

Exponentially Small Splitting of Separatrices and Transversality Associated to Whiskered Tori with Quadratic Frequency Ratio*

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Abstract. The splitting of invariant manifolds of whiskered (hyperbolic) tori with two frequencies in a nearly integrable Hamiltonian system, whose hyperbolic part is given by a pendulum, is studied. We consider a torus with a fast frequency vector $\omega/\sqrt{\varepsilon}$, with $\omega = (1, \Omega)$, where the frequency ratio Ω is a quadratic irrational number. Applying the Poincaré–Melnikov method, we carry out a careful study of the dominant harmonics of the Melnikov potential. This allows us to provide an asymptotic estimate for the maximal splitting distance and show the existence of transverse homoclinic orbits to the whiskered tori with an asymptotic estimate for the transversality of the splitting. Both estimates are exponentially small in ε , with the functions in the exponents being periodic with respect to $\ln \varepsilon$, and can be explicitly constructed from the continued fraction of Ω . In this way, we emphasize the strong dependence of our results on the arithmetic properties of Ω . In particular, for quadratic ratios Ω with a 1-periodic or 2-periodic continued fraction (called metallic and metallic-colored ratios, respectively), we provide accurate upper and lower bounds for the splitting. The estimate for the maximal splitting distance is valid for all sufficiently small values of ε , and the transversality can be established for a majority of values of ε , excluding small intervals around some transition values where changes in the dominance of the harmonics take place, and bifurcations could occur.

Key words. splitting of separatrices, transverse homoclinic orbits, Melnikov integrals, quadratic frequency ratio

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1. Introduction and setup.

1.1. Background and state of the art. This paper is dedicated to the study of the exponentially small splitting of separatrices in a perturbed 3-degree-of-freedom Hamiltonian system, associated to a 2-dimensional whiskered torus (invariant hyperbolic torus) whose frequency ratio is an arbitrary *quadratic irrational number* (i.e., a real root of a quadratic polynomial with integer coefficients).

We start with an integrable Hamiltonian H_0 having whiskered (hyperbolic) tori with coincident stable and unstable whiskers (invariant manifolds). We focus our attention on a

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torus, with a frequency vector of *fast frequencies*,

$$(1) \quad \omega_\varepsilon = \frac{\omega}{\sqrt{\varepsilon}}, \quad \omega = (1, \Omega),$$

whose *frequency ratio* Ω is a quadratic irrational number. If we consider a perturbed Hamiltonian $H = H_0 + \mu H_1$, where μ is small, in general the whiskers no longer coincide. This phenomenon has become known as *splitting of separatrices* and is related to the nonintegrability of the system and the existence of chaotic dynamics. If we assume, for the two involved parameters, a relation of the form $\mu = \varepsilon^r$ for some $r > 0$, we have a problem of singular perturbation, and in this case the splitting is *exponentially small* with respect to ε . Our aim is to detect homoclinic orbits (i.e., intersections between the stable and unstable manifolds) associated to persistent whiskered tori, provide an *asymptotic estimate* for both the *maximal splitting distance* and its *transversality*, and show the dependence of such estimates on the *arithmetic properties* of the frequency ratio Ω .

To measure the splitting, it is very usual to apply the *Poincaré–Melnikov method*, introduced by Poincaré in [34] and rediscovered much later by Melnikov [32] and Arnold [1]. By considering a transverse section to the stable and unstable perturbed whiskers, one can consider a function $\mathcal{M}(\theta)$, $\theta \in \mathbb{T}^2$, usually called a *splitting function*, giving the vector distance, with values in \mathbb{R}^2 , between the whiskers on this section along the complementary directions. The method provides a first order approximation to this function, with respect to the parameter μ , given by the (vector) *Melnikov function* $M(\theta)$, defined by an integral (see, for instance, [44]). We have

$$(2) \quad \mathcal{M}(\theta) = \mu M(\theta) + \mathcal{O}(\mu^2),$$

and hence for μ small enough the simple zeros of $M(\theta)$ give rise to simple zeros of $\mathcal{M}(\theta)$, i.e., to transverse intersections between the perturbed whiskers. In this way, we can obtain asymptotic estimates both for the maximal splitting distance as the maximum of the function $|\mathcal{M}(\theta)|$ and for the transversality of the splitting, which can be measured by the minimal eigenvalue (in modulus) of the (2×2) matrix $D\mathcal{M}(\theta^*)$ for any given zero θ^* .

An important fact, related to the Hamiltonian character of the system, is that both functions $\mathcal{M}(\theta)$ and $M(\theta)$ are gradients of scalar functions [16, 10]:

$$(3) \quad \mathcal{M}(\theta) = \nabla \mathcal{L}(\theta), \quad M(\theta) = \nabla L(\theta).$$

Such scalar functions are called *splitting potential* and *Melnikov potential*, respectively. This means that there always exist homoclinic orbits, which correspond to critical points of \mathcal{L} .

As said before, the case of fast frequencies ω_ε as in (1), with a perturbation of order $\mu = \varepsilon^r$ for a given r as small as possible, turns out to be a *singular problem*. The difficulty comes from the fact that the Melnikov function $M(\theta)$ is exponentially small in ε , and the Poincaré–Melnikov method can be directly applied only if one assumes that μ is exponentially small with respect to ε . In order to validate the method in the case $\mu = \varepsilon^r$, one has to ensure that the error term is also exponentially small, and the Poincaré–Melnikov approximation dominates it. To overcome such a difficulty in the study of the exponentially small splitting, Lazutkin [28] introduced the use of parameterizations of a complex strip of the whiskers (whose width is

defined by the singularities of the unperturbed ones) by periodic analytic functions, together with flow-box coordinates. This tool was initially developed for the Chirikov standard map [28] and allowed several authors to validate the Poincaré–Melnikov method for Hamiltonians with one and a half degrees of freedom (with 1 frequency) [25, 39, 14, 15, 21] and for area-preserving maps [13].

Later, those methods were extended to the case of whiskered tori with 2 frequencies. In this case, the arithmetic properties of the frequencies play an important role in the exponentially small asymptotic estimates of the splitting function, due to the presence of *small divisors*. This was first mentioned in [29], later detected in [41], and rigorously proved in [3] for the quasi-periodically forced pendulum, assuming a polynomial perturbation in the coordinates associated to the pendulum. Recently, a more general (meromorphic) perturbation has been considered in [24]. It is worth mentioning that, in some cases, the Poincaré–Melnikov method does not predict correctly the size of the splitting, as shown in [2], where a Hamilton–Jacobi method is instead used. This method was previously used in [38, 30, 37], where exponentially small estimates for the transversality of the splitting were obtained, excluding some intervals of the perturbation parameter ε . Similar results were obtained in [9, 8]. Moreover, the *continuation* of the transversality for all sufficiently small values of ε was shown in [9] for the concrete case of the famous *golden ratio* $\Omega = (\sqrt{5} - 1)/2$, and in [6] for the case of the *silver ratio* $\Omega = \sqrt{2} - 1$, provided certain conditions on the phases of the perturbation are fulfilled. Otherwise, homoclinic bifurcations can occur, as studied, for instance, in [42] for the Arnold example. Let us also mention that analogous estimates could also be obtained from a careful averaging out of the fast angular variables [35], at least concerning sharp upper bounds of the splitting.

In general, in the quoted papers the frequency ratio is assumed to be a given concrete quadratic number (golden, silver). A generalization to some other concrete quadratic frequency ratios was considered in [8], extending the asymptotic estimates for the maximal splitting distance, but without a satisfactory result concerning transversality. Recently, a parallel study of the cases of 2 and 3 frequencies has been considered in [4] (in the case of 3 frequencies, with a frequency vector $\omega = (1, \Omega, \Omega^2)$, where Ω is a concrete cubic irrational number), obtaining also exponentially small estimates for the maximal splitting distance.

In this paper, we consider a 2-dimensional torus whose frequency ratio Ω in (1) is a given quadratic irrational number. Our main objective is to develop a methodology, allowing us to obtain asymptotic estimates for both the maximal splitting distance and the transversality of the splitting, whose dependence on ε is described by two *piecewise-smooth functions* denoted $h_1(\varepsilon)$ and $h_2(\varepsilon)$, respectively (see Theorem 1). Such functions are periodic with respect to $\ln \varepsilon$, and their behavior depends strongly on the arithmetic properties of the frequency ratio Ω . In particular, we show that the functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$ can be constructed explicitly from the *continued fraction* of Ω , and we can deduce some of their properties like the number of corners in each period (this can be seen as an indication of the complexity of the dependence on ε of the splitting). Our goal is to show that our methods can be applied to an arbitrary quadratic ratio, and hence the results on the splitting distance and transversality generalize those of [8, 9, 4, 6]. Although we do not study here the continuation of the transversality for all values of $\varepsilon \rightarrow 0$, we stress that this could be carried out by means of a specific study in each case, as done in [42, 9, 6] for some concrete (golden, silver) ratios.

We point out that the periodicity in $\ln \varepsilon$ of the functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$ comes directly from the special properties of the continued fractions of quadratic numbers and cannot be satisfied in other cases (see [5], where the case of numbers of constant type is considered).

As we describe in section 1.3, the dependence on ε of the function $h_2(\varepsilon)$, associated to the asymptotic estimates for the transversality of the splitting, is more cumbersome than for the function $h_1(\varepsilon)$, associated to the maximal splitting distance. We stress here that, for some purposes, it is not necessary to establish the transversality of the splitting, and it can be enough to provide estimates of the maximal splitting distance. Indeed, such estimates imply the existence of splitting between the invariant manifolds, which provides a strong indication of the nonintegrability of the system near the given torus, and opens the door to the application of topological methods [23, 22] for the study of Arnold diffusion in such systems.

1.2. Setup. Here we describe the nearly integrable Hamiltonian system under consideration. In particular, we study a *singular* or *weakly hyperbolic* (*a priori stable*) Hamiltonian with 3 degrees of freedom possessing a 2-dimensional whiskered torus with fast frequencies. In canonical coordinates $(x, y, \varphi, I) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T}^2 \times \mathbb{R}^2$, with the symplectic form $dx \wedge dy + d\varphi \wedge dI$, the Hamiltonian is defined by

$$(4) \quad H(x, y, \varphi, I) = H_0(x, y, I) + \mu H_1(x, \varphi),$$

$$(5) \quad H_0(x, y, I) = \langle \omega_\varepsilon, I \rangle + \frac{1}{2} \langle \Lambda I, I \rangle + \frac{y^2}{2} + \cos x - 1,$$

$$(6) \quad H_1(x, \varphi) = h(x)f(\varphi).$$

Our system has two parameters, $\varepsilon > 0$ and μ , linked by a relation $\mu = \varepsilon^r$, $r > 0$ (the smaller r , the better). Thus, if we consider ε as the unique parameter, we have a singular problem for $\varepsilon \rightarrow 0$. See [7] for a discussion about singular and regular problems.

Recall that we are assuming a vector of fast frequencies $\omega_\varepsilon = \omega/\sqrt{\varepsilon}$ as given in (1), with a frequency vector $\omega = (1, \Omega)$ whose frequency ratio Ω is a *quadratic irrational number*; we assume without loss of generality that $0 < \Omega < 1$. It is a well-known property (and we prove it in section 2.3; see also [27, section II.2]) that any vector with quadratic ratio satisfies a *Diophantine condition*

$$(7) \quad |\langle k, \omega \rangle| \geq \frac{\gamma}{|k|} \quad \forall k \in \mathbb{Z}^2 \setminus \{0\},$$

with some $\gamma > 0$. We also assume in (5) that Λ is a symmetric (2×2) matrix, such that H_0 satisfies the condition of *isoenergetic nondegeneracy*,

$$(8) \quad \det \begin{pmatrix} \Lambda & \omega \\ \omega^\top & 0 \end{pmatrix} \neq 0.$$

For the perturbation H_1 in (6), we deal with the analytic periodic functions

$$(9) \quad h(x) = \cos x, \quad f(\varphi) = \sum_{k \in \mathcal{Z}} e^{-\rho|k|} \cos(\langle k, \varphi \rangle - \sigma_k), \quad \text{with } \sigma_k \in \mathbb{T},$$

where we introduce, in order to avoid repetitions in the Fourier series, the set

$$(10) \quad \mathcal{Z} = \{k = (k_1, k_2) \in \mathbb{Z}^2 : k_2 > 0 \text{ or } (k_2 = 0, k_1 \geq 0)\}.$$

The constant $\rho > 0$ gives the complex width of analyticity of the function $f(\varphi)$. Concerning the phases σ_k , they can be chosen arbitrarily for the purpose of this paper.

To justify the form of the perturbation H_1 chosen in (6) and (9), we stress that it facilitates the explicit computation of the Melnikov potential, which is necessary in order to show that it dominates the error term in (2), and therefore to establish the existence of splitting. Moreover, the fact that all coefficients $f_k = e^{-\rho|k|}$, in the Fourier expansion with respect to φ , are nonzero and have an exponential decay ensures that the study of the dominant harmonics of the Melnikov potential can be carried out directly from the arithmetic properties of the frequency vector ω (see section 3). Since our method is completely constructive, a perturbation with another kind of concrete harmonics f_k could also be considered (like $f_k = |k|^m e^{-\rho|k|}$), simply at the cost of more cumbersome computations in order to determine the dominant harmonics of the Melnikov potential.

It is worth recalling that the Hamiltonian defined in (4)–(9) is paradigmatic, since it is a generalization of the famous Arnold example (introduced in [1] to illustrate the transition chain mechanism in Arnold diffusion). It provides a model for the behavior of a nearly integrable Hamiltonian system near a single resonance (see [7] for a motivation) and has often been considered in the literature (see, for instance, [19, 30, 11]). We also mention that a perturbation with an exponential decay as the function $f(\varphi)$ in (9) has also been considered (see, for instance, [35]). In the present paper, our aim is to emphasize the dependence of the splitting, and its transversality, on the arithmetic properties of the frequency vector ω .

Let us describe the invariant tori and whiskers, as well as the splitting and Melnikov functions. First, it is clear that the unperturbed system H_0 (that corresponds to $\mu = 0$) consists of the pendulum, given by $P(x, y) = y^2/2 + \cos x - 1$, and 2 rotors with fast frequencies: $\dot{\varphi} = \omega_\varepsilon + \Lambda I$, $\dot{I} = 0$. The pendulum has a hyperbolic equilibrium at the origin, with separatrices that correspond to the curves given by $P(x, y) = 0$. We parameterize the upper separatrix of the pendulum as $(x_0(s), y_0(s))$, $s \in \mathbb{R}$, where

$$x_0(s) = 4 \arctan e^s, \quad y_0(s) = \frac{2}{\cosh s}.$$

Then the lower separatrix has the parameterization $(x_0(-s), -y_0(-s))$. For the rotors system (φ, I) , the solutions are $I = I_0$, $\varphi = \varphi_0 + t(\omega_\varepsilon + \Lambda I_0)$. Consequently, the Hamiltonian H_0 has a 2-parameter family of 2-dimensional whiskered tori: in coordinates (x, y, φ, I) , each torus can be parameterized as

$$\mathcal{T}_{I_0} : \quad (0, 0, \theta, I_0), \quad \theta \in \mathbb{T}^2,$$

and the inner dynamics on each torus is $\dot{\theta} = \omega_\varepsilon + \Lambda I_0$. Each invariant torus has a *homoclinic whisker*, i.e., coincident 3-dimensional stable and unstable invariant manifolds, which can be parameterized as

$$(11) \quad \mathcal{W}_{I_0} : \quad (x_0(s), y_0(s), \theta, I_0), \quad s \in \mathbb{R}, \quad \theta \in \mathbb{T}^2,$$

with the inner dynamics given by $\dot{s} = 1$, $\dot{\theta} = \omega_\varepsilon + \Lambda I_0$.

In fact, the collection of the whiskered tori for all values of I_0 is a 4-dimensional *normally hyperbolic invariant manifold*, parameterized by $(\theta, I) \in \mathbb{T}^2 \times \mathbb{R}^2$. This manifold has a 5-dimensional homoclinic manifold, which can be parameterized by (s, θ, I) , with inner dynamics

$\dot{s} = 1$, $\dot{\theta} = \omega_\varepsilon + \Lambda I$, $\dot{I} = 0$. We stress that this approach is usually considered in the study of Arnold diffusion (see, for instance, [12]).

Among the family of whiskered tori and homoclinic whiskers, we will focus our attention on the torus \mathcal{T}_0 , whose frequency vector is ω_ε as in (1), and its associated homoclinic whisker \mathcal{W}_0 .

When adding the perturbation μH_1 for $\mu \neq 0$ small enough, the *hyperbolic KAM theorem* can be applied (see, for instance, [33]) thanks to the Diophantine condition (7) and to the isoenergetic nondegeneracy (8). For μ small enough, the whiskered torus persists with some shift and deformation, as a perturbed torus $\mathcal{T} = \mathcal{T}^{(\mu)}$, as well as its local whiskers $\mathcal{W}_{\text{loc}} = \mathcal{W}_{\text{loc}}^{(\mu)}$ (a precise statement can be found, for instance, in [11, Theorem 1]).

The local whiskers can be extended along the flow, but in general for $\mu \neq 0$ the (*global*) *whiskers* do not coincide anymore, and one expects the existence of splitting between the (3-dimensional) stable and unstable whiskers, denoted $\mathcal{W}^s = \mathcal{W}^{s,(\mu)}$ and $\mathcal{W}^u = \mathcal{W}^{u,(\mu)}$, respectively. Using *flow-box coordinates* (see [11], where the n -dimensional case is considered) in a neighborhood containing a piece of both whiskers (away from the invariant torus), one can introduce parameterizations of the perturbed whiskers, with parameters (s, θ) inherited from the unperturbed whisker (11), and the inner dynamics

$$\dot{s} = 1, \quad \dot{\theta} = \omega_\varepsilon.$$

Then the distance between the stable whisker \mathcal{W}^s and the unstable whisker \mathcal{W}^u can be measured by comparing such parameterizations along the complementary directions. The number of such directions is 3 but, due to the energy conservation, it is enough to consider 2 directions, say the ones related to the action coordinates. In this way, one can introduce a (vector) *splitting function*, with values in \mathbb{R}^2 , as the difference of the parameterizations $\mathcal{J}^{s,u}(s, \theta)$ of (the action components of) the perturbed whiskers \mathcal{W}^s and \mathcal{W}^u . Initially this splitting function depends on (s, θ) , but it can be restricted to a transverse section by considering a fixed s , say $s = 0$, and we can define as in [10, section 5.2] the splitting function

$$(12) \quad \mathcal{M}(\theta) := \mathcal{J}^u(0, \theta) - \mathcal{J}^s(0, \theta), \quad \theta \in \mathbb{T}^2.$$

As said in (3), this function turns out to be the gradient of the (scalar) *splitting potential* $\mathcal{L}(\theta)$ (see [10, 16]). Notice that the *nondegenerate critical points* of \mathcal{L} correspond to simple zeros of \mathcal{M} and give rise to *transverse homoclinic orbits* to the whiskered torus.

Applying the Poincaré–Melnikov method, the first order approximation (2) of the splitting function is given by the (vector) *Melnikov function* $M(\theta)$, which is the gradient of the *Melnikov potential*: $M(\theta) = \nabla L(\theta)$. The latter one can be defined as an integral: we consider any homoclinic trajectory of the unperturbed homoclinic whisker \mathcal{W}_0 in (11), starting on the section $s = 0$, and the trajectory on the torus \mathcal{T}_0 to which it is asymptotic as $t \rightarrow \pm\infty$, and we subtract the values of the perturbation H_1 on the two trajectories. This gives an absolutely convergent integral, depending on the initial phase $\theta \in \mathbb{T}^2$ of the considered trajectories:

$$(13) \quad \begin{aligned} L(\theta) &:= - \int_{-\infty}^{\infty} [H_1(x_0(t), \theta + t\omega_\varepsilon) - H_1(0, \theta + t\omega_\varepsilon)] dt \\ &= - \int_{-\infty}^{\infty} [h(x_0(t)) - h(0)] f(\theta + t\omega_\varepsilon) dt, \end{aligned}$$

where we have taken into account the specific form (6) of the perturbation.

Our choice of the pendulum $P(x, y) = y^2/2 + \cos x - 1$ in (5), whose separatrix has simple poles, makes it possible to use the method of residues in order to compute the coefficients L_k of the Fourier expansion of the Melnikov potential $L(\theta)$. Such coefficients turn out to be exponentially small in ε (see their expression in section 3.1). For each value of ε only a finite number of *dominant harmonics* are relevant to find the nondegenerate critical points of $L(\theta)$, i.e., the simple zeros of the Melnikov function $M(\theta)$. Due to the exponential decay of the Fourier coefficients of $f(\varphi)$ in (9), it is not hard to study such dominance and its dependence on ε .

In order to give asymptotic estimates for both the maximal splitting distance and the transversality of the homoclinic orbits, the estimates obtained for the Melnikov function $M(\theta)$ have to be validated also for the splitting function $\mathcal{M}(\theta)$. The difficulty in the application of the Poincaré–Melnikov approximation (2), due to the exponential smallness in ε of the function $M(\theta)$ in our singular case $\mu = \varepsilon^r$, can be solved by obtaining upper bounds (on a complex domain) for the *error term* in (2), showing that if $r > r^*$ with a suitable r^* , its Fourier coefficients are dominated by the coefficients of $M(\theta)$ (see also [11]).

We stress that our approach can also be directly applied to other classical 1-degree-of-freedom Hamiltonians $P(x, y) = y^2/2 + V(x)$, with a potential $V(x)$ having a unique nondegenerate maximum, although the use of residues becomes more cumbersome when the complex parameterization of the separatrix has poles of higher orders (see some examples in [15]).

1.3. Main result. For the Hamiltonian system (4)–(9) with the 2 parameters linked by $\mu = \varepsilon^r$, $r > r^*$ (with some suitable r^*), and a frequency vector $\omega = (1, \Omega)$ with a quadratic ratio Ω , our main result provides exponentially small *asymptotic estimates* for some measures of the splitting. On one hand, we obtain an asymptotic estimate for the *maximal distance* of splitting, given in terms of the maximum size in modulus of the splitting function $\mathcal{M}(\theta) = \nabla \mathcal{L}(\theta)$, and this estimate is valid for all ε sufficiently small. On the other hand, we show the existence of *transverse homoclinic orbits*, given as simple zeros θ^* of $\mathcal{M}(\theta)$ (or, equivalently, as nondegenerate critical points of $\mathcal{L}(\theta)$), and we obtain an asymptotic estimate for the *transversality* of the homoclinic orbits, measured by the minimal eigenvalue (in modulus) of the matrix $D\mathcal{M}(\theta^*) = D^2\mathcal{L}(\theta^*)$, at each zero of $\mathcal{M}(\theta)$. This result on transversality is valid for “almost all” ε sufficiently small, since we have to exclude a small neighborhood of a finite number of geometric sequences where homoclinic bifurcations could take place.

With our approach, the Poincaré–Melnikov method can be validated for an exponent $r > r^*$ with $r^* = 3$, although a lower value of r^* can be given in some particular cases (see remark 1 after Theorem 1). However, such values of r^* are not optimal and could be improved using other methods, like the parameterization of the whiskers as solutions of the Hamilton–Jacobi equation (see, for instance, [30, 2]). In this paper, the emphasis is put on the generalization of the estimates to the case of an arbitrary quadratic frequency ratio Ω , rather than on the improvement of the value of r^* .

Due to the form of $f(\varphi)$ in (9), the Melnikov potential $L(\theta)$ is readily presented in its Fourier series (see section 3.1), with coefficients $L_k = L_k(\varepsilon)$ which are exponentially small in ε . We use this expansion of $L(\theta)$ in order to detect its *dominant harmonics* for every given ε . Careful control of the error term in (2) ensures that the dominant harmonics of

$L(\theta)$ correspond to the dominant harmonics of the splitting potential $\mathcal{L}(\theta)$. Such a dominance is also valid for the splitting function $\mathcal{M}(\theta)$, since the size of their Fourier coefficients \mathcal{M}_k (vector) and \mathcal{L}_k (scalar) is directly related: $|\mathcal{M}_k| = |k| \mathcal{L}_k$, $k \in \mathcal{Z}$ (recall the definition (10)).

As shown in section 4, in order to obtain an asymptotic estimate for the maximal distance of splitting, it is enough to consider the first dominant harmonic, given by some vector in \mathcal{Z} which depends on the perturbation parameter: $k = S_1(\varepsilon)$. Using estimates for this dominant harmonic \mathcal{L}_{S_1} as well as for all the remaining harmonics, we show that the dominant harmonic is large enough to ensure that it provides an approximation to the maximum size of the whole splitting function (see also [4, 5]). On the other hand, to show the transversality of the splitting, it is necessary to consider at least two dominant harmonics in order to prove the nondegeneracy of the critical points of the splitting potential (see also [8, 9]). For most values of the parameter ε , it is enough to consider the two “essential” dominant harmonics \mathcal{L}_{S_1} and \mathcal{L}_{S_2} (i.e., the two most dominant harmonics whose associated vectors $S_1(\varepsilon), S_2(\varepsilon) \in \mathcal{Z}$ are linearly independent; see section 2.2), and the transversality is then directly established.

However, one has to consider at least three harmonics for ε close to some “transition values” $\hat{\varepsilon}$, introduced below as the values at which a change in the *second* essential dominant harmonic occurs, and, consequently, the second and some subsequent harmonics have similar sizes. Such transition values turn out to be *corners* of the function $h_2(\varepsilon)$, related to the size of the second dominant harmonic (see the theorem below) and are given by a finite number of geometric sequences. The study of the transversality for ε close to a transition value, which is not considered in this paper, requires us to carry out a specific study that depends strongly on the quadratic frequency ratio Ω and on the concrete perturbation considered in (9). In some cases, the *continuation* of the transversality for all sufficiently small values $\varepsilon \rightarrow 0$ can be established under a certain condition on the phases σ_k in (9), as done in [9] and [6] for the golden and silver ratios, respectively. Otherwise, *homoclinic bifurcations* can occur when ε goes across a transition value (see, for instance, [42]).

The dependence on ε of the size of the splitting and its transversality is closely related to the arithmetic properties of the frequency vector $\omega = (1, \Omega)$, since the integer vectors $k \in \mathcal{Z}$ associated to the dominant harmonics can be found, for any ε , among the main quasi-resonances of the vector ω , i.e., the vectors k giving the “least” small divisors $|\langle k, \omega \rangle|$ (relatively to the size of $|k|$). In section 2, we develop a methodology for a complete study of the *resonant properties* of vectors with a quadratic ratio, which is one of the main goals of this paper. This methodology relies in the classification, established in [8] for any vector with a quadratic ratio Ω , of the integer vectors k into “*resonant sequences*” (see also sections 2.2 and 2.3 for definitions). Among them, the sequence of *primary resonances* corresponds to the vectors k which best fit the Diophantine condition (7), and the vectors k belonging to the remaining sequences are called *secondary resonances*.

Regarding particular cases, for the golden ratio, $\Omega = (\sqrt{5} - 1)/2$, the primary resonances can be described in terms of the Fibonacci numbers: $k = (-F_{n-1}, F_n)$ (see, for instance, [9]), and in the case of the silver ratio, $\Omega = \sqrt{2} - 1$, the primary resonances are given in terms of the Pell numbers (see [6]), which play an analogous role. In general, for a given quadratic ratio Ω the sequence of primary resonances, as well as the remaining resonant sequences, can be determined from the continued fraction of Ω , which is eventually periodic, i.e., periodic starting at some element (see section 2.1). We can construct, from the continued fraction, a

unimodular matrix T (i.e., with integer entries and determinant ± 1) having ω as an eigenvector with the associated eigenvalue

$$\lambda = \lambda(\Omega) > 1$$

(see Proposition 5 for an explicit construction). Then the iteration of the matrix $(T^{-1})^\top$ from an initial (“primitive”) vector allows us to generate any resonant sequence (see the definition (21)).

Next, we establish the *main result* of this work, providing two types of exponentially small asymptotic estimates for the splitting, as $\varepsilon \rightarrow 0$, related to the maximal distance of splitting and its transversality. The first one is given by the maximum of $|\mathcal{M}(\theta)|$, $\theta \in \mathbb{T}^2$. On the other hand, we show that for most values of ε the function $\mathcal{M}(\theta)$ has simple zeros, which correspond to transverse homoclinic orbits, and for each zero θ^* we give an estimate for the minimum eigenvalue (in modulus) of the matrix $D\mathcal{M}(\theta^*)$, as a measure for the transversality of the splitting. This generalizes the results of [8, 9].

We stress that the dependence on ε of both asymptotic estimates is given by the exponent $1/4$, and by the functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$, which are periodic with respect to $\ln \varepsilon$ and *piecewise-smooth* and, consequently, have a finite number of *corners* (i.e., jump discontinuities of the derivative) in each period. Some examples are shown in Figures 1–2 (where a *logarithmic scale* for ε is used). The oscillatory behavior of the functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$ depends strongly on the arithmetic properties of Ω , and, in fact, both functions can be explicitly constructed from the continued fraction of Ω (see section 3.2). Below, in two additional results we establish more accurately the behavior of the functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$ in the simplest cases of 1-periodic and 2-periodic continued fractions.

For positive quantities, we use the notation $f \sim g$ if we can bound $c_1 g \leq f \leq c_2 g$ with constants $c_1, c_2 > 0$ not depending on ε, μ .

Theorem 1 (main result). *Assume that the conditions described above for the Hamiltonian (4)–(9) with a quadratic frequency ratio Ω hold, that ε is small enough, and that $\mu = \varepsilon^r$, $r > 3$. Then, for the splitting function $\mathcal{M}(\theta)$, we have the following:*

- (a) $\max_{\theta \in \mathbb{T}^2} |\mathcal{M}(\theta)| \sim \frac{\mu}{\varepsilon^{1/2}} \exp \left\{ -\frac{C_0 h_1(\varepsilon)}{\varepsilon^{1/4}} \right\}$.
- (b) *The number of zeros θ^* of $\mathcal{M}(\theta)$ is 4κ with $\kappa(\varepsilon) \geq 1$ integer, and they are all simple, for any ε , except for a small neighborhood of some transition values $\hat{\varepsilon}$ belonging to a finite number of geometric sequences.*
- (c) *At each zero θ^* of $\mathcal{M}(\theta)$, the minimal eigenvalue of $D\mathcal{M}(\theta^*)$ satisfies*

$$|m^*| \sim \mu \varepsilon^{1/4} \exp \left\{ -\frac{C_0 h_2(\varepsilon)}{\varepsilon^{1/4}} \right\}.$$

The functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$, defined in (48), are piecewise-smooth and $4 \ln \lambda$ -periodic in $\ln \varepsilon$, with $\lambda = \lambda(\Omega)$ as given in Proposition 5. In each period, the function $h_1(\varepsilon)$ has at least 1 corner and $h_2(\varepsilon)$ has at least 2 corners. They satisfy for $\varepsilon > 0$ the following bounds:

$$\min h_1(\varepsilon) = 1, \quad \max h_1(\varepsilon) \leq J_1, \quad \max h_2(\varepsilon) \leq J_2, \quad h_1(\varepsilon) \leq h_2(\varepsilon),$$

with the constants

$$(14) \quad J_1 = J_1(\Omega) := \frac{1}{2} \left(\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}} \right), \quad J_2 = J_2(\Omega) := \frac{1}{2} \left(\lambda + \frac{1}{\lambda} \right).$$

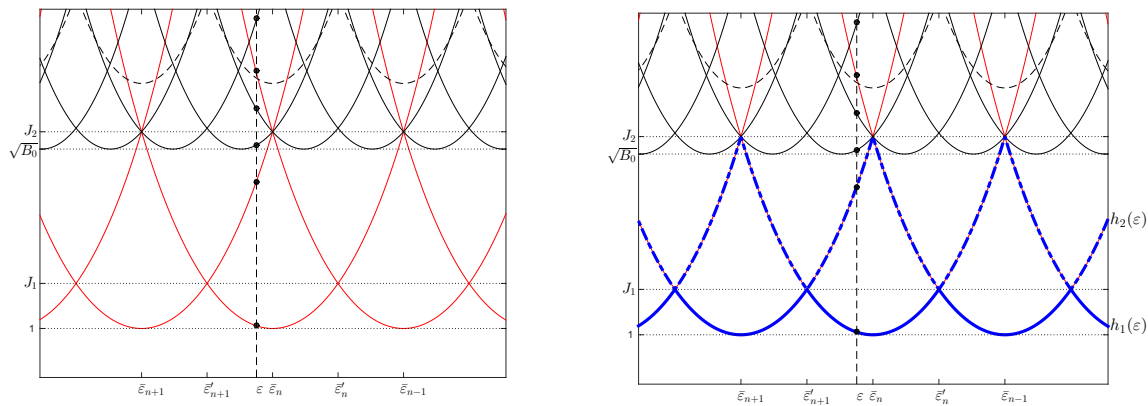


Figure 1. Dependence on ε of the functions in the exponents, for the metallic ratio $\Omega = [\overline{3}]$ (the bronze ratio), using a logarithmic scale for ε . (a) Graphs of the functions $g_{s(q,n)}^*(\varepsilon)$, associated to essential (solid lines) and nonessential (dashed lines) resonances $s(q,n)$; red lines correspond to the primary functions $\bar{g}_n(\varepsilon)$ (see section 3.1). (b) Graphs of the functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$.

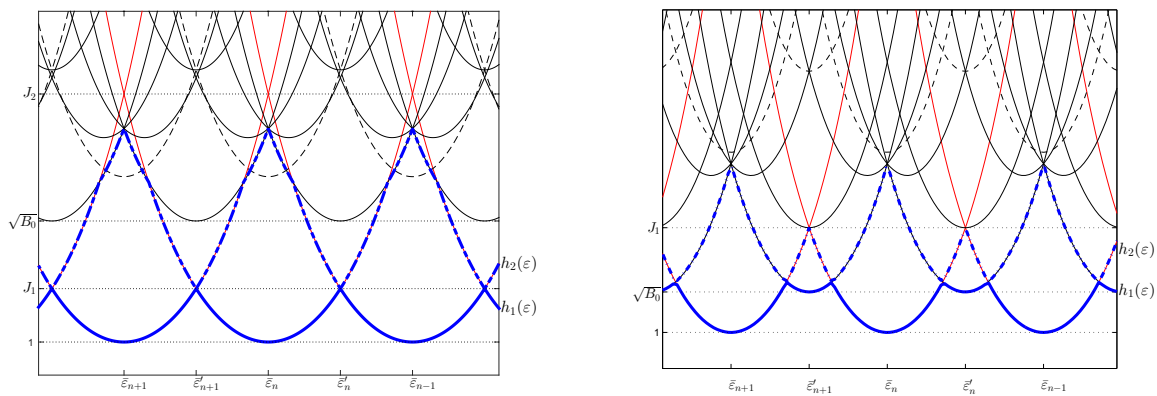


Figure 2. Graphs of the functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$ for two metallic-colored ratios. (a) $\Omega = [\overline{1,3}]$ (a golden-colored ratio); (b) $\Omega = [\overline{2,3}]$ (a silver-colored ratio).

The corners of $h_1(\varepsilon)$ are exactly the points $\check{\varepsilon}$ such that $h_1(\check{\varepsilon}) = h_2(\check{\varepsilon})$. The corners of $h_2(\varepsilon)$ are the same points $\check{\varepsilon}$, and the points $\hat{\varepsilon}$ where the results of (b)–(c) do not apply. The (integer) function $\kappa(\varepsilon)$ is piecewise-constant and $4 \ln \lambda$ -periodic in $\ln \varepsilon$, eventually with discontinuities at the transition values $\hat{\varepsilon}$. On the other hand, $C_0 = C_0(\Omega, \rho)$ is a positive constant defined in (40).

Remarks.

1. If the function $h(x)$ in (9) is replaced by $h(x) = \cos x - 1$, then the results of Theorem 1 are valid for $\mu = \varepsilon^r$ with $r > 2$ (instead of $r > 3$). The details of this improvement are not given here, since they work exactly as in [9].
2. As a consequence of the theorem, replacing $h_1(\varepsilon)$ by its upper bound $J_1 > 0$, we get

the following *lower bound* for the maximal splitting distance:

$$\max_{\theta \in \mathbb{T}^2} |\mathcal{M}(\theta)| \geq \frac{c\mu}{\sqrt{\varepsilon}} \exp \left\{ -\frac{C_0 J_1}{\varepsilon^{1/4}} \right\},$$

where c is a constant. This may be enough if our aim is only to prove the existence of the splitting of separatrices, without giving an accurate description of it.

3. The results of Theorem 1 can be partially generalized if the frequency ratio Ω is a non-quadratic *number of constant type*, i.e., whose continued fraction has bounded entries, but it is not periodic. The numbers of constant type are exactly those such that $\omega = (1, \Omega)$ satisfies a Diophantine condition with the minimal exponent, as in (7). This case has been considered in [5], where a function analogous to $h_1(\varepsilon)$, providing an asymptotic estimate for the maximal splitting distance, is defined. In this case, this function is bounded but it is no longer periodic in $\ln \varepsilon$.

For the simplest cases of continued fractions, we can obtain more accurate information on the functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$. As we show in section 2.1, we can restrict ourselves to the case of a *purely periodic* continued fraction, $\Omega = [\overline{a_1, \dots, a_m}]$ (we assume that $0 < \Omega < 1$; see section 2.1 for the notation). In particular, we consider the following two cases:

- the *metallic ratios*: $\Omega = [\overline{a}] = \frac{\sqrt{a^2+4}-a}{2}$, with $a \geq 1$;
- the *metallic-colored ratios*: $\Omega = [\overline{a, b}] = \frac{\sqrt{a^2b^2+4ab-ab}}{2a}$, with $1 \leq a < b$

(for the metallic-colored ratios, it is not necessary to consider the case $a > b$, since $[\overline{a, b}] = [a, \overline{b, a}]$).

The metallic ratios, which are limits of the sequence of quotients of consecutive terms of generalized Fibonacci sequences, have often been considered (see, for instance, [43, 17]). For some particular cases, we mention the *golden*, *silver*, and *bronze* ratios: $[\overline{1}]$, $[\overline{2}]$, and $[\overline{3}]$, respectively. The metallic-colored ratios can be subdivided in several classes, such as

- the *golden-colored ratios*: $\Omega = [\overline{1, b}]$, with $b \geq 2$;
- the *silver-colored ratios*: $\Omega = [\overline{2, b}]$, with $b \geq 3$;
- the *bronze-colored ratios*, etc.

For such types of ratios, we next provide descriptions of the functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$. Additionally, in part (b) we include a statement concerning the exact number of critical points of the splitting function $\mathcal{M}(\theta)$ for the case of metallic ratios. Such results come from an accurate analysis of the first and second essential dominant harmonics of the Melnikov potential, studying whether they are both given by primary resonances for any ε , or they can be given by secondary resonances for some intervals of ε . We point out that a rigorous analysis of the role of the secondary resonances becomes too cumbersome in some cases. For this reason, although we give *rigorous proofs* for some results (the statements of Theorem 2), we provide *numerical evidence* for others (given below as Conjecture 3) after checking them with intensive computations carried out for a large number of frequency ratios (see the proofs and justifications in section 3.3).

Theorem 2. *Under the conditions of Theorem 1, we have the following:*

- (a) *If the frequency ratio Ω is metallic, the function $h_1(\varepsilon)$ has exactly 1 corner $\tilde{\varepsilon}$ in each period and satisfies $\max h_1(\varepsilon) = h_1(\tilde{\varepsilon}) = J_1$, and the distance between consecutive corners is exactly $4 \ln \lambda$.*

- (b) If the frequency ratio Ω is golden-colored, the function $h_2(\varepsilon)$ has at least 3 corners in each period and satisfies $\min h_2(\varepsilon) = J_1$, $\max h_2(\varepsilon) < J_2$.
- (c) If the frequency ratio Ω is metallic-colored but not golden-colored, the function $h_1(\varepsilon)$ has at least 2 corners in each period and satisfies $\max h_1(\varepsilon) < J_1$.

Conjecture 3. Under the conditions of Theorem 1, we have the following:

- (a) If the frequency ratio Ω is golden-colored, the function $h_1(\varepsilon)$ has exactly 1 corner $\check{\varepsilon}$ in each period and satisfies $\max h_1(\varepsilon) = h_1(\check{\varepsilon}) = J_1$, and the distance between consecutive corners is exactly $4 \ln \lambda$.¹
- (b) If the frequency ratio Ω is metallic, the function $h_2(\varepsilon)$ has exactly 2 corners $\check{\varepsilon}$, $\hat{\varepsilon}$ in each period and satisfies $\min h_2(\varepsilon) = h_2(\check{\varepsilon}) = J_1$, $\max h_2(\varepsilon) = h_2(\hat{\varepsilon}) = J_2$, and the distance between consecutive corners is exactly $2 \ln \lambda$. Moreover, the number of zeros θ^* of $\mathcal{M}(\theta)$ is exactly 4 for any ε except for a small neighborhood of the transition values $\hat{\varepsilon}$.²

As mentioned before, the functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$ can be defined explicitly for any quadratic ratio Ω from its continued fraction (see section 3). Such functions have piecewise expressions, which are simple in the case of a metallic ratio, but in general can be very complicated, depending on the number of their corners in each period. Nevertheless, we stress that numerical justifications are required for results concerning infinite families of ratios, such as the metallic or the golden-colored ones. Instead, for a *particular* frequency ratio the results can be rigorously established, after an analysis of primary and secondary resonances that can be carried out by a finite number of computations. As an example, we provide a rigorous result on the properties of the functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$ for a concrete frequency ratio with a 3-periodic continued fraction: $\Omega = [1, 2, 2] = \frac{\sqrt{85}-5}{6}$ (the proof is also given in section 3.3). Such properties are illustrated in Figure 3.

Proposition 4. Under the conditions of Theorem 1, for the frequency ratio $\Omega = [1, 2, 2]$, the functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$ have exactly 2 corners $\check{\varepsilon}$ in each period and exactly 6 corners $\check{\varepsilon}$, $\hat{\varepsilon}$ in each period, respectively, and satisfy $1 < \min h_2(\varepsilon) < \max h_1(\varepsilon) < \sqrt{5/3} < \max h_2(\varepsilon) < J_1$. Moreover, the number of zeros θ^* of $\mathcal{M}(\theta)$ is exactly 4 for any ε except for a small neighborhood of the transition values $\hat{\varepsilon}$.

Organization of the paper. We start in section 2 by studying the arithmetic properties of frequency vectors $\omega = (1, \Omega)$ with a quadratic ratio Ω . Such properties are closely related to the continued fraction of Ω (section 2.1), which allows us to construct the iteration matrices and study the resonant properties of the vector ω (sections 2.2 and 2.3), and to provide accurate results for the cases of metallic and metallic-colored ratios (section 2.4), mainly considered in this paper. Next, in section 3 we find an asymptotic estimate for the first and second dominant harmonics of the splitting potential, which allows us to define the functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$ and study their general properties (sections 3.1 and 3.2), as well as the specific properties for 1-periodic and 2-periodic continued fractions (section 3.3) considered in Theorem 2 and Conjecture 3, and for a particular 3-periodic case in Proposition 4. Finally, in section 4 we provide rigorous bounds of the remaining harmonics, allowing us to obtain asymptotic

¹The result of part (a) has been checked numerically for golden-colored ratios $\Omega = [1, b]$, $2 \leq b \leq 10^6$.

²The results of part (b) have been checked numerically for metallic ratios $\Omega = [\bar{a}]$, $1 \leq a \leq 10^4$.

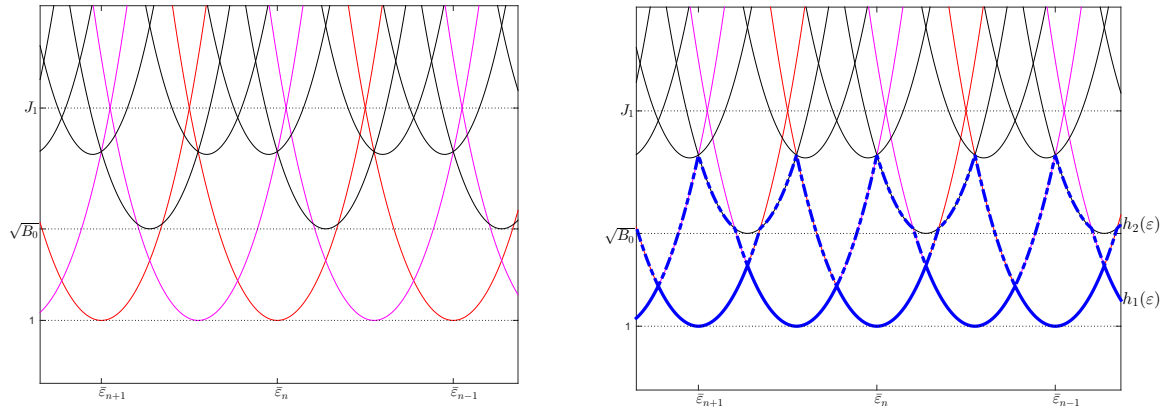


Figure 3. Graphs of the functions $g_{s(q,n)}^*(\varepsilon)$, $h_1(\varepsilon)$, and $h_2(\varepsilon)$ for $\Omega = [1, 2, 2]$: the two sequences of primary resonances correspond to the red and magenta graphs; the nonessential resonances are not represented (see the proof of Proposition 4).

estimates for both the maximal splitting distance and the transversality of the splitting, as established in Theorem 1.

Finally, we introduce some notation that we use in this paper. For positive quantities, we write $f \preceq g$ if we can bound $f \leq cg$ with some constant c not depending on ε and μ . In this way, we can write $f \sim g$ if $g \preceq f \preceq g$. On the other hand, when comparing positive sequences a_n, b_n we use an expression like “ $a_n \approx b_n$ as $n \rightarrow \infty$ ” if $\lim_{n \rightarrow \infty} (a_n/b_n) = 1$, and also “ $a_n \leq b_n$ as $n \rightarrow \infty$ ” if $\limsup_{n \rightarrow \infty} (a_n/b_n) \leq 1$.

2. Vectors with quadratic ratio.

2.1. Continued fractions of quadratic numbers. It is well known that any irrational number $0 < \Omega < 1$ has an infinite *continued fraction*

$$\Omega = [a_1, a_2, a_3, \dots] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}, \quad a_j \in \mathbb{Z}^+, \quad j \geq 1$$

(notice that the integer part is $a_0 = 0$; hence we have removed the entry “0;” from the notation). Its entries a_j are called the *partial quotients* of the continued fraction. It is also well known that the rational numbers $\frac{p_j}{q_j} = [a_1, \dots, a_j]$, $j \geq 1$, called the (*principal*) *convergents* of Ω , provide successive best rational approximations to Ω . Thus, if we consider the “vector convergents” $w(j) := (q_j, p_j)$, we obtain approximations to the direction of the vector $\omega = (1, \Omega)$ (see, for instance, [40] and [27] as general references on continued fractions).

The convergents of a continued fraction are usually computed from the standard recurrences

$$(15) \quad \begin{aligned} q_{-1} &= 0, & q_0 &= 1, & q_j &= a_j q_{j-1} + q_{j-2}, \\ p_{-1} &= 1, & p_0 &= a_0 = 0, & p_j &= a_j p_{j-1} + p_{j-2}, \quad j \geq 1. \end{aligned}$$

Alternatively, we can compute them in terms of products of unimodular matrices [5, Proposition 1],

$$(16) \quad \begin{pmatrix} q_j & q_{j-1} \\ p_j & p_{j-1} \end{pmatrix} = A_1 \cdots A_j, \quad \text{where } A_i = \mathcal{T}(a_i) := \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix}.$$

If we consider the first column, we can write $w(j) = A_1 \cdots A_j w(0)$.

An important tool in the study of continued fractions is the *Gauss map* $g : (0, 1) \rightarrow [0, 1)$, defined as $g(x) = \{\frac{1}{x}\}$, where $\{\cdot\}$ stands for the fractional part of any real number. This map acts on a given continued fraction by removing the first entry: for $\Omega = [a_1, a_2, a_3, \dots]$, we have $g(\Omega) = [a_2, a_3, \dots]$. We consider, for a given number $\Omega \in (0, 1)$, the sequence (x_j) defined by

$$(17) \quad x_0 = \Omega \quad x_j = g(x_{j-1}), \quad j \geq 1,$$

which satisfies that $x_j \neq 0$ for any j if Ω is irrational. It is clear that $x_j = [a_{j+1}, a_{j+2}, \dots]$ for any j .

In our case of a *quadratic irrational number* Ω , it is well known that the continued fraction is *eventually periodic*, i.e., periodic starting at some partial quotient. For an m -periodic continued fraction, we use the notation

$$\Omega = [b_1, \dots, b_r, \overline{a_1, \dots, a_m}].$$

In fact, as we see below, we can restrict ourselves to the numbers with *purely periodic* continued fractions, i.e., periodic starting at the first partial quotient: $\Omega = [\overline{a_1, \dots, a_m}]$. It is easy to relate such properties with the sequence (x_j) defined by the Gauss map: the continued fraction of Ω is eventually periodic (hence, Ω is quadratic) if and only if $x_{r+m} = x_r$ for some $r \geq 0$, $m \geq 1$, and it is purely periodic if and only if $x_m = x_0$ for some $m \geq 1$.

In the following proposition, which plays an essential role in the results of this paper, we see that for any given vector $\omega = (1, \Omega)$ with a quadratic ratio Ω , there exists a *unimodular* matrix $T = T(\Omega)$ having ω as an eigenvector with the associated eigenvalue $\lambda = \lambda(\Omega) > 1$. We show how we can construct both T and λ , directly from the continued fraction of Ω . Additionally, we show that applying the matrix T to a convergent $w(j)$, we get the convergent $w(j+m)$.

Proposition 5.

- (a) Let $\Omega \in (0, 1)$ be a quadratic irrational number with a purely periodic continued fraction, $\Omega = [\overline{a_1, \dots, a_m}]$, and consider the matrices $A_j = \mathcal{T}(a_j)$ as in (16). Then the matrix $T = A_1 \cdots A_m$ is unimodular and has $\omega = (1, \Omega)$ as eigenvector with eigenvalue $\lambda = \frac{1}{x_0 x_1 \cdots x_{m-1}} > 1$, where (x_j) is the sequence defined by (17). Moreover, for the convergents $w(j)$ of Ω we have that $T w(j) = w(j+m)$ for any $j \geq 0$.
- (b) Let $\hat{\Omega}$ be a quadratic irrational number with a nonpurely periodic continued fraction: $\hat{\Omega} = [b_1, \dots, b_r, \Omega]$, with Ω as in (a), and consider the matrices $B_j = \mathcal{T}(b_j)$ and $S = B_1 \cdots B_r$. Then the matrix $\hat{T} = S T S^{-1}$ is unimodular and has $\hat{\omega} = (1, \hat{\Omega})$ as eigenvector with eigenvalue λ as in (a). Moreover, for the convergents $\hat{w}(j)$ of $\hat{\Omega}$ we have that $\hat{T} \hat{w}(j) = \hat{w}(j+m)$ for any $j \geq r$.

Proof. Using the construction of the sequence (x_j) associated to Ω , we see that $\frac{1}{x_{j-1}} = a_j + x_j$, and we easily deduce the equality

$$\begin{pmatrix} 1 \\ x_{j-1} \end{pmatrix} = x_{j-1} A_j \begin{pmatrix} 1 \\ x_j \end{pmatrix}, \quad n \geq 1.$$

Iterating this equality for $j = 1, \dots, m$ and using that $x_0 = \Omega = x_m$, we obtain

$$\begin{pmatrix} 1 \\ \Omega \end{pmatrix} = x_0 x_1 \cdots x_{m-1} A_1 A_2 \cdots A_m \begin{pmatrix} 1 \\ \Omega \end{pmatrix},$$

which proves that $T\omega = \lambda\omega$, and it is clear that T is unimodular. To complete part (a), using (16) and the periodicity of the continued fraction, we have

$$Tw(j) = A_1 \cdots A_m A_1 \cdots A_j w(0) = A_1 \cdots A_{j+m} w(0) = w(j+m).$$

With similar arguments we prove part (b). Indeed, using the sequence (\hat{x}_j) associated to $\hat{\Omega}$, we see that

$$\begin{pmatrix} 1 \\ \hat{\Omega} \end{pmatrix} = \hat{x}_0 \hat{x}_1 \cdots \hat{x}_{r-1} B_1 B_2 \cdots B_r \begin{pmatrix} 1 \\ \Omega \end{pmatrix},$$

which says that the matrix S provides a unimodular linear change between the directions of the vectors ω and $\hat{\omega}$. We deduce that $\hat{T}\hat{\omega} = \lambda\hat{\omega}$. On the other hand, the matrix S also provides a relation between their respective convergents. Indeed, using (16) we see that, for $j \geq r$,

$$\hat{w}(j) = B_1 \cdots B_r A_1 \cdots A_{j-r} \hat{w}(0) = Sw(j-r)$$

(notice that $\hat{w}(0) = w(0) = (1, 0)$). Then, using (a), we deduce that

$$\hat{T}\hat{w}(j) = STw(j-r) = Sw(j) = \hat{w}(j+r). \quad \blacksquare$$

Remarks.

1. In what concerns the contents of this paper, it is enough to consider quadratic numbers with purely periodic continued fractions. As we see from the proof of this proposition, writing $S = \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix}$, we have the equality

$$\hat{\Omega} = \frac{s_3 + s_4 \Omega}{s_1 + s_2 \Omega} \quad \text{with} \quad s_1 s_4 - s_2 s_3 = \pm 1,$$

expressing the equivalence of the number $\hat{\Omega}$, with an eventually periodic continued fraction, with the number Ω with a purely periodic one. Then it can be shown that our main result (Theorem 1) applies to both numbers Ω and $\hat{\Omega}$ for ε small enough, and we only need to consider the purely periodic case. For instance, the results for the golden number $\Omega = [\bar{1}]$ also apply to the *noble* numbers $\hat{\Omega} = [b_1, \dots, b_r, \bar{1}]$. We point out that the threshold in ε of validity of the results, not considered in this paper, would depend on the nonperiodic part of the continued fraction.

2. This proposition provides a particular case of an algebraic result by Koch [26], which also applies to higher dimensions: for any given vector $\omega \in \mathbb{R}^n$ whose components generate an algebraic number field of degree n , there exists a unimodular matrix T having ω as an eigenvector with the associated eigenvalue λ of modulus greater than 1. This result is usually applied in the context of renormalization theory, since the iteration of the matrix T provides successive rational approximations to the direction of the vector ω (see, for instance, [26, 31]).

2.2. Resonant sequences. In this section and the next one, we review briefly the technique developed in [8] for classifying the quasi-resonances of a given frequency vector $\omega = (1, \Omega)$ whose ratio Ω is quadratic, and we study their relation with the convergents of the continued fraction of Ω . A vector $k \in \mathbb{Z}^2 \setminus \{0\}$ can be considered a quasi-resonance if $\langle k, \omega \rangle$ is small in modulus. To determine the dominant harmonics of the Melnikov potential, we can restrict ourselves to quasi-resonant vectors, since the effect of vectors far enough from resonances can easily be bounded.

More precisely, we say that an integer vector $k \neq 0$ is a *quasi-resonance* of ω if

$$|\langle k, \omega \rangle| < \frac{1}{2}.$$

It is clear that any quasi-resonance can be presented in the form

$$k^0(q) := (-p, q), \quad \text{with } p = p^0(q) := \text{rint}(q\Omega)$$

(we denote by $\text{rint}(x)$ the closest integer to x). Hence, we have the *small divisors* $\langle k^0(q), \omega \rangle = q\Omega - p$. We denote by \mathcal{A} the set of quasi-resonances $k^0(q)$ with $q \geq 1$ (which can be assumed with no loss of generality). We also say that $k^0(q)$ is an *essential quasi-resonance* if it is not a multiple of another integer vector (if $p \neq 0$, this means that $\gcd(q, p) = 1$), and we denote by \mathcal{A}_0 the set of essential quasi-resonances.

As stated in section 2.1, the matrix $T = T(\Omega)$ given by Proposition 5 (in both cases of purely or nonpurely periodic continued fractions) provides approximations to the direction of $\omega = (1, \Omega)$. Instead of T , we are going to use another matrix providing approximations to the orthogonal line $\langle \omega \rangle^\perp$, i.e., to the quasi-resonances of ω . Notice the following simple but important equality:

$$(18) \quad \langle (T^{-1})^\top k, \omega \rangle = \langle k, T^{-1} \omega \rangle = \frac{1}{\lambda} \langle k, \omega \rangle,$$

with $\lambda = \lambda(\Omega)$ as given by Proposition 5. With this in mind, for a quadratic ratio with an (eventually) m -periodic continued fraction, we define the matrix

$$(19) \quad U = U(\Omega) := \sigma (T^{-1})^\top, \quad \text{where } \sigma := \det T = (-1)^m$$

(the sign σ , which is not relevant, is introduced in order to have a simpler expression in (22)). It is clear from (18) that if $k \in \mathcal{A}$, then also $Uk \in \mathcal{A}$. We say that the vector $k = k^0(q) = (-p, q)$ is *primitive* if $k \in \mathcal{A}$ but $U^{-1}k \notin \mathcal{A}$. If so, we also say that q is a *primitive integer* and denote

by \mathcal{P} the set of primitive integers, with $q \geq 1$. We deduce from (18) that k is primitive if and only if the following *fundamental property* is fulfilled:

$$(20) \quad \frac{1}{2\lambda} < |\langle k, \omega \rangle| < \frac{1}{2}.$$

If a primitive $k^0(q) = (-p, q)$ is essential, we also say that q is an *essential primitive integer*, and we denote by $\mathcal{P}_0 \subset \mathcal{P}$ the set of essential primitive integers.

Now we define, for each primitive vector $k^0(q)$, the following *resonant sequences* of integer vectors:

$$(21) \quad s(q, n) := U^n k^0(q), \quad n \geq 0.$$

It turns out that such resonant sequences cover the whole set of vectors in \mathcal{A} , providing a classification for them.

Remark. A resonant sequence $s(q, n)$ generated by an essential primitive $k^0(q)$ cannot be a multiple of another resonant sequence. Indeed, in this case we would have $k^0(q) = c s(\tilde{q}, n_0)$ with $c > 1$ and $n_0 \geq 0$, and hence $k^0(q)$ would not be essential.

Let us establish a relation between the resonant sequences $s(q, n)$ and the convergents of Ω . Alternatively to the convergents $w(j) = (q_j, p_j)$ considered in section 2.1, we rather consider the “*resonant convergents*” (see also [5]),

$$v(j) := (-p_j, q_j).$$

The next lemma shows that the action of the matrix U defined in (19) on the vectors $v(j)$ is analogous to the action of T on the vectors $w(j)$ (which has been described in Proposition 5). This implies that the sequence of resonant convergents is divided into m of the resonant sequences defined in (21). We also see that the primitive vectors generating such sequences are the m first resonant convergents (belonging to \mathcal{A}).

Lemma 6.

- (a) Let Ω be a quadratic number with an (eventually) periodic continued fraction, $\Omega = [b_1, \dots, b_r, \overline{a_1, \dots, a_m}]$ (with $r \geq 0$). Then we have

$$Uv(j) = v(j + m), \quad j \geq r,$$

and hence the sequence of resonant convergents $v(j)$, for $j \geq r$, is divided into m resonant sequences.

- (b) If Ω has a purely periodic continued fraction, $\Omega = [\overline{a_1, \dots, a_m}]$, the primitive vectors among the resonant convergents are

$$\begin{array}{ll} v(1), \dots, v(m) & \text{if } a_1 = 1; \\ v(0), \dots, v(m-1) & \text{if } a_1 \geq 2. \end{array}$$

Proof. We use the following simple relation between the entries of the matrices T and U (valid in both cases $r = 0$ or $r \geq 1$):

$$(22) \quad \text{if } T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{then } U = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix},$$

where we have taken into account that $\det T = (-1)^m$. Then the equality $Tw(j) = w(j+m)$, which holds for $j \geq r$, is exactly the same as $Uv(j) = v(j+m)$, as stated in (a), by the relation between the vectors $v(j)$ and $w(j)$. As an immediate consequence, we have that the sequence of resonant convergents $v(j)$ (for $j \geq r$) is divided into m resonant sequences.

To prove (b), we first see that the small divisors associated to the resonant convergents $v(j)$ satisfy the equality

$$q_j \Omega - p_j = (-1)^j x_0 \cdots x_j, \quad j \geq 0,$$

where (x_j) is the sequence introduced in (17). This can easily be checked by induction using the recurrence (15) and the equality $\frac{1}{x_{j-1}} - a_j = x_j$.

If the continued fraction of Ω is purely periodic, recalling the expression for λ given in Proposition 5(a) and the fundamental property (20), it is clear that a resonant convergent $v(j)$ is primitive if and only if the following inequalities hold:

$$\frac{x_0 \cdots x_{m-1}}{2} < x_0 \cdots x_j < \frac{1}{2}.$$

Recall that $x_j \in (0, 1)$ for any j . Using (a), we see that such inequalities can only be fulfilled by, at most, m consecutive values of j . For $a_1 \geq 2$, the first one is $j = 0$ since $x_0 = \Omega < 1/2$, and the last one is clearly $j = m - 1$. Instead, for $a_1 = 1$ the first one is $j = 1$, since $x_0 = \Omega > 1/2$ and $x_0 x_1 = 1 - x_0 < 1/2$, and the last one is $j = m$ since $x_m = x_0 > 1/2$. ■

Remarks.

1. The matrices T and U cannot be triangular; i.e., we have $b \neq 0$ and $c \neq 0$ in (22). Indeed, this would imply that the eigenvalue λ is rational, and hence the frequency ratio Ω would also be a rational number.
2. The primitive resonant convergents given in part (b) of this proposition are all essential primitive vectors, since all the convergents p_j/q_j are reduced fractions (as a consequence of the fact that the matrices in (16) are unimodular).

2.3. Primary and secondary resonances. Now, our aim is to study which integer vectors k best fit the Diophantine condition (7). As in [8], we define the “numerators”

$$(23) \quad \gamma_k := |\langle k, \omega \rangle| \cdot |k|, \quad k \in \mathbb{Z}^2 \setminus \{0\},$$

where we use the norm $|\cdot| = |\cdot|_1$ (i.e., the sum of absolute values). As said in section 2.2, we can restrict ourselves to vectors $k = k^0(q) \in \mathcal{A}$ (with $q \geq 1$), and such vectors will be called primary or secondary resonances depending on the size of γ_k . We are also interested in studying the “separation” between both types of resonances.

Recall that the matrix T given by Proposition 5 has $\omega = (1, \Omega)$ as an eigenvector with eigenvalue $\lambda > 1$. We consider a basis ω, v_2 of eigenvectors of T , where the second vector v_2

has the eigenvalue σ/λ (of modulus < 1 ; recall that $\sigma = \det T$). For the matrix U defined in (19), let u_1, u_2 be a basis of eigenvectors with eigenvalues σ/λ and λ , respectively. Writing the entries of the matrices T and U as in (22), it is not hard to obtain expressions for such eigenvalues and eigenvectors:

$$(24) \quad \lambda = a + b\Omega, \quad \frac{\sigma}{\lambda} = d - b\Omega,$$

$$(25) \quad v_2 = (-b\Omega, c), \quad u_1 = (c, b\Omega), \quad u_2 = (-\Omega, 1).$$

We also get the quadratic equations for the frequency ratio Ω and the eigenvalue λ :

$$(26) \quad b\Omega^2 = c - (a - d)\Omega, \quad \lambda^2 = (a + d)\lambda - \sigma.$$

For any primitive integer q , recalling that we write $k^0(q) = (-p, q)$, we define the quantities

$$(27) \quad r_q := \langle k^0(q), \omega \rangle = q\Omega - p, \quad z_q := \langle k^0(q), v_2 \rangle = cq + bp\Omega.$$

Remark. As a consequence of the fact that Ω is an irrational number, one readily sees that if $q \neq \bar{q}$, then $r_q \neq r_{\bar{q}}$ and $z_q \neq z_{\bar{q}}$ (in the latter case, using also that $c \neq 0$, as seen in remark 1 after Lemma 6).

The following proposition, whose proof is given in [8] (see also [4] for a comparison with the case of 3-dimensional cubic frequencies), shows that the resonant sequences $s(q, n)$ defined in (21) have a limit behavior: the sizes of the vectors $s(q, n)$ exhibit a *geometric growth*, and the numerators $\gamma_{s(q, n)}$ tend to a “limit numerator” γ_q^* as $n \rightarrow \infty$.

Proposition 7. *Let $\omega = (1, \Omega)$ be a frequency vector whose ratio $\Omega \in (0, 1)$ is quadratic, and consider the vectors v_2 and u_2 as in (25). For any primitive integer $q \in \mathcal{P}$ (see (20)), consider the quantities r_q and z_q defined in (27). Then the resonant sequence $s(q, n)$ defined in (21) satisfies*

- (a) $|s(q, n)| = K_q \lambda^n + \mathcal{O}(\lambda^{-n})$, where $K_q := \left| \frac{z_q}{\langle u_2, v_2 \rangle} u_2 \right|$;
- (b) $\gamma_{s(q, n)} = \gamma_q^* + \mathcal{O}(\lambda^{-2n})$, where $\gamma_q^* := \lim_{n \rightarrow \infty} \gamma_{s(q, n)} = |r_q| K_q$.

Using (25)–(27), we get the following alternative expression for the limit numerators:

$$(28) \quad \gamma_q^* = \frac{\Omega(1 + \Omega)}{|c + b\Omega^2|} |\delta_q|, \quad \text{where } \delta_q := \frac{r_q z_q}{\Omega} = cq^2 - (a - d)qp - bp^2, \quad p = p^0(q).$$

It is clear that $\delta_q \neq 0$ and it is an integer. We can select the minimal of the values $|\delta_q|$ and, consequently, of the limit numerators γ_q^* , which is reached by some concrete primitive \hat{q} . We define

$$(29) \quad \delta^* := \min_{q \in \mathcal{P}} |\delta_q| = |\delta_{\hat{q}}| \geq 1, \quad \gamma^* := \min_{q \in \mathcal{P}} \gamma_q^* = \gamma_{\hat{q}}^* > 0.$$

It is easy to see, as a consequence, that $\liminf_{|k| \rightarrow \infty} \gamma_k = \gamma^* > 0$. Hence, any vector with a quadratic ratio satisfies the Diophantine condition (7), and we can consider γ^* as the *asymptotic Diophantine constant*.

As we see, all limit numerators γ_q^* are multiples of a concrete positive number. An important consequence of this fact is that it allows us to establish a classification of the vectors in \mathcal{A} . We define the *primary resonances* as the integer vectors belonging to the sequence $s_0(n) := s(\hat{q}, n)$, and *secondary resonances* the vectors belonging to any of the remaining sequences $s(q, n)$, $q \neq \hat{q}$ (recall that \hat{q} is the primitive giving the minimum in (29)). We also introduce *normalized numerators* $\tilde{\gamma}_k$ and their limits $\tilde{\gamma}_q^*$, $q \in \mathcal{P}$, after dividing by γ^* , and in this way $\tilde{\gamma}_{\hat{q}}^* = 1$. We also define a value $B_0 = B_0(\Omega)$ measuring the “separation” between the primary and the *essential* secondary resonances:

$$(30) \quad \tilde{\gamma}_k := \frac{\gamma_k}{\gamma^*}, \quad \tilde{\gamma}_q^* := \frac{\gamma_q^*}{\gamma^*} = \frac{|\delta_q|}{\delta^*}, \quad B_0 := \min_{q \in \mathcal{P}_0 \setminus \{\hat{q}\}} \tilde{\gamma}_q^*.$$

Using the fundamental property (20) and the inequality $|p - q\Omega| < 1/2$, we get the following lower bound for the limit numerators, which slightly improves the analogous bound given in [8]:

$$(31) \quad \gamma_q^* > \frac{(1 + \Omega)q - \alpha}{2\lambda}, \quad \alpha = \frac{|b|\Omega(1 + \Omega)}{2|c + b\Omega^2|}.$$

Remarks.

1. Since the lower bound (31) is increasing with respect to q , it is enough to check a finite number of cases in order to find the minimum in (29).
2. We are implicitly assuming that the primitive integer \hat{q} providing the minimum in (29) is unique. In fact, we will show in section 2.4 that this is true for the cases of metallic or metallic-colored ratios Ω introduced in section 1.3. But in other cases, the minimum could be reached by two or more primitives, and consequently there could be two or more sequences of primary resonances. For instance, for the ratio $\Omega = [\overline{1, 2, 2}]$ there are two sequences of primary resonances (see the proof of Proposition 4 in section 3.3).
3. Any primitive integer \hat{q} generating a sequence of primary resonances is essential. Indeed, if \hat{q} is not essential, then we have $k^0(\hat{q}) = c s(\bar{q}, n_0)$ with $c > 1$ and $n_0 \geq 0$, and therefore $s(\hat{q}, n) = c s(\bar{q}, n_0 + n)$, which implies by (23) that $\gamma_{\hat{q}}^* = c^2 \gamma_{\bar{q}}^*$, and the minimum in (29) would not be reached for \hat{q} .

Next, we show that the sequence of primary resonances is one (or more) of the m resonant sequences in which, by Lemma 6, the resonant convergents are divided if the continued fraction of Ω is m -periodic. In fact, we can give a lower bound for the numerators of all the remaining sequences.

Lemma 8. *For any primitive integer q such that the vectors in the sequence $s(q, n)$ are not resonant convergents, its normalized numerator satisfies $\tilde{\gamma}_q^* > \sqrt{5}/2$.*

Proof. We use some results in [40, section I.5] (namely, Theorems I.5B and I.5C) concerning the properties of the convergents of any irrational number. On one hand, for an infinite number of convergents the inequality $|q_n\Omega - p_n| < 1/(\sqrt{5}q_n)$ is satisfied, and on the other hand, if a given integer $q \geq 1$ is not a convergent and p/q is a reduced fraction, then $|q\Omega - p| \geq 1/2q$. To compare such results with our Diophantine condition (7), notice that $|k^0(q)| = q + p \approx (1 + \Omega)q$ as $q \rightarrow \infty$.

The first quoted result implies that, at least for one of the resonant sequences $s(q, n)$ whose vectors are resonant convergents, its limit numerator satisfies $\gamma_q^* \leq (1 + \Omega)/\sqrt{5}$. By the second result, if a given resonant sequence $s(q, n)$ is generated by an essential primitive q and its vectors are not resonant convergents, then $\gamma_q^* \geq (1 + \Omega)/2$. This is also true if q is not essential, by the previous remark 3. Dividing the two obtained bounds, we get the lower bound $\sqrt{5}/2$ for the normalized limit $\tilde{\gamma}_q^*$, when q does not generate a sequence of resonant convergents. ■

2.4. Results for metallic and metallic-colored ratios. Now, we provide particular arithmetical results for the (purely periodic) cases of a metallic ratio $\Omega = [\bar{a}]$ and a metallic-colored ratio $\Omega = [\bar{a}, \bar{b}]$ introduced in section 1.3.

Metallic ratios. Let us write, for a given $\Omega = [\bar{a}]$, $a \geq 1$, the matrix $T = T(\Omega)$ and the eigenvalue $\lambda = \lambda(\Omega)$, as deduced from Proposition 5(a), and the matrix $U = U(\Omega)$ from (22),

$$(32) \quad T = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & -1 \\ -1 & a \end{pmatrix}, \quad \lambda = \frac{1}{\Omega} = a + \Omega.$$

We also have from (26) the quadratic equation

$$(33) \quad \lambda^2 = a\lambda + 1.$$

By Lemma 6, all resonant convergents $v(j)$ belong to a unique resonant sequence, whose primitive vector is $v(1) = (-1, 1)$ if $a = 1$, and $v(0) = (0, 1)$ if $a \geq 2$. We deduce from Lemma 8 that this resonant sequence provides the primary resonances: in both cases $\hat{q} = 1$ and hence $s_0(n) = s(1, n)$. In the next result, we compute the separation B_0 , defined in (30), for *all* metallic ratios, providing in this way a sharp lower bound for the normalized numerators of *all* the essential secondary resonances.

Proposition 9. *Let $\Omega = [\bar{a}]$, $a \geq 1$, be a metallic ratio. Then the sequence of primary resonances is generated by the primitive integer $\hat{q} = 1$, and we have*

$$B_0 = \tilde{\gamma}_{q_1}^* = \begin{cases} 5 & \text{if } a = 1, \\ a & \text{if } a \geq 2 \end{cases} \quad \text{for } q_1 = \begin{cases} 7 & \text{if } a = 1, \\ 3 & \text{if } a = 2, \\ a \pm 1 & \text{if } a \geq 3. \end{cases}$$

Proof. We use the expression (28), taking into account the entries of the matrix T given in (32). For the primary resonances, we have $\delta_1 = \delta^* = 1$, and hence $\gamma^* = \frac{\Omega(1+\Omega)}{1+\Omega^2}$. Dividing (31) by γ^* and using that $\lambda = 1/\Omega$, we get, for the normalized numerators $\tilde{\gamma}_q^* = |\delta_q|$, the following lower bound:

$$\tilde{\gamma}_q^* > \frac{1 + \Omega^2}{2} q - \frac{\Omega}{4}.$$

If $a = 1$ (the golden ratio), one checks that the second essential primitive is $(-4, 7)$ with $\tilde{\gamma}_7^* = 5$, and $\tilde{\gamma}_q^* > 5$ for $q \geq 8$. For $a \geq 2$, assuming that $q > 2/\Omega$, we get $\tilde{\gamma}_q^* > a$. Otherwise, if $q < 2/\Omega$, since $p < q\Omega - 1/2$, we get $p = p^0(q) < 3/2$, i.e., $p = 0$ or $p = 1$. The only essential primitive with $p = 0$ is $(0, 1)$, which gives the primary resonances, and for $p = 1$ we

have an “interval” of primitives $(-1, q)$, with $\frac{a+1}{2} \leq q \leq \frac{3a}{2}$ and $q \neq a$ (we have applied (20) together with the fact that $a < 1/\Omega < a+1$). For such primitives, applying (28) we obtain $\delta_q = q^2 - aq - 1$, a quadratic polynomial in q , which is an increasing function for $q \geq a/2$, with $\delta_{a\pm 1} = \pm a$. This change of sign indicates that $\tilde{\gamma}_q^* = |\delta_q|$ is minimal for $q = a \pm 1$. This argument is valid for $a = 2$ (the silver ratio), but in this case we must exclude $q = a - 1$, which lies outside the interval considered. ■

Metallic-colored ratios. Now, we consider $\Omega = [\overline{a, b}]$, $1 \leq a < b$. Recall that, for $a = 1$, this is called a golden-colored ratio; we see below that our results are somewhat different for this particular case. We have

$$T = \begin{pmatrix} ab+1 & a \\ b & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -b \\ -a & ab+1 \end{pmatrix}, \quad \lambda = \frac{1}{1-a\Omega} = ab+1+a\Omega$$

and, from (26), the quadratic equation

$$(34) \quad \lambda^2 = (ab+2)\lambda - 1.$$

Applying Lemma 6, we see that the resonant convergents $v(j)$ are divided into 2 resonant sequences, whose respective primitive vectors are

$$(35) \quad \begin{aligned} v(1) &= (-1, 1), & v(2) &= (-b, b+1) & \text{if } a &= 1; \\ v(0) &= (0, 1), & v(1) &= (-1, a) & \text{if } a &\geq 2. \end{aligned}$$

By Lemma 8, one of the 2 sequences of resonant convergents provides the primary resonances: $s_0(n) = s(\hat{q}, n)$. We call the vectors in the second sequence the *main secondary resonances* and denote them by $s_1(n) := s(\bar{q}, n)$. Next, we find in Proposition 10 the value of $\tilde{\gamma}_{\bar{q}}^*$, i.e., the limit numerator of the main secondary resonances, and this gives a rigorous upper bound for the separation B_0 . In Conjecture 11 we present numerical evidence of the exact equality $B_0 = \tilde{\gamma}_{\bar{q}}^*$ after checking it for a large number of ratios. We point out that for a given concrete frequency ratio Ω the separation $B_0(\Omega)$ can be rigorously determined since, by the lower bound (31), it is enough to consider the limit numerators $\tilde{\gamma}_q^*$ for a finite number of essential primitive integers q .

We also find in Conjecture 11 the value of the “*second separation*,” i.e., the minimal normalized limit numerator among the essential resonant sequences whose vectors are not resonant convergents:

$$(36) \quad B_1 := \min_{q \in \mathcal{P}_0 \setminus \{\hat{q}, \bar{q}\}} \tilde{\gamma}_q^*.$$

Proposition 10. *Let $\Omega = [\overline{a, b}]$, $1 \leq a < b$, be a metallic-colored ratio. Then the sequences of primary resonances and main secondary resonances are generated, respectively, by primitive integers \hat{q}, \bar{q} given by*

$$\begin{aligned} \hat{q} &= 1, & \bar{q} &= b+1 & \text{if } a &= 1, \\ \hat{q} &= a, & \bar{q} &= 1 & \text{if } a &\geq 2. \end{aligned}$$

In both cases, the separation satisfies

$$1 < B_0 \leq \tilde{\gamma}_{\bar{q}}^* = \frac{b}{a}.$$

Proof. We use (28) in order to determine which primitives (35) generate the sequence of primary resonances. For $a = 1$, we obtain $\delta_1 = -1$ and $\delta_{b+1} = b$. For $a \geq 2$, we obtain $\delta_1 = b$ and $\delta_a = -a$. In both cases, the minimum (in modulus) is $\delta^* = a$, which is reached for $\hat{q} = 1$ if $a = 1$, and $\hat{q} = a$ if $a \geq 2$. Then we have $\bar{q} = b + 1$ if $a = 1$, and $\bar{q} = 1$ if $a \geq 2$, and we obtain $\tilde{\gamma}_{\bar{q}}^* = |\delta_{\bar{q}}/\delta_{\hat{q}}| = b/a$, which provides the upper bound: $B_0 \leq b/a$. We also have the inequality $B_0 > 1$ as a consequence of Lemma 8. ■

Conjecture 11. *In the notation of Proposition 10, the separation and the second separation are exactly*³

$$(37) \quad B_0 = \tilde{\gamma}_{\bar{q}}^* = \frac{b}{a}, \quad B_1 = \begin{cases} b + 4 & \text{if } a = 1, \\ \frac{(a-1)b + a}{a} & \text{if } a \geq 2, \end{cases}$$

which satisfy $1 < B_0 < B_1$.

Justification. Numerically, we can compute B_1 by bounding from below the limit numerators $\tilde{\gamma}_q^*$ for all the essential primitives $q \neq \hat{q}, \bar{q}$ (in view of (31), only a finite number of primitives q have to be considered). We have checked that they all satisfy $\tilde{\gamma}_q^* > b/a$ (at least for all the frequency ratios we have explored), and hence we get $B_0 = b/a$ and $B_1 > B_0$. We also obtain an expression for B_1 , given in (37) separately for the cases $a = 1$ and $a \geq 2$. □

Remark. The numerical explorations allow us to determine accurately the primitive integers q_2 such that $B_1 = \tilde{\gamma}_{q_2}^*$, i.e., giving the minimum in (36):

$$q_2 = \begin{cases} 2 & \text{if } a = 1, b = 2, \\ 3, 9 & \text{if } a = 1, b = 3, \\ 4, 11 & \text{if } a = 1, b = 4, \\ 5, 8, 13 & \text{if } a = 1, b = 5, \\ b + 3 & \text{if } a = 1, b \geq 6, \\ 2b + 3 & \text{if } a = 2, \\ a - 1, ab + a + 1 & \text{if } a \geq 3. \end{cases}$$

In each case, the primitive integers q_2 generate the “third most resonant” sequences among the nonconvergent ones (i.e., after the 2 sequences of resonant convergents). Again, we stress that it is possible to obtain this kind of result thanks to the lower bound (31), which allows us to carry out a finite number of computations for any given ratio Ω .

3. Searching for the asymptotic estimates. In order to provide asymptotic estimates for the splitting, we start with the first order approximation, given by the Poincaré–Melnikov method. Although our main result (Theorem 1) is stated in terms of the splitting function $\mathcal{M}(\theta) = \nabla \mathcal{L}(\theta)$, it is more convenient for us to work with the (scalar) splitting potential $\mathcal{L}(\theta)$, whose first order approximation is given by the Melnikov potential $L(\theta)$.

³The values of B_0 and B_1 have been checked numerically for all golden-colored ratios with $1 = a < b \leq 10^6$ and for all metallic-colored ratios with $2 \leq a < b \leq 10^3$.

In this section, we provide the constructive part of the proof, which amounts to finding, for every sufficiently small ε , the first and the second dominant harmonics of the Fourier expansion of the Melnikov potential $L(\theta)$, with exponentially small asymptotic estimates for their size, given by functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$ in the exponents. As a direct consequence of the arithmetic properties of quadratic ratios, such functions are periodic with respect to $\ln \varepsilon$. We also study, from such arithmetic properties, whether the dominant harmonics are given by primary resonances. This allows us to provide a more complete description of the functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$ in some particular cases (Theorem 2 and Conjecture 3).

The final step in the proof of our main result is considered in section 4. It requires providing bounds for the sum of the remaining terms of the Fourier expansion of $L(\theta)$, ensuring that it can be approximated by its dominant harmonics. Furthermore, to ensure that the Poincaré–Melnikov method (2) predicts correctly the size of the splitting in the singular case $\mu = \varepsilon^r$, one has to extend the results to the Melnikov function $\mathcal{M}(\theta)$ by showing that the asymptotic estimates of the dominant harmonics are large enough to overcome the harmonics of the error term in (2). This step is analogous to the one done in [9] for the case of the golden number $\Omega = [\bar{1}]$ (using the upper bounds for the error term provided in [11]).

3.1. Estimates of the harmonics of the splitting potential. We plug our functions f and h , defined in (9), into the integral (13) and get the Fourier expansion of the Melnikov potential, where the coefficients can be obtained using residues:

$$L(\theta) = \sum_{k \in \mathbb{Z} \setminus \{0\}} L_k \cos(\langle k, \theta \rangle - \sigma_k), \quad L_k = \frac{2\pi |\langle k, \omega_\varepsilon \rangle| e^{-\rho|k|}}{\sinh \left| \frac{\pi}{2} \langle k, \omega_\varepsilon \rangle \right|}.$$

We point out that the phases σ_k are the same as in (9). Using (1) and taking into account the definition of the numerators γ_k in (23), we can present each coefficient $L_k = L_k(\varepsilon)$, $k \in \mathbb{Z} \setminus \{0\}$, in the form

$$(38) \quad L_k = \alpha_k e^{-\beta_k}, \quad \alpha_k(\varepsilon) \approx \frac{4\pi\gamma_k}{|k|\sqrt{\varepsilon}}, \quad \beta_k(\varepsilon) = \rho|k| + \frac{\pi\gamma_k}{2|k|\sqrt{\varepsilon}},$$

where an exponentially small term has been neglected in the denominator of α_k . The most relevant term in this expression is β_k , which gives the exponential smallness in ε of each coefficient, and we will show that α_k provides a polynomial factor. This says that, for any given ε , the smallest exponents $\beta_k(\varepsilon)$ provide the largest (exponentially small) coefficients $L_k(\varepsilon)$ and hence the dominant harmonics. We are going to study the dependence on ε of this dominance.

We start with providing a more convenient expression for the exponents $\beta_k(\varepsilon)$, which shows that the smallest ones are $\mathcal{O}(\varepsilon^{-1/4})$ (this is directly related to the exponents $1/4$ in Theorem 1). We introduce for any given X, Y the function

$$(39) \quad G(\varepsilon; X, Y) := \frac{Y^{1/2}}{2} \left[\left(\frac{\varepsilon}{X} \right)^{1/4} + \left(\frac{X}{\varepsilon} \right)^{1/4} \right],$$

which has its minimum at $\varepsilon = X$, with $G(X; X, Y) = Y^{1/2}$ as the minimum value. Notice

that each function $G(\cdot; X, Y)$ is determined by the point $(X, Y^{1/2})$. Now, we define

$$g_k(\varepsilon) := G(\varepsilon; \varepsilon_k, \tilde{\gamma}_k), \quad \varepsilon_k := \frac{D_0 \tilde{\gamma}_k^2}{|k|^4}, \quad D_0 := \left(\frac{\pi \gamma^*}{2\rho} \right)^2,$$

and the functions $g_k(\varepsilon)$ have their minimum at $\varepsilon = \varepsilon_k$, with the minimal values $g_k(\varepsilon_k) = \tilde{\gamma}_k^{1/2}$. Recall that the asymptotic Diophantine constant $\gamma^* = \gamma_q^*$ and the normalized numerators $\tilde{\gamma}_k = \gamma_k/\gamma^*$ were introduced in (29)–(30). We deduce from (38) that

$$(40) \quad \beta_k(\varepsilon) = \frac{C_0}{\varepsilon^{1/4}} g_k(\varepsilon), \quad C_0 := (2\pi\rho\gamma^*)^{1/2},$$

and hence the lower bound $\beta_k(\varepsilon) \geq \frac{C_0 \tilde{\gamma}_k^{1/2}}{\varepsilon^{1/4}}$.

Since we are interested in obtaining *asymptotic estimates* for the splitting and its transversality, rather than lower bounds, we need to determine for any given ε the first and the second essential dominant harmonics, which can be found among the smallest values $g_k(\varepsilon)$. To this aim it is useful to consider, for a given frequency ratio Ω , the graphs of the functions $g_k(\varepsilon)$ associated to *essential* quasi-resonances $k \in \mathcal{A}_0$ (recall that the notion of “essentiality” has been introduced at the beginning of section 2.2). As an illustration, such graphs are shown in Figure 1(a) for a concrete example (the bronze ratio $\Omega = [\bar{3}]$), using a logarithmic scale for ε . Other examples are shown in Figures 2–3. The periodicity which can be noticed from the graphs can easily be explained from the classification of the integer vectors into resonant sequences (recall their definition in (21)). Indeed, for $k = s(q, n)$ belonging to a concrete resonant sequence, using the approximations for $|s(q, n)|$ and $\gamma_{s(q, n)}$ given by Proposition 7, we obtain the following approximations as $n \rightarrow \infty$:

$$(41) \quad g_{s(q, n)}(\varepsilon) \approx g_{s(q, n)}^*(\varepsilon) := G(\varepsilon; \varepsilon_{s(q, n)}^*, \tilde{\gamma}_q^*), \quad \varepsilon_{s(q, n)} \approx \varepsilon_{s(q, n)}^* := \frac{D_0 (\tilde{\gamma}_q^*)^2}{K_q^4 \lambda^{4n}},$$

which motivates the use of a logarithmic scale. We point out that the graphs shown in Figure 1(a) do not correspond to the true functions $g_{s(q, n)}(\varepsilon)$, but rather to the approximations $g_{s(q, n)}^*(\varepsilon)$, which satisfy the following scaling property:

$$(42) \quad g_{s(q, n+1)}^*(\varepsilon) = g_{s(q, n)}^*(\lambda^4 \varepsilon).$$

This gives, for any resonant sequence, the mentioned periodicity: the graph of $g_{s(q, n+1)}^*$ is a translation of $g_{s(q, n)}^*$, to distance $4 \ln \lambda$. For nonessential resonant sequences, whose vectors do not belong to \mathcal{A}_0 , we see that if $s(q, n) = c s(\bar{q}, n_0 + n)$ with $c > 1$ and $n_0 \geq 0$, then

$$(43) \quad g_{s(q, n)}^*(\varepsilon) = c g_{s(\bar{q}, n_0 + n)}^*(\varepsilon)$$

(see also the remark 3 just before Lemma 8).

In order to study the dependence on ε of the most dominant harmonics, it is useful to study the intersections between the graphs of different functions $g_k^*(\varepsilon)$, since this gives the values of ε at which a change in the dominance may take place. In the next lemma, we

consider the graphs associated to two different quasi-resonances, $k, \bar{k} \in \mathcal{A}$, and we show that only two situations are possible: they do not intersect (which says that one of them always dominates the other one), or they intersect transversely in a unique point (and in this case a unique change in the dominance takes place).

Lemma 12. *Let $k, \bar{k} \in \mathcal{A}$, with $k \neq \bar{k}$, given by $k = s(q, n)$ and $\bar{k} = s(\bar{q}, \bar{n})$, and assume that $\tilde{\gamma}_q^* \leq \tilde{\gamma}_{\bar{q}}^*$. Denoting $Z = (\varepsilon_k^*/\varepsilon_{\bar{k}}^*)^{1/4}$ and $W = (\tilde{\gamma}_{\bar{q}}^*/\tilde{\gamma}_q^*)^{1/2}$, the graphs of the functions $g_k^*(\varepsilon)$ and $g_{\bar{k}}^*(\varepsilon)$ intersect if and only if $Z < 1/W$ or $Z > W$. If so, the intersection is unique and transverse and takes place at $\varepsilon = \varepsilon_k^* \cdot \left(\frac{Z(WZ-1)}{Z-W}\right)^2$.*

Proof. First of all, we show that g_k^* and $g_{\bar{k}}^*$ cannot be the same function. By the definition (39), if $g_k^* = g_{\bar{k}}^*$, then we have $\tilde{\gamma}_q^* = \tilde{\gamma}_{\bar{q}}^*$ and $\varepsilon_k^* = \varepsilon_{\bar{k}}^*$. The latter equality implies that $K_q \lambda^n = K_{\bar{q}} \lambda^{\bar{n}}$. Using the expressions given in Proposition 7, we get the equalities $|r_q z_q| = |r_{\bar{q}} z_{\bar{q}}|$ and $z_q \lambda^n = z_{\bar{q}} \lambda^{\bar{n}}$. We deduce that $|r_{\bar{q}}/r_q| = \lambda^{\bar{n}-n}$, but we have $|r_q|, |r_{\bar{q}}| \in (1/2\lambda, 1/2)$ by the fundamental property (20). This says that $n = \bar{n}$ and hence $z_q = z_{\bar{q}}$. As seen in the remark next to the definition (27), we also get $q = \bar{q}$, which contradicts the assumption $k \neq \bar{k}$.

Now, introducing the variable $\zeta = (\varepsilon/\varepsilon_k^*)^{1/4} > 0$, we define

$$f_1(\zeta) := \frac{1}{2} \left(\zeta + \frac{1}{\zeta} \right) = \frac{g_k^*(\varepsilon)}{(\tilde{\gamma}_q^*)^{1/2}}, \quad f_2(\zeta) := \frac{W}{2} \left(\frac{\zeta}{Z} + \frac{Z}{\zeta} \right) = \frac{g_{\bar{k}}^*(\varepsilon)}{(\tilde{\gamma}_{\bar{q}}^*)^{1/2}},$$

with $W \geq 1$ by hypothesis, and it is clear from the above analysis that we cannot have $W = Z = 1$. It is straightforward to check that the graphs of f_1 and f_2 can intersect only once, transversely, at $\zeta^2 = \frac{Z(WZ-1)}{Z-W}$. Such an intersection occurs if and only if $Z < 1/W$ or $Z > W$. Then we get the result after translating from ζ to the original variable ε . ■

The sequence of primary resonances $s_0(n) = s(\hat{q}, n)$, introduced in section 2.3, plays an important role, since they give the smallest minimum values among the functions $g_k^*(\varepsilon)$, and hence they will provide the most dominant harmonics, at least for ε close to such minima. With this fact in mind, and recalling that $\tilde{\gamma}_{\hat{q}}^* = 1$, we denote

$$(44) \quad \bar{g}_n(\varepsilon) := g_{s_0(n)}^*(\varepsilon) = G(\varepsilon; \bar{\varepsilon}_n, 1) = \frac{1}{2} \left[\left(\frac{\varepsilon}{\bar{\varepsilon}_n} \right)^{1/4} + \left(\frac{\bar{\varepsilon}_n}{\varepsilon} \right)^{1/4} \right],$$

$$(45) \quad \bar{\varepsilon}_n := \varepsilon_{s_0(n)}^* = \frac{D_0}{K_{\hat{q}}^4 \lambda^{4n}}.$$

To study the periodicity with respect to $\ln \varepsilon$, we introduce intervals \mathcal{I}_n , whose “length” (in the logarithmic scale) is $4 \ln \lambda$, centered at $\bar{\varepsilon}_n$, and the left and right “halves” of such intervals,

$$(46) \quad \mathcal{I}_n := [\bar{\varepsilon}'_{n+1}, \bar{\varepsilon}'_n] = \mathcal{I}_n^+ \cup \mathcal{I}_n^-, \quad \mathcal{I}_n^+ := [\bar{\varepsilon}'_{n+1}, \bar{\varepsilon}_n], \quad \mathcal{I}_n^- := [\bar{\varepsilon}_n, \bar{\varepsilon}'_n],$$

where $\bar{\varepsilon}'_n := \sqrt{\bar{\varepsilon}_n \bar{\varepsilon}_{n-1}} = \lambda^2 \bar{\varepsilon}_n$ are the geometric means of the sequence $\bar{\varepsilon}_n$. For a given $n \geq 1$, it is easy to determine the behavior of the functions (44): for $\varepsilon \in \mathcal{I}_n^+$, the value of the function

$\bar{g}_n(\varepsilon)$ decreases from J_1 to 1, the value of the function $\bar{g}_{n+1}(\varepsilon)$ increases from J_1 to J_2 , and we have $\bar{g}_m(\varepsilon) \geq J_2$ if $m \neq n, n+1$ (recall that the values J_1 and J_2 were defined in (14)). A symmetric result holds for $\varepsilon \in \mathcal{I}_n^-$ with the functions $\bar{g}_n(\varepsilon)$ and $\bar{g}_{n-1}(\varepsilon)$ (see the red graphs in Figure 1(a) for an illustration).

3.2. Dominant harmonics of the splitting potential. In this section, we introduce the functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$, appearing in the exponents in Theorem 1, as the first and second minima for any given ε of the values $g_k^*(\varepsilon)$ among the essential quasi-resonances $k \in \mathcal{A}_0$. We study some of the properties of $h_1(\varepsilon)$ and $h_2(\varepsilon)$, which hold for an arbitrary quadratic ratio Ω . In section 3.3, we put emphasis on the dependence of such functions on the continued fraction of the frequency ratio Ω , giving a more accurate description of them for the cases of metallic and metallic-colored ratios, whose arithmetic properties have been studied in section 2.4.

Namely, we provide information on the minimum and maximum values of the functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$ and show that they are piecewise-smooth and $4 \ln \lambda$ -periodic in $\ln \varepsilon$, and we give lower bounds for the number of their corners (i.e., jump discontinuities of the derivative) in any period, say the interval $\mathcal{I}_n = [\bar{\varepsilon}'_{n+1}, \bar{\varepsilon}'_n]$, or rather the semiopen interval $(\bar{\varepsilon}'_{n+1}, \bar{\varepsilon}'_n]$, to avoid repetitions if the endpoints are corners. In fact, such properties are clear from Figures 1–3 for the concrete frequency ratios considered there, but we are going to show that they hold for an arbitrary quadratic ratio Ω .

Previously to this, let us define two functions analogous to $h_1(\varepsilon)$ and $h_2(\varepsilon)$, but taking into account *only the primary* resonances:

$$(47) \quad \bar{h}_1(\varepsilon) := \min_n \bar{g}_n(\varepsilon) = g_{N_1}^*(\varepsilon), \quad \bar{h}_2(\varepsilon) := \min_{n \neq N_1} g_n^*(\varepsilon) = g_{N_2}^*(\varepsilon),$$

with $N_i = N_i(\varepsilon)$. In other words, the two dominant harmonics among the primary resonances correspond to

$$\bar{S}_i = \bar{S}_i(\varepsilon) = s_0(N_i), \quad i = 1, 2.$$

On each concrete interval \mathcal{I}_n (see the definition (46)) one readily sees, from the properties described in the last part of section 3.1, what primary resonances provide the first and second minima: $N_1(\varepsilon) = n$ for $\varepsilon \in \mathcal{I}_n$, and $N_1(\varepsilon) = n \pm 1$ for $\varepsilon \in \mathcal{I}_n^\pm$. It is also clear that the functions $\bar{h}_1(\varepsilon)$ and $\bar{h}_2(\varepsilon)$ are piecewise-smooth and $4 \ln \lambda$ -periodic. In each period, the function $h_1(\varepsilon)$ has exactly 1 corner (at $\bar{\varepsilon}'_n$), and $h_2(\varepsilon)$ has exactly 2 corners (at $\bar{\varepsilon}'_n$ and $\bar{\varepsilon}_n$). Moreover, we have

$$\begin{aligned} \min \bar{h}_1(\varepsilon) &= \bar{h}_1(\bar{\varepsilon}_n) = 1, & \max \bar{h}_2(\varepsilon) &= \bar{h}_2(\bar{\varepsilon}_n) = J_2, \\ \max \bar{h}_1(\varepsilon) &= \min \bar{h}_2(\varepsilon) = \bar{h}_1(\bar{\varepsilon}'_n) = \bar{h}_2(\bar{\varepsilon}'_n) = J_1 \end{aligned}$$

(see also Figure 1(b) for an illustration).

Now, we define the functions $h_i(\varepsilon)$ as the minimal values of the functions $g_k^*(\varepsilon)$ among *all essential* quasi-resonances, and we denote by $S_i = S_i(\varepsilon)$ the integer vectors k at which such minima are reached:

$$(48) \quad \begin{aligned} h_1(\varepsilon) &:= \min_{k \in \mathcal{A}_0} g_k^*(\varepsilon) = g_{S_1}^*(\varepsilon), & h_2(\varepsilon) &:= \min_{k \in \mathcal{A}_0 \setminus \{S_1\}} g_k^*(\varepsilon) = g_{S_2}^*(\varepsilon), \\ h_3(\varepsilon) &:= \min_{k \in \mathcal{A}_0 \setminus \{S_1, S_2\}} g_k^*(\varepsilon) = g_{S_3}^*(\varepsilon). \end{aligned}$$

It is clear that $h_i(\varepsilon) \leq \bar{h}_i(\varepsilon)$ for any ε and $i = 1, 2$. In order to provide an accurate description of the splitting and its transversality, we have to study whether the equality between the above functions can be established for any value of ε , or at least for some intervals of ε . This amounts to studying whether the dominant harmonics can always be found among the primary resonances ($S_i = \bar{S}_i$) or, on the contrary, secondary resonances have to be taken into account.

In fact, the properties described above for the functions $\bar{h}_i(\varepsilon)$ are partially generalized to the functions $h_i(\varepsilon)$ in the next proposition, which corresponds to some parts of the statement of Theorem 1 concerning such functions. Recall that the values J_1 and J_2 were defined in (14).

Proposition 13. *The functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$ are piecewise-smooth, $4 \ln \lambda$ -periodic in $\ln \varepsilon$. In each period, the function $h_1(\varepsilon)$ has at least 1 corner and $h_2(\varepsilon)$ has at least 2 corners. They satisfy for $\varepsilon > 0$ the following bounds:*

$$\min h_1(\varepsilon) = 1, \quad \max h_1(\varepsilon) \leq J_1, \quad \max h_2(\varepsilon) \leq J_2, \quad h_1(\varepsilon) \leq h_2(\varepsilon).$$

The corners of $h_1(\varepsilon)$ are exactly the points $\tilde{\varepsilon}$ such that $h_1(\tilde{\varepsilon}) = h_2(\tilde{\varepsilon})$. The corners of $h_2(\varepsilon)$ are the same points $\tilde{\varepsilon}$, and the points $\hat{\varepsilon}$ where $h_2(\hat{\varepsilon}) = h_3(\hat{\varepsilon})$.

Proof. First of all, it is clear that the functions h_1 and h_2 are $4 \ln \lambda$ -periodic in $\ln \varepsilon$, as we see from the scaling property (42). Then we can restrict ourselves to a concrete interval, say \mathcal{I}_1 . Recalling also that $h_i(\varepsilon) \leq \bar{h}_i(\varepsilon) \leq J_i$ for any ε and $i = 1, 2$, the minimum in the definition (48) of h_i can be restricted to the integer vectors $k = s(q, n) \in \mathcal{A}_0$ such that the graph of the function $g_{s(q, n)}^*$ visits (the interior of) the rectangle $\mathcal{I}_1 \times [1, J_i]$. We are going to show that this is possible only for a finite number of integer vectors.

Indeed, recalling (39), if the graph of a function $G(\varepsilon; X, Y)$ visits $\mathcal{I}_1 \times [1, J_1]$, then $Y^{1/2} < J_1$ and $\bar{\varepsilon}_2 < X < \bar{\varepsilon}_0$; and if it visits $\mathcal{I}_1 \times [1, J_2]$, then $Y^{1/2} < J_2$ and $\bar{\varepsilon}'_3 < X < \bar{\varepsilon}'_0$. For the function $g_{s(q, n)}^*$, defined in (41), we have to consider $Y = \tilde{\gamma}_q^*$ and $X = \varepsilon_{s(q, n)}^*$. By the lower bound (31), it is clear that only a finite number of functions $g_{s(q, n)}^*$ can visit the rectangle $\mathcal{I}_1 \times [1, J_i]$, $i = 1, 2$. This implies, by Lemma 12, that only a finite number of (transverse) intersections between the graphs of $g_{s(q, n)}^*$ can take place inside the rectangles $\mathcal{I}_1 \times [1, J_i]$.

We deduce from the above considerations that the functions h_1 and h_2 are both piecewise-smooth. Indeed, we can consider a partition of \mathcal{I}_1 into subintervals such that, for ε belonging to (the interior of) each subinterval, the function h_1 coincides with only one of the functions $g_{s(q, n)}^*$, i.e., the dominant harmonic is given by $S_1(\varepsilon) = s(q, n)$, which remains constant on this subinterval. At each endpoint of such subintervals, a change in the dominant harmonic takes place, i.e., S_1 has a jump discontinuity. By Lemma 12, the endpoints of the subintervals correspond to transverse intersections between the graphs of different functions $g_{s(q, n)}^*$, which give rise to corners $\tilde{\varepsilon}$ of h_1 . A similar argument applies to the function h_2 , with a different partition, associated to the changes of the second dominant harmonic $S_2(\varepsilon)$. In fact, the values $\tilde{\varepsilon}$ are the points where $h_1(\tilde{\varepsilon}) = h_2(\tilde{\varepsilon})$, and they are corners of both functions h_1 and h_2 . In the same way, the function h_2 has additional corners $\hat{\varepsilon}$ at the points where $h_2(\hat{\varepsilon}) = h_3(\hat{\varepsilon})$.

Finally, we provide a lower bound for the number of corners $\tilde{\varepsilon}$, $\hat{\varepsilon}$ in a given period (if the endpoints of a period are corners, we count them as one single corner). Since $\bar{g}_1(\bar{\varepsilon}_1) = 1$, we have $S_1(\varepsilon) = \bar{S}_1(\varepsilon) = s_0(1)$ in some neighborhood of $\bar{\varepsilon}_1 \in \mathcal{I}_1$. Analogously, we have $S_1(\varepsilon) = \bar{S}_1(\varepsilon) = s_0(2)$ in some neighborhood of $\bar{\varepsilon}_2 \in \mathcal{I}_2$, which implies the existence of at least

one corner of h_1 with $\bar{\varepsilon}_2 < \tilde{\varepsilon} < \bar{\varepsilon}_1$ and, consequently, in any given period. On the other hand, if $\tilde{\varepsilon} < \bar{\varepsilon}$ are two consecutive corners of h_1 (and h_2), there exists at least one additional corner $\hat{\varepsilon}$ of h_2 (and h_3), since $S_1(\varepsilon)$ and $S_2(\varepsilon)$ cannot be simultaneously constant in the interval $[\tilde{\varepsilon}, \bar{\varepsilon}]$ (this would imply that $g_{S_1}^*$ and $g_{S_2}^*$ intersect at both points $\tilde{\varepsilon}$ and $\bar{\varepsilon}$, which is not possible by Lemma 12). ■

Remarks.

1. We can also deduce from the proof of this proposition some useful properties of the functions $S_i = S_i(\varepsilon)$, giving the dominant harmonics. Namely, each function $S_i(\varepsilon)$ is “piecewise-constant,” with jump discontinuities at the corners of $h_i(\varepsilon)$. Moreover, the asymptotic behavior of the functions $S_i(\varepsilon)$ as $\varepsilon \rightarrow 0$ turns out to be polynomial:

$$(49) \quad |S_i(\varepsilon)| \sim \frac{1}{\varepsilon^{1/4}}.$$

Indeed, the first dominant harmonic belongs to some resonant sequence: we can write $S_1(\varepsilon) = s(q, N)$ for some $q = q(\varepsilon)$, and for $N = N(\varepsilon)$ such that the value $\varepsilon_{s(q, N)}^*$ is the closest to ε among the sequence $\varepsilon_{s(q, n)}^*$, $n \geq 0$. Recalling (41) and the estimate $|s(q, N)| \sim \lambda^N$ given in Proposition 7(a), we get (49). An analogous argument holds for $S_2(\varepsilon)$, possibly replacing N by $N \pm 1$, and possibly belonging to a different resonant sequence $s(q, \cdot)$. Notice that it is not necessary to include q in the estimate (49) (in spite of the fact that K_q and $\tilde{\gamma}_q^*$ appear in the expression (41)), since by the arguments in the above proof (Proposition 13) only a finite number of resonant sequences $s(q, \cdot)$ can be involved.

2. A more careful look at the arguments of the previous remark says that if the dominant harmonics in a given interval \mathcal{I}_n are known, then in view of the scaling property (42) the dominant harmonics in the interval \mathcal{I}_{n+1} are the next vectors in the respective resonant sequences:

$$S_i(\varepsilon) = U S_i(\lambda^4 \varepsilon)$$

(recall that the matrix U appears in the definition of the resonant sequences in (21)).

3. Although we implicitly assume that there exists only one sequence of primary resonances (see remark 2 before Lemma 8), it is not hard to adapt the definitions and results to the case of two or more sequences of primary resonances. In this case, we would choose one of such sequences as “the” sequence $s_0(n)$ when the functions $\bar{g}_n(\varepsilon)$ are defined in (44). As an example, for the ratio $\Omega = [1, 2, 2]$ considered in Proposition 4, we show in Figure 3 the graphs of $h_1(\varepsilon)$ and $h_2(\varepsilon)$, with two sequences of primary resonances.

3.3. Dominant harmonics for metallic and metallic-colored ratios. This section is devoted to the proof of Theorem 2 and the justification of Conjecture 3, providing a more accurate description of the functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$ for the cases of 1-periodic and 2-periodic continued fractions, i.e., for metallic and metallic-colored ratios Ω , introduced in section 1.3, using some arithmetic results from section 2.4. We emphasize the different behavior of the two functions in each case.

The main issue is to discuss whether the first dominant harmonic $S_1(\varepsilon)$ and (eventually) the second one $S_2(\varepsilon)$ are given, for any ε , by primary resonances: $S_i(\varepsilon) = \bar{S}_i(\varepsilon)$. If so, the

function $h_1(\varepsilon)$ and (eventually) the function $h_2(\varepsilon)$ coincide with the functions $\bar{h}_i(\varepsilon)$ introduced in (47), whose description is very simple (as in Figure 1). Otherwise, the dominant harmonics are given by secondary resonances at least for some intervals of ε , which leads to a more complicated function $h_2(\varepsilon)$ (than $\bar{h}_2(\varepsilon)$, as in Figure 2(a)) or both complicated functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$ (as in Figures 2(b) and 3).

The proof of such results requires a careful analysis of the role of secondary resonances and is carried out rigorously for the statements of Theorem 2. Instead, for the statements of Conjecture 3, we provide evidence after having checked them numerically for a large number of frequency ratios.

Proof of Theorem 2(a)/Justification of Conjecture 3(a). Both of these results concern the behavior of $h_1(\varepsilon)$ for a metallic ratio or a golden-colored ratio. In the first case we provide a rigorous proof, and in the second case the result relies on Conjecture 11, which has been validated numerically for a large number of cases.

It will be enough to show the lower bound

$$(50) \quad \sqrt{B_0} > J_1,$$

where B_0 is the separation between the primary and the essential secondary resonances (recall the definition (30)). This lower bound ensures that the most dominant harmonic is found, for all ε , among the primary resonances: $S_1(\varepsilon) = \bar{S}_1(\varepsilon)$, and hence $h_1(\varepsilon) = \bar{h}_1(\varepsilon)$ (such facts are reflected in Figures 1(b) and 2(a)), which completes the proof, in view of the properties of the function \bar{h}_1 defined in (47).

Hence, it remains to show that the inequality (50) is fulfilled in the two cases of a metallic and a golden-colored ratio. Notice that, by the definition of J_1 in (14), we can rewrite (50) as $4B_0\lambda > (\lambda + 1)^2$.

For a metallic ratio $\Omega = [\bar{a}]$, $a \geq 1$, we know from Proposition 9 that $B_0 = 5$ if $a = 1$, and $B_0 = a$ if $a \geq 2$ (a rigorous result). Then the inequality (50) is easily checked using the quadratic equation (33).

On the other hand, for a golden-colored ratio $\Omega = [\bar{1}, \bar{b}]$, $b \geq 2$, by Conjecture 11 we have $B_0 = b$ (which has been established for $2 \leq b \leq 10^6$). Then it is easy to check the inequality (50) using in this case the quadratic equation (34). ■

Proof of Theorem 2(b). Let us consider a golden-colored ratio $\Omega = [\bar{1}, \bar{b}]$, $b \geq 2$. We know from Proposition 10 that the primary resonances $s_0(n) = s(\hat{q}, n)$ and the main secondary resonances $s_1(n) = s(\bar{q}, n)$ are generated, respectively, by $\hat{q} = 1$ and $\bar{q} = b + 1$ or, equivalently, by the vectors $v(1) = (-1, 1)$ and $v(2) = (-b, b + 1)$. To study the relative position of the graphs of the functions $g_{s_1(n)}^*$ with respect to the functions $\bar{g}_n = g_{s_0(n)}^*$, we compute

$$\frac{\varepsilon_{s_1(n)}^*}{\bar{\varepsilon}_n} = \frac{(\tilde{\gamma}_{b+1}^*)^2 K_1^4}{K_{b+1}^4} = \frac{b^2(\lambda - 1)^4}{(b\lambda)^4} = \frac{1}{\lambda^2},$$

where we have used the quadratic equation (34) and the fact that, by (27),

$$\frac{K_{b+1}}{K_1} = \frac{z_{b+1}}{z_1} = \frac{b(b+1) + b\Omega}{b + \Omega} = \frac{b\lambda}{\lambda - 1}.$$

Hence, we have seen that $\varepsilon_{s_1(n)}^* = \bar{\varepsilon}_n / \lambda^2 = \bar{\varepsilon}'_{n-1}$, i.e., the geometric means introduced in (46) (see also Figure 2(a)).

Now, let us check that

$$(51) \quad \sqrt{B_0} J_1 < J_2.$$

We know from Proposition 10 that $B_0 \leq b$ (a rigorous result). Then it is enough to see that $b\lambda(\lambda+1)^2 < (\lambda^2+1)^2$, which can be easily checked using again the quadratic equation (34).

Notice that $g_{s_1(n)}(\bar{\varepsilon}_n) = g_{s_1(n+1)}(\bar{\varepsilon}_n) = \sqrt{B_0} J_1$. We deduce from (51) that, for some interval around $\bar{\varepsilon}_n$, the second dominant harmonic S_2 is not the primary resonance $\bar{S}_2 = s_0(n-1)$ (if $\varepsilon > \bar{\varepsilon}_n$) or $\bar{S}_2 = s_0(n+1)$ (if $\varepsilon < \bar{\varepsilon}_n$), since at least a main secondary resonance is more dominant: $s_1(n)$ (if $\varepsilon > \bar{\varepsilon}_n$) or $s_1(n+1)$ (if $\varepsilon < \bar{\varepsilon}_n$) (eventually, another secondary resonance could also be the second dominant harmonic S_2). This implies that at least 3 changes in the second dominant harmonic take place in a given period, and hence the function h_2 has at least 3 corners. We also deduce that the maximum value of the function h_2 is $< J_2$, since the value J_2 can only be reached, at the points $\bar{\varepsilon}_n$, if a primary resonance is the second dominant there (again, see Figure 2(a) for an illustration, where h_2 has 4 corners in each period).

Concerning the minimum of the function h_2 , it is always reached at the points $\bar{\varepsilon}'_n$ by a primary resonance: $\bar{g}_n(\bar{\varepsilon}'_n) = \bar{g}_{n-1}(\bar{\varepsilon}'_n) = J_1$, since by the inequality (50) all secondary resonances take greater values at $\bar{\varepsilon}'_n$. ■

Proof of Theorem 2(c). Now, we consider a metallic-colored but not golden-colored ratio, $\Omega = [a, b]$, $2 \leq a < b$. In this case, we know from Proposition 10 that the primary and main secondary resonances, $s_0(n)$ and $s_1(n)$, are generated, respectively, by $\hat{q} = a$ and $\bar{q} = 1$ or, equivalently, by the vectors $v(1) = (-1, a)$ and $v(0) = (0, 1)$. As in part (c), we study the relative position of the functions $g_{s_1(n)}^*$ with respect to \bar{g}_n , by computing

$$\frac{\varepsilon_{s_1(n)}^*}{\bar{\varepsilon}_n} = \frac{(\tilde{\gamma}_1^*)^2 K_a^2}{K_1^4} = \frac{(b/a)^2 (\lambda-1)^4}{b^4} = \lambda^2,$$

where we have used the quadratic equation (34) and the fact that

$$\frac{K_a}{K_1} = \frac{z_a}{z_1} = \frac{ab + a\Omega}{b} = \frac{\lambda-1}{b}.$$

Hence, we have seen that $\varepsilon_{s_1(n)}^* = \lambda^2 \bar{\varepsilon}_n = \bar{\varepsilon}'_n$ (see also Figure 2(b)).

Next, we show that, instead of (50), we have

$$\sqrt{B_0} < J_1$$

or, equivalently, $4B_0\lambda < (\lambda+1)^2$. We know from Proposition 10 that $B_0 \leq b/a$ (a rigorous result). Then it is enough to see that $4b\lambda < a(\lambda+1)^2$, which can be checked using again the quadratic equation (34).

We deduce that, for some interval around $\bar{\varepsilon}'_n$, the most dominant harmonic is not a primary resonance, $S_1 \neq \bar{S}_1$, since at least the secondary resonance $s_1(n)$ is more dominant. This implies that at least 2 changes in the dominance take place in a given period, and

hence the function h_1 has at least 2 corners. We also deduce that the maximum value of the function h_1 is $< J_1$, since the value J_1 can only be reached at the points $\bar{\varepsilon}'_n$, provided a primary resonance is the most dominant there (again, see Figure 2(b) for an illustration). ■

Justification of Conjecture 3(b). We consider a metallic ratio $\Omega = [\bar{a}]$, $a \geq 1$, and we are going to show that for any ε the second dominant harmonic is also a primary resonance, $S_2(\varepsilon) = \bar{S}_2(\varepsilon)$, and hence $h_2(\varepsilon) = \bar{h}_2(\varepsilon)$ (see Figure 1 for an illustration). Then it is enough to use the simple properties of the function \bar{h}_2 defined in (47).

Thus, we have to check that a secondary resonance cannot be the second dominant harmonic in any interval of ε . By the periodicity, we can restrict ourselves to primitive vectors: $s(q, 0) = k^0(q)$. The function $g_{s(q,0)}^*$ reaches its minimum at the point $\varepsilon_{s(q,0)}^*$, belonging for some $n = n(q)$ to one of the intervals $\mathcal{I}_n = \mathcal{I}_n^+ \cup \mathcal{I}_n^-$ (see the definition (46)). Assume, for instance, that $\varepsilon_{s(q,0)}^* \in \mathcal{I}_n^+ = [\bar{\varepsilon}'_{n+1}, \bar{\varepsilon}_n]$. In this interval, the two dominant harmonics among the primary resonances are $\bar{S}_1 = s_0(n) = S_1$ and $\bar{S}_2 = s_0(n+1)$. By the inequality (50), the second dominant harmonic among all resonances is $S_2 = \bar{S}_2$, at least for $\varepsilon \in \mathcal{I}_n^+$ close enough to $\bar{\varepsilon}'_{n+1}$, and we have to check that this is also true on the whole interval \mathcal{I}_n^+ . Otherwise, assume that $S_2 = s(q, 0)$ (a secondary resonance) for some values $\varepsilon \in \mathcal{I}_n^+$ (far from $\bar{\varepsilon}'_{n+1}$). Then there would be an intersection between the graphs of \bar{g}_{n+1} and $g_{s(q,0)}^*$ in the interval \mathcal{I}_{n+1}^+ and, in view of the uniqueness given by Lemma 12, we would have $g_{s(q,0)}^*(\bar{\varepsilon}_n) < \bar{g}_{n+1}(\bar{\varepsilon}_n) = J_2$. A symmetric discussion can be done for the case $\varepsilon_{s(q,0)}^* \in \mathcal{I}_n^-$.

By the above considerations, we have to check that, for any essential primitive q , and denoting $n = n(q)$ as above, we have the lower bound

$$(52) \quad g_{s(q,0)}^*(\bar{\varepsilon}_n) \geq J_2.$$

Since the minimal value of the function $g_{s(q,0)}^*$ is $(\tilde{\gamma}_q^*)^{1/2}$, it is enough to consider the essential primitives such that $(\tilde{\gamma}_q^*)^{1/2} < J_2$ (by the lower bound (31), there is a finite number of such primitives). We have carried out a numerical verification of (52) for all metallic ratios with $1 \leq a \leq 10^4$.

Finally, we have to justify the statement concerning the number of zeros θ^* of the splitting function $\mathcal{M}(\theta)$ for any ε except for a small neighborhood of the transition values $\hat{\varepsilon}$. Notice that since the second dominant harmonic changes from $s_0(n-1)$ to $s_0(n+1)$ as ε goes from \mathcal{I}_n^- to \mathcal{I}_n^+ , the transition values are $\hat{\varepsilon} = \bar{\varepsilon}_n$. As mentioned above, the dominant harmonics $S_1 = \bar{S}_1 = s_0(n)$ and $S_2 = \bar{S}_2 = s_0(n \pm 1)$ are two consecutive resonant convergents, and hence we get $\kappa = 1$ in (69). As explained in section 4.2, this implies directly that the number of zeros of $\mathcal{M}(\theta)$ is exactly $4\kappa = 4$. ■

Remark. The exact equality in (52) holds true for some primitives q , as one can see in Figure 1 for the concrete case of the bronze ratio.

We point out that rigorous results can always be given if one considers a particular frequency ratio, instead of infinite families of ratios. As an illustration, we include in this section the proof of Proposition 4, concerning a frequency ratio with a 3-periodic continued fraction: $\Omega = [1, 2, 2]$.

Proof of Proposition 4. We see from Proposition 5(a), the relation (22), and the equali-

ties (24) that

$$(53) \quad T = \begin{pmatrix} 7 & 3 \\ 5 & 2 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & -5 \\ -3 & 7 \end{pmatrix}, \quad \lambda = \frac{1}{3\Omega - 2} = 7 + 3\Omega.$$

By Lemma 6, the resonant convergents $v(j)$ are divided into 3 resonant sequences, whose respective primitive vectors are

$$(54) \quad v(1) = (-1, 1), \quad v(2) = (-2, 3), \quad v(3) = (-5, 7),$$

and the primary resonances are found among such sequences, according to Lemma 8. We obtain from (28) the values $\delta_1 = -3$, $\delta_3 = 3$, and $\delta_7 = -5$, whose minimum (in modulus) is reached for the first two primitive vectors in (54). Hence, two of the sequences of resonant convergents are primary resonances, $s(1, n)$ and $s(3, n)$, with the limit numerator $\gamma^* = \gamma_1^* = \gamma_3^* = \frac{3\Omega(1+\Omega)}{5+3\Omega^2}$. The third one is secondary, $s(7, n)$, with the normalized limit numerator $\tilde{\gamma}_7^* = \gamma_7^*/\gamma^* = 5/3$. On the other hand, we find from (20) that the nonconvergent (essential) primitive vectors $k^0(q) = (-p, q)$, generating resonant sequences $s(q, n)$, are given by $q = 2, 4, 5, 11, 13, 15, 16, \dots$ with $p = \text{rint}(q\Omega)$. Their normalized limit numerators $\tilde{\gamma}_q^*$ and the amounts K_q can be explicitly obtained from (28)–(30) and Proposition 7(a), respectively.

To determine the functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$, we need to study the relative position between the graphs of different functions $g_{s(q,n)}^*(\varepsilon)$ introduced in (41). To keep the notation used in this paper, we choose one of the sequences of primary resonances as the “main” one, $s_0(n) := s(1, n)$, and we also denote $\bar{g}_n(\varepsilon)$ and $\bar{\varepsilon}_n$ as in (44)–(45) and $\bar{\varepsilon}'_n$ and \mathcal{I}_n as in (46). As in Proposition 13, we can restrict ourselves to a concrete period, say $\mathcal{I}_1 = [\bar{\varepsilon}'_2, \bar{\varepsilon}'_1]$ (containing $\bar{\varepsilon}_1$).

For the sake of simplicity, it is convenient to introduce as in Lemma 12 the variable $\zeta = (\varepsilon/\bar{\varepsilon}_1)^{1/4}$. Then the interval \mathcal{I}_1 becomes $\mathcal{J}_1 := [\lambda^{-1/2}, \lambda^{1/2}]$. In the new variable, we denote $f_{s(q,n)}^*(\zeta) := g_{s(q,n)}^*(\varepsilon)$, and also $\bar{f}_n(\zeta) := \bar{g}_n(\varepsilon)$ for the particular case $q = 1$. It is clear from the scaling property (42) that $f_{s(q,n)}^*(\zeta) = f_{s(q,1)}^*(\lambda^{n-1}\zeta)$. We can write

$$(55) \quad f_{s(q,1)}^*(\zeta) = \frac{W_q}{2} \left(\frac{\zeta}{Z_q} + \frac{Z_q}{\zeta} \right), \quad W_q := (\tilde{\gamma}_q^*)^{1/2}, \quad Z_q := \left(\frac{\varepsilon_{s(q,1)}^*}{\bar{\varepsilon}_1} \right)^{1/4}.$$

Hence, any function $f_{s(q,n)}^*(\zeta)$ has its minimum at $Z_q \lambda^{-(n-1)}$, with W_q as the minimum value. Using the definition (41), together with (27) and the first expression for λ in (53), we get

$$(56) \quad Z_q = (\tilde{\gamma}_q^*)^{1/2} \frac{K_1}{K_q} = (\tilde{\gamma}_q^*)^{1/2} \frac{5+3\Omega}{5q+3p\Omega} = (\tilde{\gamma}_q^*)^{1/2} \frac{7\lambda+1}{(5q+2p)\lambda+p}.$$

Denoting also $F_i(\zeta) := h_i(\varepsilon)$, $i = 1, 2$, our aim is to study such functions for $\zeta \in \mathcal{J}_1$. As a previous step, we can define functions $\bar{F}_i(\zeta)$, $i = 1, 2$, as in (47), but taking the minima over the *two sequences* of primary resonances, $\bar{f}_n(\zeta)$ and $f_{s(3,n)}^*(\zeta)$, i.e., for the primitives $q = 1, 3$. For such primitives, we know that $W_1 = W_3 = 1$ and, computing from (56), we see that $Z_1 = 1$

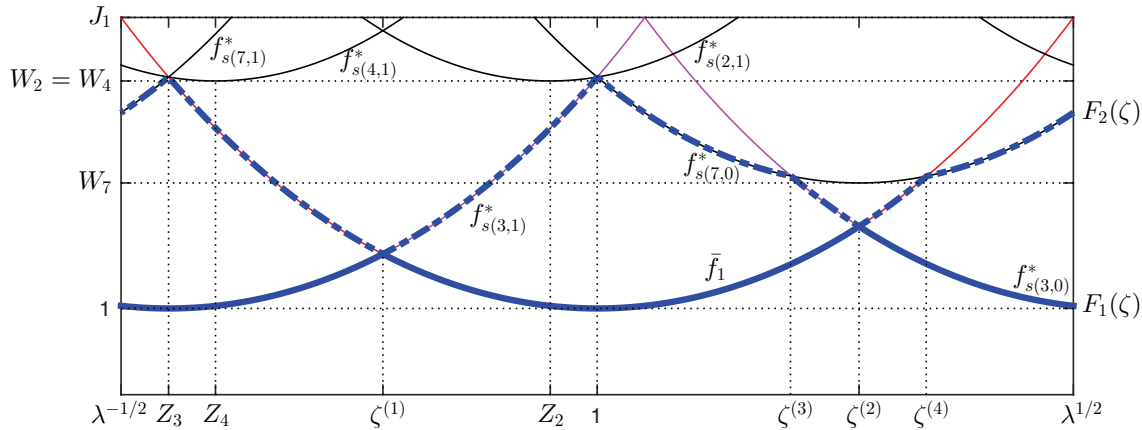


Figure 4. Graphs of the functions \bar{f}_n , $f_{s(q,n)}^*(\zeta)$, $F_1(\zeta)$, and $F_2(\zeta)$ in the rectangle $\mathcal{J}_1 \times [1, J_1]$ for $\Omega = [1, 2, 2]$, using a logarithmic scale for ζ . The relevant values W_i , Z_i are also shown, as well as the corners $\zeta^{(i)}$, which include $\zeta^{(5)} = Z_3$ and $\zeta^{(6)} = 1$ (for the notation, see the proof of Proposition 4).

and $\lambda^{-1/2} < Z_3 < \lambda^{-1/4}$. It is not hard to deduce that both of the functions $\tilde{F}_1(\zeta)$ and $\tilde{F}_2(\zeta)$ are given, for any $\zeta \in \mathcal{J}_1$, by the first and the second minimum among the 3 functions

$$(57) \quad f_{s(3,1)}^*, \quad \bar{f}_1, \quad f_{s(3,0)}^*,$$

having their minima at the points Z_3 , 1 , $Z_3\lambda$, respectively. A careful analysis of the intersections in the interval \mathcal{J}_1 (or rather the semiopen interval $(\lambda^{-1/2}, \lambda^{1/2}]$) between the graphs of the functions (57) (including also the “next” function \bar{f}_0) allows us to determine the corners of the functions $\tilde{F}_i(\zeta)$. By the symmetry of the graphs (in the logarithmic scale), such intersections are given as geometric means. Namely, we see that $\tilde{F}_1(\zeta)$ has 2 corners, at $\zeta^{(1)} := Z_3^{1/2}$ and $\zeta^{(2)} := (Z_3\lambda)^{1/2}$, and $\tilde{F}_2(\zeta)$ has 4 corners, at the same points $\zeta^{(1)}$ and $\zeta^{(2)}$, and also at $\zeta^* := Z_3\lambda^{1/2}$ and $\zeta^{**} := \lambda^{1/2}$. We deduce that $\max \tilde{F}_1(\zeta) = \bar{f}_1(\zeta^{(2)})$, $\min \tilde{F}_2(\zeta) = \bar{f}_1(\zeta^{(1)})$ (where we take into account that $\bar{f}_1(\zeta^{(1)}) < \bar{f}_1(\zeta^{(2)})$), and $\max \tilde{F}_2(\zeta) = f_{s(3,1)}^*(\zeta^*) = \bar{f}_1(\zeta^{**}) = J_1$ (recall the definition (14)). (Such facts are illustrated by Figure 4.)

For the functions $F_i(\zeta) = h_i(\varepsilon)$, we also have to take into account some secondary resonances. Since $F_i(\zeta) \leq \tilde{F}_i(\zeta)$, we can restrict our study to the graphs visiting (the interior of) the rectangle $\mathcal{J}_1 \times [1, J_1]$. Thus, we only need to consider resonant sequences $s(q, n)$ generated by primitive vectors with $W_q < J_1$, with W_q being given by (55). Using the definition (28) together with the lower bound (31) (which allows us to check a finite number of cases), we see that only a few additional primitives have to be considered: $q = 7, 2, 4$; for them, we have $W_7 = \sqrt{5/3}$ and $W_2 = W_4 = \sqrt{7/3}$. It is not hard to check that $W_7 > \max \tilde{F}_1(\zeta)$, which implies that the harmonics associated to secondary resonances cannot provide the minimum among the functions $f_{s(q,n)}^*$, and hence $F_1(\zeta) = \tilde{F}_1(\zeta)$ for any $\zeta \in \mathcal{J}_1$.

It remains to compare $F_2(\zeta)$ with $\tilde{F}_2(\zeta)$. For the secondary primitives $q = 7, 2, 4$, we get $Z_7 = \lambda^{-1}\zeta^{(2)}$ and $\lambda^{-1/2} < Z_4 < \zeta^{(1)} < Z_2 < 1$, and we see that the rectangle $\mathcal{J}_1 \times [1, J_1]$ is

visited by the graphs of the functions

$$f_{s(7,1)}^*, \quad f_{s(7,0)}^*, \quad f_{s(2,1)}^*, \quad f_{s(4,1)}^*, \quad f_{s(4,0)}^*.$$

The intersections between such graphs and those of the functions (57) can be explicitly computed by the formula given in Lemma 12. We can check in this way that the corners $\zeta^{(1)}$ and $\zeta^{(2)}$ of $F_2(\zeta)$ remain unchanged as corners of $\tilde{F}_2(\zeta)$, whereas the corners ζ^* and ζ^{**} are replaced by 4 new corners of $\tilde{F}_2(\zeta)$, which we denote by $\zeta^{(i)}$, $i = 3, 4, 5, 6$ (which become the transition points), provided by intersections of $f_{s(7,1)}^*$ and $f_{s(7,0)}^*$ with the graphs of (57), and we see that $\max F_2(\zeta) < J_1$. We point out that $f_{s(2,1)}^*$ and $f_{s(4,1)}^*$ also intersect the graphs of (57) exactly at two such points, $\zeta^{(5)} = Z_3$ and $\zeta^{(6)} = 1$, but they do not provide the second dominant harmonic in any interval of ζ . (Again, see Figure 4 for more details.)

Finally, it is not hard to check that, in each interval between consecutive transition values $\hat{\varepsilon}$, the determinant of the two vectors $s(q, n)$ providing the 2 dominant harmonics is ± 1 , which implies that the number of zeros θ^* of the splitting function $\mathcal{M}(\theta)$ is exactly 4 (see also section 4). ■

Remark. As in some parts of Theorem 2, it is not hard to provide in this example explicit expressions, in terms of λ (or Ω), for the values of $\max h_1(\varepsilon) = \bar{f}_1(\zeta^{(1)})$, $\min h_2(\varepsilon) = \bar{f}_1(\zeta^{(2)})$ and $\max h_2(\varepsilon) = \bar{f}_1(\zeta^{(5)})$, but they are somewhat cumbersome and we omit them.

4. Justification of the asymptotic estimates.

4.1. Approximation of the splitting potential by its dominant harmonics. The last part of this paper is devoted to the proof of Theorem 1, which gives exponentially small asymptotic estimates of the maximal splitting distance and the transversality of the splitting. We start by describing our approach in a few words.

Notice that Theorem 1 is stated in terms of the Fourier coefficients of the splitting function $\mathcal{M} = \nabla \mathcal{L}$ introduced in (12). We write, for the splitting potential and function,

$$(58) \quad \mathcal{L}(\theta) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \mathcal{L}_k \cos(\langle k, \theta \rangle - \tau_k), \quad \mathcal{M}(\theta) = - \sum_{k \in \mathbb{Z} \setminus \{0\}} \mathcal{M}_k \sin(\langle k, \theta \rangle - \tau_k),$$

with scalar positive coefficients \mathcal{L}_k , and vector coefficients

$$(59) \quad \mathcal{M}_k = k \mathcal{L}_k.$$

Although the Melnikov approximation (2) is in principle valid for real θ , it is standard to see that it can be extended to a complex strip of suitable width (see, for instance, [11]), from which one gets upper bounds for $|\mathcal{L}_k - \mu L_k|$ and $|\tau_k - \sigma_k|$, which imply the asymptotic estimates given below in Lemma 14, ensuring that the most dominant harmonics of the Melnikov potential $L(\theta)$ are also dominant for the splitting potential $\mathcal{L}(\theta)$. The asymptotic estimates for the maximal splitting distance and the transversality, given in Theorem 1, are determined from a few (one or two) dominant harmonics of the potential. Thus, we consider approximations on $\mathcal{L}(\theta)$ given by such dominant harmonics, together with estimates of the sum of all other harmonics, which show that they are dominated by the most dominant ones.

For the proof of part (a) of the theorem, which provides an asymptotic estimate for the maximal splitting distance, it will be enough to consider the approximation given by the first dominant harmonic. Thus, we write

$$(60) \quad \mathcal{L}(\theta) = \mathcal{L}^{(1)}(\theta) + \mathcal{F}^{(2)}(\theta), \quad \mathcal{L}^{(1)}(\theta) := \mathcal{L}_{S_1} \cos(\langle S_1, \theta \rangle - \tau_{S_1}),$$

and we give below, in Lemma 14, an estimate of the sum of all harmonics in the remainder $\mathcal{F}^{(2)}(\theta)$. This ensures that the maximal splitting distance can be approximated by the size of the coefficient of the dominant harmonic $S_1 = S_1(\varepsilon)$ (see the proof of Theorem 1(a) below).

On the other hand, for the proof of parts (b) and (c) of Theorem 1, which concern the transversality of the splitting, we need to detect simple zeros of $\mathcal{M}(\theta)$. This is not possible with the approximation (60) given by only one harmonic, and we need to consider at least two harmonics. Recalling that, by (59), each (vector) harmonic \mathcal{M}_k of $\mathcal{M}(\theta)$ lies in the direction of the integer vector k , we consider the two most dominant *essential* harmonics, given by *linearly independent* quasi-resonances $S_1 = S_1(\varepsilon)$ and $S_2 = S_2(\varepsilon)$ (recall the definition of essential quasi-resonances at the beginning of section 2.2). In fact, some nonessential harmonics cS_1 , $c = 2, \dots, m$, can be (eventually) more dominant than S_2 . In order to show, in section 4.2, that such nonessential harmonics have no effect on the transversality, we consider them separately, with specific upper bounds. We define the *index of nonessentiality* $m = m(\varepsilon) \geq 1$ as the integer satisfying

$$g_{S_1}^*(\varepsilon) < \dots < g_{mS_1}^*(\varepsilon) \leq g_{S_2}^*(\varepsilon) < g_{(m+1)S_1}^*.$$

Recall from (43) that $g_{cS_1}^* = c g_{S_1}^*$. It is clear that $m = 1$ if and only if the two most dominant harmonics are the nonessential ones, S_1 and S_2 . For instance, we see from Figure 2(a) that, for the case $\Omega = [1, 3]$, we have $m = 2$ for ε belonging to some intervals, and $m = 1$ for the remaining values of ε .

Now, we write

$$(61) \quad \begin{aligned} \mathcal{L}(\theta) &= \mathcal{L}^{(2)}(\theta) + \mathcal{F}^{(\widehat{2})}(\theta) + \mathcal{F}^{(3)}(\theta), \\ \mathcal{L}^{(2)}(\theta) &:= \mathcal{L}_{S_1} \cos(\langle S_1, \theta \rangle - \tau_{S_1}) + \mathcal{L}_{S_2} \cos(\langle S_2, \theta \rangle - \tau_{S_2}), \\ \mathcal{F}^{(\widehat{2})}(\theta) &:= \sum_{c=2}^m \mathcal{L}_{cS_1} \cos(c \langle S_1, \theta \rangle - \tau_{cS_1}), \end{aligned}$$

and $\mathcal{F}^{(3)}(\theta)$ containing all harmonics not in $\mathcal{L}^{(2)}(\theta)$ or $\mathcal{F}^{(\widehat{2})}(\theta)$ (of course, we consider $\mathcal{F}^{(\widehat{2})} = 0$ if $m = 1$). Then suitable estimates of the harmonics in $\mathcal{F}^{(\widehat{2})}(\theta)$ and $\mathcal{F}^{(3)}(\theta)$ allow us to establish the existence of simple zeros θ^* of $\mathcal{M}(\theta)$, together with an asymptotic estimate for the minimal eigenvalue of $D\mathcal{M}(\theta^*)$, which can be taken as a measure for the transversality of the splitting (see section 4.2). However, we have to exclude some intervals where the second and third essential dominant harmonics are of the same magnitude and the approximation (61) does not ensure transversality. Such intervals are small neighborhoods of the *transition values* $\widehat{\varepsilon}$, where a change in the second dominant harmonic takes place. Such transition values can be defined as the values where

$$(62) \quad h_2(\widehat{\varepsilon}) = h_3(\widehat{\varepsilon}).$$

We will use the following lemma, analogous to the one established in [8, 9], providing an asymptotic estimate for the dominant harmonics \mathcal{L}_{S_1} and \mathcal{L}_{S_2} (and an upper bound for the difference of their phases τ_{S_i} with respect to the original ones σ_{S_i}), as well as an estimate for the sum of all the harmonics in the remainders $\mathcal{F}^{(i)}$, $i = 2, \widehat{2}, 3$, appearing in (60) and (61). To unify the notation, we write $\mathcal{F}^{(i)} = \sum_{k \in \mathcal{Z}_i}(\cdots)$, defining the sets of indices

$$\mathcal{Z}_2 = \mathcal{Z} \setminus \{0, S_1\}, \quad \mathcal{Z}_{\widehat{2}} = \{2S_1, \dots, mS_1\}, \quad \mathcal{Z}_3 = \mathcal{Z} \setminus (\{0, S_1, S_2\} \cup \mathcal{Z}_{\widehat{2}}).$$

The estimate for each sum is given, due to the exponential smallness of the harmonics, in terms of the dominant harmonic in each set \mathcal{Z}_i , which we denote as $\widetilde{S}_i = \widetilde{S}_i(\varepsilon)$, $i = 2, \widehat{2}, 3$. Notice that, for $i = \widehat{2}$, the dominant harmonic is clearly $\widetilde{S}_{\widehat{2}} = 2S_1$ (nonessential), and for $i = 2, 3$, the dominant harmonic can be either $\widetilde{S}_i = S_i$ (essential) or $\widetilde{S}_i = 2S_{i-1}$ (nonessential). With this in mind, we introduce the functions

$$(63) \quad \begin{aligned} \widetilde{h}_{\widehat{2}}(\varepsilon) &:= 2h_1(\varepsilon) = g_{\widetilde{S}_{\widehat{2}}}^*(\varepsilon), \\ \widetilde{h}_i(\varepsilon) &:= \min(h_i(\varepsilon), 2h_{i-1}(\varepsilon)) = g_{\widetilde{S}_i}^*(\varepsilon), \quad i = 2, 3. \end{aligned}$$

We stress that, in the three cases, the function $\widetilde{h}_i(\varepsilon)$ is given by the minimum of the values $g_k^*(\varepsilon)$, with k belonging to the corresponding set of indices: $\widetilde{h}_i(\varepsilon) = \min_{k \in \mathcal{Z}_i} g_k^*(\varepsilon)$, $i = 2, \widehat{2}, 3$. Compared with the functions $h_i(\varepsilon)$ defined in (48), we see that nonessential harmonics are also taken into account in the definition of $\widetilde{h}_i(\varepsilon)$. Notice also that the equality (62) characterizing the transition values can be rewritten as $h_2(\widehat{\varepsilon}) = \widetilde{h}_3(\widehat{\varepsilon})$.

Recall that the coefficients \mathcal{L}_k , introduced in (58), are all positive. In fact, we are not directly interested in the splitting potential $\mathcal{L}(\theta)$, but rather in some of its derivatives (such as $\mathcal{M}(\theta)$, $\text{DM}(\theta)$). The constant C_0 in the exponentials is the one defined in (40). On the other hand, recall that the meaning of the notation “ \sim ” and “ \preceq ” was introduced at the end of section 1.3.

Lemma 14. *For ε small enough and $\mu = \varepsilon^r$ with $r > 3$, one has*

- (a) $\mathcal{L}_{S_i} \sim \mu L_{S_i} \sim \frac{\mu}{\varepsilon^{1/4}} \exp \left\{ -\frac{C_0 h_i(\varepsilon)}{\varepsilon^{1/4}} \right\}$, $|\tau_{S_i} - \sigma_{S_i}| \preceq \frac{\mu}{\varepsilon^3}$, $i = 1, 2$;
- (b) $\sum_{k \in \mathcal{Z}_i} \mathcal{L}_k \sim \frac{1}{\varepsilon^{1/4}} \mathcal{L}_{\widetilde{S}_i} \sim \frac{\mu}{\varepsilon^{1/4}} \exp \left\{ -\frac{C_0 \widetilde{h}_i(\varepsilon)}{\varepsilon^{1/4}} \right\}$, $i = 2, \widehat{2}, 3$.

Sketch of the proof. We only give the main ideas of the proof, since it is similar to analogous results in [9, Lemmas 4 and 5] and [8, Lemma 3]. At first order in μ , the coefficients of the splitting potential can be approximated, neglecting the error term in the Melnikov approximation (2), by the coefficients of the Melnikov potential, given in (38): $\mathcal{L}_k \sim \mu L_k = \mu \alpha_k e^{-\beta_k}$. As mentioned in section 3.1, the main behavior of the coefficients $L_k(\varepsilon)$ is given by the exponents $\beta_k(\varepsilon)$, which have been written in (40) in terms of the functions $g_k(\varepsilon)$. In particular, the coefficients L_{S_i} associated to the two essential dominant harmonics $k = S_i(\varepsilon)$, $i = 1, 2$, can be expressed in terms of the functions $h_i(\varepsilon)$ introduced in (48). In this way, we obtain an estimate for the factor $e^{-\beta_{S_i}}$, which provides the exponential factor in (a).

We also consider the factor α_k , with $k = S_i(\varepsilon)$. Recalling from (49) that $|S_i| \sim \varepsilon^{-1/4}$, we get from (38) that $\alpha_{S_i} \sim \varepsilon^{-1/4}$, which provides the polynomial factor in part (a).

The estimate obtained is valid for the dominant coefficient of the Melnikov potential $L(\theta)$. To complete the proof of part (a), one has to show that an analogous estimate is also valid for the splitting potential $\mathcal{L}(\theta)$, i.e., when the error term in the Poincaré–Melnikov approximation (2) is not neglected. This requires obtaining an upper bound (provided in [11, Theorem 10]) for the corresponding coefficient of the error term in (2) and showing that, in our singular case $\mu = \varepsilon^r$, it is also exponentially small and dominated by the main term in the approximation. This can be worked out straightforwardly as in [9, Lemma 5] (where the case of the golden number was considered), so we omit the details here.

The proof of part (b) is carried out in similar terms. For the dominant harmonic $k = \tilde{S}_i$ inside each set \mathcal{Z}_i , $i = 2, \hat{2}, 3$, we get $|\tilde{S}_i| \sim \varepsilon^{-1/4}$ as in (49), and an exponentially small estimate for $\mathcal{L}_{\tilde{S}_i}$ with the function $\tilde{h}_i(\varepsilon)$ defined in (63). Such estimates are also valid if one considers the whole sum in (b), since for any given ε the terms of this sum can be bounded by a geometric series, and hence it can be estimated by its dominant term (see [9, Lemma 4] for more details). ■

In regard to the proof of Theorem 1(a),(c), we need to measure the size of each perturbation $\mathcal{F}^{(i)}(\theta)$ in (60)–(61) with respect to the coefficients of the approximations $\mathcal{K}^{(j)}(\theta)$. Since by Lemma 14 the size of $\mathcal{F}^{(i)}(\theta)$ is given by the size of its dominant harmonic, we introduce the small parameters

$$(64) \quad \eta_{i,j} := \frac{\mathcal{L}_{\tilde{S}_i}}{\mathcal{L}_{S_j}} \sim \exp \left\{ -\frac{C_0(\tilde{h}_i(\varepsilon) - h_j(\varepsilon))}{\varepsilon^{1/4}} \right\}, \quad (i,j) = (2,1), (\hat{2},1), (3,1), (3,2),$$

as a measure of the perturbations $\mathcal{F}^{(i)}$ in (60)–(61), relatively to the size of the essential dominant coefficients \mathcal{L}_{S_j} (we consider $\eta_{2,1} = 0$ if the index of nonessentiality is $m = 1$). Although we define the parameters $\eta_{i,j}$ in terms of the coefficients of $\mathcal{L}(\theta)$, we can also define them from the coefficients of its derivatives, such as the splitting function $\mathcal{M}(\theta)$, in view of (59) and the fact that the respective factors have the same magnitude: $|S_j| \sim |\tilde{S}_i|$.

The parameters $\eta_{i,j}$ are always exponentially small in ε , provided we exclude some small neighborhoods where \mathcal{L}_{S_j} and $\mathcal{L}_{\tilde{S}_i}$ can have the same magnitude. For instance, we have $\eta_{3,2}$ exponentially small if ε is not very close to the transition values (62), at which the second and third dominant harmonics have the same magnitude. Analogously, the parameter $\eta_{2,1}$ is exponentially small, excluding neighborhoods where the first and second dominant harmonics have the same magnitude.

Proof of Theorem 1(a). Applying Lemma 14, we see that the coefficient of the dominant harmonic of the splitting function $\mathcal{M}(\theta)$ is greater than the sum of all other harmonics. More precisely, we have for $\varepsilon \rightarrow 0$ the estimate

$$(65) \quad \max_{\theta \in \mathbb{T}^2} |\mathcal{M}(\theta)| = |\mathcal{M}_{S_1}| (1 + \mathcal{O}(\eta_{2,1})) \sim |\mathcal{M}_{S_1}| \sim |S_1| \mathcal{L}_{S_1},$$

which implies the result, using the asymptotic estimate (49) for $|S_1|$, and the asymptotic estimate for $|\mathcal{M}_{S_1}|$, in terms of $h_1(\varepsilon)$, deduced from Lemma 14(a).

We point out that the previous argument does not apply directly when ε is close to a value where h_1 and h_2 coincide; i.e., the first and second dominant harmonics have the same

magnitude (for instance, for a metallic ratio Ω this occurs near the values $\bar{\varepsilon}'_n$; see (46) and Figure 1(b)). Eventually, more than two harmonics (but a finite number, according to the arguments given in Lemma 12) might also have the same magnitude and become dominant. In such cases, the parameter $\eta_{2,1}$ is not exponentially small, but we can replace the main term in (65) by a finite number of terms, plus an exponentially small perturbation, and by the properties of Fourier expansions the maximum value of $|\mathcal{M}(\theta)|$ can be compared to any of its dominant harmonics. ■

4.2. Nondegenerate critical points and transversality. This section is devoted to the study of the transversality of the homoclinic orbits for values of the perturbation parameter ε , not very close to the transition values $\hat{\varepsilon}$ defined in (62). For such values of ε we show that, under suitable conditions, the splitting potential $\mathcal{L}(\theta)$ has 4κ nondegenerate critical points for some integer $\kappa \geq 1$, i.e., the splitting function $\mathcal{M}(\theta)$ has 4κ simple zeros, which give rise to 4κ transverse homoclinic orbits. Such critical points are easily detected in the approximation $\mathcal{L}^{(2)}(\theta)$ introduced in (61), given by the 2 essential dominant harmonics, and using the estimates for $\mathcal{F}^{(2)}(\theta)$ and $\mathcal{F}^{(3)}(\theta)$ given in Lemma 14 we can prove the persistence of the critical points in the whole function $\mathcal{L}(\theta)$.

In fact, we make the computations easier by performing a linear change on \mathbb{T}^2 , taking $\mathcal{L}^{(2)}(\theta)$ to a very simple form. As in [8, 9], we introduce the variables

$$(66) \quad \psi_1 = \langle S_1, \theta \rangle - \tau_{S_1}, \quad \psi_2 = \langle S_2, \theta \rangle - \tau_{S_2}.$$

This change of variables is valid for ε in the interval between two consecutive transition values, in which we have two concrete essential dominant harmonics $S_1(\varepsilon)$ and $S_2(\varepsilon)$, which remain constant in this interval. In the new variables, the functions \mathcal{L} , $\mathcal{L}^{(2)}$, $\mathcal{F}^{(2)}$, $\mathcal{F}^{(3)}$ in (61) become

$$(67) \quad \mathcal{K}(\psi) = \mathcal{K}^{(2)}(\psi) + \mathcal{G}^{(2)}(\psi_1) + \mathcal{G}^{(3)}(\psi),$$

where, in particular, we have

$$(68) \quad \mathcal{K}^{(2)}(\psi) = \mathcal{L}_{S_1} \cos \psi_1 + \mathcal{L}_{S_2} \cos \psi_2.$$

It is clear that $\mathcal{K}^{(2)}$ has the 4 critical points $\psi^{*,0} := (0,0), (\pi,0), (0,\pi), (\pi,\pi)$, all nondegenerate (one maximum, two saddles, and one minimum, respectively). Regarding $\mathcal{K}(\psi)$ as a perturbation of $\mathcal{K}^{(2)}(\psi)$, we are going to show that it also has 4 critical points ψ^* , all nondegenerate, which are close to the critical points $\psi^{*,0}$ of $\mathcal{K}^{(2)}(\psi)$. We point out that, in general, the change (66) is not one-to-one on \mathbb{T}^2 , but rather “ κ -to-one,” where

$$(69) \quad \kappa = \kappa(\varepsilon) := |\det(S_1, S_2)|.$$

Hence, the number of critical points of $\mathcal{L}(\theta)$ is 4κ . It is not hard to show that $\kappa(\varepsilon)$ is $4 \ln \lambda$ -periodic in $\ln \varepsilon$. Moreover, it is “piecewise-constant” with (eventual) jump discontinuities when changes in the dominant harmonics take place.

Remark. For a metallic ratio $\Omega = [\bar{a}]$, we know from Conjecture 3(b) that $\kappa = 1$ (a result checked numerically for $1 \leq a \leq 10^4$), and hence there are exactly 4 transverse homoclinic

orbits for any ε small enough (excluding a neighborhood of the transition values $\widehat{\varepsilon}$). Although for other frequency ratios it is possible, in principle, to have $\kappa \geq 2$, we have obtained $\kappa = 1$ for all the cases we have explored.

To establish the persistence of the critical points, we are going to use the following lemma, whose proof is a simple application of the 2-dimensional fixed point theorem and is omitted here.

Lemma 15. *If $f_1, f_2 : \mathbb{T}^2 \rightarrow \mathbb{R}$ are differentiable and satisfy*

$$f_i^2 + \left(\left| \frac{\partial f_i}{\partial \psi_1} \right| + \left| \frac{\partial f_i}{\partial \psi_2} \right| \right)^2 < 1, \quad i = 1, 2,$$

then the system of equations

$$(70) \quad \sin \psi_1 = f_1(\psi), \quad \sin \psi_2 = f_2(\psi)$$

has exactly 4 solutions ψ^ , which are simple. Furthermore, if $f_1(\psi), f_2(\psi) = \mathcal{O}(\eta)$ for any $\psi \in \mathbb{T}^2$, with η sufficiently small, then the solutions of the system satisfy $\psi^* = \psi^{*,0} + \mathcal{O}(\eta)$, with $\psi^{*,0} = (0, 0), (0, \pi), (\pi, 0), (\pi, \pi)$.*

In order to apply this lemma, we introduce the following perturbation parameter, using the parameters $\eta_{i,j}$ defined in (64),

$$(71) \quad \eta := \max(\eta_{2,1}, \eta_{3,1}, \eta_{3,2}).$$

Lemma 16. *The function $\mathcal{K}(\psi)$ has exactly 4 critical points, all nondegenerate:*

$$(72) \quad \psi^* = \psi^{*,0} + \mathcal{O}(\eta), \quad \text{with } \psi^{*,0} = (0, 0), (0, \pi), (\pi, 0), (\pi, \pi).$$

At each critical point, we have

$$\det D\mathcal{K}(\psi^*) = \delta_1^* \delta_2^* \mathcal{L}_{S_1} \mathcal{L}_{S_2} (1 + \mathcal{O}(\eta)),$$

where $\delta_i^ = \cos \psi_i^{*,0} = \pm 1$, $i = 1, 2$.*

Proof. We see from (67)–(68) that the system of equations $\nabla \mathcal{K}(\psi) = 0$ can be written as in (70), with the functions

$$f_1(\psi) = \frac{1}{\mathcal{L}_{S_1}} \left(\frac{d\mathcal{G}^{(2)}}{d\psi_1} + \frac{\partial \mathcal{G}^{(3)}}{\partial \psi_1} \right), \quad f_2(\psi) = \frac{1}{\mathcal{L}_{S_2}} \cdot \frac{\partial \mathcal{G}^{(3)}}{\partial \psi_2}$$

(notice that $\mathcal{G}^{(2)}$ does not depend on ψ_2). By Lemma 14, we have $f_1(\psi), f_2(\psi) = \mathcal{O}(\eta)$, with η as given in (71). Hence, applying Lemma 15 we deduce the result for the critical points of $\mathcal{K}(\psi)$.

We also provide, for each critical point, an asymptotic estimate for the determinant of $D^2\mathcal{K}(\psi^*)$. It is clear, for the perturbed critical points, that the signs $\delta_i^* = \pm 1$ become perturbed as follows: $\cos \psi_i^* = \delta_i^* + \mathcal{O}(\eta^2)$. Writing

$$D^2\mathcal{K}(\psi) = \begin{pmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{pmatrix},$$

we have from (67) the approximations

$$(73) \quad k_{11} = \frac{\partial^2 \mathcal{K}}{\partial \psi_1^2} = -\mathcal{L}_{S_1} \left(\cos \psi_1 + \mathcal{O}(\eta_{2,1}, \eta_{3,1}) \right),$$

$$(74) \quad k_{12} = \frac{\partial^2 \mathcal{K}}{\partial \psi_1 \partial \psi_2} = \mathcal{L}_{S_1} \cdot \mathcal{O}(\eta_{3,1}) = \mathcal{L}_{S_2} \cdot \mathcal{O}(\eta_{3,2}),$$

$$(75) \quad k_{22} = \frac{\partial^2 \mathcal{K}}{\partial \psi_2^2} = -\mathcal{L}_{S_2} (\cos \psi_2 + \mathcal{O}(\eta_{3,2})),$$

and we deduce the expression for the determinant of $D^2 \mathcal{K}(\psi^*)$. ■

Now we complete the proof of part (b) of our main theorem by applying the inverse of the linear change (66) to the critical points ψ^* of $\mathcal{K}(\psi)$, in order to get the critical points θ^* of the splitting potential $\mathcal{L}(\theta)$, i.e., the zeros of the Melnikov function $\mathcal{M}(\theta)$.

Proof of Theorem 1(b). Since the linear change (66) is “ κ -to-one,” with κ as in (69), the 4 critical points ψ^* of $\mathcal{K}(\psi)$ give rise to 4κ critical points θ^* of $\mathcal{L}(\theta)$. It is clear that such critical points are also nondegenerate, and hence they are simple zeros of the splitting function $\mathcal{M}(\theta)$. ■

Remarks.

1. Recall that the vectors $S_i = S_i(\varepsilon)$ remain constant between consecutive transition values $\widehat{\varepsilon}$ (see the proof of Proposition 13). On the other hand, by (72) the points ψ^* are $\mathcal{O}(\eta)$ -close to the points $\psi^{*,0}$, where η is exponentially small. Hence, the points $\theta^* = \theta^*(\varepsilon)$ remain “nearly constant” along each interval of ε between consecutive transition values $\widehat{\varepsilon}$ and can “change” when ε goes across a value $\widehat{\varepsilon}$.
2. As a particular interesting case, we may consider the phases $\sigma_k = 0$ in the perturbation (9). In this case, our Hamiltonian system given by (4–9) is *reversible* with respect to the involution

$$(76) \quad \mathcal{R} : (x, y, \varphi, I) \mapsto (-x, y, -\varphi, I)$$

(indeed, its associated Hamiltonian field satisfies the identity $X_H \circ \mathcal{R} = -\mathcal{R} X_H$). We point out that reversible perturbations have also been considered in some related papers [18, 20, 36]. Under the reversibility (76), the whiskers are related by the involution: $\mathcal{W}^s = \mathcal{R} \mathcal{W}^u$. Hence, their parameterizations in (12) can be chosen in such a way that $\mathcal{J}^s(\theta) = \mathcal{J}^u(-\theta)$, provided the transverse section $x = \pi$ is considered in their definition. This implies that the splitting function is an odd function, $\mathcal{M}(-\theta) = -\mathcal{M}(\theta)$ (and the splitting potential $\mathcal{L}(\theta)$ is even), and, using its periodicity, one sees that $\mathcal{M}(\theta)$ has at least the following 4 zeros: $\theta^* = (0, 0), (\pi, 0), (0, \pi), (\pi, \pi)$ (notice that they do not depend on ε). Although such zeros could be nonsimple in principle, the result of Theorem 1(b) says that they are simple for any ε except for a small neighborhood of the transition values $\widehat{\varepsilon}$, at which some bifurcations of the zeros could take place.

It remains to provide, for each zero θ^* of the splitting function $\mathcal{M}(\theta)$, an asymptotic estimate for the minimal eigenvalue of the matrix $D^2 \mathcal{L}(\theta^*) = D\mathcal{M}(\theta^*)$ as a measure for the transversality of the splitting.

Proof of Theorem 1(c). Denoting $D = \det D^2\mathcal{L}(\theta^*)$ and $T = \operatorname{tr} D^2\mathcal{L}(\theta^*)$, we can present the (modulus of the) minimal eigenvalue of $D^2\mathcal{L}(\theta^*)$ in the form

$$|m^*| = \frac{2|D|}{|T| + \sqrt{T^2 - 4D}} \sim \frac{|D|}{|T|},$$

where we have taken into account that $0 \leq \sqrt{T^2 - 4D} \leq |T|$. Thus, we need to find estimates for $|D|$ and $|T|$ at the critical points θ^* of $\mathcal{L}(\theta)$ (or zeros of $\mathcal{M}(\theta)$).

By the linear change (66), we have $D^2\mathcal{L}(\theta^*) = A^\top D^2\mathcal{K}(\psi^*) A$, where A is the matrix having the vectors S_1 and S_2 as rows. In (69), we have defined $\kappa = |\det A|$. Since $\kappa = \kappa(\varepsilon)$ is piecewise-constant and periodic in $\ln \varepsilon$, it is bounded from below and from above: $\kappa \sim 1$. Applying Lemma 16, we get the asymptotic estimate

$$|D| = \mathcal{L}_{S_1} \mathcal{L}_{S_2} (1 + \mathcal{O}(\eta)) \sim \mathcal{L}_{S_1} \mathcal{L}_{S_2}.$$

On the other hand, in order to estimate T we write $D^2\mathcal{K}$ as in (73)–(75) and obtain

$$D^2\mathcal{L}(\theta) = k_{11} S_1 \cdot S_1^\top + k_{12}(S_1 \cdot S_2^\top + S_2 \cdot S_1^\top) + k_{22} S_2 \cdot S_2^\top,$$

which implies that $T = k_{11} |S_1|_2^2 + 2k_{12} \langle S_1, S_2 \rangle + k_{22} |S_2|_2^2$, where $|\cdot|_2$ denotes the usual Euclidean norm (which is equivalent to the norm $|\cdot| = |\cdot|_1$ mainly used in this paper). Now we use, at the critical points ψ^* , the estimates for the matrix $D^2\mathcal{K}(\psi^*)$ given in (73)–(75). We obtain $|k_{11}| \sim \mathcal{L}_{S_1}$ as the main entry, and $|k_{12}| \sim \mathcal{L}_{S_1} \cdot \mathcal{O}(\eta_{3,1})$, $|k_{22}| \sim \mathcal{L}_{S_2}$. Applying also the estimate (49), we obtain

$$|T| \sim \frac{1}{\sqrt{\varepsilon}} \mathcal{L}_{S_1}, \quad \text{and hence} \quad m^* \sim \frac{|D|}{|T|} \sim \sqrt{\varepsilon} \mathcal{L}_{S_2}.$$

Applying the estimate for \mathcal{L}_{S_2} given in Lemma 14, we obtain the desired estimate for the minimal eigenvalue. ■

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