CONVERGENCE, STABILITY AND ROBUSTNESS OF MULTIDIMENSIONAL OPINION DYNAMICS IN CONTINUOUS TIME

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Abstract. We analyze a continuous time multidimensional opinion model where agents have heterogeneous but symmetric and compactly supported interaction functions. We consider Filippov solutions of the resulting dynamics and show strong Lyapunov stability of all equilibria in the relative interior of the set of equilibria. We investigate robustness of equilibria when a new agent with arbitrarily small weight is introduced to the system in equilibrium. Assuming the interaction functions to be indicators, we provide a necessary condition and a sufficient condition for robustness of the equilibria. Our necessary condition coincides with the necessary and sufficient condition obtained by Blondel et al. for one dimensional opinions.

Key words. opinion dynamics, multidimensional opinions, bounded confidence.

AMS subject classifications. 37N99, 91F99.

1. Introduction. Opinion dynamics is the study of the evolution of opinions through interactions among a group of people referred to as agents. Models of opinion dynamics are based on the interaction policies between agents. These interaction policies depend on the opinions of interacting agents and their confidence bounds. Considering real life examples of interpersonal relations leads to the observation that not everyone trusts everyone else. This brings the idea of *bounded confidence* (BC) in the modeling of opinion dynamics. The BC models suggest that an agent will only be influenced by those whose opinions are closer to his/her own. BC models have been studied in discrete and continuous time setting. One of the well known discrete time BC model is known as the HK model and was introduced by Hegselmann and Krause [18, 23]. The BC model used in [18] was given as

(1.1)
$$x(t+1) = A(x(t))x(t), \quad t \in \{0, 1, 2, \ldots\},\$$

with interaction policy determined through the adjacency matrix $A(x) \in \mathbb{R}^{n \times n}$ with entries

$$a_{ij}(x) = \begin{cases} \frac{1}{|N_i(x)|}, & j \in N_i(x), \\ 0, & j \notin N_i(x), \end{cases}$$

where n is the number of agents and for i = 1, ..., n, $x_i(t) \in \mathbb{R}$ represents the opinion of the *i*th agent at time t and $N_i(x) = \{1 \leq j \leq n \mid |x_i - x_j| \leq \epsilon_i\}$ defines the *neighbors* of agent i and $\epsilon_i > 0$ is the confidence bound of the *i*th agent. Note that $|N_i(x)|$ denotes the cardinality of $N_i(x)$. In this discrete time model agents synchronously update their opinions by averaging the opinions of their neighbors. Hegselmann and Krause analyze the model for uniform confidence bounds $\epsilon_i = \epsilon$ for all agents *i* and provide sufficient conditions that lead to consensus where all agents share one opinion [18, 23]. Variations on the form of $N_i(x)$ have appeared in the literature. A particular case investigated by Mirtabatabaei and Bullo [28] is given by

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 $N_i(x) = \{1 \le j \le n \mid |x_i - x_j| \le \epsilon_j\}$ where ϵ_j is the influence bound of agent j and this model is referred to as the bounded influence (BI) model. These authors also derive some sufficient conditions for both the BC and BI models to guarantee that a trajectory converges to a steady state. We note that more generally one may take the opinions of agents to be vectors in \mathbb{R}^d . Other works that deal with the discrete time HK models may be found in [4, 6, 7, 15, 20, 25, 26, 27, 29, 31, 16]. An alternative discrete time model with asynchronous updates can be found in [34, 36].

To motivate the continuous time model, we may take the *i*th agent's opinion to be changing at a rate proportional to the difference $\frac{\sum_{j \in N_i(x(t))} x_j(t)}{|N_i(x(t))|} - x_i(t)$, between the average opinion of the neighbors and the self opinion. If the proportionality is given by a constant $\lambda > 0$ we obtain

(1.2)
$$\dot{x}_i(t) = \lambda \frac{\sum_{j \in N_i(x(t))} (x_j(t) - x_i(t))}{|N_i(x(t))|}, \quad i = 1, \dots, n$$

Alternatively, if one reweighs the opinion velocity by $|N_i(x(t))|/n$ to suggest a faster movement if there are more neighbors, then one obtains

(1.3)
$$\dot{x}_i(t) = \frac{\lambda}{n} \sum_{j \in N_i(x(t))} (x_j(t) - x_i(t)), \quad i = 1, \dots, n.$$

Additionally, one may assign weights $w_j > 0$ for agents to indicate how influential they are. After absorbing λ/n into the weights this results in the continuous time opinion model used by Blondel et al. [5] where the confidence bounds are taken to be homogeneous and equal to 1:

(1.4)
$$\dot{x}_i(t) = \sum_{j:|x_i(t)-x_j(t)|<1} w_j(x_j(t)-x_i(t)), \quad i=1,\ldots,n.$$

Blondel et al. investigate the case where opinions are taken to be scalars d = 1 and show that almost all trajectories x(t) converge to a limiting opinion x^* such that for any i, j, if $i \neq j$, then $|x_i^* - x_j^*| \geq 1$ or $x_i^* = x_j^*$. The reason for the qualification "almost all" is due to the discontinuity of the vector field in (1.4). Limiting state of opinions is viewed as a set of clusters where all agents in a given cluster share a common opinion. In the limiting state, different clusters must necessarily differ in their opinion values by more than 1, the confidence bound.

Given that almost all trajectories converge to an equilibrium of clusters, a natural question is what kind of equilibrium clusters one might expect to see "out there." The answer depends of course on assumptions on initial conditions. A natural situation to consider is where initial opinions of agents are chosen to be independent and drawn from a uniform distribution supported in a compact interval or compact and convex set in higher dimensions. There is extensive numerical evidence [20] that showed that when started from random initial conditions, the resulting equilibrium clusters tended to be separated by a distance much greater than 1 (the confidence bound) and close to 2 when the weights of clusters were roughly equal [5]. In order to explain these results, Blondel et al. also introduced a notion of robustness of equilibria which they called stability, and provided necessary and sufficient conditions [5]. In that notion, roughly speaking, an equilibrium is said to be robust/stable if after adding an agent with a sufficiently small weight and letting the system evolve, the new solution to the system can be made sufficiently close to the original equilibrium. Blondel et al.

al. [5] also conjecture that when the initial opinions are chosen to be independent and identically distributed according to a probability density function p(x) which has connected support and satisfies $0 < a \leq p(x) \leq b$ (for some $b \geq a > 0$), the probability of convergence to a robust/stable equilibrium tends to 1 as the number of agents $n \to \infty$. The intuition behind this conjecture is that when the number of agents is very large, the probability of the presence of at least one agent who would perturb the system away from an non-robust/unstable equilibrium is high (close to 1). We shall refer to this notion as *robustness* instead of stability to differentiate it from Lyapunov stability. We shall provide our own definition of robustness in Section 3 and provide a necessary condition and a sufficient condition.

There is extensive literature that focuses on the analysis of the opinion dynamics models for its consensus [3, 14, 30, 35]. Motsch and Tadmor [30] study a general class of opinion models for n number of agents with opinions of each agent considered as a vector in \mathbb{R}^d

(1.5)
$$\dot{x}_i(t) = \alpha \sum_{j=1}^n a_{ij}(x(t))(x_j(t) - x_i(t)), \quad i = 1, \dots, n,$$

where opinions are considered as vectors in \mathbb{R}^d and the adjacency matrix $A = a_{ij}$ is taken to be one of the two following forms:

(1.6)
$$a_{ij}(x) = \phi(|x_j - x_i|)/n, \quad 1 \le i, j \le n,$$

(1.7)
$$a_{ij}(x) = \frac{\phi(|x_j - x_i|)}{\sum_{k=1}^n \phi(|x_k - x_i|)}, \quad 1 \le i, j \le n.$$

Here ϕ is a nonnegative function with compact support which generalizes the indicator function that appears in (1.4) and |x| denotes the norm of $x \in \mathbb{R}^d$. We note that a_{ij} are symmetric in (1.6) and this also corresponds to (1.3). On the other hand a_{ij} are not symmetric in (1.7) and this also corresponds to (1.2). Motsch and Tadmor [30] prove that if the support of ϕ is large enough to cover the convex hull of the initial state, namely when every agent interacts with every other agent initially, this will lead to consensus regardless of whether the adjacency matrix is symmetric or not. The influence of the shape of ϕ on the likelihood of consensus is investigated via numerical simulations in [30]. The results show that *heterophilious* dynamics enhances consensus. The term heterophilious refers to the situation where agents are more influenced by others whose opinions differ greatly (but still lie within the confidence bound) than those whose opinions are closer to their own. Additionally some sufficient conditions for consensus are also provided in [30]. Jabin and Motsch [22] also consider the system (1.5) in multidimensions with nonsymmetric compactly supported interaction functions given by (1.7) and prove convergence of trajectories as $t \to \infty$ to an equilibrium.

Hendrickx and Tsitsiklis [19] study the dynamics where scalars $x_i(t)$, i = 1, ..., n, obey the equations

(1.8)
$$\dot{x}_i(t) = \sum_{j=1}^n a_{ij}(t)(x_j(t) - x_i(t)), \quad i = 1, \dots, n,$$

where the interaction function, $a_{ij}(t)$ is state independent and only a function of time. The $a_{ij}(t)$ are assumed to be nonnegative and measurable and the model represents a general "consensus seeking system" where $x_i(t)$ are some agent attributes that are scalar functions of time. Hendrickx and Tsitsiklis prove that under the assumption that the interaction functions $a_{ij}(t)$ satisfy the *cut-balance* condition, trajectories converge to a limit which is in the convex hull of initial attributes $x_i(0)$, i = 1, 2, ..., n. Moreover, Hendrickx and Tsitsiklis [19] provide sufficient conditions for the consensus and the disagreement of any two agent attributes $x_i(t)$, $x_j(t)$, $i, j = 1, 2, ..., n, i \neq j$ as $t \to \infty$. We note that, the so-called type symmetric interaction functions, where there exists K such that for all i, j, and t, it holds that $a_{ij}(t) \leq Ka_{ji}(t)$, automatically satisfy the cut-balance condition.

The models described so far consider the state as an ordered *n*-tuple (x_1, \ldots, x_n) where $x_i \in \mathbb{R}^d$ denotes the *i*th agent's opinion. The agents may alternatively be regarded as a continuum as in [5]. It is also possible to consider a model where the identity of the agents is not important. In that case the state may be modeled by a measure on \mathbb{R}^d . In this viewpoint the state space can be taken to be the space of finite Borel measures on \mathbb{R}^d . See Canuto et al. [9, 10] for an "Eulerian" approach where *d*-dimensional opinions are considered. A proof of convergence of all trajectories (as $t \to \infty$) for discrete time model is provided in [10]. An analogous result is stated without proof for the continuous time case in [9].

Contributions of this work: In our study we generalize the model in (1.4) to *d*-dimensional opinions with finite number of agents indexed by i = 1, 2, ..., n. We consider

$$\dot{x}_i(t) = \sum_{j=1}^n \xi_{ij}(|x_j(t) - x_i(t)|) w_j(x_j(t) - x_i(t)), \quad i = 1, \dots, n,$$

where $\xi_{ij} : [0, \infty) \to [0, \infty)$ are compactly supported on $[0, q_{ij}]$ for some $q_{ij} > 0$ and symmetric; $\xi_{ij} = \xi_{ji}$ for each i, j = 1, 2, ..., n. More precise assumptions on ξ_{ij} are given in §2. We note that $x_i \in \mathbb{R}^d$ is the opinion of the *i*th agent and $x_i^{\ell} \in \mathbb{R}$ is the ℓ th component of the opinion of the *i*th agent. Also, compact support here means that the function is zero outside a set whose closure is compact. This model is similar to (1.5) except for the addition of the agents' weights w_j and the assumptions on the form of the interaction functions ξ_{ij} . We also note that the symmetric case of (1.5) given by (1.6) is a special case of our model while the non-symmetric case of (1.5) given by (1.7) differs from ours.

The rest of the paper is organized as follows. In §2 we discuss what we mean by solutions, and their existence for all times $t \ge 0$ and derive some preliminary results which include the Lyapunov stability of equilibrium points in the relative interior of the set of equilibria. We also provide a discussion on how the result in [19] implies convergence of all trajectories. In §3 we re-introduce the notion of robustness of equilibria. Under the assumption that $\xi_{ij} = 1_{[0,1)}$ for all i, j, we provide two main results; a necessary condition for robustness and a sufficient condition for robustness. We also provide a detailed study of the dynamics that ensues when a new agent with zero or near zero weight is introduced into a system in equilibrium. §4 provides some numerical simulation results and in §5 we provide some concluding remarks.

2. The model, equilibria and their stability. In this section we define our model and provide some results on Filippov solutions. The main results are Lemma 2.5 about the set of equilibria and Theorem 2.8 which proves the Lyapunov stability of all equilibria in the relative interior of the set of equilibria.

2.1. Model assumptions and existence of Filippov solutions for $t \ge 0$. We consider *n* number of agents i = 1, 2, ..., n, and we assign a weight denoted by $w_i > 0$ to each agent i = 1, 2, ..., n. The weights can be interpreted as how influential the agent is, and we denote the opinion of *i*th agent at time $t \ge 0$ with a vector $x_i(t) = (x_i^1(t), x_i^2(t), ..., x_i^d(t)) \in \mathbb{R}^d$ where $d \ge 1$. We shall use |y| to denote the Euclidean norm of a vector $y \in \mathbb{R}^d$. Our model of the opinion dynamics is given by

(2.1)
$$\dot{x}_i(t) = \sum_{j=1}^n \xi_{ij}(|x_j(t) - x_i(t)|) w_j(x_j(t) - x_i(t)) = f_i(x(t)), \quad i = 1, \dots, n,$$

where the *interaction functions* $\xi_{ij} : [0, \infty) \to [0, \infty)$ are defined only for $i \neq j$, and we assume that each ξ_{ij} satisfies the following:

- 1. $\xi_{ij} = \xi_{ji}$ for each $i, j = 1, 2, \dots, n$. (Symmetry)
- 2. There exists $q_{ij} > 0$ such that for $r \ge q_{ij}$, $\xi_{ij}(r) = 0$. (Compact support) 3. If $\xi_{ij}(r) = 0$, then r = 0 or $r \ge q_{ij}$.

4. ξ_{ij} is C^1 on $[0, q_{ij})$ and the following left hand limits exist:

$$\lim_{r \to q_{ij}-} \xi_{ij}(r) \quad \text{and} \quad \lim_{r \to q_{ij}-} \xi'_{ij}(r).$$

We note that the above conditions allow for a discontinuity of ξ_{ij} at q_{ij} . Thus, $\xi_{ij}(x) = \mathbb{1}_{\{|x_i - x_j| < q_{ij}\}}(x)$ is a special case. Compactly we can write (2.1) as

(2.2)
$$\dot{x}(t) = f(x(t)),$$

where $f = (f_1, \ldots, f_n)$ is a vector field in \mathbb{R}^{nd} . In general, ξ_{ij} may have a discontinuity at q_{ij} and hence f will only be piecewise smooth and discontinuous. In this case f is C^1 on an open set which includes the open set

(2.3)
$$C_f = \{ x \in \mathbb{R}^{nd} \mid |x_i - x_j| \neq q_{ij}, \forall i, j = 1, \dots, n \}.$$

Due to the discontinuous but piecewise smooth f, one has to consider Caratheodory solutions, Krasovskii solutions or Filippov solutions (instead of classical solutions) [11, 12]. We note that the authors of [5] consider a subclass of Caratheodory solutions which they call Proper solutions. In this paper, by a solution we shall mean a Filippov solution. We recall that a Filippov solution x(t) starting from initial condition x_0 is an absolutely continuous function of t that satisfies the differential inclusion

$$\dot{x}(t) \in \mathcal{F}(x(t))$$

for almost all t and $x(0) = x_0$. The Filippov set valued map \mathcal{F} is defined by [2, 12]

(2.5)
$$\mathcal{F}(x) = \bigcap_{\delta > 0} \bigcap_{\mathrm{meas}(N) = 0} \overline{\mathrm{co}} f(B(x, \delta) \setminus N),$$

where \overline{co} denotes the convex closure of a set. We note that a Krasovskii solution is defined very similarly except no sets of Lebesgue measure zero are removed as in (2.5) [11]. In our case, because of right-continuity of ξ_{ij} , removal of the sets of measure zero in (2.5) does not make a difference and hence, Filippov solutions and Krasovskii solutions coincide.

By our assumptions, $\xi_{ij}(|x_j - x_i|)$ is bounded for $x \in \mathbb{R}^{nd}$. Hence, one can obtain a global bound of the form $|f(x)| \leq M|x|$ for all $x \in \mathbb{R}^{nd}$. Therefore by Theorem 3.3 of [33], starting from every initial condition $x_0 \in \mathbb{R}^{nd}$, solutions exist for all $t \geq 0$. We note that the upper semicontinuity requirement in Theorem 3.3 of [33] is guaranteed for the Filippov set valued map (2.5) [2]. We also refer to [11] for a detailed study of Krasovskii solutions for the case of d = 1 and $\xi_{ij} = 1_{[0,1)}$. **2.2. The Filippov set valued map.** The vector field f corresponding to our model is discontinuous. The Filipov set valued map $\mathcal{F}(x)$ of (2.5) at a given point $x \in \mathbb{R}^{nd}$ is defined by examining f(y) in all small neighborhoods of x. In particular, at a point of discontinuity x, one needs to examine the various possible continuous extensions of f. To that end we define the functions $\xi_{ij} : [0, \infty) \to [0, \infty)$ such that these agree with ξ_{ij} on $[0, q_{ij})$ and are continuous extensions of ξ_{ij} . In particular, we note that $\xi_{ij}(q_{ij}) = \xi_{ij}(q_{ij}-)$ (the left hand limit). For each possible (undirected) graph G on the vertices $\{1, 2, \ldots, n\}$, we define the continuous vector field f^G by

(2.6)
$$f_i^G(x) = \sum_{j; (i,j) \in G} \tilde{\xi}_{ij}(|x_j - x_i|) w_j(x_j - x_i), \quad i = 1, 2, \dots, n,$$

where $(i, j) \in G$ means that there is an edge between i and j in the graph G. Now, we describe how for every Filippov solution x(t), the derivative $\dot{x}(t)$ can be described in terms of f^G . Let $\mathcal{G} = \{G_1, G_2, \ldots, G_N\}$ be the set of possible graphs on vertices $\{1, 2, \ldots, n\}$. We note that at each $x \in \mathbb{R}^{nd}$ there exists $G \in \mathcal{G}$ such that $f(x) = f^G(x)$. Following the ideas in [11] we note that at each $x \in \mathbb{R}^{nd}$, there exists a nonempty subset $\mathcal{G}_x \subset \mathcal{G}$ such that the Filippov set valued map is given by

(2.7)
$$\mathcal{F}(x) = \operatorname{co}\{f^G(x) \mid G \in \mathcal{G}_x\}$$

where co is the convex hull of a set. Hence, for any solution x(t), there exist measurable functions $\alpha_i(t)$ for i = 1, ..., N such that

(2.8)
$$\dot{x}(t) = \sum_{i=1}^{N} \alpha_i(t) f^{G_i}(x(t)), \text{ for almost all } t,$$

where $0 \le \alpha_i(t) \le 1$ and $\sum_{i=1}^N \alpha_i(t) = 1$ for almost all $t \ge 0$.

Uniqueness of the solutions is not guaranteed for all initial conditions. Blondel et al. [8] show that when d = 1, and $\xi_{ij} = 1_{[0,1)}$ the system has a unique Proper solution for almost all initial conditions. If ξ_{ij} are C^1 functions on $[0, \infty)$, then one can verify that the vector field f in (2.1) is C^1 . Thus, for any given initial state, the system has a unique solution defined for all $t \ge 0$.

2.3. Equilibria. Lack of unique solutions requires a careful definition of equilibrium points. We define an equilibrium of (2.1) as follows.

DEFINITION 2.1. $x_0 \in \mathbb{R}^{nd}$ is said to be an equilibrium of the system (2.1) if and only if $x(t) = x_0$ for $t \ge 0$ is the unique solution emanating from the initial condition x_0 .

We note that $0 \in \mathcal{F}(x)$ is a necessary, but not sufficient, condition for x to be an equilibrium. On the other hand, $\mathcal{F}(x) = \{0\}$ is a sufficient, but not necessary, condition for x to be an equilibrium. To see this, consider the one dimensional vector field f(x) = 0 for $x \leq 0$ and f(x) = -1 for x > 0, for which 0 is an equilibrium, but $\mathcal{F}(0) = \{-\alpha \mid 0 \leq \alpha \leq 1\}$. Let us define

(2.9)
$$F = \{ x \in \mathbb{R}^{nd} \mid x_i = x_j \text{ or } |x_i - x_j| > q_{ij}, \ i, j = 1, \dots, n \}$$

We note that f(x) = 0 for all $x \in F$. Also since $F \subset C_f$ we have $\mathcal{F}(x) = \{f(x)\} = \{0\}$ for all $x \in F$, and thus each $x \in F$ is an equilibrium point for the system (2.1). We note that the closure \overline{F} of F is given by

(2.10)
$$\overline{F} = \{ x \in \mathbb{R}^{nd} \mid x_i = x_j \text{ or } |x_i - x_j| \ge q_{ij}, \ i, j = 1, \dots, n \}.$$

We shall show that along every Filippov solution, the (weighted) average opinion $\sum_{i=1}^{n} w_i x_i$ is constant and the (weighted) second moment $\sum_{i=1}^{n} w_i |x_i|^2$ is decreasing. First we state a lemma which states a related property which holds for each of the vector fields f^G .

LEMMA 2.2. For each graph G on $\{1, 2, ..., n\}$ and each $x \in \mathbb{R}^{nd}$ the following hold:

(2.11)
$$\sum_{i=1}^{n} w_i f_i^G(x) = 0,$$

(2.12)
$$\sum_{i=1}^{n} w_i x_i^T f_i^G(x) \le 0$$

Proof. Let G be a graph on $\{1, 2, ..., n\}$ and $x \in \mathbb{R}^{nd}$. Then

$$\sum_{i=1}^{n} w_i f_i^G(x) = \sum_{i=1}^{n} \sum_{j; (i,j) \in G} \tilde{\xi}_{ij}(|x_j - x_i|) w_i w_j(x_j - x_i) = 0.$$

We note that the sum on the right hand side of the equation above is computed for each (directed) edge $(i, j) \in G$. By the symmetry of $\tilde{\xi}_{ij}$ and the fact that $(i, j) \in G \iff (j, i) \in G$, we will have pairs of terms that are equal but opposite in sign. Hence the result follows. We may also write

$$2\sum_{i=1}^{n} w_{i}x_{i}^{T}f_{i}^{G}(x) = 2\sum_{i=1}^{n}\sum_{j; (i,j)\in G} \tilde{\xi}_{ij}(|x_{j} - x_{i}|)w_{i}w_{j}x_{i}^{T}(x_{j} - x_{i})$$

$$= \sum_{i=1}^{n}\sum_{j; (i,j)\in G} \tilde{\xi}_{ij}(|x_{j} - x_{i}|)w_{i}w_{j}x_{i}^{T}(x_{j} - x_{i})$$

$$+ \sum_{j=1}^{n}\sum_{i; (j,i)\in G} \tilde{\xi}_{ij}(|x_{j} - x_{i}|)w_{i}w_{j}x_{j}^{T}(x_{i} - x_{j})$$

$$= \sum_{i=1}^{n}\sum_{j; (i,j)\in G} \tilde{\xi}_{ij}(|x_{j} - x_{i}|)w_{i}w_{j}(x_{i}^{T} - x_{j}^{T})(x_{j} - x_{i})$$

$$= -\sum_{i=1}^{n}\sum_{j; (i,j)\in G} \tilde{\xi}_{ij}(|x_{j} - x_{i}|)w_{i}w_{j}|x_{j} - x_{i}|^{2} \leq 0.$$

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LEMMA 2.3. Let x(t) be a solution of the system (2.1). Then the weighted average opinion $\bar{x}(t) = \frac{1}{n} \sum_{i=1}^{n} w_i x_i(t)$ is constant for all $t \ge 0$. Moreover, if we set $m_2(x) = \sum_{i=1}^{n} w_i |x_i|^2$, then $m_2(x(t))$ is decreasing in t for $t \ge 0$. Proof. Let x(t) be a solution. Using (2.8) and (2.11),

$$\frac{d}{dt}\bar{x}(t) = \frac{1}{n}\sum_{i=1}^{n} w_i \dot{x}_i(t) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{N} w_i \alpha_j(t) f_i^{G_j}(x(t))$$
$$= \frac{1}{n}\sum_{j=1}^{N} \alpha_j(t)\sum_{i=1}^{n} w_i f_i^{G_j}(x(t)) = 0$$

for almost all t. By continuity of $\bar{x}(t)$ we conclude it is constant in t. Also, by (2.8) and (2.12),

$$\frac{d}{dt}m_2(x(t)) = \frac{d}{dt}\sum_{i=1}^n w_i x_i(t)^T x_i(t) = \sum_{i=1}^n 2w_i(x_i(t))^T \dot{x}_i(t)$$
$$= 2\sum_{i=1}^n \sum_{j=1}^N \alpha_j(t)w_i(x_i(t))^T f_i^{G_j}(x(t)) = 2\sum_{j=1}^N \alpha_j(t)\sum_{i=1}^n w_i(x_i(t))^T f_i^{G_j}(x(t)) \le 0,$$

for almost all t. By continuity of $m_2(x(t))$ we conclude that it is decreasing in t.

COROLLARY 2.4. Each trajectory of the dynamical system (2.1) stays in a compact set forward in time.

Proof. Let $x_0 \in \mathbb{R}^{nd}$ be any point and let x(t) be any trajectory starting from x_0 . Since m_2 is decreasing along the trajectories, $m_2(x(t)) \leq m_2(x_0)$ for all $t \geq 0$, and the result follows since $\sqrt{m_2(x)}$ provides a norm on \mathbb{R}^{nd} . \Box

The following lemma provides a useful description of the set of equilibria.

LEMMA 2.5. For each $x \in \mathbb{R}^{nd}$, we have that $0 \in \mathcal{F}(x)$ if and only if $x \in \overline{F}$. The set of equilibria contains F and is contained in \overline{F} , and when ξ_{ij} are all C^1 , \overline{F} is precisely the set of equilibria.

Proof. If $x \in \overline{F}$, then f(x) = 0, and hence clearly $0 \in \mathcal{F}(x)$. On the other hand, suppose $x \notin \overline{F}$. Then there exist i^*, j^* such that $i^* \neq j^*, x_{i^*} \neq x_{j^*}$ and $|x_{i^*} - x_{j^*}| < q_{i^*j^*}$. By continuity, for all y in all sufficiently small neighborhoods of x, we have that $y_{i^*} \neq y_{j^*}$ and $|y_{i^*} - y_{j^*}| < q_{i^*j^*}$. Then for each graph $G \in \mathcal{G}_x$ in (2.7), we have that $(i^*, j^*) \in G$ and hence

$$2\sum_{i=1}^{n} w_{i}x_{i}^{T}f_{i}^{G}(x) = -\sum_{j=1}^{n}\sum_{j; (i,j)\in G} \tilde{\xi}_{ij}(|x_{i} - x_{j}|)w_{i}w_{j}|x_{j} - x_{i}|^{2}$$
$$\leq -\tilde{\xi}_{i^{*}j^{*}}(|x_{j^{*}} - x_{i^{*}}|)w_{i^{*}}w_{j^{*}}|x_{j^{*}} - x_{i^{*}}|^{2} < 0.$$

As a result, since each $z = (z_1, \ldots, z) \in \mathcal{F}(x)$ is a convex combination of f^G , we have that for each $z \in \mathcal{F}(x)$, $\sum_{i=1}^n w_i x_i^T z_i < 0$, and hence $0 \notin \mathcal{F}(x)$. This proves the first statement.

If $x \in F$, as f is C^1 in a neighborhood of x, we have that $\mathcal{F}(x) = \{f(x)\} = \{0\}$. Hence, the set of equilibria contains F and is contained in \overline{F} . Finally, if ξ_{ij} are all C^1 , then so is f, and for $x \in \overline{F}$ we have $\mathcal{F}(x) = \{f(x)\} = \{0\}$.

We note that, by above lemma, F is precisely the *relative interior* of the set of equilibria. It is instructive to describe a natural partition of F into separated subsets which will play a role in our proof of Lyapunov stability. Let $I = \{1, 2, ..., n\}$ and \mathcal{P}_n be the collection of all partitions of I. Then we may write

$$F = \bigcup_{P \in \mathcal{P}_n} F_P,$$

where

$$(2.13) F_P = \left\{ x \in \mathbb{R}^{nd} \mid \begin{array}{c} x_i = x_j \iff \exists \ell \text{ such that } \{i, j\} \subset S_\ell \\ |x_i - x_j| > q_{ij} \iff \exists \ell \text{ such } \{i, j\} \subset S_\ell \end{array} \right\}.$$

Here, $P = \{S_1, S_2, \dots, S_t\}$ is a partition of I, that is $S_k \subset I$, $S_k \neq \emptyset$, $S_k \cap S_l = \emptyset$, $\forall k \neq l$, and $\cup_{k=1}^t S_k = I$. Correspondingly we define \overline{F}_P by

$$(2.14) \quad \overline{F}_P = \left\{ x \in \mathbb{R}^{nd} \mid \begin{array}{c} x_i = x_j \iff \exists \ell \text{ such that } \{i, j\} \subset S_\ell \\ |x_i - x_j| \ge q_{ij} \iff \not\exists \ell \text{ such } \{i, j\} \subset S_\ell \end{array} \right\}$$

We note that \overline{F}_P are closed subsets of \mathbb{R}^{nd} and $\overline{F} = \bigcup_{P \in \mathcal{P}_n} \overline{F}_P$.

For $x \in \overline{F}_P \subset \overline{F}$, where $P = \{S_1, S_2, \ldots, S_t\}$, we call the sets S_k , $k = 1, 2, \ldots, t$ clusters. Then we can say that $x \in \overline{F}_P$ has t clusters. We like to note that sometimes we shall refer to the opinions x_i such that $i \in S_k$ as a cluster.

LEMMA 2.6. For different partitions P_1 and P_2 of the index set I, the sets \overline{F}_{P_1} and \overline{F}_{P_2} are disjoint and hence separated.

Proof. Let P_1 and P_2 be two different partitions of I. Then $\exists S$ such that $S \in P_1$ and $S \notin P_2$ or vice versa. WLOG we consider the former. Let $i \in S$. Then $\exists! T \in P_2$ such that $i \in T$ and $S \neq T$. Thus, $\exists j \neq i$ such that $j \in S \setminus T$ or $j \in T \setminus S$. If $\{i, j\} \subset S, (S \in P_1)$, and $j \notin T, (T \in P_2)$,

$$\left. \begin{array}{ll} x \in \overline{F}_{P_1} & \Rightarrow x_i = x_j, \\ x \in \overline{F}_{P_2} & \Rightarrow |x_i - x_j| \geq q_{ij}. \end{array} \right\} \Rightarrow \overline{F}_{P_1} \cap \overline{F}_{P_2} = \emptyset.$$

If $\{i, j\} \subset T$, $(T \in P_2)$, and $j \notin S$, $(S \in P_1)$,

$$\begin{array}{ll} x \in \overline{F}_{P_1} & \Rightarrow |x_i - x_j| \ge q_{ij}. \\ x \in \overline{F}_{P_2} & \Rightarrow x_i = x_j. \end{array} \right\} \Rightarrow \overline{F}_{P_1} \cap \overline{F}_{P_2} = \emptyset$$

2.4. Lyapunov stability.

DEFINITION 2.7. We shall say that an equilibrium point x^* of (2.1) is strongly Lyapunov stable if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all x_0 in the δ neighborhood of x^* , all solutions starting at x_0 stay in the ϵ neighborhood of x^* forward in time.

This definition is an extension of the definition in the case of C^1 vector fields [32] and our terminology of *strong* is consistent with [12].

THEOREM 2.8. If $x^* \in F$ then x^* is Lyapunov stable.

Proof. Let $x^* \in F$. Then $x_i^* = x_j^*$ or $|x_i^* - x_j^*| > q_{ij}$, $\forall i, j$. By Lemma 2.6 there exists a unique partition $P = \{S_1, S_2, \ldots, S_t\}$ of the index set $I = \{1, 2, \ldots, n\}$ such that $x^* \in F_P$ where F_P is defined as in (2.13). Define the set $G_P \subset \mathbb{R}^{nd}$ as follows:

$$G_P = \left\{ x \in \mathbb{R}^{nd} \mid \begin{array}{c} \not\exists \ell \text{ such that } \{i, j\} \subset S_\ell \implies |x_i - x_j| > q_{ij} \\ \exists \ell \text{ such that } \{i, j\} \subset S_\ell \implies |x_i - x_j| < q_{ij} \end{array} \right\}$$

It is clear that $x^* \in G_P$, $G_P \subset C_f$ and that G_P is open. Let us define a function $V: G_P \to \mathbb{R}$ such that $V(x) = \sum_{i=1}^n w_i |x_i - x_i^*|^2$. It is easy to see that $V \in C^1(G_P)$ and $V(x) \ge 0$ for all $x \in G_P$ with $V(x) = 0 \iff x = x^*$. In what follows, by $\dot{V}(x)$, we mean DV(x)f(x). We have

$$\begin{split} \dot{V}(x) &= \sum_{i=1}^{n} 2w_i (x_i - x_i^*)^T \dot{x}_i = \sum_{i,j=1}^{n} \xi_{ij} (|x_j - x_i|) w_i w_j \Big[(x_j^* - x_i^*)^T - (x_j - x_i)^T \Big] (x_j - x_i) \\ &= \sum_{\exists \ell; \{i,j\} \subset S_\ell} \xi_{ij} (|x_j - x_i|) w_i w_j \Big[(x_j^* - x_i^*)^T - (x_j - x_i)^T \Big] (x_j - x_i) \\ &+ \sum_{\exists \ell; \{i,j\} \subset S_\ell} \xi_{ij} (|x_j - x_i|) w_i w_j \Big[(x_j^* - x_i^*)^T - (x_j - x_i)^T \Big] (x_j - x_i) \\ &= -\sum_{\exists \ell; \{i,j\} \subset S_\ell} \xi_{ij} (|x_j - x_i|) w_i w_j |x_j - x_i|^2. \end{split}$$

The last expression follows from the fact that $\forall i, j$

$$\exists \ell \text{ such that } \{i, j\} \subset S_{\ell} \implies x_i^* = x_j^*.$$
$$\exists \ell \text{ such that } \{i, j\} \subset S_{\ell} \implies \xi_{ij}(|x_j - x_i|) = 0.$$

Therefore $\dot{V}(x) \leq 0, \forall x \in G_P$. Hence, V defined on the open set G_P is a (weak) Lyapunov function for the equilibrium point x^* [32]. Since the vector field f is C^1 in G_P , the Lyapunov stability of x^* follows from a standard result for C^1 vector fields [32]. \Box

Theorem 2.8 shows that all equilibria in F are Lyapunov stable. If $x^* \in \overline{F} \setminus F$ is an equilibrium, this theorem does not apply, and the x^* may not be stable. For instance, as an example, in the case where all ξ_{ij} are C^1 , let us consider n = 2 agents with scalar opinions and the homogeneous confidence bound q = 1. The dynamics for this case is

(2.15)
$$\begin{aligned} \dot{x}_1 &= \xi_{12}(|x_2 - x_1|)w_2(x_2 - x_1), \\ \dot{x}_2 &= \xi_{21}(|x_1 - x_2|)w_1(x_1 - x_2), \end{aligned}$$

provided the weights $w_1 = w_2 = 1$ and note that $\xi_{12} = \xi_{21}$. Consider an equilibrium point $x^* = (x_1^*, x_2^*)$ such that $|x_1^* - x_2^*| = 1$. Let $x \in B_{\delta}(x^*)$ for any $\delta > 0$ such that $|x_1 - x_2| < 1$. By Lemma 2.3 the line $x_1 + x_2 = C$ is invariant under the dynamics. Since $x_1(t) = C - x_2(t)$ for all t, we may write (2.15) as follows:

$$\dot{x}_1 = -\dot{x}_2 = -\xi_{12}(|x_1 - x_2|)w_1(x_1 - x_2)$$
$$= \xi_{12}(|C - 2x_1|)w_1(C - 2x_1).$$

This is a one-dimensional ODE with the equilibrium point $x_1 = \frac{C}{2}$ ($x_2 = \frac{C}{2}$) and $x_1(t) \to \frac{C}{2}$ as $t \to \infty$. Thus the trajectory starting from $x \in B(x^*, \delta)$ (an open ball of radius δ centered at x^*) such that $|x_1 - x_2| < 1$ approaches an equilibrium on the line $x_1 = x_2$ as $t \to \infty$ showing that the equilibrium point x^* is not stable.

2.5. A more general model. While our model (2.1) is more in line with continuous time multidimensional models in the literature, one may consider the more general model where each component x_i^{ℓ} of the opinion of an agent has a different influence function ξ_{ij}^{ℓ} :

(2.16)
$$\dot{x}_i^{\ell} = \sum_{j=1}^n \xi_{ij}^{\ell} (|x_j - x_i|) (x_j^{\ell} - x_i^{\ell}), \quad i = 1, \dots, n, \ \ell = 1, \dots, d.$$

If we make the same assumptions on ξ_{ij}^{ℓ} as in the beginning of this section, most key results obtained in this section will remain valid, which we briefly describe. Existence of Filippov solutions for all $t \geq 0$ is still guaranteed. The set of C^1 vector fields described by (2.6) will be larger, as one has to consider ordered *d*-tuples of graphs on vertices $\{1, \ldots, n\}$. Thus, the role of \mathcal{G} is replaced by \mathcal{G}^d . A modified form of (2.8) holds, and Lemma 2.3 and Corollary 2.4 remain valid. The set of discontinuities (of the vector field) however, becomes more complicated. Nevertheless, with F and \overline{F} defined as above after setting q_{ij} to be the maximum of q_{ij}^{ℓ} , the set of equilibria contains F and is contained in \overline{F} . The Lyapunov stability result for equilibria in Falso follows using the same Lyapunov function. **2.6.** Convergence of trajectories. The scalar convergence result in [19] readily applies to any (Filippov) solution x(t) of our model (2.1). To see this, define $a_{ij}(t) = w_j \xi_{ij}(x(t))$ for $t \ge 0$. Then for each $\ell = 1, \ldots, d$, the function $y(t) = (x_1^\ell(t), x_2^\ell(t), \ldots, x_n^\ell(t))$ satisfies

$$\dot{y}_i(t) = \sum_{j=1}^n a_{ij}(t)(y_j(t) - y_i(t)),$$

for almost all t. With K being the ratio of the maximum and minimum among the weights, we see that $a_{ij}(t) \leq Ka_{ji}(t)$ for all t and i, j, making a_{ij} type symmetric, and the result in [19] implies convergence of y(t). Hence, the convergence of x(t) follows.

3. Robustness of equilibria. Suppose that the dynamics (2.1) is in an equilibrium state $x^* = (x_1^*, \ldots, x_n^*) \in \mathbb{R}^{nd}$. Introduce a new agent (whom we shall call the zero agent) with initial opinion x_0^* , weight δ and additional symmetric interaction functions ξ_{0j} for $j = 1, \ldots, n$ with compact supports $[0, q_{0j}]$. Consider the resulting configuration as an initial state for (2.1) with (n + 1) agents and let a resulting solution be $(\tilde{x}_0(t), \tilde{x}_1(t), \tilde{x}_2(t), \ldots, \tilde{x}_n(t))$. Define $\Delta(x_0^*, \delta; x^*) = \sup |\tilde{x}_i(t) - x_i^*|$ where the supremum is taken over $i = 1, 2, \ldots, n$, all possible solutions $\tilde{x}(t)$ starting with initial condition $(x_0^*, x_1^*, \ldots, x_n^*)$, and all times $t \geq 0$. Thus, $\Delta(x_0^*, \delta; x^*)$ is a measure of the disruption to the equilibrium x^* caused by the introduction of the zero agent with weight δ and initial opinion x_0^* .

We shall say that the equilibrium x^* is robust with respect to the initial zero opinion x_0^* provided

(3.1)
$$\lim_{\delta \to 0+} \Delta(x_0^*, \delta; x^*) = 0.$$

We shall say that the equilibrium x^* is robust almost surely and uniformly provided there exists a set Z of Lebesgue measure zero such that

(3.2)
$$\lim_{\delta \to 0^+} \sup_{x_0^* \in \mathbb{R}^d \setminus Z} \Delta(x_0^*, \delta; x^*) = 0$$

Our probabilistic terminology is justified if one considers choosing the initial opinion x_0^* at random with uniform probability density inside the union of the balls $B(x_j^*, q_{0j})$ which is a set with finite Lebesgue measure. If x_0^* is outside of these balls, then $\Delta(x_0^*, \delta; x^*) = 0$. The term "uniformly" refers to taking supremum over $\mathbb{R}^d \setminus Z$. Finally, we shall say that the equilibrium x^* is not robust provided there exists a set Z of strictly positive Lebesgue measure such that for each $x_0^* \in Z$ the limit (3.1) fails to hold.

Our definition of almost sure uniform robustness slightly differs from the stability defined in [5]. In [5] no sets of measure zero are removed and Proper solutions (instead of Filippov solutions) are considered. For scalar opinions (d = 1), Blondel et al. [5] prove that an equilibrium $x^* \in \mathbb{R}^n$ is robust if and only if for any two clusters x_i^* and x_j^* of x^* with weights w_i and w_j respectively, we have $|x_i^* - x_j^*| > 1 + \frac{\min\{w_i, w_j\}}{\max\{w_i, w_j\}}$. Here, the weight of a cluster x_i^* is defined to be the sum of the weights of all agents in the cluster x_i^* . We also note that a slightly different notion of robustness was considered in the earlier work [20].

Merger of agents within a cluster: We shall analyze our multidimensional model (2.1) concerning robustness of its equilibrium points that are in the relative interior F. In order to make the analysis tractable, from now on, we shall take ξ_{ij} to

be indicator functions of the set [0, 1). Note that, as mentioned earlier, we take |.| to be the Euclidean norm in \mathbb{R}^d . Suppose that $x^* = (x_1^*, \ldots, x_n^*)$ is an equilibrium with $k \leq n$ number of clusters. Since the influence functions ξ_{ij} are now assumed to be identical, two distinct agents $i, j \in \{1, \ldots, n\}$ with same initial opinion $(x_i^* = x_j^*)$ can differ in their opinions at time t > 0 $(x_i(t) \neq x_j(t))$ only after the system encounters a discontinuity of f at which uniqueness of solutions is lost. Since the two main results in our paper, Theorem 3.6 and Theorem 3.8 either rely on analysis of what happens to the system until the first switching or assume conditions that guarantee uniqueness, for ease of presentation we shall merge all the agents in a cluster into a single agent with the combined weight. We emphasize that when uniqueness is lost one has to exercise caution.

After such merging and renaming, we can consider the equilibrium to be $x^* = (x_1^*, x_2^*, \ldots, x_k^*) \in \mathbb{R}^{kd}$ with renamed combined weights w_1, \ldots, w_k . Each x_i^* for $i = 1, \ldots, k$ is called a cluster, and the unit open ball $B_i^* = B(x_i^*, 1)$ will be called the confidence ball of cluster i or that x_i^* . We shall call $B_{ij}^* = B_i^* \cap B_j^*$, the mutual confidence ball of x_i^* and x_j^* . More generally, given a nonempty subset $S \subset \{1, 2, \ldots, k\}$, we shall denote by B_S^* the intersection of all the balls $B(x_i^*, 1)$ where $i \in S$. We shall also denote by m_S^* , the center of mass of the clusters in S defined by $m_S^* = \frac{\sum_{i \in S} w_i x_i^*}{\sum_{i \in S} w_i}$.

In this section we present two main results: Theorem 3.6 provides a necessary condition for robustness while Theorem 3.8 provides a sufficient condition.

3.1. Dynamics with zero agent. Here we focus on some general results on the dynamics that ensues when a zero agent is introduced into a system which is in equilibrium with k clusters. Let us denote the distinct equilibrium clusters by $x_i^* \in \mathbb{R}^d$ with $i = 1, \ldots, k$ and their weights by w_i . Suppose the zero agent is introduced at initial opinion $x_0^* \in \mathbb{R}^d$ with weight $\delta \geq 0$ which is "small". We shall refer to $x^* = (x_1^*, \ldots, x_k^*)$ as the equilibrium or equilibrium clusters and x_0^* as the initial zero opinion. We shall also assume that $|x_i^* - x_j^*| > 1$ for all $i, j \in \{1, \ldots, k\}$ with $i \neq j$.

Zero agent with zero weight: It is instructive to first focus on the case $\delta = 0$ and later consider small positive perturbations to δ . When $\delta = 0$, the resulting dynamical system is effectively d dimensional as only the opinion of the zero agent will change in time while other agents' opinions are frozen at x_i^* . In other words, the dynamics in the $\mathbb{R}^{(k+1)d}$ is restricted to a d dimensional affine subspace corresponding to $x_i = x_i^*$ for $i = 1, \ldots, k$. The resulting d dimensional dynamics for the opinion $x_0(t)$ of the zero agent will follow a system that switches between linear vector fields:

(3.3)
$$\dot{x}_0(t) = \sum_{i, |x_0 - x_i^*| < 1} w_i(x_i^* - x_0(t)).$$

There will be k codimension 1 switching surfaces $(d-1 \text{ dimensional spheres in } \mathbb{R}^d)$ given by $|x_i^* - x_0|^2 = 1$ for i = 1, ..., k. These spheres divide \mathbb{R}^d into open sets O_S which correspond to all possible subsets S of $\{1, ..., k\}$. More precisely, for each subset $S \subset \{1, ..., k\}$ define the open set O_S by

(3.4)
$$O_S = \{x_0 \in \mathbb{R}^d \mid |x_0 - x_i^*| < 1 \ \forall i \in S \text{ and } |x_0 - x_i^*| > 1 \ \forall i \notin S \}.$$

Inside O_S , $x_0(t)$ evolves according to

(3.5)
$$\dot{x}_0(t) = \sum_{i \in S} w_i (x_i^* - x_0(t)).$$

When S is empty, all points in O_S are equilibria. For nonempty S this may be rewritten as

(3.6)
$$\dot{x}_0(t) = W_S(m_S^* - x_0(t)),$$

where $W_S = \sum_{i \in S} w_i$ and m_S^* is the center of mass $m_S^* = \frac{1}{W_S} \sum_{i \in S} w_i x_i^*$. Thus, when S is nonempty, inside the open set O_S the (images of the) trajectories $x_0(t)$ are straight lines (when extended beyond O_S if necessary) that join with m_S . For convenience we define $O = \bigcup_{S \subset \{1, \dots, k\}} O_S$.

We shall only consider initial zero opinions x_0^* that lie in O (as the complement of O is a set of measure zero) and consider what happens to the Filippov solutions $x_0(t)$ that start at x_0^* . Since x_0^* is in one of the open sets O_S , there is a time interval $[0, \epsilon)$ in which the solution is unique. Uniqueness may only be lost when the initially unique solution reaches a switching surface.

Consider a point $y \in \mathbb{R}^d$ that lies on precisely one switching surface; that is, there exists a unique *i* such that $|x_i^* - y| = 1$. In that case, for all sufficiently small open neighborhoods U of y, we have $U \cap O = U \cap (O_{S_1} \cup O_{S_2})$ where after a possible reordering of the cluster labels, we may assume $|y - x_l^*| = 1$, $S_1 = \{1, \ldots, l-1\}$ and $S_2 = \{1, \ldots, l\}$. Let us examine what happens when trajectory $x_0(t)$ reaches y in finite time from either O_{S_1} or O_{S_2} . The key issue is whether this trajectory may be continued uniquely. There are a few cases to consider. We refer to [13] for simple conditions that allow for unique continuation of Filippov solutions at a switching surface.

Case 1: S_1 is empty. Then l = 1 and $S_2 = \{1\}$. In this case, since all points in O_{S_1} are equilibria, $x_0(t)$ could not have arrived at y from O_{S_1} . As the vector field in O_{S_2} will be pointing away from the tangent plane to the switching surface at y, the solution could not have arrived from O_{S_2} either. Thus this represents an impossible scenario. (We note that, if such a y is considered as an initial condition, there will be multiple Filippov solutions emanating from it.)

Case 2: S_1 is nonempty. Let T_y denote the tangent hyperplane to the sphere $|x_0 - x_l^*|^2 = 1$ at y. There are five possible cases to consider depending on the position of $m_{S_1}^*$ and $m_{S_2}^*$ relative to T_y . We observe that $m_{S_2}^*$ lies on the interior of the line segment $[m_{S_1}^*, x_l^*]$ and that x_l^* does not lie on T_y .

Case 2(a): $m_{S_1}^*$ and x_l^* are on the same side of T_y , which also implies that $m_{S_2}^*$ lies on that side as well.

In this case the vector fields in O_{S_1} as well as O_{S_2} in a small enough neighborhood of y are both pointing into the same side of T_y where x_l^* is. This will satisfy the uniqueness condition for continuation of the Filippov solution. If the solution arrived at y from O, it must have arrived from O_{S_1} . Moreover, the unique continuation will carry it into O_{S_2} until next switching. In short, this corresponds to switching where agent zero enters the influence of cluster l.



FIG. 3.1. An illustration of Case 2(a)

Case 2(b): $m_{S_2}^*$ and x_l^* are on opposite sides of T_y in which case $m_{S_1}^*$ also lies on the same side as $m_{S_2}^*$.

In this case the vector fields in O_{S_1} as well as O_{S_2} in a small enough neighborhood of y are both pointing into the same side of T_y which does not contain x_l^* . This will also satisfy the uniqueness condition for continuation of the Filippov solution. If the solution arrived at y from O, it must have arrived from O_{S_2} . Moreover, the unique continuation will carry it



FIG. 3.2. An illustration of Case 2(b)

into O_{S_1} until next switching. In short, this corresponds to switching where agent zero leaves the influence of cluster l.

Case 2(c): $m_{S_1}^*$ and $m_{S_2}^*$ are on opposite sides of T_y in which case $m_{S_2}^*$ must lie on the same side as x_l^* . In this case, in all small neighborhoods of y, the vector fields in O_{S_1} and O_{S_2} are both pointing in opposite directions, and away from T_y . Thus the solution $x_0(t)$ could not have arrived at y from O_{S_1} or from O_{S_2} . Thus, this also represents an impossible scenario. (As in Case 1, we note that if y is the initial condition, then there are multiple possible solutions emanating from it.)

Case 2(d): $m_{S_1}^*$ lies on T_y . In this case in all small neighborhoods of y, the vector field in O_{S_2} is pointing away from T_y while the vector field inside O_{S_1} becomes tangential to T_y at y. Hence unique extension of Filippov solutions in not guaranteed.

Case 2(e): $m_{S_2}^*$ lies on T_y . In this case in all small neighborhoods of y, the vector field in O_{S_1} is pointing away from T_y while the vector field inside O_{S_2} becomes tangential to T_y at y. Hence unique extension of Filippov solutions in not guaranteed.

Therefore, we reach the conclusion that the trajectory $x_0(t)$ emanating from an initial zero opinion x_0^* in O will be unique for $t \ge 0$ provided that it never reaches a point in the intersection of two or more of the switching surfaces $|x_0 - x_i^*|^2 =$ 1 and never encounters Cases 2(d) or 2(e). (Note that we do not claim this to be a necessary condition for uniqueness).

Type 1 and type 2 initial zero opinions and trajectories: We shall call such a trajectory and the corresponding initial



FIG. 3.3. An illustration of Case 2(c)

zero opinion in O a regular trajectory and a regular initial zero opinion respectively. We also note that when a regular trajectory crosses a switching surface at a point y, the vector fields on either side of the switching surface are transversal to the surface and point in the same direction as described in cases 2(a,b).

Among regular trajectories there are two types to consider. The first type, which we shall call *type 1* consists of trajectories that undergo only finitely many switches. The corresponding initial zero opinions in O will be referred to as *type 1* initial zero opinions. A regular trajectory and the corresponding regular initial zero opinion will be called *type 2* when the trajectory undergoes infinitely many switches.

Zero agent with weight $\delta \geq 0$:

We turn our attention to the case when $\delta > 0$. In this case, the dynamics evolves in $\mathbb{R}^{(k+1)d}$ according to the system

(3.7)
$$\dot{x}_i = \sum_{j, |x_i - x_j| < 1} w_j (x_j - x_i), \quad i = 0, 1, \dots, k,$$

where $w_0 = \delta$. Now there are k(k+1)/2 switching surfaces given by the codimension 1 spheres $|x_i - x_j|^2 = 1$ for $0 \le i < j \le k+1$.

The vector field is piecewise C^{1} , and there are open sets $\mathcal{O}_{G} \subset \mathbb{R}^{(k+1)d}$ corresponding to every graph G on the vertices $\{0, 1, \ldots, k\}$ such that inside \mathcal{O}_{G} the vector field is linear and corresponds to the dynamics

(3.8)
$$\dot{x}_i = \sum_{j,(i,j)\in G} w_j(x_j - x_i), \quad i = 0, 1, \dots, k,$$

where $w_0 = \delta$. It is important to note that, when the zero agent is introduced into the equilibrium, the dynamics (3.7) is initially of the special form:

(3.9) Initial Dynamics
$$\begin{cases} \dot{x}_0^{\delta} = \sum_{j=1}^k w_j (x_j^{\delta} - x_0^{\delta}), \\ \dot{x}_j^{\delta} = \delta (x_0^{\delta} - x_j^{\delta}), \quad j = 1, 2, \dots, k, \end{cases}$$

with initial conditions $x_j^{\delta}(0) = x_j^*$, j = 0, 1, 2, ..., k. In this form, none of the original clusters interact with each other. The superscript δ is used to emphasize the dependence of solutions on δ . In the next two lemmas we obtain important results about initial dynamics.

LEMMA 3.1. Consider the initial dynamics (3.9) with initial condition $(x_0^*, x_1^*, \ldots, x_k^*)$. Denote the center of mass of the clusters $m_S^{\delta}(t) = \frac{1}{W} \sum_{i \in S} w_i x_i^{\delta}(t)$ where $S = \{1, 2, \ldots, k\}$ and $W = \sum_{j=1}^k w_j$. Then for $t \ge 0$ the following hold:

(3.10)
$$\begin{aligned} \dot{x}_0^\delta &= W(m_S^\delta - x_0^\delta),\\ \dot{m}_S^\delta &= \delta(x_0^\delta - m_S^\delta), \end{aligned}$$

where initial conditions are $x_0^{\delta}(0) = x_0^*$, $m_S^{\delta}(0) = \sum_{i=1}^k w_i x_i^*$. Thus the trajectories $x_0^{\delta}(t)$ and $m_S^{\delta}(t)$ lie on the line segment $[x_0^*, m_S^*]$ for all $t \ge 0$.

Moreover, if m_*^{δ} denotes the center of mass including agent zero, $m_*^{\delta} = \frac{1}{W+\delta} \sum_{i=0}^k w_i x_i^*$, then one obtains for all $t \ge 0$ that

$$(3.11) |x_0^{\delta}(t) - m_*^{\delta}|^2 \le e^{-2(W+\delta)t} |x_0^* - m_*^{\delta}|^2, |m_S^{\delta}(t) - m_*^{\delta}|^2 \le e^{-2(W+\delta)t} |m_S^* - m_*^{\delta}|^2.$$

Proof. The proof of the first part is straightforward algebra. For the second part of (3.11), noting that $W(m_S^{\delta} - x_0^{\delta}) = (W + \delta)(m_*^{\delta} - x_0^{\delta})$, one obtains that

$$\frac{d}{dt}|x_0^{\delta} - m_*^{\delta}|^2 = 2(x_0^{\delta} - m_*^{\delta})^T \dot{x}_0^{\delta} = -2(W + \delta)|x_0^{\delta} - m_*^{\delta}|^2,$$

and the result is immediate. The second estimate is proven similarly. \Box

The following lemma provides continuous perturbation results on initial dynamics in the limit $\delta \to 0+$, and forms the backbone of our Theorem 3.8. A key point of the following lemma is that the trajectory $x_0^{\delta}(t)$ of the zero agent perturbs uniformly on the infinite time interval $[0, \infty)$ when δ is perturbed from 0. The trajectories $x_j^{\delta}(t)$ of the clusters perturb uniformly only on any finite time interval [0, T], except for the case of one cluster (k = 1), in which case the perturbation is uniform on the infinite time interval.

LEMMA 3.2. Let $x^* = (x_0^*, x_1^*, \dots, x_k^*) \in K \subset \mathbb{R}^{(k+1)d}$ for some compact set K and consider initial dynamics (3.9) for $\delta \in [0, \delta_0]$. Then,

(i) $\forall T > 0$, $\lim_{\delta \to 0^+} \sup_{t \in [0,T], x^* \in K} |x_j^{\delta}(t, x^*) - x_j^0(t, x^*)| = 0$, $j = 0, 1, \dots, k$.

- (ii) $\lim_{\delta \to 0^+} \sup_{t \in [0,\infty], x^* \in K} |x_0^{\delta}(t,x^*) x_0^0(t,x^*)| = 0.$
- (iii) For k = 1,

$$\lim_{\delta \to 0^+} \sup_{t \in [0,\infty], x^* \in K} |x_j^{\delta}(t, x^*) - x_j^0(t, x^*)| = 0, \quad j = 0, 1.$$

Proof. The key idea of the proof is to establish pointwise equicontinuity of the trajectories and the use of Arzela-Ascoli theorem to show that the convergence of the trajectories as $\delta \to 0$ is uniform in t and x^* . For (ii) and (iii) where the time interval is $[0, \infty)$, we need to compactify the interval by including ∞ . As the topology on $[0, \infty]$ which makes it compact does not arise from the standard metric, we use the "topological version" of the definition of pointwise equicontinuity and the Arzela-Ascoli theorem, see for instance [1]. We note that the pointwise limit of the trajectories, for fixed t (including $t = \infty$ for (ii) and (iii)) and x^* , as $\delta \to 0$ is continuous due to continuous dependence on δ of the vector field.

Compactly we can write initial dynamics (3.9) as $\dot{x}^{\delta} = A(\delta)x^{\delta}$ where $A(\delta)$ is continuous in δ . It is straightforward to show that as $t \to \infty$, the trajectories $x_j^{\delta}(t)$ for $j = 0, 1, \ldots, k$ converge to $m_*^{\delta}(x^*) = \frac{1}{(W+\delta)} \sum_{i=0}^k w_i x_i^*$, which is continuous in δ . This allows us to extend the time domain of the trajectories continuously to include ∞ by defining $x_j^{\delta}(\infty, x^*) = m_*^{\delta}(x^*) \in \mathbb{R}^d$ so that $x_j^{\delta} : [0, \infty] \times K \to \mathbb{R}^d$ for $j = 0, 1, \ldots, k$ and regard $\{x_j^{\delta}\}$ as a family of continuous functions of t and x^* indexed by $\delta \in [0, \delta_0]$.

Since the second moment function $m_2(x) = \sum_{i=0}^k w_i |x_i|^2$ $(w_0 = \delta)$ is decreasing along the trajectories, it is clear that $\{x^{\delta}(t, x^*) | \delta \in [0, \delta_0]\}$ is uniformly bounded: there exists $N_1 > 0$ such that for i = 0, 1, ..., k

$$|x_i^{\delta}(t, x^*)| \le N_1, \quad \forall t \in [0, \infty], \ \forall x^* \in K, \ \forall \delta \ge 0.$$

Proof of (i):

First we establish the equicontinuity of the family $\{x_j^{\delta}(t, x^*) | \delta \in [0, \delta_0]\}$ for $x^* \in K$ and $t \in [0, T], T < \infty$.

For $x^*, y^* \in K$ and $t \in [0, T]$ we have

$$|x^{\delta}(t,x^*) - x^{\delta}(t,y^*)| \le ||e^{A(\delta)t}|| |x^* - y^*| \le e^{||A(\delta)||T|} |x^* - y^*|.$$

Since $A(\delta)$ is continuous, $\exists N_2 > 0$ such that $||A(\delta)|| < N_2$ for $\delta \in [0, \delta_0]$. For given $\epsilon > 0$ one can find γ_1 so that if $|x^* - y^*| < \gamma_1$ then the right hand side of above inequality is $< \epsilon/2$. Similarly,

$$|x^{\delta}(t,x^{*}) - x^{\delta}(s,x^{*})| \leq \int_{s}^{t} |\dot{x}^{\delta}(u,x^{*})| du \leq \int_{s}^{t} ||A(\delta)|| ||x^{\delta}(u,x^{*})| du \leq N_{1}N_{2}|t-s|.$$

For any given $\epsilon > 0$ one can find $\gamma_2 > 0$ so that if $|t-s| < \gamma_2$ then the right hand side of this inequality is $< \epsilon/2$. Thus, for (y^*, s) satisfying $|x^* - y^*| < \gamma_1$ and $|t-s| < \gamma_2$, we obtain that $|x^{\delta}(t, x^*) - x^{\delta}(s, y^*)| < \epsilon$, which shows equicontinuity of the family at (t, x^*) .

Moreover, by continuous dependence of solution of an ODE on parameters, for any fixed $(t, x^*) \in [0, T] \times K$ and for each $j = 0, 1, \ldots, k$ we have $x_j^{\delta}(t, x^*) \to x_j^0(t, x^*)$ as $\delta \to 0^+$. Combining with Arzela-Ascoli theorem and standard arguments, we have that

$$\lim_{\delta \to 0^+} \sup_{t \in [0,T], x^* \in K} |x_j^{\delta}(t, x^*) - x_j^0(t, x^*)| = 0, \quad j = 0, 1, \dots, k.$$

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Proof of (ii):

It is sufficient to prove that $x_0^{\delta}(t)$ is equicontinuous at $(\infty, x^*) \in [0, \infty] \times K$, for any $x^* \in K$. For $x^*, y^* \in K$ one obtains

$$|x_0^{\delta}(\infty, x^*) - x_0^{\delta}(\infty, y^*)| = |m_*^{\delta}(x^*) - m_*^{\delta}(y^*)| \le \frac{1}{W + \delta} \sum_{i=0}^k w_i |x_i^* - y_i^*| \le (k+1)|x^* - y^*|.$$

For any given $\epsilon > 0$, one can choose $\gamma > 0$ such that above inequality is $\langle \epsilon/2$. Noting that $m_*^{\delta}(x^*) = x_0^{\delta}(\infty, x^*)$, from Lemma 3.1 we obtain that

$$|x_0^{\delta}(t, x^*) - x_0^{\delta}(\infty, x^*)|^2 \le e^{-2Wt} |x_0^* - m_*^{\delta}|^2, \quad \forall x^* \in K$$

For any given $\epsilon > 0$, one can choose $T_0 > 0$ such that $\forall t > T_0$, $|x_0^{\delta}(t, x^*) - x_0^{\delta}(\infty, x^*)| < \epsilon/2$. Then, for (y^*, t) satisfying $|x^* - y^*| < \gamma$ and $t > T_0$,

$$|x_0^{\delta}(t,x^*) - x_0^{\delta}(\infty,y^*)| \le (k+1)|x^* - y^*| + e^{-Wt}|x_0^* - m_*^{\delta}| < \epsilon.$$

Moreover, $x_0^{\delta}(\infty, x^*) = \frac{1}{(W+\delta)} \sum_{i=0}^k w_i x_i^* \to x_0^0(\infty, x^*)$, as $\delta \to 0^+$. Hence, we can conclude that

$$\lim_{\delta \to 0^+} \sup_{t \in [0,\infty], x^* \in K} |x_0^{\delta}(t, x^*) - x_0^{0}(t, x^*)| = 0.$$

Proof of (iii):

The proof for the case of j = 0 is shown above. For the equicontinuity of $x_1^{\delta}(t, x^*)$ at $(\infty, x^*) \in [0, \infty] \times K$, for any $x^* \in K$, consider $x^*, y^* \in K$ and note that $x^* = (x_0^*, x_1^*)$ and $y^* = (y_0^*, y_1^*)$. Then,

$$|x_1^{\delta}(\infty, x^*) - x_1^{\delta}(\infty, y^*)| = |m_*^{\delta}(x^*) - m_*^{\delta}(y^*)| \le 2|x^* - y^*|.$$

For any $\epsilon > 0$ given, one can choose $\gamma > 0$ such that the right hand side of above inequality is $\langle \epsilon/2$. Noting that $m_*^{\delta}(x^*) = x_1^{\delta}(\infty, x^*)$ and using Lemma 3.1 (since k = 1, we have that $m_S^{\delta} = x_1^{\delta}$)

$$|x_1^{\delta}(t,x^*) - x_1^{\delta}(\infty,x^*)|^2 \le e^{-2w_1 t} |x_1^* - m_*^{\delta}(x^*)|^2, \quad \forall x^* \in K.$$

One can choose $T_0 > 0$ such that $\forall t > T_0$, $e^{-w_1 t} |x_1^* - m_*| < \epsilon/2$. Thus,

$$|x_1^{\delta}(t,x^*) - x_1^{\delta}(\infty,y^*)| \le 2|x^* - y^*| + e^{-w_1 t}|x_1^* - m_*^{\delta}(x^*)| < \epsilon$$

Hence, the result follows as before. \square

DEFINITION 3.3. Given an equilibrium $x^* = (x_1^*, \ldots, x_k^*)$, consider the k switching surfaces or spheres (in \mathbb{R}^d) given by

$$\{x_0 \in \mathbb{R}^d \mid |x_0 - x_i^*|^2 = 1\}, \quad i = 1, \dots, k$$

We shall call the equilibrium x^* generic provided the following hold:

1. $|x_i^* - x_j^*| \neq 1$ for $i, j \in \{1, \dots, k\}$.

2. no two spheres above are tangential,

3. for any nonempty subset $S \subset \{1, 2, ..., k\}$, the center of mass m_S^* does not lie on any of the above spheres.

Now, for generic equilibria, we state a lemma which shows that type 1 trajectories retain their properties of uniqueness as well as finitely many switchings for all t > 0, when the weight of zero agent is changed from 0 to small positive values and the initial opinion of zero agent is perturbed by small amounts.

LEMMA 3.4. Let $x^* = (x_1^*, \ldots, x_k^*)$ be a generic equilibrium. Suppose $\bar{x}_0^* \in O$ is an initial zero opinion of type 1 with respect to this equilibrium cluster. Then there exists $m \in \mathbb{Z}_+$, $\epsilon > 0$ and $\delta_0 > 0$ such that for all $x_0^* \in B(\bar{x}_0^*, \epsilon)$ and $\delta \in [0, \delta_0)$, the solution $x_0(t, \delta, x_0^*)$ emanating from x_0^* is unique for $t \ge 0$ and undergoes exactly m switchings only intersecting one switching surface at any switching time.

Proof. In this proof, by "smooth", we shall mean C^1 . By definition of type 1, when $\delta = 0$, the unique solution $x_0(t, 0, \bar{x}_0^*)$ emanating from \bar{x}_0^* undergoes finitely many switchings, say m, where the solution only intersects one switching surface at a given switching time. Denote the switching times by $0 < \bar{t}_1 < \bar{t}_2 \cdots < \bar{t}_m$, and let the corresponding switching surfaces be $|x_0 - x_{i_l}^*|^2 = 1$ where $i_1, \ldots, i_m \in \{1, \ldots, k\}$. We shall argue with the aid of implicit function theorem, that the solution $x_0(t, \delta, x_0^*)$ perturbs smoothly and uniquely when δ and x_0^* are perturbed from the values of 0 and \bar{x}_0^* respectively.

For i = 0, ..., k, denote the *i*th component of the solution of (3.8) with initial condition $(x_0^*, x_1^*, ..., x_k^*)$ by $\phi_i^G(t, \delta, x_0^*)$ and the *i*th component of the vector field by f_i^G . In between the switching times \bar{t}_i , the solution $x_0(t, 0, \bar{x}_0^*)$ evolves according to linear dynamics of the form (3.8). Denote the corresponding graphs by $G_0, G_1, ..., G_m$.

We first argue that the first switching time $t_1(\delta, x_0^*)$ perturbs smoothly. Define $g_1(t, \delta, x_0^*)$ by

$$g_1(t,\delta,x_0^*) = |\phi_0^{G_0}(t,\delta,x_0^*) - \phi_{i_1}^{G_0}(t,\delta,x_0^*)|^2,$$

which is a smooth function of its arguments and by our assumption $g_1(\bar{t}_1, 0, \bar{x}_0^*) = 1$. Then

$$\frac{\partial}{\partial t}g_1(\bar{t}_1, 0, \bar{x}_0^*) = 2(\phi_0^{G_0}(t, 0, \bar{x}_0^*) - \phi_{i_1}^{G_0}(t, 0, \bar{x}_0^*)^T (f_0^{G_0}(\phi^{G_0}(t, 0, \bar{x}_0^*)) - f_{i_1}^{G_0}(\phi^{G_0}(t, 0, \bar{x}_0^*))) \neq 0,$$

because of the fact that the vector fields on either sides of the switching surface at time \bar{t}_1 are transversal to the surface by our assumption of type 1.

By the implicit function theorem, there exist $\epsilon > 0$ and $\delta_0 > 0$ such that for all $x_0^* \in B(\bar{x}_0^*, \epsilon_1)$ and $\delta \in [0, \delta_0)$ the first switching time $t_1(\delta, x_0^*)$ can be uniquely defined as a smooth function of its arguments so that

$$g_1(t_1(\delta, x_0^*), \delta, x_0^*) = 1.$$

Moreover, by Lemma 3.2 on uniform perturbation of trajectories, ϵ , δ_0 can be chosen so that no other switching occurs before the supremum of $t_1(\delta, x_0^*)$ over $\delta \in [0, \delta_0)$ and $x_0^* \in B(\bar{x}_0^*, \epsilon)$. Additionally, since $\phi_i^G(t, \delta, x_0^*)$ as well as the vector field corresponding to G are smooth, we can also conclude that the switching locations perturb smoothly so that δ_0 and ϵ can be chosen so that the unique continuation of the solution beyond the first switching holds for $\delta \in [0, \delta_0)$ and $x_0^* \in B(\bar{x}_0^*, \epsilon)$.

This argument can be continued finitely many times, by shrinking ϵ and δ_0 if needed until one arrives at the resulting ϵ and δ_0 . We note for instance, that for the second switching, one defines the function

$$g_2(t,\delta,x_0^*) = |\phi_0^{G_1}(t-t_1(\delta,x_0^*),\delta,\phi_0^{G_0}(t_1(\delta,x_0^*))) - \phi_{i_2}^{G_1}(t-t_1(\delta,x_0^*),\delta,\phi_0^{G_0}(t_1(\delta,x_0^*)))|^2,$$

and considers the implicit equation $g_2(t, \delta, x_0^*) = 1$.

We remark that, in the proof of Lemma 3.4, the type 1 assumption is necessary. If a type 2 trajectory (zero weighted zero agent with infinitely many switchings) is perturbed, the above proof does not work (since the decreasing infinite sequence of ϵ and δ_0 values may limit to zero), and it is not clear that the perturbed solution will remain unique and/or perturb smoothly.

3.2. The main results. Before we present a necessary condition for robustness, we provide some preliminaries. Recall that, given a nonempty subset $S \subset \{1, 2, \ldots, k\}$, we denote by B_S^* the intersection of all the balls $B(x_i^*, 1)$ where $i \in S$. We define C_S^* , the exclusive mutual confidence ball of S by

$$C_S^* = \{ y \in B_S^* \mid |y - x_i| > 1 \ \forall i \notin S \}.$$

DEFINITION 3.5. Consider an equilibrium $x^* = (x_1^*, x_2^*, \ldots, x_k^*) \in \mathbb{R}^{kd}$ of k clusters. We say that the equilibrium satisfies the shared center of mass condition (SCMC) provided there exists $S \subset \{1, 2, \ldots, k\}$ containing at least two elements such that $m_S^* \in C_S^*$.

It can be shown [5] that the necessary and sufficient condition for robustness in the one dimensional case is equivalent to the requirement that the equilibrium x^* does not satisfy SCMC. Naturally, an interesting question is whether, this also holds in the multidimensional case (d > 1). The next theorem, proves that violation of SCMC is necessary for robustness. In other words, SCMC implies that the equilibrium is not robust.

THEOREM 3.6. Let x^* be a generic equilibrium with k clusters that satisfies the shared center of mass condition. Then, for any set $S \subset \{1, 2, ..., k\}$ with at least two elements such that $m_S^* \in C_S^*$, x^* is not robust with respect to initial zero opinions in some open ball centered at m_S^* . Therefore, x^* is not robust.

Proof. Let $S \subset \{1, 2, ..., k\}$ be a set with at least two elements such that $m_S^* \in C_S^* \subset B_S^*$. Let r < 1 be the maximum of $|m_S^* - x_i^*|$ for $i \in S$. Let $\epsilon > 0$ be such that $\epsilon < 1 - r$ and $B(m_S^*, \epsilon) \subset C_S^*$. Suppose that we introduce the zeroth agent to the system with initial zero opinion $x_0^* \in B(m_S^*, \epsilon)$ and weight $\delta > 0$. Initially, the clusters will obey initial dynamics and hence, by $(3.10) x_0^{\delta}(t)$ will lie on the line segment $[m_S^*, x_0^*] \subset C_S^*$ until the first switching. Moreover, the system has to switch in finite time since the initial dynamics if unswitched will lead $x_j^{\delta}(t) \to m_{S_0}^*$ as $t \to \infty$ for all $j \in S$, where $m_{S_0}^* = (Wm_S^* + \delta x_0^*)/(W + \delta)$ is the center of mass of clusters in S and agent zero $(W = \sum_{i \in S} w_i)$. Also, since the clusters not in S are not moving initially, and the the zero agent will lie on the line segment $[m_S^*, x_0^*]$, the first switching can not involve the zero agent and clusters not in S.

Moreover, since the each cluster $i \in S$ is moving towards $m_{S_0}^*$ until the first switching, at any time $t \ge 0$,

$$|x_i^{\delta}(t) - m_{S_0}^*| \le |x_i^* - m_{S_0}^*| \le y + r, \; \forall i \in S$$

where $y = |m_s^* - m_{s_0}^*|$ as it is illustrated in Figure 3.4. Note that $x_0^* \in B(m_S^*, \epsilon)$ and one can choose $\delta \in [0, \delta_0]$ so that $m_{S_0}^*$ is closer to m_S^* , in other words, $y < \frac{\epsilon}{2}$. Then,



FIG. 3.4. A Diagram

$$|x_i^{\delta}(t) - x_0^*| < \epsilon - y + |x_i^{\delta}(t) - m_{S_0}^*| < \epsilon - y + y + r = \epsilon + r < 1$$

$$|x_i^{\delta}(t) - m_S^*| < y + |x_i^{\delta}(t) - m_{S_0}^*| < y + y + r < \epsilon + r < 1, \quad \forall i \in S, \; \forall t \ge 0.$$

Thus, the distance from $x_i^{\delta}(t)$ to the endpoints of the line segment $[m_S^*, x_0^*]$ is less than 1 for all $t \ge 0$. Hence, we conclude that the distance from $x_i^{\delta}(t)$ to the line segment $[m_S^*, x_0^*]$ is also less than 1 for all $t \ge 0$. Therefore, the first switching cannot involve the zero agent and a cluster in S. Then the first switching has to involve one cluster $i \in S$ and (at least) another cluster $j \in \{1, 2, \ldots, k\}$, so that at the first switching time t_1^{δ} , we have that $|x_i^{\delta}(t_1^{\delta}) - x_i^{\delta}(t_1^{\delta})| = 1$. Then, we may write

$$\begin{split} 1 < |x_i^* - x_j^*| &\leq |x_i^{\delta}(t_1^{\delta}) - x_i^*| + |x_j^{\delta}(t_1^{\delta}) - x_j^*| + |x_i^{\delta}(t_1^{\delta}) - x_j^{\delta}(t_1^{\delta})| \\ \Rightarrow \quad |x_i^{\delta}(t_1^{\delta}) - x_i^*| + |x_j^{\delta}(t_1^{\delta}) - x_j^*| \geq |x_i^* - x_j^*| - 1 > 0 \\ \Rightarrow \quad \Delta(x_0^*, \delta; x^*) &= \sup_{t \geq 0, i} (|x_i^{\delta}(t) - x_i^*|) \geq \frac{1}{2} (|x_i^* - x_j^*| - 1) > 0 \\ \Rightarrow \quad \lim_{\delta \to 0} \Delta(x_0^*, \delta; x^*) \geq \frac{1}{2} (|x_i^* - x_j^*| - 1) > 0, \end{split}$$

and hence the equilibrium x^* is not robust with respect to x_0^* . We note that the solution may not be unique after the first switching at time t_1^{δ} , a fact that does not affect the above conclusion. Since $B(m_S^*, \epsilon)$ has strictly positive Lebesgue measure, the final conclusion follows. \Box

Unfortunately, it is not clear to us if the negation of SCMC (let us call it non-SCMC) is sufficient for almost sure uniform robustness. The difficulty in showing sufficiency arises from non-uniqueness of solutions as well as type 2 trajectories discussed earlier.

Before we give a set of sufficient conditions that guarantee almost sure and uniform robustness, we state a simple geometric lemma. We note that by a unit sphere with center $x_0 \in \mathbb{R}^d$, we mean the set $\{x \in \mathbb{R}^d \mid |x - x_0| = 1\}$, and by the *radius* of a sphere, we shall mean any closed line segment joining the center x_0 with a point on the sphere.

LEMMA 3.7. Let $x_1, x_2 \in \mathbb{R}^d$ and $|x_1 - x_2| > 1$. Let S_1, S_2 be unit spheres centered at x_1, x_2 , respectively. Any radius of S_1 intersects S_2 at most once if and only if $|x_1 - x_2| \ge \sqrt{2}$.

Proof.

Without loss of generality we can assume that $x_1 = 0$. Hence $|x_2| > 1$. Equation of radial lines of S_1 can be written as $x(t) = \lambda t$ where $\lambda \in \mathbb{R}^d$ is a unit norm vector and $t \in [0, \infty)$. When $0 \le t \le 1$, x(t) represents the radius of S_1 corresponding to λ . This radius intersects S_2 if and only if there exists $t \in [0, 1]$ such that $|\lambda t - x_2|^2 = 1$ or equivalently $t^2 - 2(\lambda^T x_2)t + |x_2|^2 - 1 = 0$. Solving this quadratic equation for t values, one can easily obtain $t_{\pm} = \lambda^T x_2 \pm \sqrt{(\lambda^T x_2)^2 - |x_2|^2 + 1}$. First, suppose that $|x_2| \ge \sqrt{2}$. If t_+, t_- are not real, then there is no intersection. If these are real, then



FIG. 3.5. A radius of S_1 interets S_2 twice.

since $t_+t_- = |x_2|^2 - 1 \ge 1$, either t_+, t_- are both negative (implying no intersection), $t_+ = t_- = 1$ (implying exactly one intersection), $0 < t_- < 1 < t_+$ (implying exactly one intersection) or $t_+ \ge t_- > 1$ implying no intersection.

Now, suppose that $1 < |x_2| < \sqrt{2}$. We can choose a unit vector λ such that $\lambda^T x_2$ lies in any desired nonempty subinterval of $[-|x_2|, |x_2|]$. In particular, we may choose

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 λ such that $0 < \lambda^T x_2 < |x_2|/\sqrt{2} < 1$ and $(\lambda^T x_2)^2 > |x_2|^2 - 1$. Since $|x_2| < \sqrt{2}$, we have that $|x_2|^2/2 > |x_2|^2 - 1$ and hence the above choice is feasible. Then we have

$$0 \le (\lambda^T x_2)^2 - |x_2|^2 + 1 < (\lambda^T x_2)^2 - 2(\lambda^T x_2) + 1 = (1 - (\lambda^T x_2))^2.$$

Thus, $t_{+} = \lambda^{T} x_{2} + \sqrt{(\lambda^{T} x_{2})^{2} - |x_{2}|^{2} + 1} < \lambda^{T} x_{2} + 1 - (\lambda^{T} x_{2}) = 1$. Moreover, since $t_{+}t_{-} = |x_{2}|^{2} - 1 > 0$, we have that $0 < t_{-} < t_{+} < 1$, showing the existence of a radius of S_{1} that intersects S_{2} twice. \Box

THEOREM 3.8. Let x^* be a generic equilibrium with k number of clusters that does not satisfy the shared center of mass condition. Furthermore, suppose that no three distinct closed balls \overline{B}_i have a nontrivial intersection and that for any $i \neq j$, we have that either $|m_{ij}^* - x_i^*| > \sqrt{2}$ or $|m_{ij}^* - x_j^*| > \sqrt{2}$ where $m_{ij}^* = \frac{w_i x_i^* + w_j x_j^*}{w_i + w_j}$. Then, the equilibrium x^* is robust almost surely and uniformly.

We note that the key ideas of the proof of Theorem 3.8 are as follows. First ensure that only one switching occurs with unique continuation of trajectories for almost all initial zero opinions. Secondly, ensure that the first switching time is uniformly bounded, so that part (i) of Lemma 3.2 is used prior to switching (finite time interval) and part (ii) (infinite time interval) is used after the switching. The condition that $|m_{ij}^* - x_i^*| > \sqrt{2}$ or $|m_{ij}^* - x_j^*| > \sqrt{2}$ for all pairs $i \neq j$ was enforced to ensure uniqueness of solutions beyond the first switching. If the less restrictive condition of $|x_i^* - x_j^*| > \sqrt{2}$ for all pairs $i \neq j$ is used, more detailed analysis is needed to either establish uniqueness or investigate the nature of the resulting multiple solutions involving unstable sliding modes.

Proof. We only need to consider initial zero opinions x_0^* that lie in the union $\cup_{i=1}^k B_i$, but not on any of the spheres $|x_0 - x_i^*|^2 = 1$. (The latter are a set of measure zero, and initial zero opinions outside the union $\cup_{i=1}^k \overline{B}_i$ do not perturb the equilibrium.) We also note that under the assumptions of the theorem, $|x_i^* - x_j^*| > \sqrt{2}$ for $i \neq j$.

As the initial zero opinion x_0^* can be in at most two of the balls B_i , there are only two different scenarios. In Scenario 1, without of loss of generality, $x_0^* \in B_1^*$ and $x_0^* \notin B_i^*$ for $i \neq 1$. Initially the dynamics will be given by

(3.12)
$$\begin{aligned} \dot{x}_{0}^{\delta} &= w_{1}(x_{1}^{\delta} - x_{0}^{\delta}), \\ \dot{x}_{1}^{\delta} &= \delta(x_{0}^{\delta} - x_{1}^{\delta}), \\ \dot{x}_{j}^{\delta} &= 0, \quad j = 2, 3, \dots, k, \end{aligned}$$

with initial conditions $x_0^{\delta}(0) = x_0^*$, $x_j^{\delta}(0) = x_j^*$, $\forall j = 1, 2, \ldots, k$. As long as the system follows (3.12), by Lemma 3.1, regardless of $\delta > 0$, $x_0^{\delta}(t)$ and $x_1^{\delta}(t)$ will lie on the line segments $[x_0^*, m_{01}^*)$ and $(m_{01}^*, x_1^*]$ respectively where $m_{01}^* = (\delta x_0^* + w_1 x_1^*)/(\delta + w_1)$, and moreover $x_j^{\delta}(t)$ remain equal to x_j^* for $j \ge 2$.

In fact, for all $\delta > 0$, $x_j^{\delta}(t)$, $j = 0, 1, 2, \ldots, k$ will obey the system (3.12) for all times $t \ge 0$, i.e. no switching happens. To see this, first note that since $x_0^* \in B_1^* \setminus \bigcup_{j \ne 1} \overline{B_j^*}$ and $x_1^* \in B_1^* \setminus \bigcup_{j \ne 1} \overline{B_j^*}$, the fact that intercluster distances are greater than $\sqrt{2}$, together with Lemma 3.7 implies that the line segment $[x_0^*, x_1^*] \cap \overline{B_j^*}$ is an empty set for each $j = 2, \ldots, k$. The dynamics will change from (3.12) only if $x_0^{\delta}(t)$ or $x_1^{\delta}(t)$ enter B_j^* for some $j = 2, \ldots, k$. Since $x_0^{\delta}(t)$ and $x_1^{\delta}(t)$ remain on the line segments $[x_0^*, m_{01}^*)$ and $(m_{01}^*, x_1^*]$ respectively until such a change in the dynamics, this can only happen if $[x_0^*, x_1^*]$ intersects $\overline{B_j^*}$ for some $j = 2, \ldots, k$, which is not possible. Therefore the system will follow the dynamics (3.12) for all times $t \ge 0$. By Lemma 3.2, part (iii) we can conclude that

$$\lim_{\delta \to 0} \sup_{t \ge 0, \ x_0^* \in K_1^*} |x_1^{\delta}(t, x^*) - x_1^*| = 0,$$

where $K_1^* = B_1^* \setminus \bigcup_{j \neq 1} \overline{B}_j^*$ and $x^* = (x_1^*, \ldots, x_k^*)$. It is clear that if $\mathcal{R}_2 \subset \mathbb{R}^d$ denotes the region of x_0^* values defining Scenario 1, then \mathcal{R}_2 is simply the finite union of K_1^*, \ldots, K_k^* which are defined similar to K_1^* , and thus the closure of \mathcal{R}_2 is compact. Thus by Lemma 3.2 (part (iii))



FIG. 3.6. An illustration of Scenario 1 in the proof of Theorem 3.8.

In the alternative, Scenario 2, $\exists i, j \ (i \neq j)$ such that $x_0^* \in B_{ij}^*$. WLOG we will assume that $x_0^* \in B_{12}^*$ and $|m_{12}^* - x_2^*| > \sqrt{2}$. The dynamics of the motion initially will be as follows:

$$\begin{aligned} \dot{x}_{0}^{\delta} &= w_{1}(x_{1}^{\delta} - x_{0}^{\delta}) + w_{2}(x_{2}^{\delta} - x_{0}^{\delta}) \\ \dot{x}_{1}^{\delta} &= \delta(x_{0}^{\delta} - x_{1}^{\delta}), \\ \dot{x}_{2}^{\delta} &= \delta(x_{0}^{\delta} - x_{2}^{\delta}), \\ (3.13) \qquad \dot{x}_{j}^{\delta} &= 0, \quad j = 3, 4, \dots, k, \end{aligned}$$

with initial conditions $x_0^{\delta}(0) = x_0^*$, $x_j^{\delta}(0) = x_j^*$, j = 1, 2, ..., k. We shall argue that the system has to switch and that there exists $\delta_0 > 0$ such that the first switching time $T^{\delta}(x_0^*)$ is uniformly bounded above by some $T_0 > 0$ independent of $\delta \in [0, \delta_0]$ and $x_0^* \in B_{12}^*$. To see this, let $\rho > 0$ be such that $\overline{B}(m_{12}^*, \rho) \subset B_1^* \setminus \bigcup_{j \neq 1} B_j^*$. Following Lemma 3.2 part (ii), for the dynamics (3.13) and $x_0^* \in B_{12}^*$ one can find $\delta_0 > 0$ such that $\forall \delta \in [0, \delta_0)$,

$$\sup_{t\in[0,\infty],\ x_0^*\in B_{12}^*}|x_0^{\delta}(t,x^*)-x_0^0(t,x^*)|<\rho/2.$$

Additionally, by Lemma 3.1, with $\delta = 0$, we get

$$|x_0^0(t,x^*) - m_{12}^*|^2 \le e^{-2(w_1 + w_2)t} |x_0^* - m_{12}^*|^2, \quad \forall x_0^* \in B_{12}^*.$$

Note that $|x_0^* - m_{12}^*|^2 < 2, \forall x_0^* \in B_{12}^*$. Thus, $\exists T_0 > 0$ (independent of δ), such that $\forall t \geq T_0$, and $\forall x_0^* \in B_{12}^*, |x_0^0(t, x^*) - m_{12}^*| < \rho/2$. Thus, $\forall \delta \in [0, \delta_0)$,

$$\sup_{t, x_0^*} |x_0^{\delta}(t, x^*) - m_{12}^*| \le \sup_{t, x_0^*} |x_0^{\delta}(t, x^*) - x_0^{0}(t, x^*)| + \sup_{t, x_0^*} |x_0^{0}(t, x^*) - m_{12}^*| < \rho,$$

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where supremum is taken over $t \in [T_0, \infty]$, and $x_0^* \in B_{12}^*$. Thus for $t \ge T_0$, $\delta \in [0, \delta_0)$ and $x_0^* \in B_{12}^*$ we have that $x_0^{\delta}(t, x_0^*) \in B(m_{12}^*, \rho)$.

Now, by Lemma 3.2 part (i) we have that

$$\lim_{\delta \to 0^+} \sup_{t \in [0, T_0], x_0^* \in B_{12}^*} |x_j^{\delta}(t, x_0^*) - x_j^*| = 0,$$

for j = 1, 2 and hence taking δ_0 smaller if necessary, we can conclude that for $t \in [0, T_0]$, all the assumptions of the theorem hold with $x_1^*, x_2^*, m_{12}^*, B_1^*$ and B_2^* replaced by $x_1^{\delta}(t), x_2^{\delta}(t), m_{12}^{\delta}(t), B(x_1^{\delta}(t), 1)$ and $B(x_2^{\delta}(t), 1)$ respectively, where $m_{12}^{\delta}(t) = (w_1 x_1^{\delta}(t) + w_2 x_2^{\delta}(t))/(w_1 + w_2)$.

Also from the observation $x_0^{\delta}(t, x_0^*) \in B(m_{12}^*, \rho)$ (for $t \geq T_0$), we can conclude that if no switching occurs, then agent zero has to leave the ball $B(x_2^{\delta}(t), 1)$ some time during $[0, T_0]$, which involves switching, leading to a contradiction. Thus we can conclude that a switching occurs during $[0, T_0]$ and that the only possible switching must involve one or both of the spheres $|x_0 - x_1|^2 = 1$ and $|x_0 - x_2|^2 = 1$.

We shall argue further, that, as long as the initial dynamics (3.13) persists, the zero agent cannot leave the ball $B(x_1^{\delta}(t), 1)$, showing that the first switching occurs on the sphere $|x_0 - x_2|^2 = 1$. In fact, after some algebra, one obtains that

$$\frac{d}{dt}|x_1^{\delta} - x_0^{\delta}|^2 = -2\delta|x_1^{\delta} - x_0^{\delta}|^2 - 2(w_1 + w_2)(x_1^{\delta} - x_0^{\delta})^T(m_{12}^{\delta} - x_0^{\delta}).$$

Also, from the geometry of Figure 3.7, we note that for

$$\inf_{x_0 \in B_1^* \cap B_2^*} (x_1^* - x_0)^T (m_{12}^* - x_0) > 0$$

By Lemma 3.2 part (i), by shrinking $\delta_0 > 0$ if necessary to keep $m_{12}^{\delta}(t)$ and $x_1^{\delta}(t)$ sufficiently close to m_{12}^* and x_1^* respectively (for $t \in [0, T_0]$), we can conclude that

$$(x_1^{\delta}(t) - x_0^{\delta}(t))^T (m_{12}^{\delta}(t) - x_0^{\delta}(t)) > 0, \quad \forall \delta \in [0, \delta_0) \text{ and } t \in [0, T_0].$$



FIG. 3.7. Shows $(x_1^* - x_0^{\delta})^T (m_{12}^* - x_0^{\delta}) > 0.$

This shows that the distance between $x_0^{\delta}(t)$ and $x_1^{\delta}(t)$ is decreasing until the first switching, and hence the zero agent cannot leave $B(x_1^{\delta}(t), 1)$. Hence we conclude that the first switching occurs at some time $T^{\delta}(x_0^*) < T_0$ (for $\delta \in [0, \delta_0)$ and $x_0^* \in B_{12}^*$) when the dynamics enters the switching surface $|x_0 - x_2|^2 = 1$.

Now, we argue that the solution has a unique continuation beyond the first switching time $T^{\delta}(x_0^*)$ in which the zero agent is only in the ball $B(x_1^{\delta}(t), 1)$. To see this, we must examine the vector fields on either sides of the switching surface $|x_0 - x_2|^2 = 1$, and these correspond to the dynamics (3.13) and (3.12). We first compute the time derivative

$$\begin{aligned} \frac{d}{dt} |x_0^{\delta}(t) - x_2^{\delta}(t)|^2 &= 2(x_0^{\delta}(t) - x_2^{\delta}(t))^T (\dot{x}_0^{\delta}(t) - \dot{x}_2^{\delta}(t)) \\ &= 2(w_1 + w_2)(x_0^{\delta}(t) - x_2^{\delta}(t))^T (m_{12}^{\delta}(t) - x_0^{\delta}(t)) - 2\delta |x_0^{\delta}(t) - x_2^{\delta}(t)|^2, \end{aligned}$$

which holds under the dynamics (3.13). We shall show that this is strictly positive for all sufficiently small δ at the time $T^{\delta}(x_0^*)$ of switching, noting that $|x_0^{\delta}(T^{\delta}(x_0^*)) - x_2^{\delta}(T^{\delta}(x_0^*))| = 1$. First we observe that $|x_0^{\delta}(T^{\delta}(x_0^*)) - x_1^{\delta}(T^{\delta}(x_0^*))| < 1$, and from the geometry of Figure 3.8 that $|m_{12}^* - x_0^{\delta}(T^{\delta}(x_0^*))| < |x_1^* - x_0^{\delta}(T^{\delta}(x_0^*))|$. Let $\gamma = (|m_{12}^* - x_2^*|^2 - 2)/4$ which is strictly positive due to the assumptions of the theorem. Using part (i) of Lemma 3.2 and the triangle inequality, by shrinking δ_0 if needed, we can assume that $x_1^* = x_1^* + x_2^* + x_2^* + x_2^*$

$$|x_0^{\delta}(T^{\delta}(x_0^*)) - x_1^*|^2 \le 1 + 2\gamma, \quad \forall x_0^* \in B_{12}^*.$$
 We have

FIG. 3.8. Since
$$|m_{12}^* - x_2^*| > \sqrt{2} > 1$$
,
have $d_1 > d$.

Hence, we obtain

$$\begin{aligned} & (x_0^{\delta}(T^{\delta}(x_0^*)) - x_2^*)^T (m_{12}^* - x_0^{\delta}(T^{\delta}(x_0^*))) \\ &= \frac{1}{2} |m_{12}^* - x_2^*|^2 - \frac{1}{2} |x_0^{\delta}(T^{\delta}(x_0^*)) - x_2^*|^2 - \frac{1}{2} |m_{12}^* - x_0^{\delta}(T^{\delta}(x_0^*))|^2 \\ &= \frac{1}{2} |m_{12}^* - x_2^*|^2 - \frac{1}{2} - \frac{1}{2} |m_{12}^* - x_0^{\delta}(T^{\delta}(x_0^*))|^2 \ge \gamma > 0, \end{aligned}$$

which holds for all $\delta \in [0, \delta_0)$ and $x_0 \in B_{12}^*$. By shrinking δ_0 further if necessary, we conclude that the derivative $\frac{d}{dt}|x_0^{\delta}(t) - x_2^{\delta}(t)|^2$ is strictly positive for all $\delta \in [0, \delta_0)$ and $x_0^* \in B_{12}^*$ when $t = T^{\delta}(x_0)$. This ensures unique continuation (see [13, 17, 12] for instance). To see that the unique continuation immediately enters the open set

$$\{x \in \mathbb{R}^{(k+1)d} \mid |x_0 - x_1| < 1, \text{ and } |x_i - x_j| > 1 \text{ for all other pairs } i \neq j\}$$

we compute the derivative $\frac{d}{dt}|x_0^{\delta}(t) - x_2^{\delta}(t)|^2 = 2w_1(x_0^{\delta}(t) - x_2^{\delta}(t))^T(x_1^{\delta}(t) - x_0^{\delta}(t))$, which holds under the dynamics (3.12). Following similar reasoning as above and using the fact that $|x_1^* - x_2^*|^2 > 2$, the result follows.

Thus the dynamics will switch to that of Scenario 2, at time $T^{\delta}(x_0^*) < T_0$ and thus follow (3.12) with perturbed initial conditions x_j^{**} for j = 0, 1 and 2. Moreover, as argued earlier, all the conditions of the theorem are still met for the perturbed initial conditions x^{**} . The rest of the proof follows similar to Scenario 1. We conclude that x^* is almost surely and uniformly robust. \Box



FIG. 3.9. An illustration of Scenario 2 in the proof of Theorem 3.8.

In order to extend Theorem 3.8 to obtain less restrictive sufficient conditions, one may look for cases which ensure finitely many switchings with uniqueness of solutions for almost all initial opinions. In this context, our definition and discussion of type 1 and type 2 initial zero opinions as well as Lemma 3.4 is useful. However, this lemma applies only locally in a neighborhood of a type 1 initial zero opinion. Thus, taking supremum over all initial opinions might be challenging. As type 1 initial zero opinions form an open set, a key question is the nature of the complement of this set which includes type 2 initial zero opinions. If one can show that this complement has measure zero, then it may help expand Theorem 3.8.

When we examine Theorem 3.8 in the one dimensional case where necessary and sufficient conditions are known, we note that the condition that no three distinct confidence balls have a nontrivial intersection is automatically satisfied in 1D. However, the condition that $|m_{ij}^* - x_i^*| > \sqrt{2}$ or $|m_{ij}^* - x_j^*| > \sqrt{2}$ for all $i \neq j$ or even the less restrictive condition that $|x_i^* - x_j^*| > \sqrt{2}$ for all pairs $i \neq j$ is not necessary in 1D. In fact it must be noted that the $\sqrt{2}$ condition was imposed to avoid multiple switchings in Theorem 3.8. Likewise the condition that no three distinct confidence balls have a nontrivial intersection was also imposed to avoid multiple switchings. Therefore, we feel that the conditions of Theorem 3.8 may be far from being necessary even in multiple dimensions. On the other hand, there is no reason to expect that the non-shared center of mass condition, to be not sufficient in multiple dimensions.

4. Numerical Results. In this section, we present some numerical results for the opinion dynamic model (2.1). Our simulations represent the cases for which agents have vector opinions in \mathbb{R}^2 (d = 2). We use a uniform weight $w_i = 1, \forall i = 1, 2, ..., n$ and a uniform confidence bound $q = q_{ij} = 1, \forall i, j = 1, 2, ..., N$. We take the interaction functions ξ_{ij} to be indicator functions of [0, 1). We use MATLAB ode45 for the solution of the ODEs.



FIG. 4.1. (a) Time evolution of 400 agent opinions $(x^1, x^2) \in \mathbb{R}^2$. Initial opinions are independently and uniformly distributed on a 10×10 square region. (b) Distribution of the nearest cluster distance for different population sizes.

Figure 4.1(a) shows the evolution of 400 agent opinion vectors in \mathbb{R}^2 . Each curve in Figure 4.1(a) represents the trajectory of an agent's opinion and star markings represent the limiting opinions (clusters). Figure 4.1(a) explicitly shows the evolution and the convergence of the trajectories to an equilibrium with multiple clusters.

TABLE 4.1

Some main properties of equilibria obtained from experiments with 400,800 and 1600 agents with randomly chosen vector opinions in \mathbb{R}^2 . Columns represent the following. A: Number of agents, B: Number of experiments, C: Number of clusters, D: Number of pairs satisfying pairwise SCMC, E: Diameter of cluster spread, F:Median distance to the nearest cluster, G: Spread of nearest cluster distance.

Α	В	С	D	E	F	G
400	40	24.6 ± 0.5	3.2 ± 0.3	11.93 ± 0.186	1.7120	0.9714
800	40	26.4 ± 0.6	2.6 ± 0.4	11.86 ± 0.166	1.6307	0.9303
1600	10	28.8 ± 1.5	4 ± 0.9	11.63 ± 0.401	1.5216	0.8274

We also investigated the nature of the equilibria that result from starting with n agents with randomly chosen initial conditions that are uniformly and independently distributed on a 10×10 square region. With n = 400,800 and 1600 number of agents, we performed numerical experiments multiple times and the results are presented in Table 4.1. In columns C and D we have shown the means and the 95% confidence intervals for the number of clusters and the number of pairs that satisfy the pairwise SCMC (to be precise, the number of two element subsets $S \subset \{1, \ldots, n\}$ such that $m_S^* \in C_S^*$), respectively. Column E shows the mean and the 95% confidence intervals of the "diameter of the cluster spread" defined by the largest intercluster distance. Columns F and G, for each value of n, consider the distance to the nearest neighbor cluster for agents from the combined statistics of the experiments. Column F shows the median (50th percentile) value and column G shows the "spread" as measured by the difference between the 95th and 5th percentiles.

We notice some trends as n increases. The number of pairs satisfying the SCMC shows a decrease going from n = 400 to n = 800, although it increases again for n = 1600. It was also observed that the probability of a non-robust set of limiting

equilibrium clusters is close to 1, which seems to contradict the conjecture by Blondel et al. [5] mentioned in our introduction. It has been observed numerically in the one dimensional case that as the agent number n becomes large, the resulting equilibria become robust with a probability approaching 1 [5, 20], in agreement with the conjecture. We feel that in our two dimensional case, the number of agents n is not large enough to see a result that is in agreement with this conjecture. In fact, columns C and D show that the trends are not convergent for n = 1600. We suspect that if n = 1000 showed agreement with the conjecture in one dimension, then $n = 1000^2$ number of agents will be required to see a similar level of agreement in two dimensions. This would mean much larger values of n must be used to verify the conjecture in two dimensions, which will require a faster C language implementation of the ODE solvers.

In Figure 4.1(b) we have plotted the distance to the nearest neighbor cluster in the vertical axis and the fraction of agents with a distance less than or equal to that value in the horizontal axis. The curves in Figure 4.1(b) as well as columns F and G of Table 4.1 show a trend where both the median and the spread of the distance to the nearest neighbor cluster is decreasing. It is also clear that the trend is not convergent for n = 1600. An interesting question is whether in the limiting case (as $n \to \infty$) the curve becomes nearly flat, and if so, how does the corresponding value relate to $\sqrt{2}$?

5. Concluding remarks. We analyzed the opinion dynamic model (2.1) for a general class of interaction functions ξ_{ij} . Even if we consider the more general model where for each component ℓ of the opinions of agents has different interaction functions ξ_{ij}^{ℓ} as given by (2.16), many of our results of §2 are still valid. The robustness analysis of §3 however, will be more complicated.

When $\xi_{ij} = 1_{[0,1)}$, the one dimensional necessary and sufficient condition [5] coincides with the non-SCMC (negation of SCMC) which is necessary in higher dimensions as shown by our Theorem 3.6. The open question is whether it is also sufficient? In [20] a necessary and sufficient condition without rigorous proof is described in terms of invariant sets of the dynamics of the zero agent with weight $\delta = 0$. In order to rigorously demonstrate that conclusions made by studying $\delta = 0$ carry over to the limiting case $\delta \to 0+$, one requires uniform perturbation of solutions on the infinite time interval. Our Lemma 3.2 shows that when the dynamics is linear, the uniform perturbation on infinite time interval holds only if the zero agent is interacting with only one other agent. This key lemma was used in Theorem 3.8 to obtain a sufficient condition for robustness. The discussion below this theorem outlines possible strategies for extending it. When the interaction functions are indicators but with different confidence bounds, the dynamics is still linear in between switchings and we expect that both Theorem 3.6 and 3.8 could be generalized. If one considers more general ξ_{ij} as in §2, then we believe that the uniform perturbation results in Lemma 3.2 can be generalized. However, with general ξ_{ij} , the straight line trajectories of Lemma 3.1 will change, complicating the analysis.

The notion of almost sure uniform robustness we introduced in §3 while desirable, may be more difficult to establish than the notion of robustness with respect to a specific initial zero opinion x_0^* , which is also introduced in §3. We also believe that it is most natural to consider Filippov solutions (instead of Caratheodory solutions or Proper solutions) as we have done, especially in light of the fact that in [11] the existence of sliding mode solutions for the opinion dynamics is shown. This, however, necessitates a thorough understanding of sliding mode solutions (stable and unstable) in order to undertake a complete study of robustness.

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