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Model Uncertainty in Commodity Markets*

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Abstract. Agents who acknowledge that their models are incorrectly specified are said to be ambiguity averse, and this affects the prices they are willing to trade at. Models for prices of commodities attempt to capture three stylized features: seasonal trend, moderate deviations (a diffusive factor), and large deviations (a jump factor) both of which mean-revert to the seasonal trend. Here we model ambiguity by allowing the agent to consider a class of models absolutely continuous w.r.t. their reference model, but penalize candidate models that are far from it. We show that the buyer (seller) of a forward contract introduces a negative (positive) drift in the dynamics of the spot price and enhances downward (upward) jumps so the prices they are willing to trade at are lower (higher) than that of the forward price under $\mathbb{P}$. When ambiguity averse buyers and sellers employ the same reference measure they cannot trade because the seller requires more than what the buyer is willing to pay. Finally, we observe that when ambiguity averse agents price options written on the commodity forward, the effect of ambiguity aversion is strongest when the option is at-the-money and weaker when it is deep in-the-money or deep out-of-the-money.

Key words. ambiguity aversion, Knightian uncertainty, commodities, certainty equivalent, robust pricing, indifference pricing, optimal control

AMS subject classifications. 91G20, 91G80, 49J20

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1. Introduction.

Commodities as an asset class comprise many different types of goods and services which are produced and consumed by agents to satisfy a physical need. These agents must plan ahead how much of the commodity to produce or consume, and crucial in this decision are the assumptions about future prices of the commodity. This requires the challenging task of specifying a model to capture the complex behavior of prices.

The literature has proposed many models to capture the stylized behavior of commodity prices. These models are tailored to handle the physical commitments and manage the risk that producers and consumers are exposed to and are also used by professional traders and financial investors who seek exposure to this asset class. But how confident are these market participants, whether physical or financial, about their models? If market participants acknowledge that their models are misspecified, how do they account for this in their decision making? How does this affect the prices they are willing to pay or receive for the commodity and derivatives written on the commodity?

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The commodities literature has largely overlooked the question of model uncertainty. In this paper we show how agents can incorporate model uncertainty in their decisions. They do this by choosing a reference model with probability measure $\mathbb{P}$ but acknowledge that they are not confident about their choice, so they consider alternative measures $\mathbb{Q}$, absolutely continuous with respect to $\mathbb{P}$, and decide how to choose from among the set of candidate measures.

The literature has proposed a large list of reduced-form models for energy commodities, minerals, metals, and agricultural products with different goals, such as to explain the behavior of prices, how to manage risks, and how to price derivatives written on commodities. Early papers in the field include Gibson and Schwartz (1990), who propose a two-factor model for oil prices, and Schwartz (1997), who propose different ways of modeling commodity prices and show applications using data for copper, oil, and gold; see also Schwartz and Smith (2000), and more recently Cartea and Williams (2008), who employ a two-factor model to explain the dynamics of natural gas prices and show how to value interruptible supply contracts (see also Hikspoors and Jaimungal (2008)). Concerning models of electricity, Benth et al. (2003) model electricity forward prices, while Roncoroni (2002), Cartea and Figueroa (2005), Weron (2007), Benth, Kallsen, and Meyer-Brandis (2007) propose models for wholesale electricity prices; see also Benth and Saltyte-Benth (2006), Hikspoors and Jaimungal (2007), Benth, Saltyte-Benth, and Koekebakker (2008), and Jaimungal and Surkov (2011). Finally, Hambly, Howison, and Kluge (2009) model power prices and price different types of energy derivatives, including swing options, designed to manage the risk exposure of producers and consumers.

In this paper we assume that the primitive asset is the spot commodity. We examine the effect of model uncertainty by showing how the prices that consumers are willing to pay, and producers to receive, for forward contracts and other derivatives depend on the degree of confidence that they place on their reference model. We show the effect of ambiguity in model choice for a general structural model of prices and discuss in detail two particular cases such as arithmetic and geometric reduced-form models which nest most of the models proposed by the literature. We show that when the agent becomes less confident about her model, she computes more conservative prices: the seller will require higher prices and the consumer will seek to pay lower prices. More specifically, for forward contracts, the ambiguity averse buyer and seller modify (i) the drift of reference model—the buyer introduces a downward drift and the seller an upward drift; and (ii) the jump component by tilting the jump measure—the buyer enhances downward jumps and dampens upward jumps, and the seller enhances upward jumps and dampens downward jumps.

Our results also show that when the buyer and seller employ the same reference measure they will not trade with each other because the ambiguity averse seller requires a price higher than what the ambiguity averse buyer is willing to pay. This spread varies across time and shrinks as the delivery of the forward contract approaches—there is less uncertainty around the price of the underlying commodity as the forward contract enters the delivery period.

We also show the effect of ambiguity aversion on option prices. In particular we examine a bull-spread written on forward contracts. We see that the effect of ambiguity is strongest when the option is at-the-money and weakest when it is trading deep out-of-the-money or deep in-the-money. The insights are the following. In the neighborhood of at-the-money, agents' ambiguity about the reference model is more important since any movement in the price of the underlying commodity (and hence price of the forward contract) has a considerable effect on the value of the option. Thus, agents are more conservative around at-the-money values and this implies that the buyer decreases and the seller increases the prices they are willing to trade at, hence the spreads (when both agents employ the same reference model) are wide in this region and tighter outside it.

Closest to our work is that of Bannor et al. (2013), where the authors investigate model risk in the context of parameter uncertainty in energy markets. We note that our approach is different because agents consider a larger class of alternative models. This class consists of all models described by a probability measure where the only requirement is that the measures are equivalent to the reference measure $\mathbb{P}$.

Model uncertainty has been used in several other settings in the literature. For applications in portfolio optimization and consumption problems, see, for instance, Hansen and Sargent (2001), Uppal and Wang (2003), Hansen and Sargent (2007), and Guidolin and Rinaldi (2013); in credit derivatives, Jaimungal and Sigloch (2012); in algorithmic trading, Cartea, Jaimungal, and Donnelly (2014); and in real-options Cartea and Jaimungal (2015).

The rest of this paper is organized as follows. In section 2 we describe a structural reference model and two particular cases which are commonly used in the commodities literature, and we show how agents price forward contracts. In section 3 we show how the agents incorporate model uncertainty in their decisions. We show the effect of ambiguity aversion on the spot price model and how this affects the forward prices they are willing to trade at. In section 4 we discuss the implications for an arithmetic model of commodity prices, and in section 5 we discuss the implications for a geometric model. In section 6 we investigate how the prices of options written on forward contracts of the commodity are affected by ambiguity aversion. Finally, section 7 concludes, and in the appendix we collect proofs.

2. Reference model: No model uncertainty.

Our goal is to shed light onto how agents incorporate model uncertainty in their decisions; thus, to isolate the effect of model ambiguity, we assume that agents are risk-neutral and ambiguity averse. Furthermore, to stream the discussion of the effect of ambiguity, in this section we assume that agents are not ambiguous about their reference model, and in section 3 we develop the mathematical framework used by them to incorporate model uncertainty.

We focus on two types of agents, the buyer and the seller, who can be interpreted as the consumer and producer of the commodity, and their reference models are specified by the measures $\mathbb{P}^i$, where $i = \{+, -\}$ denotes buyer or seller, respectively. The price of the spot commodity is denoted by $S = (S_t)_{0\le t\le T}$ and a forward contract is a commitment between the seller and the buyer; the seller is committed to delivering $\int_{T_1}^{T_2} S_u\,du$ of the commodity over the delivery period $[T_1, T_2]$, and the buyer is committed to paying an amount $F(t, T_1, T_2)$ per unit of the commodity. They decide their willingness to charge (pay) for delivery of the commodity at a future date based on their risk preferences, which we assumed to be risk-neutral. Thus, the agents arrive at the forward price by evaluating

$$F^{\pm}(t, T_1, T_2) = \mathbb{E}_t^{\mathbb{P}^{\pm}}\left[\frac{1}{T_2 - T_1}\int_{T_1}^{T_2} S_u\,du\right], \tag{1}$$

where $\mathbb{E}_t^{\mathbb{P}^{\pm}}$ are the expectation operators under the reference measures $\mathbb{P}^{\pm}$, conditioned on information up until time t.

2.1. Reference model: Spot dynamics and forward prices. In general, the dynamics of commodity spot prices consist of three distinctive features: seasonal pattern, short-term price movements that do not stray too far from the seasonal component, and short-lived jumps or spikes which capture large price deviations from the seasonal component. The relevance of each one of these stylized facts depends on the type of commodity (energy, metal, agricultural) and, within a given type, there are other factors which affect the behavior of prices, for example, market structure, regulation, access to storage, and types of contracts traded at exchanges such as forward contracts with financial settlement—therefore giving access to financial speculators.

The reference models used by agents consist of three components: (i) deterministic seasonal component, (ii) short-term component, and (iii) large-deviation component. Here we focus on one reference model so for clarity we do not index the probability measure $\mathbb{P}$ with $i = \{+, -\}$.

(i) The seasonal pattern is given by $\theta(t)$, assumed to be deterministic and calculated from historical data in addition to information that is known by market participants which affects the future average prices of the commodity; see Benth, Biegler-König, and Kiesel (2013).

(ii) The short-term factor $X = (X_t)_{0\le t\le T}$ is an Ornstein–Uhlenbeck (OU) process and satisfies the SDEs

$$dX_t = -\kappa_D\, X_t + \sqrt{v_t}\, dW_t^X\,, \tag{2}$$

$$dv_t = \kappa_v\,(\vartheta - v_t)\; dt + \eta\,\sqrt{v_t}\, dW_t^v\,, \tag{3}$$

where κ_D is the speed at which short-term deviations in the commodity market subside, v_t is (instantaneous) variance, $\boldsymbol{W}_t = \left(W_t^X,\ W_t^v\right)'$ is a two-dimensional standard Brownian motion with $d[W^v\,,\,W^X]_t = \rho\, dt$, and $'$ denotes the transpose operator.

Several articles in the literature show that the volatility is stochastic and that it is an important feature to include when valuing contingent claims; see, e.g., Hikspoors and Jaimungal (2008) and Trolle and Schwartz (2009). For this reason, we include stochastic volatility in the diffusive component of our price model.

(iii) The large-deviation factor, denoted by $Y = (Y_t)_{0\le t\le T}$, is an OU-type process which satisfies

$$dY_t = -\kappa_J\, Y_t\, dt + d\hat{J}_t\,, \tag{4}$$

where $\kappa_J > 0$ is the mean-reverting rate parameter. Here $\hat{J} = (\hat{J}_t)_{0\le t\le T}$ is a compensated pure jump process

$$\hat{J}_t = \int_0^t \int_{-\infty}^{\infty} z\,(\mu - \nu)(dz, ds)\,, \tag{5}$$

with arrival rate λ and $\mu(dz, ds) = \lambda\, G(dz)\, ds$ is the compensator (or mean-measure) of the Poisson random measure (PRM) $\mu(dz, ds)$, where $G(dz)$ denotes the distribution of the jump size.

As usual, we work on a completed filtered probability space $(\Omega, \mathcal{F}_{0\le t\le T}, \mathbb{P})$, where $\mathbb{P}$ is the reference measure, and $\mathcal{F}_t$ is the natural filtration generated by the triplet $(W^X, W^v, \hat{J})$. And recall that we denote the spot price of the commodity by $S = (S_t)_{0\le t\le T}$.

In commodities it is possible, and in fact not unusual, to observe negative prices. It is not uncommon to be in situations where oversupply of the commodity, combined with limited storage capacity (or impossible to store, such as electricity), forces producers to pay consumers to take the physical commodity—it is too expensive for producers to dispose of or keep the commodity. For instance, gas and electricity are commodities for which the market has borne negative prices, and this is an example of the extreme risks that producers face. Below we discuss two of the most popular reduced-form models used in the literature, arithmetic and exponential, and then we show a general structural model.

Arithmetic model of spot prices. The arithmetic model allows spot prices to become negative:

$$S_t = \theta(t) + X_t + Y_t \,, \tag{6}$$

where $\theta(t)$ is the seasonal component, and X_t and Y_t are as in (2) and (4). Arithmetic models of this type were introduced by Benth, Kallsen, and Meyer-Brandis (2007) to model electricity prices. A further advantage of arithmetic models is that computing forward prices over a delivery period is easier than in an exponential model.

Exponential model of spot prices. The exponential model ensures nonnegativity of prices:

$$S_t = e^{\theta(t)+X_t+Y_t} \,, \tag{7}$$

where $\theta(t)$ is the seasonal component, and X_t and Y_t are as in (2) and (4)—see, for instance, Schwartz and Smith (2000), Cartea and Figueroa (2005), Weron (2007), Hambly, Howison, and Kluge (2009), and Benth, Saltyte-Benth, and Koekebakker (2008).

Structural spot price models. The seasonality, diffusive, and jumps factors can also be viewed as underlying supply or demand drivers in a structural model of spot prices. For example, for system load or fuel prices, write in general

$$S_t = P(t, \theta(t), X_t, Y_t) \,, \tag{8}$$

where P captures the price/demand curve. Both the arithmetic and geometric models fall within this class of models, and we adopt this notation throughout the remainder of the article and specify the specific form of P only when necessary; see Weron (2007), Cartea and Villaplana (2008), Figuerola-Ferretti and Gonzalo (2010), Carmona, Coulon, and Schwarz (2013), and Carmona and Coulon (2014).

2.2. Forward prices in reference model. As discussed above, the price of a forward contract with delivery period $[T_1, T_2]$ is given by (1). Here we define the function

$$H(t, x, v, y) = \mathbb{E}^{\mathbb{P}}_{t,x,v,y}\left[\frac{1}{T_2 - T_1}\int_{t\vee T_1}^{T_2} S_u \, du\right] \,, \qquad t \le T_2 \,, \tag{9}$$

with S_t given above by (6) or (7), and $\mathbb{E}^{\mathbb{P}}_{t,x,y}[\cdot]$ denotes $\mathbb{P}$-expectation conditional on $(X_t = x, Y_t = y)$. The function H coincides with the price of the forward contract (1) when $t \le T_1$, and it coincides with the contract known as "balance of the month" on the interval $t \in (T_1, T_2]$, which is traded for certain commodities. We are mainly interested in $t \le T_1$, i.e., the price of forward contract before start of delivery.

Using standard techniques, one can show that $H(t,x,v,y)$ satisfies the following partial-integro-differential equation (PIDE):

$$(\partial_t + \mathcal{L})\, H + \lambda \int_{-\infty}^{\infty} \Delta_z H\, G(dz) + P(t,\theta(t),x,v,y)\, \mathbb{1}_{t\geq T_1} = 0\,, \tag{10}$$

subject to the terminal condition

$$H(T_2,x,v,y) = 0 \qquad \forall\, x,v,y\,,$$

and where $\mathcal{L}$ is the $\mathbb{P}$-generator of (X_t, v_t, Y_t^c), i.e.,

$$\mathcal{L} = -\kappa_D\, x\, \partial_x + \tfrac{1}{2} v\, \partial_{xx} - \kappa_v\, v\, \partial_v + \tfrac{1}{2}\eta^2\, v\, \partial_{vv} + \rho\, \sigma\, \eta\, v\, \partial_{xv} - (\lambda\, \psi + \kappa_J\, y)\, \partial_y\,, \tag{11}$$

the shift operator Δ_z acts as

$$\Delta_z H(t,x,v,y) = H(t,x,v,y+z) - H(t,x,v,y)\,, \tag{12}$$

and the constant (expected jump size)

$$\psi = \int_{-\infty}^{\infty} z\, G(dz)\,. \tag{13}$$

Below we provide two explicit solutions in the reference model for the arithmetic and geometric cases.

Theorem 1 (forward price in arithmetic reference model). *Suppose that the spot price is given by (6), i.e., $P(t,\theta,x,y) = \theta + x + y$; then under the reference model $\mathbb{P}$, the value function H is affine and is explicitly*

$$H(t,x,v,y) = \frac{1}{T_2 - T_1}\left(A(t) + B(t)\, x + D(t)\, y\right), \quad t \leq T_2\,, \tag{14}$$

where $A(t)$, $B(t)$, and $D(t)$ are deterministic functions of time:

$$\begin{aligned}
A(t) &= \int_t^{T_2} \theta(u)\, \mathbb{1}_{u\geq T_1}\, du\,,\\
B(t) &= \frac{1 - e^{-\kappa_D (T_2 - T_1)}}{\kappa_D}\, e^{-\kappa_D (T_1 - t)}\, \mathbb{1}_{t<T_1} + \frac{1 - e^{-\kappa_D (T_2 - t)}}{\kappa_D}\, \mathbb{1}_{t\geq T_1}\,,\\
D(t) &= \frac{1 - e^{-\kappa_J (T_2 - T_1)}}{\kappa_J}\, e^{-\kappa_J (T_1 - t)}\, \mathbb{1}_{t<T_1} + \frac{1 - e^{-\kappa_J (T_2 - t)}}{\kappa_J}\, \mathbb{1}_{t\geq T_1}\,.
\end{aligned}$$

Proof. This can be shown by directly computing the expectation in (9) or solving the PIDE (10) by making an ansatz of the form (14). Note that the price of the forward contract is given by $H(t,x,y)$ restricted to $t \leq T_1$. ∎

In the proposition below we show a result similar to that in Proposition 1 where we assume that the dynamics of the spot commodity are exponential.

Theorem 2 (forward price in exponential reference model). *Suppose that the spot price is given by* (7)*, i.e.,* $P(t,\theta,x,y) = \exp\{\theta + x + y\}$*; then under the reference model* $\mathbb{P}$*, the value function* H *is exponential affine and is explicitly*

$$H(t,x,v,y) = \tfrac{1}{T_2-T_1}\int_{T_1}^{T_2} \exp\left\{\Theta(t,u) + x\,e^{-\kappa_D(u-t)} + v\,C(t,u) + y\,e^{-\kappa_J(u-t)}\right\} du\,, \quad t \le T_2\,,$$

where $C(t,u)$ *solves the Riccati ODE*

$$\partial_t C(t,u) + \tfrac{1}{2}\,e^{-2\kappa_D(u-t)} + \left(\rho\,\sigma\,\eta\,e^{-\kappa_D(u-t)} - \kappa_v\right)\,C(t,u) + \tfrac{1}{2}\,\eta^2\,C^2(t,u) = 0\,, \quad C(u,u) = 0, \tag{15}$$

and

$$\begin{aligned}\Theta(t,u) = \theta(u) &+ \kappa_v\vartheta\int_t^u C(t,\tau)\,d\tau \\ &+ \lambda\int_0^{u-t}\left(\Psi\left(y\,e^{-\kappa_J\tau}\right) - 1\right)\,d\tau - \frac{\lambda}{\kappa_J}\,\psi\,\left(1 - e^{-\kappa_J(u-t)}\right),\end{aligned} \tag{16}$$

where $\Psi(u) = \int_{-\infty}^{\infty} e^{uz}\,G(dz)$ *is the moment generation function of the jump distribution.*

Proof. See Appendix A.1. ∎

There are two cases when $C(t,u)$ in the above result can be obtained in closed-form: (i) if volatility is constant, so that $\eta = \kappa_v = 0$, and (ii) if X_t is a Brownian motion, so that $\kappa_D = 0$.

In case (i), the ODE (15) reduces to a linear ODE

$$\partial_t C(t,u) + \tfrac{1}{2}e^{-2\kappa_D(u-t)} = 0, \qquad C(u,u) = 0\,,$$

hence

$$C(t,u) = \frac{1 - e^{-2\kappa_D(u-t)}}{2\kappa_D}\,.$$

In case (ii), the ODE (15) reduces to a constant coefficient Riccati ODE

$$\partial_t C(t,u) + \tfrac{1}{2} + (\rho\sigma\eta - \kappa_v)\,C(t,u) + \tfrac{1}{2}\eta^2\,C^2(t,u) = 0, \qquad C(u,u) = 0\,,$$

whose solution can be expressed as

$$C(t,u) = \frac{1}{\eta^2}\,\frac{1 - e^{-\frac{1}{2}\eta^2(\ell^+-\ell^-)(u-t)}}{\ell^+ - \ell^-\,e^{-\frac{1}{2}\eta^2(\ell^+-\ell^-)(u-t)}}\,,$$

where

$$\ell^{\pm} = \frac{1}{\eta^2}\left((\kappa_v - \rho\sigma\eta) \pm \sqrt{(\kappa_v - \rho\sigma\eta)^2 - \eta^2}\right)\,.$$

3. Model uncertainty and ambiguity aversion. Agents trading the physical commodity face a wide range of risky decisions over different time scales. Over long horizons, decisions to permanently increase production, by investing in new technologies or increasing capacity, have a long-term impact on the operations and financial performance of their business. And, over shorter time scales, decisions about how much of the commodity to produce or consume are also critical to manage the risks they are exposed to. Moreover, professional investors who seek financial exposure to this asset class must also manage the risk in financial contracts which can span a large time horizon depending on the type of commodity.

To manage these risks, stakeholders face the challenge of formulating a framework for the dynamics of prices. This framework is devised so that agents acknowledge that their models are misspecified; thus prices and decisions must reflect this ambiguity. They incorporate this ambiguity by considering alternative models which are specified by a measure $\mathbb{Q}$ which is equivalent to the reference measure $\mathbb{P}$. We denote by $\mathcal{Q}$ the set of equivalent measures that the agent considers—later in (20), (24), and (25) we provide more details on this set. Next, the agent must rank the alternatives and to do this she includes a penalty function to measure the "cost" of rejecting the reference measure $\mathbb{P}$ and accepting a candidate model $\mathbb{Q}$ to compute the price of a derivative. For instance, if the agent is very confident about the reference model, any "small" deviation from the reference measure $\mathbb{P}$ is heavily penalized, i.e., it is very costly to choose an alternative. On the other hand, if the agent is extremely ambiguous about her choice of the reference measure, considering other models will only result in a very small penalty.

Therefore, a buyer (seller) of the commodity who includes ambiguity with respect to the reference measure solves

$$F^{\pm}(t,T_1,T_2) = \pm \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}\left[\pm\frac{1}{T_2-T_1}\int_{T_1}^{T_2} S_t\,dt + \mathcal{H}(\mathbb{Q}|\mathbb{P})\right], \tag{17}$$

where $\mathcal{H}(\mathbb{Q}|\mathbb{P})$ is the penalty function, i.e., the cost from choosing the candidate measure $\mathbb{Q}$ over the reference model $\mathbb{P}$.

A popular choice for the penalty function is the relative entropy

$$\mathcal{H}_{t,T}(\mathbb{Q}|\mathbb{P}) = \frac{1}{\gamma}\log\frac{d\mathbb{Q}}{d\mathbb{P}}, \tag{18}$$

where $\gamma > 0$ is a constant which reflects the agent's degree of confidence in the reference model. In the limiting case $\gamma \to 0$, the agent is ambiguity neutral and therefore rejects any alternative model. If the agent is extremely ambiguous about the reference model, then γ is very large. In the extreme case $\gamma \to \infty$ the agent considers the worst case scenario—the seller (buyer) picks the model that yields the highest (lowest) forward price.

In our model setup, buyers and sellers are ambiguity averse to the three factors that drive the spot commodity which are given above by (2), (3), and (4). Moreover, we assume that the degree of ambiguity aversion shown by the agent is specific to each factor. For example, Bannor et al. (2013) find that in wholesale electricity markets, jump risk is by far the most important source of model risk; see also Stahl et al. (2012). Thus here we focus on alternative measures that treat model uncertainty specific to each factor, where the degree of ambiguity aversion specific to the OU diffusive factor and stochastic volatility is encoded in the ambiguity matrix Φ and to the OU-type jump factor is ε (more details are provided below).

We next consider how the agent incorporates ambiguity aversion specific only to the diffusive and stochastic volatility factor, then only to the jump factor, and finally we consider ambiguity aversion to all three factors.

Ambiguity aversion to diffusive factors. The agent considers alternative models which are characterized by the Radon–Nikodym derivative

$$\frac{d\mathbb{Q}^{\boldsymbol{\alpha}}}{d\mathbb{P}} = \exp\left\{-\frac{1}{2}\int_0^T \boldsymbol{\alpha}'_u\,\boldsymbol{\Sigma}^{-1}\,\boldsymbol{\alpha}_u\,du - \int_0^T \boldsymbol{\alpha}'_u\,d\boldsymbol{W}_u\right\}, \tag{19}$$

where $\boldsymbol{W}_t = \left(W_t^X,\ W_t^v\right)'$ and $\boldsymbol{\alpha}_t$ is a two-dimensional $\mathcal{F}_t$-adapted process, and

$$\boldsymbol{\Sigma} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \qquad \text{so that} \qquad \boldsymbol{W}_t^{\boldsymbol{\alpha}} = -\int_0^t \boldsymbol{\alpha}_u\,du + \boldsymbol{W}_t$$

are $\mathbb{Q}^{\boldsymbol{\alpha}}$-standard Brownian motions.

This change of measure is parameterized by an $\mathcal{F}$-predictable process $\boldsymbol{\alpha}_t$ which changes the drift of the reference model. In addition, the set of candidate measures is

$$\mathcal{Q}^1 = \left\{\mathbb{Q}^{\alpha}(\alpha)\ :\ \boldsymbol{\alpha} \text{ is } \mathcal{F}\text{-predictable and } \mathbb{E}^{\mathbb{P}}\left[\int_0^T \boldsymbol{\alpha}'_u\,\boldsymbol{\Sigma}^{-1}\,\boldsymbol{\alpha}_u\,du\right] < +\infty\right\}, \tag{20}$$

and the entropic penalization specific to the diffusive factor in the model is

$$\mathcal{H}^{\boldsymbol{\Phi}}(\mathbb{Q}^{\boldsymbol{\alpha}}|\mathbb{P}) = \mathbb{E}^{\mathbb{Q}^{\boldsymbol{\alpha}}}\left[\frac{1}{2}\int_0^T \boldsymbol{\alpha}'_u\,\boldsymbol{\Phi}^{-1}\,\boldsymbol{\alpha}_u\,du\right],$$

where $\boldsymbol{\Phi}$ is an ambiguity matrix with inverse given by

$$\boldsymbol{\Phi}^{-1} = \phi\,\boldsymbol{\Sigma}^{-1} + \phi_x\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \phi_v\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{21}$$

Here $\phi \geq 0$ is an ambiguity parameter common to both the diffusive and volatility factors, $\phi_x \geq 0$ and $\phi_v \geq 0$ are the ambiguity parameters specific to the diffusive and volatility factors, respectively, and recall that when the agent is very confident about the reference model the ambiguity parameters are zero.

Ambiguity aversion to jump factor. Similarly, the agent considers alternative models to the jump factor parameterized by an $\mathcal{F}$-predictable random field $g_t(\cdot)$ which are characterized by the Radon–Nikodym derivative

$$\frac{d\mathbb{Q}^g}{d\mathbb{P}} = \exp\left\{-\int_0^T\int_{-\infty}^{\infty}(e^{g_u(z)}-1)\,\nu(dz,du) + \int_0^T\int_{-\infty}^{\infty} g_u(z)\,\mu(dz,du)\right\}. \tag{22}$$

The $\mathbb{Q}^g$-compensator of $\mu(dz,dt)$ is

$$\nu_{\mathbb{Q}}(dz,dt) = e^{g_t(z)}\nu(dz,dt), \tag{23}$$

$$\begin{tikzcd}
 & \mathbb{Q}^{\boldsymbol{\alpha}} \arrow[dr, "{\varepsilon, g_t(z)}"] & \\
\mathbb{P} \arrow[ur, "{\boldsymbol{\Phi}, \boldsymbol{\alpha}_t}"] \arrow[dr, "{\varepsilon, g_t(z)}"'] \arrow[rr, dotted] & & \mathbb{Q}^{\boldsymbol{\alpha},g} \\
 & \mathbb{Q}^{g} \arrow[ur, "{\boldsymbol{\Phi}, \boldsymbol{\alpha}_t}"'] &
\end{tikzcd}$$

Figure 1. *The two natural alternative routes from the reference measure $\mathbb{P}$ to a candidate measure $\mathbb{Q}^{\alpha,g}$ in which diffusion, mean measure of jump components have been altered.*

and we choose the class of candidate measures to be

$$\mathcal{Q}^2 = \left\{ \mathbb{Q}^g(g) \,:\, g \text{ are } \mathcal{F}\text{-predictable and } \mathbb{E}^{\mathbb{P}}\left[\int_0^T \int_{-\infty}^{\infty} (g_u(z))^2 \,\nu(dz,du) \right] < +\infty \right\}, \tag{24}$$

and the penalty function is

$$\mathcal{H}^{\varepsilon}(\mathbb{Q}^{\lambda}|\mathbb{P}) = \mathbb{E}^{\mathbb{Q}^g}\left[\frac{1}{\varepsilon} \int_0^T \int_{-\infty}^{\infty} \left(e^{g_u(z)}(g_u(z)-1)+1 \right) \nu(dz,du) \right].$$

Ambiguity aversion to diffusive, volatility, and jump factors. Now that we have specified the set of candidate models for each individual factor, we next discuss the set of candidate measures which the agent considers when accounting for ambiguity to both factors at the same time. Thus we seek a change of measure such that

$$\mathbb{P} \xrightarrow{\boldsymbol{\Phi},\varepsilon} \mathbb{Q}^{\boldsymbol{\alpha},g},$$

and in Figure 1 we depict how to progressively obtain this measure change, which can be done in two canonical ways, to obtain

$$\begin{aligned}
\frac{d\mathbb{Q}^{\boldsymbol{\alpha},g}}{d\mathbb{P}} = \exp\Bigg\{ &-\tfrac{1}{2}\int_0^T \boldsymbol{\alpha}_u' \,\boldsymbol{\Sigma}^{-1}\,\boldsymbol{\alpha}_u \,du - \int_0^T \boldsymbol{\alpha}_u' \,d\boldsymbol{W}_u \\
&- \int_0^T \int_{-\infty}^{\infty} (e^{g_u(z)}-1)\,\nu(dz,du) + \int_0^T \int_{-\infty}^{\infty} g_u(z)\,\mu(dz,du) \Bigg\}.
\end{aligned}$$

Since the PRM driving the jumps in spot prices and the OU process are mutually independent, a Radon–Nikodym derivative that generates an equivalent measure can be factored into the product of two independent components. Figure 1 shows that this decomposition can be viewed as two consecutive measure changes which yield the desired total measure change. This total measure change can be reached via two canonical paths: (i) alter only the diffusive components, and then alter the jump component, or (ii) alter the jump component, and then alter the diffusive components. That is,

$$\frac{d\mathbb{Q}^{\boldsymbol{\alpha},g}}{d\mathbb{P}} = \frac{d\mathbb{Q}^{\boldsymbol{\alpha}}}{d\mathbb{P}} \cdot \frac{d\mathbb{Q}^{\boldsymbol{\alpha},g}}{d\mathbb{Q}^{\boldsymbol{\alpha}}} \qquad \text{or} \qquad \frac{d\mathbb{Q}^{\boldsymbol{\alpha},g}}{d\mathbb{P}} = \frac{d\mathbb{Q}^{g}}{d\mathbb{P}} \cdot \frac{d\mathbb{Q}^{\boldsymbol{\alpha},g}}{d\mathbb{Q}^{g}},$$

and the set of candidate measures is such that

$$\mathcal{Q} = \Bigg\{ \mathbb{Q}^{\boldsymbol{\alpha},g} : \boldsymbol{\alpha}_t \text{ and } g_t \text{ are } \mathcal{F}_t\text{-predictable}, \ \mathbb{E}^{\mathbb{P}}\left[\int_0^T \boldsymbol{\alpha}'_u\, \boldsymbol{\Sigma}^{-1}\, \boldsymbol{\alpha}_u\, du\right] < +\infty\,, \\ \mathbb{E}^{\mathbb{P}}\left[\int_0^T \int_{-\infty}^{\infty} g_u^2\, \nu(dz,du)\right] < +\infty \Bigg\}. \tag{25}$$

Equipped with the set of measures that the agent considers as alternative models, we proceed to solving the control problem faced by the buyer and seller. The buyer and seller assign the robust price

$$H^{\pm}(t,x,y) = \pm \inf_{\mathbb{Q}^{\boldsymbol{\alpha},\lambda}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}^{\boldsymbol{\alpha},\lambda}}_{t,x,y}\left[\pm\frac{1}{T_2-T_1}\int_{T_1\vee t}^{T_2} S_u\, du + \mathcal{H}^{\boldsymbol{\Phi},\varepsilon}(\mathbb{Q}^{\boldsymbol{\alpha},\lambda}|\mathbb{P})\right] \tag{26}$$

to the forward contract, where the cost from deviating from the reference model is given by the penalization

$$\begin{aligned}&\mathcal{H}^{\boldsymbol{\Phi},\varepsilon}(\mathbb{Q}^{\boldsymbol{\alpha},\lambda}|\mathbb{P})\\ &= \mathbb{E}^{\mathbb{Q}^{\boldsymbol{\alpha},g}}\left[\frac{1}{2}\int_0^{T_2} \boldsymbol{\alpha}'_u\, \left(\boldsymbol{\Phi}^{\pm}\right)^{-1}\, \boldsymbol{\alpha}_u\, du + \frac{1}{\varepsilon^{\pm}}\int_0^{T_2}\int_{-\infty}^{\infty}\left(1+e^{g_u(z)}\,(g_u(z)-1)\right)\nu_{\mathbb{P}}(dz,du)\right],\end{aligned} \tag{27}$$

where in the matrix $\boldsymbol{\Phi}^{\pm}$ the entries are $\phi_x^{\pm}$ and $\phi_v^{\pm}$ for buyer $(+)$ and seller $(-)$, respectively.

It is important to point out that the above penalty is not a relative entropy since each component in (27) measures distances between various measures yet the optimization problem is given by an expectation under the single measure $\mathbb{Q}^{\boldsymbol{\alpha},g}$. Moreover, only in the particular case $\varepsilon = \phi$ and $\phi_x = \phi_v = 0$ does the penalty reduce to relative entropy.

For clarity of exposition, throughout the rest of the paper we drop the superscript $\pm$ in the ambiguity aversion parameters. Only when relevant will we stress different ambiguity parameters for the buyer and seller.

3.1. Dynamic programming equation. Here we discuss how the agents choose a candidate measure and, for simplicity of notation, we assume that the buyer and seller use the same reference model. To solve the optimal control problems in (26), the dynamic programming principle holds, and the value functions $H^{\pm}(t,x,v,y)$ are the unique solutions of the Hamilton–Jacobi–Bellman (HJB) equations

$$\begin{aligned}0 =& (\partial_t + \mathcal{L})H^{+} + \inf_{\boldsymbol{\alpha}}\left\{\sqrt{v}\,\boldsymbol{\alpha}'\,\boldsymbol{\Omega}\,\boldsymbol{\mathcal{D}}H^{+} + \tfrac{1}{2}\boldsymbol{\alpha}'\,\boldsymbol{\Phi}^{-1}\,\boldsymbol{\alpha}\right\}\\ &+ \lambda \inf_{g\in\mathcal{G}}\int_{-\infty}^{\infty}\left\{\Delta_z H^{+}\, e^{g(z)} + \frac{1}{\varepsilon}\left(1+e^{g(z)}\,(g(z)-1)\right)\right\} G(dz) \qquad \text{(Buyer case)}\\ &+ P(t,\theta(t),x,y)\,\mathbb{1}_{t\geq T_1}\,,\end{aligned} \tag{28}$$

and

$$\begin{aligned}
0 = {} & (\partial_t + \mathcal{L})H^- + \sup_{\boldsymbol{\alpha}} \left\{ \sqrt{v}\, \boldsymbol{\alpha}'\, \boldsymbol{\Omega}\, \boldsymbol{\mathcal{D}} H^- - \tfrac{1}{2} \boldsymbol{\alpha}'\, \boldsymbol{\Phi}^{-1}\, \boldsymbol{\alpha} \right\} \\
& + \lambda \sup_{g \in \mathcal{G}} \int_{-\infty}^{\infty} \left\{ \Delta_z H^- \, e^{g(z)} - \frac{1}{\varepsilon} \left(1 + e^{g(z)}\, (g(z) - 1) \right) \right\} G(dz) \\
& + P(t, \theta(t), x, y)\, \mathbb{1}_{t \geq T_1}\,,
\end{aligned} \tag{29}$$

(Seller case)

subject to the terminal conditions

$$H^{\pm}(T_2, x, v, y) = 0\,,$$

where

$$\boldsymbol{\Omega} = \begin{pmatrix} 1 & 0 \\ 0 & \eta \end{pmatrix}, \quad \boldsymbol{\mathcal{D}} H = \begin{pmatrix} \partial_x H \\ \partial_v H \end{pmatrix},$$

the shift operator $\Delta_z H(t, x, v, y)$ is as in (12), and $\mathcal{L}$ denotes the $\mathbb{P}$-generator of $\{X_t, v_t, Y_t^c\}$, which is given by (11) and the constant ψ (see (13)) is the expected jump size.

For a general spot price model, these equations cannot be solved in closed-form. However, they can be solved in feedback form, and numerical schemes can be applied. To this end, the proposition below shows the nonlinear PIDE which the candidate value function satisfies.

Proposition 1 (forward price PIDE with model uncertainty). *Let the spot commodity price be given by* (8). *The candidate solution for the buyer and seller forward prices solves the following PIDEs:*

$$\begin{aligned}
& (\partial_t + \mathcal{L})H^{\pm} \mp \tfrac{1}{2} \boldsymbol{\mathcal{D}} H^{\pm\prime}\, \boldsymbol{\Omega}'\, \boldsymbol{\Phi}^{\pm}\, \boldsymbol{\Omega}\, \boldsymbol{\mathcal{D}} H^{\pm} \\
& \qquad \pm \frac{\lambda}{\varepsilon^{\pm}} \int_{-\infty}^{\infty} \left(1 - e^{\mp \varepsilon^{\pm} \Delta_z H^{\pm}} \right) G(dz) + P(t, \theta(t), x, y)\, \mathbb{1}_{t \geq T_1} = 0\,,
\end{aligned} \tag{30}$$

subject to the terminal condition

$$H^{\pm}(T_2, x, v, y) = 0 \quad \forall x, v, y\,.$$

Moreover the optimal controls in feedback form are

$$\alpha^{\pm,*} = \mp \sqrt{v}\, \boldsymbol{\Phi}^{\pm}\, \boldsymbol{\Omega}\, \boldsymbol{\mathcal{D}} H^{\pm} \qquad \textit{and} \qquad g^{\pm,*}(z) = \mp\, \varepsilon^{\pm}\, \Delta_z H^{\pm}\,. \tag{31}$$

Proof. See Appendix A.2. ■

In the next section, we study the arithmetic model, which can be solved explicitly, and in section 5 we study the geometric model by performing numerical experiments, as well as a perturbation expansion.

4. Robust modeling in the arithmetic model.

In this section, we investigate the explicit form of the value function when the price function is arithmetic, i.e., when $P(t, \theta, x, y) = \theta + x + y$. The proposition below contains the main result of this section.

Proposition 2. *The solution of the buyer/seller HJB equations* (28) *and* (29) *in the arithmetic price model* $P(t, \theta, x, y) = \theta + x + y$, *admits the ansatz*

$$H^{\pm}(t, x, y, v) = A^{\pm}(t; \boldsymbol{\Phi}, \boldsymbol{\varepsilon}) + B(t)\, x \mp C(t; \boldsymbol{\Phi})\, v + D(t)\, y\,,$$

where

$$A^{\pm}(t) = \int_t^{T_2} \left\{ \theta(u)\, \mathbb{1}_{u>T_1}\, du \mp \kappa_v\, \vartheta\, C(u) - \lambda\, \psi\, D(u) \pm \lambda \frac{1-\Psi\left(\mp\, \boldsymbol{\varepsilon}\, D(u)\right)}{\boldsymbol{\varepsilon}} \right\} du\,, \tag{32a}$$

$$B(t) = \frac{1-e^{-\kappa_D (T_2-T_1)}}{\kappa_D}\, e^{-\kappa_D (T_1-t)}\, \mathbb{1}_{t<T_1} + \frac{1-e^{-\kappa_D (T_2-t)}}{\kappa_D}\, \mathbb{1}_{t\ge T_1}\,, \tag{32b}$$

$$D(t) = \frac{1-e^{-\kappa_J (T_2-T_1)}}{\kappa_J}\, e^{-\kappa_J (T_1-t)}\, \mathbb{1}_{t<T_1} + \frac{1-e^{-\kappa_J (T_2-t)}}{\kappa_J}\, \mathbb{1}_{t\ge T_1}\,, \tag{32c}$$

and $C(t;\boldsymbol{\Phi})$ satisfies the Riccati ODE

$$0 = \partial_t C(t;\boldsymbol{\Phi}) - \kappa_v C(t;\boldsymbol{\Phi}) + \tfrac{1}{2} \begin{pmatrix} B(t) & -C(t;\boldsymbol{\Phi}) \end{pmatrix} \tilde{\boldsymbol{\Phi}} \begin{pmatrix} B(t) \\ -C(t;\boldsymbol{\Phi}) \end{pmatrix}, \qquad C(T_2;\boldsymbol{\Phi}) = 0\,, \tag{33}$$

where $\tilde{\boldsymbol{\Phi}} = \boldsymbol{\Omega}'\,\boldsymbol{\Phi}\,\boldsymbol{\Omega}$, ψ is the expected jump size (13), and

$$\Psi(a) = \int_{-\infty}^{\infty} e^{az}\, G(dz) \tag{34}$$

is the moment generating function of the jump distribution.

Moreover, the optimal controls are

$$\boldsymbol{\alpha}_t^{\pm,*} = \mp\sqrt{v_t}\; \boldsymbol{\Phi}\,\boldsymbol{\Omega} \begin{pmatrix} B(t) \\ \mp\, C(t;\boldsymbol{\Phi}) \end{pmatrix} \qquad \textit{and} \qquad g_t^{\pm,*}(z) = \mp\boldsymbol{\varepsilon}\, D(t)\, z\,. \tag{35}$$

Proof. See Appendix A.3. ∎

Proposition 3. *The solution to C in (33) is nonnegative.*

Proof. See Appendix A.4 ∎

Next, we provide a verification theorem showing that the optimal solutions we find are the solutions to the original control problem.

Theorem 3 (verification theorem). *The candidate solution of the HJB equations provided in Proposition 2 are the solutions of the original control problems (26), when the price function is arithmetic, and are the value functions the agents seek.*

Proof. See Appendix A.5. ∎

4.1. The effect of ambiguity aversion on diffusive and jump factors. To understand the intuition behind our result we discuss the effect of model uncertainty on both factors.

Ambiguity aversion to diffusive factor. Assume that agents are ambiguous about their reference model for the diffusion factor and confident about the jump factor so that ϕ, $\phi_{x,v} > 0$ (see $\boldsymbol{\Phi}^{-1}$ above in (21)) and $\varepsilon \downarrow 0$ in Proposition 2. Thus the buyer and seller choose candidate models where the drifts of the diffusive components are given by

$$\boldsymbol{\alpha}_t^{\pm,*} = \mp\sqrt{v_t}\; \boldsymbol{\Phi}\,\boldsymbol{\Omega} \begin{pmatrix} B(t) \\ \mp\, C(t;\,\boldsymbol{\Phi}) \end{pmatrix} := \mp\, \sqrt{v_t} \begin{pmatrix} E^{\pm,*}(t;\,\boldsymbol{\Phi}) \\ L^{\pm,*}(t;\,\boldsymbol{\Phi}) \end{pmatrix},$$

and recall that $B(t) \ge 0$ and $C(t;\,\boldsymbol{\Phi}) \ge 0$. Since C is very small compared to B, we have that $E^{\pm,*} \sim (\boldsymbol{\Phi\Omega})_{11}\, B(t)$ and $L^{\pm,*} \sim (\boldsymbol{\Phi\Omega})_{21}\, B(t)$ and so both are increasing, and positive, up to T_1—note they are the same for buyer and seller, but the sign out front multiplying $\sqrt{v_t}$ makes their contributions different.

Thus, under the optimal measure, the diffusive and volatility factors satisfy

$$dX_t = \kappa_D \left(\mp \frac{1}{\kappa_D} E^{\pm,*}(t)\, v_t - X_t \right) dt + \sqrt{v_t}\, dW_t^{X,*}\,, \tag{36}$$

$$dv_t = \kappa_v^{\pm,*}(t)\, \left(\vartheta^{\pm,*}(t) - v_t\right)\, dt + \eta\, \sqrt{v_t}\, dW_t^{v,*}\,, \tag{37}$$

where

$$\kappa_v^{\pm,*}(t) = \kappa_v \pm \eta\, L^{\pm,*}(t;\mathbf{\Phi})\,, \tag{38}$$

$$\vartheta^{\pm,*}(t) = \left(1 \pm \frac{\eta}{\kappa_v} L^{\pm,*}(t;\mathbf{\Phi}) \right)^{-1} \vartheta\,. \tag{39}$$

Next, equipped with the dynamics (36) and (37) we see how agents' ambiguity affects the mean-reverting diffusive factor X_t and the variance factor v_t. The insights are as follows.

Effect of ambiguity on mean reversion level of X_t. In the optimal measure, the mean reversion level of X_t becomes time dependent and is negative for the buyer (positive for the seller), and recall that under the reference measure the mean reverting level is zero. Moreover, this deviation from the zero mean reverting level of the reference model is more pronounced as we approach the start of the delivery period and also more pronounced the higher the value of the ambiguity parameter. Hence, the buyer (seller) expects future prices to decrease (increase) relative to the ambiguity neutral agent.

Effect of ambiguity on speed and level of mean reversion of v_t. In the optimal measure, the agents modify the dynamics of the variance model in two ways. One, the speed of mean reversion κ_v in the reference model is modified to one which is time dependent and $\kappa_v^{+,*}(t) > \kappa_v$ for the buyer and $\kappa_v^{-,*}(t) < \kappa_v$ for the seller. In addition, the level to which v_t reverts also becomes time dependent under the optimal measure: $\vartheta^{+,*}(t) < \vartheta_v$ for the buyer, and $\vartheta^{-,*}(t) > \vartheta_v$ for the seller. Hence, the buyer (seller) expects the volatility of volatility to decrease (increase), as well as the expected level of volatility, relative to the ambiguity neutral agent. This would result in decreased (increased) option prices with positive Vega for the buyer (seller) relative to an ambiguity neutral agent.

Ambiguity aversion to jump factor. Similarly, if the agents are ambiguous only to the jump factor in their reference model ($\varepsilon > 0$ and $\phi = \phi_x = \phi_v = 0$ in Proposition 2), then they choose candidate measures where the arrival and size of jumps are modified according to

$$g_t^{\pm,*}(z) = \mp\, \varepsilon\, D(t)\, z\,, \tag{40}$$

and recall that in the alternative measure considered by the agents the jump measure is $\nu_{\mathbb{Q}^g}(dz, dt) = e^{\,g_t(z)}\, \nu_{\mathbb{P}}(dz, dt)$.

The ambiguity averse seller adjusts the arrival rate and jump size as follows. Under the optimal measure $\mathbb{Q}^{g,*}$, the jump component, positive jumps are larger, and their arrival more frequent, than in the reference measure. In addition, the frequency and size of negative jumps are altered so that compared to the reference measure they arrive less often and are of smaller size. In this way, the seller computes conservative prices—i.e., higher than those obtained under the reference measure.

The ambiguity averse buyer also becomes more conservative. He picks an optimal measure under which positive jumps arrive less often and are of smaller size, and negative jumps arrive more frequently and are larger. Hence, the buyer's willingness to pay for the commodity is at prices which are lower than those computed under the reference model.

Furthermore, the effect of ambiguity aversion on the diffusive and jump factors is proportional to the degree of ambiguity. The higher the degree of ambiguity aversion, the more conservative the behavior shown by the agents when computing prices at which they are willing to trade. Moreover, for fixed ε, $\mathbf{\Phi}$, the effect of model uncertainty intensifies as t approaches the start of delivery T_1. In the next corollary we show that in a model where buyer and seller use the same reference measure and either one is ambiguity averse to at least one of the factors, a trade between them would never occur.

Corollary 1 (absence of trade between buyer and seller under model uncertainty). *If the buyer and seller employ the same reference model, and at least one of them is ambiguity averse with respect to either the diffusive or the jump component, they will never trade between them.*

Explicitly, the bid-ask spread $\Upsilon(t,x,v,y) := H^-(t,x,v,y) - H^+(t,x,v,y)$ *is*

$$\Upsilon(t,x,v,y) = \kappa_v\,\vartheta \int_t^{T_2} \left[C(u;\mathbf{\Phi}^-) + C(u;\mathbf{\Phi}^+)\right]\,du \tag{41a}$$

$$+\,\lambda \int_t^{T_2} \left(\frac{1-\Psi\left(-\varepsilon^+ D(u)\right)}{\varepsilon^+} - \frac{\Psi\left(\varepsilon^- D(u)\right)-1}{\varepsilon^-}\right)\,du \tag{41b}$$

$$+\left[C(t;\mathbf{\Phi}^-) + C(t;\mathbf{\Phi}^+)\right]\,v\,. \tag{41c}$$

Moreover we have that $\Upsilon(t,x,v,y) \geq 0$*, where equality holds if and only if* $\mathbf{\Phi}^{\pm} = \mathbf{0}$ *and* $\varepsilon^{\pm} = 0$.

Proof. See Appendix A.6. ∎

The above result shows that the buyer will never accept the seller's ask price. It also shows that the current value of the diffusion component X_t and jump component Y_t do not affect the spread. The current level of volatility, however, does affect the spread through the term (41c). This term can be viewed as the volatility premium (discount) that agents add/subtract to forward prices. Moreover, note that the spread is stochastic due to the presence of the variance component in (41c).

4.2. Disentangling the effect of ambiguity aversion on jumps. In the setup of our model jumps arrive according to a homogeneous Poisson process, and it is possible to disentangle the effect of ambiguity aversion on the arrival rate of jumps from the distribution of the jump size. This is achieved by integrating

$$\nu^*(dz,dt) = e^{g_t^{\pm,*}(z)}\nu^{\mathbb{P}}(dy,dt) = \lambda\,e^{\mp\boldsymbol{\varepsilon}\,D(t)\,z}\,G(dz)\,dt$$

over jump sizes, to obtain the optimal jump intensity

$$\lambda^*(t) = \lambda \int_{-\infty}^{\infty} e^{\mp\boldsymbol{\varepsilon}^{\pm}\,D(t)\,z}\,G(dz) \tag{42}$$

and the optimal jump distribution

$$G^*(t,dz) = \frac{e^{\mp\boldsymbol{\varepsilon}^{\pm}\,D(t)\,z}\,G(dz)}{\displaystyle\int_{-\infty}^{\infty} e^{\mp\boldsymbol{\varepsilon}^{\pm}\,D(t)\,z}\,G(dz)}\,. \tag{43}$$

4.2.1. Double exponential. An interesting case to work out explicitly is when the jump distribution is double exponential. Specifically, suppose that

$$G(dz) = \left\{ p\, a^+\, e^{-a^+ z}\, \mathbb{1}_{z>0} + (1-p)\, a^-\, e^{-a^- |z|}\, \mathbb{1}_{z\leq 0} \right\} dz\,, \tag{44}$$

where $p \in [0,1]$ and $a^{\pm} > 0$, so that when a jump arrives, it is positive (negative) with probability p $(1-p)$ and exponentially distributed with mean size $\frac{1}{a^+}$ $(\frac{1}{a^-})$, respectively.

In this case, the optimal measure remains in the class of double exponential distributions, albeit a time dependent one. Explicitly, (42) becomes

$$\lambda^{\pm,*}(t) = \lambda \left\{ p\, \frac{a^+}{a^+ \pm \varepsilon\, D(t)} + (1-p)\, \frac{a^-}{a^- \mp \varepsilon\, D(t)} \right\}$$

and (43) becomes

$$G^{\pm,*}(t,dz) = \left\{ p^{+,*}(t)\, e^{-a^{\pm+,*}(t)\, z} \mathbb{1}_{z>0} + (1-p^{+,*}(t))\, e^{-a^{\pm-,*}(t)\, |z|} \mathbb{1}_{z>0} \right\} dz\,,$$

where the parameters for the positive and negative jump sizes are

$$a^{\pm+,*} = a^+ \pm D(t), \qquad a^{\pm-,*} = a^- \mp D(t)\,,$$

and the probability of an up jump is

$$p^{\pm,*}(t) = \left(1 + \frac{1-p}{p}\, \frac{a^-}{a^+}\, \frac{a^+ \pm \varepsilon\, D(t)}{a^- \mp \varepsilon\, D(t)} \right)^{-1}\,.$$

To illustrate how the agents modify the arrival rate and jump sizes when they are ambiguity averse, we use the explicit formulae derived here and show in Figure 2 the optimal: arrival rate λ^*, probability of upward jumps p^*, and mean jump sizes. The reference model parameters for the jump factor are

$$\lambda = 2/\text{week}, \quad \kappa_J = 5, \quad p = 0.8, \quad a^+ = 20, \quad a^- = 10\,. \tag{45}$$

The jump ambiguity parameter and start and end of forward contract are

$$\varepsilon = 10^{-3}\,, \qquad T_1 = 1\,\text{week}\,, \qquad T_2 = 2\,. \tag{46}$$

The overall message of Figure 3, as already discussed in section 4 for the arithmetic case, is that the buyer (seller) enhances downward (dampens) jumps and dampens (enhances) upward jumps. These effects are intensified as we approach the start of the delivery period of the forward contract. The figure shows in detail how each component of the jump factor is modified by ambiguity aversion. The top left panel in the figure shows the agents' optimal arrival rates. When start of delivery is far away, the optimal rates $\lambda^{\pm,*}$ coincide with that of the reference measure, but as delivery of the forward contract approaches, i.e., closer to T_1, the optimal arrival rates diverge: the seller's increases and the buyer's decreases.

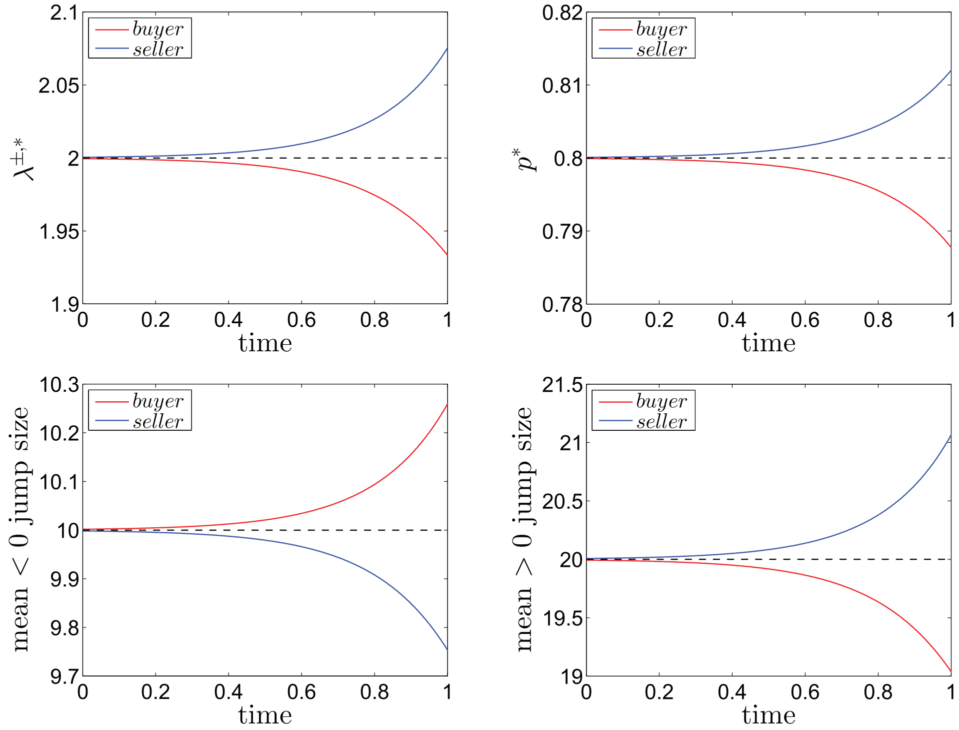

Figure 2. *Effect of ambiguity aversion on jump factor.*

Moreover, the top right panel depicts the probability of positive jumps which exhibits a similar time-dependent behavior as that of the optimal arrival rate. The bottom panel shows the mean jump sizes under the optimal measure. For example, we see that the buyer modifies the reference measure so that the mean size of negative jumps is larger (in absolute value) under the optimal measure, and the mean size of positive sizes is smaller in the optimal measure. As with the optimal arrival rate, all these effects are intensified as we approach the start of the delivery period of the forward contract.

4.3. Simulations: Arithmetic model. Here we present numerical results and simulations to show how ambiguity aversion affects the different components of the model and how agents compute robust forward prices. We assume that the seasonal component is $\theta = 20$, the jump parameters are as in (45), the ambiguity and delivery period parameters are

$$\varepsilon = \{1, 2, 4\} \times 10^{-3}\,, \quad \phi = \{1, 2, 4\} \times 10^{-9}\,, \quad \phi_x = \phi_v = 0\,, \quad T_1 = 1\,\text{week}\,, \quad T_2 = 2\,\text{weeks}\,,$$

and the diffusion parameters are

$$\kappa_D = 0.5, \quad \vartheta = 50^2, \quad \kappa_v = 25, \quad \text{and} \quad \eta = 300\,.$$

The top panel of Figure 3 shows a sample path of the spot price and the forward prices of the buyer and seller. We observe that the effect of jumps in the underlying commodity has close to no effect on the robust forward prices if the start of delivery is far away (see jump at

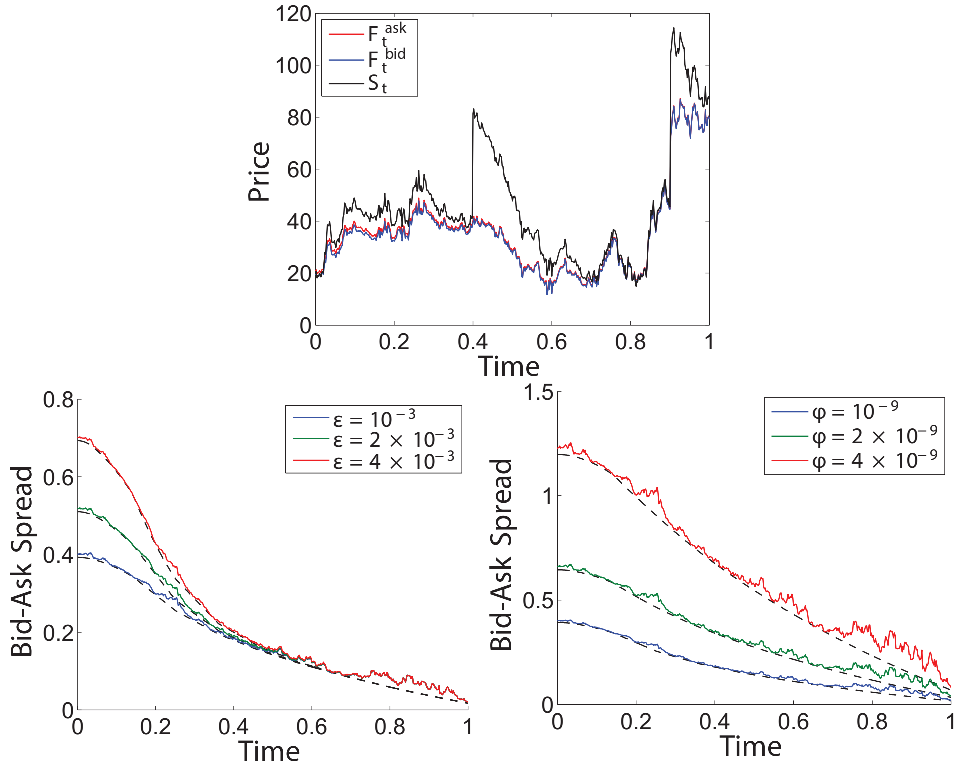

Figure 3. *Forward price is unaffected by the early jump, but it responds to the late jump. Below, dashed lines show spreads when volatility is constant ($\kappa_v = \eta = 0$).*

around 0.4 week), but nearer the start of delivery, jumps in the spot price do cause jumps in forward prices.

The left-hand figure in the bottom panel shows the bid-ask spread of the robust forward contract (seller minus buyer; see Corollary 1) when the agents are ambiguous to only the jump factor. We see that as the degree of ambiguity aversion to the jump component increases, the spread increases. The right-hand figure shows a similar result when the agents are ambiguity averse to only the diffusive component. Again, we see that the higher the degree of ambiguity aversion, the wider the spread. Both figures in the bottom panel show that as time approaches the start of delivery, the spreads narrow because there is less uncertainty about the price of the underlying commodity as the forward contract enters into the delivery period.

Furthermore, as already seen in the discussion following Corollary 1, which provides an analytical expression of the spread, the stochastic behavior of the spread is due to stochastic variance; see (41c). Thus, if variance is constant (set volatility of the volatility to zero), the spread becomes deterministic—this case is depicted with dashed lines in the bottom panel.

5. Robust modeling in the geometric model.

5.1. Computing forward prices: Numerical method. From looking at the form of the optimal controls in Proposition 1 we see that the overall effect of ambiguity aversion in the geometric model is similar to that of the arithmetic case and can be summarized as follows.

The buyer and seller choose optimal measures under which the spot price of the commodity drifts up for the seller and down for the buyer, and the combined effect of arrival and size of jumps exhibits an asymmetric behavior: the buyer enhances jumps down and dampens upward jumps, and the seller enhances jumps up and dampens jumps down—see optimal controls (31).

However, another interesting feature that we observe in the geometric model, which is not present in the arithmetic one, is that there are cross effects between the jump and diffusive state variables. We use a numerical scheme to examine these cross effects and illustrate the effects in the figures below. For ease of computation, we assume that volatility is constant, so we only investigate the cross effects when the agents are ambiguous only to the diffusive and jump factor. We assume

$$\theta(t) = \log(20)\,, \qquad \lambda = 2/\text{week},$$

and jump sizes have a double exponential distribution (44) with parameters

$$p = 0.8, \quad a^{+} = \log(1.2), \quad a^{-} = -\log(0.9)\,.$$

The diffusion parameters and delivery period are

$$\kappa_D = 0.5, \quad \kappa_J = 2\,, \quad \text{and} \quad T_1 = 1\,\text{week}, \qquad T_2 - T_1 = 1\,\text{day},$$

and the ambiguity parameters are

$$\varepsilon = 0.04\,, \quad \phi = 0.02\,.$$

Recall that under the reference measure the drift of the diffusive component is 0. When implementing a numerical scheme, we use a discrete version of the jump distribution, where the process can jump only to points on the grid. With this approximation, the mean jump $\psi = \int_{-\infty}^{\infty} z\,G(dz)$ under the reference model is 0.1037. This should be used for comparison with the mean jump under the optimal measure shown below.

Here we use an explicit finite difference scheme to solve the nonlinear PIDEs. We then use the numerical solutions to understand how ambiguity aversion affects forward prices. In particular we look at the optimal controls $\alpha^*(t)$ and the effect of the control $g_t^*(z)$ via the average jump size in the optimal measure for a range of values of the variables x and y.

For the set of parameters we choose here, the numerical results show the cross effects the state variables x, diffusion, and y, jump, have on the drift and mean jump adjustment. Overall, the larger x and y are, the stronger the adjustment in the drift under the optimal measure relative to the drift in the reference measure, which is zero. The top left panel of Figure 4 shows that when x and y are large and positive, the buyer assumes a larger upward drift adjustment than when x and y are both large and negative. Similarly, the top right panel shows that when x and y are large and positive, the seller assumes a larger downward drift adjustment than when x and y are both large and negative.

The bottom panels in Figure 4 show the cross effects that x, y have on the optimal arrival rates $\lambda^{\pm,*}$ at $t = 0$. In both pictures we see that when x and y are large (positive) the optimal arrival rate of jumps is furthest away from the arrival rate in the reference model ($\lambda = 2$):

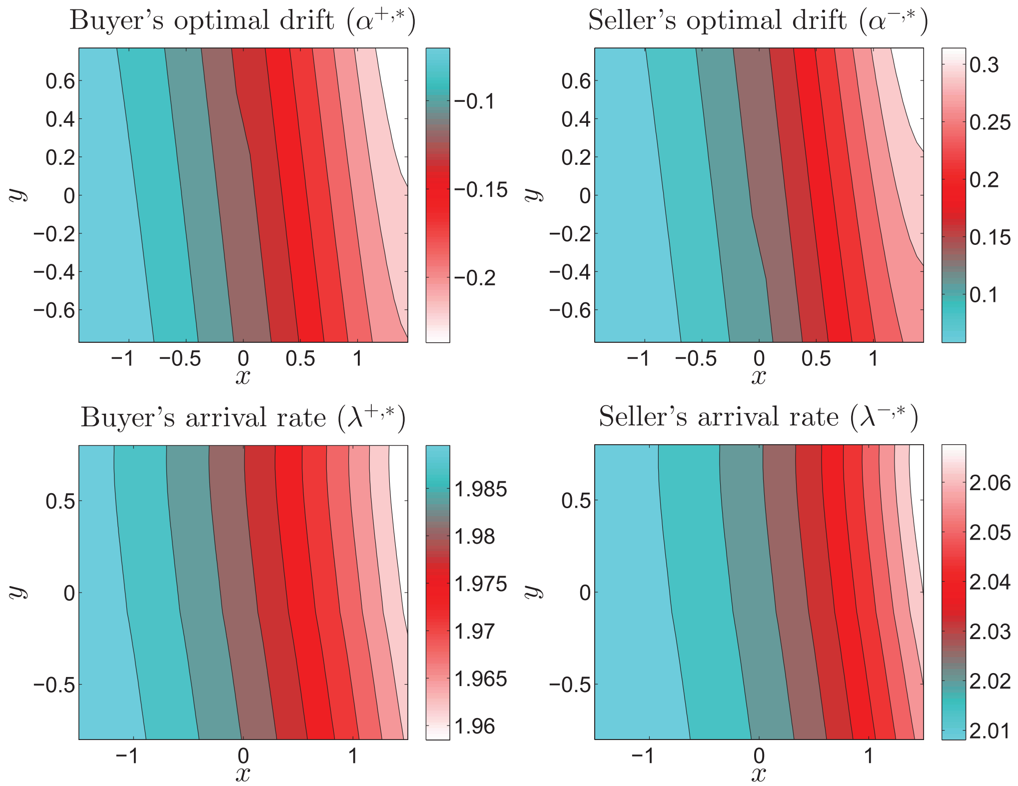

Figure 4. *Cyan: close to reference model; red: away from reference model; white: far from reference model.*

the buyer's is below and the seller's is above. As the pair x, y decreases, their optimal arrival rates get closer to the arrival rate in the reference model.

Furthermore, the ambiguity averse agent modifies the jump sizes in a similar way (in the interest of space we omit the figures). For example, as the pair x, y increases, the absolute difference between the mean jump sizes in the optimal measure and the reference model increases.

Finally, we perform a simulation of underlying drivers, X_t and Y_t, and show in Figure 5 how the robust forward prices and spreads are affected when the agents are ambiguity averse. In the simulations we employ the above set of parameters but use a wider range of ambiguity aversion parameters. The effect of ambiguity aversion in the geometric case is similar to that of the arithmetic discussed above.

The top panel of the figure shows a sample path of the spot price and the forward prices of the buyer and seller. We observe that the effect of jumps in the underlying commodity has a marginal effect on the forward prices if the start of delivery is far away (see jump at around 0.4 week), but nearer the start of delivery jumps in the spot price do cause jumps in forward prices.

In the bottom panel, the first figure shows the bid-ask spread of the forward contract when the agents are ambiguous to only the jump factor. As the degree of ambiguity aversion to the jump component increases, the spread increases. The right-hand figure shows a similar

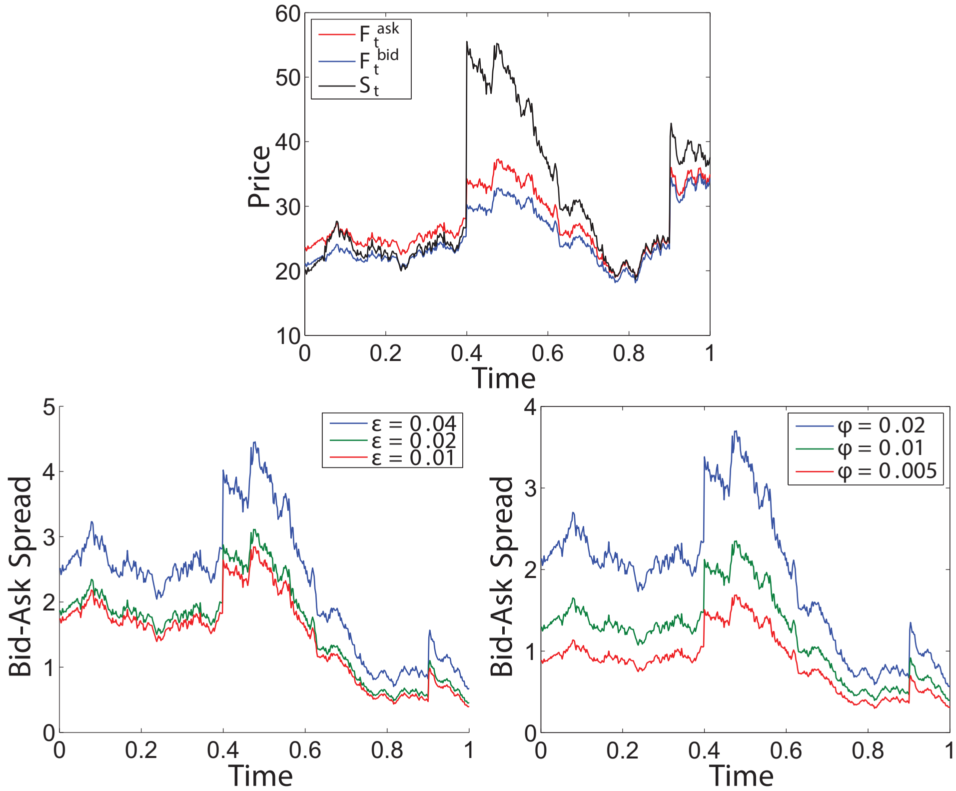

Figure 5. *Forward price is moderately affected by the early jump, but it responds strongly to the late jump.*

result when the agents are ambiguity averse to only the diffusive component. Again, we see that the higher the degree of ambiguity aversion, the wider the spread. Both figures in the bottom panel show that as time approaches start of delivery, the spreads narrow because there is less uncertainty around the commodity price as the forward contract enters into the delivery period. Furthermore, unlike in the arithmetic case shown in Figure 3, the spread in the exponential model is affected by jumps in the forward price itself, that is, the spread is affected by the Y_t driver.

5.2. Computing forward prices: Perturbation method.

Here we employ a regular perturbation method (see Fouque, Jaimungal, and Lorig (2011)) to approximate the forward prices that the buyer and seller compute. For simplicity, we set $\phi_x = \phi_v = 0$ and instead focus on joint ambiguity through ϕ and jump ambiguity through ε. In this case, we obtain a perturbation expansion of the robust forward price by writing

$$\begin{aligned} H(t,x,v,y) &= H^{(0)}(t,x,v,y) \\ &\quad + \underbrace{\phi\, H^{\pm,(1,0)}(t,x,v,y)}_{\text{diffusion adjustment}} + \overbrace{\varepsilon\, H^{\pm,(0,1)}(t,x,v,y)}^{\text{jump adjustment}} + O\left(\varsigma^2\right) \end{aligned} \tag{47}$$

for the buyer and seller, where $\varsigma^2 = \max(\phi^2, \phi\varepsilon, \varepsilon^2)$. Here, the zeroth order component $H^{(0)}(t,x,v,y)$ is the forward price under the reference measure, briefly discussed in Theorem 2, and repeated here for convenience:

$$H^{(0)}(t,x,v,y) = \int_{T_1}^{T_2} \exp\left\{\Theta(t,u) + x\, e^{-\kappa_D(u-t)} + v\, C(t,u) + y\, e^{-\kappa_J(u-t)}\right\} du\,, \tag{48}$$

where $\Theta(t,u)$ is given by (16) and $C(t,u)$ solves (15). The diffusion $H^{(1,0)}(t,x,v,y)$ and jump $H^{(0,1)}(t,x,v,y)$ adjustments account for the corrections that capture the effect of ambiguity on the prices.

Theorem 4 (approximation formula for forward contract). *Suppose that the value function admits the perturbation expansion (47). Then, the diffusion and jump ambiguity corrections to the forward price are given by*

$$H^{\pm,(1,0)}(t,x,y) = \mp\mathbb{E}^{\mathbb{P}}\left[\frac{1}{2}\int_t^{T_2} \left(\boldsymbol{\mathcal{D}}H^{(0)\prime}\,\boldsymbol{\Omega}'\boldsymbol{\Sigma}\boldsymbol{\Omega}\,\boldsymbol{\mathcal{D}}H^{(0)}\right)(u,X_u,v_u,Y_u)\,du \,\middle|\, \mathcal{F}_t\right], \tag{49a}$$

$$H^{\pm,(0,1)}(t,x,y) = \mp\mathbb{E}^{\mathbb{P}}\left[\frac{1}{2}\int_t^{T_2}\int_{-\infty}^{\infty} \left(\Delta_z H^{(0)}(u,X_u,v_u,Y_u)\right)^2 G(dz)\,du \,\middle|\, \mathcal{F}_t\right], \tag{49b}$$

where explicitly

$$\partial_x H^{(0)}(t,x,v,y) = \int_{T_1}^{T_2} e^{-\kappa_D(u-t)}\, L(t,u,x,v,y)\,du \geq 0\,, \tag{50a}$$

$$\partial_v H^{(0)}(t,x,v,y) = \int_{T_1}^{T_2} C(t,u)\, L(t,u,x,v,y)\,du \geq 0\,, \tag{50b}$$

$$\Delta_z H^{(0)}(t,x,v,y) = \int_{T_1}^{T_2} \left(\exp\left\{z\, e^{-\kappa_J(u-t)}\right\} - 1\right) L(t,u,x,v,y)\,du\,, \tag{50c}$$

and

$$L(t,u,x,v,y) = \exp\left\{\Theta(t,u) + x\, e^{-\kappa_D(u-t)} + v\, C(t,u) + y\, e^{-\kappa_J(u-t)}\right\}. \tag{50d}$$

Proof. See Appendix A.7. ■

Moreover, it is straightforward to obtain the optimal controls employing the zeroth order approximation of the price of the forward contract, that is, using $H^{\pm} \approx H^{(0)}$.

Proposition 4 (approximate optimal feedback control for geometric model). *Using the approximation (47) to value forward contracts, we obtain*

$$\boldsymbol{\alpha}^{\pm,*}(t) = \mp\,\boldsymbol{\Phi}\,\boldsymbol{\Omega}\,\boldsymbol{\mathcal{D}}H^{(0)}(t,X_t,v_t,Y_t) + O(\varsigma^2) \tag{51}$$

and

$$g_t^{\pm,*}(z) = \mp\,\varepsilon\,\Delta_z H^{(0)}(t,X_t,v_t,Y_t) + O(\varsigma^2)\,, \tag{52}$$

where $\boldsymbol{\mathcal{D}}H^{(0)}$ and $\Delta_z H^{(0)}$ are as in (50). Furthermore,

$$\mathit{sign}\left\{\Delta_z H^{(0)}\right\} = \mathit{sign}\,\{z\}\,.$$

Proof. The optimal controls (51) and (52) follow immediately from Proposition 1 applied to the approximation of H provided in Theorem 4. The result on the sign of $\Delta_z H^{(0)}$ follows immediately from (50c) and the fact that L in (50d) is clearly positive. ■

Using this perturbation approach it is clear how the buyer and seller bias the drift of the diffusive component as well as the arrival rate and size of jumps. For example, $\alpha^+(t) < 0$ so the buyer includes a downward trend in the spot price. This downward trend is stronger the higher the degree of ambiguity aversion specific to the diffusive component of the reference model.

In addition, we see that for positive jumps $z > 0$ the control $g_t^{+,*}(z) < 0$ so that, compared to jumps in the reference measure, positive jumps are reduced, and for negative jumps $z < 0$ the control $g_t^{+,*}(z) > 0$, so the arrival rate and size of upward jumps is larger than that in the reference model. We also see that this bias in the jump measure is stronger when the price of the commodity is high, i.e., high x and/or high y.

Finally, it is straightforward to see the cross effects between the jump and diffusive components. Not only do the controls depend on the pair (x, y), but the drift affects x, which in turn affects y because $g_t^*(z)$ depends on x too.

6. Option pricing.

In commodities, forwards or futures contracts are the most heavily traded. Trading the spot commodity directly is not customary and most of the time not feasible due to the perishability, or nonstorability of the commodity. There are other derivatives contracts, such as options on forwards/futures, Asian-style options, swing options, interruptible contracts, etc. In this section we focus on how options written on forward contracts are affected by ambiguity.

Let us consider an option which pays $\varphi\left(H(T_0, X_{T_0}, v_{T_0}, Y_{T_0})\right)$ at time $T_0 \le T_1$, i.e., an option maturing at T_0 on the forward contract (which delivers over the period $[T_1, T_2]$). Denote the robust certainty equivalent value of this option by $\mathfrak{H}(t, x, v, y; \varphi)$; then an easy computation shows that

$$\mathfrak{H}^\pm(t, x, v, y; \varphi) = \pm \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}_{t,x,v,y}\left[\pm\varphi\left(H(T_0, X_{T_0}, v_{T_0}, Y_{T_0})\right) + \mathcal{H}^{\boldsymbol{\Phi},\varepsilon}(\mathbb{Q}^{\boldsymbol{\alpha},\lambda}|\mathbb{P})\right], \tag{53}$$

where $H(t, x, v, y)$ denotes the robust certainty equivalent value of the forward contract investigated in the previous sections, and for simplicity we assume that the risk-free rate is zero.

Applying the dynamic programming principle shows that $\mathfrak{H}$ satisfy a similar HJB equation as H, but with no source term and a modified boundary terminal condition. Specifically, $\mathfrak{H}^\pm$ satisfies (for $t \in [0, T_0]$)

$$\begin{aligned} 0 = (\partial_t + \mathcal{L})\mathfrak{H}^\pm &\pm \inf_{\boldsymbol{\alpha}}\left\{ \pm\sqrt{v}\,\boldsymbol{\alpha}'\,\boldsymbol{\Omega}\,\boldsymbol{\mathcal{D}}\mathfrak{H}^\pm + \tfrac{1}{2}\boldsymbol{\alpha}'\,\boldsymbol{\Phi}^{-1}\,\boldsymbol{\alpha}\right\} \\ &\pm \lambda \inf_{g\in\mathcal{G}} \int_{-\infty}^{\infty}\left\{\pm\Delta_z\mathfrak{H}^\pm\, e^{g(z)} + \frac{1}{\varepsilon}\left(1 + e^{g(z)}\,(g(z)-1)\right)\right\} G(dz) \end{aligned} \tag{54}$$

with terminal condition

$$\mathfrak{H}^\pm(T_0, x, v, y; \varphi) = \varphi\left(H(T_0, x, v, y)\right). \tag{55}$$

Proposition 5 (option price with model uncertainty). *The candidate solution for the buyer and seller option prices solve the following PIDEs:*

$$(\partial_t + \mathcal{L})\mathfrak{H}^\pm \mp \tfrac{1}{2}\,\boldsymbol{\mathcal{D}}\mathfrak{H}^{\pm\prime}\,\boldsymbol{\Omega}'\,\boldsymbol{\Phi}\,\boldsymbol{\Omega}\,\boldsymbol{\mathcal{D}}\mathfrak{H}^\pm \pm \frac{\lambda}{\varepsilon}\int_{-\infty}^{\infty}\left(1 - e^{\mp\varepsilon\Delta_z\mathfrak{H}^\pm}\right) G(dz) = 0, \tag{56}$$

subject to the terminal condition

$$\mathfrak{H}^{\pm}(T_0, x, v, y; \varphi) = \varphi\left(H(T_0, x, v, y)\right) \quad \forall x, v, y\,.$$

Moreover the optimal controls in feedback form are

$$\alpha^{\pm,*} = \mp\sqrt{v}\,\boldsymbol{\Phi}\,\boldsymbol{\Omega}\,\boldsymbol{\mathcal{D}}\mathfrak{H}^{\pm} \qquad \textit{and} \qquad g^{\pm,*}(z) = \mp\,\varepsilon\,\Delta_z\mathfrak{H}^{\pm}\,. \tag{57}$$

Proof. The proof follows the same steps as Proposition 2. We omit the details for brevity. ■

The interpretation of the optimal controls is similar to the cases described above. Agents become conservative and compute robust prices which are below or above the price under the reference model. Ambiguity averse agents do this by modifying the drift of the diffusion process and tilting the jump measure of the reference model. In the arithmetic and geometric cases discussed above the sign of the optimal controls was clear, but here the signs of $\alpha^{\pm,*}$ and $g^{\pm,*}$ depend on the sign of $\boldsymbol{\mathcal{D}}\mathfrak{H}^{\pm}$ which can be interpreted as the sign of the Delta and the Vega of the option. For example, when these two are positive, the buyer (seller) introduces a negative (positive) drift in the diffusion of the diffusive factor commodity. And the buyer (seller) enhances (depresses) negative jumps and depresses (enhances) positive jumps.

Finally, as in the geometric cases, agents' optimal measures show cross effects between the diffusive and jump state variables. We explore these in the next subsection.

6.1. Numerical results. In this section we examine the effect of ambiguity aversion on the value a bull-spread option and a bear-spread struck around the money. Specifically, the bull-spread is a long call struck at $K_1 = 15$ and a short call struck at $K_2 = 25$, so that $\varphi(F) = (F - K_1)_+ - (F - K_2)_+$, where the operation $(U)_+$ denotes the maximum between U and 0. We use the same model and ambiguity parameters as in section 5.1 and assume that volatility is constant. The bear-spread consists of a short call struck at $K_1 = 15$ and a long call struck at $K_2 = 25$—note that the bear-spread is a short position in a bull-spread plus a bond. We discuss in detail the results for the bull-spread and then discuss the bear-spread.

Effect of ambiguity aversion on bull-spread. The first picture in Figure 6 shows the midprice, computed as the average price of the buyer and seller, as a function of x and y. The heatmap shows that when x is large the option is in-the-money and the midprice is highest, and when x is negative and large in absolute value, the option is out-of-the-money and the midprice is low. The second heatmap shows the spread, computed as the difference between the seller's and buyer's prices, as a function of x and y. We see that when the option is in- or out-of-the money (high or low x, y), the spread is lowest at around \$0.05, and as the pair x, y move inward to zero, i.e., the option is at-the-money, the spread widens, to about \$0.15. Clearly, when the option is in the neighborhood of at-the-money (between the strikes K_1 and K_2, where the option's Delta and Vega are positive) agents' ambiguity about the reference model is more important since any movement in the price of the underlying commodity (and hence price of the forward contract) has a considerable effect on the value of the option. Thus, agent's are more conservative around at-the-money values and, as we have shown, this implies that the buyer decreases and the seller increases the prices they are willing to trade at.

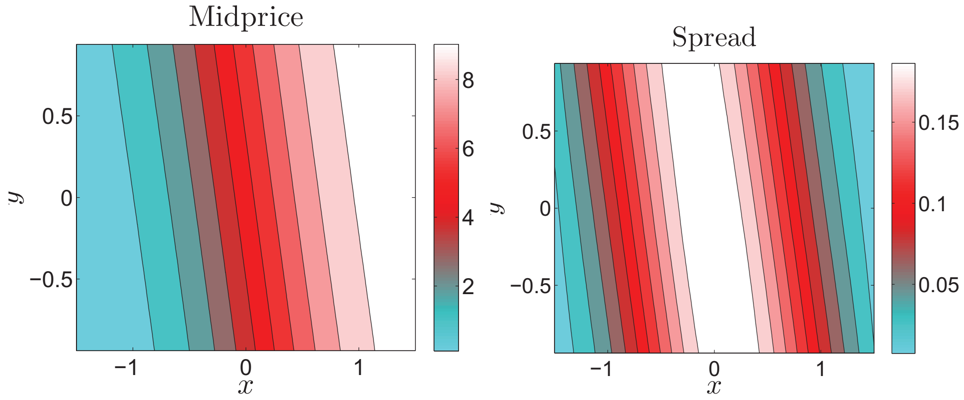

Figure 6. *The midprice and spread for the option at $t = 0$. Cyan: close to reference model; red: away from reference model; white: far from reference model.*

The above result, that the spread widens when the option is in the neighborhood of at-the-money, can also be seen through the effect that ambiguity aversion has on the drift of the diffusion process and on the jump measure. Figure 7 shows heatmaps of the optimal drift and mean jump sizes. The top panel of the figure shows that the effect of the drift is strongest when the option is in the neighbourhood of at-the-money, for both the buyer and the seller. This confirms why the spread in the previous figure was also largest in this region.

Moreover, the bottom panel of Figure 7 shows the optimal arrival rates for the buyer and the seller. When x is in the neighborhood of zero, the buyer's (seller's) arrival rate is furthest away from the arrival rate under the reference model (recall that in the reference measure $\lambda = 2$). As the pair x, y becomes larger in absolute value (i.e., southwest and northeast regions), the arrival rates get closer to that of the reference model. The effect on the jump sizes is similar and omitted here in the interest of space.

Effect of ambiguity aversion on bear-spread. The ambiguity averse buyer and seller of the bear-spread choose pricing measures with effects similar to those discussed above for the bull-spread. The most important driver is whether the option is in-the-money or out-of-the-money. The spread in prices between the two agents is smallest when the option is deep out-of-the-money or deep in-the-money. Moreover, the buyer (seller) of the bear-spread behaves similarly to the seller (buyer) of the bull-spread, although they are not identical because the robust pricing is in general nonlinear.

7. Conclusions.

We show how ambiguity averse agents price commodity derivatives and demonstrate that as agents becomes less confident about their models, they compute more conservative prices: sellers will require higher prices and consumers will seek to pay lower prices. Our framework allows us to single out the main drivers that explain how and why prices under the reference model differ from those computed when the agent is ambiguity averse. For example, when pricing forward contracts, the ambiguity averse buyer and seller modify (i) the drift of reference model—the buyer introduces a downward drift and the seller an upward drift; and (ii) the jump component by tilting the jump measure—the buyer enhances jumps down and dampens upward jumps, and the seller enhances jumps up and dampens jumps down.

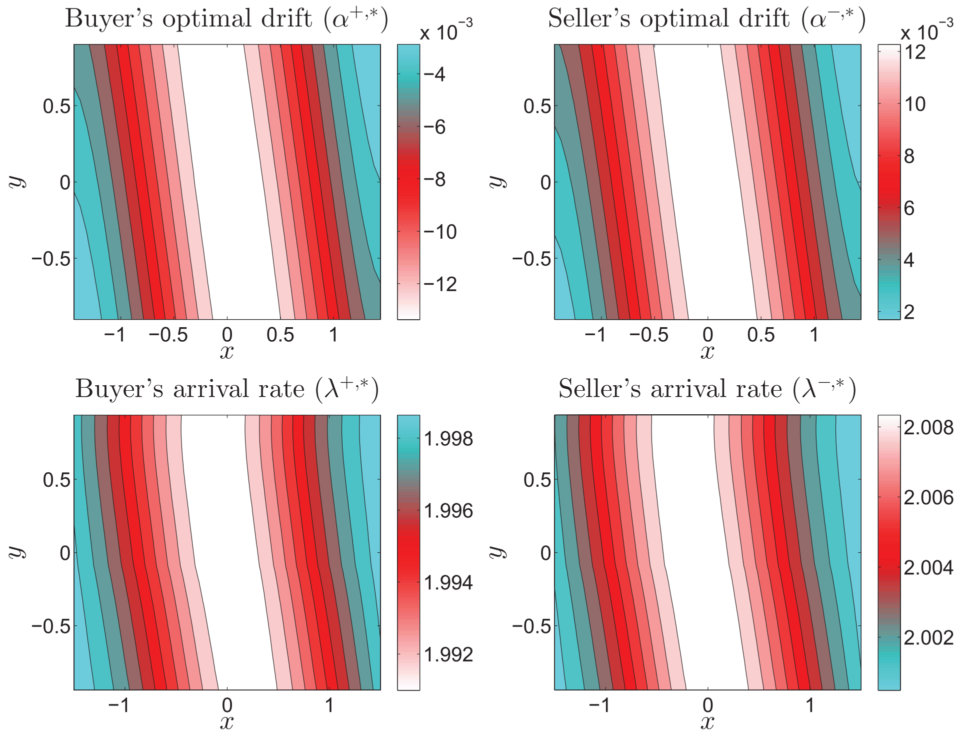

Figure 7. *Optimal drifts and arrival rates at $t = 0$. Cyan: close to reference model; red: away from reference model; white: far from reference model.*

Moreover, we examine the effect of model uncertainty when pricing a bull-spread and a bear-spread written on forward contracts. We see that the effect of ambiguity is strongest when the option is at-the-money and weakest when it is trading deep out-of-the-money or deep in-the-money. The insights are the following. In the neighborhood of at-the-money, agents' ambiguity about the reference model is more important since any movement in the price of the underlying commodity has a considerable effect on the value of the option.

Finally, our results also show that when the buyer and seller employ the same reference measure they will not trade between them because the ambiguity averse seller requires a price higher than what the ambiguity averse buyer is willing to pay. This spread varies across time and shrinks as the delivery of the forward contract approaches—there is less uncertainty around the price of the underlying commodity as the forward contract enters the delivery period.

Appendix A. Proofs.

Appendix A.1. Proof of Theorem 2. This result is somewhat standard, and we only provide an outline here for brevity. It follows by first interchanging the order of integration and expectation in (9). Next, we need to compute $h(t, x, v, y) = \mathbb{E}_{t,x,v,y}[S_u]$. The usual techniques imply that h satisfies the PIDE

$$(\partial_t + \mathcal{L})h + \lambda \int_{-\infty}^{\infty} \Delta_z h \, dz = 0$$

with terminal condition $h(u,x,v,y) = e^{\theta(u)+x+y}$. Now, since the operator $\mathcal{L}$ is affine, and the terminal condition is exponential affine, it suggests the ansatz $h(t,x,v,y) = e^{A(t)+B(t)\,x+C(t)\,v+D(t)\,y}$. Inserting this ansatz into the PIDE above and collecting terms proportional to 1, x, v, and y lead to four ODEs whose solutions are given by the results in the theorem. ■

Appendix A.2. Proof of Proposition 1. First note that $\boldsymbol{\Phi}^{-1}$ is a positive definite matrix. Therefore, we can apply the first order condition in $\boldsymbol{\alpha}$ to (28) and (29) directly. The result of this computation shows that the optimal control for the diffusion ambiguity $\boldsymbol{\alpha}^*$ in feedback form is the one given in (31).

Next, we show that $g_t^{\pm,*}(z) = \mp\varepsilon\,\Delta_z H^{\pm}$ is the optimal control in feedback form for the jump ambiguity. To this end, let

$$h^{\pm}(g) := A\,e^{g} \pm \frac{1}{\varepsilon}(1+e^{g}(g-1))$$

for $g, A \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}_+$. These functions correspond to the integrand of the jump terms for the buyer/seller (the integrals in the second line of (28) and (29), respectively). The first order condition applied to these functions yield the equation

$$e^{g^{\pm,*}}\left(A \pm \frac{g^{\pm,*}}{\varepsilon}\right) = 0\,,$$

which implies there is only one critical point $g^{\pm,*} = \mp\varepsilon\,A$ and at this critical point we have

$$h^{\pm}(g^{\pm,*}) = \pm\frac{1-e^{\mp\varepsilon\,A}}{\varepsilon}\,.$$

Since there exists only one critical point, we need only check whether the function truly attains a minimum for h^+ (and a maximum for h^-) at the critical point. For the case of $h^+(g)$, note that as $\lim_{g\to+\infty} h^+(g) = +\infty$, and $\lim_{g\to-\infty} h^+(g) = \frac{1}{\varepsilon}$. Furthermore, note that $h^+(g^{+,*}) \le \frac{1}{\varepsilon}$ since $\varepsilon \ge 0$ and attains equality only if $\varepsilon \downarrow 0$. Hence $g^{+,*}$ is the minimizer of $h^+(g)$. Similarly, $g^{-,*}$ is the maximizer of $h^-(g)$.

We next need to show that indeed $g_t^+(z) = - \mp\, \varepsilon\,\Delta_z H$ is a minimizer of the functional

$$\mathfrak{G}(\Delta_z H, g) = \int_{-\infty}^{\infty}\left\{\Delta_z H\, e^{g(z)} + \frac{1}{\varepsilon}\,\left(1+e^{g(z)}\,\left(g(z)-1\right)\right)\right\} G(dz)$$

over all $g \in \mathcal{G}$. The case of $g_t^-(z)$ follows similarly.

To this end, take any $f \in \mathcal{G}$, and define a class of functions $f_\epsilon : \mathbb{R} \to \mathbb{R}$ for $\epsilon \in [0,1]$ via

$$f_\epsilon(z) = \log\left(\epsilon\,(e^{f(z)} - e^{g^{+,*}(z)}) + e^{g^{+,*}(z)}\right).$$

Clearly, $f_\epsilon \in \mathcal{G}$, and $f_0 = g^{+,*}(z)$ (the function which we claim is optimal) while $f_1 = f(z)$ (the arbitrary alternate). Thus f_ϵ is a homotopy deformation from $g^{+,*}$ to f. Next, let $\mathfrak{G}_\epsilon(\Delta_z H) := \mathfrak{G}(\Delta_z H,\, f_\epsilon)$, i.e., the functional we aim to minimize evaluated along the deformation. We will show that

$$\mathfrak{G}_0(\Delta_z H) \le \mathfrak{G}_1(\Delta_z H)$$

for any $f \in \mathcal{G}$. It is sufficient to demonstrate that $\mathfrak{G}_\epsilon(\Delta_z H)$ has nonzero second-derivative for all $\epsilon \in [0,1]$. Substituting in the expression for f_ϵ, and letting $\ell(z) := e^{f(z)} - e^{g^{+,*}(z)}$, we have

$$G_\epsilon(\Delta_z H) = \int_{-\infty}^{\infty} \Big\{ \Delta_z H \left(\epsilon\, \ell(z) + e^{g^{+,*}(z)}\right) \\ + \frac{1}{\varepsilon}\left[1 + \left(\epsilon\, \ell(z) + e^{g^{+,*}(z)}\right)\left(\log\left(\epsilon\, \ell(z) + e^{g^{+,*}(z)}\right) - 1\right)\right]\Big\}\, G(dz)\,.$$

Hence,

$$\frac{d}{d\epsilon} G_\epsilon(\Delta_z H) = \int_{-\infty}^{\infty} \left\{ \Delta_z H\, \ell(z) + \frac{1}{\varepsilon}\left[\ell(z) \log\left(\epsilon\, \ell(z) + e^{g^{+,*}(z)}\right)\right]\right\} G(dz)$$

and

$$\frac{d^2}{d\epsilon^2} G_\epsilon(\Delta_z H) = \frac{1}{\varepsilon}\int_{-\infty}^{\infty} \frac{(\ell(z))^2}{\epsilon\, \ell(z) + e^{g^{+,*}(z)}}\, G(dz) = \frac{1}{\varepsilon}\int_{-\infty}^{\infty} (\ell(z))^2\, e^{-f_\epsilon(z)}\, G(dz) \geq 0\,.$$

Furthermore, note that in light of the first order condition, $\frac{d}{d\epsilon} G_\epsilon(\Delta_z H)\big|_{\epsilon=0} = 0$. Therefore, $g^{+,*}(z) = f_0(z) = -\varepsilon\, \Delta_z H$ is the optimal control in feedback form and upon substituting back we obtain the nonlinear PIDE (30), and similarly for the seller. ■

Appendix A.3. Proof of Proposition 2. From Proposition 1, $H^\pm$ solve the nonlinear PIDEs:

$$(\partial_t + \mathcal{L})H^\pm \mp \tfrac{1}{2}\, \boldsymbol{\mathcal{D}} H^{\pm\prime}\, \boldsymbol{\Omega}'\, \boldsymbol{\Phi}^\pm\, \boldsymbol{\Omega}\, \boldsymbol{\mathcal{D}} H^\pm \\ \pm \frac{\lambda}{\varepsilon^\pm}\int_{-\infty}^{\infty}\left(1 - e^{\mp\varepsilon^\pm \Delta_z H^\pm}\right) G(dz) + (\theta(t) + x + y)\, \mathbb{1}_{t\geq T_1} = 0\,,$$

subject to the terminal conditions

$$H^\pm(T_2, x, v, y) = 0 \qquad \forall x, v, y\,.$$

The terminal condition, together with the affine form of the potential term in the equation above, suggests to the ansatz $H^\pm(t,x,y) = A^\pm(t) + B(t)x \mp C(t)\, v + D(t)y$. Inserting this ansatz into the nonlinear PIDE, collecting terms proportional to 1, x, v, and y leads to a system of four ODEs for the buyer (the seller is similar):

$$\partial_t A^\pm(t) \mp \kappa_v\, \vartheta\, C(t) - \lambda\, \psi\, D(t) \pm \lambda \int_{-\infty}^{\infty} \frac{1 - e^{\mp\varepsilon\, D(t)}}{\varepsilon} + \theta(t) \mathbb{1}_{t\geq T_1} = 0\,, \tag{A.1a}$$

$$\partial_t B(t) - \kappa_D B(t) + \mathbb{1}_{t\geq T_1} = 0\,, \tag{A.1b}$$

$$\partial_t C(t) - \kappa_v\, C(t) + \tfrac{1}{2}\begin{pmatrix} B(t) & -C(t)\end{pmatrix} \boldsymbol{\Omega}'\boldsymbol{\Phi}\boldsymbol{\Omega} \begin{pmatrix} B(t) \\ -C(t)\end{pmatrix} = 0\,, \tag{A.1c}$$

$$\partial_t D(t) - \kappa_J D_t + \mathbb{1}_{t\geq T_1} = 0\,, \tag{A.1d}$$

subject to the terminal conditions

$$A(T_2) = B(T_2) = C(T_2) = D(T_2) = 0\,.$$

The solution to (A.1a), (A.1b), and (A.1d) are trivial and shown explicitly in (32). Since $B(t)$ is bounded, so is $C(t)$ and therefore $A(t)$. Moreover, $D(t)$ is bounded. Therefore, the solution is indeed the solution to the nonlinear PIDE.

Moreover, by direct substitution of the ansatz for H into the optimal controls provided in feedback form in Proposition 1, we arrive at the optimal controls in (35).

Appendix A.4. Proof of Proposition 3. First, write $\tau = T_2 - t$ and let $\tilde{C}(\tau) = C(T_2 - \tau; \boldsymbol{\Phi})$; then

$$\partial_\tau \tilde{C}(\tau) = -\kappa_v\, \tilde{C}(\tau) + \tfrac{1}{2}\begin{pmatrix} B(T_2-\tau) & -\tilde{C}(\tau)\end{pmatrix} \boldsymbol{\Omega}'\, \boldsymbol{\Phi}\, \boldsymbol{\Omega} \begin{pmatrix} B(T_2-\tau) \\ -\tilde{C}(\tau)\end{pmatrix},$$

and $\tilde{C}(0) = 0$. Hence, since $\boldsymbol{\Omega}'\, \boldsymbol{\Phi}\, \boldsymbol{\Omega}$ is a positive definite matrix,

$$\partial_\tau \tilde{C}(\tau) \geq -\kappa_v\, \tilde{C}(\tau)\,. \tag{A.2}$$

Next, suppose that $\exists\ \tau_*$ s.t. $\tilde{C}(\tau_*) \geq 0$; then the inequality (A.2) implies that $\tilde{C}(\tau) \geq e^{-\kappa_v(\tau-\tau_*)}\,\tilde{C}(\tau_*) \geq 0$ for all $\tau \geq \tau_*$. Hence, we must have $\tilde{C}(\tau) \geq 0$ for all $\tau \geq \tau_*$.

Finally, since $B(T_2) = C(T_2; \boldsymbol{\Omega}) = 0$, $\partial_\tau \tilde{C}(0) = 0$, and therefore from (A.2), either $\tilde{C}(\tau) = 0$ for all τ, or $\tilde{C}(\tau)$ becomes positive within a neighborhood of $\tau = 0$. From the explicit form of B in (32b), we have that for $\tau \ll 1$,

$$B(T_2 - \tau) = \tau + O(\tau^2)\,,$$

and therefore

$$\tilde{C}(\tau) = \tfrac{1}{4}\,[\boldsymbol{\Omega}'\boldsymbol{\Phi}\boldsymbol{\Omega}]_{11}\,\tau^2 + O(\tau^3)\,.$$

This shows that there is a neighborhood around $\tau = 0$ where $\tilde{C} \geq 0$. Combining this with the previous observation completes the proof. ∎

Appendix A.5. Proof of Theorem 3 (verification theorem). We focus on the buyer case, as the seller case is similar. Let $\widehat{H}(t,x,y)$ be a candidate solution to the optimal control problem and satisfying (28). From Ito's lemma, we have

$$\begin{aligned}\widehat{H}(T, X_T, v_T, Y_{T_-}) = \widehat{H}(t,x,v,y) &+ \int_t^T (\partial_t + \mathcal{L}) H(u, X_u, v_u, Y_u)\, du \\ &+ \int_t^T \sqrt{v_u}\, \boldsymbol{\mathcal{D}} H(u, X_u, v_u, Y_u)'\, \boldsymbol{\Omega}\, d\boldsymbol{W}_u \\ &+ \int_t^T \int_{-\infty}^{\infty} [H(u, X_u, v_u, Y_u + z) - H(u, X_u, v_u, Y_u)]\ \mu(dz, du)\,.\end{aligned}$$

Let $\boldsymbol{\alpha}_t$ and $g_t(z)$ be arbitrary controls in the admissible set, and let $\mathbb{Q}^{\boldsymbol{\alpha},g} \in \mathcal{Q}$ denote the measure corresponding to this control. The $\mathbb{Q}^{\boldsymbol{\alpha},g}$-expectation of the above expression implies

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^{\boldsymbol{\alpha},g}}_{t,x,v,y}\left[\widehat{H}(T, x_T, y_{T_-})\right] = \widehat{H}(t,x,v,y) &+ \mathbb{E}^{\mathbb{Q}^{\boldsymbol{\alpha},g}}_{t,x,v,y}\left[\int_t^T (\partial_t + \mathcal{L}) H(u, X_u, v_u, Y_u)\, du\right] \\ &+ \mathbb{E}^{\mathbb{Q}^{\boldsymbol{\alpha},g}}_{t,x,v,y}\left[\int_t^T \sqrt{v_u}\, \boldsymbol{\mathcal{D}} H(u, X_u, v_u, Y_u)'\, \boldsymbol{\Omega}\, (d\boldsymbol{W}^{\boldsymbol{\alpha},g}_u + \boldsymbol{\alpha}_u\, du)\right] \\ &+ \mathbb{E}^{\mathbb{Q}^{\boldsymbol{\alpha},g}}_{t,x,v,y}\left[\int_t^T \int_{-\infty}^{\infty} \Delta_z H(u, X_u, v_u, Y_u)\, \mu(dz, du)\right]\,.\end{aligned} \tag{A.3}$$

From (28), we see that

$$
\begin{aligned}
0 \leq\ & (\partial_t + \mathcal{L})\widehat{H}(t,x,v,y) + \sqrt{v}\,\boldsymbol{\alpha}_u'\Omega\boldsymbol{\mathcal{D}}\widehat{H} + \tfrac{1}{2}\boldsymbol{\alpha}_u'\boldsymbol{\Phi}^{-1}\boldsymbol{\alpha}_u \\
& + \lambda\int_{-\infty}^{\infty}\left\{\Delta_z\widehat{H}e^{g(z)} + \frac{1}{\varepsilon}\left(1 + e^{g_u(z)}(g_u(z)-1)\right)\right\}G(dz) + P(t,\theta(t),x,y)\mathbb{1}_{t\geq T_1}\,.
\end{aligned}
$$

Combining this inequality with

$$
\begin{aligned}
& \mathbb{E}_{t,x,v,y}^{\mathbb{Q}^{\boldsymbol{\alpha},g}}\left[\int_t^T\int_{-\infty}^{\infty}\Delta_z H(u,X_u,v_u,Y_u)\,\mu(dz,du)\right] \\
& \quad = \mathbb{E}_{t,x,v,y}^{\mathbb{Q}^{\boldsymbol{\alpha},g}}\left[\int_t^T\int_{-\infty}^{\infty}\Delta_z H(u,X_u,v_u,Y_u)\,\lambda\, e^{g_u(z)}G(dz)\,du\right],
\end{aligned}
$$

we obtain from (A.3),

$$
\begin{aligned}
\mathbb{E}_{t,x,v,y}^{\mathbb{Q}^{\boldsymbol{\alpha},g}}\left[\widehat{H}(T,x_T,y_{T_-})\right] \geq\ & \widehat{H}(t,x,v,y) \\
& - \mathbb{E}_{t,x,v,y}^{\mathbb{Q}^{\boldsymbol{\alpha},g}}\left[\int_t^T\left\{\tfrac{1}{2}\boldsymbol{\alpha}_u'\boldsymbol{\Phi}^{-1}\boldsymbol{\alpha}_u + P(u,\theta(u),X_u,Y_u)\mathbb{1}_{t\geq T_1}\right\}du\right] \\
& - \mathbb{E}_{t,x,v,y}^{\mathbb{Q}^{\boldsymbol{\alpha},g}}\left[\int_t^T\left\{\frac{\lambda}{\varepsilon}\left(1+e^{g_u(z)}(g_u(z)-1)\right)\right\}G(dz)\,du\right].
\end{aligned}
$$

Rearranging this inequality, taking $T \to T_2$, applying the terminal condition $\widehat{H}(T_2,x,v,y) = 0$, and recalling that $S_t = P(t,\theta(t),X_t,Y_t)$, we obtain

$$
\widehat{H}(t,x,v,y) \leq \mathbb{E}_{t,x,v,y}^{\mathbb{Q}^{\boldsymbol{\alpha},g}}\left[\int_{T_1}^{T_2} S_u\,du + \mathcal{H}^{\boldsymbol{\Phi},\varepsilon}(\mathbb{Q}^{\boldsymbol{\alpha},g}|\mathbb{P})\right].
$$

Therefore,

$$
\begin{aligned}
\widehat{H}(t,x,v,y) & \leq \inf_{\mathbb{Q}^{\boldsymbol{\alpha},g}\in\mathcal{Q}}\mathbb{E}_{t,x,v,y}^{\mathbb{Q}^{\boldsymbol{\alpha},g}}\left[\int_{T_1}^{T_2} S_u\,du + \mathcal{H}^{\boldsymbol{\Phi},\varepsilon}(\mathbb{Q}^{\boldsymbol{\alpha},g}|\mathbb{P})\right] \\
& = H(t,x,v,y)\,.
\end{aligned}
$$

All inequalities become equalities when $\boldsymbol{\alpha}_t = \boldsymbol{\alpha}_t^*$ and $g_t(z) = g_t^*(z)$ and the result follows. ■

Appendix A.6. Proof of Corollary 1. For Proposition 2, direct computation shows that the bid-ask spread is given by Υ in (41). Next, from Proposition 3, $C(t,u;\boldsymbol{\Phi})$ is positive, and hence we need only show that the term (41b) is nonnegative. Recall that $\Psi(\cdot)$ is the moment generating function of the jump size; hence to show that (41b) is nonnegative, it is sufficient to show that (for t fixed)

$$
\int_{-\infty}^{\infty}\left[\frac{e^{\varepsilon^- D(t)z}-1}{\varepsilon^-} + \frac{e^{-\varepsilon^+ D(t)z}-1}{\varepsilon^+}\right]G(dz) \geq 0 \qquad \forall\varepsilon^-,\varepsilon^+\in(0,+\infty)\,. \tag{A.4}
$$

We show this by demonstrating that

$$\mathcal{G}(y;\varepsilon^+,\varepsilon^-) := \mathcal{G}^-(y;\varepsilon^-) + \mathcal{G}^+(y;\varepsilon^+) \geq 0 \qquad \forall y \in \mathbb{R}\,;\varepsilon^{\pm} \geq 0\,,$$

where

$$\mathcal{G}^-(y;\varepsilon^-) := \frac{e^{\varepsilon^- y} - 1}{\varepsilon^-} - y \qquad \text{and} \qquad \mathcal{G}^+(y;\varepsilon^+) := \frac{e^{-\varepsilon^+ y} - 1}{\varepsilon^+} + y\,.$$

Note that

$$\lim_{\varepsilon^+\downarrow 0} \mathcal{G}^+(y;\varepsilon^+) = \lim_{\varepsilon^-\downarrow 0} \mathcal{G}^-(y;\varepsilon^-) = 0\,.$$

Next,

$$\frac{d}{d\varepsilon^{\pm}}\mathcal{G}^{\pm}(y;\varepsilon^{\pm}) = \mp y\left(e^{\mp\varepsilon^{\pm} y} - 1\right) \geq 0 \qquad \forall y \in \mathbb{R}\,,\ \varepsilon^{\pm} \geq 0\,.$$

Therefore, $\mathcal{G}^{\pm}(y;\varepsilon^{\pm}) \geq 0$, and hence, $\mathcal{G}(y;\varepsilon^+,\varepsilon^-) \geq 0$, for all $y \in \mathbb{R}$, $\varepsilon^{\pm} \geq 0$. Therefore, (A.4) is nonnegative, since it is the integral of $\mathcal{G}(D(t)\,z;\varepsilon^+,\varepsilon^-)$ with respect to the jump distribution measure $G(dz)$. This completes the proof. ■

Appendix A.7. Proof of Proposition 4. Consider the buyer case, as the seller case is similar. Recall that $\boldsymbol{\Phi} = \phi\boldsymbol{\Sigma}$ (since we set $\phi_x = \phi_v = 0$). The two expansions

$$\frac{1 - e^{\varepsilon\,(a+\phi\,b+\varepsilon\,c)}}{\varepsilon} = a + \tfrac{1}{2}\varepsilon\,a^2 + O\left(\varsigma^2\right) \quad \forall |a|, |b|, |c| < +\infty$$

and

$$(\boldsymbol{a} + \phi\,\boldsymbol{b} + \varepsilon\,\boldsymbol{c})'\boldsymbol{\Omega\Phi\Omega}(\boldsymbol{a} + \phi\,\boldsymbol{b} + \varepsilon\,\boldsymbol{c}) = \phi\,\boldsymbol{a}'\,\boldsymbol{\Omega\Sigma\Omega}\,\boldsymbol{a} + O\left(\varsigma^2\right)\,,$$

where $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are 2×2 matrices with finite entries, will be used to obtain the corrections. Next, insert the perturbation expansion (47) for H into the HJB equation (30). Then, expanding and collecting terms we have

$$\begin{aligned}
0 = &\left\{(\partial_t + \mathcal{L})H^{(0)} + \lambda\int_{-\infty}^{\infty} \Delta_z H^{(0)}\,G(dz) + P(t,\theta(t),x,y)\,\mathbb{1}_{t\geq T_1}\right\}\\
&+ \phi\left\{(\partial_t + \mathcal{L})H^{(1,0)} + \lambda\int_{-\infty}^{\infty} \Delta_z H^{(1,0)}\,G(dz) - \tfrac{1}{2}\boldsymbol{\mathcal{D}}H^{(0)\prime}\,\boldsymbol{\Omega}'\boldsymbol{\Sigma\Omega}\,\boldsymbol{\mathcal{D}}H^{(0)}\right\}\\
&+ \varepsilon\left\{(\partial_t + \mathcal{L})H^{(0,1)} + \lambda\int_{-\infty}^{\infty} \Delta_z H^{(0,1)}\,G(dz) - \tfrac{1}{2}\int_{-\infty}^{\infty}(\Delta_z H^{(0)})^2\,G(dz)\right\}\\
&+ O\left(\varsigma^2\right)\,,
\end{aligned}$$

subject to the terminal condition

$$H^{(0)}(T_2,x,v,y) = H^{(1,0)}(T_2,x,v,y) = H^{(0,1)}(T_2,x,v,y) = 0\,.$$

Setting each of the terms in curly braces to vanish individually leads to the three PIDEs

$$0 = (\partial_t + \mathcal{L})H^{(0)} + \lambda\int_{-\infty}^{\infty} \Delta_z H^{(0)}\,G(dz) + P(t,\theta(t),x,y)\,\mathbb{1}_{t\geq T_1}\,, \tag{A.5a}$$

$$0 = (\partial_t + \mathcal{L})H^{(1,0)} + \lambda\int_{-\infty}^{\infty} \Delta_z H^{(1,0)}\,G(dz) - \tfrac{1}{2}\boldsymbol{\mathcal{D}}H^{(0)\prime}\,\boldsymbol{\Omega}'\boldsymbol{\Sigma\Omega}\,\boldsymbol{\mathcal{D}}H^{(0)}\,, \tag{A.5b}$$

$$0 = (\partial_t + \mathcal{L})H^{(0,1)} + \lambda\int_{-\infty}^{\infty} \Delta_z H^{(0,1)}\,G(dz) - \tfrac{1}{2}\int_{-\infty}^{\infty}(\Delta_z H^{(0)})^2\,G(dz)\,. \tag{A.5c}$$

Equation (A.5a) is the PIDE for the value of the forward contract in the reference model, and from Theorem 2 we arrive at (48).

Next, (A.5b) is a linear PIDE for $H^{(1,0)}$ where the term $-\frac{1}{2}\boldsymbol{\mathcal{D}}H^{(0)\prime}\,\boldsymbol{\Omega}'\boldsymbol{\Sigma}\boldsymbol{\Omega}\,\boldsymbol{\mathcal{D}}H^{(0)}$ acts as a source (or potential). Since the source is known, a priori, we can apply a Feynman–Kac formula to represent its solution as (49a). Note that the linear operator acting on $H^{(1,0)}$ is the full $\mathbb{P}$-generator of the joint processes (X_t, v_t, Y_t) and not just the generator of the continuous part of processes. Similarly, (A.5b) is a linear PIDE for $H^{(0,1)}$, where this time the source term is $-\frac{1}{2}\int_{-\infty}^{\infty}(\Delta_z H^{(0)})^2\,G(dz)$, and the generator is once again the full $\mathbb{P}$-generator of the triplet (X_t, v_t, Y_t). Once again, a Feynman–Kac formula leads to the representation in (49b). ∎

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