On the Stabilizability of the Burgers Equation by Receding Horizon Control.

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ON THE STABILIZABILITY OF THE BURGERS' EQUATION BY RECEDING HORIZON CONTROL [*]

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Abstract. A receding horizon framework for stabilization of a class of infinite-dimensional controlled systems is presented. No terminal costs and constraints are used to ensure asymptotic stability of the controlled system. The key assumption is a stabilizability assumption, which can be guaranteed, for example, for the Burgers' equations with periodic and with homogeneous Neumann boundary conditions. Numerical experiments validate the theoretical results. Comparisons to the case with terminal penalties acting as control Lyapunov functions are included.

Key words. receding horizon control, model predictive control, asymptotic stability, infinite-dimensional systems.

AMS subject classifications. 49N35, 93C20, 93D20

1. Introduction. We consider the optimal control problem which consists in minimizing

$$\text{(1)} \qquad J_\infty(u, y_0) := \int_0^\infty \ell(y(t), u(t)) dt$$

subject to

$$\text{(2)} \qquad \begin{cases} \frac{d}{dt} y(t) = f(y(t)) + Bu(t) & \text{for } t > 0, \\ y(0) = y_0, \end{cases}$$

where $f(0) = 0$, $\ell(0, 0) = 0$. The state $y(t)$ and the control $u(t)$ are respectively elements of spatially dependent function spaces H and U, and B is the control operator. Furthermore, the incremental cost function $\ell(\cdot, \cdot)$ is assumed to be uniformly positive definite in both the state and control variables.

One strategy to solve problem (1)-(2) numerically employs the receding horizon control (RHC) which is also known as model predictive control (MPC). This method consists in obtaining a suboptimal solution of the infinite horizon problem by solving a series of finite horizon problems on a family of intervals which are arranged in an temporally increasing manner and which cover $[0, \infty)$. Since proceeding in this manner, the solution of (1)-(2) is not obtained, the question of justifying the RHC technique arises. This is typically addressed by analyzing whether the RHC control meets the control objective which is formulated within (1)-(2). Frequently this control objective is given by the stabilization problem. Due to replacing the infinite prediction horizon by a family of finite ones, the asymptotic stability of the receding horizon control scheme is not a-priori guaranteed. However, the succinct use of the structure of the dynamical system under consideration together with possible terminal costs and/or terminal constraints for the finite horizon problems can ensure asymptotic stability of RHC control under appropriate assumptions.

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In the past three decades, numerous results have been published on receding horizon control for finite-dimensional systems [1, 2, 13, 17, 19, 23, 24, 28] and the many references therein. Only more recently the case of infinite-dimensional systems was considered as well [3, 18, 21, 22]. In [21] a general framework to stabilize infinite-dimensional dynamical systems by receding horizon control is proposed. The stability of the receding horizon control is ensured by adding control Lyapunov functions as terminal cost to the finite horizon problems. More recently several authors, see e.g [17, 18, 19, 23] managed to prove the asymptotic stability of the RHC even without use of control Lyapunov functions and terminal constraints. So far, this framework has been well studied for finite-dimensional dynamical systems [23, 30] and discrete time dynamical systems [17, 18, 19]. But as far as we know, for infinite-dimensional systems with continuous time dynamical systems there still does not exist a rigorous theory. In this paper we make a step in this direction. The present work is inspired by, but different from [30], since we treat partial rather than ordinary differential equations. As a consequence Barbalat's lemma, which relies on the finite dimensionality of the state space, is not applicable. Moreover, we consider systems which are only locally rather than globally stabilizable.

To briefly recapture the receding horizon approach, we choose a sampling time $\delta > 0$ and an appropriate prediction horizon $T > \delta$. Then sampling instances $t_k := k\delta$ for $k = 0, 1, \ldots$ are defined. At every sampling instance t_k, an open-loop optimal control problem is solved over a finite prediction horizon $[t_k, t_k + T]$. The optimal control thus obtained is applied to steer the system from time t_k with the initial state $y_{rh}(t_k)$ until time $t_{k+1} := t_k + \delta$ at which point, a new measurement of state is assumed to be available. The process is repeated starting from the new state: we obtain a new optimal control and a new predicted state trajectory by shifting the prediction horizon forward in time. Throughout, we denote the receding horizon state- and control variables by $y_{rh}(\cdot)$ and $u_{rh}(\cdot)$, respectively. Also, $(y_T^*(\cdot; y_0, t_0), u_T^*(\cdot; y_0, t_0))$ stands for the optimal state and control of the optimal control problem with finite time horizon T, and initial function y_0 at initial time t_0. We next summarize the resulting Algorithm 1.

The importance of RHC rests not only on speed up of the solution process, but also on its use as a state feedback mechanism which can be activated at each instance time t_k. In practice this requires to include a dynamic observer. We leave this aspect to future work.

The applicability of our theoretical work will be demonstrated for the Burgers' equation. This is a nonlinear partial differential equation (PDE) that combines both nonlinear propagation and diffusion effects. It shares some important features with the Navier-Stokes equation. The Burgers' equation has the origin as a steady state. It is asymptotically stable in the case of homogeneous Dirichlet boundary conditions. For homogeneous Neumann boundary conditions and periodic boundary conditions this is not the case. Control theory for the Burgers' equation was investigated, both theoretically and numerically, by many authors. From among them we only mention [4, 5, 6, 9, 10, 11, 12, 20, 25, 26, 27, 34].

The remainder of the paper is structured as follows. In section 2 based on Bellman's principle we develop an abstract setting which estimates the value of the cost J_∞ evaluated along the receding horizon control and trajectory in terms of the minimal value functional associated to (1)-(2). These results are applied in section 3 to stabilization of the Burgers' equation with different types of boundary conditions. Section 4 contains numerical experiments which highlight the effect to the ratio $\frac{T}{\delta}$ on the stabilizing effect of the RHC strategy. Moreover comparisons are carried out

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Algorithm 1 Receding Horizon Algorithm

Input: Let the prediction horizon T, the sampling time $\delta < T$, and the initial state $y_0 \in H$ be given.

1. Set $k := 0$, $t_0 := 0$, and $y_{rh}(t_0) := y_0$.

2. Find the optimal pair $(y_T^*(\cdot; y_{rh}(t_k), t_k), u_T^*(\cdot; y_{rh}(t_k), t_k))$ over the time horizon $[t_k, t_k + T]$ by solving the finite horizon open-loop problem

$$\min_{u \in L^2(t_k, t_k+T; U)} J_T(u, y_{rh}(t_k)) := \min_{u \in L^2(t_k, t_k+T; U)} \int_{t_k}^{t_k+T} \ell(y(t), u(t))dt,$$

$$\text{s.t} \begin{cases} \frac{d}{dt}y(t) = f(y(t)) + Bu(t) & \text{for } t \in (t_k, t_k + T), \\ y(t_k) = y_{rh}(t_k) \end{cases}$$

3. Set

$$u_{rh}(\tau) := u_T^*(\tau; y_{rh}(t_k), t_k) \quad \text{for all } \tau \in [t_k, t_k + \delta),$$
$$y_{rh}(\tau) := y_T^*(\tau; y_{rh}(t_k), t_k) \quad \text{for all } \tau \in [t_k, t_k + \delta],$$
$$t_{k+1} := t_k + \delta,$$
$$k := k + 1.$$

4. Go to step 2.

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comparing the effect of RHC control with and without terminal control penalty.

2. Stability of the receding horizon method. Let $V \subset H = H^* \subset V^*$ be a Gelfand triple of real Hilbert spaces with V densely contained in H. Further let U denote the control space which is also assumed to be a real Hilbert space. For any $T > 0$ and $y_0 \in H$ we consider the controlled dynamical system

$$(3) \qquad \begin{cases} \frac{d}{dt}y(t) = f(y(t)) + Bu(t) & \text{for } t \in (0, T), \\ y(0) = y_0, \end{cases}$$

where f is a continuous function from V to V^*, $f(0) = 0$, and $B \in \mathcal{L}(U, V^*)$. Here $\mathcal{L}(U, V^*)$ denotes the space of all continuous linear operators from U to V^*. Throughout the paper, it is assumed that for any triple $(T, y_0, u) \in \mathbb{R}^+ \times H \times L^2(0, T; U)$ there exists a unique $y \in W(0, T)$, where

$$(4) \qquad W(0, T) = L^2(0, T; V) \cap H^1(0, T; V^*),$$

satisfying

$$y(t) - y(0) = \int_0^t (f(y(s)) + Bu(s))ds \quad \text{in } V^*$$

for $t \in [0, T]$. For sufficient conditions on f we refer to e.g. [32], Chapter II.3. We recall that $W(0, T)$ is continuously embedded in $C([0, T]; H)$, see e.g. [32, 36].

To define the optimal control problems we introduce the continuous incremental function $\ell : H \times U \to \mathbb{R}^+$ satisfying

$$(5) \qquad \ell(y, u) \geq \alpha_\ell(\|y\|_H^2 + \|u\|_U^2)$$

for a number $\alpha_\ell > 0$ independent of $y \in H$ and $u \in U$, and $\ell(0,0) = 0$. For every $T > 0$ and $y_0 \in H$ consider the finite horizon optimal control problem

$$\min_{u \in L^2(0,T;U)} J_T(u, y_0) := \min_{u \in L^2(0,T;U)} \int_0^T \ell(y(t), u(t))dt,$$

(P_T)   subject to

$$\begin{cases} \frac{d}{dt} y(t) = f(y(t)) + Bu(t) & \text{for } t \in (0, T), \\ y(0) = y_0. \end{cases}$$

Throughout we fix a neighborhood $\mathcal{N}_0$ of the origin in H. We assume that

(A1)   (P_T) admits an optimal pair $(y_T^*(\cdot; y_0, 0), u_T^*(\cdot; y_0, 0))$ for any
            $y_0 \in \mathcal{N}_0$ and $T > 0$.

Conditions on ℓ and f which imply (A1) are well-known from e.g. [33]. The functional J_∞ is defined as J_T in (P_T) with T replaced by ∞. With (A1) holding the following definition is well posed.

DEFINITION 1. *For any $y_0 \in \mathcal{N}_0$ the infinite horizon value function $V_\infty(\cdot)$ is defined as the extended real valued function*

$$V_\infty(y_0) := \inf_{u \in L^2(0,\infty;U)} \{J_\infty(u, y_0) \text{ subject to } (3)\}.$$

Similarly, the finite horizon value function $V_T(\cdot)$ is defined by

$$V_T(y_0) := \min_{u \in L^2(0,T;U)} \{J_T(u, y_0) \text{ subject to } (3)\}.$$

The following notion of local stabilizability will be used.

DEFINITION 2 (Local stabilizability). *The dynamical system (3) is called locally stabilizable if for every positive T and initial function $y_0 \in \mathcal{N}_0$ there exists a control $\hat{u}(\cdot, y_0) \in L^2(0, T; U)$ with*

(6) $$V_T(y_0) \leq J_T(\hat{u}, y_0) \leq \gamma(T) \|y_0\|_H^2,$$

where $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, non-decreasing and bounded function.

If $\mathcal{N}_0$ can be chosen to be all of H then we call (3) globally stabilizable. We shall require the following two assumptions:

(A2)   The dynamical system (3) is locally stabilizable for the neighborhood $\mathcal{N}_0$.

For every $T > 0$ there exists a constant $c_T \geq 0$ such that for every

(A3) $y_0 \in \mathcal{N}_0$, and u with $\|u\|_{L^2(0,T;U)} \leq \sqrt{\gamma(T)/\alpha_\ell} \, \|y_0\|_H$ we have

$$\|y(t)\|_H^2 \leq \|y_0\|_H^2 + c_T \int_0^t \|y(s)\|_H^2 ds + c_T \int_0^t \|u(s)\|_U^2 \, ds \quad \text{for all } t \in [0, T].$$

Below $\mathcal{B}_{d_1}(0)$ denotes a ball in H centered at 0 with radius d_1.

LEMMA 3. *If* (A1)-(A3) *hold, and* $T > \delta > 0$, *then there exists a neighborhood* $\mathcal{B}_{d_1}(0) \subset \mathcal{N}_0$ *with* $d_1 = d_1(T) > 0$ *such that for every* $y_0 \in \mathcal{B}_{d_1}(0)$ *the following inequalities hold*

(7)
$$V_T(y_T^*(\delta; y_0, 0)) \leq \int_\delta^{\tilde{t}} \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt$$
$$+ \gamma(T + \delta - \tilde{t})\|y_T^*(\tilde{t}; y_0, 0)\|_H^2 \quad \text{for all } \tilde{t} \in [\delta, T],$$

and

(8)
$$\int_{\tilde{t}}^T \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt \leq \gamma(T - \tilde{t})\|y_T^*(\tilde{t}; y_0, 0)\|_H^2 \quad \text{for all } \tilde{t} \in [0, T].$$

Proof. First observe that due to (5) for every $y_0 \in \mathcal{N}_0$ and $\tilde{t} \in [0, T]$ we have by Bellman's principle

$$\alpha_\ell \int_0^{\tilde{t}} (\|y_T^*(t; y_0, 0)\|_H^2 + \|u_T^*(t; y_0, 0)\|_U^2)dt \leq \int_0^{\tilde{t}} \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt$$
$$= V_T(y_0) - V_{T-\tilde{t}}(y_T^*(\tilde{t}; y_0, 0)),$$

and as a consequence $\|u_T^*\|_{L^2(0,T;U)}^2 \leq \frac{\gamma(T)}{\alpha_\ell}\|y_0\|_H^2$. By (6), (A3), and the above inequality we have

$$\|y_T^*(\tilde{t}; y_0, 0)\|_H^2 \leq \|y_0\|_H^2 + c_T \int_0^{\tilde{t}} \|y_T^*(t; y_0, 0)\|_H^2 dt + c_T \int_0^{\tilde{t}} \|u_T^*(t; y_0, 0)\|_U^2 dt$$
$$\leq \|y_0\|_H^2 + \frac{c_T}{\alpha_\ell}(V_T(y_0) - V_{T-\tilde{t}}(y_T^*(\tilde{t}; y_0, 0)))$$
$$\leq \|y_0\|_H^2 + \frac{c_T}{\alpha_\ell}V_T(y_0) \leq (1 + \frac{c_T}{\alpha_\ell}\gamma(T))\|y_0\|_H^2.$$

Since $\mathcal{N}_0$ is a neighborhood of zero, it follows that there exists a ball $\mathcal{B}_{\delta_1}(0) \subseteq \mathcal{N}_0$. Choosing $d_1 := \sqrt{(1 + \frac{c_T}{\alpha_\ell}\gamma(T))^{-1}\delta_1^2}$ we obtain that for every $y_0 \in \mathcal{B}_{d_1}(0)$ we have

$$y_T^*(\tilde{t}; y_0, 0) \in \mathcal{N}_0 \qquad \text{for all } \tilde{t} \in [0, T].$$

We turn to the verification of (7). For simplicity of notation, we denote $y_T^*(\delta; y_0, 0)$ by $y^*(\delta)$, where $y_0 \in \mathcal{B}_{d_1}(0)$. Due to Bellman's optimality principle, we have for every $\tilde{t} \in [\delta, T]$

(9)
$$V_T(y^*(\delta)) = \int_\delta^{T+\delta} \ell(y_T^*(t; y^*(\delta), \delta), u_T^*(t; y^*(\delta), \delta))dt$$
$$= \int_\delta^{\tilde{t}} \ell(y_T^*(t; y^*(\delta), \delta), u_T^*(t; y^*(\delta), \delta))dt + V_{T+\delta-\tilde{t}}(y_T^*(\tilde{t}; y^*(\delta), \delta)).$$

By optimality of $y_T^*(\cdot; y^*(\delta), \delta)$ as a solution on $[\delta, T + \delta]$ with initial state $y^*(\delta) \in \mathcal{N}_0$ at $t = \delta$ we obtain

$$V_T(y^*(\delta)) \leq \int_\delta^{\tilde{t}} \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt + V_{T+\delta-\tilde{t}}(y_T^*(\tilde{t}; y_0, 0))$$
$$\leq \int_\delta^{\tilde{t}} \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt + \gamma(T + \delta - \tilde{t})\|y_T^*(\tilde{t}; y_0, 0)\|_H^2,$$

where for the last inequality we used (6).

To prove the second inequality let $\tilde{t} \in [0, T]$ be arbitrary. By Bellman's principle and (6), we have

$$V_T(y_0) = \int_0^{\tilde{t}} \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt + \int_{\tilde{t}}^T \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt$$

$$(10) \qquad = \int_0^{\tilde{t}} \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt + V_{T-\tilde{t}}(y_T^*(\tilde{t}; y_0, 0))$$

$$\leq \int_0^{\tilde{t}} \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt + \gamma(T - \tilde{t})\|y_T^*(\tilde{t}; y_0, 0)\|_H^2.$$

Therefore,

$$\int_{\tilde{t}}^T \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt \leq \gamma(T - \tilde{t})\|y_T^*(\tilde{t}; y_0, 0)\|_H^2 \quad \text{for all } \tilde{t} \in [0, T],$$

as desired.   □

LEMMA 4. *Suppose that for some initial function* $y_0 \in H$, *properties* (7) *and* (8) *of Lemma 3 hold. Then for the choice of*

$$\theta_1 := 1 + \frac{\gamma(T)}{\alpha_\ell(T - \delta)}, \qquad \theta_2 := \frac{\gamma(T)}{\alpha_\ell \delta},$$

we have the following estimates

$$(11) \qquad V_T(y_T^*(\delta; y_0, 0)) \leq \theta_1 \int_\delta^T \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt,$$

and

$$(12) \qquad \int_\delta^T \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt \leq \theta_2 \int_0^\delta \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt.$$

Proof. To verify the inequality (11) recall that $y_T^*(\cdot; y_0, 0) \in C([0, T]; H)$. Hence there is a $\bar{t} \in [\delta, T]$ such that

$$\bar{t} = \arg\min_{t \in [\delta, T]} \|y_T^*(t; y_0, 0)\|_H^2.$$

By (7) we have

$$V_T(y_T^*(\delta; y_0, 0))$$

$$\leq \int_\delta^{\bar{t}} \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt + \gamma(T + \delta - \bar{t})\|y_T^*(\bar{t}; y_0, 0)\|_H^2$$

$$(13) \qquad \leq \int_\delta^T \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt + \gamma(T)\|y_T^*(\bar{t}; y_0, 0)\|_H^2$$

$$\leq \int_\delta^T \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt + \frac{\gamma(T)}{T - \delta} \int_\delta^T \|y_T^*(t; y_0, 0)\|_H^2 dt.$$

Furthermore, by (5)

$$(14) \qquad \int_\delta^T \|y_T^*(t; y_0, 0)\|_H^2 dt \leq \frac{1}{\alpha_\ell} \int_\delta^T \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt.$$

Now by using (13) and (14), we have

$$V_T(y_T^*(\delta; y_0, 0)) \leq (1 + \frac{\gamma(T)}{\alpha_\ell(T - \delta)}) \int_\delta^T \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt.$$

Turning to (12) we define

$$\hat{t} = \arg \min_{t \in [0, \delta]} \|y_T^*(t; y_0, 0)\|_H^2.$$

Then by (8) we have

(15)
$$\begin{aligned}
\int_\delta^T \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt &\leq \int_{\hat{t}}^T \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt \\
&\leq \gamma(T - \hat{t})\|y_T^*(\hat{t}; y_0, 0)\|_H^2 \\
&\leq \gamma(T)\|y_T^*(\hat{t}; y_0, 0)\|_H^2 \\
&\leq \frac{\gamma(T)}{\delta} \int_0^\delta \|y_T^*(t; y_0, 0)\|_H^2 dt.
\end{aligned}$$

Moreover, we have

(16) $$\frac{\gamma(T)}{\delta} \int_0^\delta \|y_T^*(t; y_0, 0)\|_H^2 dt \leq \frac{\gamma(T)}{\alpha_\ell \delta} \int_0^\delta \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt.$$

By (15) and (16) we can estimate

$$\int_\delta^T \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt \leq \frac{\gamma(T)}{\alpha_\ell \delta} \int_0^\delta \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt. \qquad \square$$

PROPOSITION 5. *Suppose that* (A1)-(A3) *hold and that $\delta > 0$. Then there exist $T^* > \delta$ and $\alpha \in (0, 1)$ such that the following inequality is satisfied*

(17) $$V_T(y_T^*(\delta; y_0, 0)) \leq V_T(y_0) - \alpha \int_0^\delta \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt$$

for every $T \geq T^*$ and $y_0 \in \mathcal{B}_{d_1(T)}(0)$ with $d_1(T)$ defined in Lemma 3.

Proof. Since for θ_1 and θ_2 defined in Lemma 4 we have

$$1 - \theta_2(\theta_1 - 1) = 1 - \frac{\gamma^2(T)}{\alpha_\ell^2 \delta(T - \delta)},$$

and

$$\frac{\gamma^2(T)}{\alpha_\ell^2 \delta(T - \delta)} \to 0 \text{ as } T \to \infty,$$

there exist $T^* > \delta$ and $\alpha \in (0, 1)$ such that $1 - \theta_2(\theta_1 - 1) \geq \alpha$ for all $T \geq T^*$. Next let $T \geq T^*$ and $y_0 \in \mathcal{B}_{d_1(T)}(0)$. Then from the definition of $V_T(y_0)$, Lemma 3, and

Lemma 4 we have

$$
V_T(y_T^*(\delta; y_0, 0)) - V_T(y_0) = V_T(y_T^*(\delta; y_0, 0)) - \int_0^T \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt
$$

$$
\leq (\theta_1 - 1) \int_\delta^T \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt - \int_0^\delta \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt
$$

$$
\leq (\theta_2(\theta_1 - 1) - 1) \int_0^\delta \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt
$$

$$
\leq -\alpha \int_0^\delta \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt.
$$

This implies (17). $\square$

THEOREM 6 (Suboptimality). *Suppose that* (A1)-(A3) *hold and let a sampling time $\delta > 0$ be given. Then there exist numbers $T^* > \delta$, and $\alpha \in (0, 1)$, such that for every fixed prediction horizon $T \geq T^*$, the receding horizon control u_{rh} obtained from* Algorithm 1 *satisfies*

$$
(18) \qquad \alpha V_\infty(y_0) \leq \alpha J_\infty(u_{rh}, y_0) \leq V_T(y_0) \leq V_\infty(y_0)
$$

for all $y_0 \in \mathcal{B}_{d_2}(0)$ with some $d_2 = d_2(T) > 0$.

Proof. The right and left inequalities are obvious, therefore we only need to verify the middle one. For fixed $\delta > 0$ choose T^* and α according to Proposition 5. Define $d_2 := \sqrt{((1 + \frac{c_T}{\alpha \alpha_\ell} \gamma(T))^{-1} d_1^2}$ where $T \geq T^*$ and d_1 is defined in Lemma 3. We proceed by induction with respect to the receding horizon sampling index k.

First, since $d_2 < d_1$ the assumptions of Proposition 5 are applicable due to Lemma 3, and we have

$$
(19) \qquad V_T(y_{rh}(t_1)) \leq V_T(y_0) - \alpha \int_0^{t_1} \ell(y_{rh}(t), u_{rh}(t))dt,
$$

and also

$$
\|y_{rh}(t_1)\|_H^2 \overset{(A3)}{\leq} \|y_{rh}(0)\|_H^2 + c_T \int_0^{t_1} (\|y_{rh}(t)\|_H^2 + \|u_{rh}(t)\|_U^2)dt
$$

$$
\overset{(5)}{\leq} \|y_0\|_H^2 + \frac{c_T}{\alpha_\ell} \int_0^{t_1} \ell(y_{rh}(t), u_{rh}(t))dt
$$

$$
\overset{(19)}{\leq} \|y_0\|_H^2 + \frac{c_T}{\alpha \alpha_\ell}(V_T(y_0) - V_T(y_{th}(t_1)))
$$

$$
\leq \|y_0\|_H^2 + \frac{c_T}{\alpha \alpha_\ell} V_T(y_0) \overset{(6)}{\leq} (1 + \frac{c_T}{\alpha \alpha_\ell} \gamma(T)) \|y_0\|_H^2 \leq d_1^2.
$$

Proceeding by induction we assume that

$$
(20) \qquad y_{rh}(t_k) \in \mathcal{B}_{d_1}(0) \quad \text{for all } k = 0, \dots, k',
$$

and that

$$
(21) \qquad V_T(y_{rh}(t_{k'})) \leq V_T(y_0) - \alpha \int_0^{t_{k'}} \ell(y_{rh}(t), u_{rh}(t))dt
$$

for $k' \in \mathbb{N}$.

Since $y_{rh}(t_{k'}) \in \mathcal{B}_{d_1}(0)$, by Lemma 3 and Proposition 5 we have

$$(22) \qquad V_T(y_{rh}(t_{k'+1})) \leq V_T(y_{rh}(t_{k'})) - \alpha \int_{t_{k'}}^{t_{k'+1}} \ell(y_{rh}(t), u_{rh}(t))dt.$$

Combined with (21) this gives

$$(23) \qquad V_T(y_{rh}(t_{k'+1})) \leq V_T(y_0) - \alpha \int_0^{t_{k'+1}} \ell(y_{rh}(t), u_{rh}(t))dt.$$

Moreover, by repeated use of (A3) which is applicable by (20) and due to (5), (6), and (23), we have

$$
\begin{aligned}
\|y_{rh}(t_{k'+1})\|_H^2 &\overset{(A3)}{\leq} \|y_{rh}(t_{k'})\|_H^2 + c_T \int_{t_{k'}}^{t_{k'+1}} (\|y_{rh}(t)\|_H^2 + \|u_{rh}(t)\|_U^2)dt \\
&\overset{(A3)}{\leq} \|y_{rh}(0)\|_H^2 + c_T \int_0^{t_{k'+1}} (\|y_{rh}(t)\|_H^2 + \|u_{rh}(t)\|_U^2)dt \\
&\overset{(5)}{\leq} \|y_{rh}(0)\|_H^2 + \frac{c_T}{\alpha_\ell} \int_0^{t_{k'+1}} \ell(y_{rh}(t), u_{rh}(t))dt \\
&\overset{(23)}{\leq} \|y_0\|_H^2 + \frac{c_T}{\alpha \alpha_\ell}(V_T(y_0) - V_T(y_{th}(t_{k'+1}))) \\
&\leq \|y_0\|_H^2 + \frac{c_T}{\alpha \alpha_\ell} V_T(y_0) \overset{(6)}{\leq} (1 + \frac{c_T}{\alpha \alpha_\ell} \gamma(T))\|y_0\|_H^2 \leq d_1^2.
\end{aligned}
$$

Hence $y_{rh}(t_{k'+1}) \in \mathcal{B}_{d_1}(0)$ which concludes the induction step. Taking the limit $k' \to \infty$ we find

$$(24) \qquad \alpha J_\infty(u_{rh}, y_0) = \alpha \int_0^\infty \ell(y_{rh}(t), u_{rh}(t))dt \leq V_T(y_0),$$

which concludes the proof. $\qquad \square$

Remark 7. If (3) is globally stabilizable, i.e. (6) holds with $\mathcal{N}_0$ replaced by H and if also (A1) is satisfied for all $y_0 \in H$, then Theorem 6 holds for all $y_0 \in H$, without the need of (A3). In fact (A3) was only used in the proof of Lemma 3 for the construction of $\mathcal{B}_{d_1}(0)$ which is not needed any more if (A2) holds globally.

In the following Theorem, we will show that the value function $V_{T-\delta}$ exponentially decays along the receding horizon trajectory y_{rh}.

THEOREM 8 (Exponential decay). *Suppose that* (A1)-(A3) *hold and let a sampling time* $\delta > 0$ *be given. Then there exist numbers* $T^* > \delta$, $\alpha > 0$ *such that for every prediction horizon* $T \geq T^*$, *and every* $y_0 \in \mathcal{B}_{d_2}(0)$ *with* $d_2(T) > 0$, *the receding horizon trajectory* $y_{rh}(\cdot)$ *satisfies*

$$(25) \qquad V_T(y_{rh}(t_k)) \leq e^{-\zeta t_k} V_T(y_0),$$

where ζ *is a positive number depending on* α, δ, *and* T *but independent of* y_0. *Moreover, for every positive* t *we have*

$$(26) \qquad V_{T-\delta}(y_{rh}(t)) \leq ce^{-\zeta t} V_T(y_0)$$

with a positive constant c *depending on* α, δ, *and* T *but independent of* y_0.

Proof. Let $\delta > 0$ be arbitrary. Then according to Theorem 6 and (17), there exists a positive number T^* such that for every $T \geq T^*$ and $y_0 \in \mathcal{B}_{d_2}(0)$ with $d_2 > 0$ we have

$$(27) \quad V_T(y_{rh}(t_{k+1})) - V_T(y_{rh}(t_k)) \leq -\alpha \int_{t_k}^{t_{k+1}} \ell(y_{rh}(t), u_{rh}(t))dt \quad \text{for every } k \in \mathbb{N},$$

with a positive $\alpha < 1$. Moreover, by using (11) and (12) we have

$$
\begin{aligned}
V_T(y_{rh}(t_{k+1})) \leq & \theta_1 \int_{t_{k+1}}^{t_k+T} \ell(y_T^*(t; y_{rh}(t_k), t_k), u_T^*(t; y_{rh}(t_k), t_k)) \, dt \\
(28) \qquad \leq & \theta_1 \theta_2 \int_{t_k}^{t_{k+1}} \ell(y_T^*(t; y_{rh}(t_k), t_k), u_T^*(t; y_{rh}(t_k), t_k)) \, dt \\
= & \theta_1 \theta_2 \int_{t_k}^{t_{k+1}} \ell(y_{rh}(t), u_{rh}(t)) \, dt,
\end{aligned}
$$

where $\theta_1 > 0$ and $\theta_2 > 0$ are defined in Lemma 4 and the last equality follows from Step 3 in Algorithm 1. Now by using (27) and (28) we obtain

$$V_T(y_{rh}(t_{k+1})) - V_T(y_{rh}(t_k)) \leq \frac{-\alpha}{\theta_1 \theta_2} V_T(y_{rh}(t_{k+1})) \quad \text{for every } k \in \mathbb{N}.$$

Therefore, by defining $\eta := (1 + \frac{\alpha}{\theta_1 \theta_2})^{-1}$ for every $k \in \mathbb{N}$ we can write

$$(29) \qquad V_T(y_{rh}(t_k)) \leq \eta V_T(y_{rh}(t_{k-1})) \leq \eta^2 V_T(y_{rh}(t_{k-2})) \leq \cdots \leq \eta^k V_T(y_0).$$

Now by defining $\zeta := \frac{|\ln \eta|}{\delta}$, we obtain the inequality (25).

Turning to the inequality (26) with $t > 0$ arbitrary, then there exists an index k such that $t \in [t_k, t_{k+1}]$. Now since $T - \delta \leq T + t_k - t$ and by using Bellman's optimality principle we have

$$
\begin{aligned}
V_{T-\delta}(y_{rh}(t)) \leq & V_{T+t_k-t}(y_{rh}(t)) \\
(30) \qquad = & V_T(y_{rh}(t_k)) - \int_{t_k}^{t} \ell(y_T^*(s; y_{rh}(t_k), t_k), u_T^*(s; y_{rh}(t_k), t_k)) \, ds \\
\leq & V_T(y_{rh}(t_k)).
\end{aligned}
$$

By using (29) and (30) we obtain

$$V_{T-\delta}(y_{rh}(t)) \leq V_T(y_{rh}(t_k)) \leq \frac{\eta^{k+1}}{\eta} V_T(y_0) = \frac{1}{\eta} e^{-\zeta t_{k+1}} V_T(y_0) \leq \frac{1}{\eta} e^{-\zeta t} V_T(y_0). \qquad \square$$

Remark 9. The above result is similar to the result obtained in [21] (Theorem 2.4), if the value function $V_{T-\delta}$ is considered as a control Lyapunov function G. At every iteration k of Algorithm 1 for every open-loop optimal control problem we have

$$
\begin{aligned}
\min_{u \in L^2(t_k, t_k+T; U)} & J_T(u, y_{rh}(t_k)) \\
= & \int_{t_k}^{T+t_k} \ell(y_T^*(t; y_{rh}(t_k), t_k), u_T^*(t; y_{rh}(t_k), t_k))dt \\
= & \int_{t_k}^{\delta+t_k} \ell(y_T^*(t; y_{rh}(t_k), t_k), u_T^*(t; y_{rh}(t_k), t_k))dt + V_{T-\delta}(y_T^*(\delta + t_k; y_{rh}(t_k), t_k)) \\
= & \int_{t_k}^{t_{k+1}} \ell(y_{rh}(t), u_{rh}(t))dt + V_{T-\delta}(y_{rh}(t_{k+1})).
\end{aligned}
$$

This means that the terminal cost $V_{T-\delta}$ is implicitly added to the objective function of every open-loop optimal control problem. Indeed $V_{t-\delta}$ can be interpreted as an approximation of the infinite horizon value function V_∞ which is incorporated in the objective function of every open-loop problem.

Remark 10. Note that the inequality (26) does not imply the asymptotic stability of the receding horizon control law defined by Algorithm 1 within the neighborhood $\mathcal{B}_{d_2}(0)$, unless the finite horizon value function $V_{t-\delta}$ is uniformly positive on the level-sets of $V_{t-\delta}$. That is, for every positive $r > 0$ we have

$$V_{T-\delta}(y) \geq C\|y\|_H^2 \quad \text{for all } y \in \Pi_r,$$

where C is a positive constant depending on the time horizon $T - \delta$ and Π_r is defined by

$$\Pi_r := \{y \in H \mid V_{T-\delta}(y) \leq r\}.$$

In the case of finite-dimensional controlled systems, the above condition was investigated in [31]. However for infinite-dimensional controlled systems, this condition only holds in special cases [15].

3. Stabilization of Burgers' Equation. Here we apply results of the previous section to the stabilization of the viscous Burgers' equation with periodic and homogeneous Neumann boundary conditions, respectively. For these boundary conditions the origin of the uncontrolled system is stable but not asymptotically stable. In the case of homogeneous Dirichlet boundary conditions, on the other hand, the origin is asymptotically stable.

We shall investigate Assumptions (A1)-(A3) and show that in the case of periodic boundary conditions Algorithm 1 provides globally stabilizing controls, while for Neumann boundary conditions we obtain locally stabilizing controls.

3.1. Burgers' Equation with Periodic Boundary Conditions. For an arbitrary finite horizon $T > 0$, we consider the controlled Burgers' equation with the periodic boundary conditions of the form

$$(31) \quad \begin{cases} \frac{d}{dt}y(t) = \mu y_{xx}(t) - y(t)y_x(t) + Bu(t) & \text{in } (0,T) \times (0,1), \\ y(t,0) = y(t,1), \quad y_x(t,0) = y_x(t,1) & \text{on } (0,T), \\ y(0,\cdot) = y_0 & \text{in } (0,1). \end{cases}$$

Throughout $\mu > 0$ and $y_0 \in L^2(0,1)$ are fixed, and the control operator B is the extension-by-zero operator given by

$$(Bu)(x) = \begin{cases} u(x) & x \in \hat{\Omega}, \\ 0 & x \in (0,1)\backslash\hat{\Omega}, \end{cases}$$

where the control domain $\hat{\Omega}$ is a nonempty open subset of $(0,1)$.

For the function space setting of (31) we introduce the spaces

$$V := \{y \in H^1(0,1) \mid y(0) = y(1)\}, \quad H := L^2(0,1),$$

and

$$W(0,T) := \{\phi : \phi \in L^2(0,T;V), \quad \frac{d}{dt}\phi \in L^2(0,T;V^*)\},$$

where V^* is the adjoint space of V. The spaces H and V are endowed with the usual norms $\|\cdot\|_H := \|\cdot\|_{L^2(0,1)}$ and $\|\cdot\|_V := \|\cdot\|_{H^1(0,1)}$. Further $\langle\cdot,\cdot\rangle_H$ and $\langle\cdot,\cdot\rangle_{V^*,V}$ denote the inner product in H and the duality pairing between V and V^*. We recall that $W(0,T)$ is continuously embedded into $C([0,T];H)$, see e.g. [32]. It will be convenient to define the continuous trilinear form $b : V \times V \times V \to \mathbb{R}$ by

$$b(\varphi, \psi, \phi) = \int_0^1 \varphi\psi_x\phi\,dx.$$

We shall frequently use the property that

$$(32) \qquad b(y,y,y) = \int_0^1 y_x y^2\,dx = \frac{1}{3}(y^3(1) - y^3(0)) = 0 \quad \text{for all } y \in V.$$

It is well-known that for every control $u \in L^2(0,T;L^2(\hat\Omega))$, equation (31) admits a unique weak solution $y \in W(0,T)$, i.e. y satisfies $y(0) = y_0$ in H, and for a.e. $t \in (0,T)$,

$$(33) \quad \frac{d}{dt}\langle y(t), \varphi\rangle_{V^*,V} + \mu\langle y(t), \varphi\rangle_V - \mu\langle y(t), \varphi\rangle_H + b(y(t), y(t), \varphi) = \langle Bu(t), \varphi\rangle_H$$

holds for all $\varphi \in V$. Using (32) and Gronwall's lemma it can easily be shown that there exists a constant C_T such that

$$(34) \qquad |y(\cdot; y_0, u)|_{W(0,T)} \leq C_T(|y_0|_H + |u|_{L(0,T;L^2(\hat\Omega))}),$$

where $y(\cdot; y_0, u)$ indicates the dependence of the solution on y_0 and u. The running cost will be taken of the form

$$(35) \qquad l(y,u) := \frac{1}{2}\|y\|_H^2 + \frac{\beta}{2}\|u\|_{L^2(\hat\Omega)}^2,$$

where $\beta > 0$.

We have now specified all items of the finite horizon problem (P_T). Using (34) it follows from standard subsequential limit arguments that (P_T) with the control system given by (31) admits a solution for each $y_0 \in H$. In particular (A1) holds with $\mathcal{N}_0 = H$. In the following lemma we show that Assumption (A2) holds as well.

LEMMA 11 (Global stabilizability). *For each $T > 0$ and initial state $y_0 \in H$ there exists a control $\hat{u}(\cdot; y_0) \in L^2(0,T;L^2(\hat\Omega))$ such that*

$$(36) \qquad V_T(y_0) \leq J_T(\hat{u}, y_0) \leq \gamma(T)\|y_0\|_H^2,$$

for a continuous, non-decreasing and bounded function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$.

Proof. Set $\hat{u}(t) := -y(t)|_{\hat\Omega}$ and consider

$$(37) \qquad \begin{cases} \frac{d}{dt}y(t) = \mu y_{xx}(t) - y(t)y_x(t) - By(t)|_{\hat\Omega} & \text{in } (0,T) \times (0,1), \\ y(t,0) = y(t,1), \quad y_x(t,0) = y_x(t,1) & \text{on } (0,T), \\ y(0,\cdot) = y_0 & \text{in } (0,1). \end{cases}$$

Taking the inner product of the first equation of (37) with $y(t)$, we have for a.e. $t \in (0,T)$

$$(38) \quad \begin{aligned} \frac{1}{2}\frac{d}{dt}\|y(t)\|_H^2 &+ \mu(\|y_x(t)\|_H^2 - y_x(t,1)y(t,1) + y_x(t,0)y(t,0)) \\ &+ b(y(t), y(t), y(t)) + \|y(t)\|_{L^2(\hat\Omega)}^2 = 0. \end{aligned}$$

Taking into account the boundary conditions and (32) one can express (38) as

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|_H^2 + \mu\|y_x(t)\|_H^2 + \|y(t)\|_{L^2(\hat{\Omega})}^2 = 0.$$

One can easily show that $\|y\|_1^2 := \|y_x\|_H^2 + \frac{1}{\mu}\|y\|_{L^2(\hat{\Omega})}^2$ is a norm which is equivalent to the H^1-norm [29], Page 26. Thus there exist positive constants $c_2 > c_1 > 0$ such that

$$c_1\|y\|_V^2 \leq \|y\|_1^2 \leq c_2\|y\|_V^2 \quad \text{for all } y \in V,$$

and consequently

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|_H^2 + \mu c_1\|y(t)\|_V^2 \leq 0,$$

and therefore,

$$\frac{d}{dt}\|y(t)\|_H^2 + 2\mu c_1\|y(t)\|_H^2 \leq 0 \quad \text{for all } t \in [0,T].$$

Multiplying both sides of the above equation by $e^{2\mu c_1 t}$ and integrating from 0 to t we obtain

$$\|y(t)\|_H^2 \leq \|y_0\|_H^2 e^{-2\mu c_1 t} \quad \text{for all } t \in [0,T].$$

By integrating the above inequality over the interval $[0,T]$, we obtain

$$(39) \qquad \int_0^T \|y(t)\|_H^2 dt \leq \frac{1}{2\mu c_1}(1 - e^{-2\mu c_1 T})\|y_0\|_H^2.$$

By the definition of the value function $V_T(\cdot)$ and (39) we have

$$V_T(y_0) \leq \int_0^T (\frac{1}{2}\|y(t)\|_H^2 + \frac{\beta}{2}\| - y(t)\|_{L^2(\hat{\Omega})}^2)\,dt \leq \frac{1+\beta}{4\mu c_1}(1 - e^{-2\mu c_1 T})\|y_0\|_H^2,$$

and (A2) follows with $\gamma(T) := \frac{1+\beta}{4\mu c_1}(1 - e^{-2\mu c_1 T})$, and $\mathcal{N}_0 = H$. $\qquad\square$

From Lemma 11, we infer that Assumption (A2) holds globally. Thus by Remark 7 we can directly apply Theorem 6 without addressing (A3) and conclude that: for any arbitrary sampling time δ, there exists a positive T^* such that for every $T \geq T^*$ the receding horizon control u_{rh} is globally suboptimal (within H) with suboptimality factor $\alpha > 0$, and we have

$$(40) \qquad \alpha V_\infty(y_0) \leq \alpha J_\infty(u_{rh}, y_0) \leq V_T(y_0) \leq V_\infty(y_0)$$

for every $y_0 \in H$. Now it remains for us to show that the receding horizon control u_{rh} computed by Algorithm 1 is globally stabilizing. This property will be verified by means of the following theorem.

THEOREM 12. *Let $y_0 \in H$ and $\delta > 0$ be arbitrary, and apply Algorithm 1 for the stabilization of the Burgers' equation (31) with a prediction horizon $T \geq T^*$, where T^* is introduced by Proposition 5. Then the receding horizon trajectory y_{rh} satisfies $\lim_{t\to\infty}\|y_{rh}(t)\|_H = 0$.*

Proof. First, we show that

(41) $$\|y_{rh}\|_{L^\infty(0,\infty;H)} \le \nu\|y_0\|_H$$

for a constant $\nu > 0$.

Due to (35), (36), and (40), we have

$$\frac{\alpha\min\{1,\beta\}}{2}\int_0^\infty \left(\|y_{rh}(t)\|_H^2 + \|u_{rh}(t)\|_{L^2(\hat\Omega)}^2\right)dt$$
$$\le \alpha J_\infty(u_{rh}(\cdot),y_0) \le V_T(y_0) \le \gamma(T)\|y_0\|_H^2.$$

Therefore by choosing $\sigma_1 := \frac{2\gamma(T)}{\alpha\min\{1,\beta\}}$, we obtain

(42) $$\int_0^\infty \|y_{rh}(t)\|_H^2 + \|u_{rh}(t)\|_{L^2(\hat\Omega)}^2 dt \le \sigma_1\|y_0\|_H^2.$$

Moreover, the receding horizon state given by Algorithm 1 satisfies $y_{rh} \in C([0,\infty),H)$, for every $k \in \mathbb{N}$ we have

(43) $$y_{rh}\mid_{(t_k,t_{k+1})} \in L^2(t_k,t_{k+1};V), \quad \frac{d}{dt}y_{rh}\mid_{(t_k,t_{k+1})} \in L^2(t_k,t_{k+1};V^*),$$

and y_{rh} is the solution of

$$\begin{cases} \frac{d}{dt}y(t) = \mu y_{xx}(t) - y(t)y_x(t) + Bu_{rh}(t) & \text{in } (t_k,t_{k+1})\times(0,1), \\ y(t,0) = y(t,1), \quad y_x(t,0) = y_x(t,1) & \text{on } (t_k,t_{k+1}), \\ y(t_k,\cdot) = y_{rh}(t_k) \text{ for } k>0, \text{ and } y(0,\cdot) = y_0 \text{ for } k=0 & \text{in } (0,1). \end{cases}$$

By multiplying the above equation by $y_{rh}(\cdot)$ and integrating over the interval $(0,1)$, we have

$$\frac{1}{2}\frac{d}{dt}\|y_{rh}(t)\|_H^2 + \mu\|(y_{rh})_x(t)\|_H^2 = \langle Bu_{rh}(t),y_{rh}(t)\rangle_H \quad \text{a.e. } t \in (t_k,t_{k+1}).$$

From Cauchy-Schwarz and Young's inequalities we infer that

$$\frac{d}{dt}\|y_{rh}(t)\|_H^2 + 2\mu\|(y_{rh})_x(t)\|_H^2 \le \|u_{rh}(t)\|_{L^2(\hat\Omega)}^2 + \|y_{rh}(t)\|_H^2 \quad \text{a.e. } t \in (t_k,t_{k+1}).$$

Integrating from t_k to t for every $t \in (t_k,t_{k+1})$ we have

$$\|y_{rh}(t)\|_H^2$$
$$\le \|y_{rh}(t_k)\|_H^2 + \int_{t_k}^t \|u_{rh}(s)\|_{L^2(\hat\Omega)}^2 ds + \int_{t_k}^t \|y_{rh}(s)\|_H^2 ds \quad \text{for all } t \in (t_k,t_{k+1}).$$

By the same estimate as above for the interval (t_{k-1},t_k) we have

(44) $$\|y_{rh}(t_k)\|_H^2 \le \|y_{rh}(t_{k-1})\|_H^2 + \int_{t_{k-1}}^{t_k} \|u_{rh}(s)\|_{L^2(\hat\Omega)}^2 ds + \int_{t_{k-1}}^{t_k} \|y_{rh}(s)\|_H^2 ds.$$

Moreover by the above two estimates we have

$$\|y_{rh}(t)\|_H^2 \le \|y_{rh}(t_{k-1})\|_H^2 + \int_{t_{k-1}}^t \|u_{rh}(s)\|_{L^2(\hat\Omega)}^2 ds + \int_{t_{k-1}}^t \|y_{rh}(s)\|_H^2 ds.$$

By repeating the above argument for $k-2, k-3, \ldots, 0$, one can show that

$$
\begin{aligned}
\|y_{rh}(t)\|_H^2 &\leq \|y_{rh}(t_k)\|_H^2 + \int_{t_k}^t \|u_{rh}(s)\|_{L^2(\hat{\Omega})}^2 ds + \int_{t_k}^t \|y_{rh}(s)\|_H^2 \, ds \\
&\leq \|y_0\|_H^2 + \int_0^t \|u_{rh}(s)\|_{L^2(\hat{\Omega})}^2 ds + \int_0^t \|y_{rh}(s)\|_H^2 \, ds \\
&\leq \|y_0\|_H^2 + \int_0^\infty \|u_{rh}(s)\|_{L^2(\hat{\Omega})}^2 ds + \int_0^\infty \|y_{rh}(s)\|_H^2 \, ds \\
&\leq (1 + \sigma_1)\|y_0\|_H^2,
\end{aligned}
$$

where in the last line (42) has been used. Choosing $\nu := \sqrt{1 + \sigma_1}$ we obtain (41).

Next we are in the position to prove

$$
\lim_{t \to \infty} \|y_{rh}(t)\|_H^2 = 0.
$$

For every $t'' \geq t'$ we have

$$
\begin{aligned}
\|y_{rh}(t'')\|_H^2 - \|y_{rh}(t')\|_H^2 &= \int_{t'}^{t''} \frac{d}{dt} \|y_{rh}(t)\|_H^2 \, dt \\
&= 2 \int_{t'}^{t''} \langle y_{rh}(t), \mu(y_{rh})_{xx}(t) - (y_{rh})_x(t) y_{rh}(t) + B u_{rh}(t) \rangle_{V,V^*} \, dt \\
&= -2\mu \int_{t'}^{t''} \|(y_{rh})_x(t)\|_H^2 dt + 2 \int_{t'}^{t''} \langle B u_{rh}(t), y_{rh}(t) \rangle_H \, dt \\
&\leq 2 \int_{t'}^{t''} \|u_{rh}(t)\|_{L^2(\hat{\Omega})} \|y_{rh}(t)\|_H \, dt \\
&\leq 2 \Big(\int_{t'}^{t''} \|u_{rh}(t)\|_{L^2(\hat{\Omega})}^2 dt \Big)^{\frac{1}{2}} \Big(\int_{t'}^{t''} \|y_{rh}(t)\|_H^2 dt \Big)^{\frac{1}{2}},
\end{aligned}
$$

and thus

$$
(45) \qquad \|y_{rh}(t'')\|_H^2 - \|y_{rh}(t')\|_H^2 \leq 2\sqrt{\sigma_1} \nu \|y_0\|_H^2 (t'' - t')^{\frac{1}{2}}.
$$

For the last inequality, (41) and (42) have been used. Moreover, from (35), (36), and (40) we have

$$
\frac{\alpha}{2} \int_0^\infty \|y_{rh}(t)\|_{L^2(\hat{\Omega})}^2 \leq \alpha J_\infty(u_{rh}, y_0) \leq V_T(y_0) \leq \gamma(T) \|y_0\|_H^2 < \infty.
$$

This estimate implies that

$$
(46) \qquad \lim_{t \to \infty} \int_{t-L}^t \|y_{rh}(s)\|_H^2 ds = 0
$$

for all $L > 0$. Suppose to the contrary that

$$
\lim_{t \to \infty} \|y_{rh}(t)\|_H^2 \neq 0.
$$

Then there exists an $\epsilon > 0$ and a sequence $\{t_n\}_{n=1}^\infty$ with $t_n > 0$ and $\lim_{n \to \infty} t_n = \infty$ for which

$$
(47) \qquad \|y_{rh}(t_n)\|_H^2 > \epsilon \quad \text{for all } n = 1, 2, \ldots.
$$

It follows from (45) and (47) that for every $L > 0$ and $n = 1, 2, \dots$

$$\int_{t_n - L}^{t_n} \|y_{rh}(t)\|_H^2 dt$$

(48)
$$= \int_{t_n - L}^{t_n} \|y_{rh}(t_n)\|_H^2 dt - \int_{t_n - L}^{t_n} \left(\|y_{rh}(t_n)\|_H^2 - \|y_{rh}(t)\|_H^2 \right) dt,$$

$$> L\epsilon - 2\sqrt{\sigma_1}\nu \|y_0\|_H^2 \int_{t_n - L}^{t_n} (t_n - t)^{\frac{1}{2}} dt = L\epsilon - \frac{4}{3}\sqrt{\sigma_1}\nu \|y_0\|_H^2 L^{\frac{3}{2}}.$$

Setting $\omega := \frac{4}{3}\sqrt{\sigma_1}\nu \|y_0\|_H^2$, and choosing $L := (\frac{\epsilon}{2\omega})^2$, we obtain

$$\int_{t_n - L}^{t_n} \|y_{rh}(t)\|_H^2 dt > \frac{L\epsilon}{2} \quad \text{for all } n = 1, 2, \dots.$$

This contradicts (46). Hence $\lim_{t \to \infty} \|y_{rh}(t)\|_H^2 = 0$ and the proof is complete. □

3.2. Burgers' Equation with Homogeneous Neumann Boundary Conditions. Here we consider the controlled Burgers' equation with homogenous Neumann boundary conditions of the form

(49)
$$\begin{cases} \frac{d}{dt}y(t) = \mu y_{xx}(t) - y(t)y_x(t) + Bu(t) & \text{in } (0, T) \times (0, 1), \\ y_x(t, 0) = y_x(t, 1) = 0 & \text{on } (0, T), \\ y(0, \cdot) = y_0 & \text{in } (0, 1). \end{cases}$$

We can utilize the same notation as in subsection 3.1, except for the energy space which is now chosen to be

$$V := \{ y \in H^1(0, 1) \mid y_x(0) = y_x(1) = 0 \}.$$

Again $V \subset H \subset V^*$ is a Gelfand triple and $W(0, T)$ is continuously embedded in $C([0, T]; H)$.

The significant difference between (31) and (49) is given by the fact that in the case of periodic boundary conditions the nonlinearity is conservative, i.e. we have that $b(\phi, \phi, \phi) = 0$ for all $\phi \in V$, which is not the case for Neumann boundary conditions. As a consequence we have to rely on the local version of the results of section 2.

Again we use the weak or variational solution concept of (33). Due to the fact that the nonlinearity is not conservative the verification of a global weak solution is not trivial. We have the following result.

LEMMA 13. *For every $T > 0$, and for every $y_0 \in H$, and $u \in L^2(0, T; L^2(\hat{\Omega}))$ there exists a unique solution $y(\cdot; y_0, u) \in W(0, T)$ to (49). Moreover there exists a constant C_T such that*

$$|y(\cdot; y_0, u)|_{W(0,T)} \leq C_T (1 + |y_0|_H + |u|_{L(0,T;L^2(\hat{\Omega}))}),$$

for all $y_0 \in H$, and $u \in L^2(0, T; L^2(\hat{\Omega}))$.

For the proof we refer to [35]. For the step that the local solution can be extended to a global one we prefer the argument given in [16] for which it is useful to recall that for a measurable function, which will be u in our case, the function $E \to \int_E |u(t)|_{L^2(\hat{\Omega})} dt$, with E a measurable subset of $(0, T)$, is absolutely continuous.

The running cost will again be taken to be of the form (35). It is now standard to argue the existence of a solution to (P_T) with the control system given by (49). In particular (A1) holds with $\mathcal{N}_0 = H$. In the following lemmas we show that Assumptions (A2) and (A3) hold as well.

LEMMA 14 (Local stabilizability). *There exists a neighborhood $\mathcal{B}_{\delta_1}(0) \subset H$ such that for every $T > 0$ and every $y_0 \in \mathcal{B}_{\delta_1}(0)$ there exists a control $\hat{u}(\cdot, y_0) \in L^2(0, T; L^2(\hat{\Omega}))$ with*

$$V_T(y_0) \le J_T(\hat{u}, y_0) \le \gamma(T) \|y_0\|_H^2,$$

where $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, non-decreasing and bounded function.

Proof. Setting $\hat{u}(t) := -y(t)|_{\hat{\Omega}}$ in the first equation of (49), multiplying $y(t)$ and taking the L^2-scalar product we obtain

$$(50) \qquad \frac{1}{2}\frac{d}{dt}\|y(t)\|_H^2 + \mu\|y_x(t)\|_H^2 + b(y(t), y(t), y(t)) + \|y(t)\|_{L^2(\hat{\Omega})}^2 = 0$$

As in the case of periodic boundary conditions one can argue that $\|y\|_1^2 := \|y_x\|_H^2 + \frac{1}{\mu}\|y\|_{L^2(\hat{\Omega})}^2$ is an equivalent norm to H^1-norm, see e.g. [29], Page 26, and hence there exist positive constants $c_2 > c_1 > 0$ such that $c_1\|y\|_V^2 \le \|y\|_1^2 \le c_2\|y\|_V^2$ for all $y \in V$. The nonlinearity satisfies the following equality

$$b(y, y, y) = \int_0^1 y_x y^2 dx \le \|y\|_{L^\infty(0,1)}\|y_x\|_H\|y\|_H \le c_a\|y\|_1^2\|y\|_H \qquad \text{for all } y \in V,$$

where c_a, depends on the embedding constant of V into $L^\infty(0, 1)$ and c_1. From (50) we therefore deduce that

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|_H^2 + \mu\|y_x(t)\|_H^2 + \|y(t)\|_{L^2(\hat{\Omega})}^2 \le c_a\|y(t)\|_1^2\|y(t)\|_H,$$

and consequently

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|_H^2 + \mu\|y(t)\|_1^2 \le c_a\|y(t)\|_1^2\|y(t)\|_H.$$

Now let us choose $\|y_0\|_H$ sufficiently small, say $\|y_0\|_H \le \frac{\mu}{4c_a}$. Then by continuity of the solution for a short interval of time $[0, T^*]$, we have $\|y(t)\|_H \le \frac{\mu}{2c_a}$ for all $t \in [0, T^*]$ and further

$$\frac{d}{dt}\|y(t)\|_H^2 + \mu c_1\|y(t)\|_H^2 \le 0 \qquad \text{for all } t \in [0, T^*].$$

Multiplying both sides of the above equation by $e^{\mu c_1 t}$ and integrating from 0 to t we obtain

$$(51) \qquad \|y(t)\|_H^2 \le \|y_0\|_H^2 e^{-\mu c_1 t} \le (\frac{\mu}{4c_a})^2 \qquad \text{for all } t \in [0, T^*].$$

Repeating the above argument implies that $\|y(t)\|_H \le \frac{\mu}{4c_a}$ will remain small for all $t \in [0, T]$ and moreover we have

$$\|y(t)\|_H^2 \le \|y_0\|_H^2 e^{-\mu c_1 t} \qquad \text{for all } t \in [0, T].$$

Integration over $[0, T]$ implies that

$$(52) \qquad \int_0^T \|y(t)\|_H^2 dt \le \frac{1}{\mu c_1}(1 - e^{-\mu c_1 T})\|y_0\|_H^2.$$

By the definition of the value function $V_T(\cdot)$ and (52) we have

$$V_T(y_0) \le \int_0^T \frac{1}{2}\|y(t)\|_H^2 + \frac{\beta}{2}\|y(t)\|_{L^2(\hat{\Omega})}^2 dt \le \frac{1+\beta}{2\mu c_1}(1 - e^{-\mu c_1 T})\|y_0\|_H^2,$$

where $\gamma(T) := \frac{1+\beta}{2\mu c_1}(1 - e^{-\mu c_1 T})$ is a nondecreasing, continuous and bounded function, as desired, and $\delta_1 := \frac{\mu}{4 c_a}$. □

LEMMA 15. *Assumption* (A3) *holds for* (49) *with* $\mathcal{N}_0 = \mathcal{B}_{\delta_1}(0)$ *defined in Lemma* 14.

Proof. For every $y_0 \in \mathcal{B}_{\delta_1}(0)$ we have from (49) that

$$(53) \quad \begin{aligned} &\frac{1}{2}\frac{d}{dt}\|y(t)\|_H^2 + \mu\|y(t)\|_V^2 \\ &\qquad \le \mu\|y(t)\|_H^2 + |b(y(t), y(t), y(t))| + |\langle y(t), Bu(t)\rangle_H| \quad \text{a.e } t \in [0, T]. \end{aligned}$$

From Agmon's inequality we recall that there exists a constant c_A such that

$$\|\phi\|_{L^\infty(0,1)} \le c_A\|\phi\|_H^{\frac{1}{2}}\|\phi\|_V^{\frac{1}{2}} \quad \text{for all } \phi \in V,$$

and consequently there exits a constant c_I such that

$$b(\phi, \phi, \phi) \le \|\phi\|_{L^\infty(0,1)}\|\phi\|_H\|\phi\|_V \le c_A\|\phi\|_H^{\frac{3}{2}}\|\phi\|_V^{\frac{3}{2}} \le \mu\|\phi\|_V^2 + c_I\|\phi\|_H^6 \quad \text{for all } \phi \in V.$$

Utilizing the above inequality and (53) we obtain

$$\frac{d}{dt}\|y(t)\|_H^2 \le 2\mu\|y(t)\|_H^2 + 2c_I\|y(t)\|_H^6 + 2\|y(t)\|_H\|u(t)\|_{L^2(\hat{\Omega})} \quad \text{for a.e } t \in [0, T].$$

Upon integration we obtain

$$\|y(t)\|_H^2 \le \|y(0)\|_H^2 + (2\mu + 1 + 2c_I\|y\|_{C([0,T];H)}^4)\int_0^t \|y(s)\|_H^2\, ds + \int_0^t \|u(s)\|_{L^2(\hat{\Omega})}^2 ds.$$

By Lemma 13 the family

$$\{\|y(\cdot; y_0, u)\|_{C([0,T];H)} \,|\, y_0 \in \mathcal{B}_{\delta_1}(0), \|u\|_{L^2(0,T;L^2(\hat{\Omega}))} \le \sqrt{2\gamma(T)/\min\{1, \beta\}}\, \|y_0\|_H\}$$

is bounded, and hence the desired estimate follows. □

Now we are in the position that we can apply Theorem 6 and it remains for us to show that the receding horizon control u_{rh} computed by Algorithm 1 is stabilizing. This will be accomplished in the following theorem. It uses the quantifier $d_2(T)$ for the size of the neighborhood of the initial data. Recall that $d_2(T)$ depends on $\gamma(T)$, which was given explicitly in the proof of Lemma 14 and on c_T, the existence of which was provided in the proof of Lemma 15.

THEOREM 16. *Let a sampling time* $\delta > 0$ *be given, and apply Algorithm* 1 *for the stabilization of the Burgers' equation* (49) *with a prediction horizon* $T \ge T^*$, *where* T^* *is introduced by Proposition* 5. *Then we have* $\lim_{t\to\infty}\|y_{rh}(t)\|_H = 0$ *provided that* $|y_0|_H \le d_2(T)$.

Proof. We recall that $\delta_1 = \frac{\mu}{4c_a}$ depends on embedding constants and was introduced in the proof of Lemma 14. Furthermore, we have $d_2(T) \leq d_1(T) \leq \delta_1$. To verify the claim we can follow for the most part the proof of Theorem 12. Again we first show that there exists some $\nu > 0$ such that

$$(54) \qquad \|y_{rh}\|_{L^\infty(0,\infty;H)} \leq \nu \|y_0\|_H,$$

for each $y_0 \in \mathcal{B}_{d_2}(0)$. By construction we have that $y_{rh} \in C([0,\infty), H)$, that (43) holds, and that

$$(55) \qquad \int_0^\infty \|y_{rh}(t)\|_H^2 + \|u_{rh}(t)\|_{L^2(\hat\Omega)}^2 dt \leq \sigma_1 \|y_0\|_H^2,$$

where $\sigma_1 := \frac{2\gamma(T)}{\alpha \min\{1,\beta\}}$. For any $k = 0, 1, \dots$ we have

$$\frac{d}{2dt}\|y_{rh}(t)\|_H^2 + \mu\|(y_{rh})_x(t)\|_H^2 + b(y_{rh}(t), y_{rh}(t), y_{rh}(t))$$
$$= \langle Bu_{rh}(t), y_{rh}(t)\rangle_H \quad \text{a.e. } t \in (t_k, t_k + 1).$$

Furthermore, d_2 in Theorem 6 has been chosen in such way that for every $t > 0$, the receding horizon trajectory $y_{rh}(t)$ stays in the neighborhood $\mathcal{B}_{\delta_1}(0)$. In other word, we have

$$(56) \qquad \|y_{rh}(t)\|_H \leq \delta_1 = \frac{\mu}{4c_a} < \frac{\mu}{2c_a} \qquad \text{for all } t > 0,$$

where c_a defined in Lemma 14. Now by Cauchy-Schwarz and Young's inequalities, by (56) and the definition of $\|\cdot\|_1$, we infer that

$$b(y_{rh}(t), y_{rh}(t), y_{rh}(t))$$
$$\leq \|y_{rh}(t)\|_{L^\infty(0,1)}\|y_{rh}(t)\|_H\|(y_{rh})_x(t)\|_H$$
$$\leq c_a\|y_{rh}(t)\|_H\|y_{rh}(t)\|_1^2$$
$$\leq \frac{\mu}{2}\|y_{rh}(t)\|_1^2 \leq \frac{\mu}{2}\|(y_{rh})_x(t)\|_H^2 + \frac{1}{2}\|y_{rh}(t)\|_H^2 \quad \text{for a.e. } t \in (t_k, t_{k+1}).$$

Thus for every $k \in \mathbb{N}$ we have

$$\frac{d}{dt}\|y_{rh}(t)\|_H^2 + \mu\|(y_{rh})_x(t)\|_H^2 \leq \|u_{rh}(t)\|_{L^2(\hat\Omega)}^2 + 2\|y_{rh}(t)\|_H^2 \quad \text{for a.e. } t \in (t_k, t_{k+1}),$$

and therefore for $t \in (t_k, t_{k+1})$

$$(57) \qquad \begin{aligned} \|y_{rh}(t)\|_H^2 + \mu \int_{t_k}^t \|(y_{rh})_x(s)\|_H^2 ds \\ \leq \|y_{rh}(t_k)\|_H^2 + \int_{t_k}^t \|u_{rh}(s)\|_{L^2(\hat\Omega)}^2 ds + 2\int_{t_k}^t \|y_{rh}(s)\|_H^2 ds. \end{aligned}$$

By the same estimate as above for the interval (t_{k-1}, t_k) we have

$$(58) \qquad \begin{aligned} \|y_{rh}(t_k)\|_H^2 + \mu \int_{t_{k-1}}^{t_k} \|(y_{rh})_x(s)\|_H^2 ds \\ \leq \|y_{rh}(t_{k-1})\|_H^2 + \int_{t_{k-1}}^{t_k} \|u_{rh}(s)\|_{L^2(\hat\Omega)}^2 ds + 2\int_{t_{k-1}}^{t_k} \|y_{rh}(s)\|_H^2 ds. \end{aligned}$$

By summing (57) and (58) we have

$$\|y_{rh}(t)\|_H^2 + \mu \int_{t_{k-1}}^t \|(y_{rh})_x(s)\|_H^2 ds$$

$$\leq \|y_{rh}(t_{k-1})\|_H^2 + \int_{t_{k-1}}^t \|u_{rh}(s)\|_{L^2(\hat\Omega)}^2 ds + 2\int_{t_{k-1}}^t \|y_{rh}(s)\|_H^2 ds.$$

Repeating the above argument for $k-2, k-3, \ldots, 0$, it follows that

$$\|y_{rh}(t)\|_H^2 \leq \|y_{rh}(t_k)\|_H^2 + \int_{t_k}^t \|u_{rh}(s)\|_{L^2(\hat\Omega)}^2 ds + 2\int_{t_k}^t \|y_{rh}(s)\|_H^2 ds$$

$$\leq \|y_0\|_H^2 + \int_0^\infty \|u_{rh}(s)\|_{L^2(\hat\Omega)}^2 ds + 2\int_0^\infty \|y_{rh}(s)\|_H^2 ds$$

$$\leq (1 + 2\sigma_1)\|y_0\|_H^2,$$

where for the last inequality (55) has been used. Choosing $\nu := \sqrt{1 + 2\sigma_1}$ we obtain (54).

Now we are in the position to prove

$$\lim_{t\to\infty} \|y_{rh}(t)\|_H^2 = 0.$$

For every $t'' \geq t'$ we have

$$\|y_{rh}(t'')\|_H^2 - \|y_{rh}(t')\|_H^2 = \int_{t'}^{t''} \frac{d}{dt}\|y_{rh}(t)\|_H^2 dt$$

$$= 2\int_{t'}^{t''} \langle y_{rh}(t), \mu(y_{rh})_{xx}(t) - (y_{rh})_x(t)y_{rh}(t) + Bu_{rh}(t)\rangle_{V,V^*} dt$$

$$\leq -2\mu \int_{t'}^{t''} \|(y_{rh})_x(t)\|_H^2 dt + 2\int_{t'}^{t''} c_a\|y_{rh}(t)\|_H\|y_{rh}(t)\|_1^2 dt$$

$$+ 2\int_{t'}^{t''} \langle Bu_{rh}(t), y_{rh}(t)\rangle_H dt$$

$$\leq -\mu \int_{t'}^{t''} \|(y_{rh})_x(t)\|_H^2 dt + \int_{t'}^{t''} \|y_{rh}(t)\|_H^2 dt + 2\int_{t'}^{t''} \langle Bu_{rh}(t), y_{rh}(t)\rangle_H dt$$

$$\leq 2\int_{t'}^{t''} \|u_{rh}(t)\|_{L^2(\hat\Omega)}\|y_{rh}(t)\|_H dt + \int_{t'}^{t''} \|y_{rh}(t)\|_H^2 dt$$

$$\leq 2\Big(\int_{t'}^{t''} \|u_{rh}(t)\|_{L^2(\hat\Omega)}^2 dt\Big)^{\frac{1}{2}} \Big(\int_{t'}^{t''} \|y_{rh}(t)\|_H^2 dt\Big)^{\frac{1}{2}}$$

$$+ \Big(\int_{t'}^{t''} \|y_{rh}(t)\|_H^2 dt\Big)^{\frac{1}{2}} \Big(\int_{t'}^{t''} \|y_{rh}(t)\|_H^2 dt\Big)^{\frac{1}{2}}$$

$$\leq 3\sqrt{\sigma_1}\nu\|y_0\|_H^2 (t'' - t')^{\frac{1}{2}},$$

where (54) and (55) were used to obtain the last inequality. Now the proof can be completed as the one for Theorem 12, except that the factor 2 in (48) has to be replaced by the factor 3 which appeared in the last estimate. $\qquad\square$

4. Numerical Results. We present numerical results to illustrate the theoretical findings of the previous sections. For the Burgers' equation with periodic -, or homogeneous Neumann boundary conditions, every constant function is a steady state of the uncontrolled equation. Hence the origin is stable, but it is not asymptotically stable. Consequently it is of interest to force the state of these equations to the steady state by an external control which is computed on the basis of the receding horizon control.

Our numerical experiments will also include a comparison of the performance of the receding horizon control scheme with and without terminal penalty term. The latter case was investigated in the previous section, the former one in [21], where it was shown that the quadratic penalty term $G(y) = \frac{1}{2}\|y\|^2_{L^2(\Omega)}$, can be used as a control Lyapunov function for the Navier-Stokes eqaution.

Our numerical tests utilize the following Algorithm 2, where G is chosen as one of the two cost functionals:

$$(59) \qquad \text{Zero: } G(y) = 0, \text{ or } \text{Quadratic: } G(y) = \frac{1}{2}\|y\|^2_{L^2(0,1)}.$$

For $G(y) = 0$, Algorithm 2 essentially coincides with Algorithm 1, except for the fact that we need to terminate our computations at some $T_\infty < \infty$.

Algorithm 2

Input: Let a final computational time horizon T_∞, and an initial state $y_0 \in L^2(0,1)$ be given.

1. Choose a prediction horizon $T < T_\infty$, and a sampling time $\delta \in (0, T]$.
2. Consider a grid $0 = t_0 < t_1 \cdots < t_r = T_\infty$ on the interval $[0, T_\infty]$, where $t_i := i\delta$ for $i = 0, \ldots, r$.
3. Solve successively the open-loop subproblem on $[t_i, t_i + T]$:

$$(60) \qquad \min \frac{1}{2}\int_{t_i}^{t_i+T}\|y(t)\|^2_{L^2(0,1)}dt + \frac{\beta}{2}\int_{t_i}^{t_i+T}\|u(t)\|^2_{L^2(\hat\Omega)}dt + G(y(t_i + T)),$$

subject to Burgers' equations (49) (or (31)) for the initial condition

$$y(t_i) = y^*_T(t_i) \text{ if } i \geq 1 \text{ and } y(t_i) = y_0 \text{ if } i = 0,$$

where $y^*_T(\cdot)$ is the solution to the preceding subproblem on $[t_{i-1}, t_{i-1} + T]$.
4. The receding horizon optimal pair $(y_{rh}(\cdot), u_{rh}(\cdot))$ is obtained by concatenation of the optimal pairs $(y^*_T(t), u^*_T(t))$ of the finite horizon subproblems on $[t_i, t_{i+1}]$ for $i = 0, \ldots, r - 1$.

The numerical simulations were carried out on the MATLAB platform. Throughout, the spatial discretization was done by the standard Galerkin method based on piecewise linear and continuous basis functions with mesh-size $h = 0.0125$. The ordinary differential equations resulting after spatial discretization were solved by the implicit Euler method with step-size $\Delta t = 0.0125$, where the nonlinear systems of equations within the implicit Euler method were solved by Newton's method. Every open-loop problem was solved by applying the Barzilai-Borwein (BB) gradient steps [7] with a nonmonotone line search [14] on the reduced problem in the "first optimize,

then discretize" manner. For every open-loop problem, the optimization algorithm was terminated when L^2-norm of the gradient for the reduced objective function was less than the tolerance 10^{-6}. Furthermore in all examples, we set $\delta = 0.25$ and $\beta = 10^{-3}$.

For every example, we implemented the receding horizon strategy for different choices of the prediction horizon T, and the two terminal costs G in (59). Furthermore, in order to have a measure for the performance of the receding horizon strategy, we consider

1. $J_{T_\infty}(u_{rh}, y_0) := \frac{1}{2} \int_0^{T_\infty} \|y_{rh}(t)\|_{L^2(0,1)}^2 dt + \frac{\beta}{2} \int_0^{T_\infty} \|u_{rh}(t)\|_{L^2(\hat{\Omega})}^2 dt$,
2. $\|y_{rh}\|_{L^2(Q)}$ with $Q := (0, T_\infty) \times (0, 1)$,
3. $\|y_{rh}(T_\infty)\|_{L^2(0,1)}$,
4. iter : The *total* number of iterations (BB-gradient steps) that the optimizer needs for all open-loop problems on the intervals $(t_i, t_i + T)$ for $i = 0, \ldots, r-1$.

EXAMPLE 1. We considered the Burgers' equation (31) with periodic boundary conditions. We chose $y_0(x) = \exp(-20(x-0.5)^2)$ as the initial function, $\mu = 10^{-3}$ as a viscosity parameter, and $T_\infty = 15$. Further the receding horizon control acts only on the set

$$\hat{\Omega} = (0.1, 0.2) \cup (0.4, 0.6) \cup (0.8, 0.9) \subset (0, 1).$$

Figures 1(a) and 1(b) depict, respectively, the solution and the evolution of the $L^2(0,1)$-norm for the state of the uncontrolled Burgers' equation (31). For the uncontrolled solution y^u we have

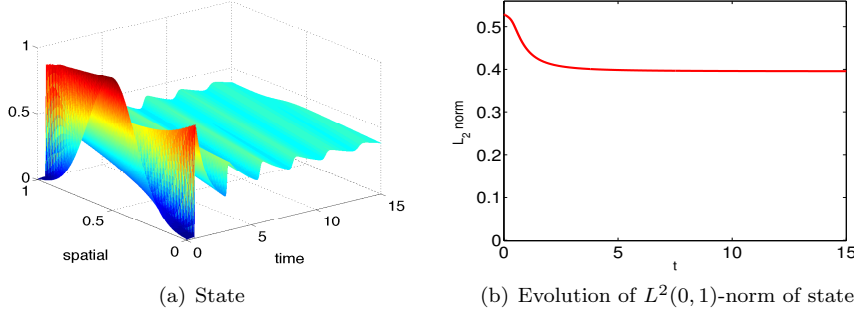$$\|y^u\|_{L^2(Q)} = 1.5768, \qquad \|y^u(T_\infty)\|_{L^2(0,1)} = 0.3958.$$

(a) State   (b) Evolution of $L^2(0,1)$-norm of state

FIG. 1. *Uncontrolled solution for Example 1*

The results of Algorithm 2 for different choices of T and G and the fixed sampling time $\delta = 0.25$ are summarized in Table 1. Figure 2 shows the results for the receding horizon pairs (y_{rh}, u_{rh}) if $G = 0$ and $T = 1$.

As expected, increasing the prediction horizon T results in a decrease of the stabilization measures $\|y_{rh}\|_{L^2(Q)}$ and $\|y_{rh}(T_\infty)\|_{L^2(0,1)}$, both for quadratic and zero terminal penalties. The quadratic terminal penalty term results in smaller values of the stabilization measures, with the difference in the $\|y_{rh}\|_{L^2(Q)}$-norm less pronounced than in the $\|y_{rh}(T_\infty)\|_{L^2(0,1)}$-norm. Using a non-trivial terminal penalty results in higher iteration numbers for the optimizer. In view of the fact that the choice of

G	Prediction Horizon	J_{T_∞}	$\|y_{rh}\|_{L^2(Q)}$	$\|y_{rh}(T_\infty)\|_{L^2(0,1)}$	iter
	$T = 1$	0.021891	0.1799	2.62×10^{-5}	11861
Quadratic	$T = 0.5$	0.023196	0.1820	3.95×10^{-5}	8352
	$T = 0.25$	0.027547	0.1828	1.32×10^{-4}	6041
	$T = 1$	0.021886	0.1805	8.41×10^{-5}	6220
Zero	$T = 0.5$	0.021893	0.1818	2.10×10^{-4}	3139
	$T = 0.25$	0.021943	0.1856	4.26×10^{-4}	1467

TABLE 1
Numerical results for Example 1

(a) RH State

(b) $L^2(0,1)$-norm of RHS

(c) RH Control
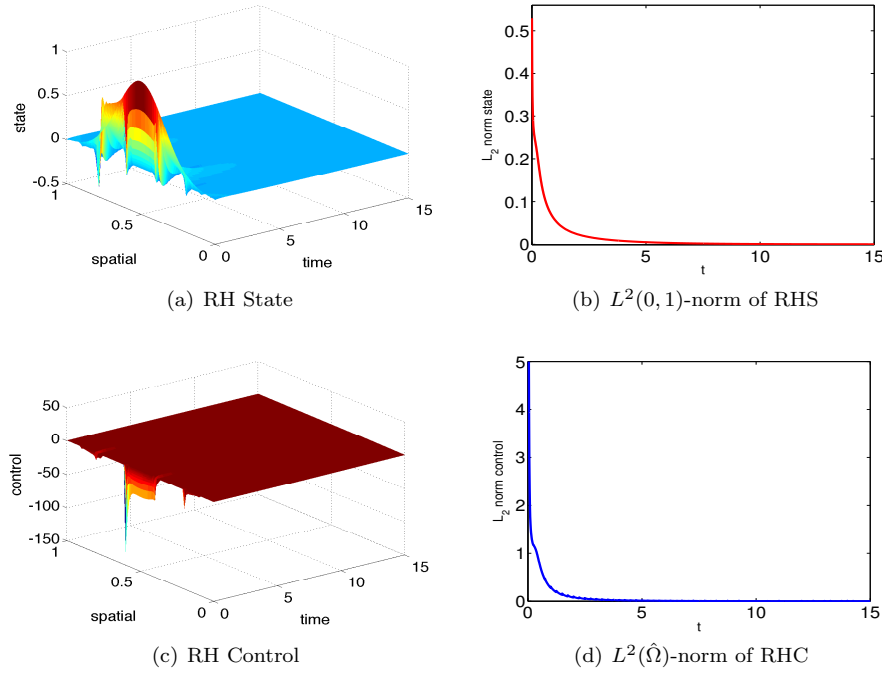
(d) $L^2(\hat{\Omega})$-norm of RHC

FIG. 2. *Receding horizon trajectories for Example 1*

T has only little effect on the stabilization measures, but significant effect on the number of iterations in the optimization algorithm, small T is preferable for this class of problems. It should also be of interest to search for methods which adaptively tune the prediction horizon.

EXAMPLE 2. Here we considered the stabilization of the Burgers' equation (49) with homogeneous Neumann boundary conditions. We set $y_0(x) = \cos(\pi x)$ as the initial function and chose $T_\infty = 10$. The spatial support for the controls is

$$\hat{\Omega} = (0, 0.15) \cup (0.85, 1) \subset (0, 1).$$

Furthermore, $\mu = 0.01$. Note that for this small viscosity parameter and the above anti-symmetric initial function, the uncontrolled numerical solution of (49) approaches

a non-constant, time independent steady state, see [8]. The uncontrolled solution y^u is illustrated in Figure 3 and we have

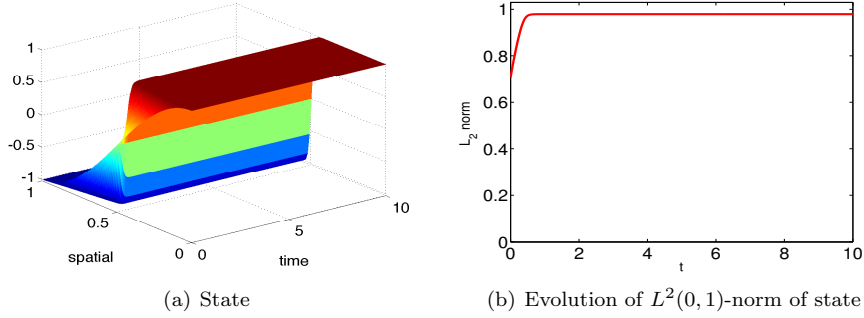$$\|y^u\|_{L^2(Q)} = 3.08, \qquad \|y^u(T_\infty)\|_{L^2(0,1)} = 0.9791.$$

(a) State

(b) Evolution of $L^2(0,1)$-norm of state

FIG. 3. *Uncontrolled solution for Example* 2

Table 2 reveals the numerical results of Algorithm 2 for different choices of the prediction horizon T and the terminal cost G. Figure 4 shows the results for the receding horizon pairs (y_{rh}, u_{rh}) in the case that zero terminal cost and $T = 1$ were chosen.
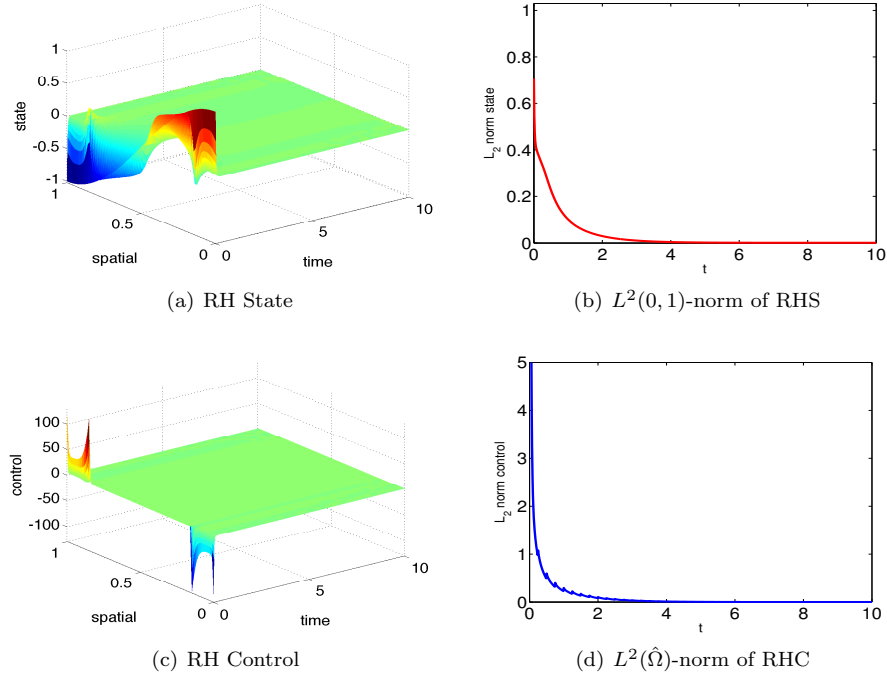

(a) RH State

(b) $L^2(0,1)$-norm of RHS

(c) RH Control

(d) $L^2(\hat{\Omega})$-norm of RHC

FIG. 4. *Receding horizon trajectories for Example* 2

G	Prediction Horizon	J_{T_∞}	$\|y_{rh}\|_{L^2(Q)}$	$\|y_{rh}(T_\infty)\|_{L^2(0,1)}$	iter
	$T = 1$	0.053394	0.2820	3.38×10^{-6}	5890
Quadratic	$T = 0.5$	0.056004	0.2792	9.99×10^{-6}	3957
	$T = 0.25$	0.060580	0.2788	9.74×10^{-6}	3285
	$T = 1$	0.053058	0.2835	7.23×10^{-6}	3382
Zero	$T = 0.5$	0.052961	0.2873	1.46×10^{-5}	1698
	$T = 0.25$	0.053717	0.2977	4.00×10^{-5}	903

TABLE 2
Numerical results for Example 2.

Concerning the effect of different choices of T and G, the same observations as in Example 1 apply.

EXAMPLE 3. In this example, we dealt with the stabilization of a *noisy* Burgers' equation with homogeneous Neumann boundary conditions. We chose $y_0(x) = \exp(-20(x - 0.5)^2)$ as the initial function, $\mu = 0.01$ as a viscosity parameter, and $T_\infty = 10$. Furthermore, the noise was simulated by generating uniformly distributed random numbers within the range $[-4, 4]$. It was added to the right-hand side of the equation (49) at the spatial-temporal grid points. The results correspond to uncontrolled solutions are reported in Table 3.

Problem types	$\|y^u\|_{L^2(Q)}$	$\|y^u(T_\infty)\|_{L^2(0,1)}$
Uncontrolled state without noise	0.6291	0.0829
Uncontrolled state with noise	0.7945	0.1436

TABLE 3
Uncontrolled solutions for Example 3

In Figure 5, we show the results for uncontrolled solution with noise and without noise. The control acts only on the set

$$\hat{\Omega} = (0.1, 0.3) \cup (0.7, 0.9) \subset (0, 1).$$

In implementations of Algorithm 2 on every interval $[t_i, t_i + T]$, first an open-loop optimal control u_T^* was computed for every subproblem without noise. Then the optimal control u_T^* is used to steer the noisy Burgers' equation. This process was repeated for every interval $[t_i, t_i + T]$ with $i = 0, \ldots, r - 1$. Table 4 (rep. Table 5) represents the results of Algorithm 2 applied on the Burgers' equation (49) with noise (resp. without noise) for different choices of the prediction horizon T and the terminal cost G. In Figures 6 and 7, we show the results for the receding horizon pairs (y_{rh}, u_{rh}) in the case that zero terminal cost and $T = 1$ were chosen.

(a) State

(b) $L^2(0,1)$-norm of the state

(c) Noisy state

(d) $L^2(0,1)$-norm of the noisy state
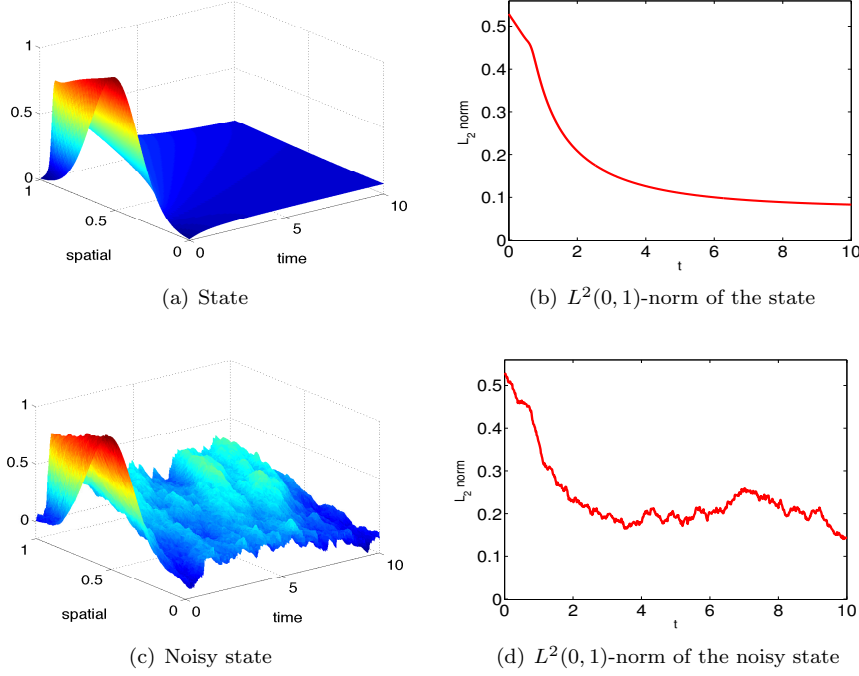
FIG. 5. *Uncontrolled solutions for* Example 3

G	Prediction Horizon	J_{T_∞}	$\|y_{rh}\|_{L^2(Q)}$	$\|y_{rh}(T_\infty)\|_{L^2(0,1)}$	iter
Quadratic	$T = 1$	0.049456	0.2996	0.0426	4327
	$T = 0.5$	0.052292	0.3031	0.0418	3489
	$T = 0.25$	0.060291	0.3063	0.0441	2739
Zero	$T = 1$	0.049545	0.3009	0.0438	2463
	$T = 0.5$	0.050112	0.3046	0.0462	1479
	$T = 0.25$	0.053008	0.3163	0.0512	880

TABLE 4

Numerical results corresponding to the noisy equation for Example 3

G	Prediction Horizon	J_{T_∞}	$\|y_{rh}\|_{L^2(Q)}$	$\|y_{rh}(T_\infty)\|_{L^2(0,1)}$	iter
Quadratic	$T = 1$	0.044212	0.2828	2.56×10^{-6}	4215
	$T = 0.5$	0.046700	0.2855	9.70×10^{-6}	3346
	$T = 0.25$	0.054818	0.2892	5.58×10^{-6}	2636
Zero	$T = 1$	0.044212	0.2839	9.26×10^{-6}	2532
	$T = 0.5$	0.044626	0.2870	2.79×10^{-5}	1450
	$T = 0.25$	0.047149	0.2983	1.51×10^{-4}	835

TABLE 5

Numerical results corresponding to the equation without noise for Example 3

(a) RH State

(b) $L^2(0, 1)$-norm of RHS

(c) RH Control

(d) $L^2(\hat{\Omega})$-norm of RHC
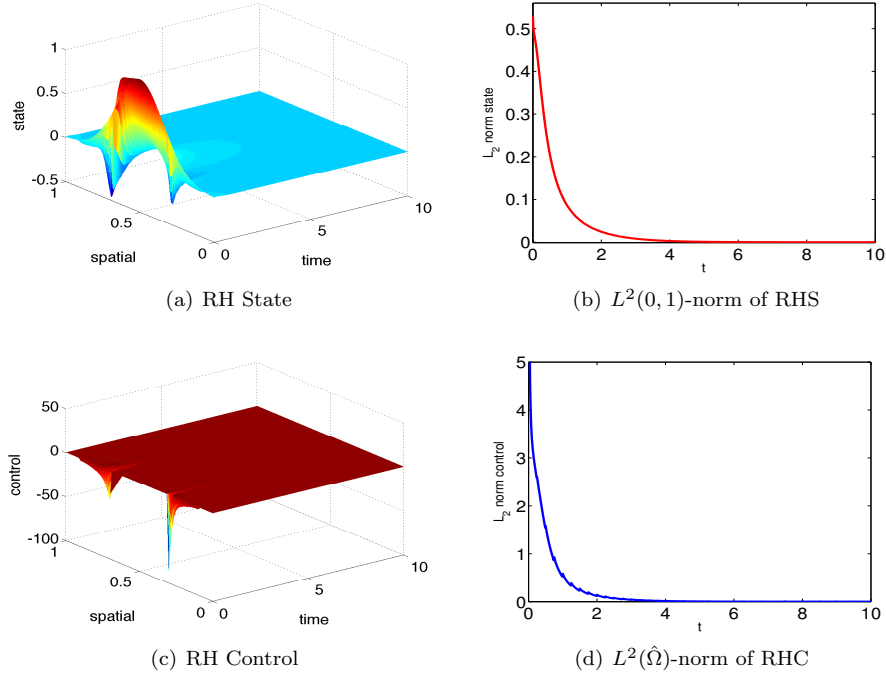
FIG. 6. *Receding horizon trajectories for* Example 3 *without noise*

(a) RH State

(b) $L^2(0, 1)$-norm of RHS

(c) RH Control

(d) $L^2(\hat{\Omega})$-norm of RHC
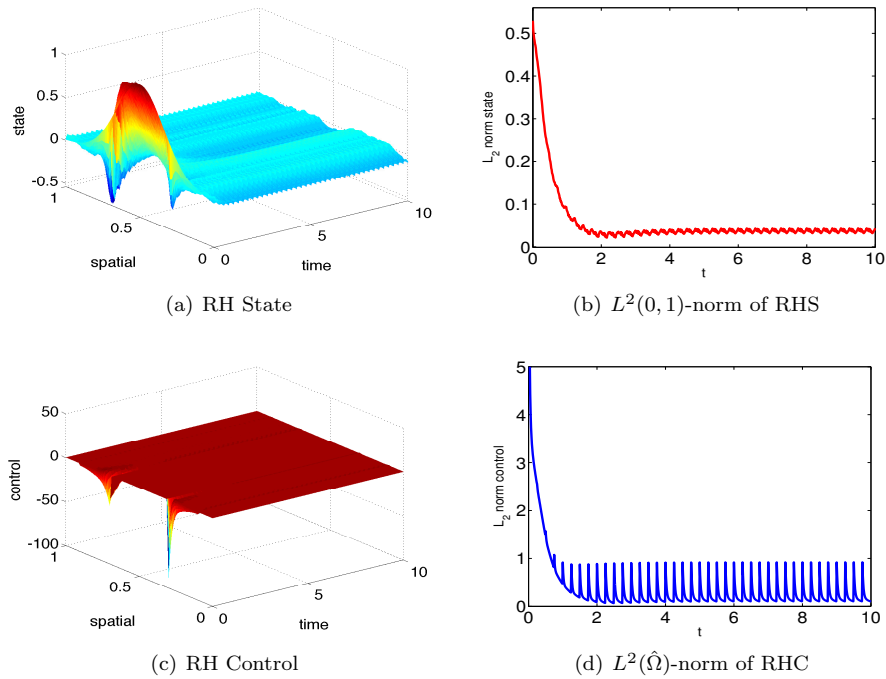
FIG. 7. *Receding horizon trajectories for* Example 3 *with presence of noise*

From Tables 4 and 5 we note that the stabilization quantifiers for the quadratic and zero terminal penalties differ less in the case with noise than without noise. Comparing Figures 6(d) and 7(d) we note the effect on the required control action due to noise in the equation.

Consistently over all numerical results it can be observed that a longer prediction horizon leads to smaller values of J_{T_∞}. Concerning the total number of iterations, it can be seen that for the problems under consideration Algorithm 2 with zero terminal cost requires significantly fewer iterations than in the case with quadratic terminal cost.

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