0$ denotes a generic constant depending only on the bounds of the coefficients $f, g, H, G, h$, and the duration $T>0$, which is allowed to vary from line to line. Now, noting that $L_{t}$ is a super-martingale with $L_{0}=1$, we deduce from (4.8) that

$$
\begin{align*}
\mathbb{E}^{\mathbb{Q}^{0}}\left[\int_{0}^{T} L_{t}\left|X_{t}^{1}\right|^{2} d t\right] & \leq C\left\{1+\mathbb{E}^{\mathbb{Q}^{0}}\left[\int_{0}^{T} \int_{0}^{t} L_{t}\left|P_{s}\right|^{2} d s d t\right]\right\}  \tag{4.9}\\
& \leq C\left\{1+\mathbb{E}^{\mathbb{Q}^{0}}\left[\int_{0}^{T} L_{s}\left|P_{s}\right|^{2} d s\right]\right\}
\end{align*}
$$

Consequently we have

$$
\begin{align*}
\mathbb{E}^{\mathbb{Q}^{0}}\left[\int_{0}^{T} L_{t}\left[H_{t} X_{t}+G_{t} P_{t}\right]^{2} d t\right] & \leq C \mathbb{E}^{\mathbb{Q}^{0}}\left[\int_{0}^{T} L_{t}\left[\left|X_{t}^{1}\right|^{2}+\left|\alpha_{t}\right|^{2}+\left|P_{t}\right|^{2}\right] d t\right]  \tag{4.10}\\
& \leq C\left\{1+\mathbb{E}^{\mathbb{Q}^{0}}\left[\int_{0}^{T} L_{t}\left(\left|\alpha_{t}\right|^{2}+\left|P_{t}\right|^{2}\right) d t\right]\right\}
\end{align*}
$$

Continuing, let us recall that the processes $P$ and $\alpha$ satisfy (4.1) and (4.7), respectively. By Itô's formula we see that

$$
\left\{\begin{align*}
d\left|\alpha_{t}\right|^{2} & =\left[2 f_{t}\left|\alpha_{t}\right|^{2}+1\right] d t+2 \alpha_{t} d B_{t}^{1}  \tag{4.11}\\
d\left|P_{t}\right|^{2} & =\left[2 M_{t}\left|P_{t}\right|^{2}+2 P_{t} h_{t}+S_{t}^{2} H_{t}^{2}\right] d t+2 S_{t} H_{t} P_{t} d Y_{t}
\end{align*}\right.
$$

where $M_{t} \triangleq\left(f_{t}+g_{t}\right)-S_{t} H_{t}\left(H_{t}+G_{t}\right), t \geq 0$. Next, we define, for $\delta>0$ and $t \in[0, T]$,

$$
X_{t}^{\delta}=\frac{X_{t}}{\left[1+\delta\left|X_{t}\right|^{2}\right]^{1 / 2}} ; \quad P_{t}^{\delta}=\frac{P_{t}}{\left[1+\delta\left|P_{t}\right|^{2}\right]^{1 / 2}}
$$

Then $\left|X_{t}^{\delta}\right| \leq\left|X_{t}\right| \wedge \delta^{-\frac{1}{2}}$ and $\left|P_{t}^{\delta}\right| \leq\left|P_{t}\right| \wedge \delta^{-\frac{1}{2}}, \forall t$; and it is not hard to show that $\lim _{\delta \rightarrow 0} X^{\delta}=X$, $\lim _{\delta \rightarrow 0} P^{\delta}=P$, uniformly on $[0, T]$, in probability. Now, define

$$
\begin{equation*}
d L_{t}^{\delta}=L_{t}^{\delta}\left[H_{t} X_{t}^{\delta}+G_{t} P_{t}^{\delta}\right] d Y_{t}, \quad L_{0}^{\delta}=1 \tag{4.12}
\end{equation*}
$$

Since $X^{\delta}$ and $P^{\delta}$ are now bounded, $L^{\delta}$ is a martingale and $\mathbb{E}^{\mathbb{Q}^{0}}\left[L_{t}^{\delta}\right]=1, t \in[0, T]$. Furthermore, by the stability of SDEs one shows that, possibly along a subsequence, $L_{t}^{\delta}$ converges to $L_{t}, \mathbb{Q}^{0}$-a.s., $t \in[0, T]$.

Noting (4.11) and applying Itô's formula we have, for $t \in[0, T]$

$$
L_{t}^{\delta}\left|\alpha_{t}\right|^{2}=x^{2}+\int_{0}^{t} L_{s}^{\delta}\left[2 f_{s}\left|\alpha_{s}\right|^{2}+1\right] d s+\int_{0}^{t} 2 L_{s}^{\delta} \alpha_{s} d B_{s}^{1}+\int_{0}^{t} L_{s}^{\delta}\left|\alpha_{s}\right|^{2}\left[H_{s} X_{s}^{\delta}+G_{s} P_{s}^{\delta}\right] d Y_{s}
$$

Since $\alpha$ has finite moments for all orders (see (4.7)), the boundedness of $X^{\delta}$ and $P^{\delta}$ then renders the two stochastic integrals on the right hand side above both true martingales. Thus, taking expectations on both sides above, and applying Gronwall's inequality, we get

$$
\begin{equation*}
\mathbb{E}^{\mathbb{Q}^{0}}\left[L_{t}^{\delta}\left|\alpha_{t}\right|^{2}\right] \leq C, \quad \forall t \in[0, T], \tag{4.13}
\end{equation*}
$$

where $C$ is a constant independent of $\delta$. Applying Fatou's Lemma we then get that

$$
\begin{equation*}
\mathbb{E}^{\mathbb{Q}^{0}}\left[L_{t}\left|\alpha_{t}\right|^{2}\right] \leq \varliminf_{\delta \rightarrow 0} \mathbb{E}^{\mathbb{Q}^{0}}\left[L_{t}^{\delta}\left|\alpha_{t}\right|^{2}\right] \leq C . \tag{4.14}
\end{equation*}
$$

Finally, noting (4.11) and applying Itô's formula again we have

$$
\begin{align*}
d L_{t}^{\delta}\left|P_{t}\right|^{2}= & L_{t}^{\delta}\left[2 M_{t}\left|P_{t}\right|^{2}+2 P_{t} h_{t}+S_{t}^{2} H_{t}^{2}\right] d t+2 S_{t} H_{t} L_{t}^{\delta} P_{t} d Y_{t}  \tag{4.15}\\
& +L_{t}^{\delta}\left|P_{t}\right|^{2}\left[H_{t} X_{t}^{\delta}+G_{t} P_{t}^{\delta}\right] d Y_{t}+2 S_{t} H_{t} L_{t}^{\delta} P_{t}\left[H_{t} X_{t}^{\delta}+G_{t} P_{t}^{\delta}\right] d t .
\end{align*}
$$

By similar arguments as before, and noting that $\left|X_{t}^{\delta}\right| \leq\left|X_{t}\right|$ and $\left|P^{\delta}\right| \leq\left|P_{t}\right|$, one shows

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}^{0}}\left[L_{t}^{\delta}\left|P_{t}\right|^{2}\right] & \leq C\left\{1+\mathbb{E}^{\mathbb{Q}^{0}}\left[\int_{0}^{t} L_{s}^{\delta}\left[\left|P_{s}\right|^{2}+\left|X_{s}\right|^{2}\right] d s\right]\right\} \\
& \leq C\left\{1+\mathbb{E}^{\mathbb{Q}^{0}}\left[\int_{0}^{t} L_{s}^{\delta}\left[\left|P_{s}\right|^{2}+\left|X_{s}^{1}\right|^{2}+\left|\alpha_{s}\right|^{2}\right] d s\right]\right\}, \quad t \in[0, T] .
\end{aligned}
$$

This, together with (4.9), implies that

$$
\mathbb{E}^{\mathbb{Q}^{0}}\left[L_{t}^{\delta}\left|P_{t}\right|^{2}\right] \leq C\left\{1+\mathbb{E}^{\mathbb{Q}^{0}}\left[\int_{0}^{t} L_{s}^{\delta}\left[\left|P_{s}\right|^{2}+\left|\alpha_{s}\right|^{2}\right] d s\right]\right\}, \quad t \in[0, T] .
$$

Applying the Gronwall inequality and recalling (4.13) we then obtain

$$
\begin{equation*}
\mathbb{E}^{\mathbb{Q}^{0}}\left[L_{t}^{\delta}\left|P_{t}\right|^{2}\right] \leq C\left\{1+\mathbb{E}^{\mathbb{Q}^{0}}\left[\int_{0}^{t} L_{s}^{\delta}\left|\alpha_{s}\right|^{2} d s\right]\right\} \leq C, \quad t \in[0, T] . \tag{4.16}
\end{equation*}
$$

By Fatou's lemma one again shows that $\mathbb{E}^{\mathbb{Q}^{0}}\left[L_{t}\left|P_{t}\right|^{2}\right] \leq C$, for all $t \in[0 . T]$. This, together with (4.14) and (4.10), leads to (4.6). The proof is now complete.

We can now complete the proof of existence. Since $L_{t}$ is a $\left(\mathbb{F}, \mathbb{Q}^{0}\right)$ martingale, we define a probability measure $\mathbb{P}$ by $\left.\frac{d \mathbb{P}}{d \mathbb{Q}^{0}}\right|_{\mathcal{F}_{T}}=L_{T}$, and apply the Girsanov Theorem so that $\left(B^{1}, B^{2}\right)$ is a $\mathbb{P}$-Brownian motion on $[0, T]$. Now, by looking at the first two equations of (4.1), we see that $\left(\Omega, \mathcal{F}, \mathbb{P}, X, Y, B^{1}, B^{2}\right)$ would be a weak solution to (3.1) if we can show that

$$
\begin{equation*}
P_{t}=\mathbb{E}^{\mathbb{P}}\left[X_{t} \mid \mathcal{F}_{t}^{Y}\right], \quad t \in[0, T], \quad \mathbb{P} \text {-a.s. } \tag{4.17}
\end{equation*}
$$

To prove (4.17) we proceed as follows. We consider the following linear filtering problem on the space $(\Omega, \mathcal{F}, \mathbb{P})$ :

$$
\begin{cases}d \alpha_{t}=f_{t} \alpha_{t} d t+d B_{t}^{1}, & \alpha_{0}=x  \tag{4.18}\\ d \beta_{t}=H_{t} \alpha_{t} d t+d B_{t}^{2}, & \beta_{0}=0\end{cases}
$$

Denote $\widehat{\alpha}_{t}=\mathbb{E}^{\mathbb{P}}\left[\alpha_{t} \mid \mathcal{F}_{t}^{\beta}\right], t \geq 0$. Then by linear filtering theory, we know that $\widehat{\alpha}$ satisfies the following SDE:

$$
\begin{equation*}
d \widehat{\alpha}_{t}=f_{t} \widehat{\alpha}_{t} d t+S_{t} H_{t}\left\{d \beta_{t}-H_{t} \widehat{\alpha}_{t} d t\right\}, \quad \widehat{\alpha}_{0}=x, \tag{4.19}
\end{equation*}
$$

where $S_{t}$ satisfies (3.3) (or (4.1)). On the other hand, from (4.18) we see that $\alpha$ is $\mathbb{F}^{B^{1}}$-adapted, and from (4.1) we see that $P$ is $\mathbb{F}^{Y}$-adapted, therefore we can apply Lemma 3.7 to conclude that $\mathbb{E}^{\mathbb{P}}\left[\alpha_{t} \mid \mathcal{F}_{t}^{Y}\right]=\mathbb{E}^{\mathbb{P}}\left[\alpha_{t} \mid \mathcal{F}_{t}^{\beta}\right]=\widehat{\alpha}_{t}, t \in[0, T]$.

Now let us define $\widetilde{P}_{t}=\mathbb{E}^{\mathbb{P}}\left[X_{t} \mid \mathcal{F}_{t}^{Y}\right], t \geq 0$. Recall that $X=X^{1}+\alpha$, where $X^{1}$ satisfies a randomized ODE (4.7) and is obviously $\mathbb{F}^{Y}$-adapted, we see that $\widetilde{P}=X^{1}+\widehat{\alpha}$, and it satisfies the SDE:

$$
\begin{equation*}
d \widetilde{P}_{t}=\left[f_{t} \widetilde{P}_{t}+g_{t} P_{t}+h_{t}\right] d t+S_{t} H_{t}\left\{d \beta_{t}-H_{t} \widehat{\alpha}_{t} d t\right\}, \quad t \geq 0 ; \quad \widetilde{P}_{0}=x \tag{4.20}
\end{equation*}
$$

Note that $X=X^{1}+\alpha$ and $\tilde{P}=X^{1}+\widehat{\alpha}$ we see that

$$
d \beta_{t}-H(t) \widehat{\alpha}_{t} d t=H_{t}\left(\alpha_{t}-\widehat{\alpha}_{t}\right) d t+d B_{t}^{2}=d Y_{t}-\left[H_{t} \widetilde{P}_{t}+G_{t} P_{t}\right] d t .
$$

Then (4.20) implies that

$$
\begin{equation*}
d \widetilde{P}_{t}=\left[f_{t} \widetilde{P}_{t}+g_{t} P_{t}+h_{t}\right] d t+S_{t} H_{t}\left\{d Y_{t}-\left[H_{t} \widetilde{P}_{t}+G_{t} P_{t}\right] d t\right\} . \tag{4.21}
\end{equation*}
$$

Define $\Delta P_{t}=P_{t}-\widetilde{P}_{t}$, then it follows from (4.1) and (4.21) that

$$
d \Delta P_{t}=\left[f_{t}-S_{t} H_{t}^{2}\right] \Delta P_{t} d t, \quad \Delta P_{0}=0
$$

Thus $\Delta P_{t} \equiv 0, \mathbb{P}$-a.s., for any $t \geq 0$. That is, (4.17) holds, proving the existence.
It is worth noting that the weak solution that we have constructed is actually a $\mathbb{Q}^{0}$-weak solution.

### 4.2 Uniqueness

Again we need only consider the solutions with deterministic initial state. We first note that if $\left(\Omega^{0}, \mathcal{F}^{0}, \mathbb{P}, \mathbb{F}^{0}, X, Y, B^{1}, B^{2}\right)$ is a $\mathbb{Q}^{0}$-weak solution to (3.1), then we can assume without loss of generality that $\mathbb{F}^{0}=\mathbb{F}^{B^{1}, B^{2}}$, hence Brownian. Next, we can define $\widetilde{P}_{t}=\mathbb{E}^{\mathbb{P}}\left[X_{t} \mid \mathcal{F}_{t}^{Y}\right]$. We are to show that $\widetilde{P}_{t}$ satisfies an SDE of the form as that in (4.1) under $\mathbb{Q}^{0}$, from which we shall derive the $\mathbb{Q}^{0}$-pathwise uniqueness.

To this end, we recall that, as a $\mathbb{Q}^{0}$-weak solution, one has $\mathbb{P} \sim \mathbb{Q}^{0}$. Define a $\mathbb{P}$-martingale $Z_{t} \triangleq \mathbb{E}^{\mathbb{P}}\left[\left.\frac{d \mathbb{Q}^{0}}{d \mathbb{P}} \right\rvert\, \mathcal{F}_{t}\right], t \geq 0$. Since $\left(B^{1}, Y\right)$ is a $\mathbb{P}$-semi-martingale with decomposition:

$$
\begin{equation*}
B_{t}^{1}=B_{t}^{1} ; \quad Y_{t}=\int_{0}^{t}\left(H_{s} X_{s}+G_{s} \widetilde{P}_{s}\right) d s+B_{t}^{2}, \quad t \geq 0 \tag{4.22}
\end{equation*}
$$

By Girsanov-Meyer Theorem (see, e.g., [25, Theorem III-20]), it is a $\mathbb{Q}^{0}$-semi-martingale with the decomposition $\left(B_{t}^{1}, Y_{t}\right)=\left(N_{t}^{1}, N_{t}^{2}\right)+\left(A_{t}^{1}, A_{t}^{2}\right)$, where $N=\left(N^{1}, N^{2}\right)$ is a $\mathbb{Q}^{0}$-local martingale of the form

$$
N_{t}^{1}=B_{t}^{1}-\int_{0}^{t} \frac{1}{Z_{s}} d\left[Z, B^{1}\right]_{s}, \quad N_{t}^{2}=B_{t}^{2}-\int_{0}^{t} \frac{1}{Z_{s}} d\left[Z, B^{2}\right]_{s}, \quad t \geq 0
$$

and $A=\left(A^{1}, A^{2}\right)$ is a finite variation process. Since by assumption $\left(B^{1}, Y\right)$ is $\mathbb{Q}^{0}$-Brownian motion, we have $A \equiv 0$. In other words, it must hold that

$$
\begin{equation*}
B_{t}^{1}=B_{t}^{1}-\int_{0}^{t} \frac{1}{Z_{s}} d\left[Z, B^{1}\right]_{s}, \quad Y_{t}=B_{t}^{2}-\int_{0}^{t} \frac{1}{Z_{s}} d\left[Z, B^{2}\right]_{s}, \quad t \geq 0 . \tag{4.23}
\end{equation*}
$$

Consider now a $(\mathbb{F}, \mathbb{P})$-martingale $d M_{t}=Z_{t}^{-1} d Z_{t}$. Since $\mathbb{F}$ is Brownian, applying Martingale Representation Theorem we see that there exists a process $\theta=\left(\theta^{1}, \theta^{2}\right) \in L_{\mathbb{F}}^{2}([0, T])$ such that $d M_{t}=\theta_{t}^{1} d B_{t}^{1}+\theta_{t}^{2} d B_{t}^{2}, t \in[0, T]$. Thus (4.23) amounts to saying that

$$
\left[M, B^{1}\right]_{t}=\int_{0}^{t} \theta_{s}^{1} d s \equiv 0 ; \quad Y_{t}=B_{t}^{2}-\left[M, B^{2}\right]_{t}=B_{t}^{2}-\int_{0}^{t} \theta_{s}^{2} d s, \quad t \geq 0
$$

Comparing this to (4.22) we have $\theta^{1} \equiv 0$ and $\theta^{2} \equiv-(H X+G \widetilde{P})$. That is, $Z=L^{-(H X+G \widetilde{P})}$, the solution to the SDE:

$$
d Z_{t}=Z_{t} d M_{t}=-Z_{t}\left(H_{t} X_{t}+G_{t} \widetilde{P}_{t}\right) d B_{t}^{2}, \quad t \in[0, T], \quad Z_{0}=1,
$$

and hence it can be written as the Doléans-Dade stochastic exponential:

$$
\begin{equation*}
Z_{t}=\exp \left\{-\int_{0}^{t}\left(H_{s} X_{s}+G_{s} \widetilde{P}_{s}\right) d B_{s}^{2}-\frac{1}{2} \int_{0}^{t}\left|H_{s} X_{s}+G_{s} \widetilde{P}_{s}\right|^{2} d s\right\} \tag{4.24}
\end{equation*}
$$

Let us now consider again the following filtering problem on probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

$$
\begin{cases}d \alpha_{t}=f_{t} \alpha_{t} d t+d B_{t}^{1}, & \alpha_{0}=x  \tag{4.25}\\ d \beta_{t}=\left[H_{t} \alpha_{t}\right] d t+d B_{t}^{2}, & \beta_{0}=0 .\end{cases}
$$

As before, we know that $\widehat{\alpha}_{t}=\mathbb{E}^{\mathbb{P}}\left[\alpha_{t} \mid \mathcal{F}_{t}^{\beta}\right]$ satisfies the SDE :

$$
\begin{equation*}
d \widehat{\alpha}_{t}=f_{t} \widehat{\alpha}_{t} d t+S_{t} H_{t}\left\{d \beta_{t}-H_{t} \widehat{\alpha}_{t} d t\right\}, \quad \widehat{\alpha}_{0}=x, \tag{4.26}
\end{equation*}
$$

where $S_{t}$ satisfies (3.3). Since $L^{-(H X+G \widetilde{P})}=Z$ is a $\mathbb{P}$-martingale, we can apply Lemma 3.7 again to conclude that $\mathbb{E}^{\mathbb{P}}\left[\alpha_{t} \mid \mathcal{F}_{t}^{Y}\right]=\mathbb{E}^{\mathbb{P}}\left[\alpha_{t} \mid \mathcal{F}_{t}^{\beta}\right]=\widehat{\alpha}_{t}$.

Now let $X_{t}=\alpha_{t}+X_{t}^{1}$, and $Y_{t}=\beta_{t}+Y_{t}^{1}$ as before, where $X^{1}$ satisfies the ODE (4.7), and $Y^{1}$ satisfies the ODE

$$
\begin{equation*}
d Y_{t}^{1}=\left[H_{t} X_{t}^{1}+G_{t} \widetilde{P}_{t}\right] d t, \quad Y_{0}^{1}=0 . \tag{4.27}
\end{equation*}
$$

Furthermore, since $X_{t}^{1}$ is $\mathbb{F}^{Y}$-adapted, we have $\widetilde{P}_{t}=X_{t}^{1}+\widehat{\alpha}_{t}$. Combining (4.7) and (4.26) we see that $\widetilde{P}_{t}$ satisfies the SDE:

$$
\begin{equation*}
d \widetilde{P}_{t}=\left[\left(f_{t}+g_{t}\right) \widetilde{P}_{t}+h_{t}\right] d t+S_{t} H_{t}\left\{d \beta_{t}-H_{t} \widehat{\alpha}_{t} d t\right\}, \quad \widetilde{P}_{0}=x \tag{4.28}
\end{equation*}
$$

Since $\beta=Y-Y^{1}$, we derive from (4.27) that

$$
d \beta_{t}-H_{t} \widehat{\alpha}_{t} d t=d Y_{t}-d Y_{t}^{1}-H_{t} \widehat{\alpha}_{t} d t=d Y_{t}-\left[H_{t} \widetilde{P}_{t}+G_{t} \widetilde{P}_{t}\right] d t
$$

and (4.28) becomes

$$
\begin{equation*}
d \widetilde{P}_{t}=\left[\left(f_{t}+g_{t}\right) \widetilde{P}_{t}+h_{t}\right] d t+S_{t} H_{t}\left\{d Y_{t}-\left[\left(H_{t}+G_{t}\right) \widetilde{P}_{t}\right] d t\right\}, \quad t \in[0, T], \quad \widetilde{P}_{0}=x . \tag{4.29}
\end{equation*}
$$

That is, $\widetilde{P}_{t}$ satisfies the same SDE as $P_{t}$ does in (4.1) on the reference space $\left(\Omega, \mathcal{F}, \mathbb{Q}^{0}\right)$.
To finish the argument, let $\left(\Omega, \mathcal{F}, \mathbb{P}^{i}, \mathbb{F}, X^{i}, Y^{i}, B^{1, i}, B^{2, i}\right), i=1,2$ be any two $\mathbb{Q}^{0}$-weak solutions, and define $\widetilde{P}_{t}^{i} \triangleq \mathbb{E}^{\mathbb{P}^{i}}\left[X_{t}^{i} \mid \mathcal{F}_{t}^{Y^{i}}\right], t \geq 0, i=1,2$. Then the arguments above show that $\left(X^{i}, B^{2, i}, \widetilde{P}^{i}\right), i=1,2$, are two solutions to the linear system of SDEs (4.1), under $\mathbb{Q}^{0}$. Thus if $\left(B^{1,1}, Y^{1}\right) \equiv\left(B^{1,2}, Y^{2}\right)$ under $\mathbb{Q}^{0}$, then we must have $\left(X^{1}, B^{2,1}, \tilde{P}^{1}\right) \equiv\left(X^{2}, B^{2,2}, \tilde{P}^{2}\right)$, under $\mathbb{Q}^{0}$, which in turn shows, in light of (4.24), that $\mathbb{P}^{1}=\mathbb{P}^{2}$. This proves the $\mathbb{Q}^{0}$-pathwise uniqueness of solutions to (3.1).

## 5 Necessary Conditions for Optimal Trading Strategy

In this section we study the optimization problem (2.7). We still denote the price dynamics observable by the insider to be $V=\left\{V_{t}: t \geq 0\right\}$, and assume that it satisfies the SDE:

$$
\left\{\begin{array}{l}
d V_{t}=\left[f_{t} V_{t}+g_{t} \mathbb{E}^{\mathbb{P}}\left[V_{t} \mid \mathcal{F}_{t}^{Y}\right]+h_{t}\right] d t+\sigma_{t}^{v} d B_{t}^{v}, \quad t \in[0, T],  \tag{5.1}\\
V_{0}=v \sim N\left(v_{0}, s_{0}\right) ;
\end{array}\right.
$$

and we assume that the demand dynamics observable by the market makers, denoted by $Y=$ $\left\{Y_{t}: t \geq 0\right\}$, satisfies the SDE

$$
\begin{equation*}
d Y_{t}=\left[\beta_{t}\left(V_{t}-\mathbb{E}^{\mathbb{P}}\left[V_{t} \mid \mathcal{F}_{t}^{Y}\right]\right)\right] d t+\sigma_{t}^{z} d B_{t}^{z}, \quad t \in[0, T] ; \quad Y_{0}=0 . \tag{5.2}
\end{equation*}
$$

We should note that in (5.1) and (5.2) the probability $\mathbb{P}$ should be understood as one defined on the canonical space $\left(\Omega^{0}, \mathcal{F}^{0}, \mathbb{F}^{0}\right)$, on which the solution to (5.1) and (5.2) is $\mathbb{Q}^{0}$-pathwisely unique. For notational simplicity, from now on we shall denote $\mathbb{E}=\mathbb{E}^{\mathbb{P}}$, when there is no danger of confusion. Moreover, note that

$$
\mathbb{E}\left[P_{t}\left(V_{t}-P_{t}\right)\right]=\mathbb{E}\left[\mathbb{E}\left[\left(V_{t}-P_{t}\right) P_{t} \mid \mathcal{F}_{t}^{Y}\right]\right]=0,
$$

and that all the coefficients are now assumed to be deterministic, we can apply Proposition 3.8 to write the problem (2.7) as

$$
\begin{align*}
J(\beta)= & \int_{0}^{T} \beta_{t} \mathbb{E}\left[\left(V_{t}-P_{t}\right) P_{T}\right] d t \\
= & \phi_{2}(T, 0) \int_{0}^{T} \beta_{t} \phi_{1}(t, 0)\left\{s_{0} \phi_{3}(T, 0)\right.  \tag{5.3}\\
& \left.+\int_{0}^{t}\left[\left(l_{r}^{2}+\left(\sigma_{r}^{v}\right)^{2}\right) \phi_{1}^{2}(0, r) \phi_{3}(T, r)-\phi_{1}(0, r) \phi_{2}(0, r) l_{r}^{2}\right] d r\right\} d t
\end{align*}
$$

where $\phi_{i}, i=1,2,3$, and $l, k$ are defined by (3.20) and (3.16), respectively, with $H=\beta$. Now by integration by parts we can easily check that

$$
\begin{aligned}
\int_{0}^{t}\left[\left(l_{r}^{2}+\right.\right. & \left.\left.\left(\sigma_{r}^{v}\right)^{2}\right) \phi_{1}^{2}(0, r) \phi_{3}(T, r)\right] d r=\int_{0}^{t} \phi_{3}(T, r) d\left[S_{r} \phi_{1}^{2}(0, r)\right] \\
& =\phi_{3}(T, t) S_{t} \phi_{1}^{2}(0, t)-\phi_{3}(T, 0) s_{0}+\int_{0}^{t} \phi_{1}(0, r) \phi_{2}(0, r) l_{r}^{2} d r
\end{aligned}
$$

we thus have $J(\beta)=\phi_{2}(T, 0) \int_{0}^{T} \beta_{t} S_{t} \phi_{1}(0, t) \phi_{3}(T, t) d t$, and the original optimal control problem (2.7) is equivalent to the following

$$
\begin{equation*}
\sup _{\beta} \bar{J}(\beta)=\sup _{\beta} \int_{0}^{T} \beta_{t} S_{t} \phi_{1}(0, t) \phi_{3}(T, t) d t . \tag{5.4}
\end{equation*}
$$

Before we proceed any further let us specify the "admissible strategy" and the standing assumptions on the coefficients that will be used throughout this section. We note that the assumptions will be slightly stronger than Assumption 3.1.

Assumption 5.1. (i) All coefficients $f, g, h, \sigma^{v}$, and $\sigma^{z}$ are deterministic, continuous functions on $[0, T]$, such that $\sigma_{t}^{z} \geq c, \sigma_{t}^{v} \geq c$ for all $t \in[0, T]$ for some constant $c>0$;
(ii) the trading intensity $\beta$ is continuous on $[0, T), \beta_{t}>0$ for all $t \in[0, T)$, and $\lim _{t \rightarrow T^{-}} \beta_{t}>0$ exists (it may be $+\infty$ ). Consequently, $\underline{\beta} \triangleq \inf _{t \in[0, T]} \beta_{t}>0$.

Remark 5.2. (i) In practice it is not unusual to assume that $\lim _{t \rightarrow T^{-}} \beta_{t}=\infty$, which amounts to saying that the insider is desperately trying to maximize the advantage of the asymmetric information (cf. e.g., [1]). We shall actually prove that this is the case for the optimal strategy, provided the Assumption 5.1-(ii) holds. In what follows we say that a trading intensity $\beta$ is admissible if it satisfies Assumption 5.1.(ii). By a slight abuse of notations we still denote all admissible trading intensities by $\mathscr{U}_{a d}$.
(ii) For $\beta \in \mathscr{U}_{a d}$, the well-posedness of CMFSDEs (5.1) and (5.2) should be understood in the sense of Theorem 3.11, and we shall consider its (unique) $\mathcal{Q}^{0}$-solution.

We note that the solution of the Riccati equation (3.17) $S$, as well as the functions $\phi_{1}$ and $\phi_{3}$ defined by (3.20), depends on the choice of trading intensity $\beta$. We shall at times denote them by $S^{\beta}, \phi_{1}^{\beta}$, and $\phi_{3}^{\beta}$, respectively, to emphasize their dependence on $\beta$. The following lemma is simple but useful for our analysis.

Lemma 5.3. Let Assumption 5.1 be in force. Then for any $\beta \in \mathscr{U}_{\text {ad }}$,
(i) the Riccati equation (3.17) has a solution $S=S^{\beta}$ defined on $[0, T)$, such that $S_{t}^{\beta}>0$, for all $t \in[0, T)$. Furthermore, there exists a constant $C_{\beta}>0$, depending on the bounds of the coefficients and $\underline{\beta}$ in Assumption 5.1, such that $S_{t}^{\beta} \leq s_{0}+C_{\beta} t, t \in[0, T]$;
(ii) $\left\|S^{\beta}\right\|_{\infty} \leq e^{K T}\left(s_{0}+K T\right)$, where $K \triangleq\left\|\sigma^{v}\right\|_{\infty}^{2}+2\|f\|_{\infty}$;
(iii) $S_{T^{-}}^{\beta} \triangleq \lim _{t \rightarrow T^{-}} S_{t}^{\beta}<\infty$, that is, the solution $S^{\beta}$ can be extended continuously to $[0, T]$;
(iv) $S_{T^{-}}^{\beta}=0$ if and only if $\lim _{t \rightarrow T^{-}} \phi_{1}^{\beta}(t, 0)=0$.

Proof. Let $\beta \in \mathscr{U}_{a d}$ be given, and denote $S=S^{\beta}$ and $\phi_{1}=\phi_{1}^{\beta}$, etc., throughout the proof for simplicitly.
(i) First note that if $S_{t}=0$ for some $t<T$, we define $\tau=\inf \left\{t \in[0, T) ; S_{t}=0\right\}$. Then, from (3.17) we see that at $\tau$ it holds that $\left.\frac{d S_{t}}{d t}\right|_{t=\tau}=\left|\sigma_{\tau}^{v}\right|^{2}>0$. But on the other hand by definition of $\tau$ we must have $\frac{S_{\tau}-S_{\tau-h}}{h}=-\frac{S_{\tau-h}}{h}<0$ for $h>0$ small enough, a contradiction. That is, $S_{t}>0$, $\forall t \in[0, T)$.

Next, let us denote, for $(t, s, \beta) \in[0, T] \times(0, \infty) \times[0, \infty)$, the right side of (3.17) by

$$
G(t, s, \beta) \triangleq\left(\sigma_{t}^{v}\right)^{2}+2 f_{t} s-\left[\frac{\beta s}{\sigma_{t}^{z}}\right]^{2}
$$

Then for any $\beta \in \mathscr{U}_{a d}$, it holds that

$$
\begin{equation*}
G\left(t, s, \beta_{t}\right)=-\frac{\beta_{t}^{2}}{\left|\sigma_{t}^{z}\right|^{2}}\left[s-\frac{f_{t}\left|\sigma_{t}^{z}\right|^{2}}{\beta_{t}^{2}}\right]^{2}+\frac{f_{t}^{2}\left|\sigma_{\sigma}^{z}\right|^{2}}{\beta_{t}^{2}}+\left|\sigma_{t}^{v}\right|^{2} \leq \max _{t \in[0, T]}\left\{\frac{f_{t}^{2}\left|\sigma_{t}^{z}\right|^{2}}{\underline{\beta}^{2}}+\left|\sigma_{t}^{v}\right|^{2}\right\} \triangleq C_{\beta}, \tag{5.5}
\end{equation*}
$$

thanks to Assumption [5.1. Thus $S_{t} \leq s_{0}+C_{\beta} t$, for all $t \in[0, T]$, proving (i).
(ii) To find the bound that is independent of the choice of $\beta$, we note that

$$
\frac{d S_{t}}{d t}=G\left(t, S_{t}, \beta_{t}\right) \leq\left(\sigma_{t}^{v}\right)^{2}+2\left|f_{t}\right| S_{t} \leq K\left(1+S_{t}\right), \quad \forall t \in[0, T]
$$

where $K \triangleq\left\|\sigma^{v}\right\|_{\infty}^{2}+2\|f\|_{\infty}$. Thus the result then follows from the Gronwall's inequality.
(iii) Since $G$ is quadratic in $s$, and $\lim _{s \rightarrow \infty} G(t, s, \beta)=-\infty$, it is easy to see from (5.5) that, for any given $\beta \in \mathscr{U}_{a d}$,

$$
\begin{equation*}
\max _{t, s} G^{+}\left(t, s, \beta_{t}\right)=\max _{t, s} G\left(t, s, \beta_{t}\right) \leq C_{\beta} . \tag{5.6}
\end{equation*}
$$

On the other hand, we write

$$
S_{t}-s_{0}=\int_{0}^{t} G\left(r, S_{r}, \beta_{r}\right) d r=\int_{0}^{t} G^{+}\left(r, S_{r}, \beta_{r}\right) d r-\int_{0}^{t} G^{-}\left(r, S_{r}, \beta_{r}\right) d r \triangleq I^{+}(t)-I^{-}(t)
$$

where $I^{ \pm}$are defined in an obvious way. Since $I^{+}(\cdot)$ and $I^{-}(\cdot)$ are monotone increasing, both limits $I^{+}\left(T^{-}\right)$and $I^{-}\left(T^{-}\right)$exist, which may be $+\infty$. But (5.6) implies that $I^{+}\left(T^{-}\right)<\infty$, and by (i), $I^{-}(t)=I^{+}(t)-S_{t}+s_{0}<I^{+}(t)+s_{0}$, for all $t \in[0, T)$, we conclude that $I^{-}\left(T^{-}\right)<\infty$ as well. That is, $S_{T^{-}} \geq 0$ exists.
(iv) We rewrite the equation (3.17) as follows (recall the definition of $\phi_{1}$ (3.20)),

$$
\begin{align*}
S_{t} & =\exp \left(\log S_{t}\right)=s_{0} \exp \left\{\int_{0}^{t} \frac{d S_{t}}{S_{t}}\right\}=s_{0} \exp \left\{\int_{0}^{t}\left(2 f_{t}+\frac{\left(\sigma_{t}^{v}\right)^{2}}{S_{t}}-\frac{\beta_{t}^{2} S_{t}}{\left(\sigma_{t}^{z}\right)^{2}}\right) d t\right\}  \tag{5.7}\\
& =s_{0} \phi_{1}(t, 0) \exp \left\{\int_{0}^{t}\left(f_{t}+\frac{\left(\sigma_{t}^{v}\right)^{2}}{S_{t}}\right) d t\right\} .
\end{align*}
$$

Thus, the result follows easily from (iii). This completes the proof.
In the rest of the section we shall try to solve the optimization problem (5.4). We first note that by definition the quantity $S$ and hence $\phi_{1}$ and $\phi_{3}$ all depend on the choice of trading intensity function $\beta$. Therefore (5.4) is essentially a problem of calculus of variation. We shall proceed by first looking for the first order necessary conditions, and then find the conditions that are sufficient for us to determine the optimal strategy.

To begin with, let us denote, for any differentiable functional $F: \mathbb{C}([0, T]) \mapsto \mathbb{C}([0, T])$ and any $\beta, \xi \in \mathbb{C}([0, T])$, the directional derivative of $F$ at $\beta$ in the direction $\xi$ by

$$
\begin{equation*}
\nabla_{\xi} F(\beta)_{t}=\left.\frac{d}{d y} F\left(\beta_{t}+y \xi_{t}\right)\right|_{y=0} \tag{5.8}
\end{equation*}
$$

We first give some useful directional derivatives that will be used frequently in the sequel. Recall the solution $S$ to the Riccati equation (3.17), and the functions $\phi_{1}$ and $\phi_{3}$, defined by (3.20). Note that they are all functionals of the trading intensity $\beta \in \mathbb{C}([0, T])$.

Lemma 5.4. Let $\xi=\left\{\xi_{t}\right\}$ be an arbitrary continuous function on $[0, T]$. Then the following identities hold, provided all the directional derivatives exist:
(i) $\nabla_{\xi} \beta_{t}=\xi_{t}$;
(ii) $\nabla_{\xi} S_{t}=-\phi_{1}^{2}(t, 0) \int_{0}^{t} \nabla_{\xi} \beta_{r} \alpha_{r} S_{r} \phi_{1}^{2}(0, r) d r$; where $\alpha_{t}=\frac{2 \beta_{t} S_{t}}{\left(\sigma_{t}^{2}\right)^{2}}$;
(iii) $\nabla_{\xi} \phi_{1}(t, 0)=-\phi_{1}(t, 0) \int_{0}^{t}\left[\nabla_{\xi} \beta_{r} \alpha_{r}+\nabla_{\xi} S_{r} \rho_{r}\right] d r$; where $\rho_{t}=\frac{\beta_{t}^{2}}{\left(\sigma_{t}^{2}\right)^{2}}$;
(iv) $\nabla_{\xi} \phi_{1}(0, t)=\phi_{1}(0, t) \int_{0}^{t}\left[\nabla_{\xi} \beta_{r} \alpha_{r}+\nabla_{\xi} S_{r} \rho_{r}\right] d r ;$
$(v) \nabla_{\xi} \phi_{3}(T, t)=\int_{t}^{T}\left\{\nabla_{\xi} \phi_{1}(r, 0) \phi_{2}(0, r) k_{r}+\phi_{1}(r, 0) \phi_{2}(0, r)\left[\nabla_{\xi} \beta_{r} \alpha_{r}+\nabla_{\xi} S_{r} \rho_{r}\right]\right\} d r$.
Proof. (i) is obvious. (iii)-(v) follows directly from chain rule. We only prove (ii).
To see this, recall (3.17). We have

$$
S_{t}(\beta+y \xi)=s_{0}+\int_{0}^{t}\left(\sigma_{r}^{v}\right)^{2} d r+\int_{0}^{t}\left\{2 f_{r} S_{r}(\beta+y \xi)-\left[\frac{S_{r}(\beta+y \xi)\left(\beta_{r}+y \xi_{r}\right)}{\sigma_{t}^{z}}\right]^{2}\right\} d r
$$

and thus

$$
\begin{aligned}
\nabla_{\xi} S_{t} & =\left.\int_{0}^{t} \frac{d}{d y}\left\{2 f_{r} S_{r}(\beta+y \xi)-\left[\frac{S_{r}(\beta+y \xi)\left(\beta_{r}+y \xi_{r}\right)}{\sigma_{r}^{z}}\right]^{2}\right\}\right|_{y=0} d r \\
& =\int_{0}^{t}\left\{2 f_{r} \nabla_{\xi} S_{r}-\frac{2 S_{r} \beta_{r}}{\sigma_{r}^{z}}\left[\frac{\beta_{r} \nabla_{\xi} S_{r}}{\sigma_{r}^{z}}+\frac{S_{r} \xi_{r}}{\sigma_{r}^{z}}\right]\right\} d r
\end{aligned}
$$

Denote $S^{\nabla} \triangleq \nabla_{\xi} S$, we see that it satisfies an ODE

$$
\frac{d}{d t} S_{t}^{\nabla}=2\left[f_{t}-\frac{\beta_{t}^{2} S_{t}}{\left(\sigma_{t}^{z}\right)^{2}}\right] S_{t}^{\nabla}-\frac{2 \beta_{t} S_{t}^{2} \xi_{t}}{\left(\sigma_{t}^{z}\right)^{2}}=2\left[f_{t}-k_{t}\right] S_{t}^{\nabla}-\frac{2 \beta_{t} S_{t}^{2} \xi_{t}}{\left(\sigma_{t}^{z}\right)^{2}}, \quad S_{0}^{\nabla}=0
$$

Solving it and noting that $\xi=\nabla_{\xi} \beta$ and $\alpha_{t}=\frac{2 \beta_{t} S_{t}}{\left(\sigma_{t}^{2}\right)^{2}}$ we obtain

$$
S_{t}^{\nabla}=-\int_{0}^{t} \frac{2 \beta_{r} S_{r}^{2} \xi_{r}}{\left(\sigma_{r}^{z}\right)^{2}} \exp \left\{\int_{r}^{t} 2\left(f_{u}-k_{u}\right) d u\right\} d r=-\phi_{1}^{2}(t, 0) \int_{0}^{t} \nabla_{\xi} \beta_{r} \alpha_{r} S_{r} \phi_{1}^{2}(0, r) d r
$$

proving (ii), whence the lemma.
We are now ready to prove the following necessary conditions of optimal strategies for the original control problem (2.7).

Theorem 5.5. Assume that Assumption 5.1 is in force. Suppose that $\beta \in \mathscr{U}_{a d}$ is an optimal strategy of the problem (2.7), then
(1) it holds that

$$
\frac{\phi_{1}(t, 0)\left(\sigma_{t}^{z}\right)^{2}}{2 \beta_{t} S_{t}} \phi_{3}(T, t)+\frac{1}{S_{t}} \Phi_{t}=\int_{t}^{T}\left[\beta_{r} \phi_{1}(r, 0) \phi_{3}(T, r)+\frac{\beta_{r}^{2}}{\left(\sigma_{t}^{z}\right)^{2}} \Phi_{r}\right] d r
$$

where

$$
\Phi_{t}=\phi_{1}^{2}(t, 0) \int_{0}^{t} \beta_{r} s_{r} \phi_{1}(0, r) d r \int_{t}^{T} g_{r} \phi_{1}(r, 0) \phi_{2}(0, r) d r .
$$

(2) Furthermore, $\lim _{t \rightarrow T^{-}} \beta_{t}=\infty$, and consequently, $\lim _{t \rightarrow T^{-}} S_{t}=0$. In particular,

$$
P_{T}=\mathbb{E}^{\mathbb{P}}\left[V_{T} \mid \mathcal{F}_{T}^{Y}\right]=V_{T}, \quad \mathbb{P} \text {-a.s. }
$$

Proof. Suppose that $\beta \in \mathscr{U}_{a d}$ is an optimal strategy of the problem (2.7). Then it is also an optimal of the problem (5.4). Thus for any function $\xi \in \mathbb{C}([0, T])$, it holds that

$$
\nabla_{\xi} \bar{J}(\beta)=\left.\frac{d \bar{J}(\beta+y \xi)}{d y}\right|_{y=0}=0
$$

or equivalently,

$$
\begin{align*}
0=\int_{0}^{T} & {\left[\nabla_{\xi} \beta_{t} S_{t} \phi_{1}(0, t) \phi_{3}(T, t)+\beta_{t} \nabla_{\xi} S_{t} \phi_{1}(0, t) \phi_{3}(T, t)+\beta_{t} S_{t} \nabla_{\xi} \phi_{1}(0, t) \phi_{3}(T, t)\right.} \\
& \left.+\beta_{t} S_{t} \phi_{1}(0, t) \nabla_{\xi} \phi_{3}(T, t)\right] d t . \tag{5.9}
\end{align*}
$$

Then substituting $\nabla_{\xi} \beta, \nabla_{\xi} S$, and $\nabla_{\xi} \phi_{i}, i=1,2,3$ in Lemma 5.4 into (5.9) and changing the order of integration if necessary we obtain that, formally,

$$
\begin{equation*}
\frac{\phi_{1}(t, 0)\left(\sigma_{t}^{z}\right)^{2}}{2 \beta_{t} S_{t}} \phi_{3}(T, t)+\frac{1}{S_{t}} \Psi_{t}=\int_{t}^{T}\left[\beta_{r} \phi_{1}(r, 0) \phi_{3}(T, r)+\frac{\beta_{r}^{2}}{\left(\sigma_{t}^{z}\right)^{2}} \Psi_{r}\right] d r \tag{5.10}
\end{equation*}
$$

where

$$
\begin{equation*}
\Psi_{t}=\phi_{1}^{2}(t, 0)\left[\phi_{1}(t, 0) \phi_{2}(0, t)-\phi_{3}(T, t)\right] \int_{0}^{t} \beta_{r} S_{r} \phi_{1}(0, r) d r . \tag{5.11}
\end{equation*}
$$

To justify the identity (5.10) we now show that both sides of (5.10) are finite for any $\beta \in \mathscr{U}_{a d}$ (note that it is possible that $\beta_{t} \rightarrow \infty$, as $t \rightarrow T^{-}$). To this end, we first note that $\phi_{1}(t, 0)$ and $\phi_{2}(r, s)$ are bounded, and

$$
\begin{equation*}
\phi_{3}(T, t)=\phi_{1}(t, 0) \phi_{2}(0, t)-\phi_{1}(T, 0) \phi_{2}(0, T)-\int_{t}^{T} g_{r} \phi_{1}(r, 0) \phi_{2}(0, r) d r, \tag{5.12}
\end{equation*}
$$

thus it is also bounded, and clearly $\lim _{t \rightarrow T} \phi_{3}(T, t)=0$. Furthermore, we rewrite (5.11) as

$$
\begin{equation*}
\Psi_{t}=\phi_{1}^{2}(t, 0) G(t, T) \int_{0}^{t} \beta_{r} S_{r} \phi_{1}(0, r) d r \tag{5.13}
\end{equation*}
$$

where $G(t, T) \triangleq \phi_{1}(t, 0) \phi_{2}(0, t)-\phi_{3}(T, t)=\phi_{1}(T, 0) \phi_{2}(0, T)+\int_{t}^{T} g_{r} \phi_{1}(r, 0) \phi_{2}(0, r) d r$, thanks to (5.12). We claim that the following integral

$$
\begin{equation*}
\int_{0}^{T}\left[\beta_{r} \phi_{1}(r, 0) \phi_{3}(T, r)+\frac{\beta_{r}^{2}}{\left(\sigma_{t}^{z}\right)^{2}} \Psi_{r}\right] d r \triangleq I_{1}+I_{2}, \tag{5.14}
\end{equation*}
$$

where $I_{1}$ and $I_{2}$ are defined in an obvious way, is well-defined. Indeed, Assumption 5.1 and the boundedness of $S$ imply that, modulo a universal constant,

$$
\begin{equation*}
\beta_{t} \phi_{1}(t, 0) \phi_{3}(T, t) \sim \beta_{t} \exp \left\{\int_{0}^{t}\left(f_{u}-\frac{\beta_{u}^{2} S_{u}}{\left|\sigma_{u}^{z}\right|^{2}}\right) d u\right\} \sim \frac{\beta_{t}}{\exp \left(\int_{0}^{t} \beta_{u}^{2} d u\right)} . \tag{5.15}
\end{equation*}
$$

On the other hand, by (5.13) it is easy to see that $\beta_{t}^{2} \Psi_{t} \sim \beta_{t}^{2} \phi_{1}^{2}(t, 0) \int_{0}^{t} \beta_{r} S_{r} \phi_{1}(0, r) d r$, and

$$
\begin{align*}
\int_{0}^{t} \beta_{r} S_{r} \phi_{1}(0, r) d r & =\int_{0}^{t} \beta_{r} S_{r} \exp \left\{\int_{0}^{r}\left[-f_{u}+\frac{\beta_{u}^{2} S_{u}}{\left|\sigma_{u}^{z}\right|^{2}}\right] d u\right\} d r \\
& \leq C \int_{0}^{t} \beta_{r} S_{r} \exp \left\{\int_{0}^{r} \frac{\beta_{u}^{2} S_{u}}{\left|\sigma_{u}^{z}\right|^{2}} d u\right\} d r  \tag{5.16}\\
& =C \int_{0}^{t} \frac{\left|\sigma_{r}^{z}\right|^{2}}{\beta_{r}} d\left[\exp \left\{\int_{0}^{r} \frac{\beta_{u}^{2} S_{u}}{\left|\sigma_{u}^{z}\right|^{2}} d u\right\} \leq C \exp \left\{\int_{0}^{t} \frac{\beta_{u}^{2} S_{u}}{\left|\sigma_{u}^{z}\right|^{2}} d u\right\} .\right.
\end{align*}
$$

Here in the above $C>0$ is a generic constant, which depends only on the bounds of the coefficients and $\underline{\beta}$ in Assumption 5.1, and is allowed to vary from line to line. Thus, similar to (5.15), we derive from (5.13) and (5.16) that

$$
\begin{equation*}
\beta_{t}^{2} \Psi_{t} \leq C \beta_{t}^{2} \phi_{1}(t, 0) G(t, T) \exp \left\{\int_{0}^{t} \frac{\beta_{u}^{2} S_{u}}{\left|\sigma_{u}^{z}\right|^{2}} d u\right\} \sim \beta_{t}^{2} \phi_{1}(t, 0) \sim \frac{\beta_{t}^{2}}{\exp \left(\int_{0}^{t} \beta_{r}^{2} d r\right)} \tag{5.17}
\end{equation*}
$$

But notice that $\beta_{t} \leq 1+\beta_{t}^{2}$, and that for any $\delta>0$, we have

$$
\int_{0}^{T-\delta} \frac{\beta_{t}^{2}}{\exp \left\{\int_{0}^{t} \beta_{u}^{2} d u\right\}} d t \leq 1-e^{-\int_{0}^{T-\delta} \beta_{u}^{2} d u} \leq 1
$$

we can easily derive from (5.15) and (5.17) that both $I_{1}$ and $I_{2}$ in (5.14) are finite, proving the claim.

We can now use the identity (5.10) to prove both conclusions of the theorem. We begin by observing that $\lim _{t \rightarrow T^{-}} S_{t}=0$ must hold. In fact, multiplying $S_{t}$ on both sides of (5.10) and then taking limits $t \rightarrow T^{-}$, and noting that $\lim _{t \rightarrow T^{-}} \phi_{3}(T, t)=0$, we conclude that $\lim _{t \rightarrow T^{-}} \Psi_{t}=0$. But then the equation (5.11), together with the fact that $\lim _{t \rightarrow T^{-}} \phi_{3}(T, t)=0$ but $\lim _{t \rightarrow T^{-}} \phi_{2}(0, t) \neq 0$, implies that $\lim _{t \rightarrow T^{-}} \phi_{1}(t, 0)=0$, and hence $\lim _{t \rightarrow T^{-}} S_{t}=0$, thanks to Lemma 5.3,

We now claim that $\lim _{t \rightarrow T^{-}} \beta_{t}=\infty$. Indeed, suppose not, then we have

$$
\lim _{t \rightarrow T^{-}} \frac{d S_{t}}{d t}=\lim _{t \rightarrow T^{-}} G\left(t, S_{t}, \beta_{t}\right)=\lim _{t \rightarrow T^{-}}\left(\sigma_{t}^{v}\right)^{2} \geq c>0
$$

which is a contradiction, since $S_{t}>0, \forall t \in[0, T)$, and $\lim _{t \rightarrow T^{-}} S_{t}=0$, proving the claim.
Now note that $S_{t}$ is the variance of the process $V_{t}-P_{t}, t \geq 0$, the facts that $\lim _{t \rightarrow T^{-}} S_{t}=0$ and that both processes $V$ and $P$ are continuous lead to that $V_{T}=P_{T}, \mathbb{P}$-a.s.. This completes the proof of part (2).

It remains to prove part (1). But note that $\phi_{1}(T, 0)=0$, we see from (5.13) that $\Psi_{t}=\Phi_{t}$, as $G(t, T)=\int_{t}^{T} g_{r} \phi_{1}(r, 0) \phi_{2}(0, r) d r$. Thus (1) follows directly from (5.10).

Remark 5.6. (i) Theorem 5.5 amounts to saying that, similar to the static information case (cf. e.g., [1, 17]), the optimal strategy $\beta$ for the optimization problem (2.7) should also maximize the advantage on the asymmetry of information near the terminal time $T$, i.e., $\lim _{t \rightarrow T^{-}} \beta_{t}=\infty$. Also, in equilibrium, such asymmetry of the information will disappear at the terminal time since $S_{T}=\lim _{t \rightarrow T^{-}} S_{t}=0$, that is, $P_{T}=\mathbb{E}^{\mathbb{P}}\left[V_{T} \mid \mathcal{F}_{T}^{Y}\right]=V_{T}, \mathbb{P}$-a.s., despite the fact that the insider only observes $V_{t}$ at time $t<T$ (see also, e.g., [4, 10-12]).
(ii) Although Theorem 5.5 gives only the necessary condition of the optimal strategy, the well-posedness of the optimal closed-loop system is guaranteed by Theorem 3.11. In other words, combining Theorem 5.5 and Theorem 3.11 we have proved the existence of the Kyle-Back equilibrium for the problem (2.3)-(2.7) under our assumptions.

## 6 Worked-out Cases and Examples

In general, it is not easy to find out an closed-form optimal strategy for the original problem (2.7), although we somehow predicted its behavior in Theorem 5.5. In this subsection we consider a special case for which the optimal strategy can be found explicitly. We show that it does possess the properties that we presented in the last sections, which in a sense justifies our results. More precisely, we shall consider a case where the market price does not impact the underlying asset price, namely, $g_{t} \equiv 0$ in equation (3.15). Recall from Lemma 5.4(ii) that $\alpha_{t}=\frac{2 \beta_{t} S_{t}}{\left(\sigma_{t}^{2}\right)^{2}}, t \in[0, T]$.
Theorem 6.1. Assume that all the assumptions of Theorem 5.5 are in force and further that the coefficient $g_{t} \equiv 0$ in the dynamics (3.15). Define $\alpha_{0} \geq 0$ so that

$$
\frac{\alpha_{0}^{2}}{4} \triangleq \frac{s_{0}+\int_{0}^{T} \phi_{2}^{2}(0, r)\left(\sigma_{r}^{v}\right)^{2} d r}{\int_{0}^{T}\left(\sigma_{t}^{z}\right)^{2} d t}
$$

Then, the solution to the optimization problem 2.7) is given as follows:
(i) the optimal strategy is given by $\beta_{t}=\frac{\alpha_{0} \phi_{2}(t, 0)\left(\sigma_{t}^{z}\right)^{2}}{2 S_{t}}$;
(ii) the error variance of price $P_{t}$

$$
\begin{equation*}
S_{t}=\phi_{2}^{2}(t, 0) \int_{t}^{T}\left(\frac{\alpha_{0}^{2}}{4}\left(\sigma_{r}^{z}\right)^{2}-\phi_{2}^{2}(0, r)\left(\sigma_{r}^{v}\right)^{2}\right) d r, \quad t \in[0, T] ; \tag{6.1}
\end{equation*}
$$

(iii) the corresponding expected payoff is given by

$$
J(\beta)=\frac{\alpha_{0} \phi_{2}(T, 0)}{2} \int_{0}^{T}\left(\sigma_{t}^{z}\right)^{2} d t ;
$$

(iv) the market price is given by

$$
P_{t}=\mathbb{E}^{\mathbb{P}}\left[V_{t} \mid \mathcal{F}_{t}^{Y}\right]=\phi_{2}(t, 0)\left[v_{0}+\int_{0}^{t} \phi_{2}(0, r) h_{r} d r+\frac{\alpha_{0}}{2} Y_{t}\right], \quad t \in[0, T] ;
$$

(v) finally, it holds that $\lim _{t \rightarrow T^{-}} \beta_{t}=\infty, \lim _{t \rightarrow T^{-}} S_{t}=0$, and in particular,

$$
\begin{equation*}
P_{T}=\mathbb{E}^{\mathbb{P}}\left[V_{T} \mid \mathcal{F}_{T}^{Y}\right]=V_{T}, \quad \mathbb{P} \text {-a.s. } \tag{6.2}
\end{equation*}
$$

Proof. Let $\beta \in \mathscr{U}_{a d}$ be an optimal strategy. Then by Theorem [5.5, we should have $\lim _{t \rightarrow T^{-}} S_{t}=$ 0 , and hence $\phi_{1}(T, 0)=\lim _{t \rightarrow T^{-}} \phi_{1}(t, 0)=0$, thanks to Lemma5.3.

Since $g_{t} \equiv 0$, we have

$$
\begin{equation*}
\phi_{3}(T, t)=\phi_{1}(t, 0) \phi_{2}(0, t)-\phi_{1}(T, 0) \phi_{2}(0, T)=\phi_{1}(t, 0) \phi_{2}(0, t) ; \tag{6.3}
\end{equation*}
$$

and (5.10) now reads

$$
\begin{equation*}
\frac{\phi_{1}^{2}(t, 0) \phi_{2}(0, t)\left(\sigma_{t}^{z}\right)^{2}}{2 \beta_{t} S_{t}}=\int_{t}^{T} \beta_{r} \phi_{1}^{2}(r, 0) \phi_{2}(0, r) d r . \tag{6.4}
\end{equation*}
$$

Now recall from Lemma [5.4(ii) that $\alpha_{t}=\frac{2 \beta_{t} S_{t}}{\left(\sigma_{t}^{2}\right)^{2}}$, we can rewrite (6.4) as

$$
\begin{equation*}
\phi_{12}(t)=\alpha_{t} \int_{t}^{T} \beta_{r} \phi_{12}(r) d r, \quad t \in[0, T], \tag{6.5}
\end{equation*}
$$

where $\phi_{12}(t) \triangleq \phi_{1}^{2}(t, 0) \phi_{2}(0, t)$. Now differentiating with respect to $t$ on both sides of (6.5) we obtain that

$$
\begin{equation*}
\frac{d \phi_{12}(t)}{d t}=-\alpha_{t} \beta_{t} \phi_{12}(t)+\frac{d \alpha_{t}}{d t} \int_{t}^{T} \beta_{r} \phi_{12}(r) d r=\left[-\alpha_{t} \beta_{t}+\frac{1}{\alpha_{t}} \frac{d \alpha_{t}}{d t}\right] \phi_{12}(t) . \tag{6.6}
\end{equation*}
$$

Now since $g \equiv 0$, we see from (3.20) that $\phi_{1}(t, 0)=\exp \left(\int_{0}^{t}\left(f_{r}-k_{r}\right) d r\right), \phi_{2}(0, t)=\exp \left(-\int_{0}^{t} f_{r} d r\right)$, where, by (3.16), $k_{t}=\frac{\beta_{t}^{2} S_{t}}{\left(\sigma_{t}^{2}\right)^{2}}$. We can easily compute that

$$
\frac{d \phi_{12}(t)}{d t}=2\left[f_{t}-k_{t}\right] \phi_{12}(t)-f_{t} \phi_{12}(t)=\left[f_{t}-2 k_{t}\right] \phi_{12}(t)
$$

Plugging this into (6.6) and noting that, by definition, $2 k_{t}=\alpha_{t} \beta_{t}$, we obtain that

$$
\begin{equation*}
\frac{d \alpha_{t}}{d t}=f_{t} \alpha_{t}, \quad \alpha_{0}=\frac{2 \beta_{0} s_{0}}{\left(\sigma_{0}^{z}\right)^{2}} \tag{6.7}
\end{equation*}
$$

Solving the above ODE we get

$$
\begin{equation*}
\frac{2 \beta_{t} S_{t}}{\left(\sigma_{t}^{z}\right)^{2}}=\alpha_{t}=\alpha_{0} \exp \left\{\int_{0}^{t} f_{r} d r\right\}=\alpha_{0} \phi_{2}(t, 0) \tag{6.8}
\end{equation*}
$$

Consequently we obtain that $\beta_{t}=\frac{\alpha_{0} \phi_{2}(t, 0)\left(\sigma_{t}^{z}\right)^{2}}{2 S_{t}}$, proving (i).
(ii) Note that $S$ satisfies the ODE (3.17) and $\phi_{2}(0, t)=\phi_{2}^{-1}(t, 0)=\alpha_{0} / \alpha_{t}$, where $\alpha$ satisfies (6.7), it is easy to check that

$$
\begin{aligned}
\frac{d}{d t}\left(\phi_{2}^{2}(0, t) S_{t}\right) & =\phi_{2}^{2}(0, t)\left[\left(\sigma_{t}^{v}\right)^{2}+2 f_{t} S_{t}-\left(\frac{\beta_{t} S_{t}}{\sigma_{t}^{z}}\right)^{2}\right]+S_{t} \alpha_{0}^{2} \frac{d}{d t}\left[\frac{1}{\alpha_{t}^{2}}\right] \\
& =\phi_{2}^{2}(0, t)\left[\left(\sigma_{t}^{v}\right)^{2}+2 f_{t} S_{t}-\left(\frac{\beta_{t} S_{t}}{\sigma_{t}^{z}}\right)^{2}\right]-2 f_{t} S_{t} \phi_{2}^{2}(0, t) \\
& =\phi_{2}^{2}(0, t)\left(\sigma_{t}^{v}\right)^{2}-\frac{\alpha_{0}^{2}\left(\sigma_{t}^{z}\right)^{2}}{4}
\end{aligned}
$$

Integrating both sides from $t$ to $T$, and noting that $S_{T^{-}}=0$ and $\phi_{2}(0, t)=\phi_{2}^{-1}(t, 0)$, we derive (6.1). Furthermore, setting $t=0$ in (6.1), one can then solve

$$
\frac{\alpha_{0}^{2}}{4} \triangleq \frac{S_{0}+\int_{0}^{T} \phi^{2}(0, r)\left(\sigma_{r}^{v}\right)^{2} d r}{\int_{0}^{T}\left(\sigma_{t}^{z}\right)^{2} d t}
$$

proving (ii).
(iii) Combining the expression of $\beta, S$, as well as equations (5.3) and (6.3), the corresponding expected payoff is

$$
\begin{aligned}
J(\beta) & =\phi_{2}(T, 0) \bar{J}(\beta)=\phi_{2}(T, 0) \int_{0}^{T} \beta_{t} S_{t} \phi_{1}(0, t) \phi_{3}(T, t) d t \\
& =\phi_{2}(T, 0) \int_{0}^{T} \beta_{t} S_{t} \phi_{1}(0, t)\left[\phi_{1}(t, 0) \phi_{2}(0, t)\right] d t \\
& =\phi_{2}(T, 0) \int_{0}^{T}\left[\frac{\alpha_{0} \phi_{2}(t, 0)\left(\sigma_{t}^{z}\right)^{2}}{2 S_{t}}\right] S_{t} \phi_{2}(0, t) d t=\frac{\alpha_{0} \phi_{2}(T, 0)}{2} \int_{0}^{T}\left(\sigma_{t}^{z}\right)^{2} d t
\end{aligned}
$$

(iv) It follows from (3.23) and (6.8) that the market price follows the dynamics

$$
P_{t}=\mathbb{E}^{\mathbb{P}}\left[V_{t} \mid \mathcal{F}_{t}^{Y}\right]=\phi_{2}(t, 0)\left[v_{0}+\int_{0}^{t} \phi_{2}(0, r) h_{r} d r+\frac{\alpha_{0}}{2} Y_{t}\right], \quad t \in[0, T]
$$

(v) Finally, again by Lemma 5.3, one has $\lim _{t \rightarrow T^{-}} \beta_{t}=\infty$ and $\lim _{t \rightarrow T^{-}} S_{t}=0$. In particular, $P_{T}=\mathbb{E}\left[V_{t} \mid \mathcal{F}_{T}^{Y}\right]=V_{T}, \mathbb{P}$-a.s.

We note that Theorem 6.1 contains several previously known results as special cases. We list them as follows.

1. Static case. In this case $V_{t} \equiv v$, where $v \sim N\left(v_{0}, S_{0}\right)$.

Setting $f \equiv 0, g \equiv 0, h \equiv 0, \sigma^{v} \equiv 0$ in Theorem 6.1 we have

$$
\phi_{2}(t, 0) \equiv 1, \quad \text { and } \quad \alpha_{0}=\left(\frac{4 S_{0}}{\int_{0}^{T}\left(\sigma_{t}^{z}\right)^{2} d t}\right)^{1 / 2}=2 \lambda,
$$

where $\lambda \triangleq \sqrt{S_{0}}\left\{\int_{0}^{T}\left(\sigma_{t}^{z}\right)^{2} d t\right\}^{-1 / 2}$ is the so-called price sensitivity or Kyle's $\lambda$ (cf. [1]). The optimal strategy is given by

$$
\beta_{t}=\frac{\alpha_{0}\left(\sigma_{t}^{z}\right)^{2}}{2 S_{t}}=\frac{\left(\int_{0}^{T}\left(\sigma_{t}^{z}\right)^{2} d t\right)^{1 / 2}\left(\sigma_{t}^{z}\right)^{2}}{S_{0}^{1 / 2} \int_{t}^{T}\left(\sigma_{r}^{z}\right)^{2} d r}
$$

and the corresponding expected payoff is given by

$$
J(\beta)=S_{0}^{1 / 2}\left(\int_{0}^{T}\left(\sigma_{t}^{z}\right)^{2} d t\right)^{1 / 2}
$$

Furthermore, one can easily check that, for $t \in[0, T]$, the corresponding market price is given by

$$
P_{t}=\mathbb{E}^{\mathbb{P}}\left[V_{t} \mid \mathcal{F}_{t}^{Y}\right]=v_{0}+\lambda Y_{t},
$$

and the corresponding mean square error is

$$
S_{t}=\frac{S_{0} \int_{t}^{T}\left(\sigma_{r}^{z}\right)^{2} d r}{\int_{0}^{T}\left(\sigma_{t}^{z}\right)^{2} d t}
$$

In particular, $S_{T}=0$, which implies that $V_{0}=P_{T}, \mathbb{P}$-a.s. These results coincide with those of [1].
If we further assume that $T=1$ and $\sigma_{t}^{z} \equiv \sigma$, where $\sigma>0$ is a constant, then the optimal trading intensity becomes

$$
\beta_{t}=\frac{\sigma}{\sqrt{S_{0}}(1-t)}, \quad 0 \leq t<1
$$

the corresponding expected payoff is $J(\beta)=\sigma \sqrt{S_{0}}$, and the corresponding market price is $P_{t}=$ $v_{0}+\frac{\sqrt{S_{0}}}{\sigma} Y_{t}, 0 \leq t<T$. We recover the results of Back [2].
2. Long-lived information case. We now compare our results with that of Back-Pedersen [4] (see also Danilova [17), in which the insider continuously observes the dynamics of $V_{t}$ that is assumed to be a martingale.

By setting $T=1, f \equiv 0, g \equiv 0, h \equiv 0$ and $\sigma^{z} \equiv 1$, Theorem 6.1 implies $\frac{\alpha_{0}^{2}}{4}=1$, assuming $S_{0} \triangleq 1-\int_{0}^{1}\left(\sigma_{s}^{v}\right)^{2} d s$, and the optimal trading intensity is

$$
\beta_{t}=\frac{1}{1-t-\int_{t}^{1}\left(\sigma_{s}^{v}\right)^{2} d s}, \quad 0 \leq t<1
$$

The corresponding expected payoff is $J(\beta)=1$, and the corresponding market price is $P_{t}=v_{0}+Y_{t}$, $0 \leq t<T$.

Acknowledgment. We would like to express our sincere gratitude to the anonymous referees for their careful reading of the manuscripts and very insightful suggestions, which helped us to improve the quality of the paper significantly.

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