ON THE TURING MODEL COMPLEXITY OF INTERIOR POINT METHODS FOR SEMIDEFINITE PROGRAMMING

ETIENNE DE KLERK AND FRANK VALLENTIN

ABSTRACT. It is known that one can solve semidefinite programs to within fixed accuracy in polynomial time using the ellipsoid method (under some assumptions). In this paper it is shown that the same holds true when one uses the short-step, primal interior point method. The main idea of the proof is to employ Diophantine approximation at each iteration to bound the intermediate bit-sizes of iterates.

1. INTRODUCTION

Semidefinite programming is used in several polynomial-time algorithms, like the celebrated Goemans-Williamson [3] approximation algorithm for the maximum cut problem, the algorithm for computing the stability number of a perfect graph [4], and many others (see e.g. [2]). To give a rigorous proof of the polynomial-time complexity of such algorithms, one requires a known theorem, due to Grötschel, Lovász, and Schrijver [4], on the Turing model complexity of solving semidefinite programs to fixed precision (under some assumptions). In [4], this theorem is proved constructively by using the ellipsoid method of Yudin and Nemirovski [20] (inspired by the earlier proof of Khachiyan [8] of the polynomial-time solvability of linear programming), but our aim here is to do so by using the theory of interior point methods. Perhaps surprisingly, such a proof has not yet been given to the best of the authors' knowledge.

For example, in Chapter 2 of the recent book [2] it is stated that:

[...] the ellipsoid method is the only known method that provably

yields polynomial runtime [for semidefinite programming] in the Turing machine model [...]

The complexity theorem in question may be stated as follows.

Theorem 1.1 (Grötschel, Lovász, Schrijver [4]). Consider the semidefinite program

(1)
$$\begin{aligned} \text{val} &= \inf \quad \langle C, X \rangle \\ X \in \mathcal{S}^n \text{ is positive semidefinite,} \\ \langle A_j, X \rangle &= b_j \text{ for } j = 1, \dots, m, \end{aligned}$$

with rational input C, A_1, \ldots, A_m , and b_1, \ldots, b_m , and where S^n denotes the set of $n \times n$ symmetric matrices. Denote by

 $\mathcal{F} = \{ X \in \mathcal{S}^n : X \text{ is positive semidefinite, } \langle A_j, X \rangle = b_j \text{ for } j = 1, \dots, m \}$

Date: July 16, 2015.

Key words and phrases. Semidefinite programming, interior point method, Turing model complexity, Ellipsoid method.

The second author was partially supported by VIDI grant 639.032.917 from the Netherlands Organization for Scientific Research (NWO).

the set of feasible solutions. Suppose we know a rational point $X_0 \in \mathcal{F}$ and positive rational numbers r, R so that

$$X_0 + B(X_0, r) \subseteq \mathcal{F} \subseteq X_0 + B(X_0, R),$$

where $B(X_0, r)$ is the ball of radius r, centered at X_0 , in the d-dimensional subspace

$$L = \{ X \in \mathcal{S}^n : \langle A_j, X \rangle = 0 \text{ for } j = 1, \dots, m \}.$$

For every positive rational number $\epsilon > 0$ one can find in polynomial time a rational matrix $X^* \in \mathcal{F}$ such that

$$\langle C, X^* \rangle - \operatorname{val} \le \epsilon,$$

where the polynomial is in n, m, $\log_2 \frac{R}{r}$, $\log_2(1/\epsilon)$, and the bit size of the data X_0 , C, A_1, \ldots, A_m , and b_1, \ldots, b_m .

Here $\langle X, Y \rangle = \text{Trace}(XY)$ denotes the trace inner product for symmetric matrices, and hence, when we talk about the ball $B(X_0, r)$ or $B(X_0, R)$ we work with the associated Frobenius norm

$$||X||_F = \langle X, X \rangle^{1/2}.$$

We will show that the analysis by Renegar [12] of the short step interior point algorithm, together with applying Diophantine approximation at every step to ensure that the bit size stays small, leads to a proof of Theorem 1.1.

There is also a practical aspect to the results in this paper. Semidefinite programming is increasingly used in computer-assisted proofs. Thus new theoretical results have been obtained in this way for binary code sizes [14], crossing numbers of graphs [1], binary sphere packings [15], and other problems. To obtain rigorous proofs, it is necessary to give a formal verification of the relevant semidefinite programming bound. Usually this is done by computing dual bounds using floating point arithmetic, and then showing rigorously that the corresponding dual solutions are feasible. This type of "reverse engineering" can be quite cumbersome; see e.g. the discussion in [15, Section 5.3]. Moreover, the semidefinite programs involved are often numerically ill-conditioned, and it may be difficult or impossible to obtain a near-optimal solution with off-the-shelf solvers; see e.g. [9]. It is therefore of practical interest to understand what may be done in polynomial time when using exact arithmetic. We note that there already exists an arbitrary precision solver, SDPA-GMP (see [19] and the references therein) that uses the GNU multi-precision linear algebra library. The algorithmic ideas presented here may potentially be used to enhance such a solver to improve its performance, by ensuring that it runs in polynomial time, i.e. that the intermediate bit-sizes do not become excessively large.

Finally, one should note that there have been several papers studying the complexity of interior point methods using *finite precision arithmetic* (allowing only a fixed number of bits for calculations); see e.g. [16, 18, 6]. For the Turing model complexity though, the only results known to us concern interior point methods for linear programming; see e.g. the original paper by Karmarkar [7], or the review in the book of Wright [17].

2. Preliminaries

In this section we set up the notation for the paper. Since we follow Renegar's proof we mainly use his notation.

On the Turing model complexity of interior point methods for semidefinite programs

2.1. SDP problem structure and notation.

- We will denote matrices (and matrix variables) by capital letters, and general vectors (or variables) by lower case letters.
- By S^n we denote the $\binom{n+1}{2}$ -dimensional vector space of symmetric matrices which is endowed with the trace inner product $\langle X, Y \rangle = \text{Trace}(XY)$. The corresponding norm is the Frobenius norm

$$||X||_F = \langle X, X \rangle^{1/2} = \sum_{i=1}^n \lambda_i(X)^2,$$

where $\lambda_i(X)$ is the *i*-th largest eigenvalue of the symmetric matrix X. By $\mathcal{S}_{\geq 0}^n$ we denote the closed convex cone of positive semidefinite matrices, and $\mathcal{S}_{\geq 0}^n$ is the open cone of positive definite matrices. If the matrix size is clear from the context, we will sometimes write $X \succ 0$ (resp. $X \succeq 0$) instead of $X \in \mathcal{S}_{\geq 0}^n$ (resp. $X \in \mathcal{S}_{\geq 0}^n$).

• The semidefinite program (1) defines the linear operator $A: \mathcal{S}^n \to \mathbb{R}^m$ componentwise by

$$(AX)_j = \langle A_j, X \rangle$$
, with $j = 1, \dots, m$.

Its adjoint operator $A^* \colon \mathbb{R}^m \to \mathcal{S}^n$ is

$$A^*y = \sum_{j=1}^m y_j A_j,$$

where we take the adjoint with respect to the trace inner product. From now on we assume that A is surjective. Hence, the adjoint A^* is injective, and the matrices A_1, \ldots, A_m are linearly independent.

The kernel of A is the linear subspace

$$L = \ker A = \{ X \in \mathcal{S}^n : AX = 0 \}$$

and the matrices A_1, \ldots, A_m form a basis of the orthogonal complement L^{\perp} . The orthogonal projection onto the subspace L is given by

$$\pi_L = I_{\mathcal{S}^n} - A^* (AA^*)^{-1} A,$$

where $I_{\mathcal{S}^n}$ is the identity operator for \mathcal{S}^n .

• We may (and will) assume that $C \in L$, without loss of generality. Indeed, every feasible $X \in F$ may be written as $X = X_0 + \Delta X$ for some $\Delta X \in L$, so that

Thus we may replace C by $\pi_L(C)$ if necessary. Moreover, the bit-size of $\pi_L(C)$ is bounded by a polynomial in the bit-size of C and A, due to Theorem 2.3 below.

2.2. **Polynomial-time operations.** For ease of reference, we will use the framework in the book of Schrijver [13] when discussing complexity. In particular, we use the same definition for the bit-size of rational numbers, vectors and matrices as in [13, §2.1], and we will denote bit-size by size(·). In particular, for relatively prime $p, q \in \mathbb{Z}$, we define the bit-size of the rational number p/q as:

$$\operatorname{size}(p/q) = 1 + \lceil \log_2 |p| + 1 \rceil + \lceil \log_2 |q| + 1 \rceil.$$

The bit size of a rational vector $(p_1/q_1, \ldots, p_n/q_n)$ is defined as the sum of the bit sizes of its components plus n. Similarly, the bit size of an $m \times n$ matrix is defined as the sum of the bit sizes of its components plus $m \times n$.

Diophantine approximation. We will perform a "rounding" procedure at the end of each iteration to reduce the bit-size of the iterate, and will use *Diophantine approximation* for this.

Theorem 2.1 (cf. Corollary 6.2a in [13]). Let α and $0 < \epsilon \leq 1$ be given rational numbers. Then one may find, in time polynomial in the bit size of α , integers p and q such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{\epsilon}{q} \text{ and } 1 \le q \le \frac{1}{\epsilon}, \ |p| \le \lceil |\alpha| \rceil q.$$

The underlying algorithm is the *continued fraction method*; see page 64 in [13] for a description of the algorithm.

As an immediate corollary, one may approximate a rational vector $\alpha \in \mathbb{Q}^n$ componentwise by a rational vector $(p_1/q_1, \ldots, p_n/q_n)$ such that

(2)
$$\left\| (\alpha_1, \dots, \alpha_n) - \left(\frac{p_1}{q_1}, \dots, \frac{p_n}{q_n} \right) \right\|_2 < \epsilon \sum_{i=1}^n \frac{1}{q_i}, \ \forall i: \ 1 \le q_i \le \frac{1}{\epsilon}, \ |p_i| \le \lceil |\alpha_i| \rceil q_i,$$

in time polynomial in the bit-size of the vector α .

We restate this result in a form that we will need later.

Corollary 2.2. Given a rational vector $\alpha \in \mathbb{Q}^n$ and rational $\epsilon > 0$, one may compute in time polynomial in size(α) integers p_1, \ldots, p_n and q_1, \ldots, q_n such that

(3)
$$\left\| (\alpha_1, \dots, \alpha_n) - \left(\frac{p_1}{q_1}, \dots, \frac{p_n}{q_n} \right) \right\|_2 < \epsilon,$$

such that

size
$$(p_1/q_1, \dots, p_n/q_n) \le n\left(6 + \log_2\left(\frac{n^2 \lceil \|\alpha\|_{\infty} \rceil}{\epsilon^2}\right)\right)$$

Proof. Assume the integers $p_i, q_i \ (i \in \{1, ..., n\})$ satisfy (2). For each *i* one has

$$|p_i| \le \lceil |\alpha_i| \rceil q_i \le \lceil \|\alpha\|_{\infty} \rceil q_i \le \lceil \|\alpha\|_{\infty} \rceil \frac{1}{\epsilon}$$

Thus

size
$$(p_i/q_i) = 1 + \lceil \log_2 |p_i| + 1 \rceil + \lceil \log_2 |q_i| + 1 \rceil$$

 $\leq 1 + \left\lceil \log_2 \frac{\lceil \|\alpha\|_{\infty} \rceil}{\epsilon} + 1 \right\rceil + \left\lceil \log_2 \frac{1}{\epsilon} + 1 \right\rceil$
 $\leq 5 + \log_2 \frac{\lceil \|\alpha\|_{\infty} \rceil}{\epsilon^2}.$

As a consequence

size
$$(p_1/q_1,\ldots,p_n/q_n) = n + \sum_{i=1}^n \operatorname{size}(p_i/q_i) \le n \left(6 + \log_2\left(\frac{\lceil \|\alpha\|_{\infty}\rceil}{\epsilon^2}\right)\right).$$

Using (2), $\sum_{i=1}^{n} \frac{1}{q_i} \leq n$, and replacing ϵ by ϵ/n completes the proof.

Linear algebra. Each iteration of the short-step interior point algorithm involves some linear algebra operations, and we will use the following results to ensure that this may be done in polynomial time.

Theorem 2.3. The following operations on matrices may be performed in polynomial time (in the bit sizes of the matrices and vectors):

- (1) Matrix addition and multiplication;
- (2) Matrix inversion;
- (3) Solving linear systems with Gaussian elimination;
- (4) Computing an orthogonal basis (using Gaussian elimination and Gram-Schmidt orthogonalization) of a nullspace $\{x : Ax = 0\}$ where the rational matrix A is given.

For a proof, see e.g. Theorem 3.3 and Corollary 3.3a in [13].

The last item implies that we may compute an orthogonal basis for L (the nullspace of A), so that we may represent any feasible point $X \in \mathcal{F}$ as $X = X_0 + \sum_{i=1}^d x_i B_i$, say, where the x_i are scalars and the B_i 's are suitable symmetric matrices of size polynomial in the input size that form an orthogonal basis for L. We may also assume without loss of generality that $||B_i||_F \leq 1$ for each i. This is important, since we will study perturbations (roundings) of the form $\overline{X} = X + \Delta X$, where $X \in \mathcal{F}$ and $\Delta X \in L$. Writing $\Delta X = \sum_{i=1}^d \Delta x_i B_i$, one then has $||\Delta X||_F \leq ||\Delta x||_2$. In other words, we may bound the size of the perturbation in \mathcal{S}^n in terms of the corresponding perturbation in \mathbb{R}^d .

2.3. Self-concordant barrier functions. We will use the definition of self-concordant functions due to Renegar [12], that is more suited to our purposes than the original definition of Nesterov and Nemirovski [11]. In what follows, f is a convex functional with open convex domain D_f (contained in a finite-dimensional, real affine space), and the gradient and Hessian of f at $x \in D_f$ will be denoted by g(x)and H(x) respectively. Note that the gradient and Hessian depend on the inner product we choose for the underlying vector space; see §1.2 and §1.3 in Renegar [12] for more details.

Definition 2.4 (cf. §2.2.1 in [12]). Assume $f : D_f \to \mathbb{R}$ (with D_f open and convex) is such that $H(x) \succ 0$ for all $x \in D_f$. Then f is called self-concordant if:

- (1) For all $x \in D_f$ one has $B_x(x, 1) \subseteq D_f$;
- (2) For all $y \in B_x(x, 1)$ one has

$$1 - \|y - x\|_x \le \frac{\|v\|_y}{\|v\|_x} \le \frac{1}{1 - \|y - x\|_x} \text{ for all } v \ne 0,$$

where $||v||_x := \langle v, H(x)v \rangle^{\frac{1}{2}}$ is called the intrinsic (or local) norm of v, and $B_x(x, 1)$ is the unit ball, centered at x, with respect to the intrinsic norm.

A self-concordant functional f is called a $\mathit{self-concordant}\ barrier$ if there is a finite value ϑ_f so that

$$\vartheta_f = \sup_{x \in D_f} \|H^{-1}g(x)\|_x,$$

that is, the intrinsic norm (at x) of the Newton step $n(x) := -H(x)^{-1}g(x)$ is always upper bounded by ϑ_f . The *analytic center* of D_f is defined as the (unique) minimizer of f. (The analytic center exists if and only if D_f is bounded.)

The self-concordant barrier function of the semidefinite program (1) is

(4)
$$f(X) = -\ln \det X \text{ with domain } D_f = \mathcal{S}_{\succ 0}^n \cap \{X \in \mathcal{S}^n : AX = b\}.$$

For this barrier function one has $\vartheta_f \leq n$; see [12, §2.3.1]. Its gradient (with respect to the trace inner product) is

$$g(X) = -\pi_L(X^{-1}),$$

and its Hessian is

$$H(X)Y = \pi_L(X^{-1}YX^{-1}) \quad \text{with} \quad Y \in L.$$

The local norm for $Y \in L$ at $X \in D_f$ is defined as

$$||Y||_X = \langle Y, H(X)Y \rangle^{1/2}.$$

For easy reference, we note that the self-concordance of the function f in (4) implies that for all $X \in D_f$ we have $B_X(X,1) \subseteq D_f$ and that for all $Y \in B_X(X,1)$ we have

(5)
$$1 - \|Y - X\|_X \le \frac{\|V\|_Y}{\|V\|_X} \le \frac{1}{1 - \|Y - X\|_X} \text{ for all } V \in L \setminus \{0\},$$

where $B_X(Y, r)$ denotes the open ball of radius r centered at Y in the local norm $\|\cdot\|_X$.

2.3.1. *Properties of self-concordant functions.* We will need the following three technical results (and one corollary) on self-concordant functions.

Theorem 2.5 (Theorem 2.2.3 in [12]). Assume f self-concordant and $x \in D_f$. If z minimizes f and $z \in B_x(x, 1)$ then

$$x^{+} := x - H(x)^{-1}g(x)$$

satisfies

$$||x^+ - z||_x \le \frac{||x - z||_x^2}{1 - ||x - z||_x}.$$

A useful, and immediate, corollary is the following.

Corollary 2.6. Under the assumptions of Theorem 2.5, one has

$$||n(x)||_{x} := ||H(x)^{-1}g(x)||_{x} \le \frac{||x-z||_{x}}{1-||x-z||_{x}}.$$

Proof. By definition,

$$\begin{aligned} \|n(x)\|_{x} &= \|x^{+} - x\|_{x} \\ &\leq \|x^{+} - z\|_{x} + \|z - x\|_{x} \\ &\leq \frac{\|x - z\|_{x}^{2}}{1 - \|x - z\|_{x}} + \|x - z\|_{x} \quad \text{(by Theorem 2.5)} \\ &= \frac{\|x - z\|_{x}}{1 - \|x - z\|_{x}}, \end{aligned}$$

as required.

The other two technical results are the following.

Theorem 2.7 (Theorem 2.2.4 in [12]). Assume f self-concordant and $x \in D_f$ such that $||n(x)||_x \leq 1$. Then

$$||n(x^+)||_{x^+} \le \left(\frac{||n(x)||_x}{1-||n(x)||_x}\right)^2.$$

Theorem 2.8 (Theorem 2.2.5 in [12]). Assume f self-concordant and $x \in D_f$ such that $||n(x)||_x \leq 1/4$. Then f has a minimizer z and

$$||z - x^+||_x \le \frac{3||n(x)||_x^2}{(1 - ||n(x)||_x)^3}$$

Thus (triangle inequality):

$$||x - z||_x \le ||n(x)||_x + \frac{3||n(x)||_x^2}{(1 - ||n(x)||_x)^3}$$

3. The short-step, logarithmic barrier algorithm

We consider a generalisation of our SDP problem, given by

$$\operatorname{val} := \min_{x \in \operatorname{cl}(D_f)} \langle c, x \rangle,$$

where c is a given vector, f is a self-concordant barrier with open domain D_f , and $cl(D_f)$ denotes the closure of D_f . As before, the gradient and Hessian of f at $x \in D_f$ are respectively denoted by g(x) and H(x).

For the SDP problem (1), $f(X) = -\ln \det(X)$ with domain $D_f = \{X \succ 0 : X \in \mathcal{F}\}$, but Algorithm 1 below is valid for a general self-concordant barrier.

Define, for given $\eta > 0$,

$$f_{\eta}(x) := \eta \langle c, x \rangle + f(x),$$

and denote by $n_{\eta}(x) = -H(x)^{-1}(\eta c + g(x))$ the (projected) Newton direction at x for f_{η} .

The analytic curve, parameterized by $\eta > 0$, where η is mapped to the unique minimizer of f_{η} , is called the *central path*.

The complexity of the short step algorithm is described in the following theorem, that is originally due to Nesterov and Nemirovski [11].

Theorem 3.1 (cf. p. 47 in [12]). The short step algorithm terminates after at most

$$k = \left\lceil 10\sqrt{\vartheta_f} \ln\left(\frac{7\vartheta_f}{6\eta_1\epsilon}\right) \right\rceil$$

7

Algorithm 1 Short step algorithm

Require: an $x_1 \in D_f$ and $\eta_1 > 0$ such that $||n_{\eta_1}(x_1)||_{x_1} \leq \frac{1}{4}$. An accuracy parameter $\epsilon > 0$. $k \leftarrow 1$ **while** $\frac{\vartheta_f}{\eta_k} > \epsilon$ **do** Set $x_{k+1} = x_k + n_{\eta_k}(x_k)$ Set $\eta_{k+1} = \left(1 + \frac{1}{8\sqrt{\vartheta_f}}\right)\eta_k$ $k \leftarrow k+1$. **end while**

iterations. The output is a feasible point x_k such that

 $\langle c, x_k \rangle - \operatorname{val} \leq \epsilon.$

Some remarks on the steps in the algorithm.

• For the SDP problem (1), the projected Newton direction is obtained by first solving the following linear system:

(6)

$$My = v$$

where

$$M_{ij} = \operatorname{Trace}(XA_iXA_j), \ (i, j \in \{1, \dots, m\})$$

and

$$v_i = -b_i + \eta \operatorname{Trace}(A_i X C X), \ (i \in \{1, \dots, m\}).$$

(We drop the subscript k that refers to the iteration number here for convenience.) Subsequently, the projected Newton direction is given by

(7)
$$n_{\eta}(X) = X(A^*y)X + X - \eta XCX$$

The matrix M is positive definite (and hence nonsingular) under the assumption that $\{A_1, \ldots, A_m\}$ are linearly independent. One may bound the sizes of M and v in (6) as follows:

$$size(M_{ij}) \leq size(XA_i) + size(A_jX) \\ \leq n(size(X) + size(A_i)) + n(size(X) + size(A_j)),$$

so that

$$\operatorname{size}(M) \le m^2(1+2n\operatorname{size}(X)) + 2mn\sum_{i=1}^m \operatorname{size}(A_i)$$

Similarly,

ŝ

$$\operatorname{size}(v) \le m + 2mn\operatorname{size}(X) + mn\operatorname{size}(C) + 2n\sum_{i=1}^{m}\operatorname{size}(A_i) + \operatorname{size}(b) + m\operatorname{size}(\eta).$$

As a consequence, the projected Newton direction may be computed in time polynomial in the bit sizes of X, η and the data A, b and C. Thus one may perform a constant number of iterations in polynomial time. We will show how to truncate the current iterate X at the end of each iteration, using Diophantine approximation, in order to guarantee that the bit-size of the iterates remains suitably bounded throughout. On the Turing model complexity of interior point methods for semidefinite programs

- The square root $\sqrt{\vartheta_f}$ that appears in the statement of the algorithm may be replaced by any larger number, e.g. $\lceil \sqrt{\vartheta_f} \rceil$. The only change to the complexity is that $\sqrt{\vartheta_f}$ should then be replaced by the corresponding larger value in the statement of Theorem 3.1.
- By construction, each iterate x_k satisfies $||n_{\eta_k}(x_k)||_{x_k} \leq \frac{1}{4}$, and after the Newton step one therefore has

$$\|n_{\eta_k}(x_{k+1})\|_{x_{k+1}} \le \frac{1}{2}$$

by Theorem 2.7. As a result, after setting $\eta_{k+1} = \left(1 + \frac{1}{8\sqrt{\vartheta_f}}\right)\eta_k$, one again has $\|n_{\eta_{k+1}}(x_{k+1})\|_{x_{k+1}} \leq \frac{1}{4}$; see [12, p. 46] for details. Since we will apply rounding (using Diophantine approximation) to the iterates later on, we will need to ensure that (8) still holds after rounding x_{k+1} .

• An issue that needs to be resolved is the initialization question, i.e. finding $x_1 \in D_f$ and $\eta_1 > 0$ (of suitable bit size) such that $||n_{\eta_1}(x_1)||_{x_1} \leq \frac{1}{4}$. This is addressed in the next section.

4. INITIALIZATION

Assume now — again in the setting of a general self-concordant barrier f — that we only know a rational starting point $x' \in D_f$. We will use a two phase procedure, where we first solve an auxiliary problem to obtain a suitable starting point for the short step algorithm. The procedure here follows Renegar [12, §2.4].

Auxiliary problem. For a given parameter $\nu > 0$, we consider the auxiliary problem where we minimize:

$$f'_{\nu}(x) := -\nu \langle g(x'), x \rangle + f(x).$$

Note that x' is on the central path of the auxiliary problem and corresponds to $\nu = 1$.

Now use the short step algorithm, reducing ν at each iteration via

$$\nu_{k+1} = \left(1 - \frac{1}{8\sqrt{\vartheta_f}}\right)\nu_k.$$

Remarks:

(8)

- The central path of the auxiliary problem passes through x' and converges to the analytic center of D_f as $\nu \downarrow 0$.
- Once ν is small enough, we may use the current value of x as a starting point for the original short step algorithm.
- After

$$k \ge 10\sqrt{\vartheta_f} \ln\left(\frac{7}{6\epsilon'}\right),$$

iterations, we have $\nu_k \leq \epsilon'$, by Theorem 3.1.

- In the SDP case of problem (1), one has $x' = X_0$ and $g(x') = -\pi_L(X_0^{-1})$, that has bit-size polynomial in the input size, by Theorem 2.3.
- A suitable choice for ϵ' that provides a starting point for the second phase depends on the (Minkowski) symmetry of D_f around x'.

Definition 4.1 (Symmetry of D around x). Let D be a bounded open convex set and $x \in D$. Let $\mathcal{L}(x, D)$ denote the set of lines that pass through x. For any $\ell \in \mathcal{L}(x, D)$, let $r(\ell)$ denote the ratio of the shorter to the longer line segments $\ell \cap (D \setminus \{x\})$. Finally define the symmetry of D around x as

$$\operatorname{sym}(x, D) := \inf_{\ell \in \mathcal{L}(x, D)} r(\ell).$$

A suitable value for ϵ' is now given by

(9)
$$\epsilon' = \frac{1}{18\vartheta_f (1 + 1/\operatorname{sym}(x', D_f))}.$$

At this point one may start the short step algorithm using x_1 equal to the last iterate produced by solving the auxiliary problem, and

(10)
$$\eta_1 = \frac{1}{12 \|H(x_1)^{-1}c\|_{x_1}} \ge \frac{1}{12} \left(\sup_{x \in D_f} \langle c, x \rangle - \text{val} \right).$$

See $\S2.4$ in [12] for more details and proofs.

The combined complexity of this two-phase procedure is given by the following theorem. The proof is easily extracted from the proof of Theorem 2.4.1 in [12].

Theorem 4.2 (cf. Theorem 2.4.1 in [12]). Assume $f \in SCB$ and D_f bounded. Assume a starting point $x' \in D_f$. If $0 < \epsilon < 1$, then within

$$10\sqrt{\vartheta_f} \ln\left(\frac{294\vartheta_f^2}{\epsilon} \left(\frac{1}{1 + \operatorname{sym}(x', D_f)}\right)\right)$$

iterations, all points x computed thereafter satisfy

$$\langle c, x \rangle - \operatorname{val} \le \epsilon \left(\sup_{x \in D_f} \langle c, x \rangle - \operatorname{val} \right).$$

For the SDP problem (1) we now assume, as in Theorem 1.1, that we have a rational $X_0 \in \mathcal{F}$, and that we know rational r > 0 and R > 0 so that $X_0 + B(X_0, r) \subset \mathcal{F} \subset X_0 + B(X_0, R)$. Note that this implies:

(11)
$$\operatorname{sym}(X_0, \mathcal{F}) \ge \frac{r}{R}$$

5. An upper bound on the norm of the dual central path

In this section we give an upper bound on the norm of the dual central path. Our analysis is based on a standard argument for the existence and uniqueness of the central path; see e.g. [10, Proof of Theorem 10.2.1].

Recall that the *(primal-dual) central path* is the curve $\eta \mapsto (X(\eta), S(\eta), y(\eta))$, with $\eta > 0$, defined as the unique solution of

$$AX = b, \ A^*y + S = C, \ XS = \frac{1}{\eta}I, \ X \succ 0, \ S \succ 0,$$

where I denotes the identity matrix.

Lemma 5.1. Under the assumptions stated in Theorem 1.1 we have

(12)
$$||S(\eta)||_F \le \frac{\sqrt{n}}{(1-1/e)r} \left(\langle X_0, C+2||C||_{\infty}I \rangle + \frac{n}{r\eta^2} \right),$$

11

where

$$||C||_{\infty} = \max_{i=1,\dots,n} \sum_{j=1}^{n} |C_{ij}|$$

is the maximum row sum norm of C.

Proof. By assumption X_0 is a strictly feasible solution of the primal and without loss of generality we may assume that $S_0 = C + 2 \|C\|_{\infty} I$ is a strictly feasible solution of the dual; otherwise we add the constraint

$$\langle I, X \rangle \le \langle I, X_0 \rangle + \sqrt{nR}$$

to the semidefinite program (1) which is redundant since

$$\langle I, X - X_0 \rangle \le (\langle I, I \rangle \langle X - X_0, X - X_0 \rangle)^{1/2} \le \sqrt{nR}.$$

Note that S_0 is indeed positive definite, since it is strictly diagonally dominant.

We may characterize $S(\eta)$ as the unique minimizer of the function

$$S \mapsto \langle X_0, S \rangle - \frac{1}{\eta} \ln \det S$$

over the set $\{S: S = C - A^*y, S \succ 0, y \in \mathbb{R}^m\}$.

As in [10, Proof of Theorem 10.2.1], we define the set

$$\mathcal{U} = \left\{ S : S = C - A^* y, \ S \succ 0, \ y \in \mathbb{R}^m, \\ \langle X_0, S \rangle - \frac{1}{\eta} \ln \det S \le \langle X_0, S_0 \rangle - \frac{1}{\eta} \ln \det S_0 \right\}.$$

Clearly, \mathcal{U} contains $S(\eta)$.

If $\sigma > 0$ denotes the smallest eigenvalue of X_0 , then, for all $S \in \mathcal{U}$:

$$\sigma \langle I, S \rangle - \frac{1}{\eta} \ln \det S \leq \langle X_0, S_0 \rangle - \frac{1}{\eta} \ln \det S_0,$$

because $\sigma \langle I, S \rangle \leq \langle X_0, S \rangle$. Now we write the previous inequality in terms of the eigenvalues $\lambda_i(S)$ of S:

$$\sum_{i=1}^{n} \left(\sigma \lambda_i(S) - \frac{1}{\eta} \ln \lambda_i(S) \right) \le \langle X_0, S_0 \rangle - \frac{1}{\eta} \ln \det S_0.$$

Defining the function

$$\phi(\lambda) = \sigma \lambda - \frac{1}{\eta} \ln \lambda, \quad \text{for } \lambda > 0,$$

which is convex and has minimizer $\lambda^* = \frac{1}{\sigma\eta}$ with minimum value $\phi(\lambda^*) = \frac{1}{\eta} \left(1 - \ln \frac{1}{\sigma\eta}\right)$, one has

$$\phi(\lambda_i(S)) \le \langle X_0, S_0 \rangle - \frac{1}{\eta} \ln \det S_0 - (n-1)\phi(\lambda^*) \quad \text{for } i = 1, \dots, n$$

By the convexity of ϕ and by approximating ϕ about the point $e\lambda^*$ we have

$$\phi(\lambda) \ge \phi(e\lambda^*) + \phi'(e\lambda^*)(\lambda - e\lambda^*) = (1 - 1/e)\sigma\lambda - \frac{1}{\eta}\ln\frac{1}{\sigma\eta}.$$

Hence,

$$\begin{aligned} \lambda_i(S) &\leq \frac{1}{(1-1/e)\sigma} \left(\langle X_0, S_0 \rangle - \frac{1}{\eta} \ln \det S_0 - (n-1)\phi(\lambda^*) + \frac{1}{\eta} \ln \frac{1}{\sigma\eta} \right) \\ &\leq \frac{1}{(1-1/e)\sigma} \left(\langle X_0, S_0 \rangle - \frac{2n-1}{\eta} + \frac{n}{\sigma\eta^2} \right) \\ &\leq \frac{1}{(1-1/e)r} \left(\langle X_0, S_0 \rangle + \frac{n}{r\eta^2} \right) \quad \text{for } i = 1, \dots, n, \end{aligned}$$

where the first inequality follows from det $S_0 \ge 1$ and $\ln x \le x - 1$, and where the second inequality follows because $\sigma \ge r$.

The last estimate now immediately implies the statement of the lemma:

$$\|S(\eta)\|_F \le \frac{\sqrt{n}}{(1-1/e)r} \left(\langle X_0, S_0 \rangle + \frac{n}{r\eta^2} \right).$$

Note that the bound on $||S(\eta)||_F$ depends on the value of η . It is therefore necessary to consider the range of values that η can take (and ν during the first phase of the auxiliary problem). During the first phase (auxiliary problem), initially $\nu_1 = 1$, which is subsequently decreased via $\nu_{k+1} = \left(1 - \frac{1}{8\sqrt{\vartheta_f}}\right)\nu_k$. It is simple to show that during each iteration k of the first phase,

$$1 \ge \nu_k \ge \epsilon',$$

where ϵ' is defined in (9), which in turn implies

(13)
$$1 \ge \nu_k \ge \frac{1}{18n(1+R/r)}$$

where we have used (9) and (11).

Similarly, during each iteration k of the second phase

$$\frac{1}{12} \left(\sup_{X \in D_f} \langle C, X \rangle - \operatorname{val} \right) \le \eta_k \le \frac{\vartheta_f}{\epsilon},$$

which implies

(14)
$$\frac{1}{6}r\|C\|_F \le \eta_k \le \frac{n}{\epsilon},$$

since $B(X_0, r) \subseteq \mathcal{F}$ and $\vartheta_f \leq n$.

6. Rounding the current iterate

We will round the current iterate $X \in \mathcal{F}$ (we again drop the subscript for convenience) at the end of each iteration to obtain a feasible $\overline{X} = X + \Delta X$, say, with suitably bounded bit-size, and where the "rounding error" $\Delta X \in L$ satisfies $\|\Delta X\|_X \leq \tilde{\epsilon}$ for some suitable value $\tilde{\epsilon} > 0$.

After the Newton step, but before the update of η , we assume that

$$\|X - X(\eta)\|_X \le c'$$

where c' > 0 is a known constant.

13

By the definition of self-concordance:

$$\begin{split} \|\bar{X} - X(\eta)\|_{\bar{X}} &\leq \frac{1}{1 - \|\Delta X\|_X} \|X + \Delta X - X(\eta)\|_X \\ &\leq \frac{1}{1 - \tilde{\epsilon}} \|X + \Delta X - X(\eta)\|_X \\ &\leq \frac{1}{1 - \tilde{\epsilon}} \|X - X(\eta)\|_X + \frac{1}{1 - \tilde{\epsilon}} \|\Delta X\|_X \\ &\leq \frac{c' + \tilde{\epsilon}}{1 - \tilde{\epsilon}}. \end{split}$$

Thus, if $\tilde{\epsilon} = \frac{1}{16}$, and $c' = \frac{1}{32}$ then $\|\bar{X} - X(\eta)\|_{\bar{X}} \leq \frac{1}{10}$. Consequently, by Corollary 2.6, one has $\|n_{\eta}(\bar{X})\|_{\bar{X}} \leq \frac{1}{9}$, as required (recall (8)). We may ensure that $\|X - X(\eta)\|_{X} \leq \frac{1}{32}$ during the course of the algorithm by taking an extra centering step. Indeed, if we still denote the iterate by X after an

extra centering step, one has $||n_{\eta}(X)||_X \leq 1/64$ (by Theorem 2.7). Consequently, by Theorem 2.8, one has

$$||X - X(\eta)||_X \le ||n_\eta(X)||_X + \frac{3||n_\eta(X)||_X^2}{(1 - ||n_\eta(X)||_X)^3} < \frac{1}{32}.$$

Note that $\bar{X} \succ 0$ since $||X - \bar{X}||_X \leq \frac{1}{16} < 1$, and the definition of self-concordance guarantees that the unit ball in the X-norm centered at X is contained in the positive definite cone.

The task is therefore to find $\bar{X} = X + \Delta X$ with bounded bit-size and so that $\|\Delta X\|_X \le \frac{1}{16}.$

It will be more convenient to bound the $X(\eta)$ -norm of ΔX than the X-norm. As a first observation, using the definition of self-concordance,

$$\begin{split} \|X - X(\eta)\|_{X(\eta)} &\leq \frac{\|X - X(\eta)\|_X}{1 - \|X(\eta) - X\|_X} \\ &\leq \frac{\|X - X(\eta)\|_X}{1 - \frac{1}{32}} \\ &= \frac{32}{31} \|X - X(\eta)\|_X. \end{split}$$

Invoking the definition of self-concordancy once more, we obtain:

$$\begin{aligned} |\Delta X||_{X} &\leq \frac{\|\Delta X\|_{X(\eta)}}{1 - \|X(\eta) - X\|_{X(\eta)}} \\ &\leq \frac{\|\Delta X\|_{X(\eta)}}{1 - \frac{32}{31} \|X(\eta) - X\|_{X}} \\ &\leq \frac{\|\Delta X\|_{X(\eta)}}{1 - \frac{32}{31} \cdot \frac{1}{32}} \\ &= \frac{31}{30} \|\Delta X\|_{X(\eta)}. \end{aligned}$$

Thus if we show that $\|\Delta X\|_{X(\eta)} \leq \frac{30}{31 \times 16}$ then we guarantee that $\|\Delta X\|_X \leq \frac{1}{16}$.

Note that

$$\begin{split} \|\Delta X\|_{X(\eta)}^2 &\leq \langle \Delta X, X(\eta)^{-1} \Delta X X(\eta)^{-1} \rangle \\ &= \eta^2 \langle \Delta X, S(\eta) \Delta X S(\eta) \rangle \\ &\leq \eta^2 \|\Delta X\|_F^2 \|S(\eta)\|_F^2, \end{split}$$

where the inner product is the Euclidean (trace) inner product, and we have used the sub-multiplicativity of the Frobenius norm.

Recall that $||S(\eta)||_F$ is bounded by (12) (Lemma 5.1).

We may now use Diophantine approximation so that (15)

$$\|\Delta X\|_F \le \frac{30}{31 \times 16} \left(\nu \frac{\sqrt{n}}{(1-1/e)r} \left(\langle X_0, -\pi_L(X_0^{-1}) + 2\|\pi_L(X_0^{-1})\|_{\infty}I \rangle + \frac{n}{r\nu^2} \right) \right)^{-1},$$

during the first phase of the algorithm, and

(16)
$$\|\Delta X\|_F \le \frac{30}{31 \times 16} \left(\eta \frac{\sqrt{n}}{(1-1/e)r} \left(\langle X', C+2 \|C\|_{\infty} I \rangle + \frac{n}{r\eta^2} \right) \right)^{-1},$$

during the second phase, where X' is the last iterate produced by the first phase.

Due to the upper and lower bounds on ν in (13), (15) will hold if $\|\Delta X\|_F \leq \epsilon_1$, where

$$\frac{1}{\epsilon_1} := \frac{17\sqrt{n}}{(1-1/e)r} \left(\langle X_0, -\pi_L(X_0^{-1}) + 2 \| \pi_L(X_0^{-1}) \|_{\infty} I \rangle + \frac{n(18n(1+R/r))^2}{r} \right),$$

during the first phase, and (16) will hold if, during the second phase,

$$\|\Delta X\|_F \le \left(\frac{17(\sqrt{n})^3}{(1-1/e)r\epsilon} \left(\langle X', C+2\|C\|_{\infty}I\rangle + \frac{36n}{r^3\|C\|_F^2}\right)\right)^{-1}.$$

To obtain a right-hand-side expression in terms of the input data only, we may use $||X' - X_0||_F \leq R$. Thus we find that the last inequality will hold if $||\Delta X||_F \leq \epsilon_2$, where

$$\frac{1}{\epsilon_2} := \frac{17(\sqrt{n})^3}{(1-1/e)r\epsilon} \left((R + \|X_0\|_F) \|C + 2\|C\|_{\infty} I\|_F + \frac{36n}{r^3 \|C\|_F^2} \right).$$

Setting $\bar{\epsilon} = \min\{\epsilon_1, \epsilon_2\}$, implies that $\log_2\left(\frac{1}{\bar{\epsilon}}\right)$ is bounded by a polynomial in the input size.

Performing Diophantine approximation in the *d*-dimensional space L yields a rational \bar{X} so that $\|\Delta X\|_F \leq \bar{\epsilon}$ and

(17)
$$\operatorname{size}(\bar{X}) \le d\left(6 + \log_2\left(\frac{d^2\lceil R \rceil}{\bar{\epsilon}^2}\right)\right),$$

by Corollary 2.2.

Thus the size of \bar{X} is always bounded by a certain polynomial in the input size.

7. Summary and conclusion

To summarize, we list the complete procedure in Algorithm 3. The main subroutine (used twice) is a short step algorithm with extra centering step and Diophantine approximation, shown as Algorithm 2.

In particular, we have shown the following.

Algorithm 2 Short step algorithm with extra centering and Diophantine approximation

Require:

• Problem data (A, b, c);

- an $x_1 \in D_f$ and $\eta_1 > 0$ such that $||n_{\eta_1}(x_1)||_{x_1} \leq \frac{1}{4}$;
- an accuracy parameter $\varepsilon > 0$;
- an update parameter $\theta > 0$;

 $k \leftarrow 1$ while $\frac{(1-\theta)}{\eta_k} > (1-\theta)\varepsilon$ do Set $x^+ = x_k + n_{\eta_k}(x_k)$ Set $x_{k+1} = x^+ + n_{\eta_k}(x^+)$ Round x_{k+1} using Diophantine approximation, so that size (x_{k+1}) is bounded as in (17), and $\|n_{\eta_k}(x_{k+1})\|_{x_{k+1}} \le \frac{1}{9}$ Set $\eta_{k+1} = \theta \cdot \eta_k$ $k \leftarrow k+1$ end while

Algorithm 3 Two-phase short step algorithm with Diophantine approximation

Require:

- SDP problem data (A, b, c) and $X_0 \in \mathcal{F}$;
- an accuracy parameter $\epsilon > 0$;
- rational R > r > 0 as in Theorem 1.1.

First phase (auxiliary problem):

Set $c = -\pi_L(X_0^{-1}), \ \eta_1 = 1, \ x_1 = X_0, \ \varepsilon = \frac{1}{18\vartheta_f(1+R/r)}, \ \theta = 1 + \frac{1}{8\sqrt{\vartheta_f}}$

Call Algorithm 2 with input $(A,b,c,x_1,\eta_1,\varepsilon,\theta)$

Second phase:

Set c = C, η_1 as in (10), x_1 equal to the last iterate of the first phase, $\varepsilon = \epsilon/\vartheta_f$, $\theta = 1 - \frac{1}{8\sqrt{\vartheta_f}}$

Call Algorithm 2 with input $(A, b, c, x_1, \eta_1, \varepsilon, \theta)$

Theorem 7.1. Under the assumptions of Theorem 1.1, Algorithm 3 computes in polynomial time a rational matrix $X^* \in \mathcal{F}$ such that

$$\langle C, X^* \rangle - \operatorname{val} \le \epsilon \left(\max_{X \in \mathcal{F}} \langle C, X \rangle - \operatorname{val} \right),$$

where the polynomial is in n, m, $\log_2 r$, $\log_2 R$, $\log_2(1/\epsilon)$, and the bit size of the data X_0 , C, A_1, \ldots, A_m , and b_1, \ldots, b_m .

The analysis presented here may also be performed for more practical variants of the interior point method, such as the long-step (large update) method; see e.g. Chapter 2 in [12]. Moreover, since all computations in Algorithm 3 involve linear algebra only (Diophantine approximation may also be implemented as such), there are definite practical perspectives for implementing Algorithm 3 (or a more practical variant), using arbitrary precision packages, like the GNU Multiple Precision Arithmetic Library (GMP) (https://gmplib.org/), that is already used in the solver SDPA-GMP [19].

E. de Klerk, F. Vallentin

References

- E. de Klerk, D.V. Pasechnik, A. Schrijver. Reduction of symmetric semidefinite programs using the regular *-representation. Math. Program., Ser. B, 109 (2007), 613–624.
- [2] B. Gärtner and J. Matoušek, Approximation Algorithms and Semidefinite Programming, Springer, 2012.
- M.X. Goemans and D.P. Williamson. Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming, Journal of the ACM 42 (1995), 1115–1145.
- [4] M. Grötschel, L. Lovász, and A. Schrijver, The ellipsoid method and its consequences in combinatorial optimization, Combinatorica 1 (1981), 169–197.
- [5] M. Grötschel, L. Lovász, and A. Schrijver, Geometric Algorithms and Combinatorial Optimization, Springer, 1988.
- [6] M. Gu, Primal-dual interior-point methods for semidefinite programming in finite precision, SIAM J. Optim. 10 (2000) 462–502.
- [7] N.K. Karmarkar, A new polynomial-time algorithm for linear programming, Combinatorica 4 (1984), 373–395.
- [8] L. Khachiyan. A polynomial time algorithm in linear programming, Soviet Mathematics Doklady 20 (1979), 191–194.
- [9] H.D. Mittelmann and F. Vallentin, High accuracy semidefinite programming bounds for kissing numbers, Experiment. Math. 19 (2010), 174–178.
- [10] R.D.C. Monteiro and M.J. Todd, *Path-following methods*, pages 268–306 in Handbook of Semidefinite Programming: Theory, Algorithms, and Applications (H. Wolkowicz, R. Saigal, L. Vandenberghe (eds.)), Kluwer, 2000.
- [11] Yu. Nesterov and A.S. Nemirovski, Interior point polynomial algorithms in convex programming. SIAM, 1994.
- [12] J. Renegar, A Mathematical View of Interior-Point Methods in Convex Optimization, SIAM, 2001.
- [13] A. Schrijver, Theory of Linear and Integer Programming, John Wiley 1998.
- [14] A. Schrijver, New code upper bounds from the Terwilliger algebra, IEEE Trans. Inf. Th. 51 (2005), 2859–2866.
- [15] D. de Laat, F.M. de Oliveira Filho, and F. Vallentin, Upper bounds for packings of spheres of several radii, Forum Math. Sigma 2 (2014), e23 (42 pages).
- [16] J.R. Vera, Ill-Posedness and Finite Precision Arithmetic: A Complexity Analysis for Interior Point Methods, pages 424–433 in Foundations of Computational Mathematics (F. Cucker, S. Smale (ed.)), Springer, 1997.
- [17] S.J. Wright, Primal-dual interior point methods, SIAM, 1997.
- [18] S.J. Wright, Effects of Finite-Precision Arithmetic on Interior-Point Methods for Nonlinear Programming, SIAM J. Optim. 12 (2001), 36–78.
- [19] M. Yamashita, K. Fujisawa, M. Fukuda, K. Kobayashi, K. Nakata, and M. Nakata, *Latest developments in the SDPA family for solving large-scale SDPs*, pp. 687–713 in Handbook on Semidefinite, Conic and Polynomial Optimization (M.F. Anjos, J.B. Lasserre (ed.)), Springer, 2012.
- [20] D. Yudin and A.S. Nemirovski, Informational complexity and effective methods of solution of convex extremal problems, Economics and mathematical methods 12 (1976), 357–369.

E. DE KLERK, DEPARTMENT OF ECONOMETRICS AND OPERATIONS RESEARCH, FACULTY OF ECONOMIC SCIENCES, TILBURG UNIVERSITY, 5000 LE TILBURG, THE NETHERLANDS

E-mail address: E.deKlerk@uvt.nl

F. VALLENTIN, MATHEMATISCHES INSTITUT, UNIVERSITÄT ZU KÖLN, WEYERTAL 86–90, 50931 KÖLN, GERMANY

E-mail address: frank.vallentin@uni-koeln.de